Random 3D surfaces and their asymptotic behavior

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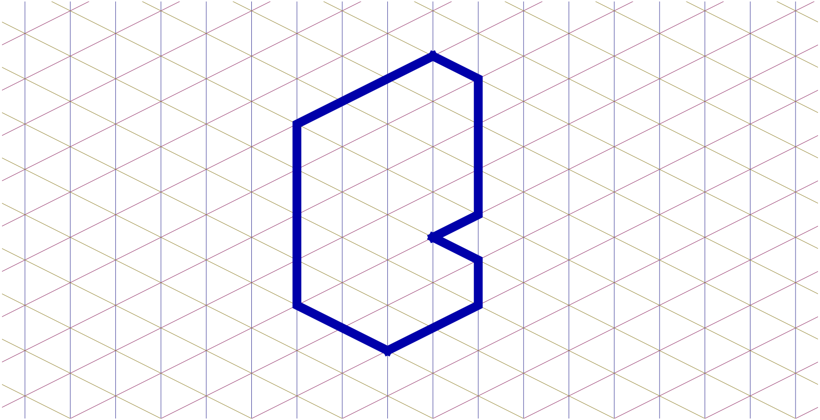
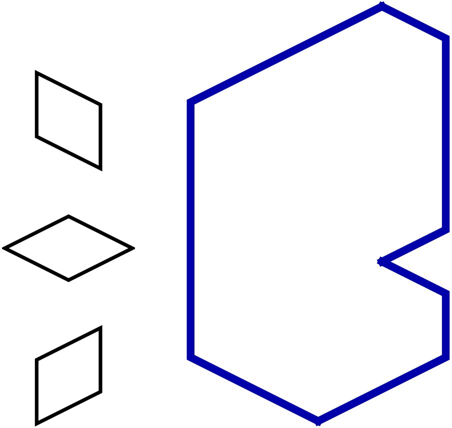
Lozenge Tilings

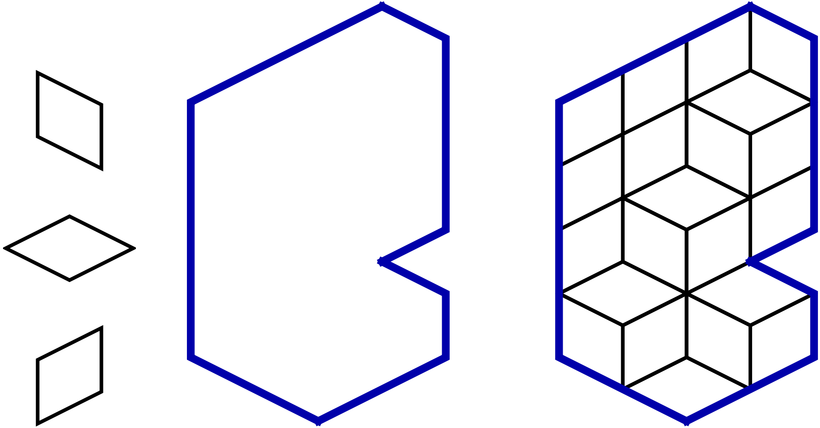
Polygon on the triangular lattice



Lozenge tilings of a polygon

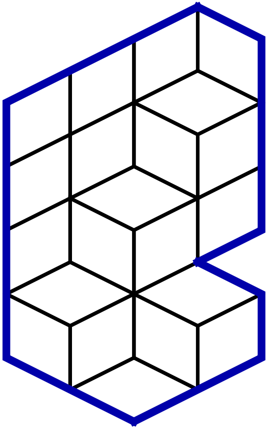


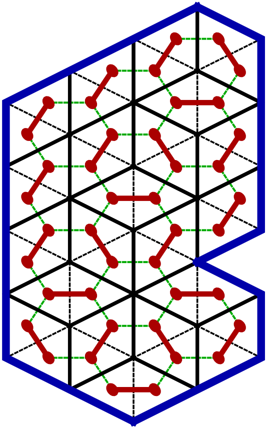
Lozenge tilings of a polygon



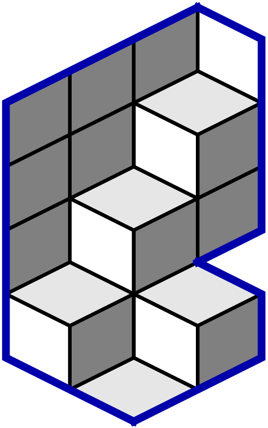
Remark

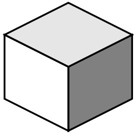
Lozenge tilings $\Longleftrightarrow$ Dimer Coverings





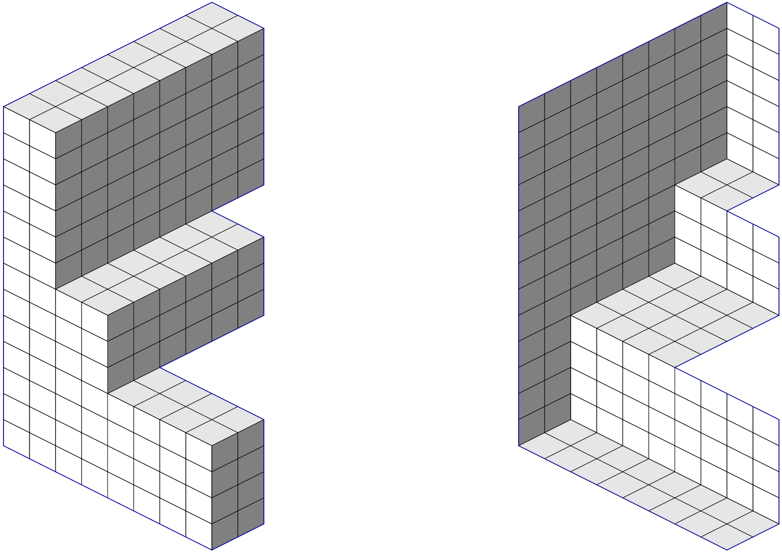
3D stepped surfaces with "polygonal" boundary conditions



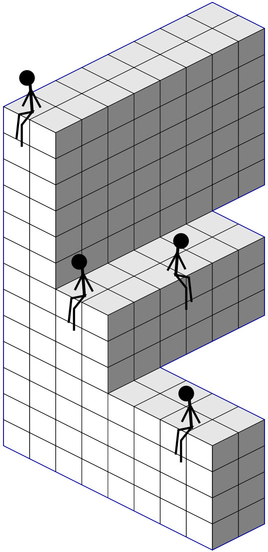
Unit cube = 

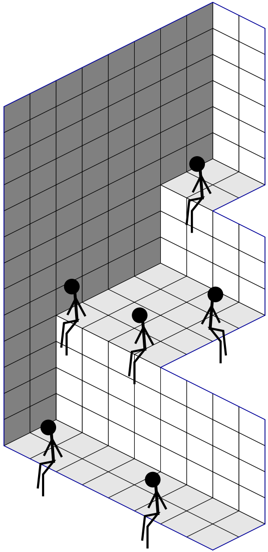
(polygon = **projection** of the boundary of 3D surfaces on the plane $x + y + z = 1$)

3D surfaces in a box. "Full" and "Empty" configurations



3D surfaces in a box. "Full" and "Empty" configurations





Two models of random tilings

1. Uniformly random tilings:

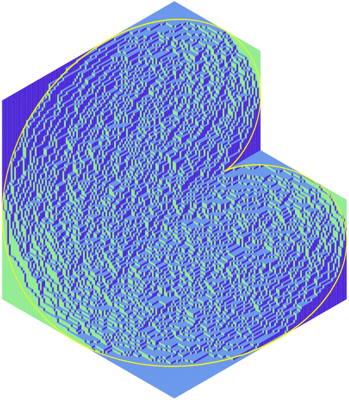
$$Prob\{\text{a tiling}\} = \frac{1}{\text{total \# of tilings}}$$

2. q-deformation ($0 < q < 1$):

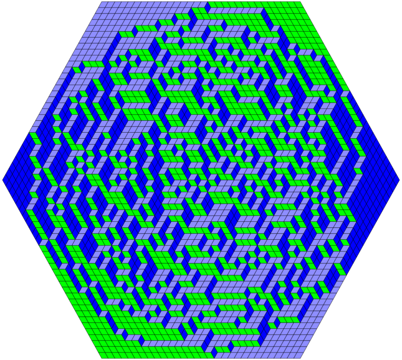
$$Prob\{\text{a tiling}\} = \frac{q^{\text{volume under the 3D surface}}}{Z(q)}$$

How very "large" tilings look like?

Fix a polygon $\mathcal{P}$ and let the mesh $= N^{-1} = \varepsilon \to 0$ (**hydrodynamic scaling**). For q-measure let also $q = q_0^{\varepsilon} \to 1$.



