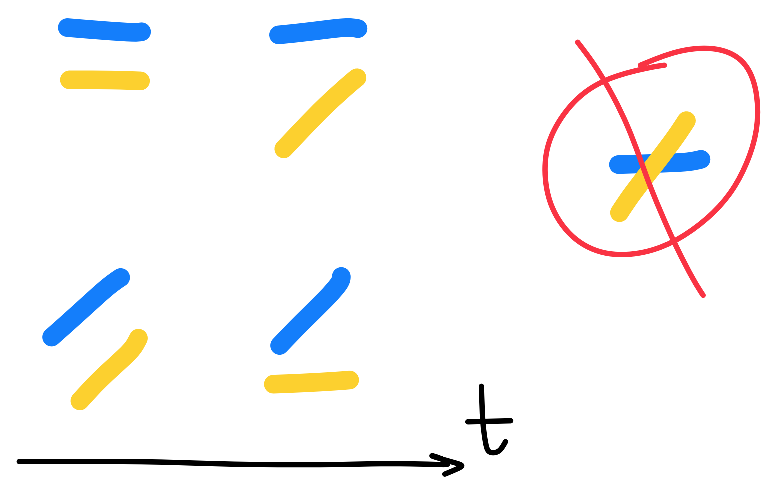
Rewriting History in Integrable Stochastic Particle Systems

Leonid Petrov
University of Virginia

joint with Axel Saenz (Oregon State University), arXiv:2212.01643

A tale of two cars on a one-lane road

Two cars (discrete time TASEP with Bernoulli jumps)



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One-lane highway: no passing. Consider two systems: FS and SF

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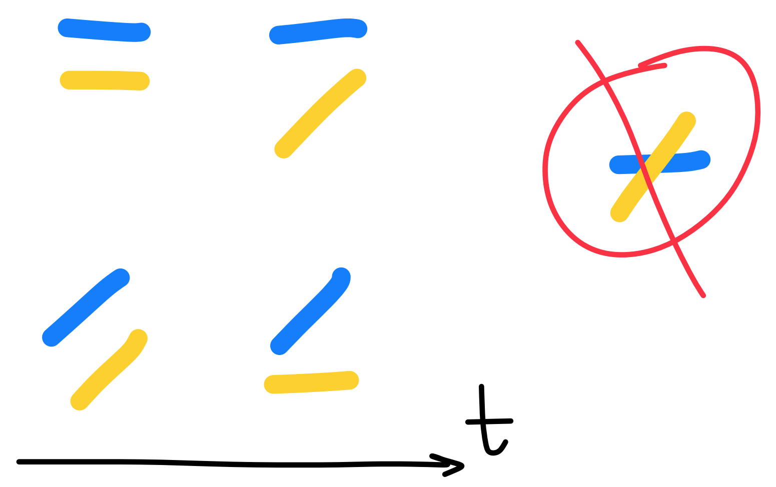

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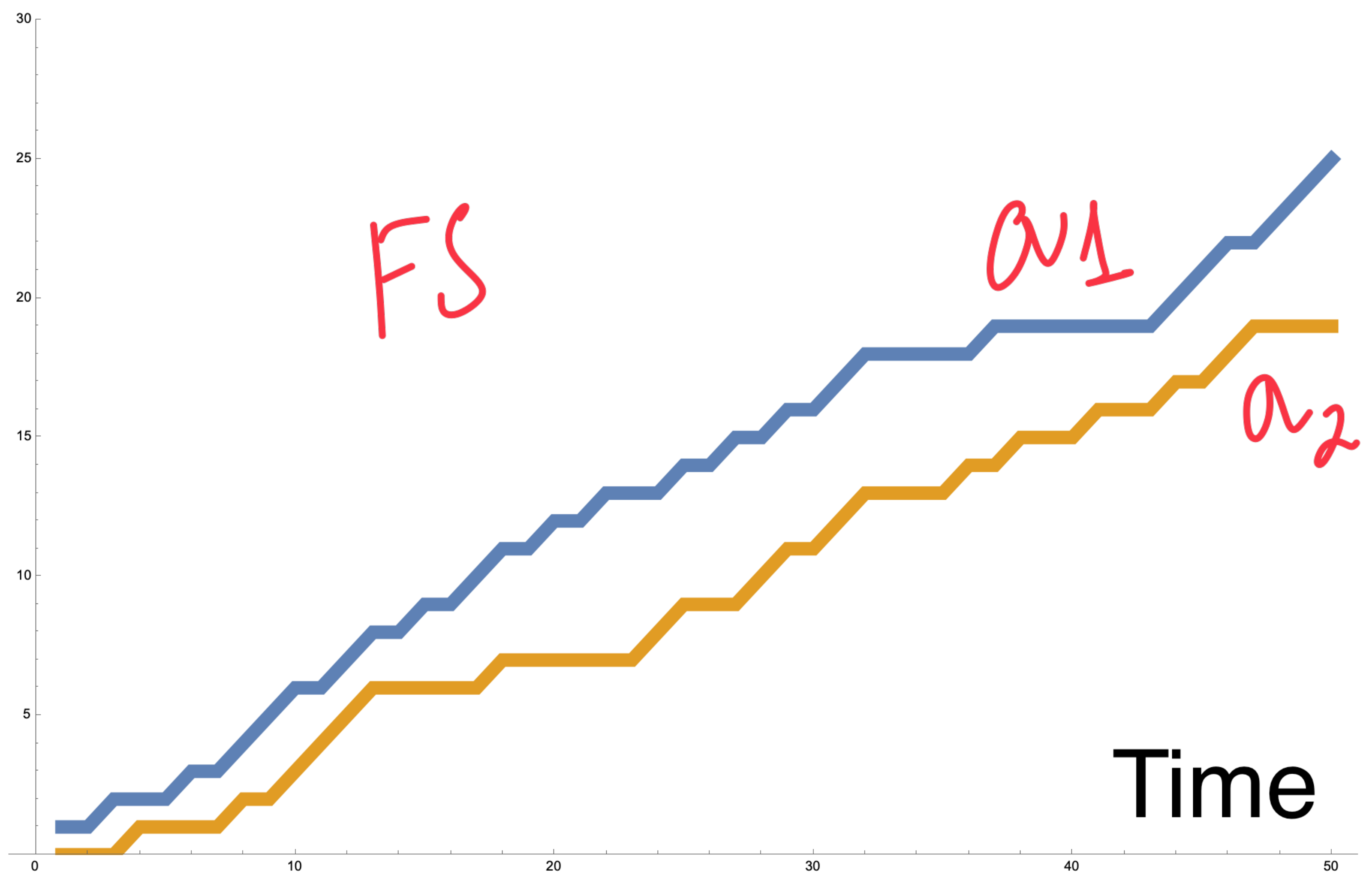

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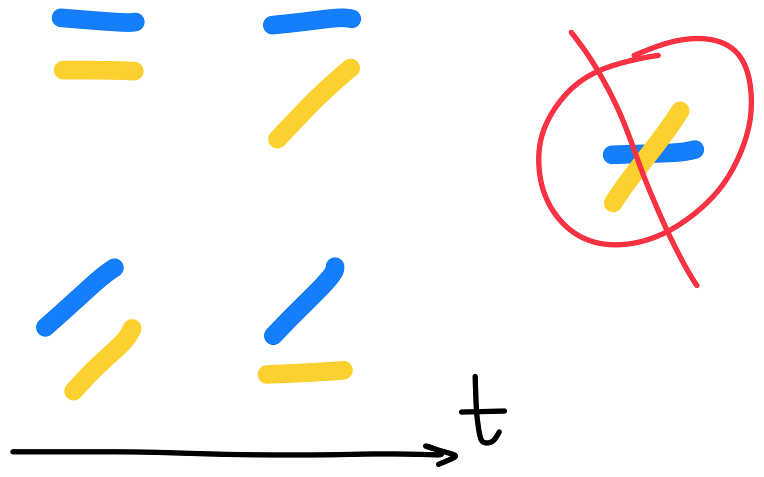

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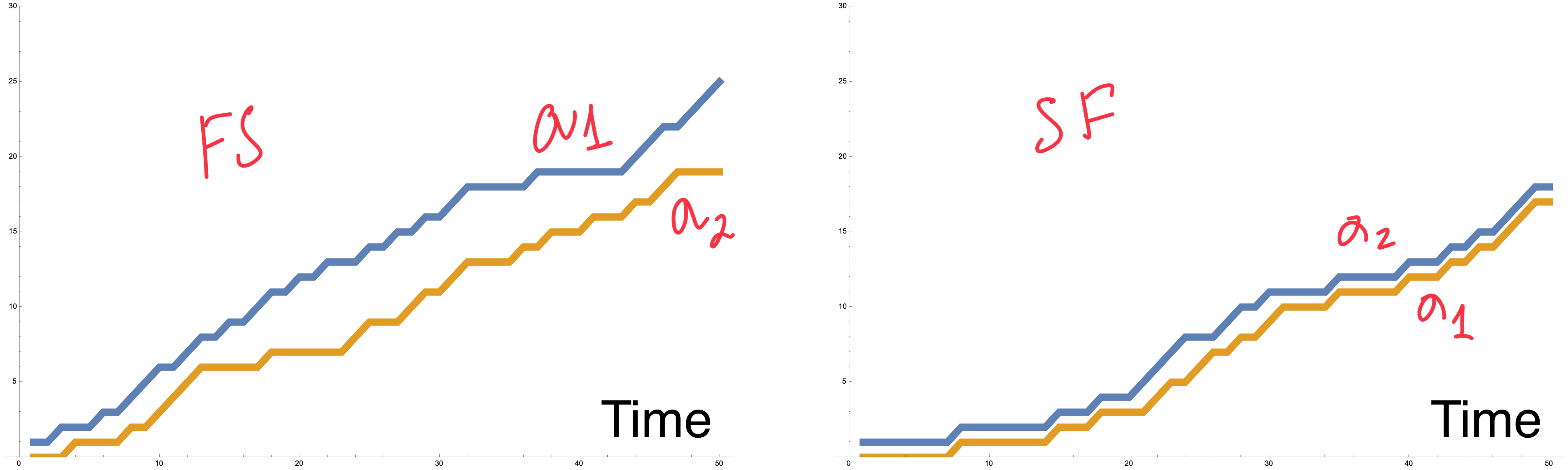

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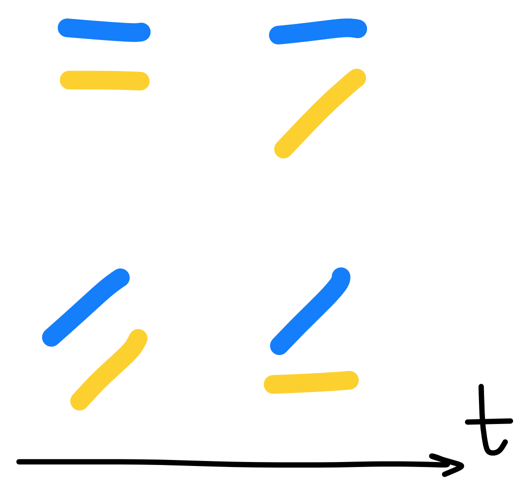

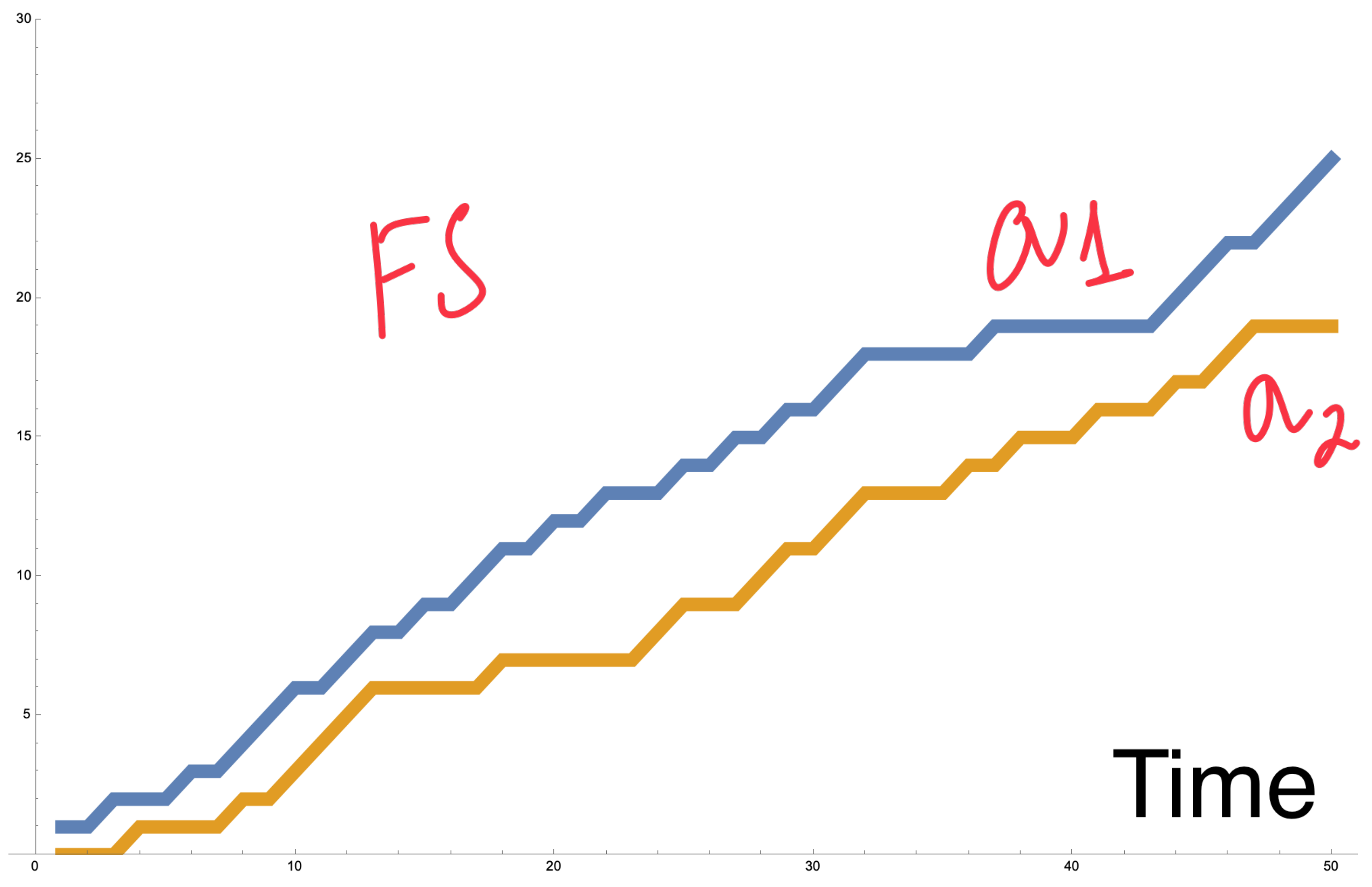

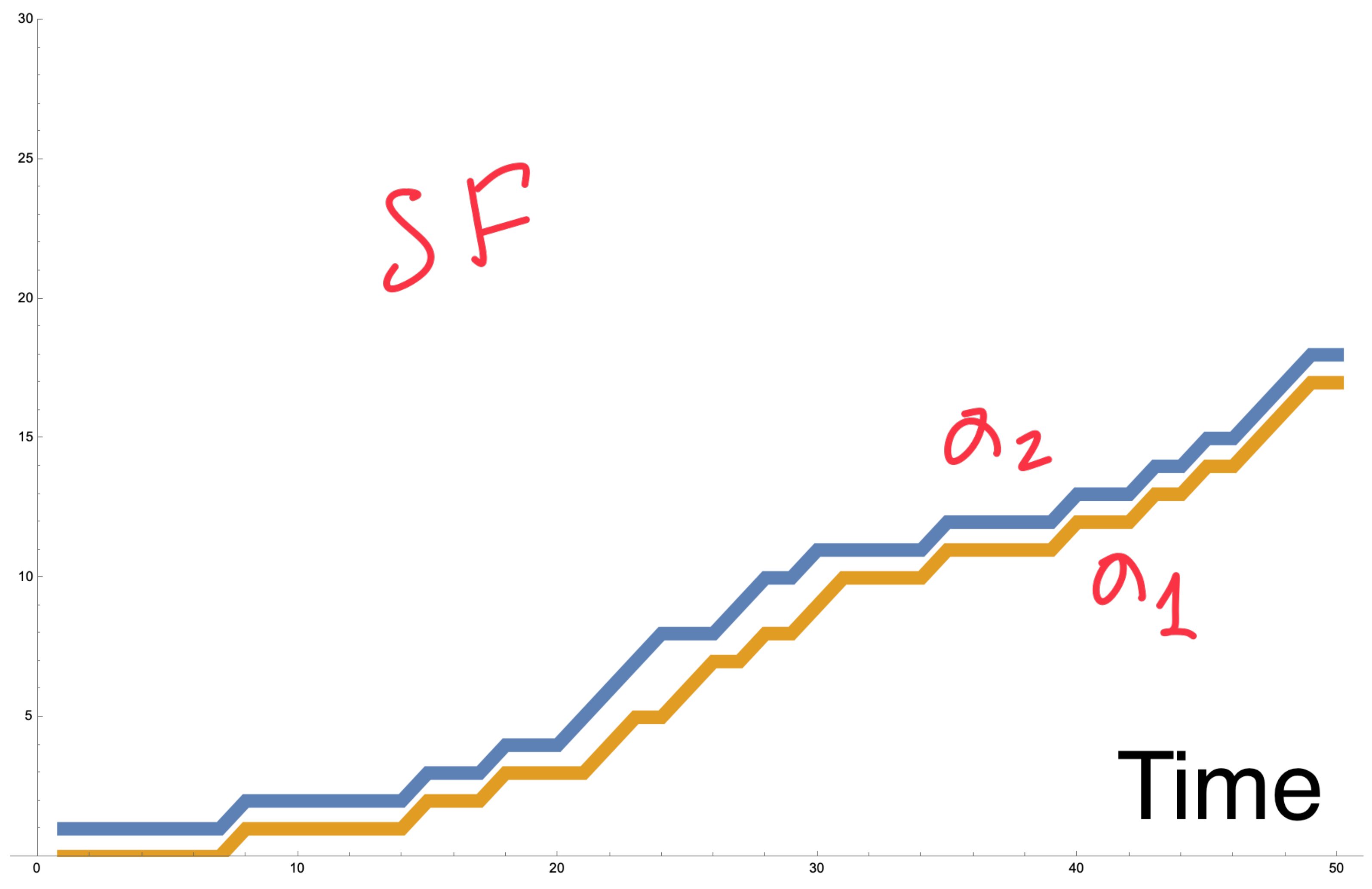

The long-time speed of the car ahead (blue) depends on which car is first;

for the car behind (yellow) it **does not** depend on the order

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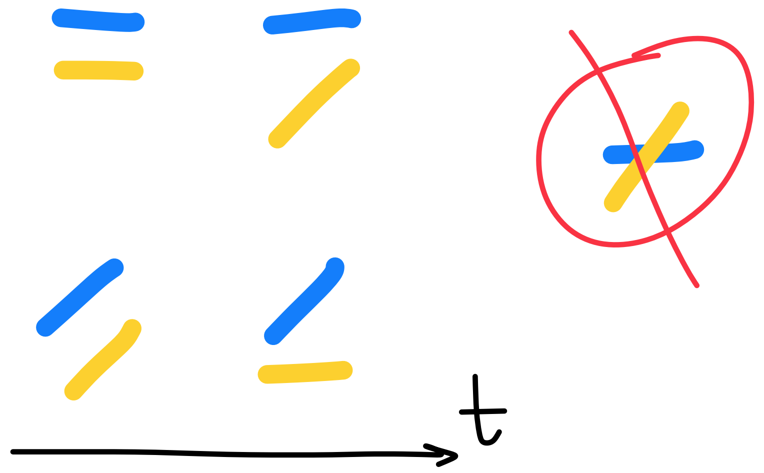

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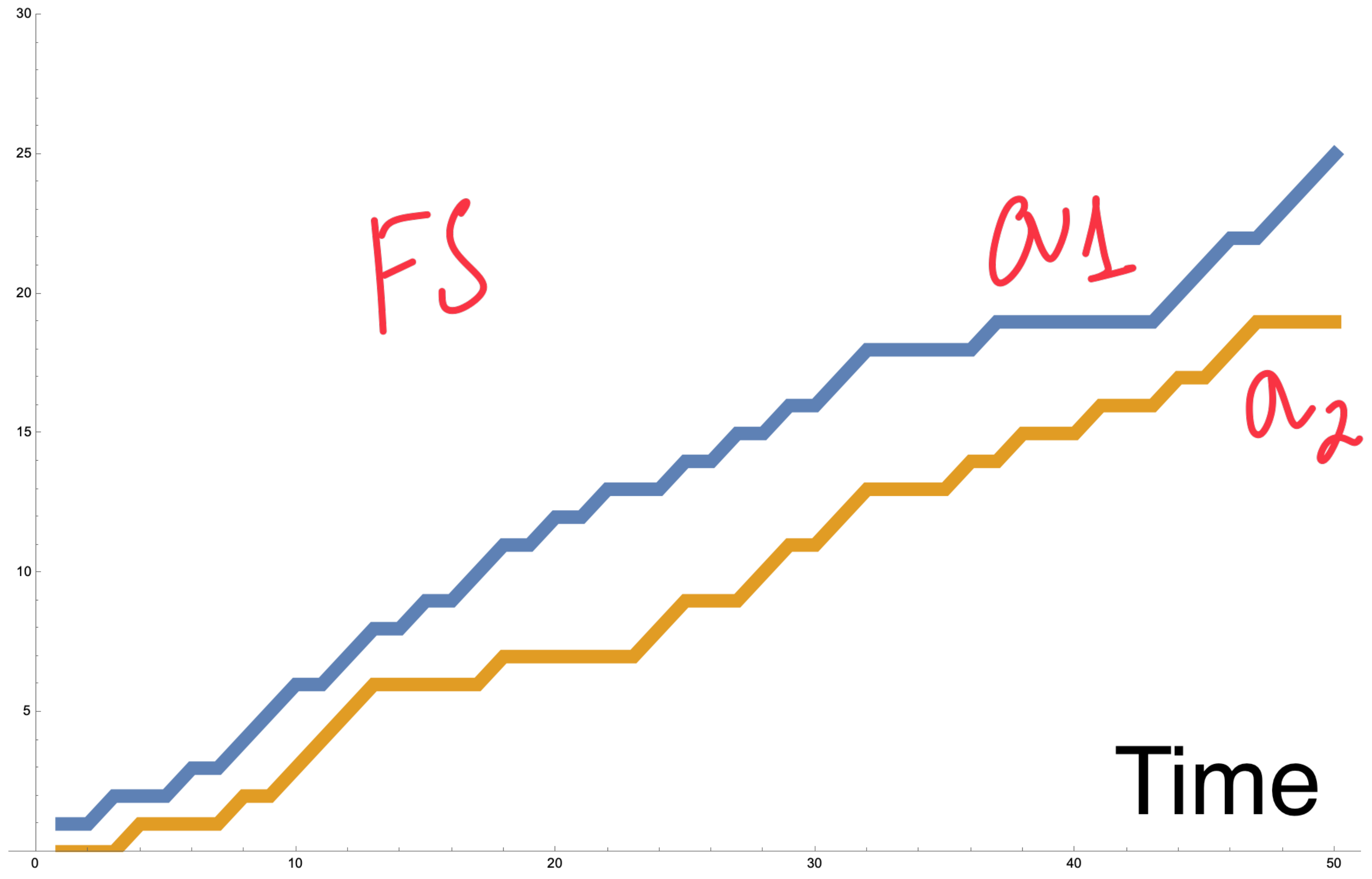

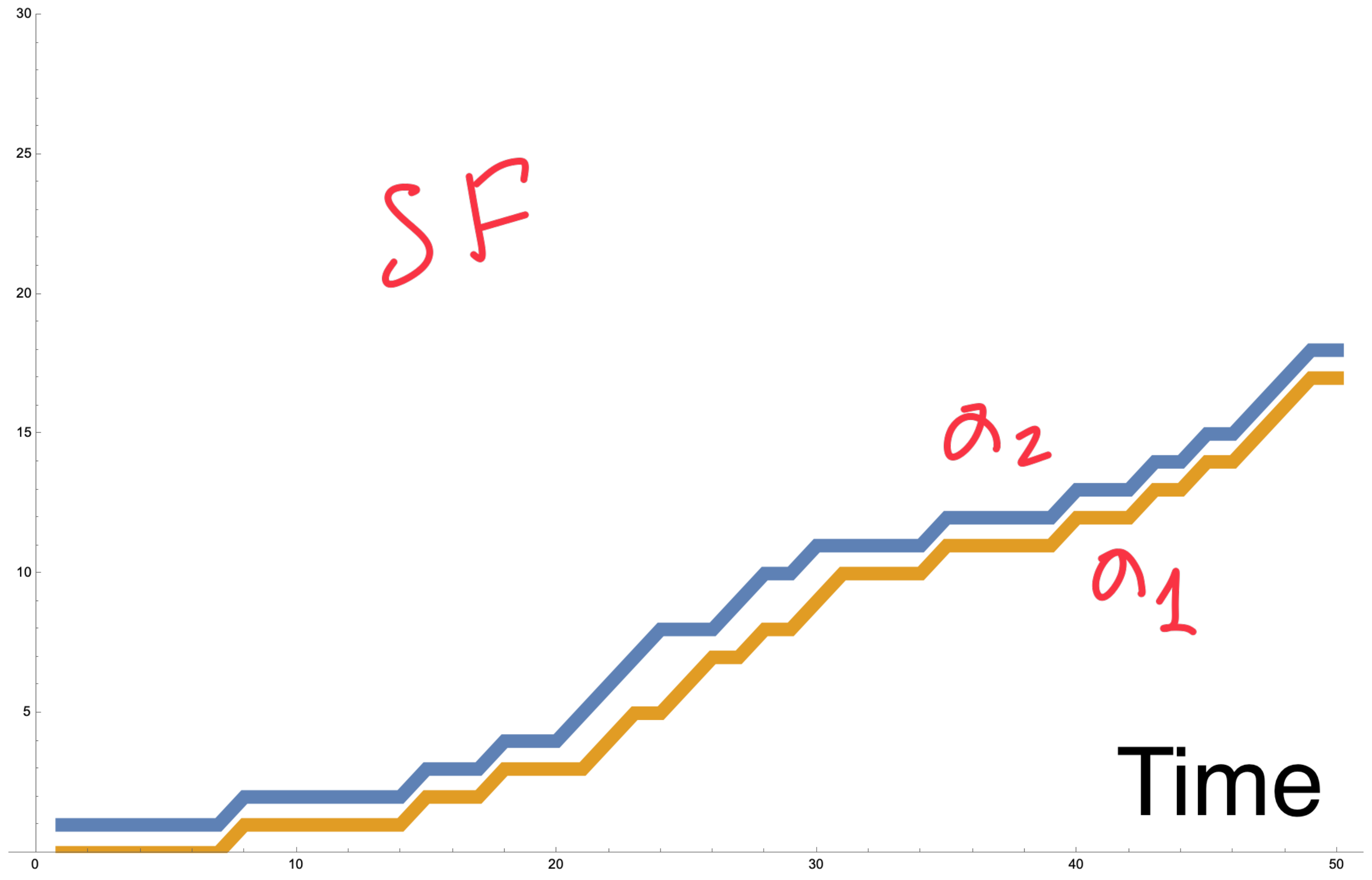

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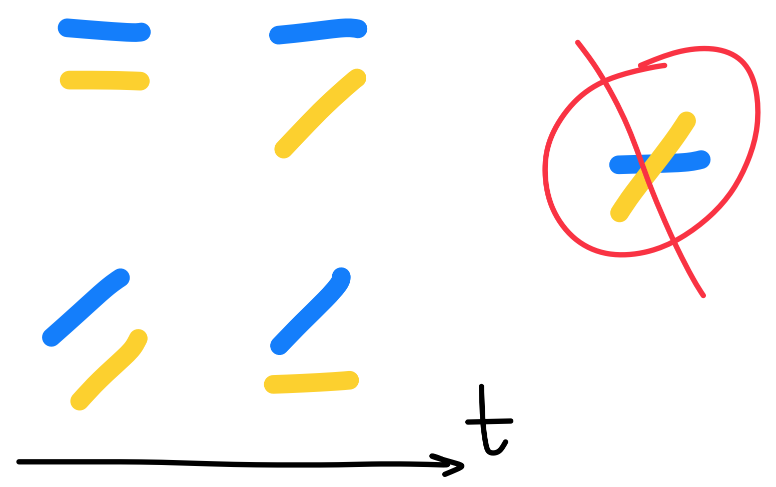

