RANDOM POLYMERS & SYMMETRIC FUNCTIONS

https://arxiv.org/abs/2003.14260

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I Strict – weak Beta polymer

Let $|x_i|, |y_j| < S$, $S > 0$, $x_i + y_j > 0$ for all i, j

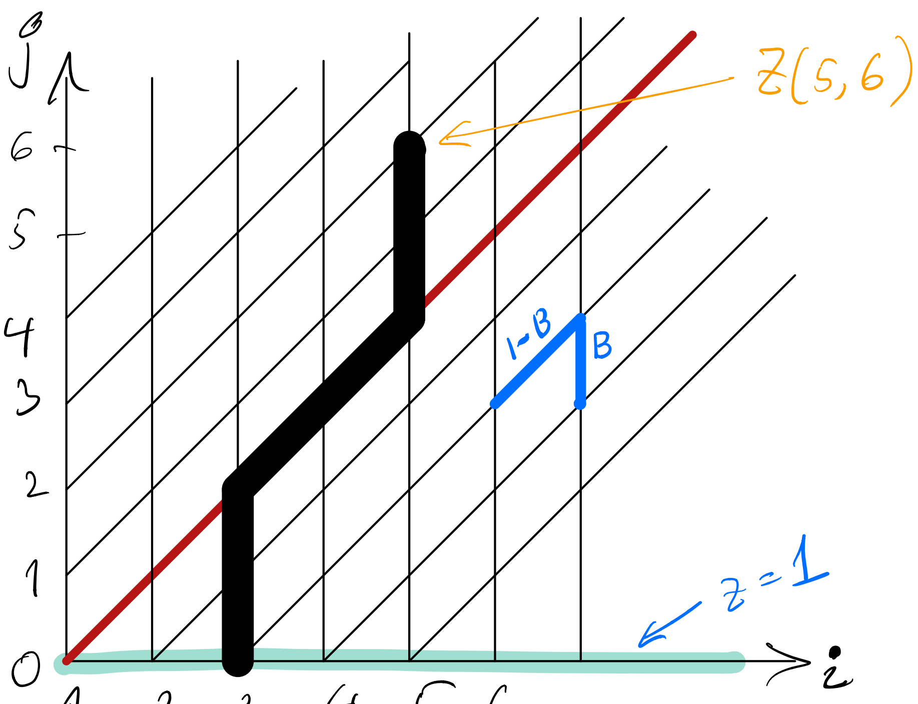
$B_{ij} \sim \mathrm{Beta}(x_i + y_j, S - y_j)$ — independent Beta R.V.

Beta (x, y)

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} t^{x-1}(1-t)^{y-1}$$

$t \in [0,1]$

$$\begin{cases} Z(i,j) = Z(i,j-1)B_{i,j} + Z(i-1,j-1)(1-B_{i,j}) & \text{for } 1 < i \le j; \\ Z(1,j) = Z(1,j-1)B_{1,j} & \text{for } j > 0; \\ Z(i,0) = 1 & \text{for } i > 0. \end{cases}$$

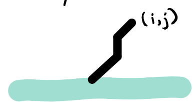


$Z(i,j)$ — strict-weak Beta polymer partition function from the bottom line to the point (i,j)

Note: $Z(i,j) = 1$ for $j \le i$.

1 2 3 4 5 6

$$Z(i,j) = \sum_{\text{paths } \pi} \prod_{e \in \pi} weight(e)$$



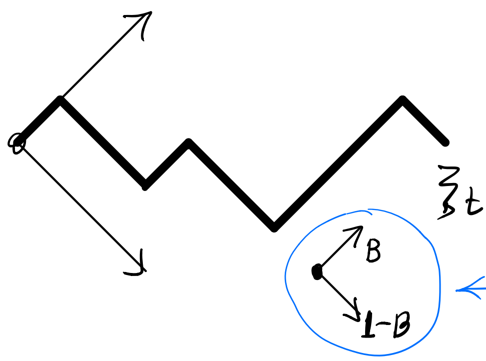
[Barraquand - Corwin 2015]: $x_i \equiv x$, $y_j \equiv y$