[Kenyon-Okounkov '07]

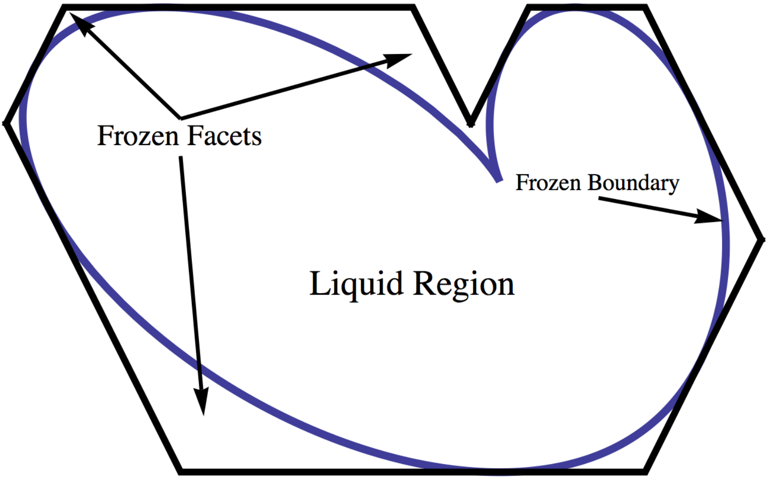


Algorithm of [Borodin-Gorin '09]

Limit shape and frozen boundary for general polygonal domains

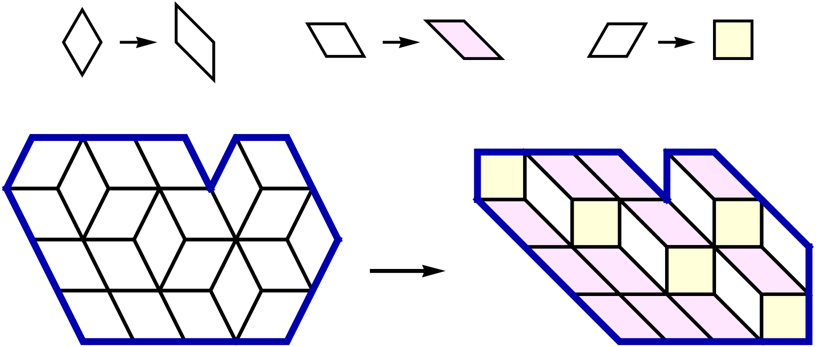
[Cohn–Larsen–Propp '98], [Cohn–Kenyon–Propp '01], [Kenyon-Okounkov '07]

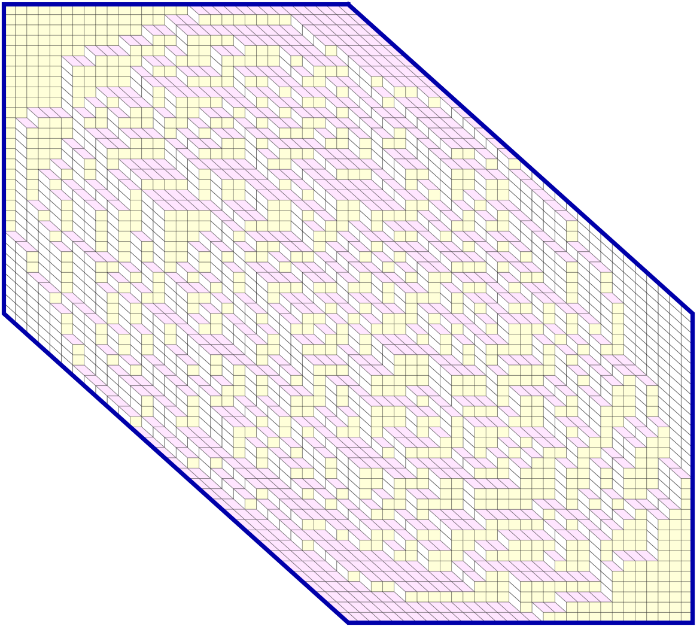
- (LLN) As the mesh goes to zero, random 3D stepped surfaces concentrate around a **deterministic limit shape surface**
- The limit shape develops **frozen facets**
- There is a connected **liquid region** where all three types of lozenges are present
- The limit shape surface and the separating **frozen boundary curve** are algebraic
- The frozen boundary is **tangent** to all sides of the polygon


Frozen Facets
Frozen Boundary
Liquid Region

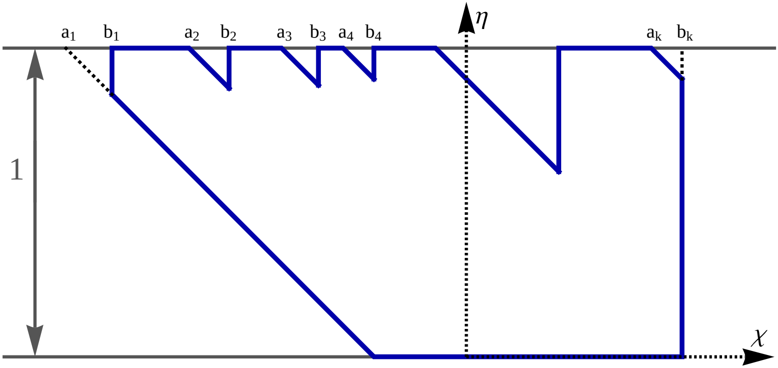
Gelfand-Tsetlin-type (**GT-type**) Polygons

Affine transform of lozenges





GT-type polygons in (χ, η) plane

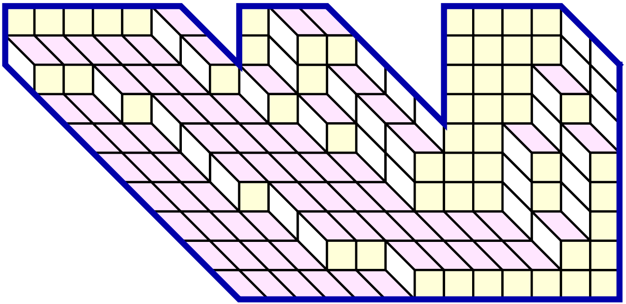


Polygon $\mathcal{P}$ has $3k$ sides, $k = 2, 3, 4, \dots$

$+$ condition $\displaystyle\sum_{i=1}^{k}(b_i - a_i) = 1$ (a_i, b_i — **fixed parameters**)

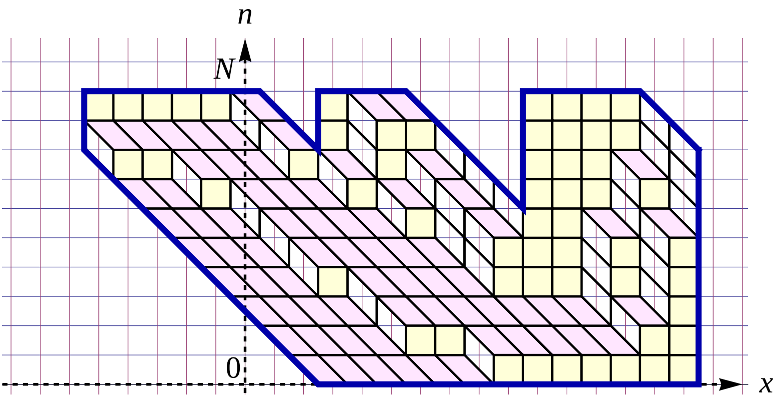
($k = 2$ — hexagon with sides A, B, C, A, B, C)

Tilings of GT-type polygons as interlacing particle configurations



Take a tiling of a GT-type polygon $\mathcal{P}$

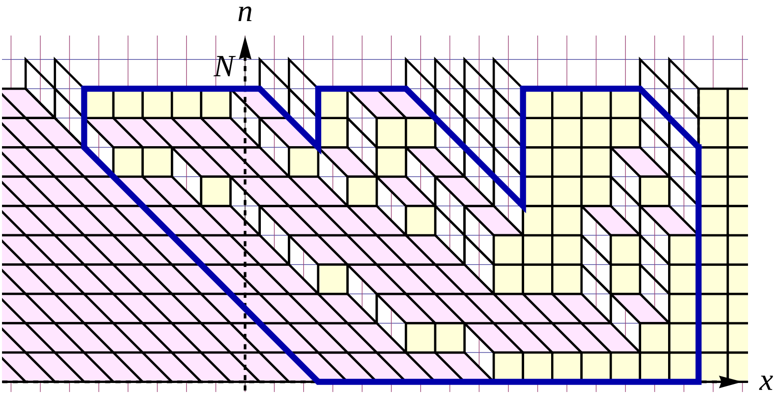
Tilings of GT-type polygons as interlacing particle configurations



Let $N := \varepsilon^{-1} \in \mathbb{Z}$ (where ε = mesh of the lattice)