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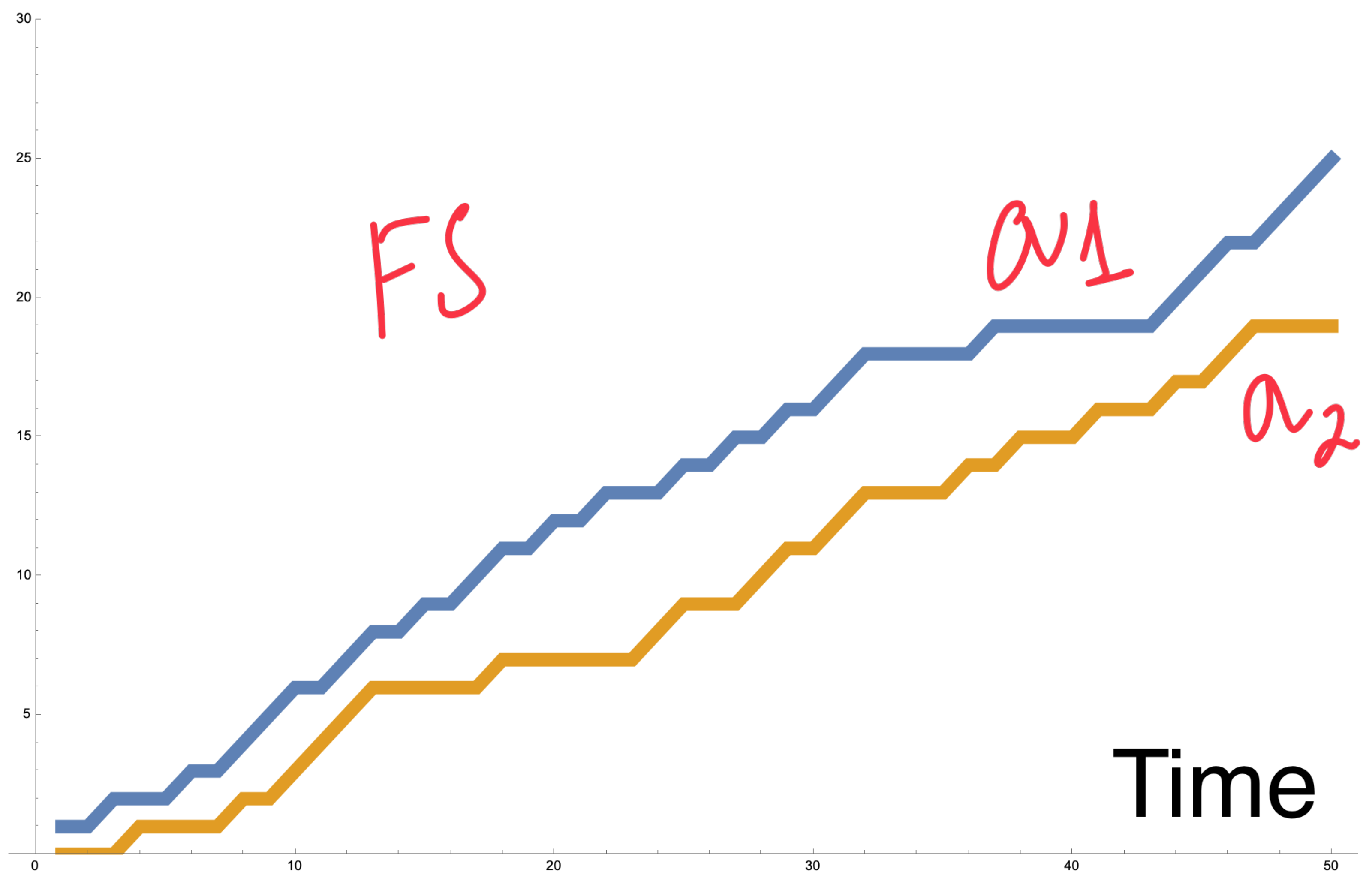

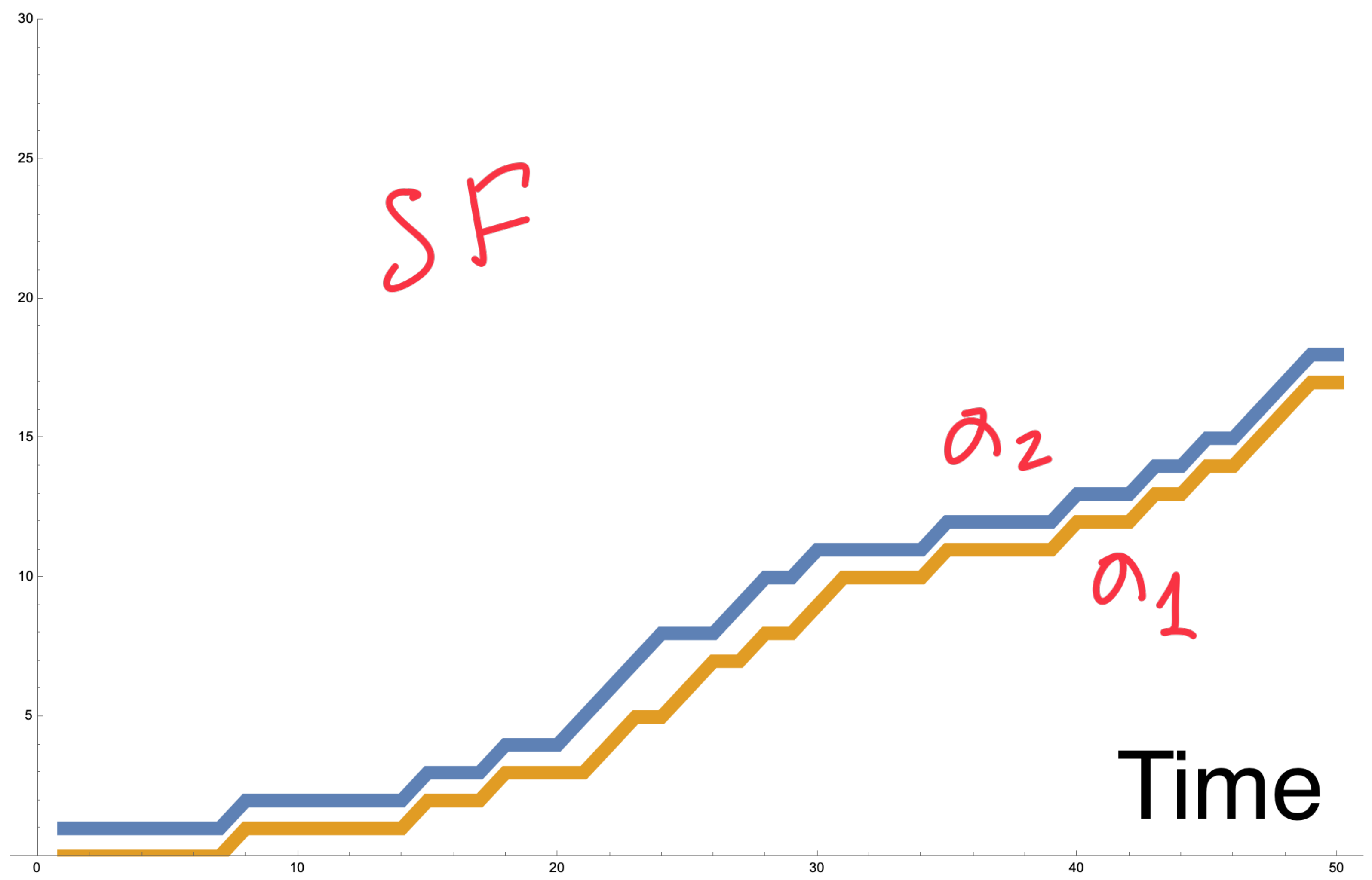

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follows from Robinson-Schensted-Knuth correspondence which encodes unused jumps

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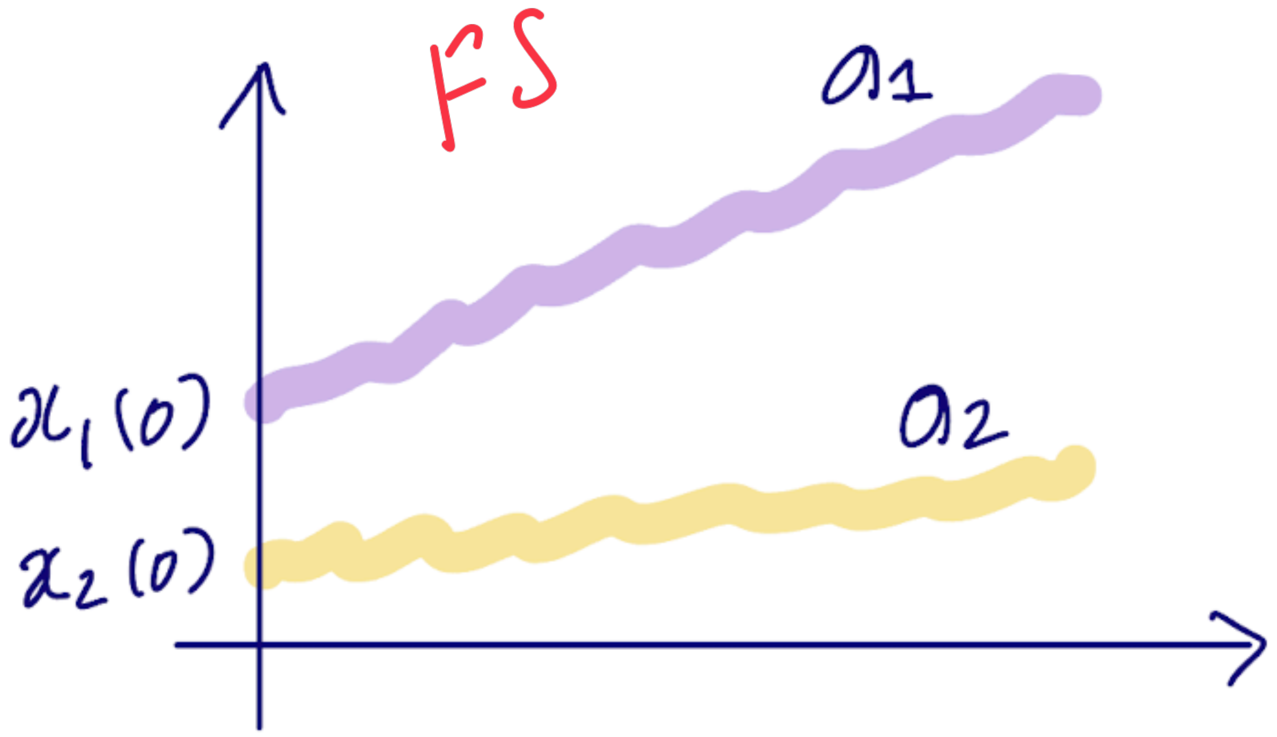

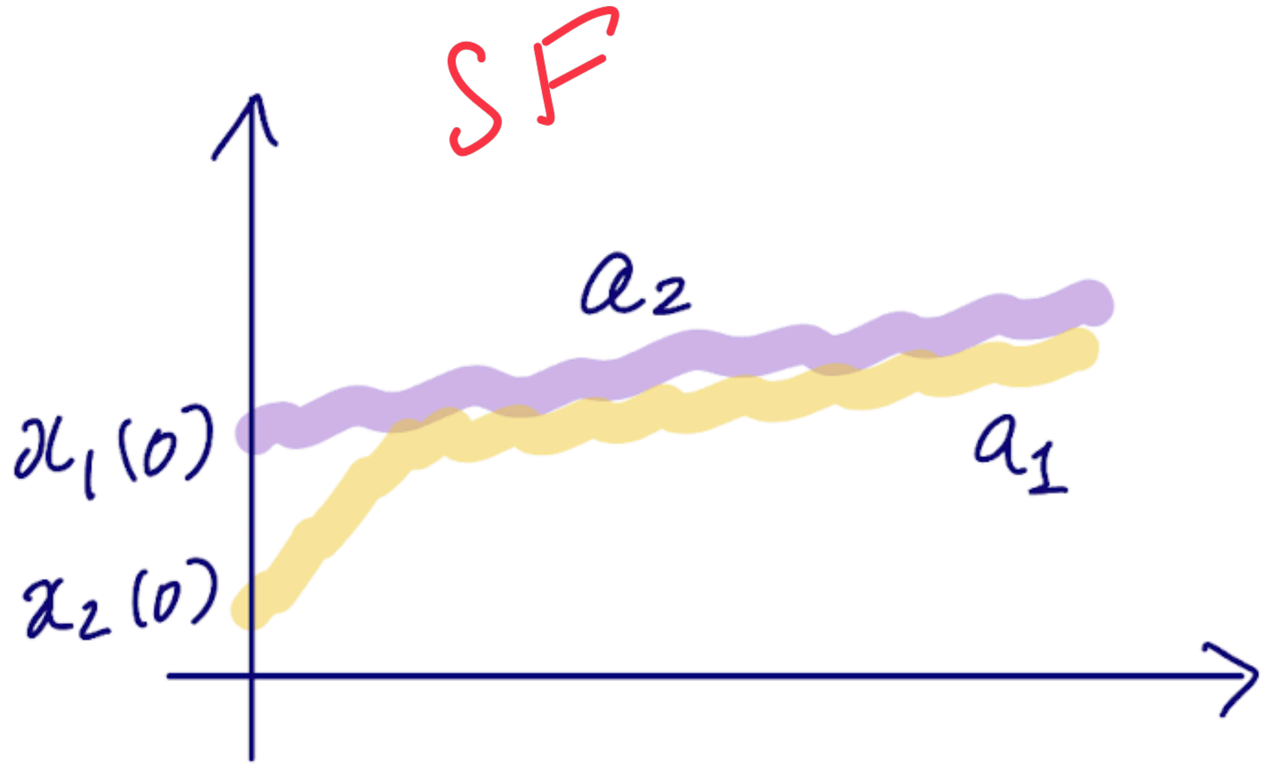

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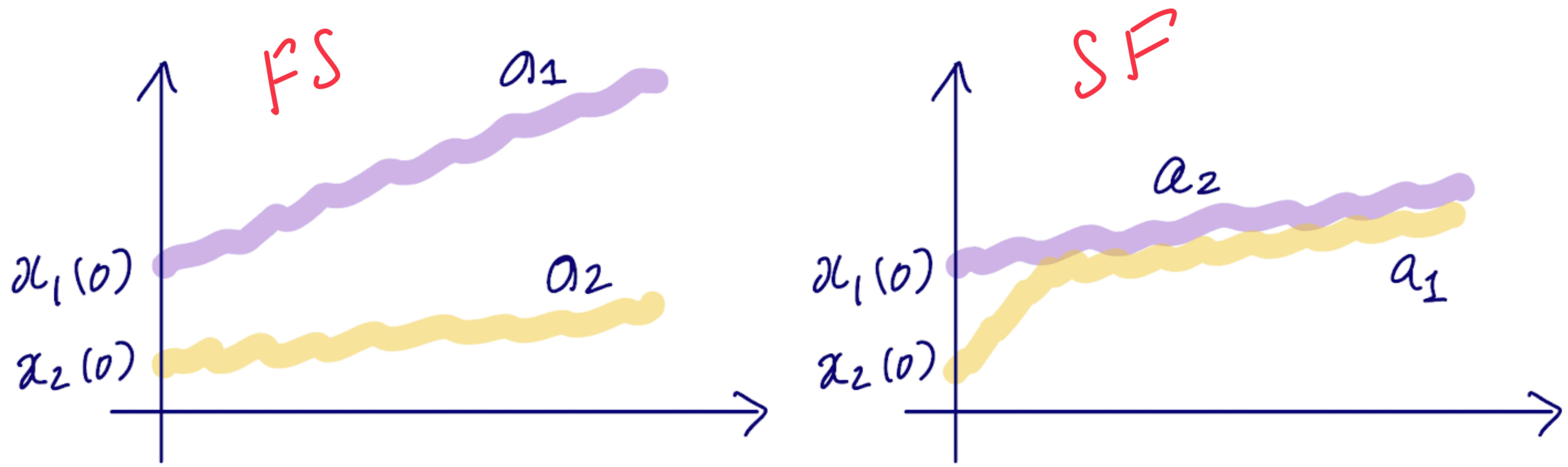

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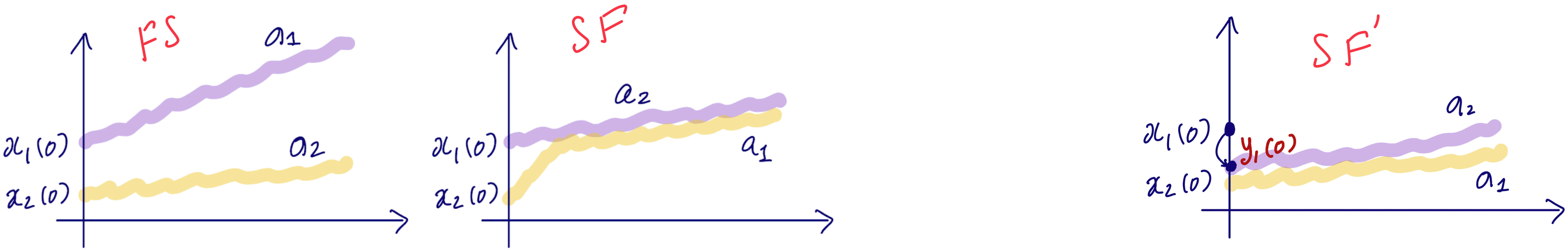

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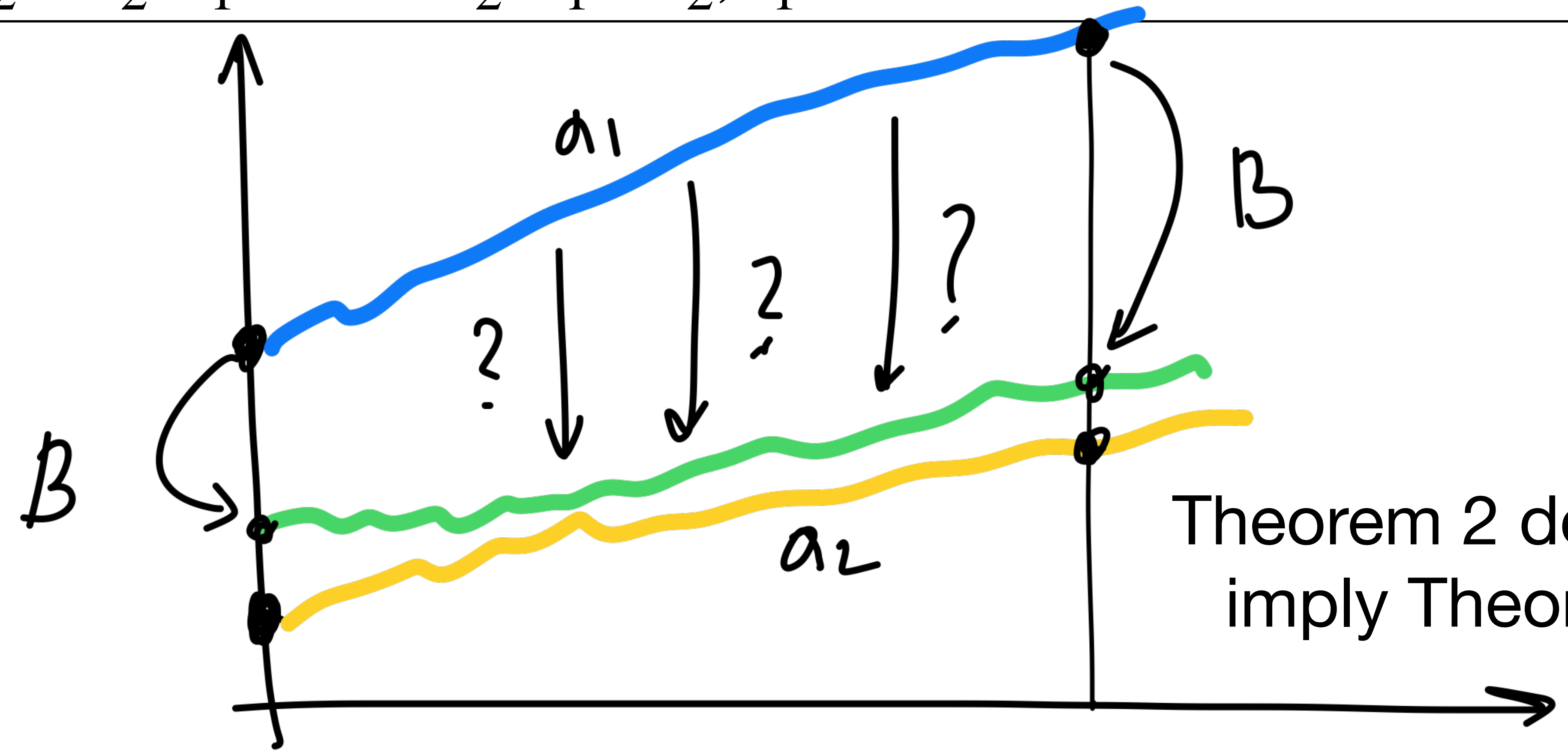

Theorem 2 does not imply Theorem 1

Proof via Yang-Baxter equation

Jumps and Yang-Baxter equation

Translate into the “vertex model” language. In vertex models, time runs **up**. $g = x_1 - x_2 - 1$ is the **gap.** Down and left are inputs, up and right are outputs.

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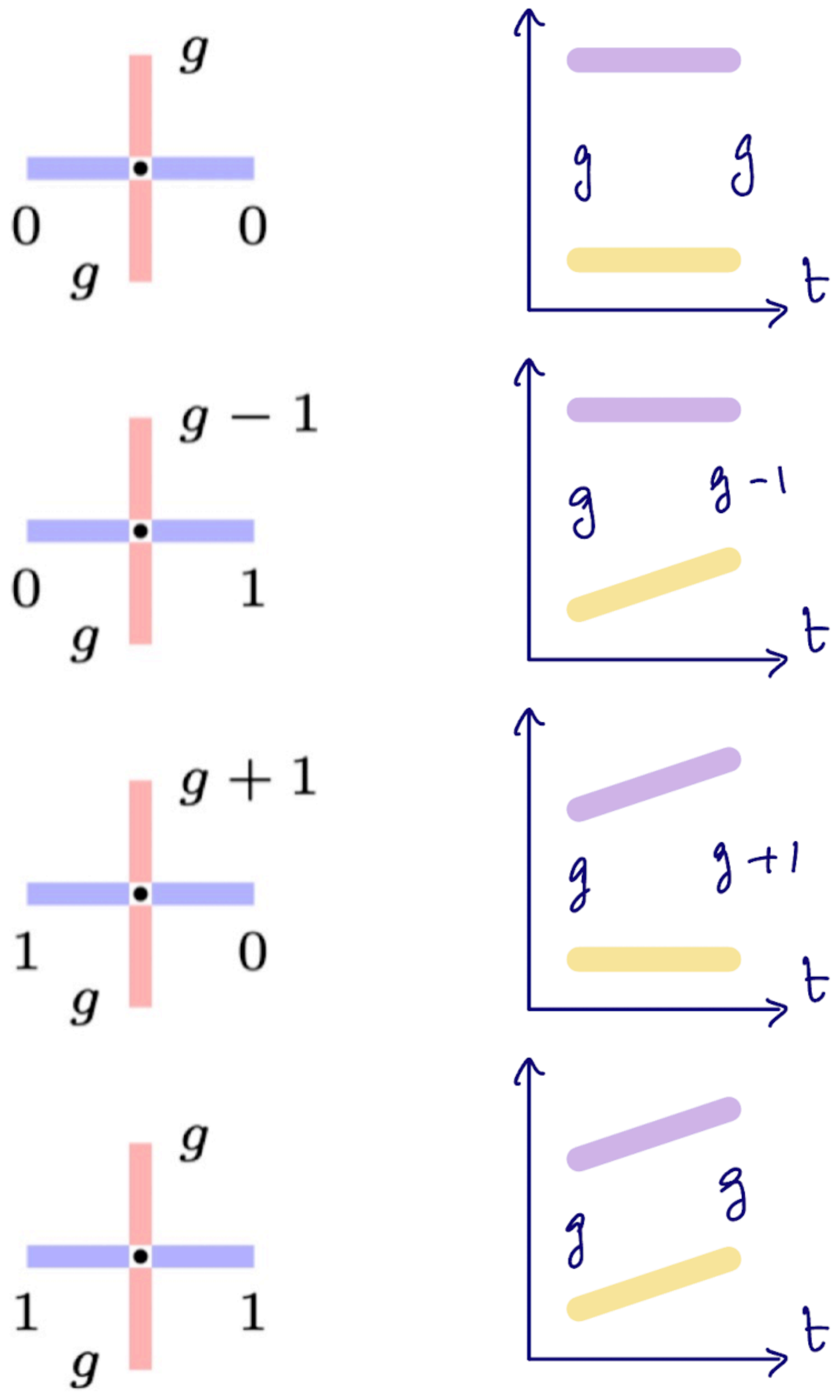