- Beta polymer as a limit of q-Hahn TASEP as $q \to 1$
- $\oint$ for $\mathbb{E}[Z(i,j)^k]$ via duality, and $\mathbb{E}[e^{-uZ(i,j)}]$
- KPZ class asymptotics $\leftarrow$ $\dfrac{Z(n,\alpha n) - h_\alpha n}{c_\alpha n^{1/3}} \xrightarrow{n\to\infty} F_2$ (*)
- Connection to random walks in random environment (RWRE):

$$Z(t,n) = \text{Prob}_{(env)}\left[\xi_t \geq t - 2n + 2\right]$$

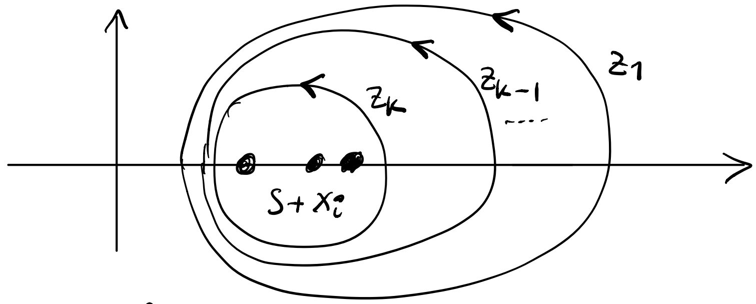
$B \sim \text{Beta}(x+y, s-y)$

(*) $\Leftrightarrow$ asymptotics of the RWRE in large deviations regime



$\leftarrow$ prob. to go up / down depend on the random environment

Example



$$\mathbb{E}\left(Z(t,n)^k\right) = \frac{1}{(2\pi i)^k} \oint \cdots \oint \prod_{i<j} \frac{z_i - z_j}{z_i - z_j - 1} \times \prod_{j=1}^{k} \left[\prod_{i=1}^{n} \frac{z_j}{z_j - s - x_i} \prod_{i=1}^{t} \frac{z_j - s + y_i}{z_j} \, dz_j \right].$$

$$\mathbb{E}\left(e^{-uZ}\right) = \sum_{k \geq 0} \frac{(-u)^k}{k!} \mathbb{E}\left(Z^k\right)$$

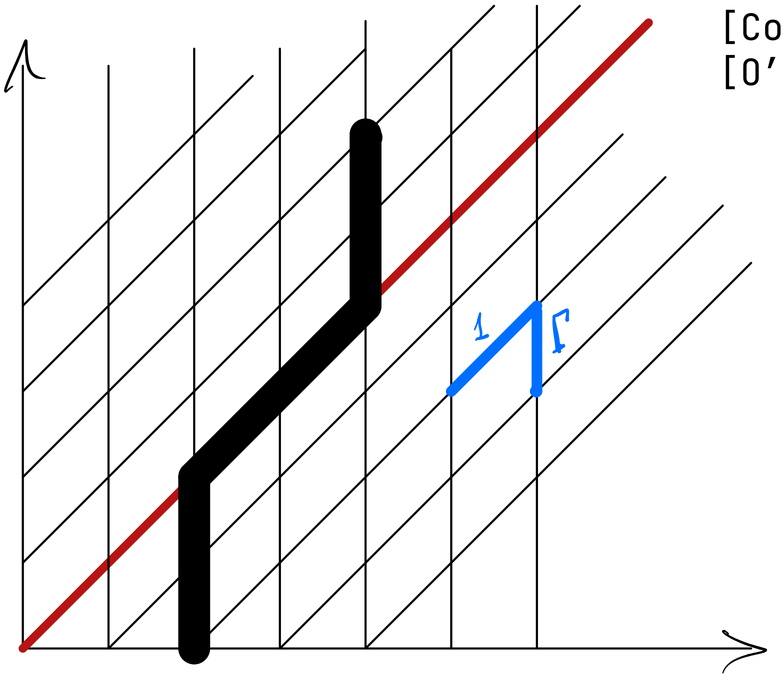
II Beta $\to$ Gamma

$s \cdot \mathrm{Beta}(x_i + y_j, s - y_j) \longrightarrow \mathrm{Gamma}(x_i + y_j)$ as $s \to \infty$