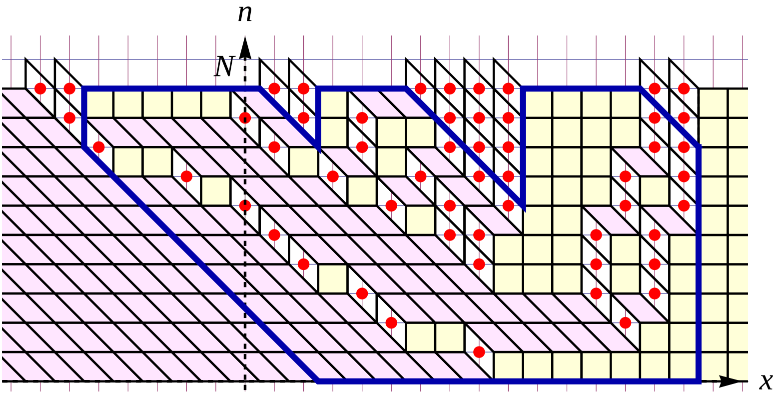
Introduce scaled *integer* coordinates (= scale the polygon)
$x = N\chi$, $n = N\eta$ (so $n = 0, \ldots, N$)

Tilings of GT-type polygons as interlacing particle configurations



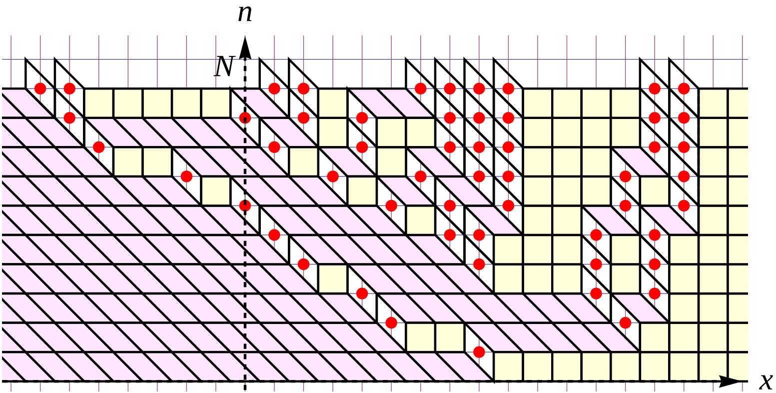
Trivially extend the tiling to the strip $0 \leq n \leq N$ with N small triangles on top

Tilings of GT-type polygons as interlacing particle configurations



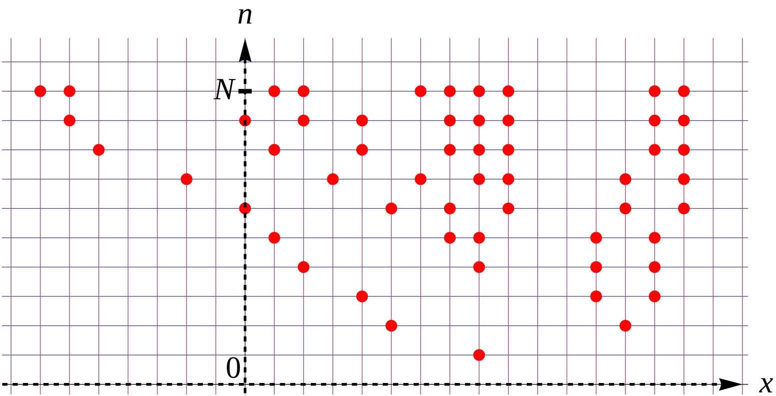
Place a particle in the center of every lozenge of type ⍂

Tilings of GT-type polygons as interlacing particle configurations



Erase the polygon. . .

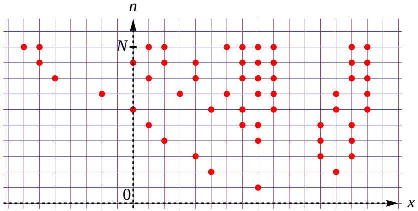
Tilings of GT-type polygons as interlacing particle configurations



. . . and the lozenges!

(though one can always reconstruct everything back)

Gelfand-Tsetlin schemes



We get a **random integer (particle) array**

$$\{\mathbf{x}_j^m \colon m = 1, \ldots, N;\ j = 1, \ldots, m\} \in \mathbb{Z}^{N(N+1)/2}$$

satisfying **interlacing constraints**

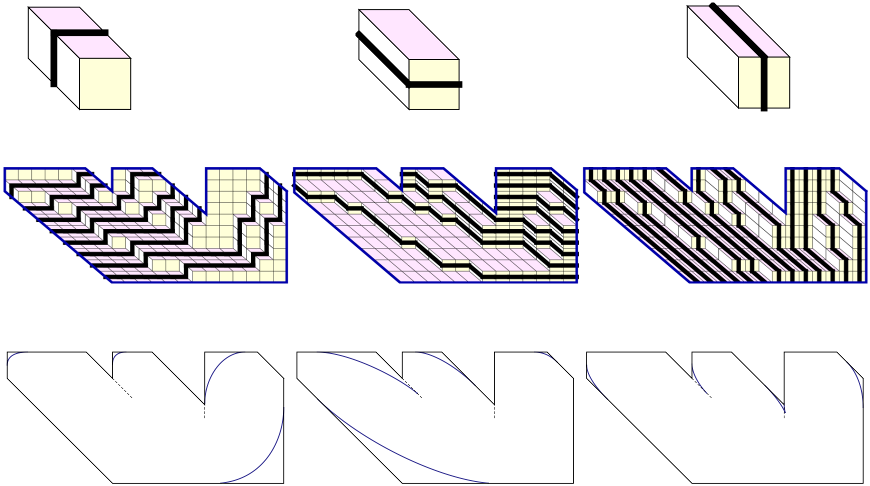
$$\mathbf{x}_{j+1}^m < \mathbf{x}_j^{m-1} \le \mathbf{x}_j^m \qquad \text{(for all possible } m, j\text{)}$$

and with certain **fixed top (N-th) row**: $\mathbf{x}_N^N < \ldots < \mathbf{x}_1^N$

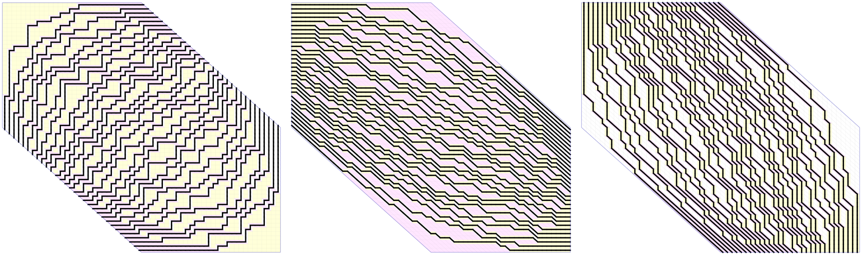
(determined by N and parameters $\{a_i, b_i\}_{i=1}^k$ of the polygon).

Local Asymptotic Behavior of
Uniformly Random Tilings
of GT-type Polygons:
Edge, Bulk

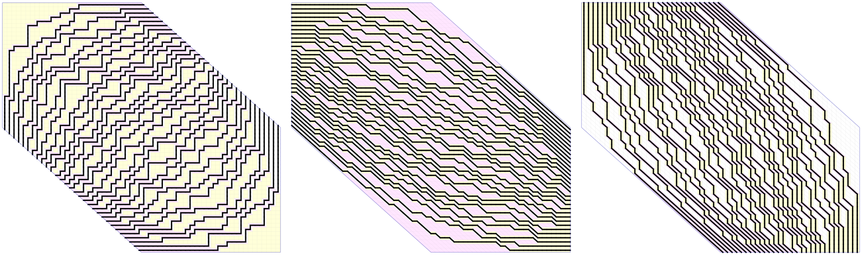
Local behavior at the edge:
3 directions of nonintersecting paths



Limit shape $\Rightarrow$ outer paths of every type concentrate around the corresponding direction of the frozen boundary:



Limit shape $\Rightarrow$ outer paths of every type concentrate around the corresponding direction of the frozen boundary:



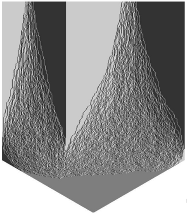
Theorem 1 [P. '12]. Edge behavior for GT-type polygons

Fluctuations $O(\varepsilon^{1/3})$ in tangent and $O(\varepsilon^{2/3})$ in normal direction ($\varepsilon = \frac{1}{N}$ = mesh of the triangular lattice)

Thus scaled fluctuations are governed by the (space-time) Airy process at **not tangent nor turning** point $(\chi, \eta) \in$ **boundary**

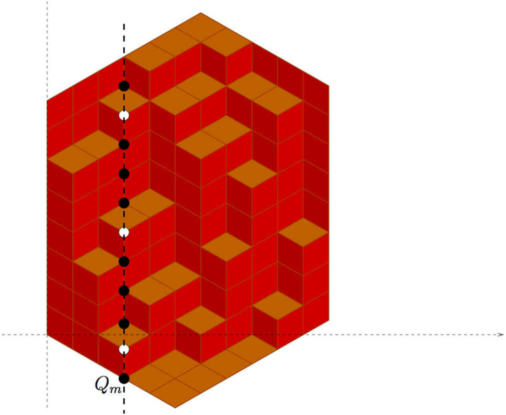
Appearance of Airy-type asymptotics

- Edge asymptotics in many spatial models (from the **Kardar–Parisi–Zhang universality class**) are governed by the Airy process
- First appearances — the static case:
random matrices (in part., Tracy-Widom distribution F_2),
random partitions (in part., the longest increasing subsequence)
- Dynamical Airy process:
PNG droplet growth, [Prähofer–Spohn '02]
- Random tilings of infinite polygons:
[Okounkov-Reshetikhin '07]