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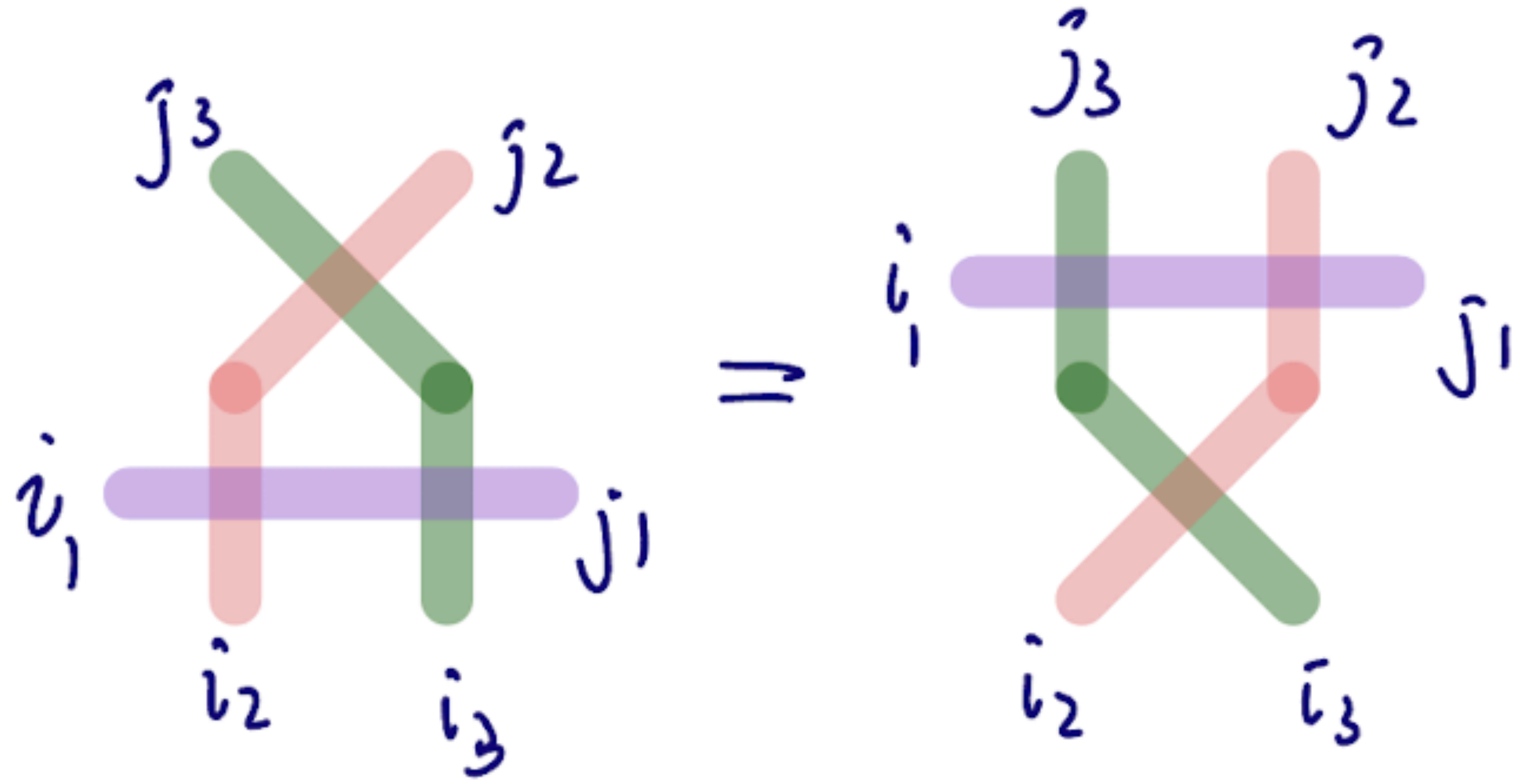
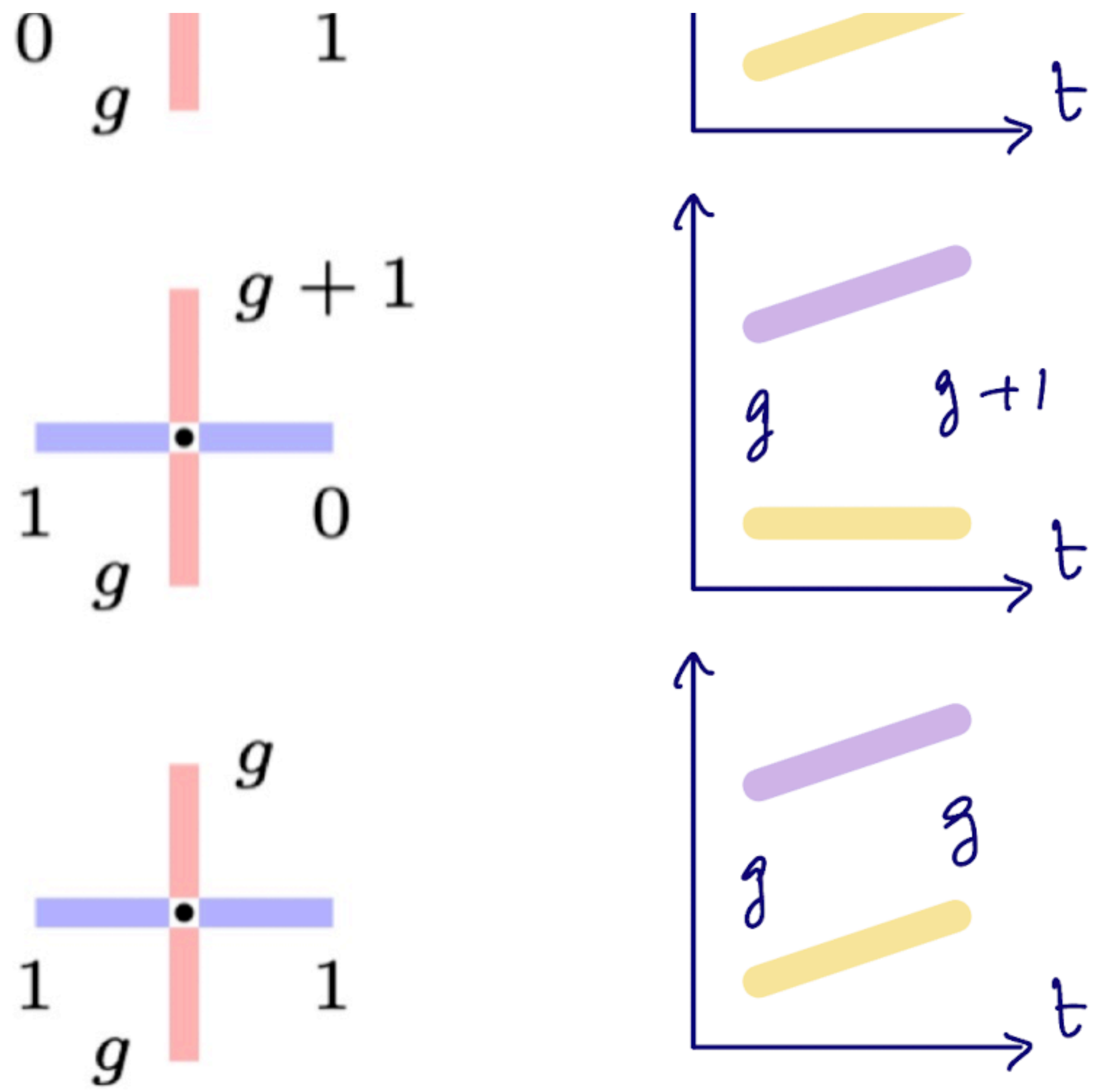
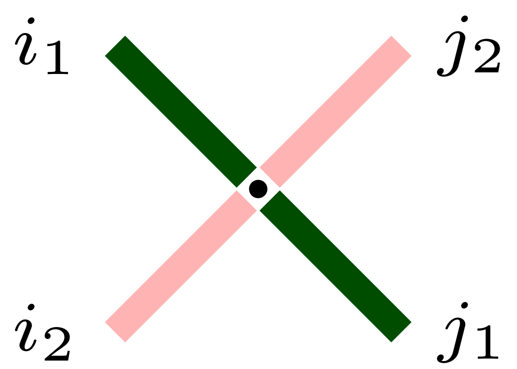
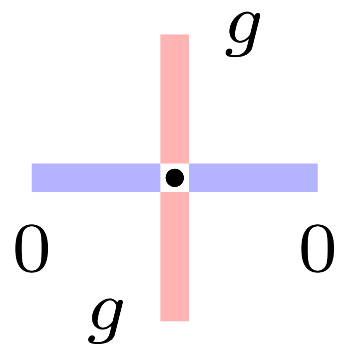




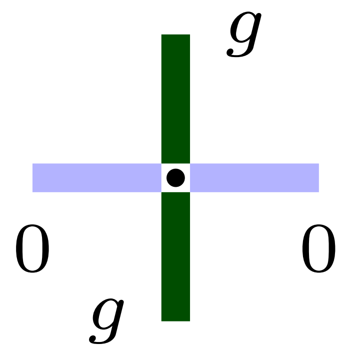
Theorem. Vertex weights satisfy the **Yang-Baxter equation**



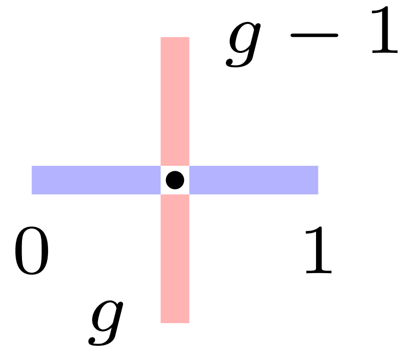
$$(a_2/a_1)^{j_2}(1 - \mathbf{1}_{j_2<j_1} a_2/a_1)\, \mathbf{1}_{j_2 \le j_1}$$



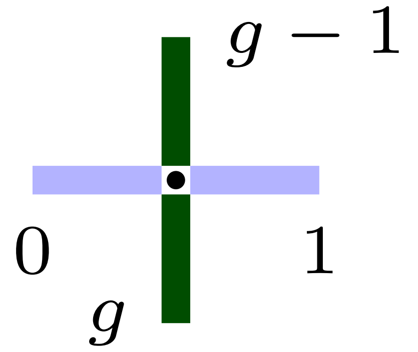
$$\frac{1}{1+a_1}$$



$$\frac{1}{1+a_2}$$



$$\frac{a_1}{1+a_1}$$



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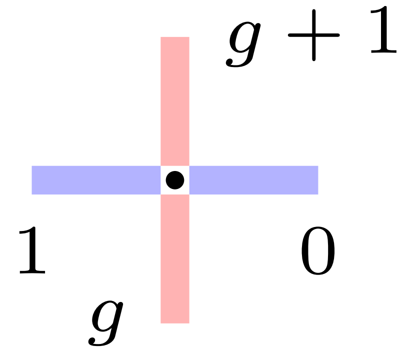

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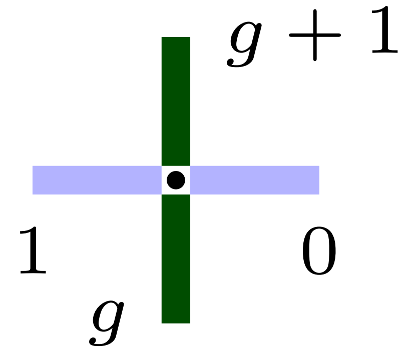

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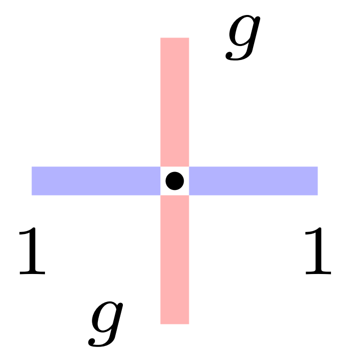

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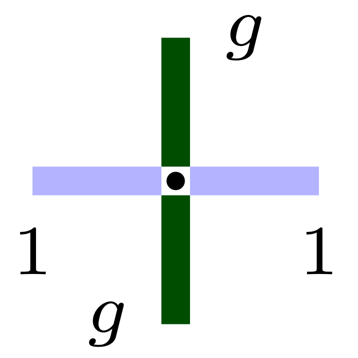

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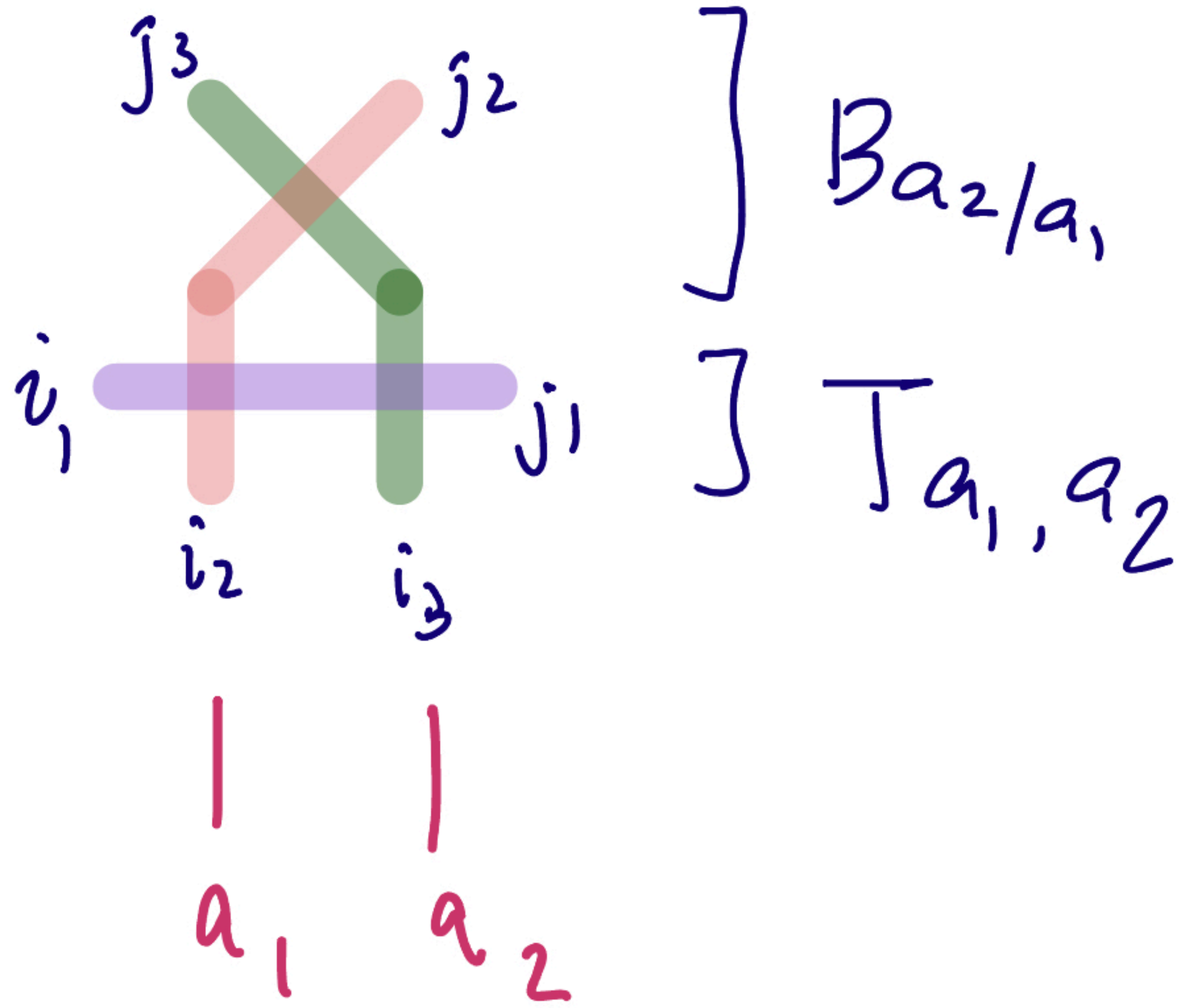
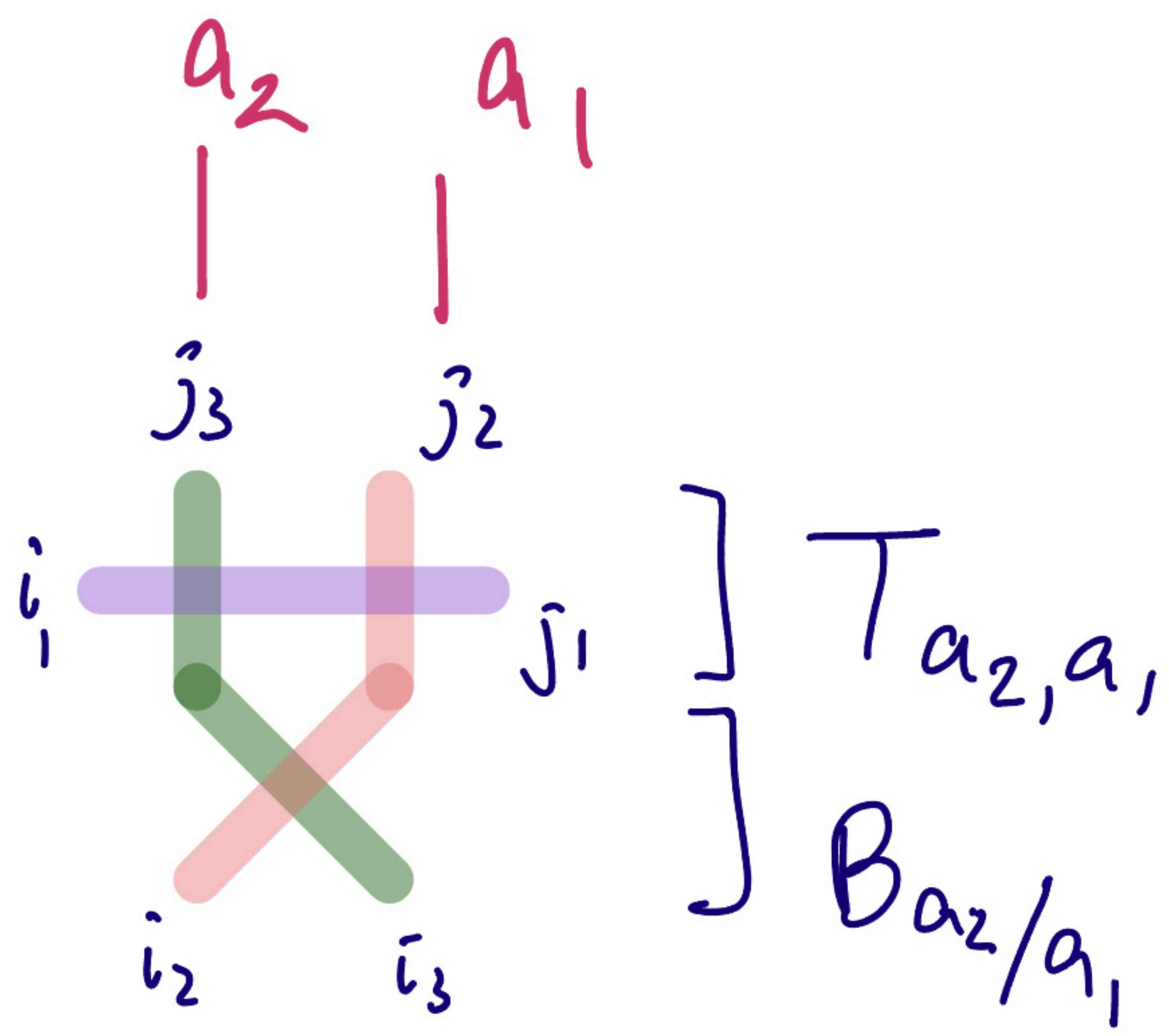
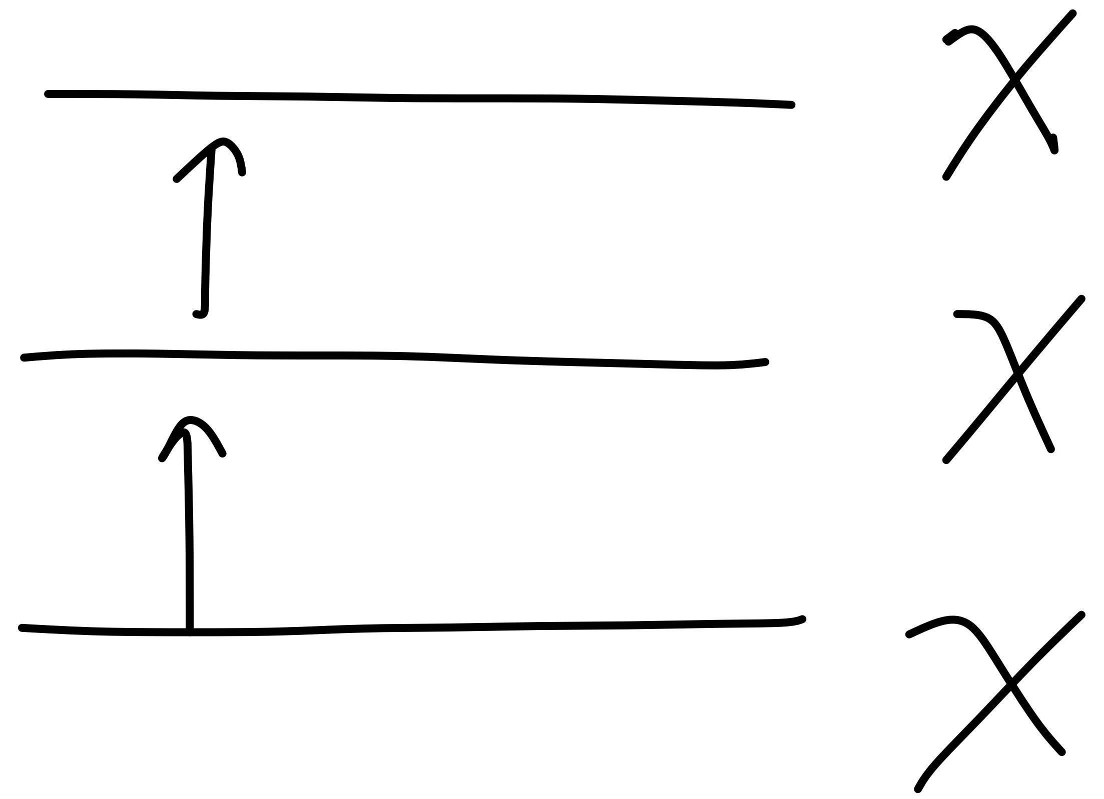
Intertwining relation proof





X - space of particle configurations



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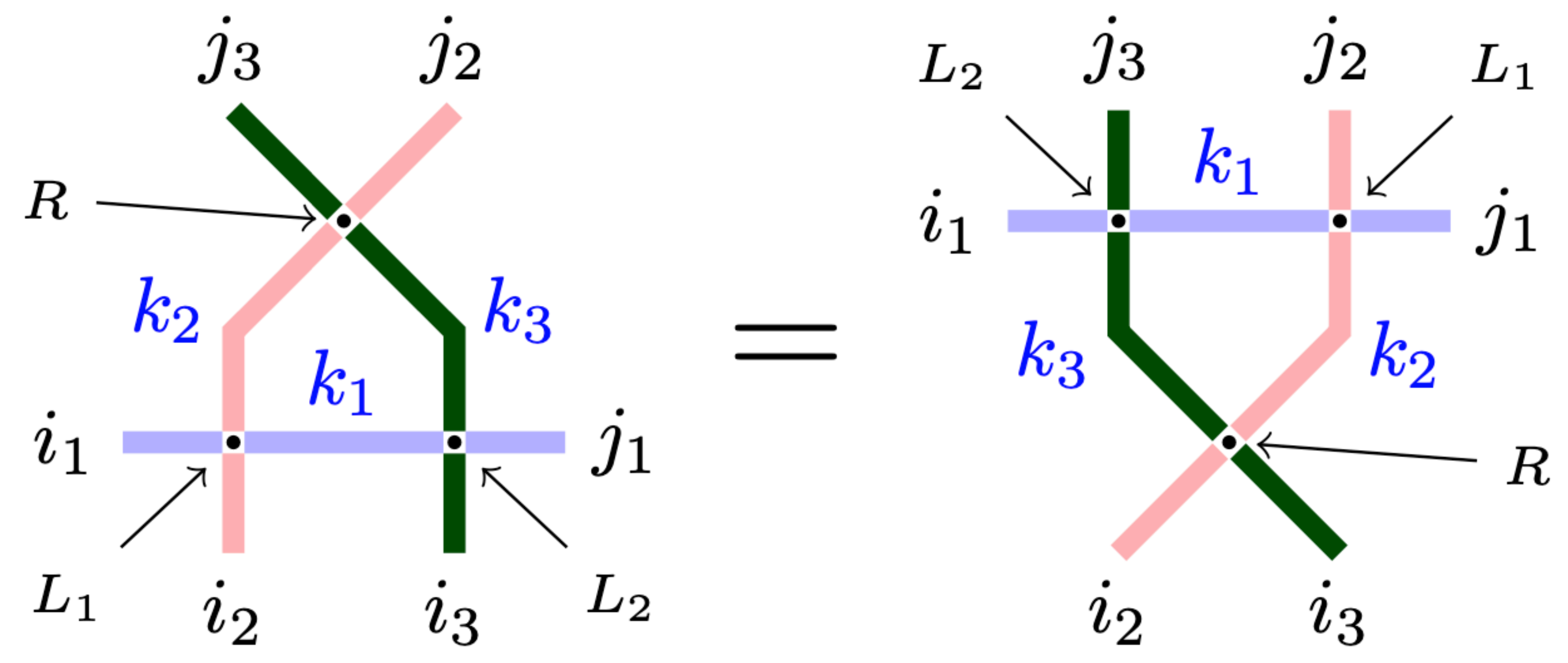
Intertwining relation for general stochastic R matrices (for specialists)



$$L^{(1)}_{u,s}(g,0;g,0) = \frac{1-q^g su}{1-su}, \qquad L^{(1)}_{u,s}(g,0;g-1,1) = \frac{-su(1-q^g)}{1-su};$$

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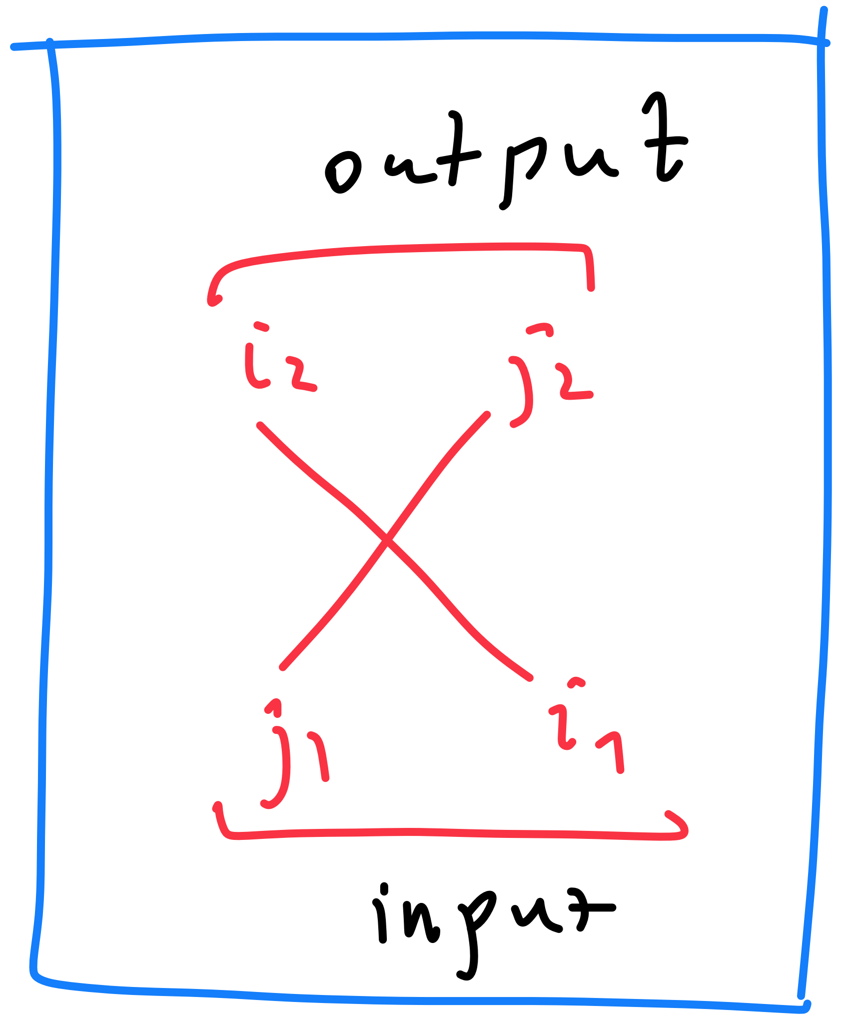
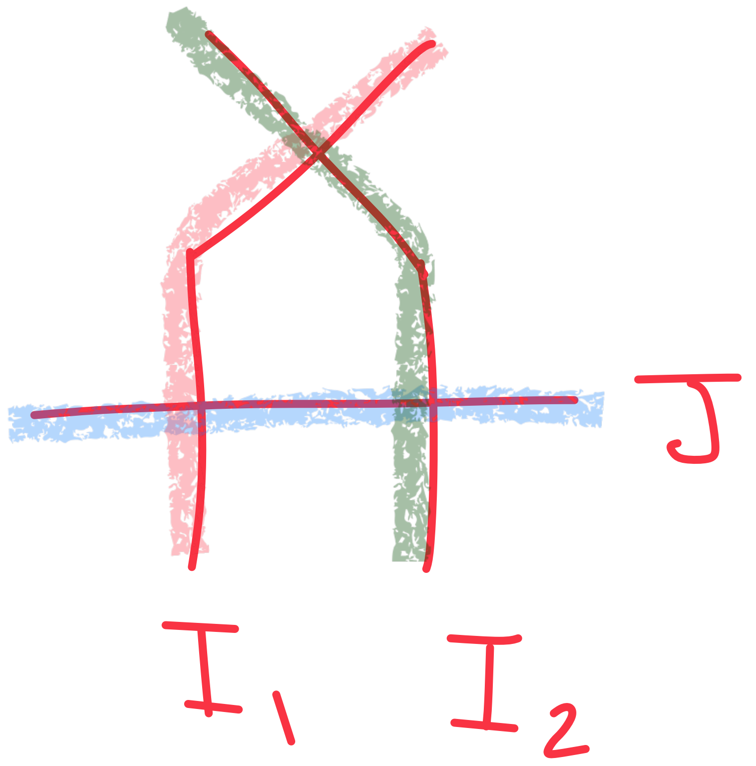
$R = R_{u_2/u_1, s_1, s_2} = L^{(I_1)}_{\frac{s_1 u_2}{u_1}, s_2}$