So $B_{ij} \simeq \Gamma_{ij}/s$, $1 - B_{ij} \simeq 1 - \Gamma_{ij}/s \to 1$

We get weight 1 per horizontal edge.

$\Rightarrow$ Strict-weak Gamma polymer



[Corwin-Seppalainen-Shen],
[O'Connell-Ortmann], early 2010s

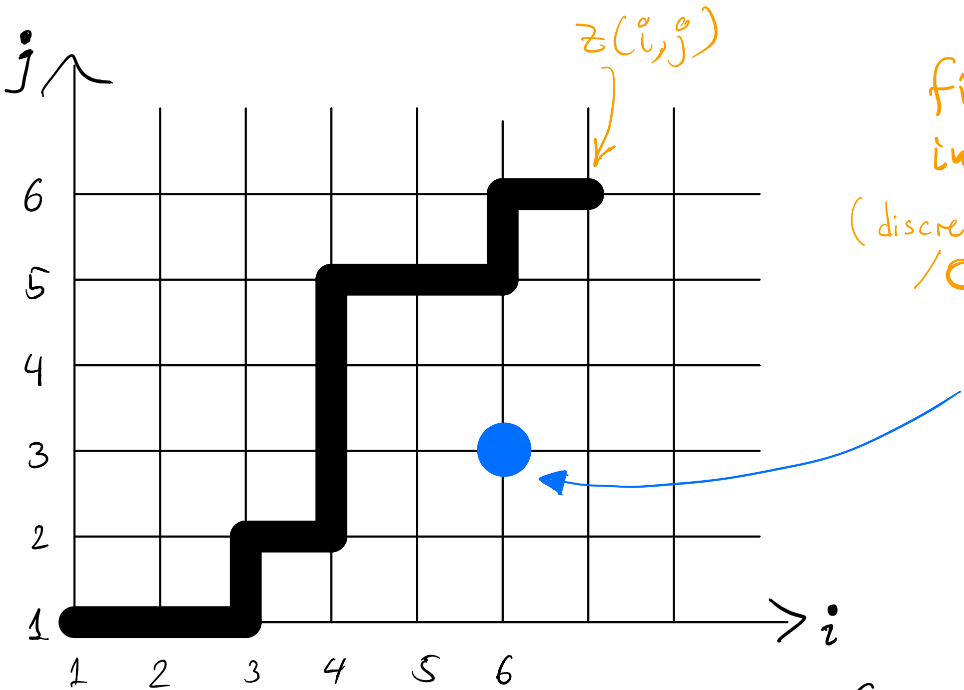
- $\oint$ for $\mathbb{E}(Z(t,n)^k)$ via duality, and $\mathbb{E}[e^{-uZ}]$
- KPZ class asymptotics
- integrability via $\mathfrak{gl}_n$ Whittaker functions

III Log-Gamma Polymer (regular $\mathbb{Z}^2$)

[Seppalainen 2009]

first fully discrete integrable polymer

(discretizes the semi-discrete / /O'Connell-Yor polymer)