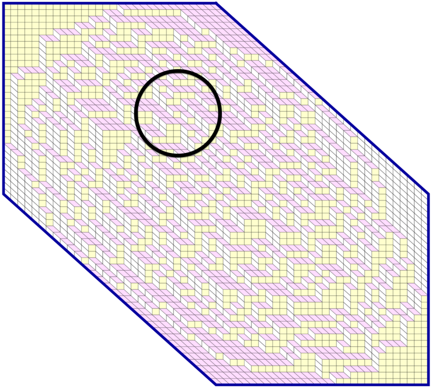
Finite polygons (our setting)

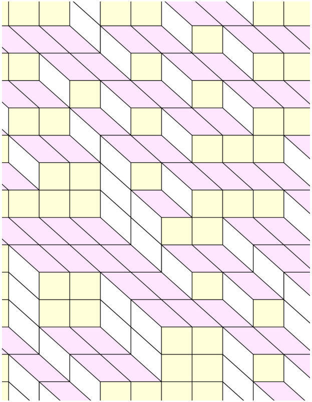
Hexagon case: [Baik-Kriecherbauer-McLaughlin-Miller '07], static case (in cross-sections of ensembles of nonintersecting paths), using orthogonal polynomials



Theorem 2 [P. '12]. Bulk asymptotics for GT-type polygons

Zooming around a point $(\chi, \eta) \in \mathcal{P}$, we asymptotically see a unique translation invariant ergodic Gibbs measure on tilings of the whole plane **with given proportions of lozenges** of all types [Sheffield '05], [Kenyon-Okounkov-Sheffield '06]



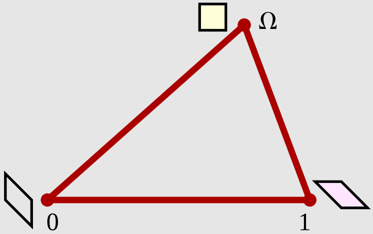


Theorem 2 [P. '12] (cont.). Proportions of lozenges

There exists a function $\Omega = \Omega(\chi, \eta)\colon \mathcal{P} \to \mathbb{C}$, $\Im\Omega \ge 0$ (*complex slope*) such that asymptotic proportions of lozenges

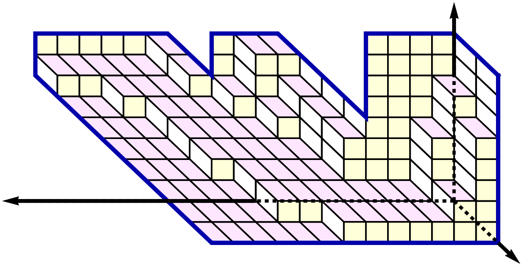
$$(p_{\lozenge}, p_{\square}, p_{\diamond}), \qquad p_{\lozenge} + p_{\square} + p_{\diamond} = 1$$

(seen in a large box under the ergodic Gibbs measure) are the normalized angles of the triangle in the complex plane:



Predicting the limit shape from bulk local asymptotics

$(p_{◇}, p_{□}, p_{▱})$ — normal vector to the limit shape surface in 3D coordinates like this:



Theorem 2 [P. '12] (cont.). Limit shape prediction

The limit shape prediction from local asymptotics coincides with the true limit shape of [Cohn–Kenyon–Propp '01], [Kenyon-Okounkov '07].

Bulk local asymptotics: previous results related to Theorem 2

- [Baik-Kriecherbauer-McLaughlin-Miller '07], [Gorin '08] — for uniformly random tilings of the hexagon = boxed plane partitions (using orth. poly)

- [Borodin-Gorin-Rains '10] — for more general measures on boxed plane partitions (using orth. poly)

- [Kenyon '08] — for rather general boundary conditions (= regions) *not allowing frozen parts of the limit shape*

- Many other random 3D stepped surface (lozenge tiling) models also show this local behavior (**universality**)

Theorem 3 [P. '12]. The complex slope $\Omega(\chi, \eta)$

The function $\Omega\colon \mathcal{P} \to \mathbb{C}$ satisfies the differential *complex Burgers equation* [Kenyon-Okounkov '07]

$$\Omega(\chi,\eta)\frac{\partial\Omega(\chi,\eta)}{\partial\chi} = -(1-\Omega(\chi,\eta))\frac{\partial\Omega(\chi,\eta)}{\partial\eta},$$

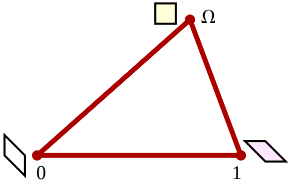
and the algebraic equation (it reduces to a degree k equation)

$$\Omega \cdot \prod_{i=1}^{k} \left((a_i - \chi + 1 - \eta)\Omega - (a_i - \chi)\right) = \prod_{i=1}^{k} \left((b_i - \chi + 1 - \eta)\Omega - (b_i - \chi)\right). \quad (1)$$

For (χ, η) in the liquid region, $\Omega(\chi, \eta)$ is the only solution of (1) in the upper half plane.

Parametrization of frozen boundary

(χ, η) approach the frozen boundary curve $\Longleftrightarrow$ $\Omega(\chi, \eta)$ approaches the real line and becomes double root of the algebraic equation (1) thus yielding two equations on Ω, χ, and η.



We take slightly different real parameter for the frozen boundary curve:

$$t := \chi + \frac{(1-\eta)\Omega}{1-\Omega}.$$

Theorem 4 [P. '12]. Explicit rational parametrization of the frozen boundary curve $(\chi(t), \eta(t))$

$$\chi(t) = t + \frac{\Pi(t) - 1}{\Sigma(t)}; \qquad \eta(t) = \frac{\Pi(t)(\Sigma(t) - \Pi(t) + 2) - 1}{\Pi(t)\Sigma(t)},$$

where

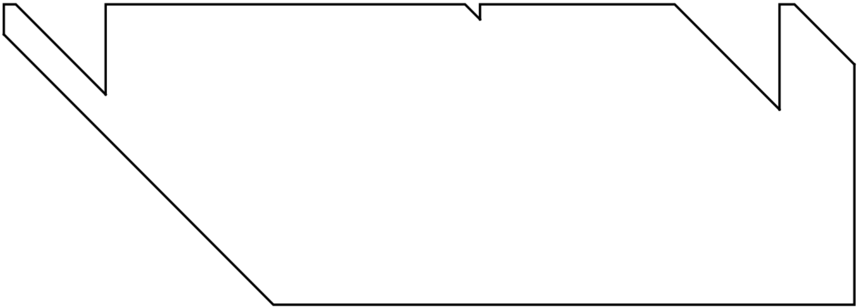
$$\Pi(t) := \prod_{i=1}^{k} \frac{t - b_i}{t - a_i}, \qquad \Sigma(t) := \sum_{i=1}^{k} \left(\frac{1}{t - b_i} - \frac{1}{t - a_i} \right),$$

with parameter $-\infty \leq t < \infty$. As t increases, the frozen boundary is passed in the clockwise direction (so that the liquid region stays to the right).

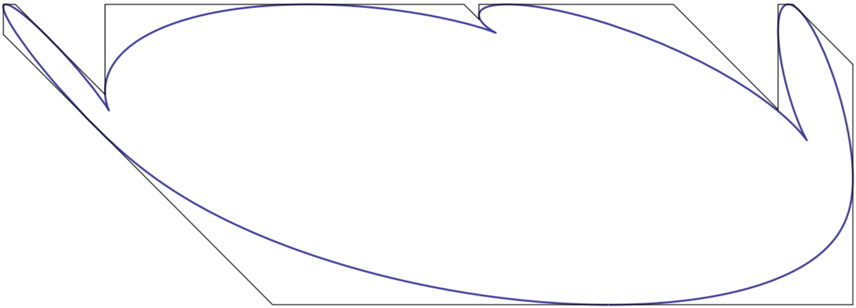
Tangent direction to the frozen boundary is given by