Stochastic:



$s_1 = q^{-I_1/2}$ (fusion)



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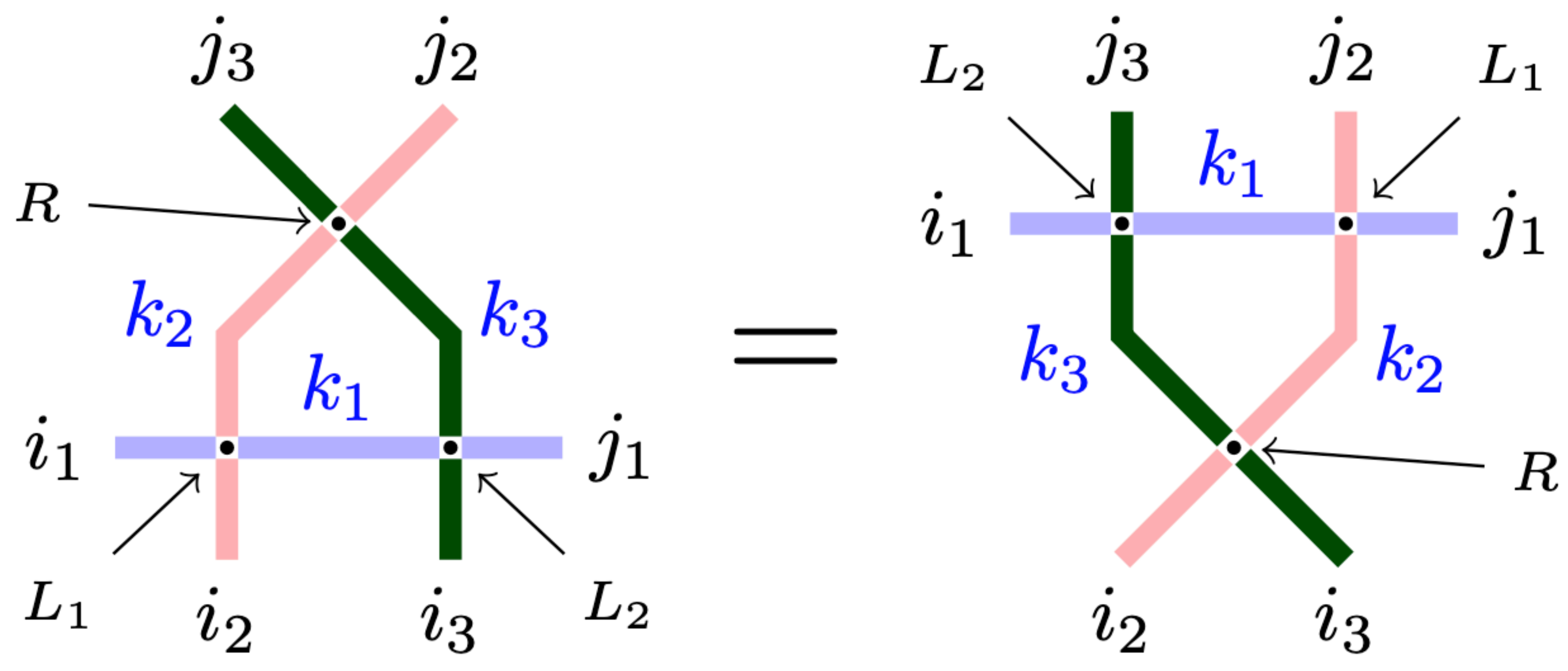

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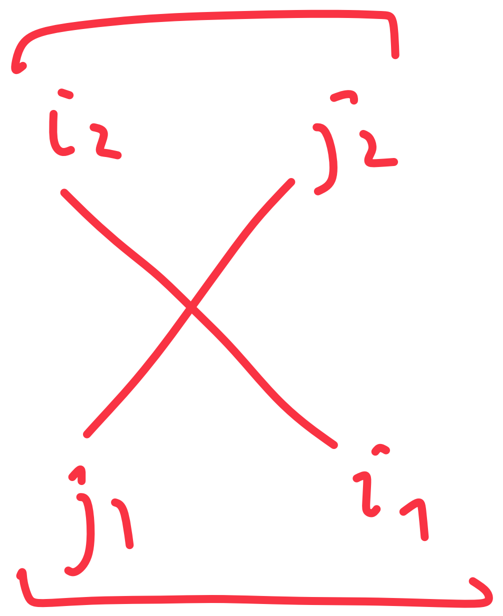

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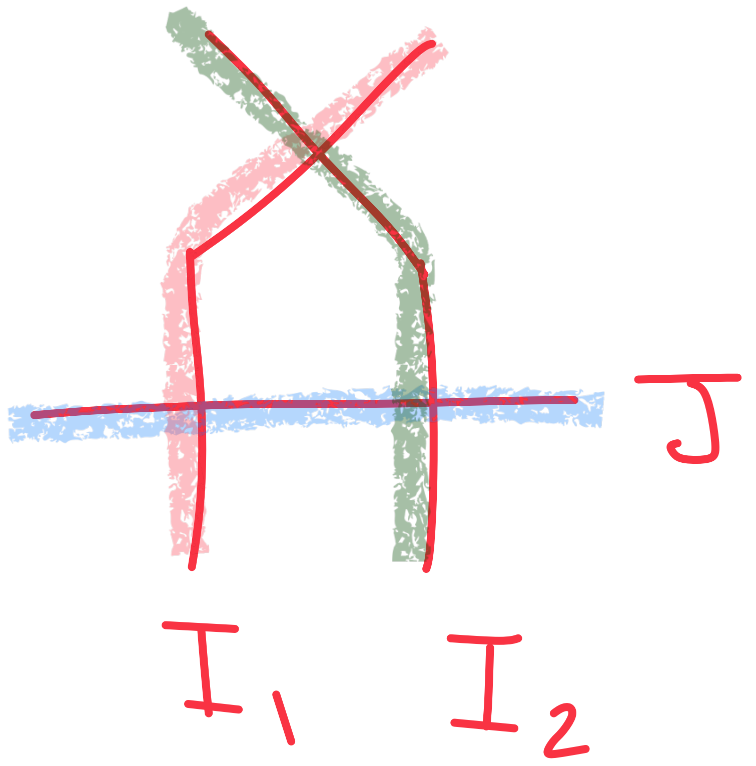
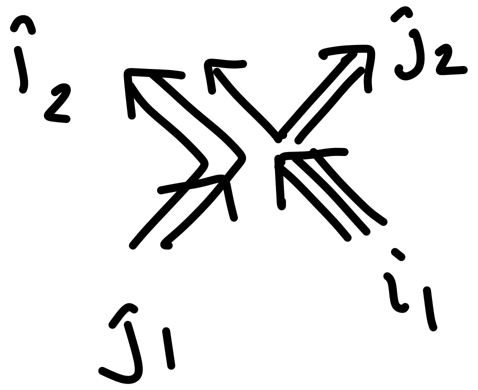


q-Hahn (monotone)

$$\boxed{u_2/u_1 = s_2/s_1}$$

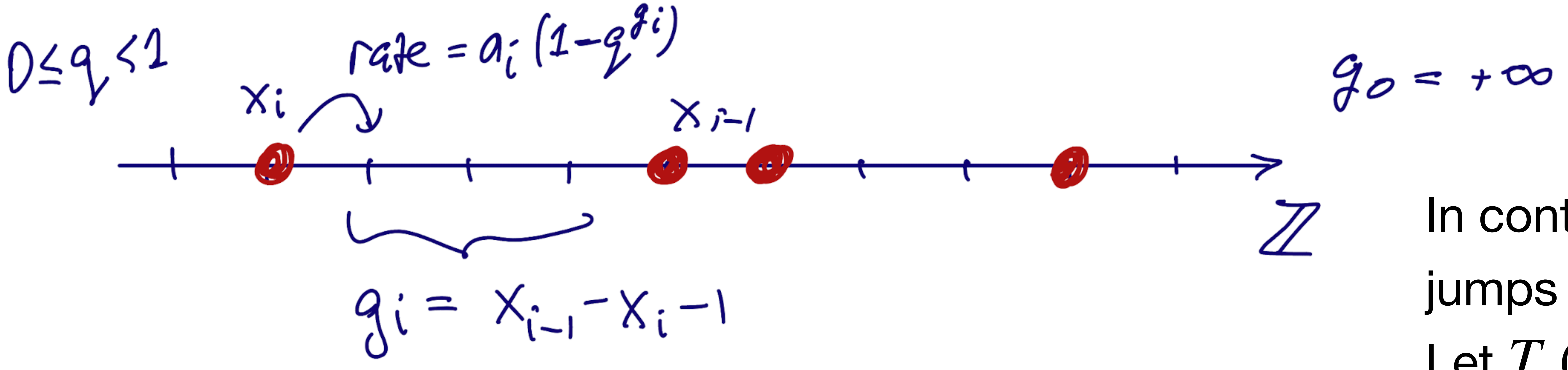
$\Rightarrow$ specialization



$$\boxed{j_2 \le i_1}$$

General intertwining and Lax equations

Intertwining for the q-TASEP in continuous time (add q to make life harder)



In continuous time, each particle x_i jumps forward at rate $a_i(1 - q^{gap_i})$. Let $T_{\mathbf{a}}(t)$ be the q-TASEP semigroup

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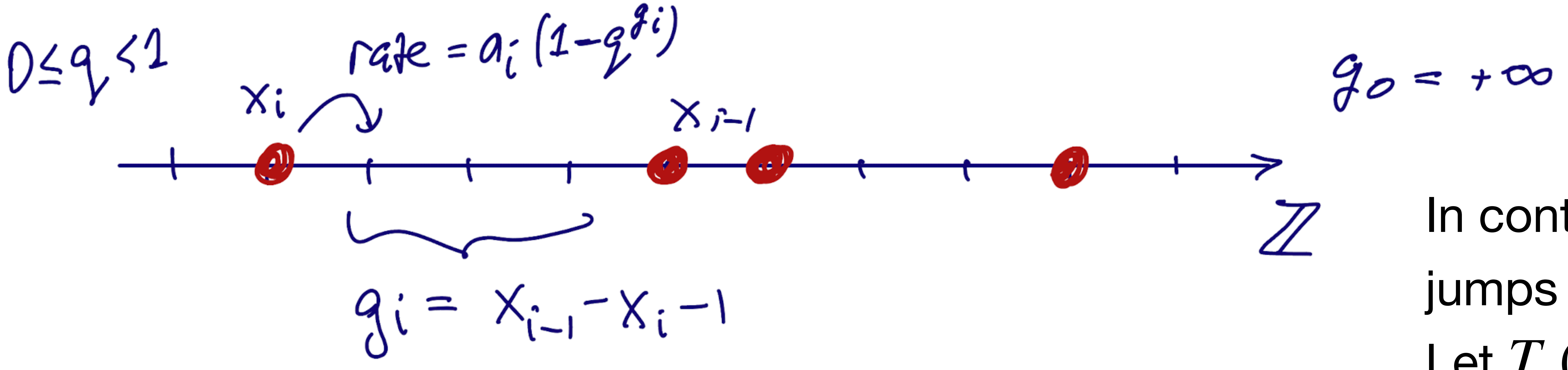
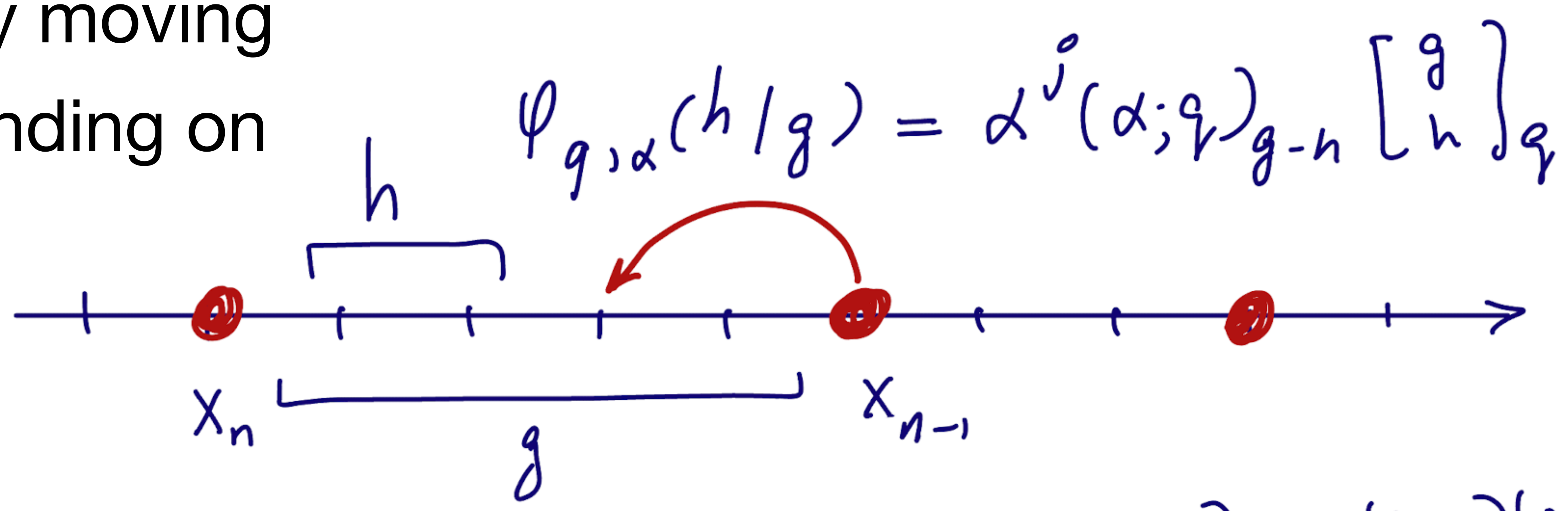


In continuous time, each particle x_i jumps forward at rate $a_i(1 - q^{gap_i})$. Let $T_{\mathbf{a}}(t)$ be the q-TASEP semigroup

Let B_{n-1} be the operator of randomly moving particle x_{n-1} back closer to x_n, depending on $\alpha = a_n/a_{n-1} < 1$.



$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

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$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}$$

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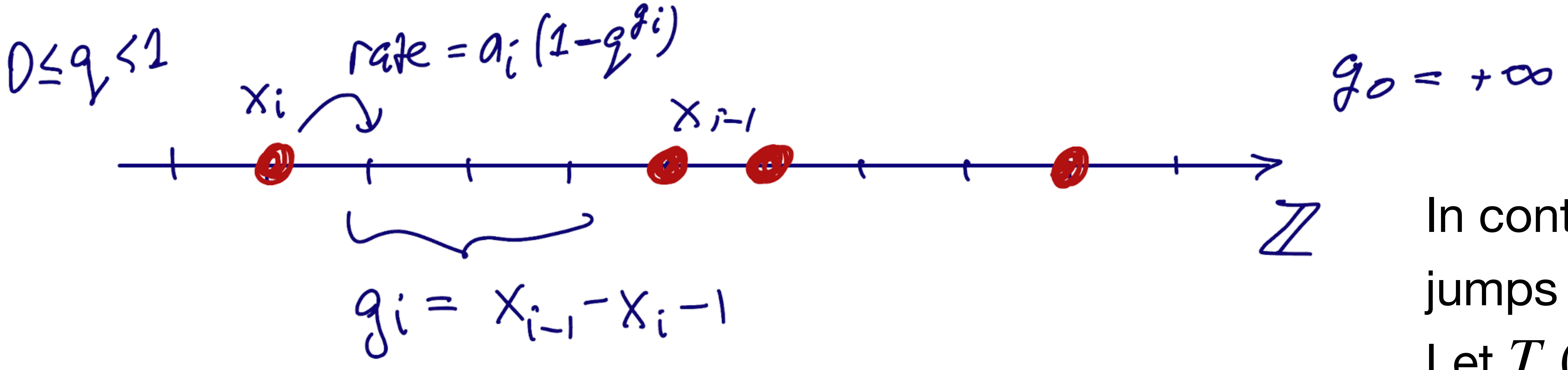

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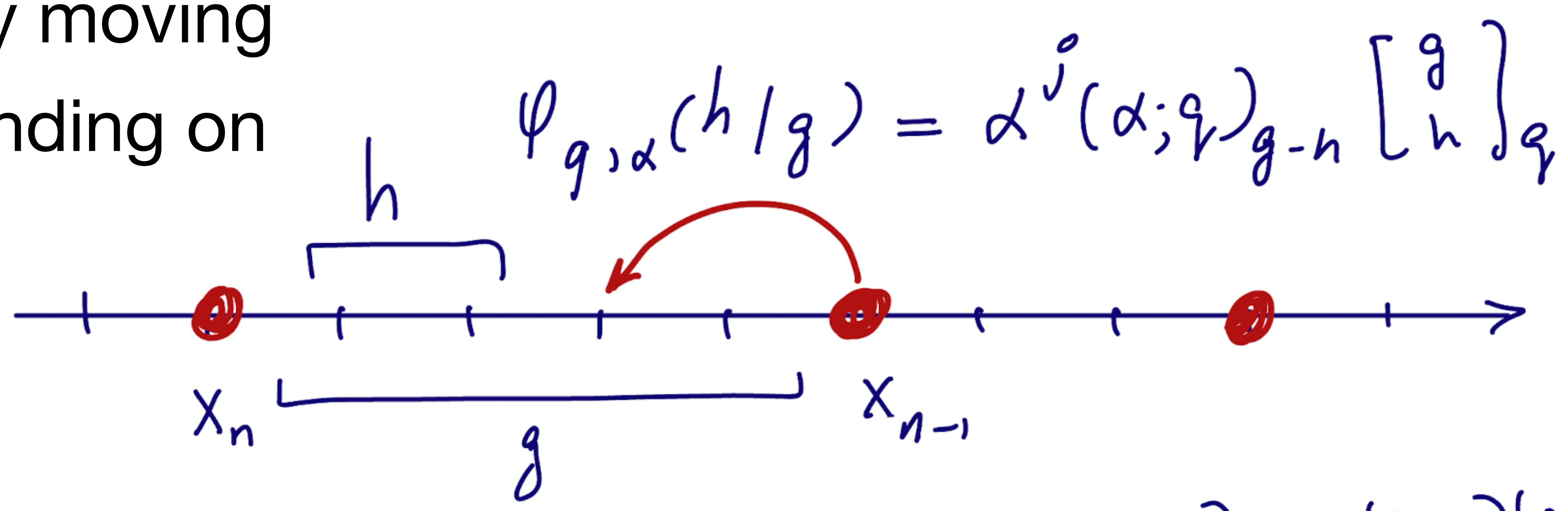

Theorem (P.-Saenz 2022). We have the intertwining $T_{\mathbf{a}}(t)B_{n-1} = B_{n-1}T_{\sigma_{n-1}\mathbf{a}}(t)$, where σ swaps $a_{n-1} \leftrightarrow a_n$.

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

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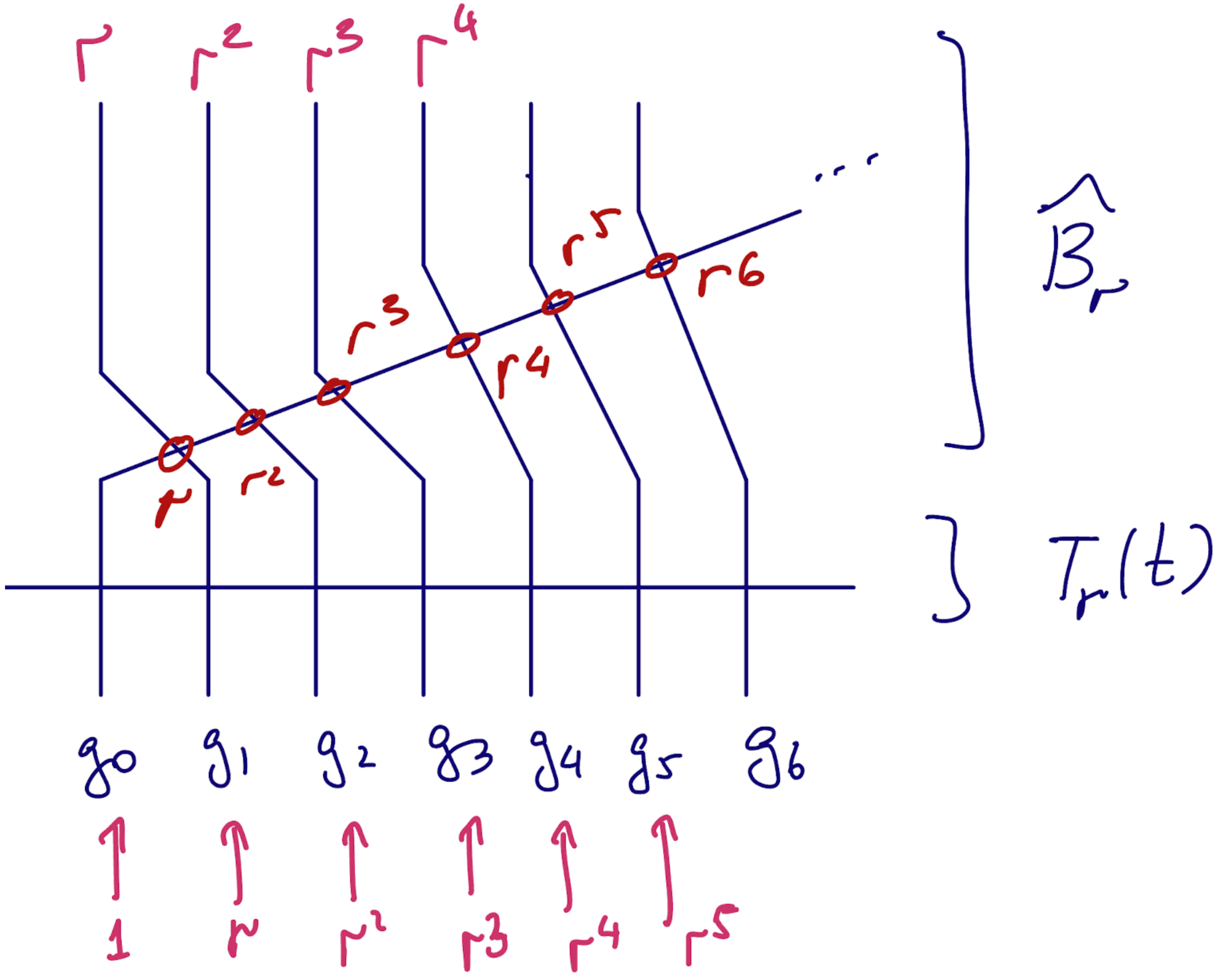
Application to densely packed configurations: [P.-Saenz 2019], [P. 2019]

From intertwining to Lax equations for q-TASEP

Let $a_n = r^n$. Let us organize the tower of applications of $B_1, B_2, B_3, \ldots$



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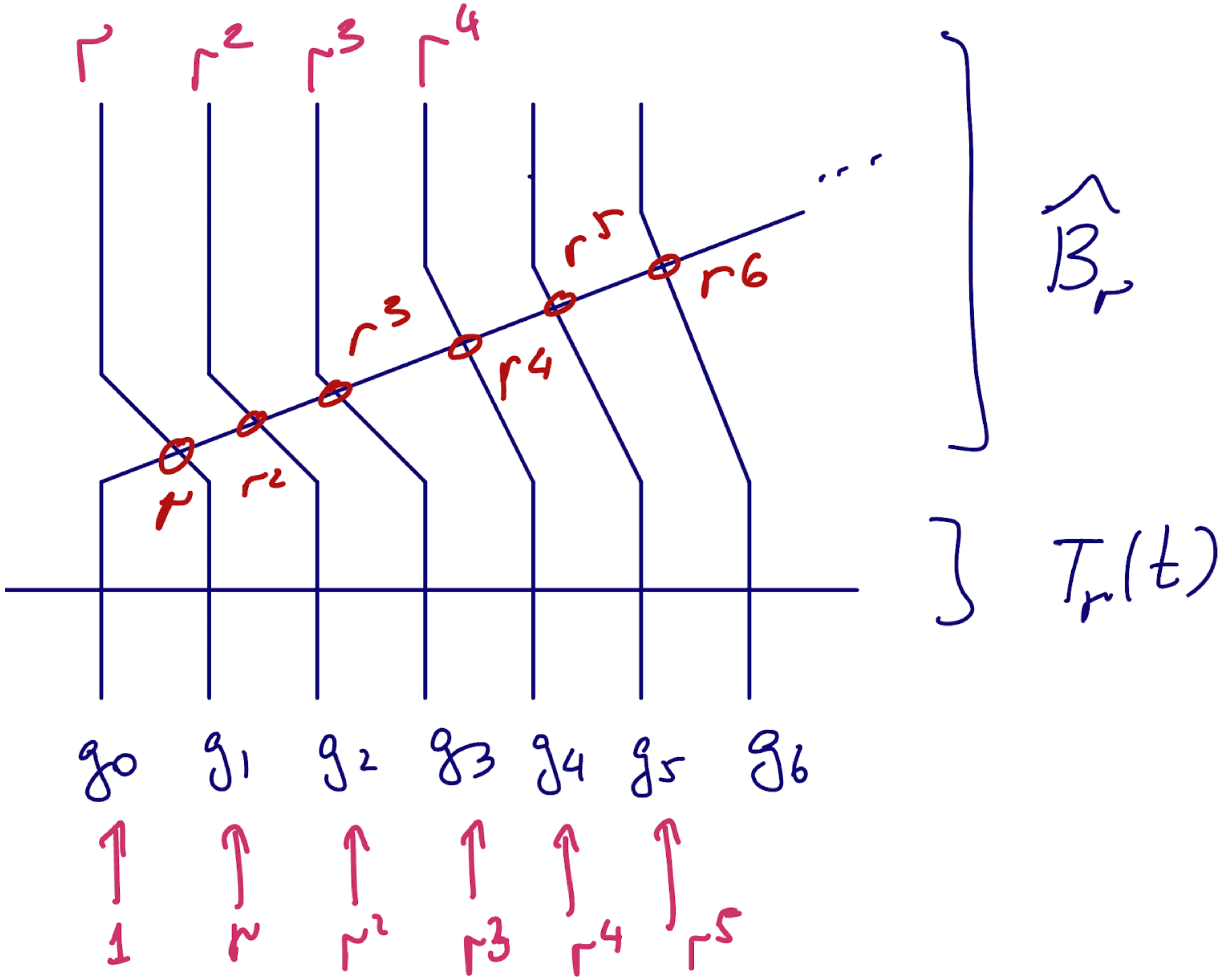

Iterated intertwining leads to $T_r(t)\hat{B}_r = \hat{B}_r T_r(rt)$, which is a **time change** in the continuous time q-TASEP