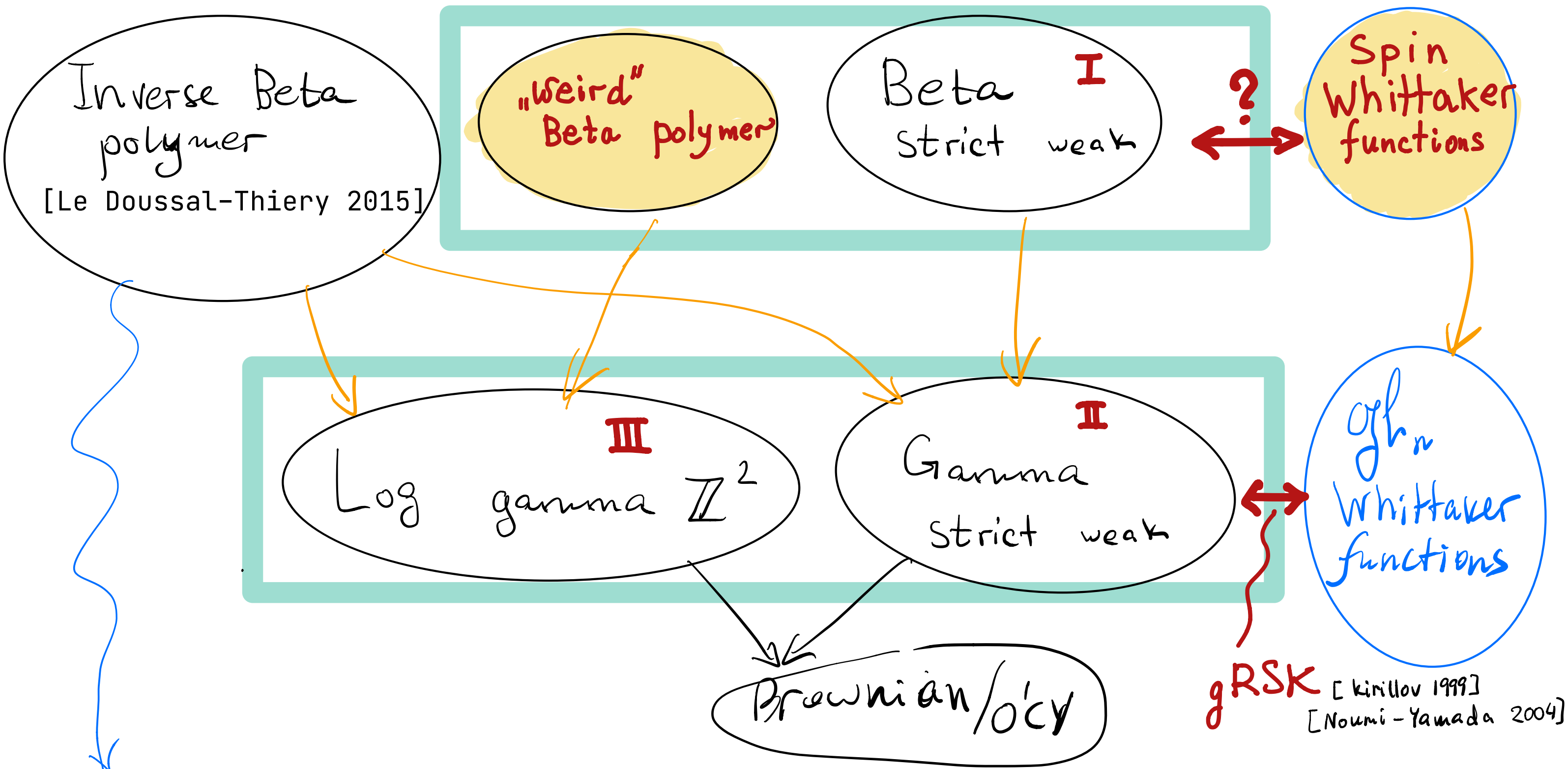
$\mathrm{Gamma}^{-1}(x_i+y_j)$

pdf $\dfrac{t^{-1-x_i-y_j}\, e^{-1/t}}{\Gamma(x_i+y_j)}$, $t\geq 0$.