$$\frac{\dot{\chi}(t)}{\dot{\eta}(t)} = \frac{\Pi(t)}{1 - \Pi(t)}.$$

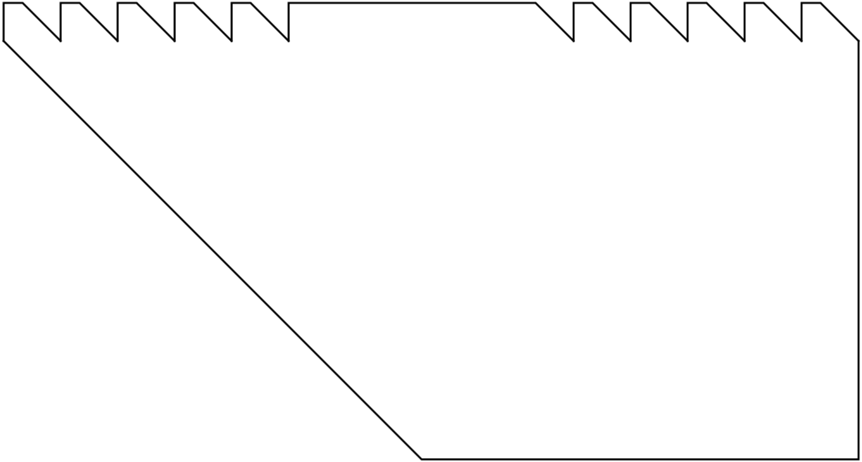
Frozen boundary examples



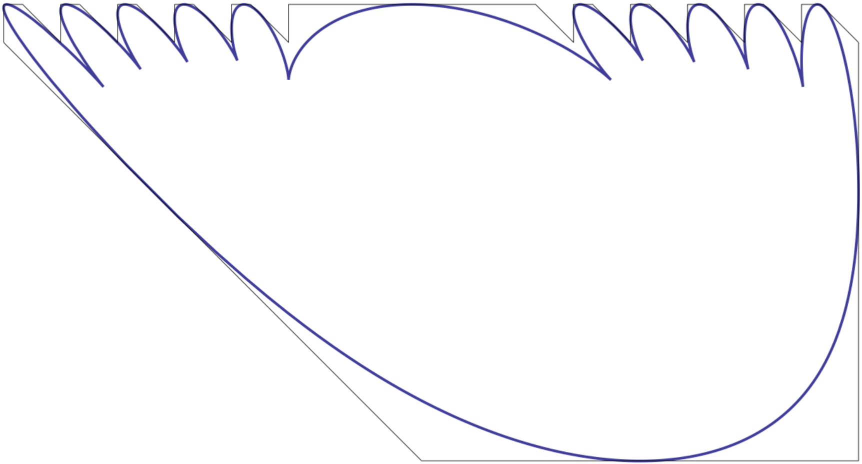
Frozen boundary examples

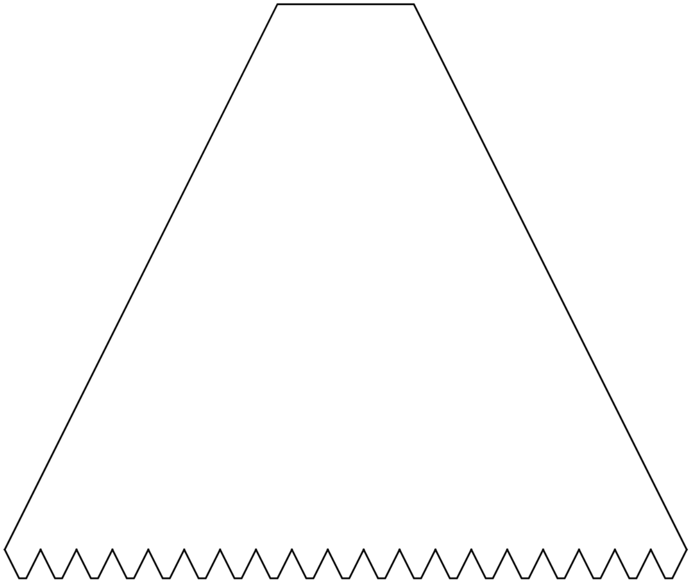


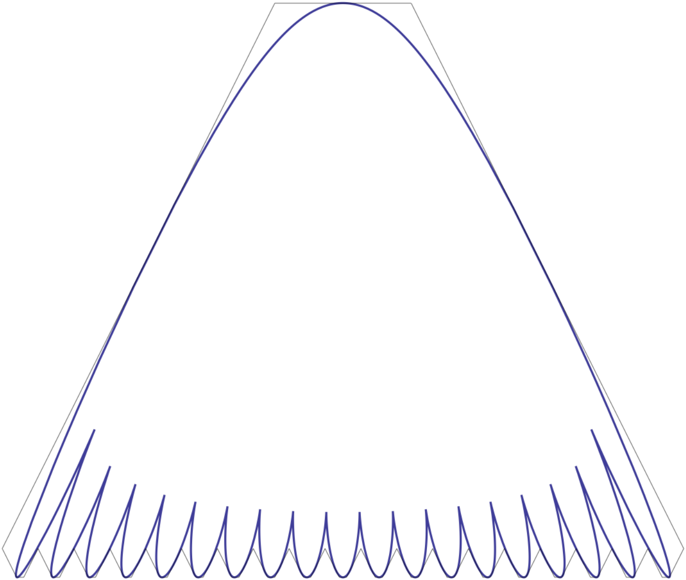
Frozen boundary examples



Frozen boundary examples





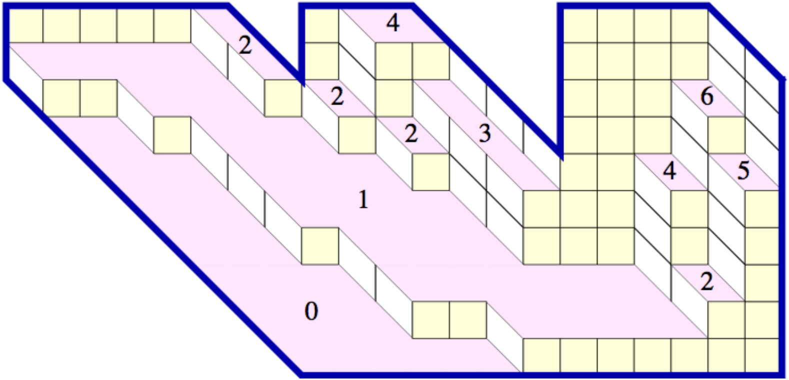


Global Fluctuations of the Height Function of **Uniformly Random Tilings** of GT-type Polygons:

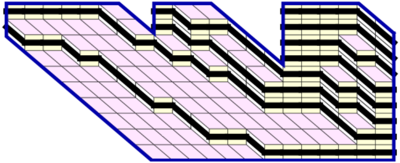
Gaussian Free Field

Height function of a tiling

$$h(x,n) := \sum_{m:\ m \leq n} 1\{\text{there is a lozenge of type } \lozenge \text{ or } \square \text{ at } (x,m)\}.$$



Level lines of the height function — one of the three families of nonintersecting paths:



Limit shape [Cohn–Kenyon–Propp '01], [Kenyon–Okounkov '07]

Almost surely, as $N \to \infty$, we have $\dfrac{h_N([\chi N], [\eta N])}{N} \to h(\chi, \eta)$

Fluctuations of the height function around its mean

$\sqrt{\pi}\Big\{ h_N([\chi N], [\eta N]) - \mathbb{E}\left(h_N([\chi N], [\eta N]) \right) \Big\}$ — random field indexed by points of the liquid region

Theorem 5 [P. '12]. CLT for fluctuations of the height function of **uniformly random tilings**

Random field of fluctuations

$$\sqrt{\pi}\Big\{ h_N([\chi N],[\eta N]) - \mathbb{E}\left(h_N([\chi N],[\eta N])\right)\Big\}$$

converges to a certain Gaussian Free Field on $\mathcal{D}$:

$$\sqrt{\pi}\int_{\mathcal{D}} \phi(\chi,\eta)\Big(h_N([\chi N],[\eta N]) - \mathbb{E}\left(h_N([\chi N],[\eta N])\right)\Big) d\chi d\eta \to$$
$$\int_{\mathcal{D}} \phi(\chi,\eta)\,\mathrm{GFF}_{\mathcal{D}}(\chi,\eta))\, d\chi d\eta$$

(weak convergence) for any smooth test function ϕ with zero boundary conditions.

Complex structure on $\mathcal{D}$

There is a **bijective** parametrization of the frozen boundary with parameter $-\infty \leq t < \infty$

Continue t to the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \colon \Im z > 0\}$:

$$t(\chi, \eta) = \chi + (1-\eta)\frac{\Omega(\chi,\eta)}{1-\Omega(\chi,\eta)}, \qquad (\chi,\eta) \in \textbf{liquid region } \mathcal{D}$$

$t \colon \mathcal{D} \to \mathbb{H}$ — **diffeomorphism**

Green function on $\mathcal{D}$

$$\mathcal{G}_{\mathcal{D}}((\chi_1,\eta_1),(\chi_2,\eta_2)) := -\frac{1}{2\pi}\ln\left|\frac{t(\chi_1,\eta_1)-t(\chi_2,\eta_2)}{t(\chi_1,\eta_1)-\overline{t(\chi_2,\eta_2)}}\right|$$

(pullback of the Green function for the Laplace operator on $\mathbb{H}$ with Dirichlet boundary conditions)

Covariances of the Gaussian Free Field $\mathrm{GFF}_{\mathcal{D}}$ on $\mathcal{D}$

$$\mathbb{E}\big(\langle \mathrm{GFF}_{\mathcal{D}}, \phi_1\rangle \langle \mathrm{GFF}_{\mathcal{D}}, \phi_2\rangle\big) = \int_{\mathcal{D}\times\mathcal{D}} \phi_1(\chi_1,\eta_1)\phi_2(\chi_2,\eta_2)\cdot \mathcal{G}_{\mathcal{D}}\big((\chi_1,\eta_1),(\chi_2,\eta_2)\big)\, d\chi_1 d\eta_1 d\chi_2 d\eta_2$$

Covariances for distinct (χ_j, η_j):

$$\mathbb{E}(\mathrm{GFF}_{\mathcal{D}}(\chi_1,\eta_1)\ldots \mathrm{GFF}_{\mathcal{D}}(\chi_s,\eta_s))$$

$$= \begin{cases} \sum_\sigma \prod_{i=1}^{s/2} \mathcal{G}_{\mathcal{D}}\big((\chi_{\sigma(2i-1)},\eta_{\sigma(2i-1)}),(\chi_{\sigma(2i)},\eta_{\sigma(2i)})\big) & s \text{ even;} \\ 0, & s \text{ odd,} \end{cases}$$

sum is taken over all pairings σ on $\{1,\ldots,s\}$.