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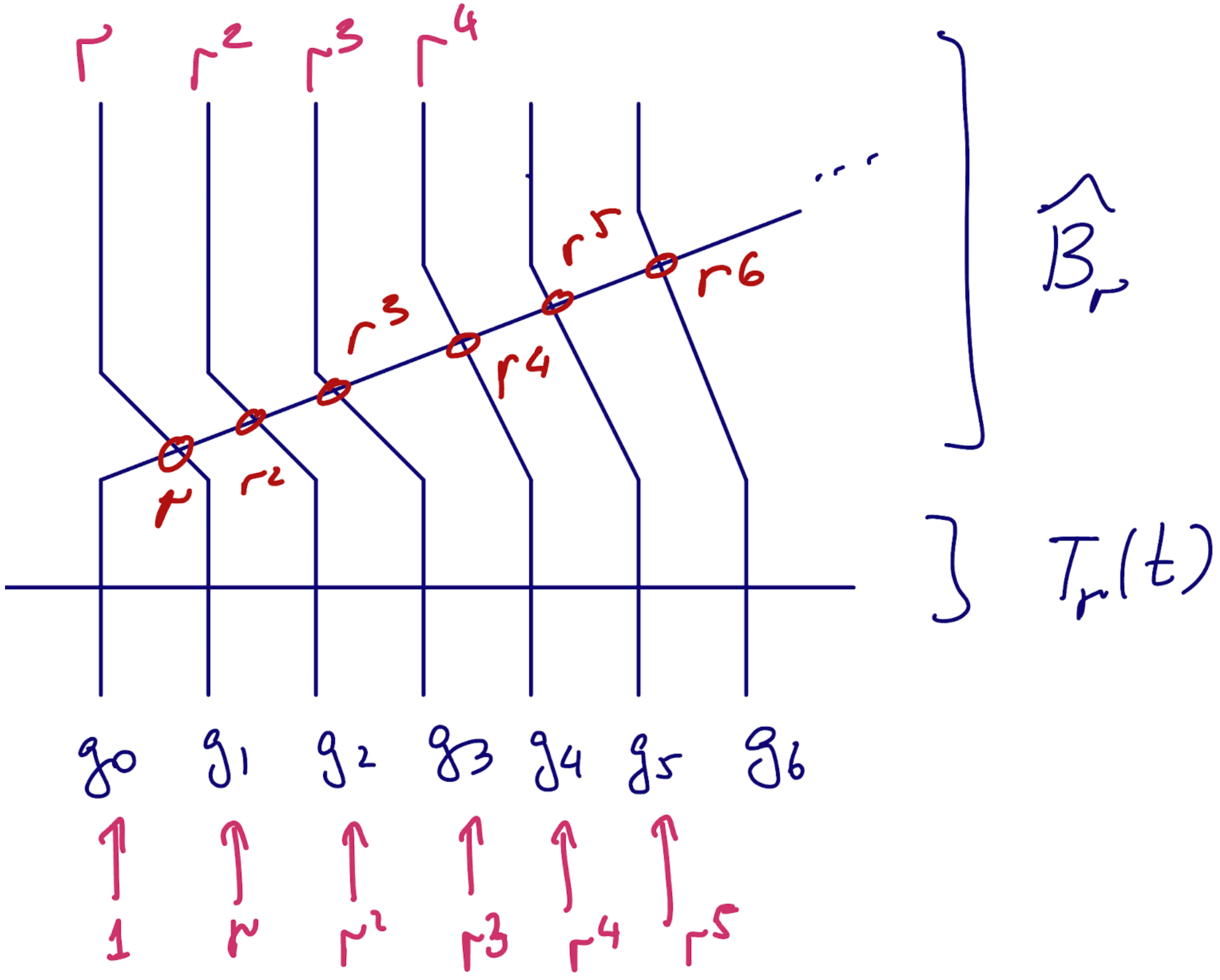

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Take a continuous time Poisson limit of $(\hat{B}_r)^{\tau/(1-r)}$. We get a Markov semigroup $B(\tau)$, and intertwining

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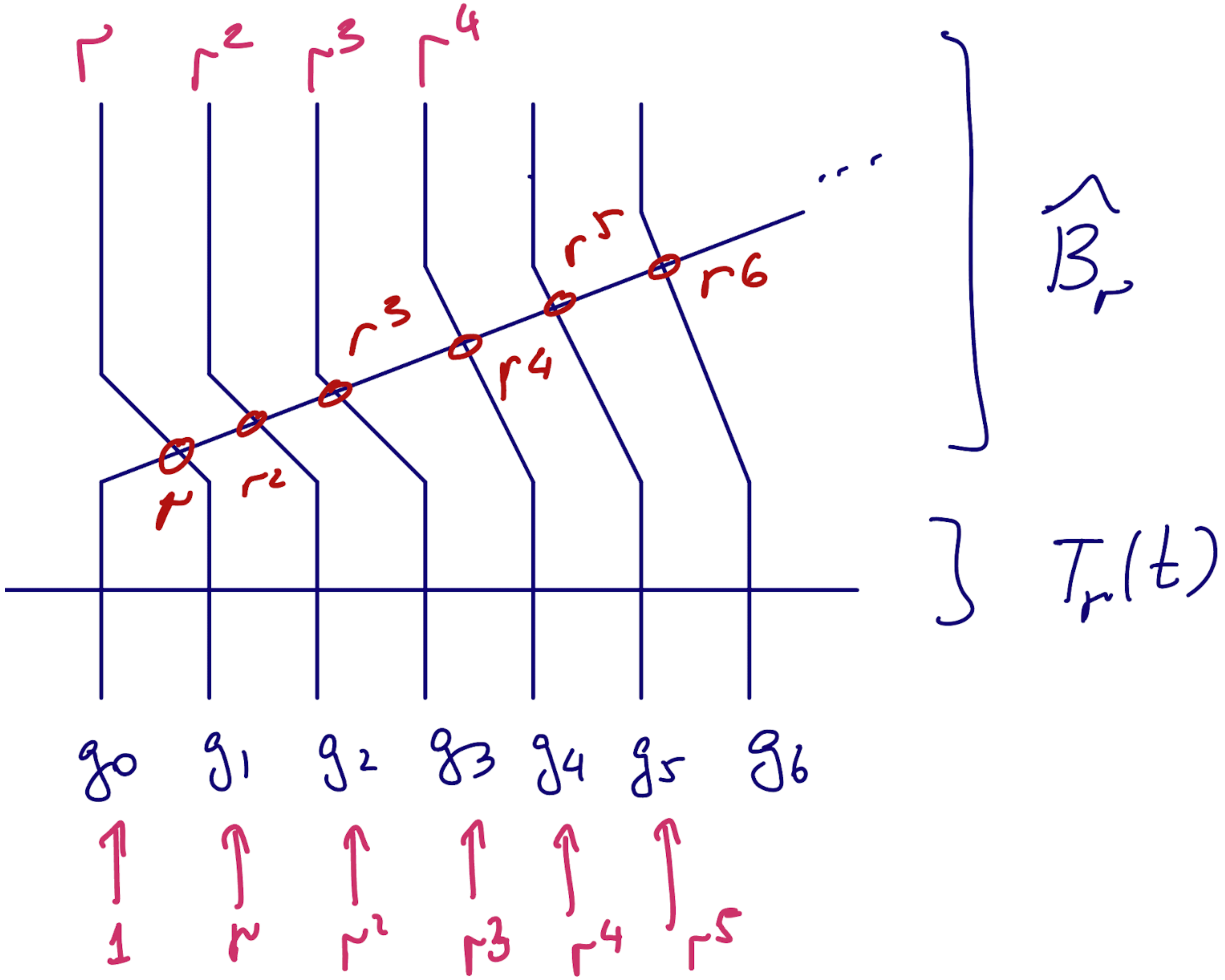

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Note: Outside the q-Hahn case $u_2/u_1 = s_2/s_1$, this cross-vertex system B_r does **not** preserve the empty configuration!

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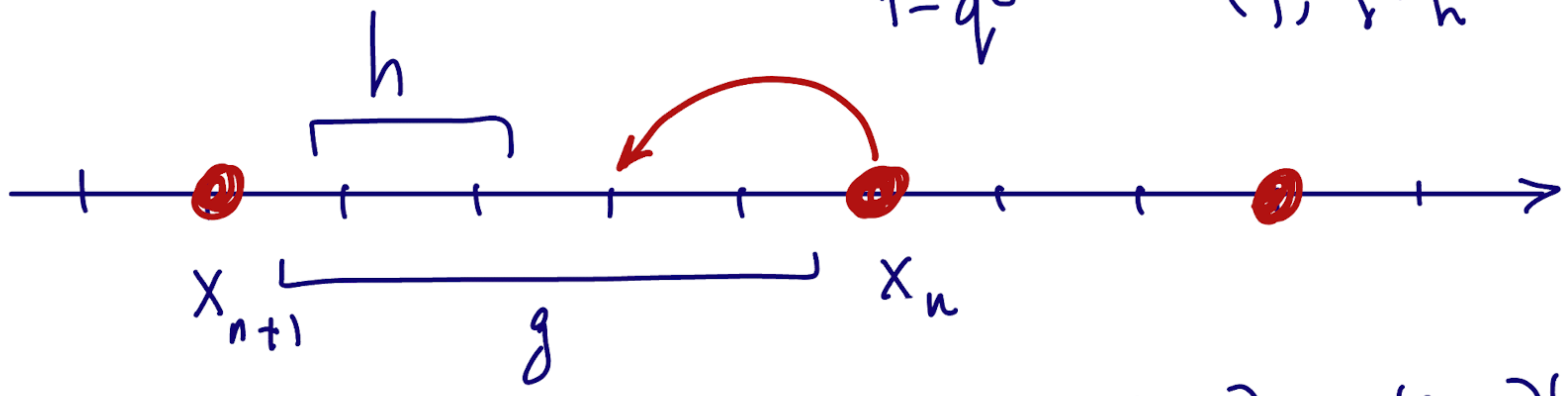
- The Lax equation should give access to all multipoint observables for q-TASEP, but this information is not that easy to extract… [Quastel-Remenik 2019] show KP equations for KPZ fixed point

The backwards dynamics B for q-TASEP and TASEP

The backwards dynamics $B(\tau)$ is a continuous time Markov process on particle configurations which are densely packed to the left. Each particle x_n jumps back independently in continuous time:

$$\text{rate} = \frac{n}{1-q^{g-h}} \cdot \frac{(q;q)_g}{(q;q)_h}, \quad h < g$$



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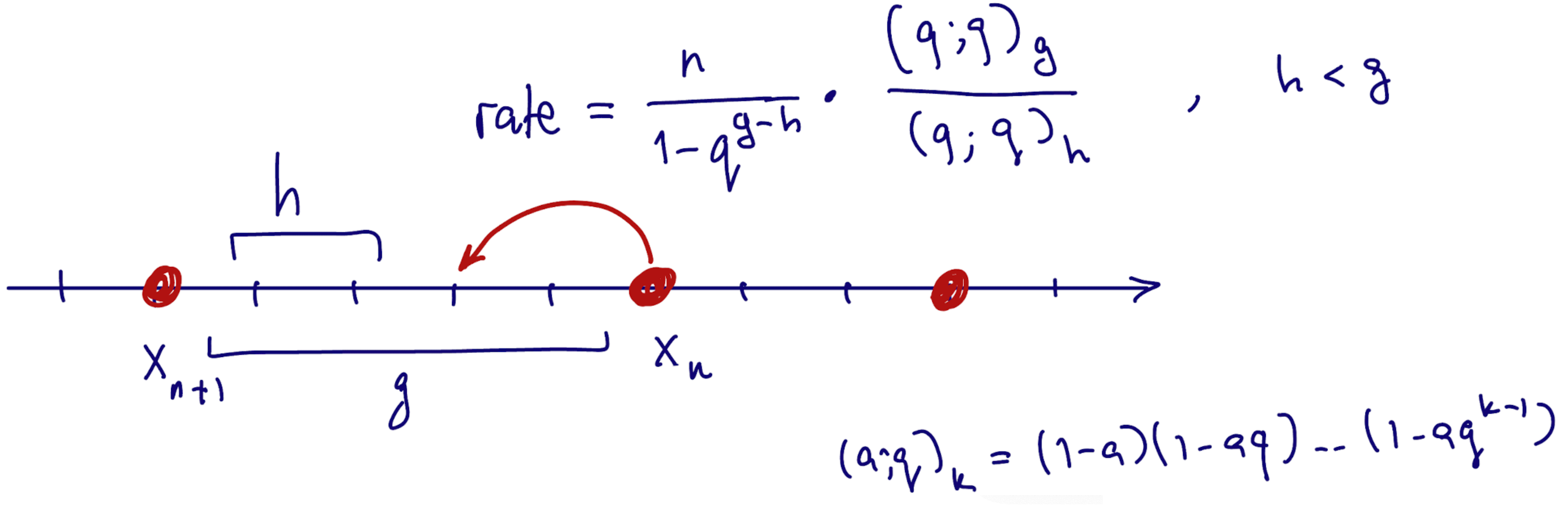
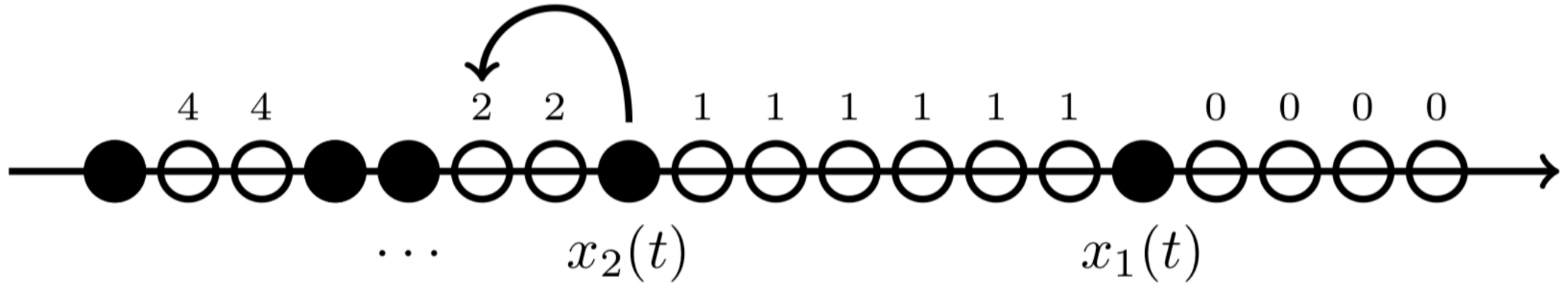


Case $q = 0$

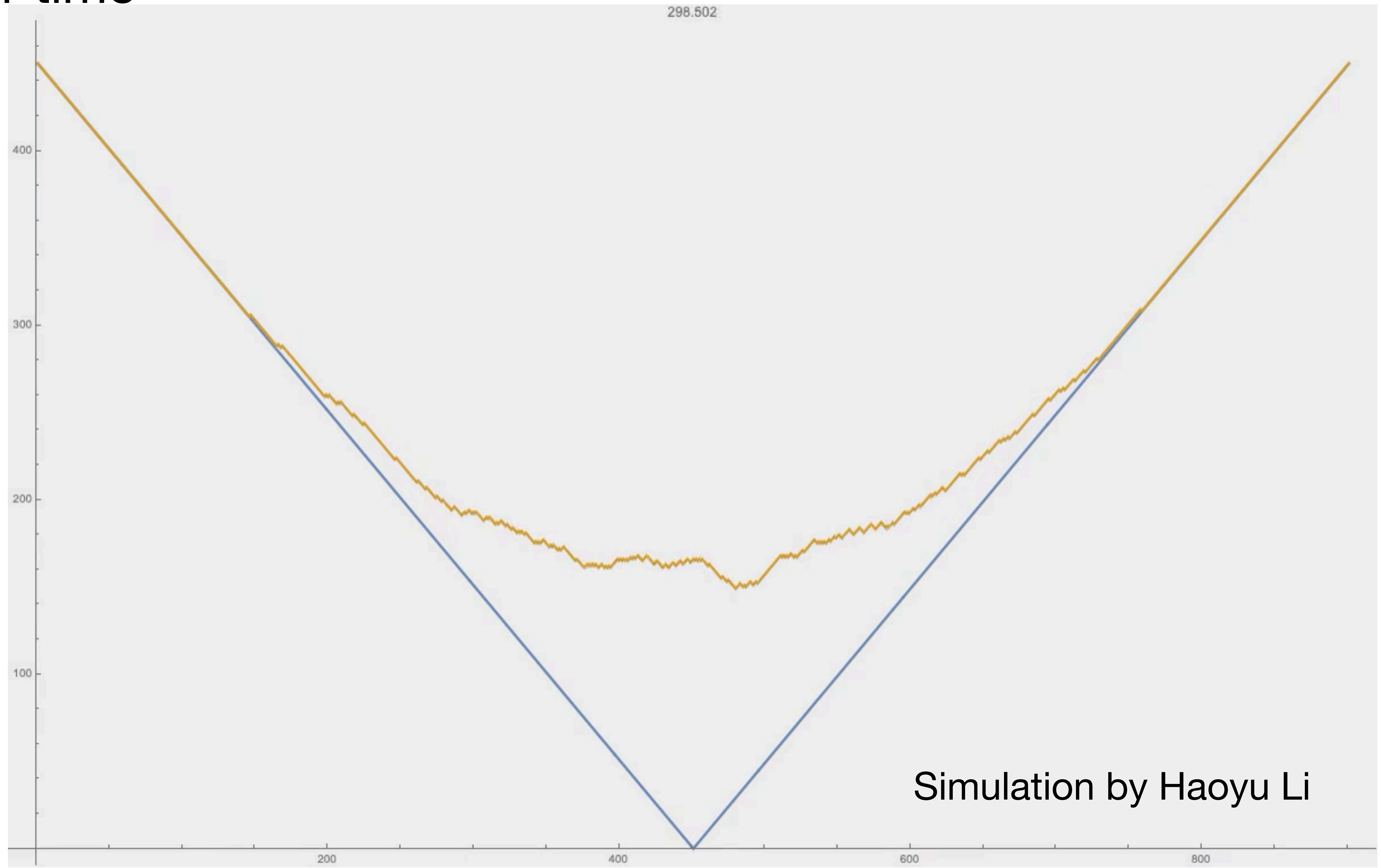


- Each hole has an independent exponential clock with rate equal to the number m of particles to its right, $\mathbb{P}(\text{wait} > s) = e^{-m \cdot s}$, $s > 0$.
- When the clock at a hole rings, the leftmost of the particles that are to the right of the hole instantaneously jumps into this hole
- Because total rate of jump is proportional to the size of the gap, this is a discrete space inhomogeneous version of the Hammersley process **[Hammersley '72], [Aldous-Diaconis '95]**

Running TASEP back in time

Theorem **[P.-Saenz ’19]**. $\delta_{step}T(t)B(\tau) = \delta_{step}T(e^{-\tau}t)$, which means that if we run TASEP from the step (densely packed) initial configuration, and then run the backwards process, then the result is a TASEP distribution at an earlier time

TASEP as a growth process
- the graph shows the height function



Simulation by Haoyu Li

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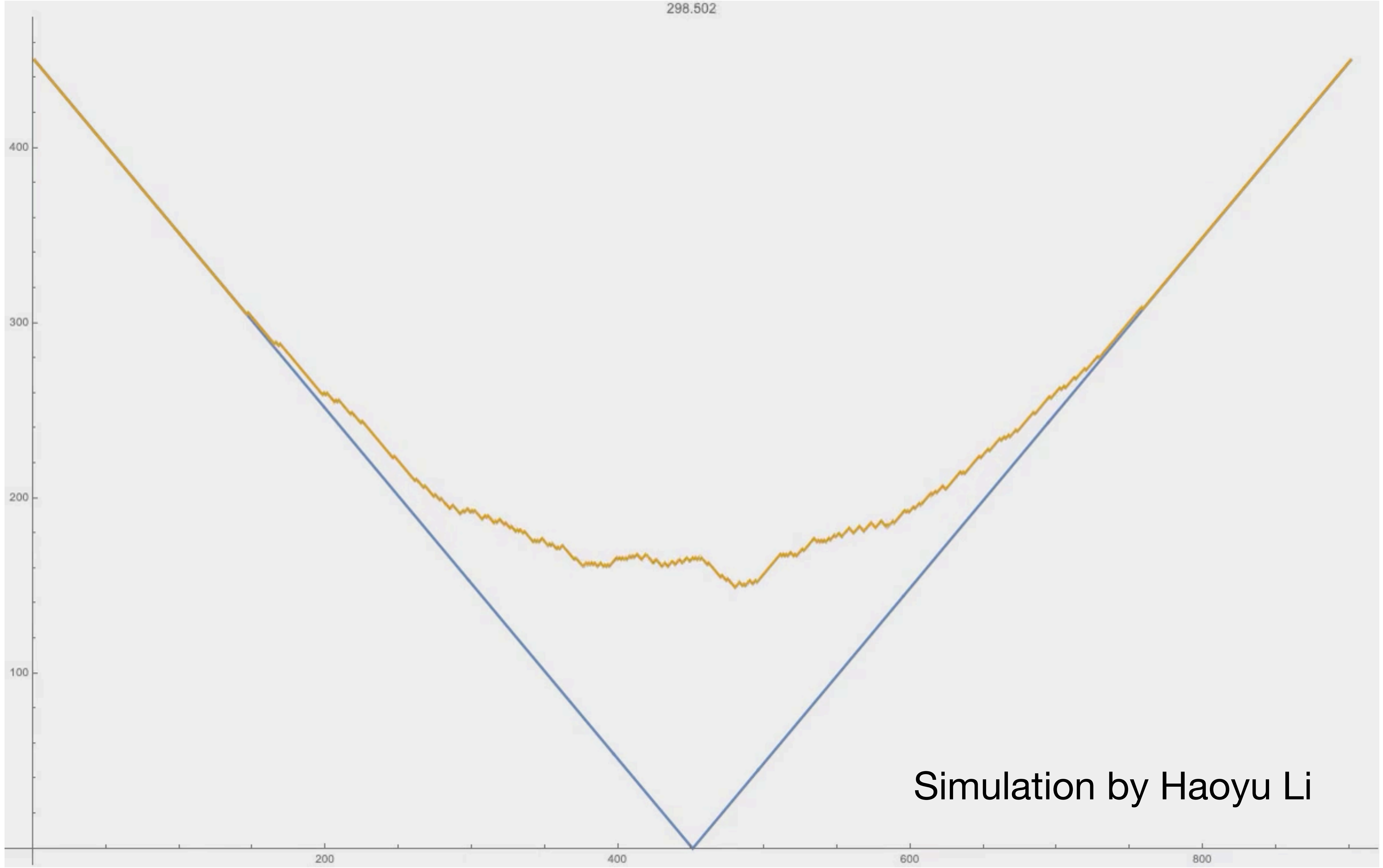

Coupling and rewriting history

Bijectivisation of the Yang-Baxter equation

Let A, B be finite sets and $\sum_{a \in A} w(a) = \sum_{b \in B} w(b)$ (with positive terms)

A **bijectivisation** (**coupling**) of this identity is a family of transition probabilities
$p(a \to b)$ and $p'(b \to a)$, satisfying

$$w(a)p(a \to b) = w(b)p'(b \to a)$$

for all $a \in A,\ b \in B$.

If all probabilities are equal to 0 or 1 and $|A| = |B|$, then this is a usual bijection.

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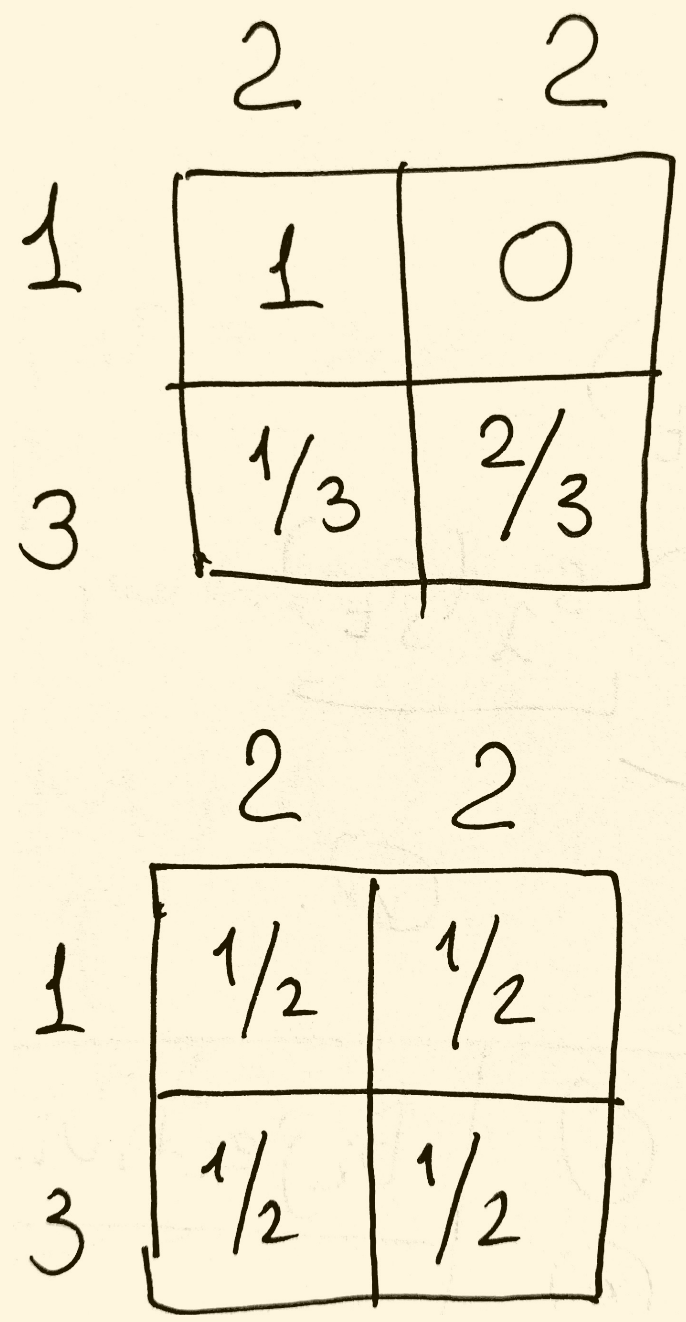
Example: **1 + 3 = 2 + 2**



(maximally dependent)

(independent)

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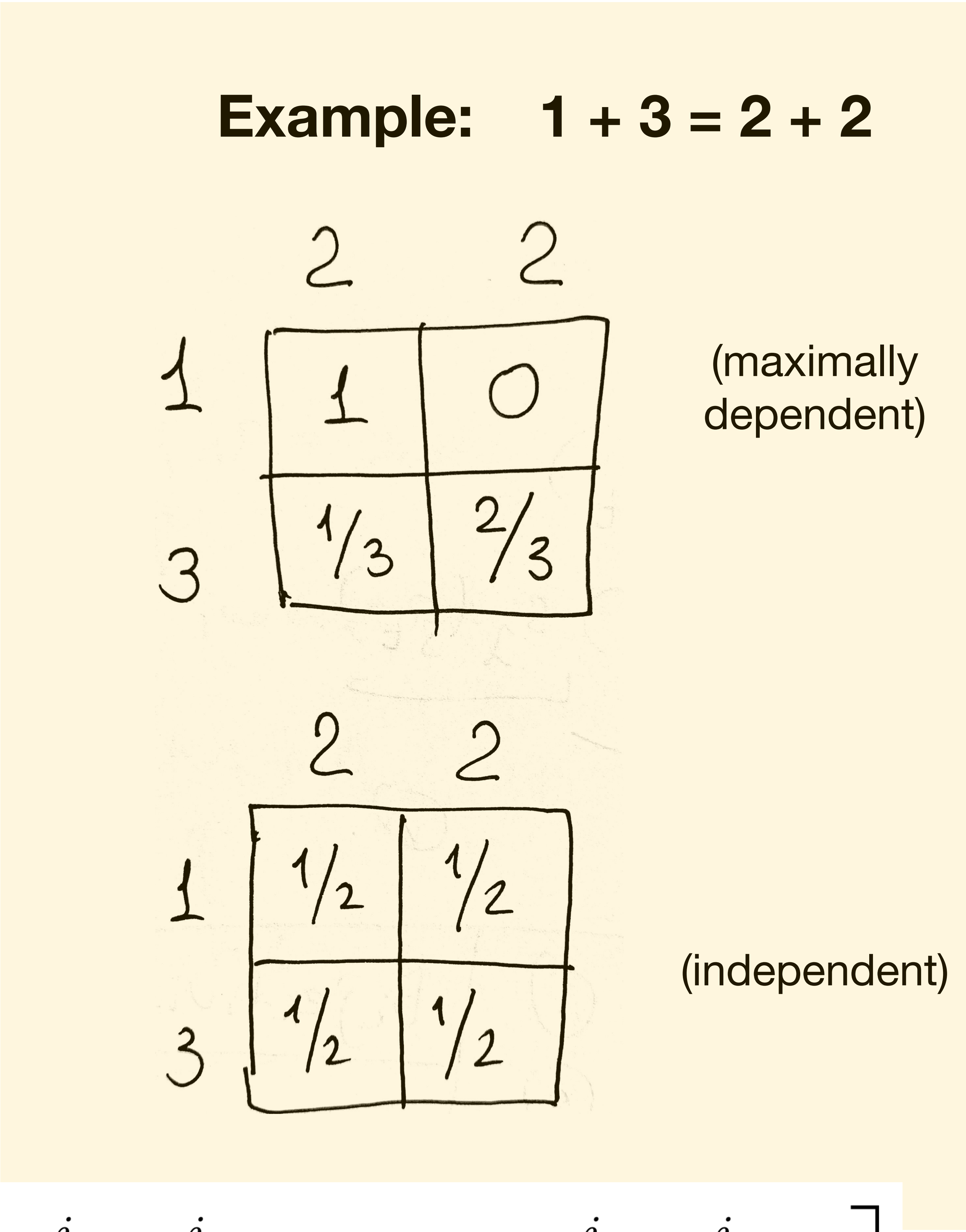