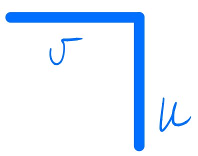
- $\oint$ for some moments
- $\mathbb{E}(e^{-uZ})$
- KPZ class asymptotics
- integrability via

gl_n Whittaker functions

Scheme of polymers + our additions



Remark: Inverse Beta is on $\mathbb{Z}^2$,



$u \sim \mathrm{Beta}^{-1}(\gamma, \beta) \quad \in (1, \infty)$

$v = u - 1 \quad \in (0, \infty)$

(Connection to symmetric functions unclear)

IV — „Weird" Beta polymer

[Bufetov – M·P. 2019]

[Corwin-Matveev-P. 2018], [Mucciconi-P. 2020]

$$\widetilde{Z}(i,j) = \begin{cases} 1 & \text{for } j=0, \\ \widetilde{Z}(1,j-1)\widetilde{B}_{1,j} \quad (*) & \text{for } i=1, \\ W^{>}_{i,j}\widetilde{Z}(i,j-1) + (1-W^{>}_{i,j})\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) > \widetilde{Z}(i-1,j), \\ (1-W^{<}_{i,j})\widetilde{Z}(i,j-1) + W^{<}_{i,j}\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) < \widetilde{Z}(i-1,j), \end{cases}$$

$$\widetilde{B}_{1,j} \sim \mathcal{Beta}^{-1}(X_1+Y_j, S-Y_j)$$

$q \to 1$ limit of q-Hahn TASEP or ${}_4\varphi_3$ vertex model

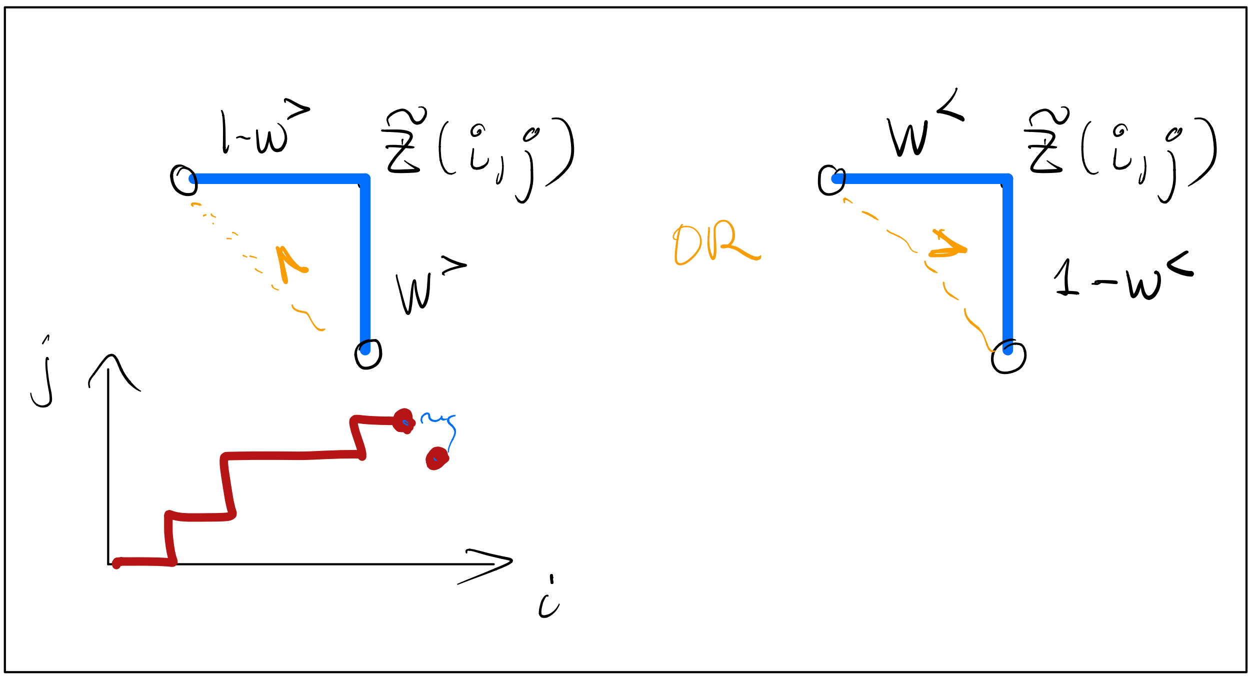
$$W^{>}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i-1,j)-\widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i,j-1)-\widetilde{Z}(i-1,j-1)}, X_i+Y_j, S-Y_j\right),$$