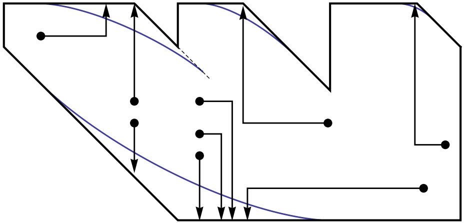
“Wick”

How to get CLT:

For distinct $(\chi_1, \eta_1), \ldots, (\chi_s, \eta_s)$ in the liquid region we show

$$\lim_{N\to\infty} \pi^{s/2}\, \mathbb{E}\left(\prod_{j=1}^{s} \left(h_N([\chi_j N],[\eta_j N]) - \mathbb{E}\, h_N([\chi_j N],[\eta_j N])\right)\right)$$
$$= \begin{cases} \sum_\sigma \prod_{i=1}^{s/2} \mathcal{G}_\mathcal{D}\big((\chi_{\sigma(2i-1)}, \eta_{\sigma(2i-1)}), (\chi_{\sigma(2i)}, \eta_{\sigma(2i)})\big) & s \text{ even;} \\ 0, & s \text{ odd,} \end{cases}$$

(sum over all pairings)



$+$ an estimate on multipoint covariances when some of the points coincide

GFF-type fluctuations in random tilings: previous results

- [Kenyon "Height Fluctuations..." '08] — Fluctuations are governed by the GFF for uniformly random tilings of rather general regions *not allowing frozen parts of the limit shape*
- [Borodin–Ferrari '08] and [Duits '11] — other random tiling models (with dynamics and of infinite regions)
- [Kuan '11] — certain ensembles of tilings of the whole upper half plane related to representations of orthogonal groups

All these papers use Kasteleyn/determinantal structure.

Also: [Borodin-Gorin '13], [Borodin-Bufetov '13] — GFF fluctuations for random matrices and related models, using **Macdonald processes** technique [Borodin–Corwin '11].

Remark: Glauber dynamics

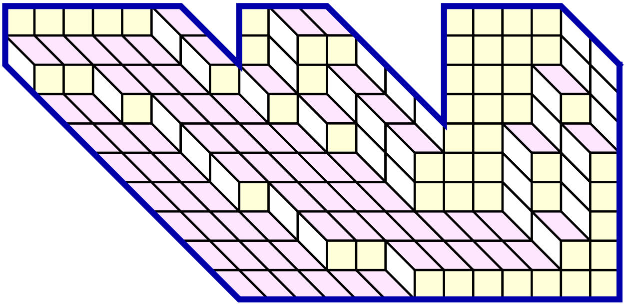
— a way to **sample** uniformly random and q^{vol} tilings.

Rule (for uniform): Add/delete a box independently at random according to exponential clocks of rate 1. Uniform measure is the unique invariant measure for the Glauber dynamics.

[Toninelli–Laslier '13] use results of [P. '12] (determinantal structure and exact asymptotics) to prove the rate $N^{2+o(1)}$ of convergence when there are no frozen facets (see also references in [Toninelli–Laslier '13])

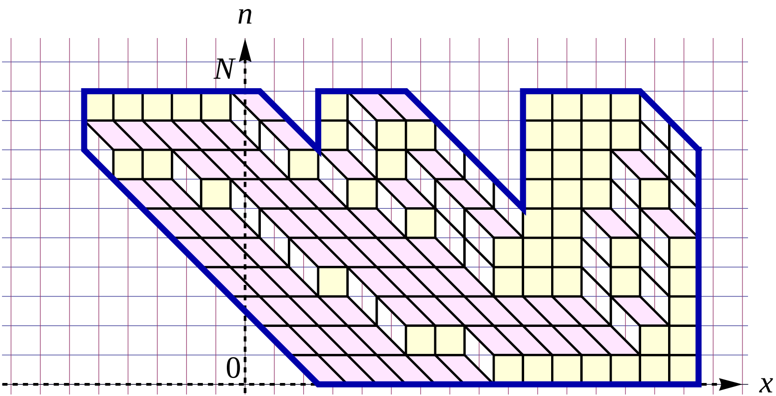
Patricle configurations and determinantal structure

Tilings of GT-type polygons as interlacing particle configurations



Take a tiling of a GT-type polygon $\mathcal{P}$

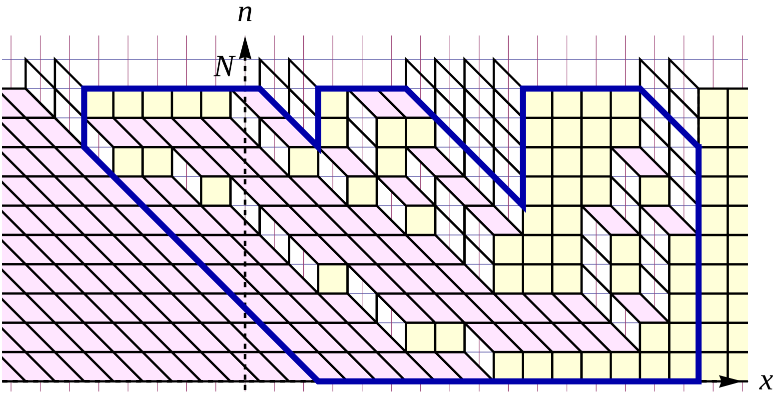
Tilings of GT-type polygons as interlacing particle configurations



Let $N := \varepsilon^{-1} \in \mathbb{Z}$ (where $\varepsilon =$ mesh of the lattice)

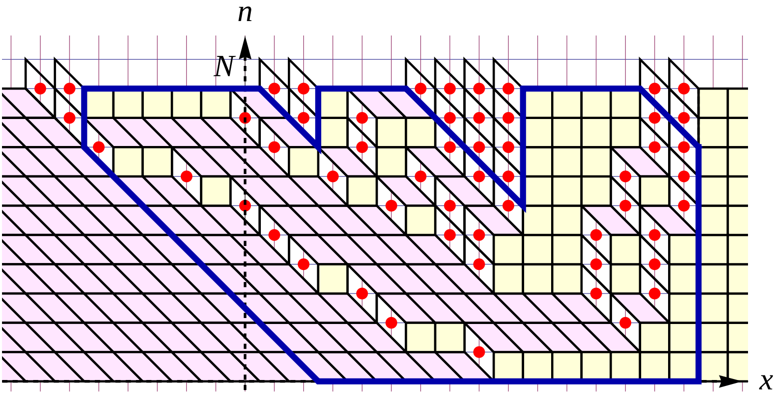
Introduce scaled *integer* coordinates ($=$ scale the polygon)
$x = N\chi$, $n = N\eta$ (so $n = 0, \ldots, N$)

Tilings of GT-type polygons as interlacing particle configurations



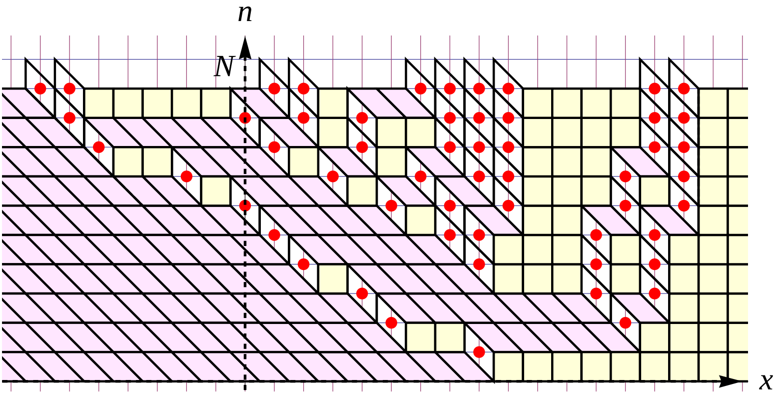
Trivially extend the tiling to the strip $0 \leq n \leq N$ with N small triangles on top

Tilings of GT-type polygons as interlacing particle configurations



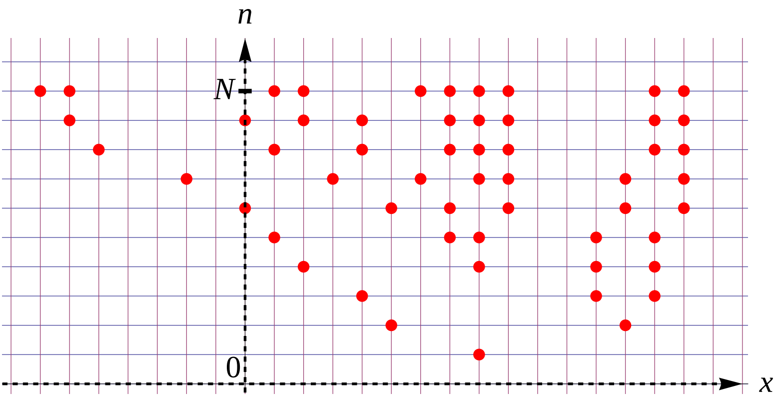
Place a particle in the center of every lozenge of type ⍂

Tilings of GT-type polygons as interlacing particle configurations



Erase the polygon. . .