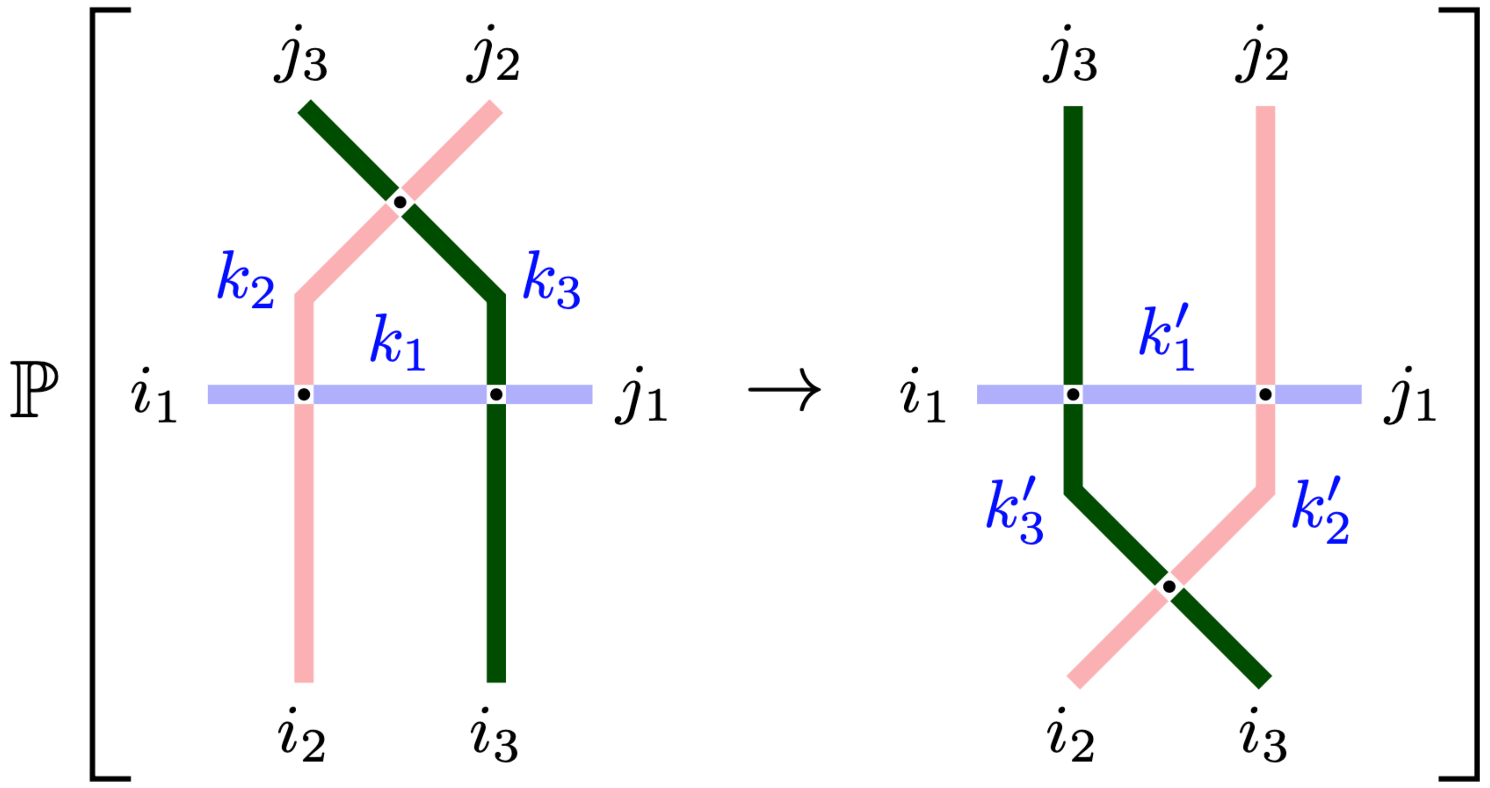

Rewriting history processes

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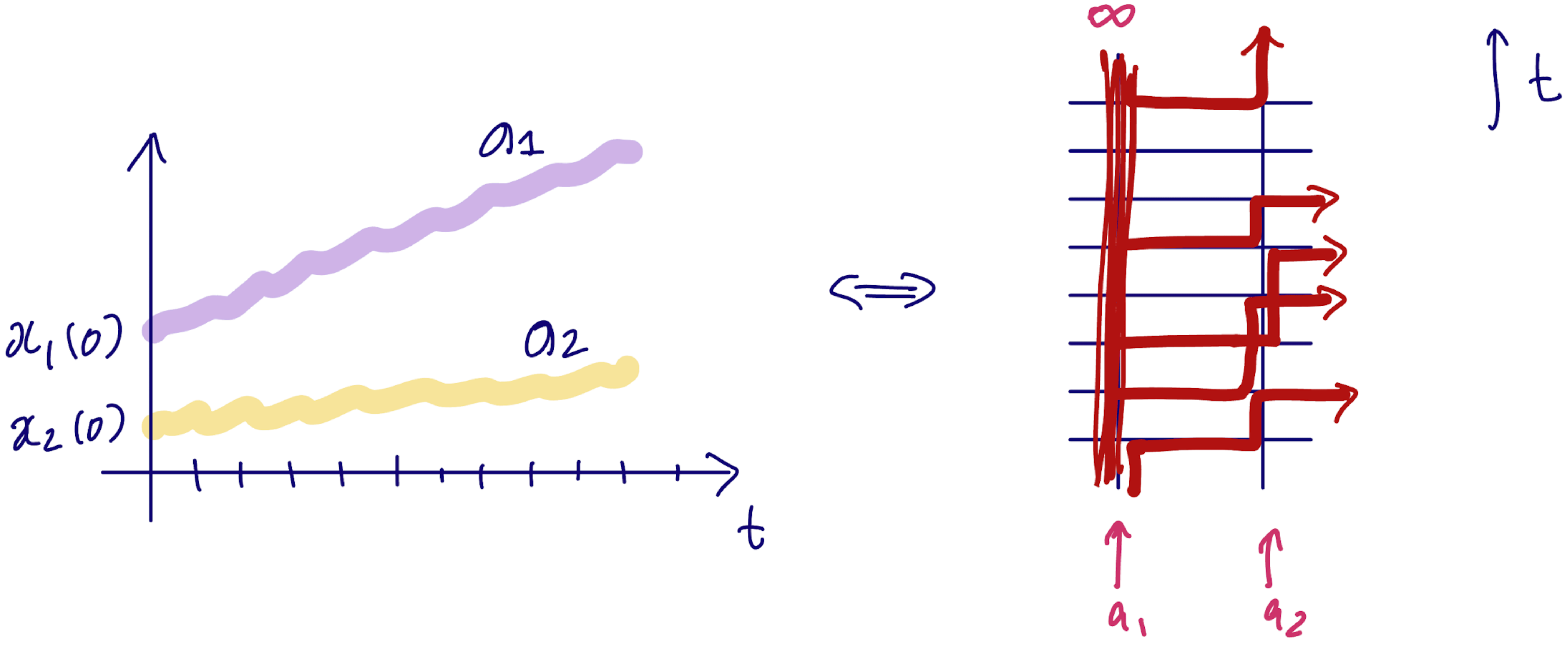

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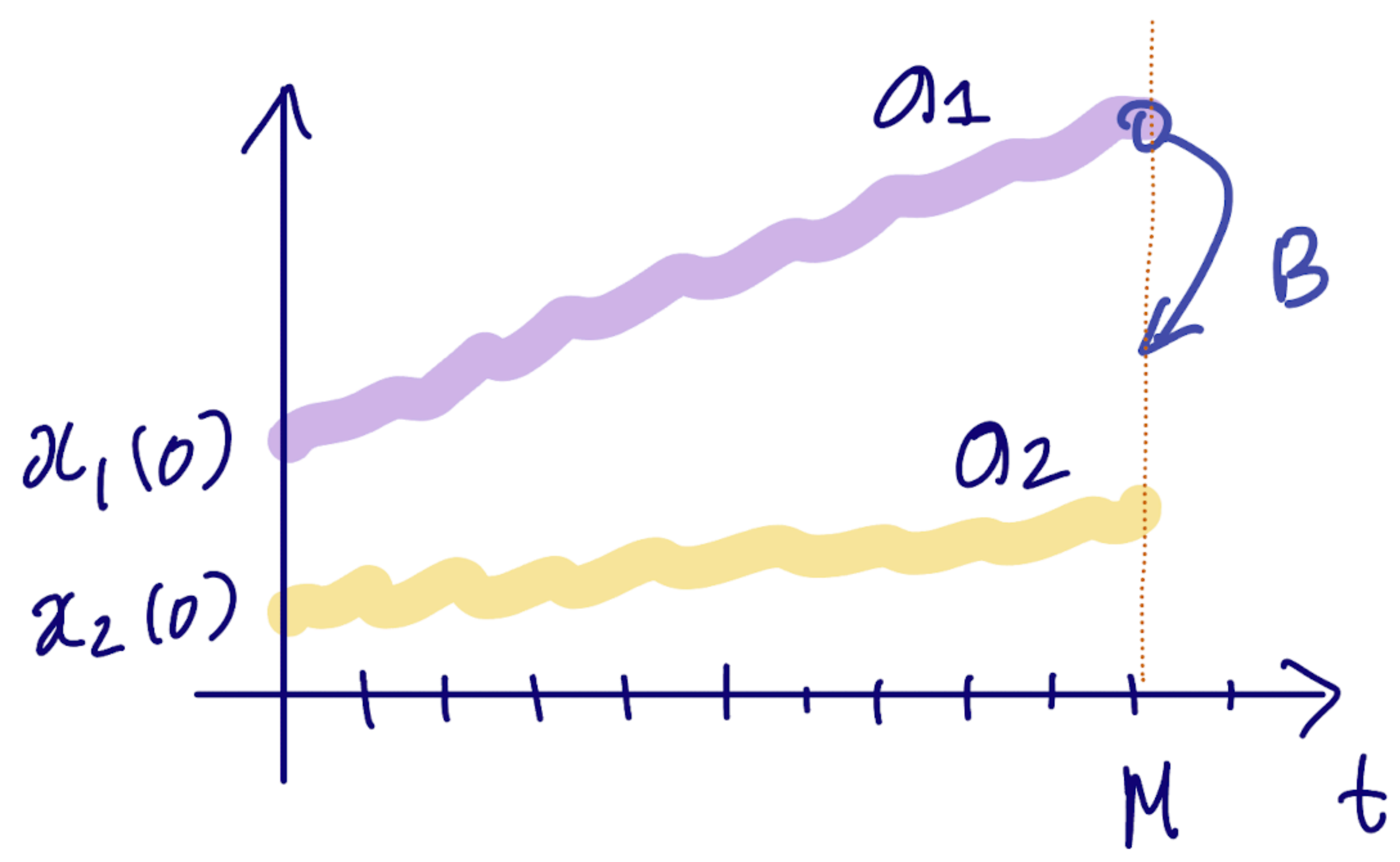
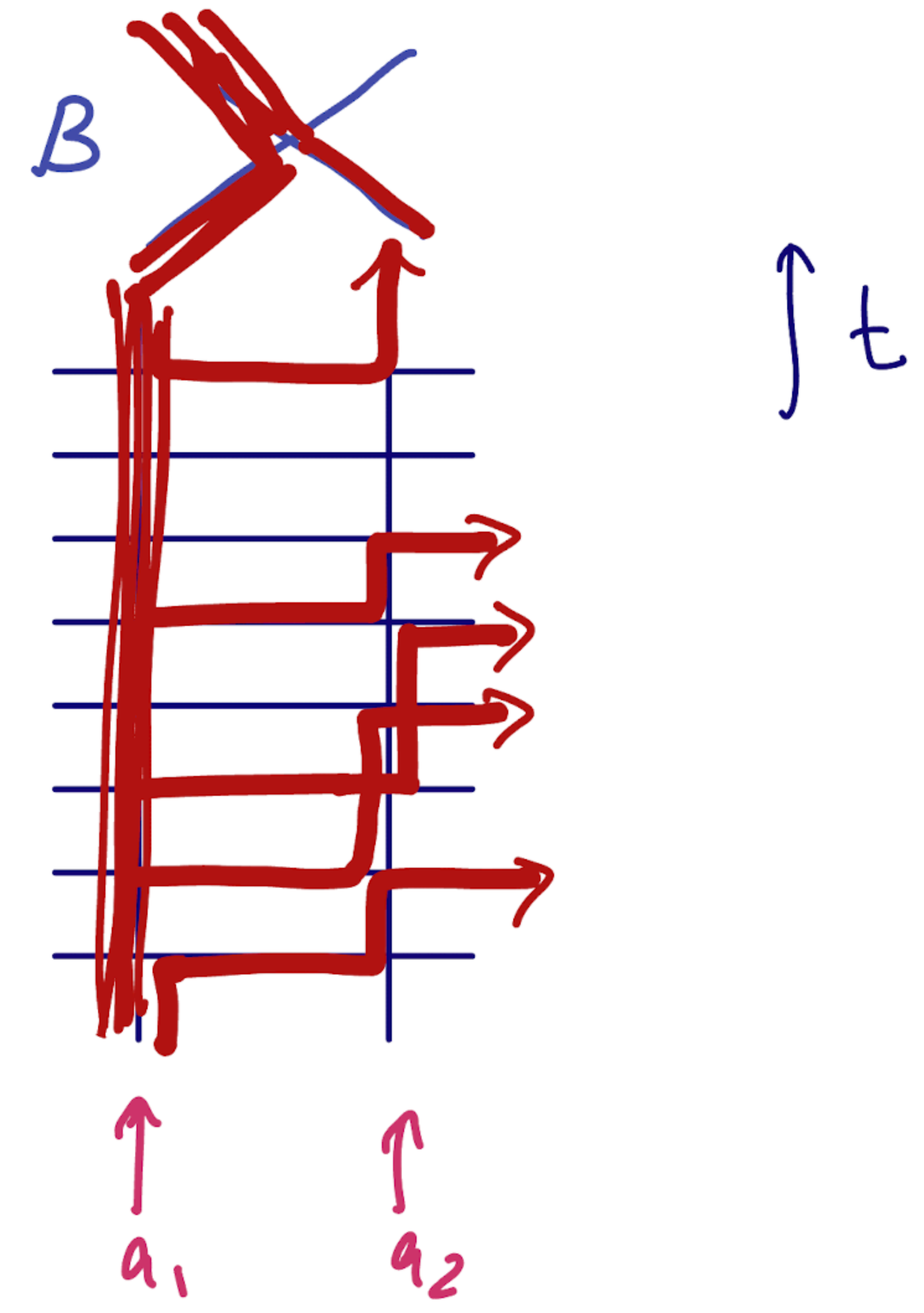


$\Longleftrightarrow$



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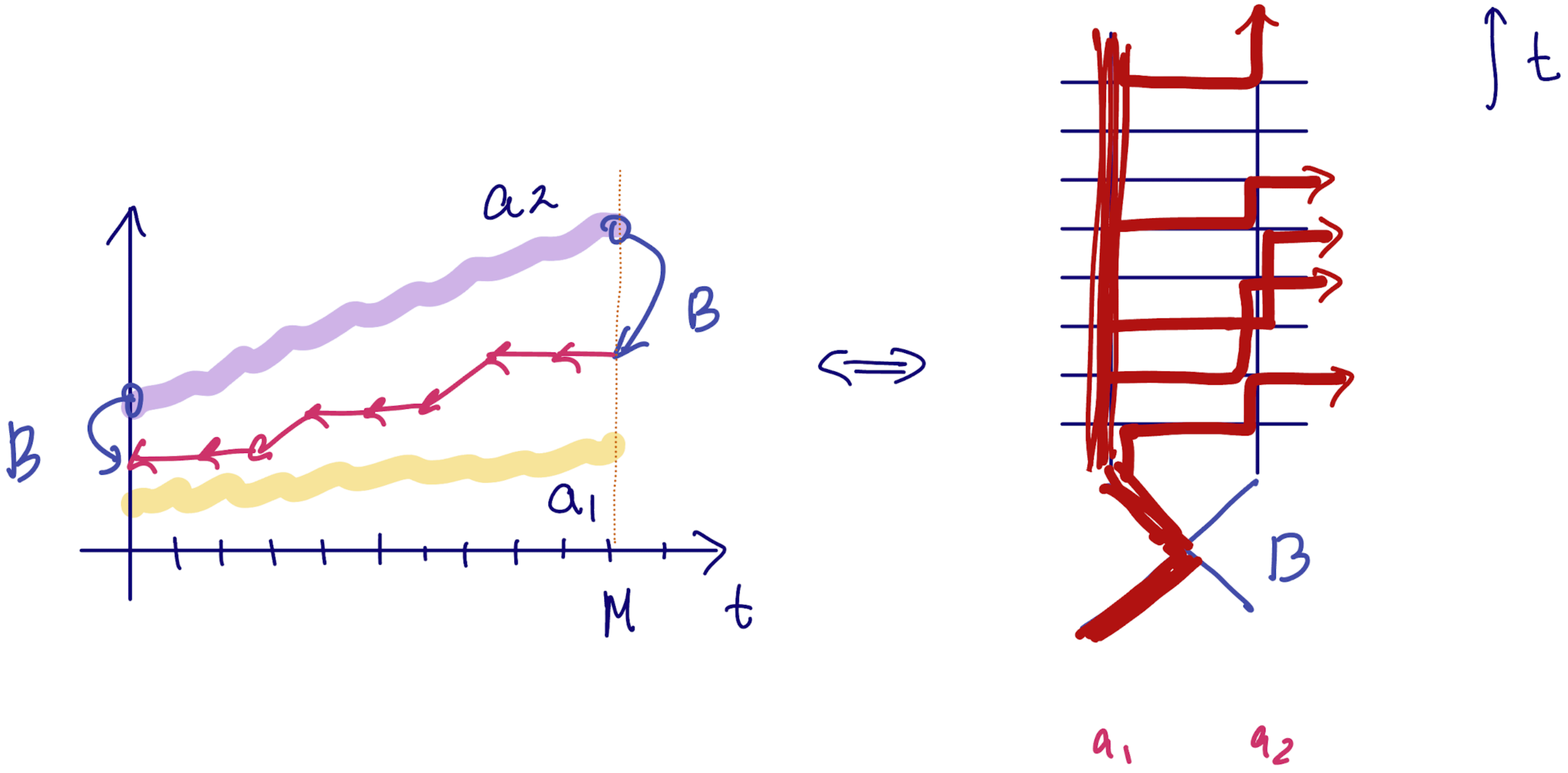
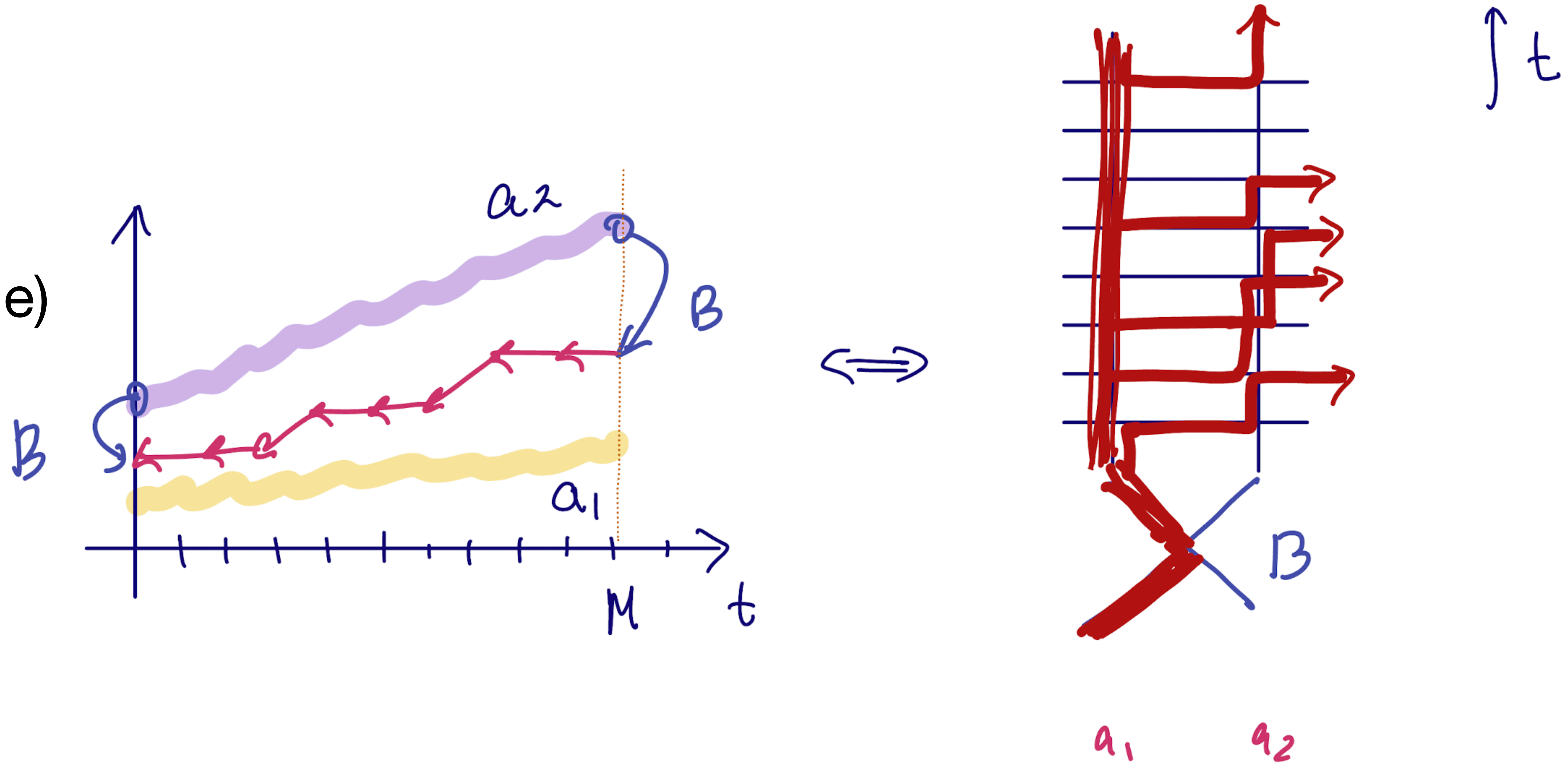


Rewriting history processes

We get two processes for rewriting history:

- from future to past (in the figure)
- from past to future (discussed in next slide)



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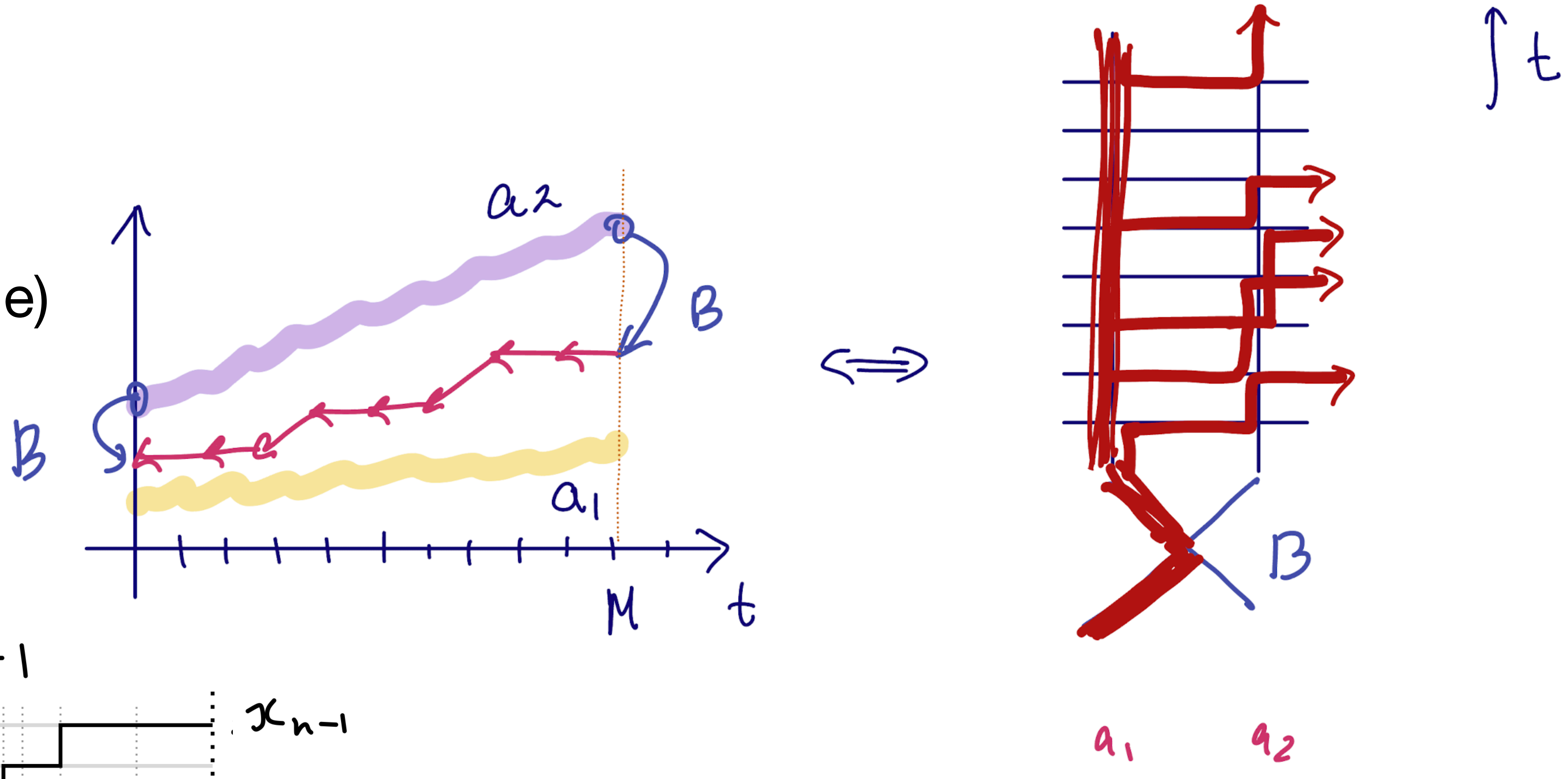