$$W^{<}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i,j-1)-\widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i-1,j)-\widetilde{Z}(i-1,j-1)}, X_i+Y_j, S-X_i\right).$$

Where NBB is a random variable on [0,1] with density

$$\mathcal{NBB}(r,p,m,n)[x] = \frac{(1-p)^r x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)}\, {}_2F_1\left(\begin{matrix} r, n+m \\ n \end{matrix} \middle| p(1-x)\right),$$

Beta$(m, n+k)$, where $k \sim$ NegBinom(r, p)



- $\oint$ for some moments at the q-level

[Corwin-Matveev-P. 2018]

- Analogue of

$\mathbb{E}[e^{-uZ}]$ at the q-level

[Bufetov-Mucciconi-P. 2019]

Proposition [Mucciconi-P. 2020]

As $S \to \infty$, $\tilde{Z} \longrightarrow$ Log-Gamma $\mathbb{Z}^2$ polymer (X_i, Y_j)

& Cases in $(*)$ disappear!

V Spin Whittaker Functions.

- Symmetric functions behind Beta and „weird" Beta polymers
- Lifts of gl_n Whittaker functions
- $q \to 1$ scaling limit of spin q-Whittaker poly's

Main open questions on sW:

- properties of sW like orthogonality
- connection to Beta polymers via "gRSK"
- Can we include inverse Beta polymer into the picture with symmetric functions?

Definition ("combinatorial formula"; "spin Givental integral")

Let $x_1,\dots,x_N, s\in\mathbb{R}$ be such that

$$|x_i|<s, \quad s>0$$

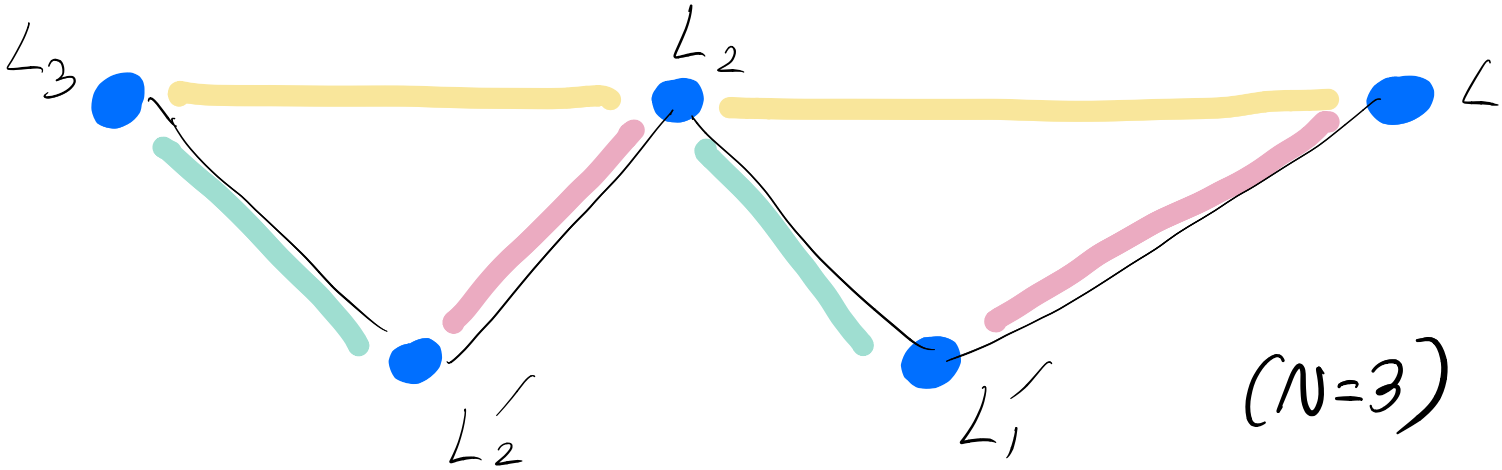
$$\underbrace{f_{x_1,\dots,x_N}}_{\text{"variables"}}(\overbrace{L_N,\dots,L_1}^{\text{"labels"}}) \doteq \int_{\vec{L}'\prec\vec{L}} f_{x_1,\dots,x_{N-1}}(L'_{N-1},\dots,L'_1)\, f_{x_N}(\vec{L}';\vec{L})\,\frac{d\vec{L}'}{L'_1\cdots L'_{N-1}}$$