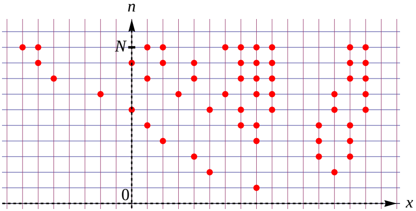
Tilings of GT-type polygons as interlacing particle configurations



...and the lozenges!

(though one can always reconstruct everything back)

Gelfand-Tsetlin schemes



We get a **random integer (particle) array**

$$\{\mathbf{x}_j^m \colon m = 1, \dots, N;\ j = 1, \dots, m\} \in \mathbb{Z}^{N(N+1)/2}$$

satisfying **interlacing constraints**

$$\mathbf{x}_{j+1}^m < \mathbf{x}_j^{m-1} \le \mathbf{x}_j^m \qquad \text{(for all possible } m, j\text{)}$$

and with certain **fixed top (N-th) row**: $\mathbf{x}_N^N < \dots < \mathbf{x}_1^N$ (determined by N and parameters $\{a_i, b_i\}_{i=1}^k$ of the polygon).

Determinantal structure

Determinantal structure

Correlation functions

Fix some (pairwise distinct) positions $(x_1, n_1), \dots, (x_s, n_s)$,

$$\rho_s(x_1, n_1; \dots; x_s, n_s) := Prob\{\text{there is a particle of random configuration } \{\mathbf{x}_j^m\} \text{ at position } (x_\ell, n_\ell),\ \ell = 1, \dots, s\}$$

Determinantal correlation kernel ($q = 1$ and $0 < q < 1$)

There is a function $K_q(x_1, n_1; x_2, n_2)$ (*correlation kernel*) s.t.

$$\rho_s(x_1, n_1; \dots; x_s, n_s) = \det[K_q(x_i, n_i; x_j, n_j)]_{i,j=1}^s$$

Determinantal structure: Existence

Uniformly random tilings of **general** polygons have determinantal structure: this follows from Kasteleyn theory, cf. [Kenyon "Lectures on dimers" '09]

Determinantal kernel is the inverse of the Kasteleyn (= honeycomb graph incidence) matrix

Problem:
in general there is **no** explicit formula for the kernel

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Problem:
in general there is **no** explicit formula for the kernel

- **Hexagon**: orthogonal polynomials [Johansson '05, ...]
- **GT-type polygons**: double contour integral [P. '12]

Determinantal structure for GT-type polygons

Correlation kernel $K_q(x_1, n_1; x_2, n_2)$ is expressed as double contour integral:

1. $q = 1$: of **elementary** functions
2. $0 < q < 1$: there is a q-**hypergeometric function** ${}_2\phi_1$ under the integral

Formula for the kernel for $q = 1$ is obtained in the $q \nearrow 1$ limit.

We are able to use the kernel only in the $q = 1$ case to study asymptotics of uniformly random tilings

Theorem 6 [P. '12]. Kernel for $q = 1$

$$K_{q=1}(x_1, n_1; x_2, n_2) = -1_{n_2<n_1} 1_{x_2 \le x_1} \frac{(x_1 - x_2 + 1)_{n_1-n_2-1}}{(n_1 - n_2 - 1)!}$$

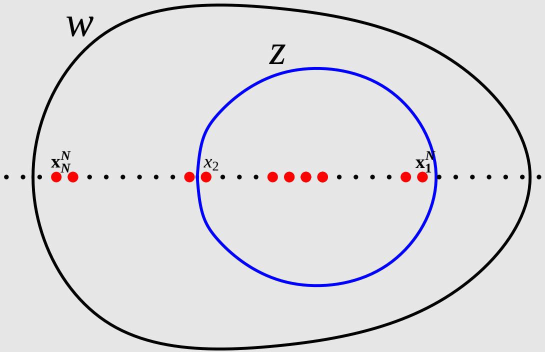
$$+ \frac{(N - n_1)!}{(N - n_2 - 1)!} \frac{1}{(2\pi \mathrm{i})^2} \times$$

$$\times \oint_{\{w\}} \oint_{\{z\}} \frac{dz dw}{w - z} \cdot \frac{(z - x_2 + 1)_{N-n_2-1}}{(w - x_1)_{N-n_1+1}} \cdot \prod_{r=1}^{N} \frac{w - \mathbf{x}_r^N}{z - \mathbf{x}_r^N}$$

where $1 \le n_1 \le N$, $1 \le n_2 \le N - 1$, $x_1, x_2 \in \mathbb{Z}$, and $(a)_m := a(a + 1) \dots (a + m - 1)$

Theorem 6 [P. '12] (cont.). Contours of integration for K

- Both contours are counter-clockwise.
- $\mathrm{Int}\{z\} \ni x_2, x_2+1, \ldots, \mathbf{x}_1^N$, $\quad \mathrm{Int}\{z\} \not\ni x_2-1, x_2-2, \ldots, \mathbf{x}_N^N$
- $\{w\}$ contains $\{z\}$, $\quad \mathrm{Int}\{w\} \ni x_1, x_1-1, \ldots, x_1-(N-n_1)$



reminder: integrand contains $\dfrac{(z-x_2+1)_{N-n_2-1}}{(w-x_1)_{N-n_1+1}} \displaystyle\prod_{r=1}^{N} \frac{w-\mathbf{x}_r^N}{z-\mathbf{x}_r^N}$

Connection to other known kernels

The kernel $K_{q=1}(x_1, n_1; x_2, n_2)$ generalizes some known kernels arising in the following models:

1. Certain cases of the general Schur process [Okounkov-Reshetikhin '03]
2. Extremal characters of the infinite-dimensional unitary group $\Rightarrow$ certain ensembles of random tilings of the entire upper half plane [Borodin-Kuan '08], [Borodin '10]
3. Eigenvalue minor process of random Hermitian $N \times N$ matrices with fixed level N eigenvalues $\Rightarrow$ random continuous interlacing arrays of depth N [Metcalfe '11]

All these models can be obtained from **uniformly random** tilings of GT-type polygons via suitable degenerations

Theorem 7 [P. '12] Kernel for $0 < q < 1$

$$K_q(x_1, n_1; x_2, n_2) = -1_{n_2<n_1} 1_{x_2 \le x_1} q^{n_2(x_1-x_2)} \frac{(q^{x_1-x_2+1}; q)_{n_1-n_2-1}}{(q;q)_{n_1-n_2-1}}$$

$$+ \frac{(q^{N-1}; q^{-1})_{N-n_1}}{(2\pi \mathrm{i})^2} \oint dz \oint \frac{dw}{w} \times$$

$$\times \frac{q^{n_2(x_1-x_2)} z^{n_2}}{w-z} \frac{(zq^{1-x_2+x_1}; q)_{N-n_2-1}}{(q;q)_{N-n_2-1}} \times$$

$$\times \, {}_2\phi_1(q^{-1}, q^{n_1-1}; q^{N-1} \mid q^{-1}; w^{-1}) \prod_{r=1}^{N} \frac{w - q^{\mathbf{x}_r^N - x_1}}{z - q^{\mathbf{x}_r^N - x_1}},$$

where $(a;q)_m := (1-a)(1-qa)\dots(1-q^{m-1}a)$.