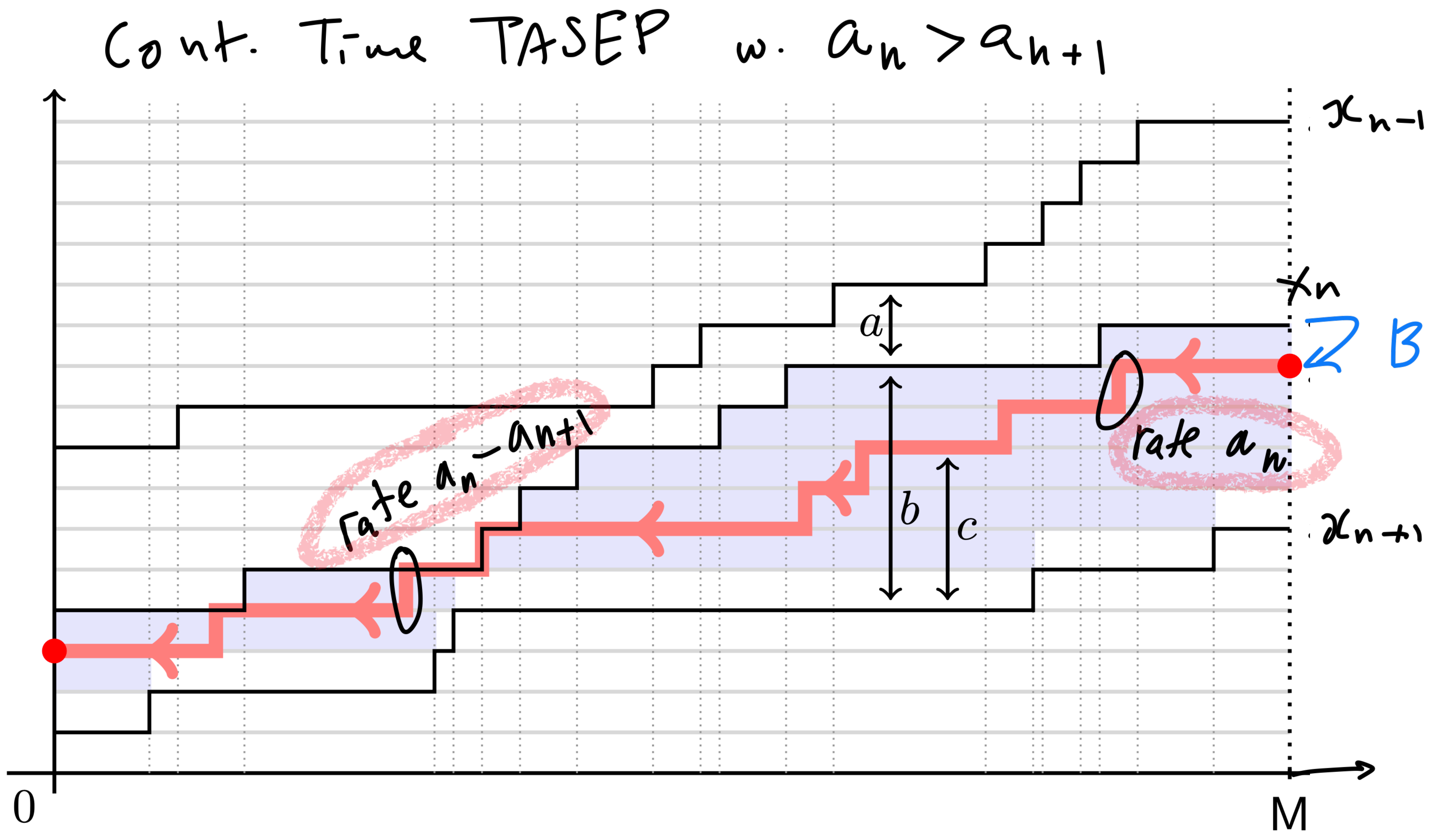

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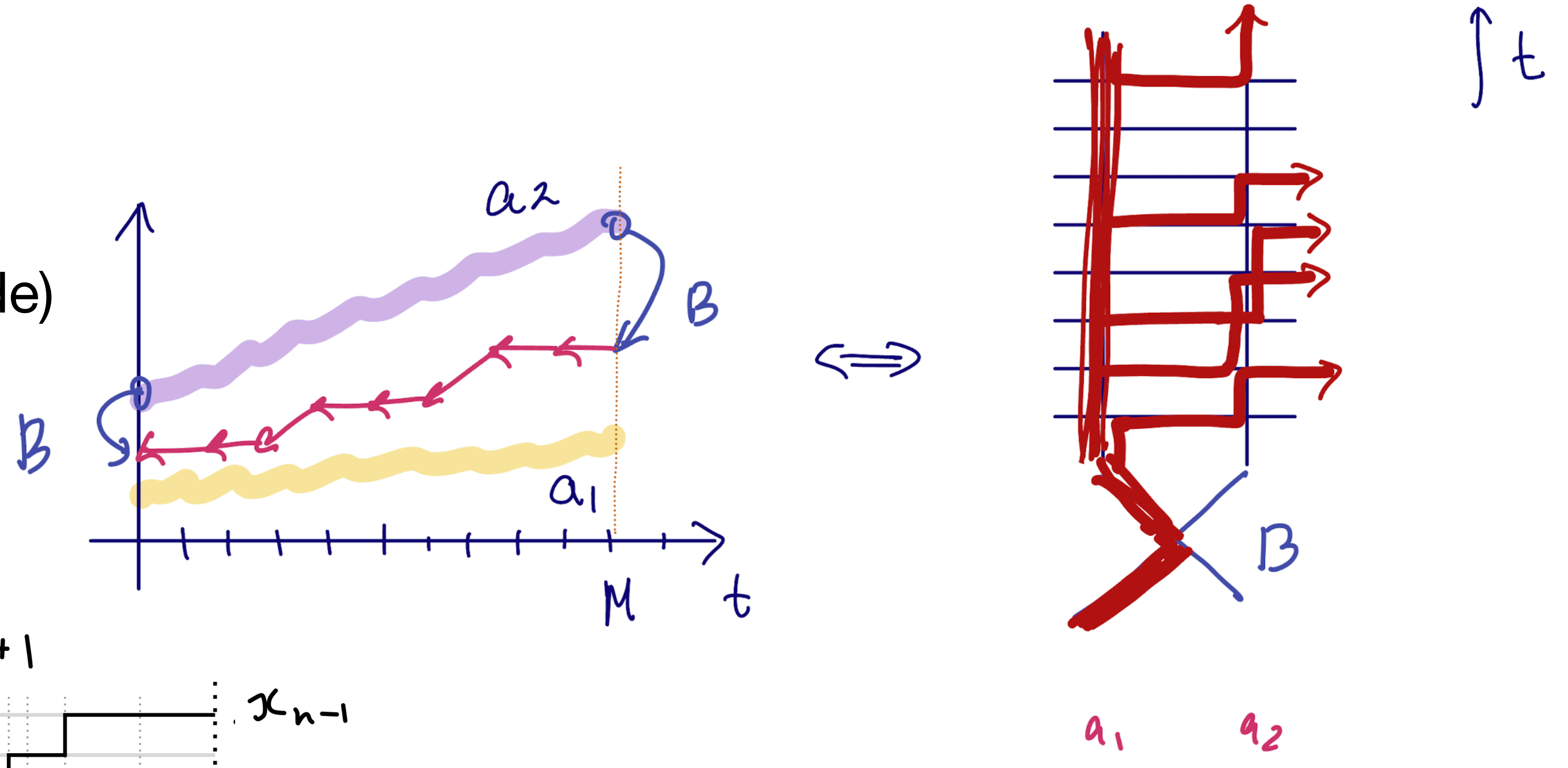
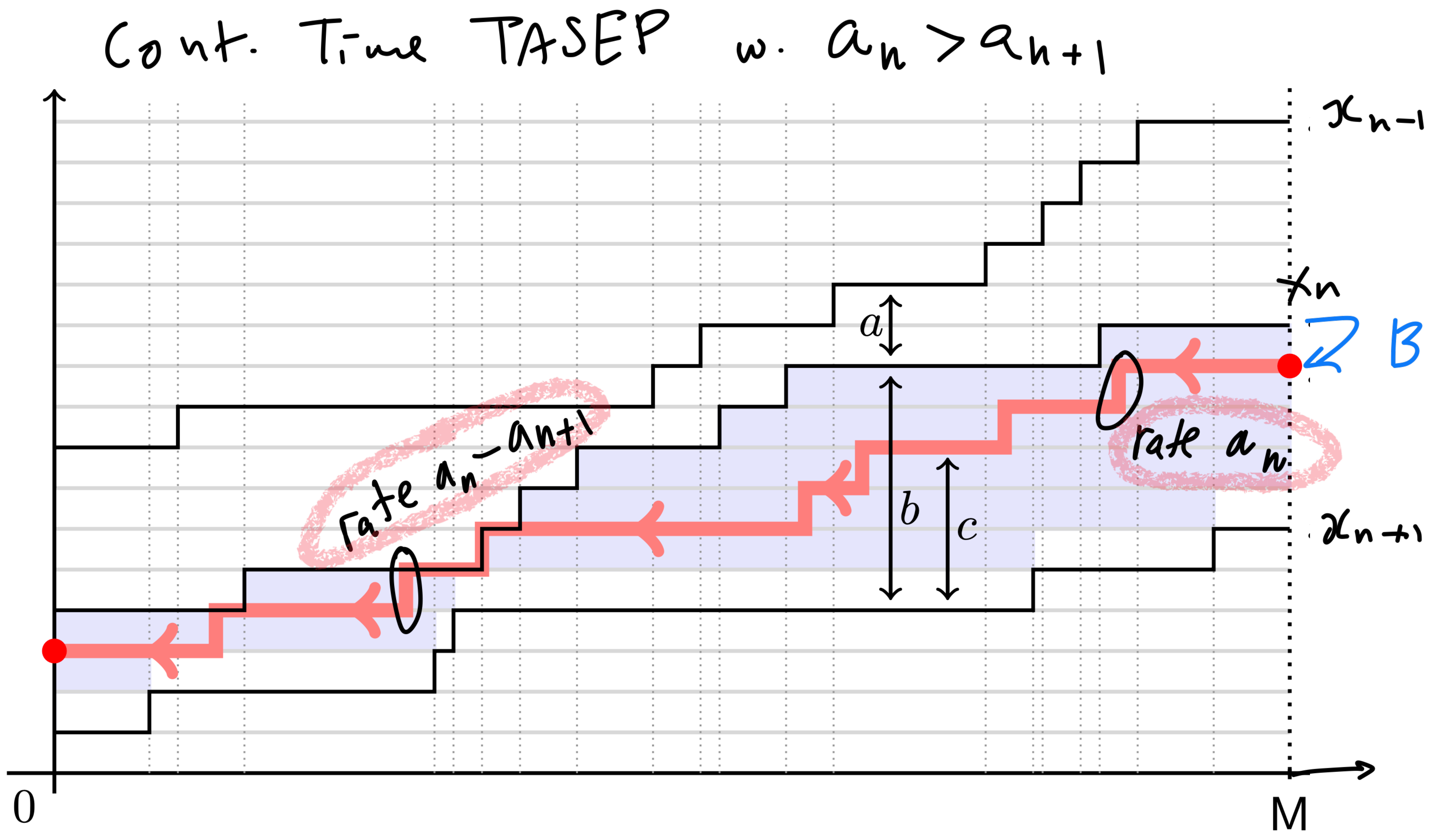




Theorem [P.-Saenz 2022]. Move x_n back by geometric jump B. Then run a random walk x_n' in the chamber between x_n, x_{x+1}, in reverse time, with jump rates down $a_n - a_{n+1}\mathbf{1}_{b=c}$.

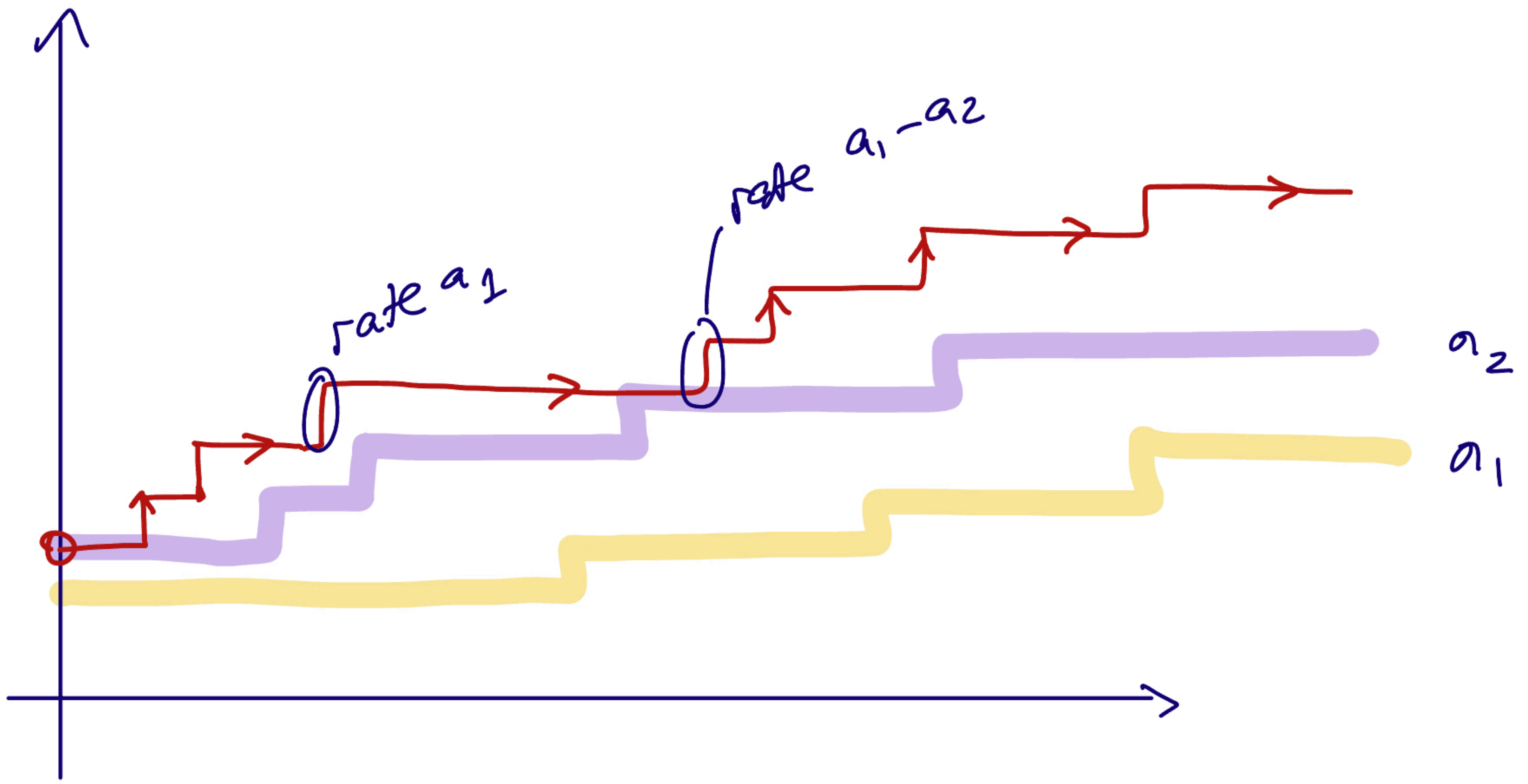
Then the new trajectories are distributed as a TASEP with speeds $(\ldots, a_{n-1}, a_{n+1}, a_n, a_{n+2}, \ldots)$

Last neat (new!) statement about the Poisson process

Rewriting history from past to future for the first particle:

Start from SF system, then the joint distribution of the red and yellow trajectories is **the same** as the FS system, i.e. TASEP with speeds $a_1 > a_2$.

Rewriting history is independent from the second particle (TASEP feature only)



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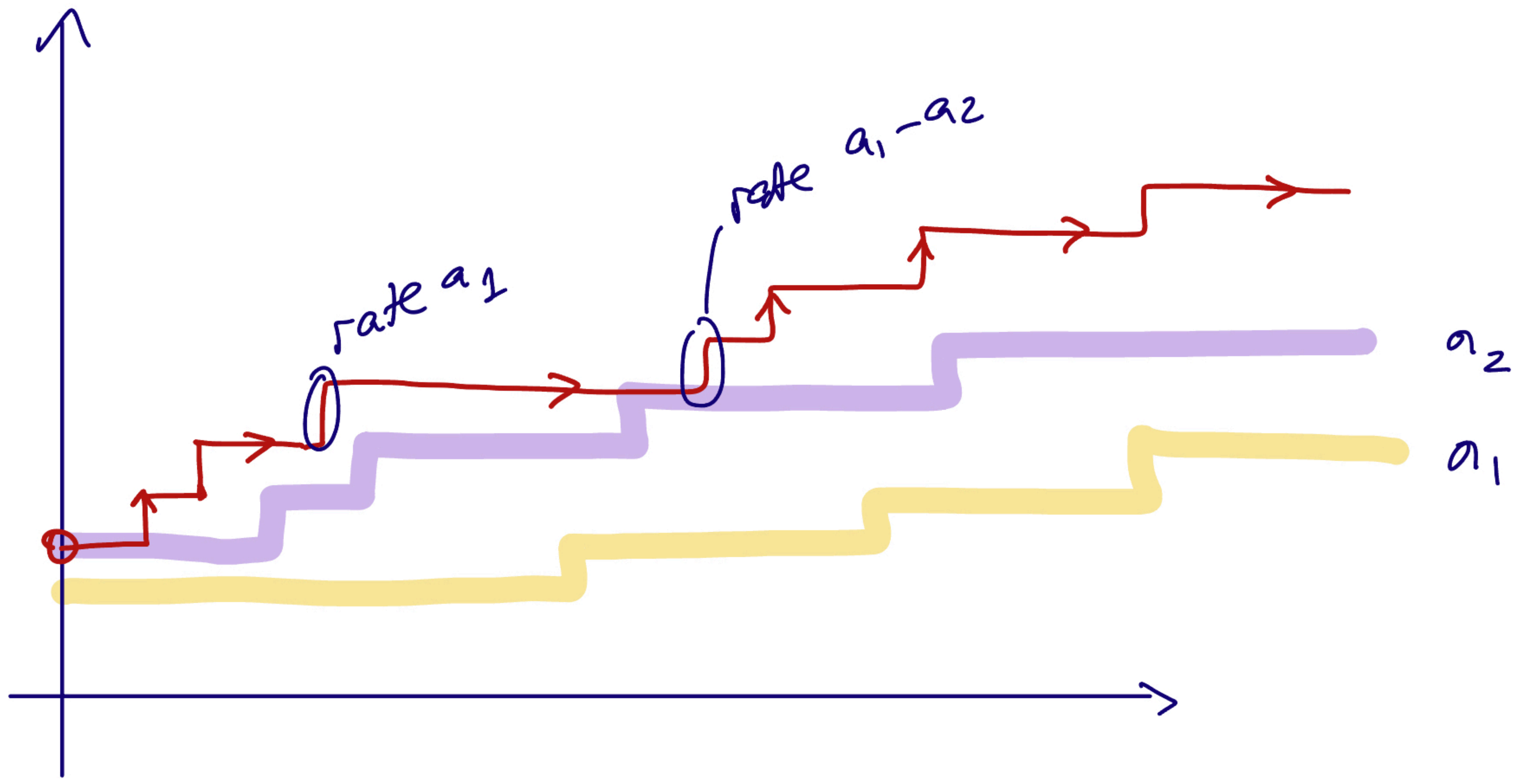

Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

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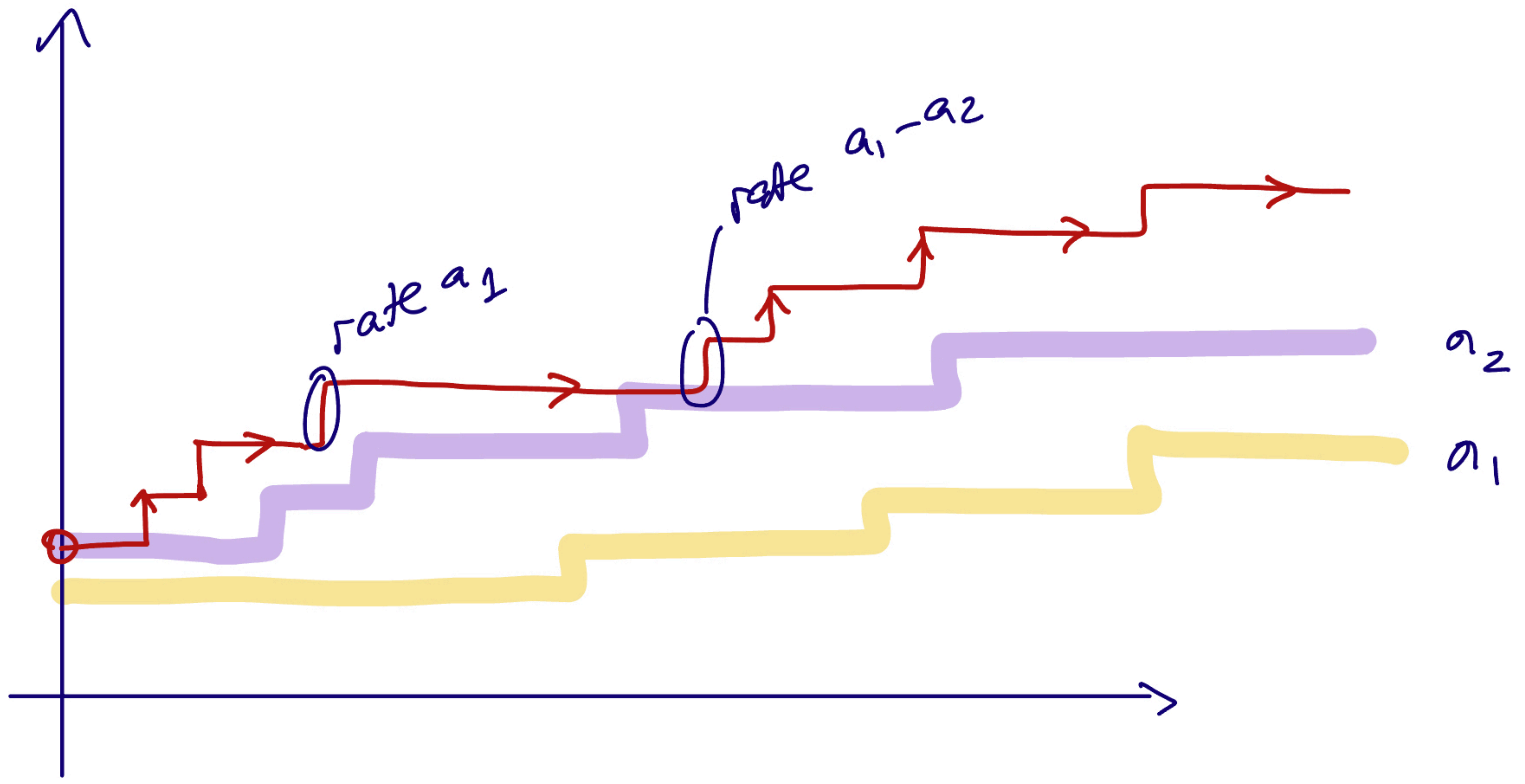

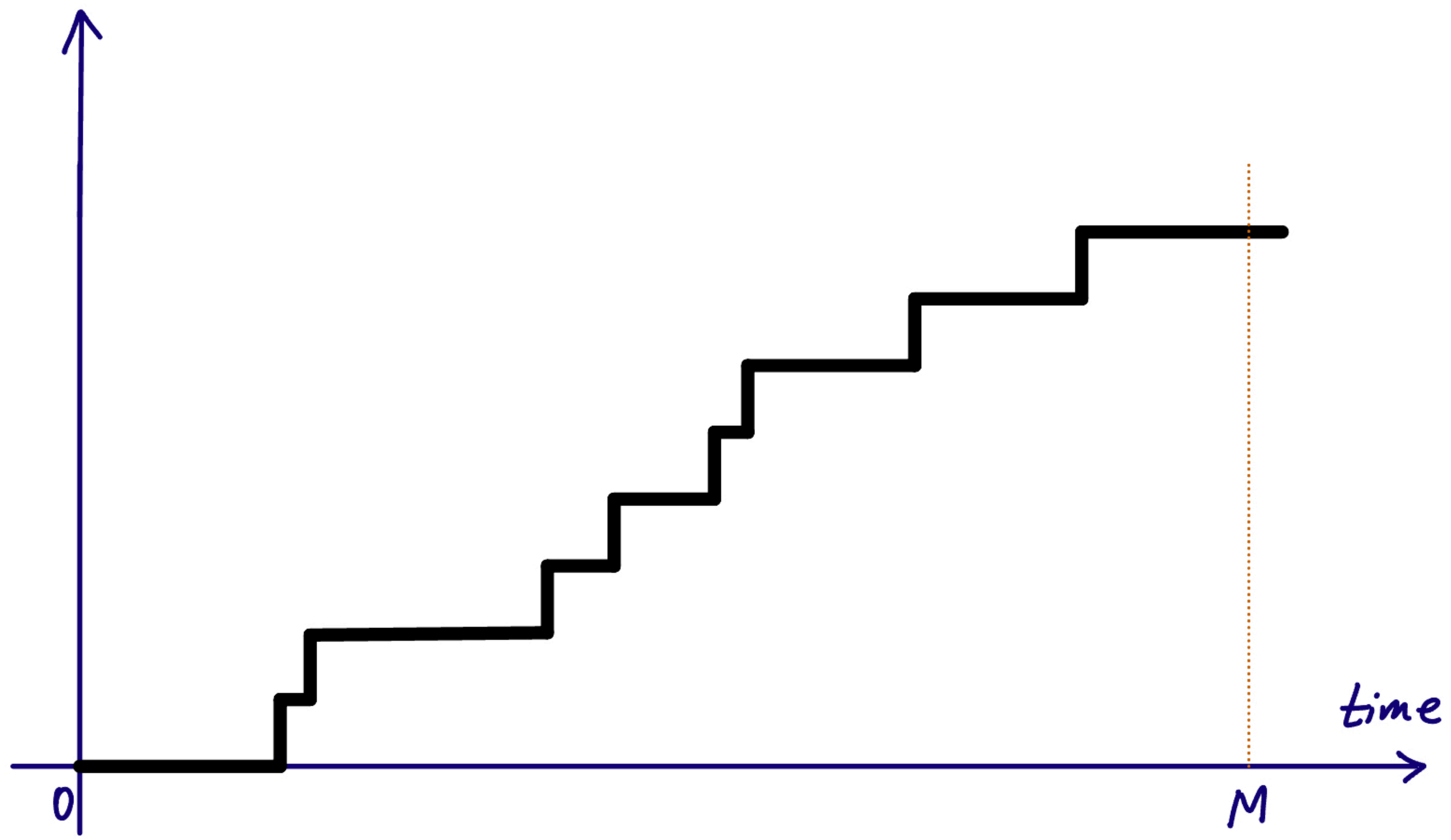

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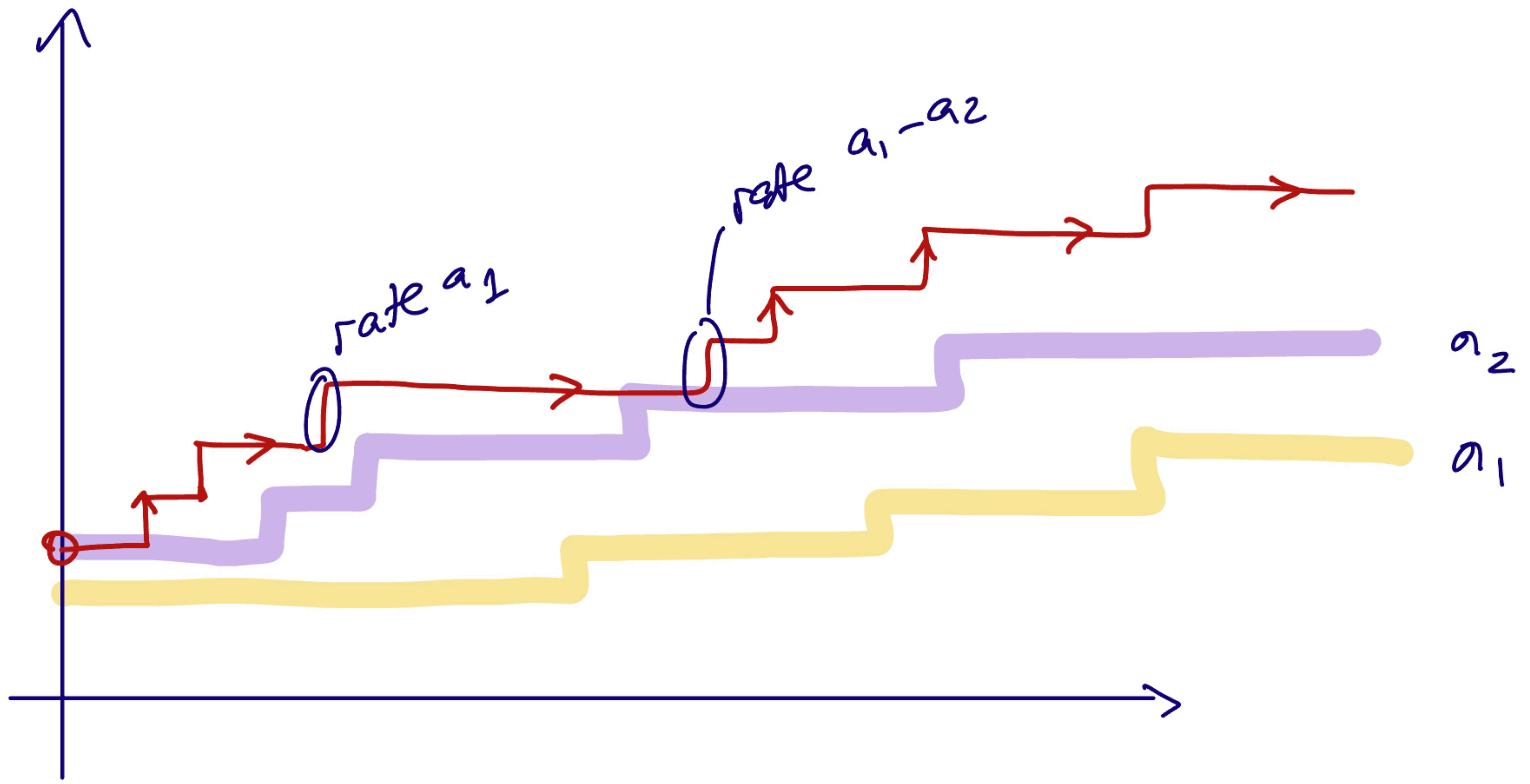