$L_i, L'_i\in\mathbb{R}$

$$\boxed{s_\lambda(x_1\dots x_N)=\sum_\mu s_{\lambda/\mu}(x_N)\,s_\mu(x_1\dots x_{N-1})}$$

$x_N^{|\lambda|-|\mu|}$

$$1\le L_N\le L'_{N-1}\le L_{N-1}\le\dots\le L'_1\le L_1$$



$(N=3)$

$$f_x(\vec{L}';\vec{L}) = \left(\frac{\Gamma(2s)}{\Gamma(s+x)\Gamma(s-x)}\right)^{N-1}\left(\frac{L'_1\cdots L'_{N-1}}{L_1L_2\cdots L_N}\right)^{x}$$

$$\times\left(1-\frac{L'_1}{L_1}\right)^{s-x-1}\left(1-\frac{L_2}{L'_1}\right)^{s+x-1}\left(1-\frac{L_2}{L_1}\right)^{1-2s}$$

$$\times\left(1-\frac{L'_2}{L_2}\right)^{s-x-1}\left(1-\frac{L_3}{L'_2}\right)^{s+x-1}\left(1-\frac{L_3}{L_2}\right)^{1-2s}\times\cdots$$

Examples

$$f_x(L) = L^{-x} \qquad (N=1)$$

$(1 \le L_2 \le L_1)$

$$f_{x,y}(L_2, L_1) \qquad (N=2)$$

$$= \left(\frac{L_1}{L_2}\right)^s L_2^{-x-y} \, {}_2F_1\left(\begin{matrix} s+x, & s+y \\ & 2s \end{matrix} \,\middle|\, 1 - \frac{L_1}{L_2}\right)$$

relies on the classic Beta integral for ${}_2F_1$:

Def'n: ${}_2F_1(a, b ; c ; z) = \sum_{n=0}^{\infty} \frac{a(a+1)\cdots(a+n-1)\, b(b+1)\cdots(b+n-1)}{n!\; c(c+1)\cdots(c+n-1)} z^n.$

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \, {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \,\middle|\, z\right)$$

$$= \int_0^1 u^{b-1}(1-u)^{c-b-1} \underbrace{(1-zu)^{-a}}_{\text{added to the usual Beta function integral}} du$$

Also using Mellin-Barnes $\int$:

$$f_{x,y}(L_2, L_1) = \left(\frac{L_1}{L_2}\right)^s L_2^{-x-y} \frac{\Gamma(2s)}{\Gamma(s+x)\Gamma(s+y)} \qquad (N=2)$$

$$\cdot \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{\Gamma(t+s+x)\Gamma(t+s+y)\Gamma(-t)\left(\frac{L_1}{L_2}-1\right)^t}{\Gamma(t+2s)} dt.$$