Asymptotic analysis of the kernel gives local asymptotics and fluctuations ($q = 1$)

Write the kernel as:

$$K_{q=1}(x_1, n_1; x_2, n_2) \sim \text{additional summand}$$
$$+\frac{1}{(2\pi i)^2} \oint \oint f(w,z) \frac{e^{N[S(w;\frac{x_1}{N},\frac{n_1}{N})-S(z;\frac{x_2}{N},\frac{n_2}{N})]}}{w-z} dwdz$$

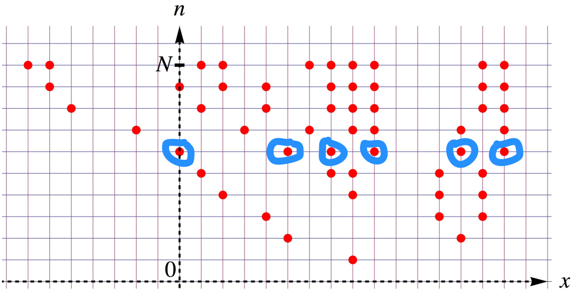
($f(w,z)$ — some "regular" part having a limit), where

$$S(w;\chi,\eta) = (w-\chi)\ln(w-\chi)$$
$$-(w-\chi+1-\eta)\ln(w-\chi+1-\eta) + (1-\eta)\ln(1-\eta)$$
$$+\sum_{i=1}^{k}\Big[(b_i-w)\ln(b_i-w)-(a_i-w)\ln(a_i-w)\Big].$$

Then investigate critical points of the *action* $S(w;\chi,\eta)$ and transform the contours of integration [Okounkov "Symmetric functions and random partitions" '02]

Projections of measures on interlacing arrays onto a fixed row

Projections of uniform and q^{vol} measures onto the fixed K-th row



Joint distribution of particles $\mathbf{x}_1^K, \ldots, \mathbf{x}_K^K$ of the Kth row ($K < N$) is described in a **much simpler** form, **both** for $q = 1$ and $0 < q < 1$

Projections, $q = 1$ (uniform measure)

Partition function:

$$Z(q=1) =: Z_N(\mathbf{x}_1^N, \ldots, \mathbf{x}_N^N) = \prod_{1 \le i < j \le N} \frac{\mathbf{x}_i^N - \mathbf{x}_j^N}{j - i}$$

Theorem 8 [P. '12]. Joint distribution on level K for $q = 1$

$$P_K(\mathbf{x}_1^K, \ldots, \mathbf{x}_K^K) = Z_K(\mathbf{x}_1^K, \ldots, \mathbf{x}_K^K) \cdot \det[A_i(\mathbf{x}_j^K)]_{i,j=1}^K,$$

where $$A_i(x) = \frac{N-K}{2\pi \mathrm{i}} \oint_{\{z\}} \frac{(z-x+1)_{N-K-1}}{(z+i)_{N-K+1}} \prod_{r=1}^{N} \frac{z+r}{z-\mathbf{x}_r^N} dz.$$

Contour contains $x, x+1, x+2, \ldots$.

An equivalent statement was obtained earlier by a more complicated technique in [Borodin-Olshanski, 12].

Projections, $0 < q < 1$ (measure q^{vol})

Partition function:

$$Z(q) = {}_qZ_N(\mathbf{x}_1^N, \dots, \mathbf{x}_N^N) = \prod_{1 \le i < j \le N} \frac{q^{\mathbf{x}_i^N} - q^{\mathbf{x}_j^N}}{q^{-i} - q^{-j}}$$

Theorem 9 [P. '12]. Joint distribution on level K, $0 < q < 1$

$${}_qP_K(\mathbf{x}_1^K, \dots, \mathbf{x}_K^K) = {}_qZ_K(\mathbf{x}_1^K, \dots, \mathbf{x}_K^K) \cdot q^{???} \cdot \det[{}_qA_i(\mathbf{x}_j^K)]_{i,j=1}^K,$$

where ${}_qA_i(x)$ is given by

$$(-1)^{N-K} \frac{1 - q^{N-K}}{2\pi \mathrm{i}} \oint_{\{z\}} dz \frac{(zq^{1-x}; q)_{N-K-1}}{\prod_{r=i}^{N-K+i}(z - q^{-r})} \prod_{r=1}^{N} \frac{z - q^{-r}}{z - q^{\mathbf{x}_r^N}}.$$

Contour contains $q^x, q^{x+1}, q^{x+2}, \dots$.

Applications

‘**Representation-theoretic**’ (‘**projective**’) limit transition in random tilings, as opposed to hydrodynamic scaling,

equivalent to:

1. ($q = 1$) Description of characters of the infinite-dimensional unitary group (celebrated Edrei-Voiculescu Theorem)

   [Edrei, Voiculescu, Vershik-Kerov, Boyer, Okounkov-Olshanski, Borodin-Olshanski, P.]

2. ($0 < q < 1$) A q-analogue, related to the q-Gelfand-Tsetlin graph and q-Toeplitz matrices [Gorin ’10].

3. ‘Random matrix type’ limits [Gorin-Panova ‘13], [Bufetov-Gorin ‘13]

'Representation-theoretic' limit

Let $N \to \infty$ together with top row particles
$\mathbf{x}_1^N(N), \ldots, \mathbf{x}_N^N(N)$,
but let us look at a **fixed** finite level $K < N$.

Question

Describe all the ways in which the particles $\mathbf{x}_1^N(N), \ldots, \mathbf{x}_N^N(N)$ can behave so that on level K we see a nontrivial (weak) limit of projected measures as $N \to \infty$.

It can be addressed using the contour integral formulas above.

‘Representation-theoretic’ limit, $q = 1$

Necessary conditions: each quantity $\mathbf{x}_i^N(N) + i$, as well as their sum $\sum_{i=1}^N (\mathbf{x}_i^N(N) + i)$, to grow at most linearly in N.

All possible sequences $\mathbf{x}^N(N)$ depend on infinitely many continuous parameters.

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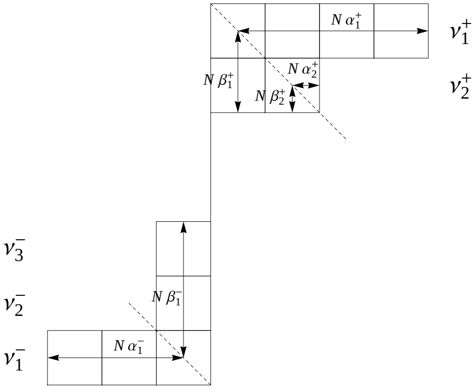
All possible sequences $\mathbf{x}^N(N)$ depend on infinitely many continuous parameters.



Here $\nu_i = \mathbf{x}_i^N + i$, $i = 1, \dots, N$.

All rows and columns of both Young diagrams, as well as the numbers of boxes must grow at most linearly in N.

‘Representation-theoretic’ limit, $0 < q < 1$

Allowed behavior of the top row particles is radically different: **stabilization of particles** (in suitable coordinates)

$$\lim_{N\to\infty} \left(\mathbf{x}^N_{N+1-j}(N) + (N+1-j)\right) = n_j, \qquad j = 1, 2, \ldots .$$

(n_j — infinitely many discrete parameters)

Limiting $(\nu_1, \nu_2, \ldots, \nu_N)$ looks like a **one-sided infinite staircase**

Prospectives related to projection formulas

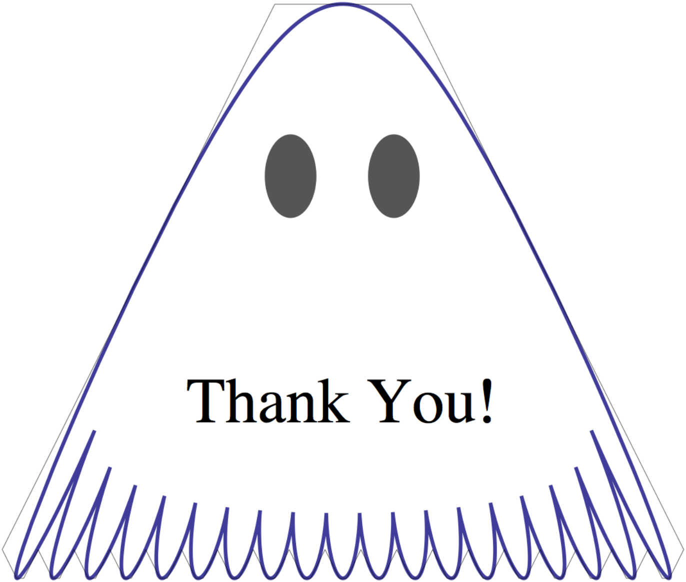
- Other regimes of top-row particles, other (scaling?) limits on Kth level
- Behavior of correlation kernels (both $q = 1$ and $0 < q < 1$) under 'representation-theoretic' limit transition: new ensembles for $0 < q < 1$
- More general measures than q^{vol}: same technique gives a $K \times K$ determinantal formula for

$$\frac{s_{\nu/\varkappa}(q^{t_1}, \dots, q^{t_{N-K}})}{s_\nu(1, q, \dots, q^{N-1})}, \qquad t_i \text{ — } \textbf{any}$$

$\nu = (\nu_1 \geq \ldots \geq \nu_N)$, $\varkappa = (\varkappa_1 \geq \ldots \geq \varkappa_K)$, and s_ν and $s_{\nu/\varkappa}$ are Schur and skew Schur polynomials.

($t_i = i - 1$ corresponds to q^{vol})

(**two-sided infinite staircase** in other example)


Thank You!