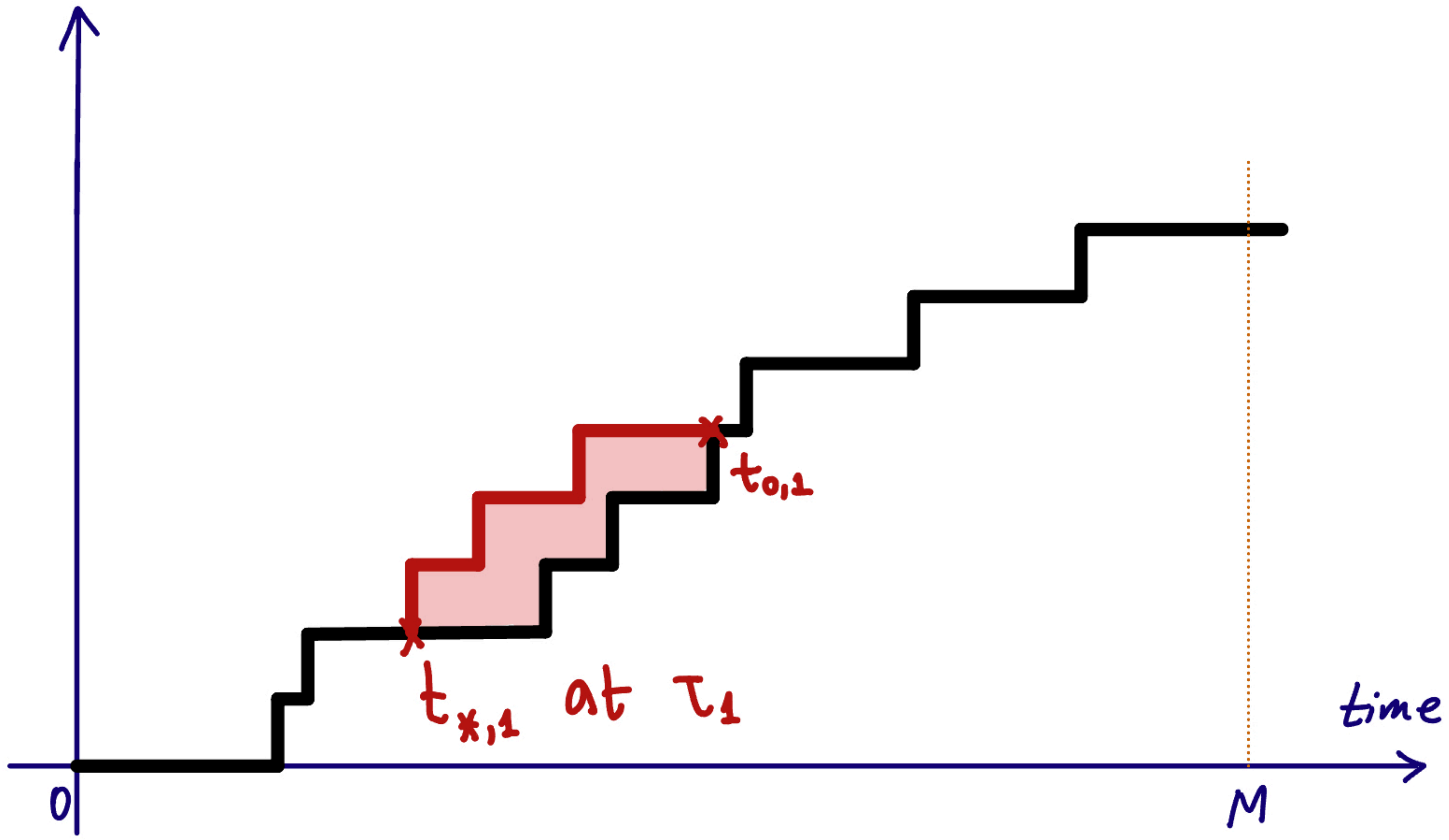

Take a limit $a_1, a_2 \to 1$. Then we get a time-inhomogeneous, continuous time dynamics which increases the slope of a Poisson random walk which initially had slope 1. At time τ, dynamics starts a new slab at a random location t_*, and then the slab has slope $\tau + 1$.

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Last neat (new!) statement about the Poisson process

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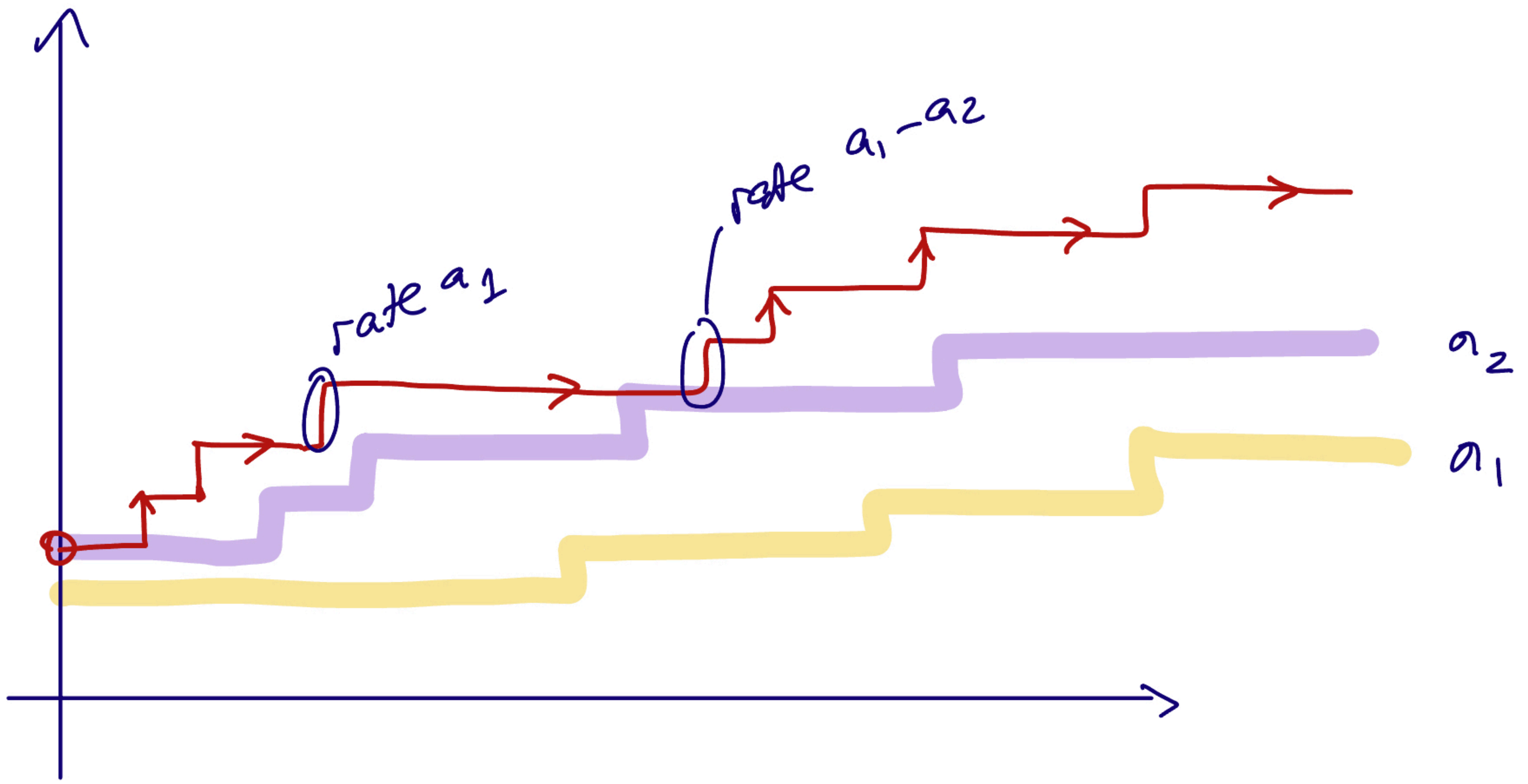

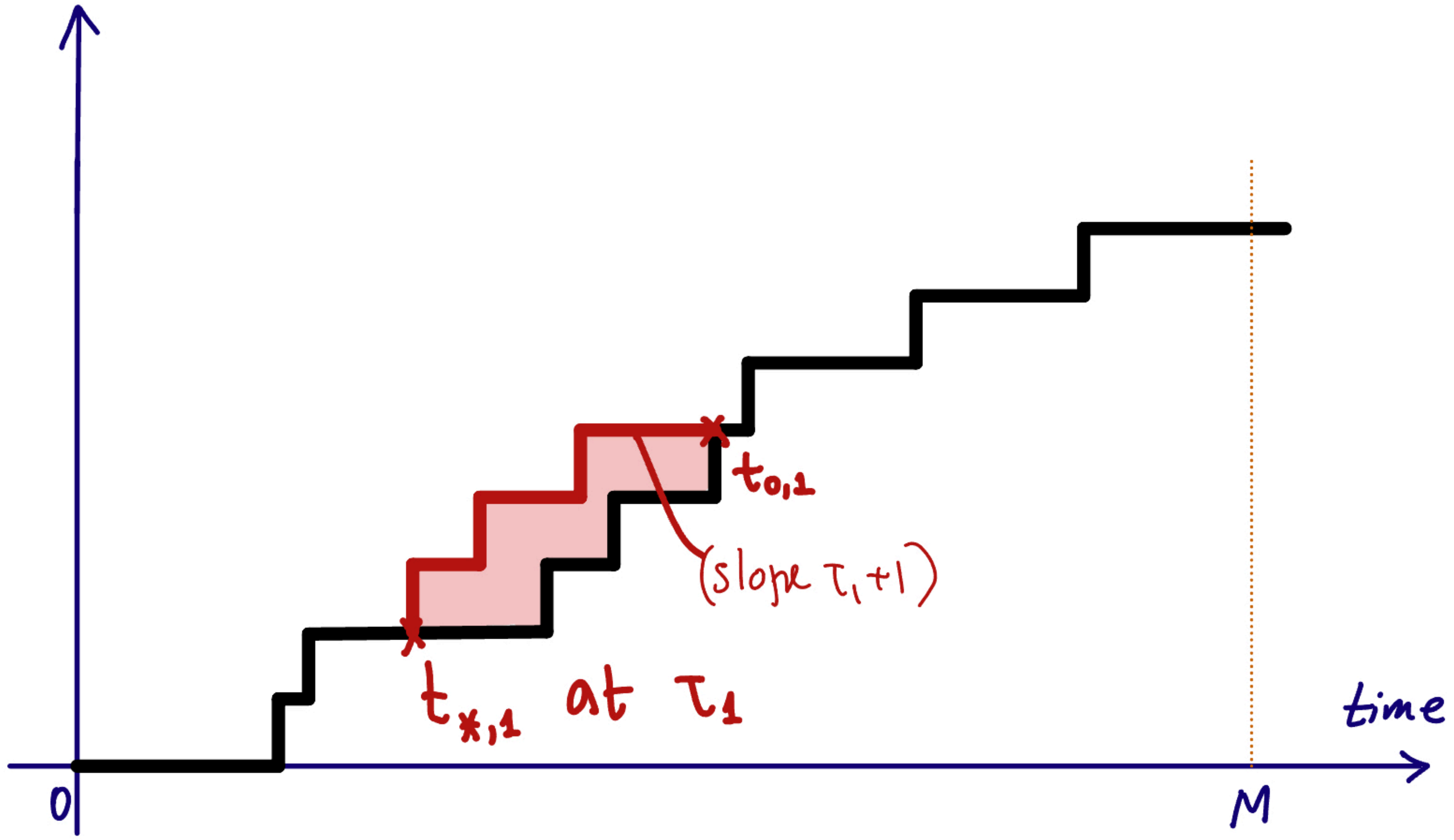

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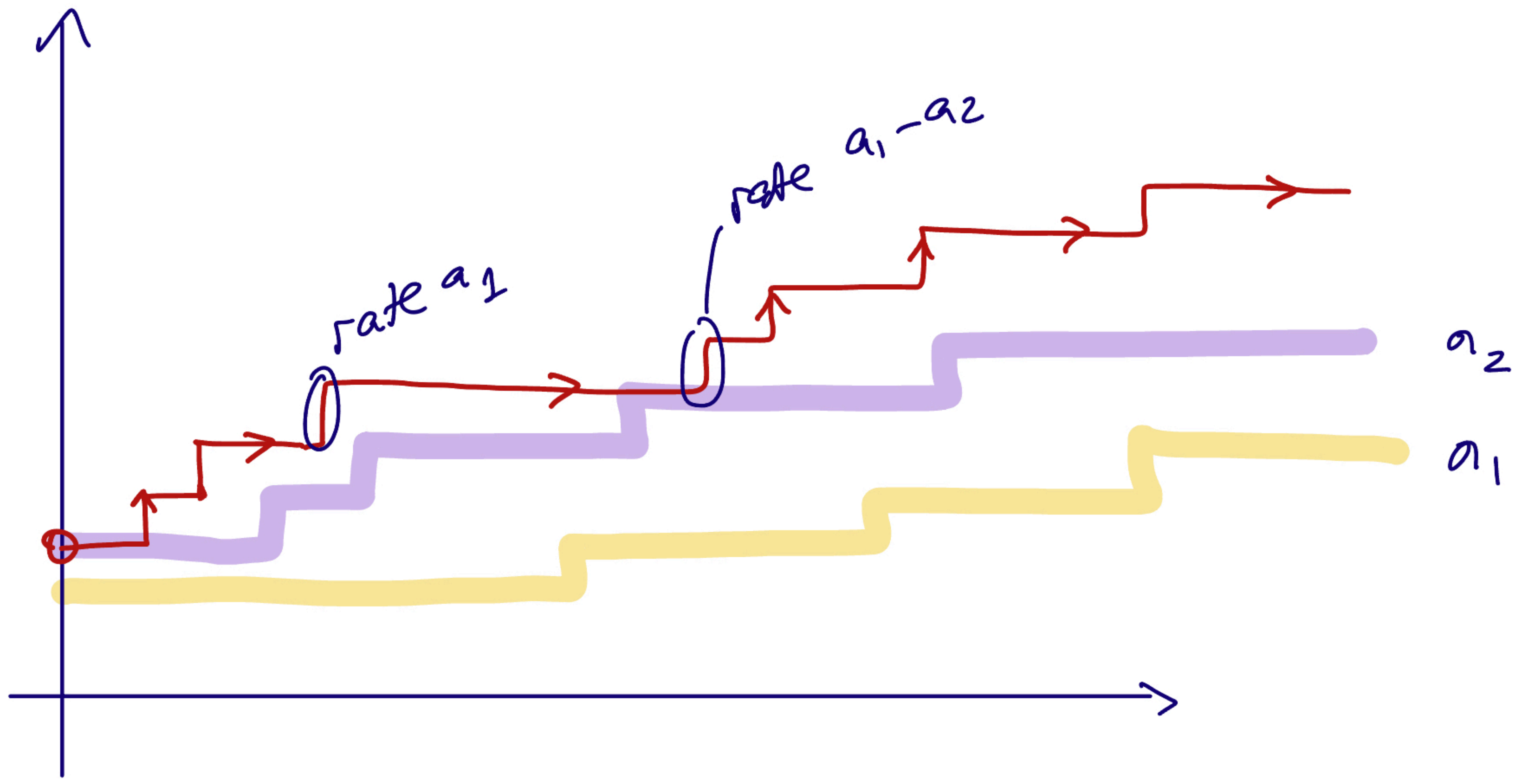

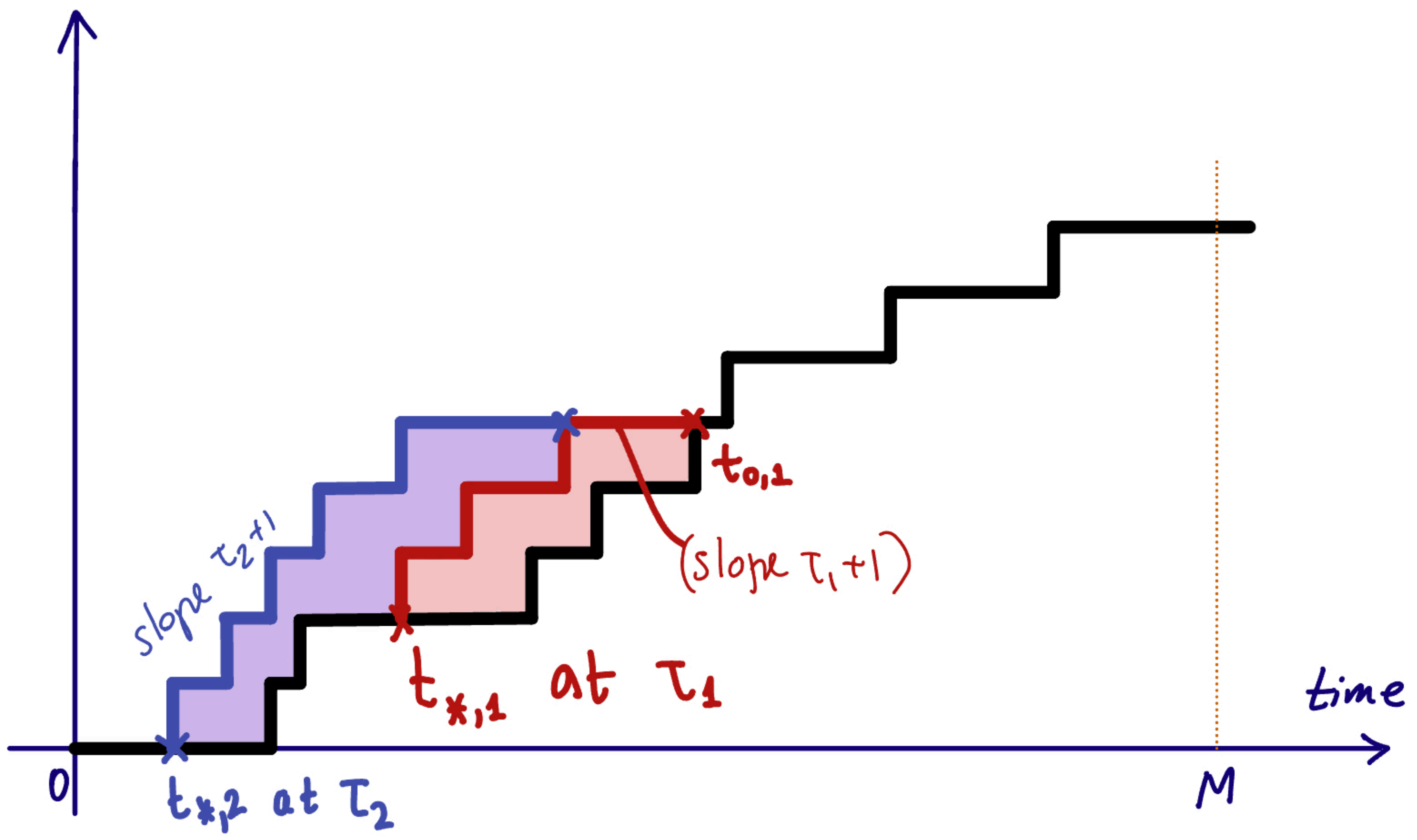

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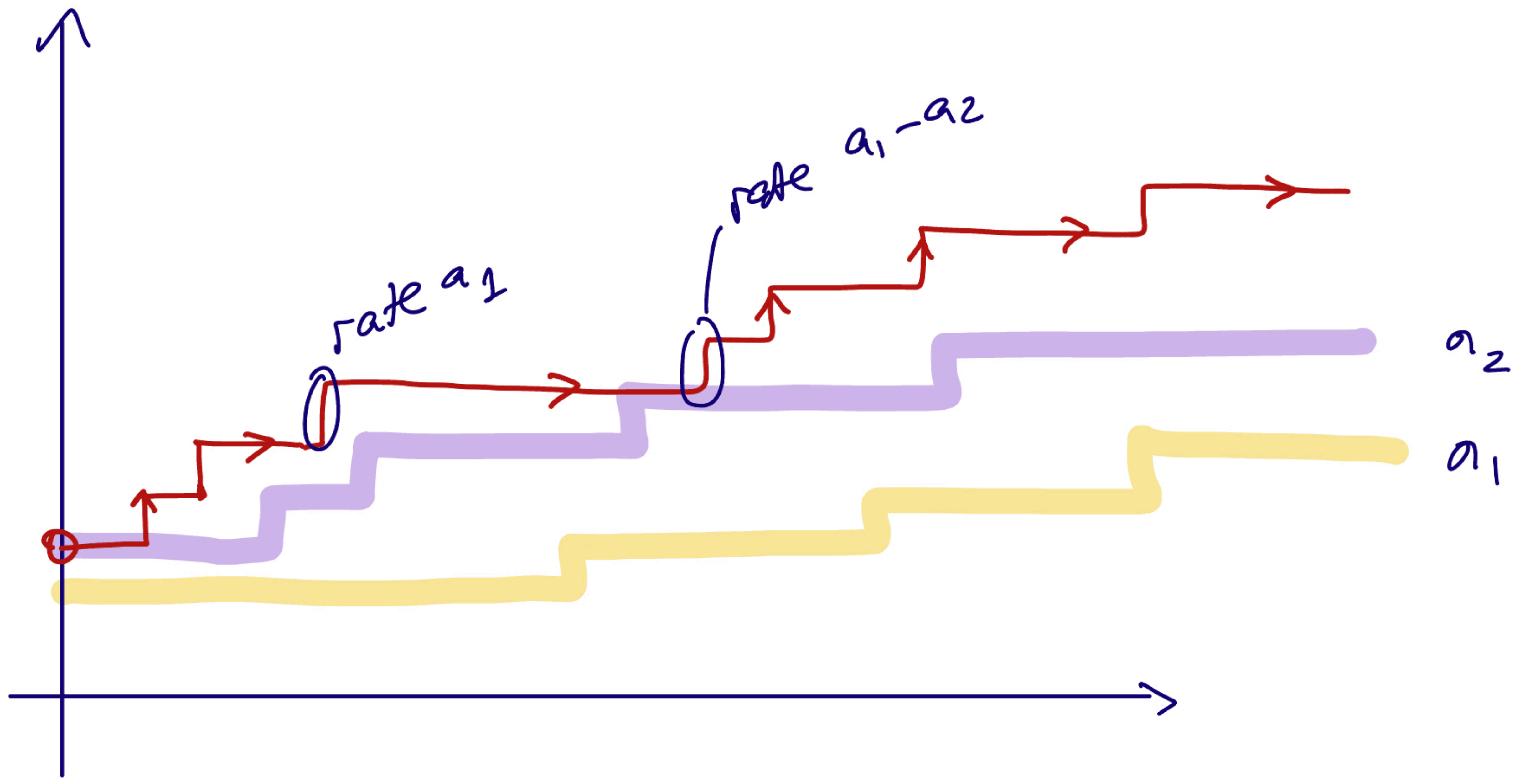

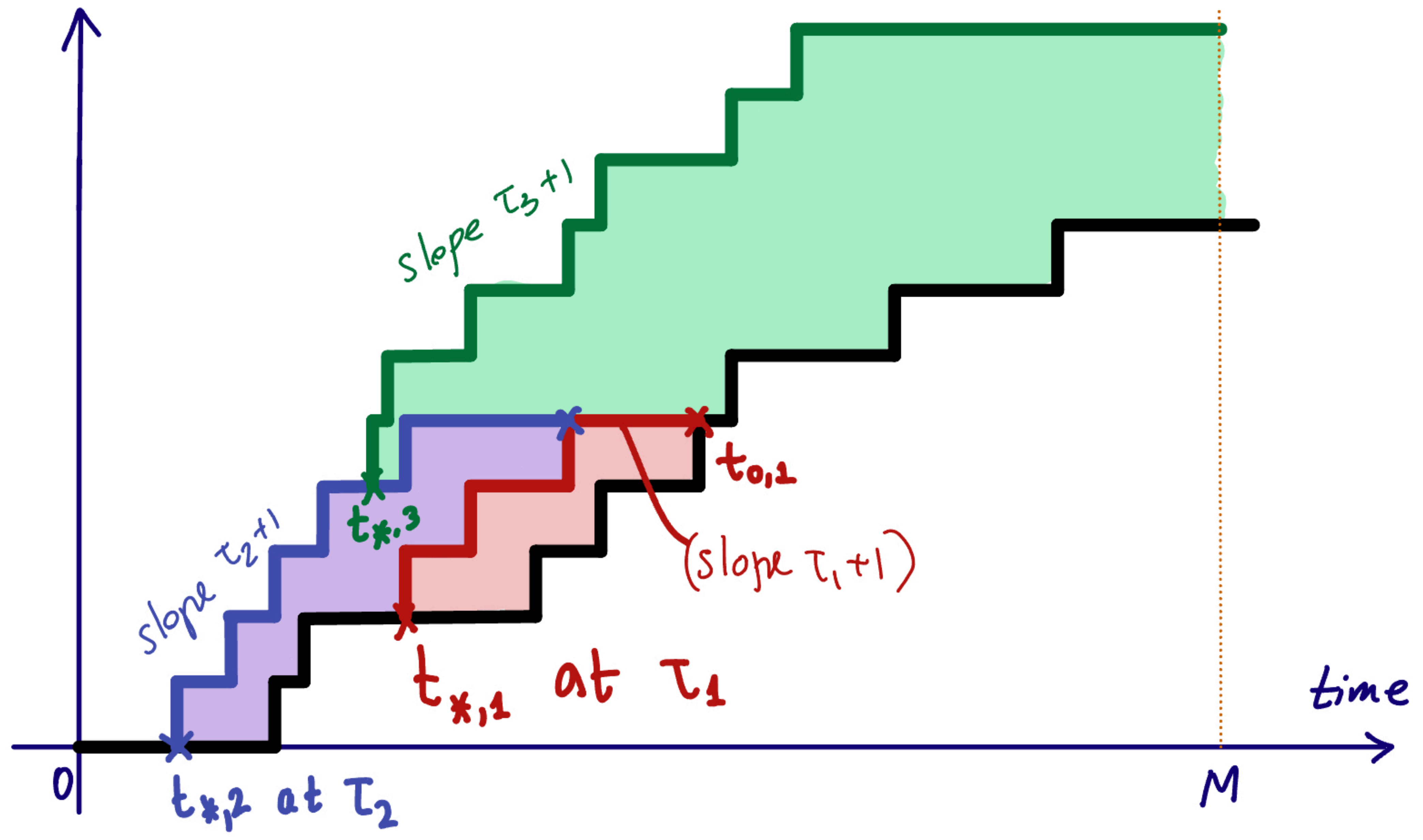

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- We discover Lax equations for q-TASEP and TASEP (their role should be better explored)
- We also construct neat couplings for the whole trajectories, and get a new result about the Poisson process

Thank you for your attention!

In stochastic particle systems, there’s a way
To rewrite history with each passing day.
A single particle, its fate made clear,
Can undo what’s been done and make it reappear.

The laws of probability and chaos at play
Can be bent to our will, if we but obey.
The deterministic systems in our control,
Will yield to a new order, as it starts to unfold.

The particles and their interactions will dictate,
The outcome of our systems, no matter their state.
With the tools of integrability, we can rewrite,
The future of our systems with a single bite.

- openAI