$(N=3)$

$$\mathfrak{f}_{x,y,z}(L_3,L_2,L_1) = \frac{\Gamma(2s)}{\Gamma(s+x)\Gamma(s+y)}\frac{\Gamma(2s)}{\Gamma(s-z)\Gamma(s+z)}\frac{\Gamma(2s)}{\Gamma(s+z)}L_3^{-2s-x-y-z}L_2^sL_1^s$$
$$\times \frac{1}{(2\pi\mathbf{i})^3}\int_{(\mathbf{i}\mathbb{R})^3} dt\,du_1\,du_2\,(L_1-L_2)^{u_1}(L_2-L_3)^{t-u_1+u_2}L_3^{-t-u_2}$$
$$\times \frac{\Gamma(t+s+x)\Gamma(t+s+y)\Gamma(u_1-t)\Gamma(u_1+s+z)\Gamma(-u_1)}{\Gamma(t+2s)\Gamma(2s+u_1)}$$
$$\times \frac{\Gamma(s+t-u_1-z)\Gamma(2s+t+x+y+u_2)\Gamma(s+z+u_2)\Gamma(-u_2)}{\Gamma(2s+t+x+y)\Gamma(2s+t-u_1+u_2)}.$$

$(N=4)$

$$\mathfrak{f}_{x,y,z,w}(L_4,L_3,L_2,L_1) = \frac{1}{(2\pi\mathbf{i})^6}\int_{(\mathbf{i}\mathbb{R})^3} dt\,du_1\,du_2\,dv_1\,dv_2\,dv_3$$
$$L_1^sL_2^sL_3^sL_4^{-3s-t-u_2-x-y-z-w-v_3}(L_1-L_2)^{v_1}(L_2-L_3)^{u_1-v_1+v_2}(L_3-L_4)^{t-u_1+u_2-v_2+v_3}$$
$$\frac{\Gamma(2s)}{\Gamma(s+x)\Gamma(s+y)}\frac{\Gamma(2s)}{\Gamma(s-z)\Gamma(s+z)}\frac{\Gamma(2s)}{\Gamma(s+z)}$$
$$\frac{\Gamma(2s)}{\Gamma(s+w)}\frac{\Gamma(2s)}{\Gamma(s-w)\Gamma(s+w)}\frac{\Gamma(2s)}{\Gamma(s-w)\Gamma(s+w)}$$
$$\times \frac{\Gamma(t+s+x)\Gamma(t+s+y)\Gamma(u_1-t)\Gamma(u_1+s+z)}{\Gamma(t+2s)\Gamma(2s+u_1)}$$
$$\times \frac{\Gamma(s+t-u_1-z)\Gamma(2s+t+x+y+u_2)\Gamma(s+z+u_2)\Gamma(-u_2)}{\Gamma(2s+t+x+y)\Gamma(2s+t-u_1+u_2)}$$
$$\frac{\Gamma(-u_1+v_1)\Gamma(s+w+v_1)\Gamma(-v_1)}{\Gamma(2s+v_1)}$$
$$\frac{\Gamma(s+u_1-v_1-w)}{\Gamma(-t+u_1-u_2)}\frac{\Gamma(-t+u_1-u_2+v_2)\Gamma(s+w+v_2)\Gamma(-v_2)}{\Gamma(2s+u_1-v_1+v_2)}$$
$$\frac{\Gamma(s+t-u_1+u_2-v_2-w)}{\Gamma(2s+t-u_1+u_2-v_2)}$$
$$\times \frac{\Gamma(2s+t-u_1+u_2-v_2)}{\Gamma(3s+t+u_2+x+y+z)}\frac{\Gamma(s+w+v_3)\Gamma(3s+t+u_2+x+y+z+v_3)\Gamma(-v_3)}{\Gamma(2s+t-u_1+u_2-v_2+v_3)}$$

General N: work in progress

Most properties of sW functions come as $q \to 1$ limits from spin q-Whittaker level.

At spin q-Whittaker level, properties follow from Yang-Baxter equation.

Properties. 1) **Prop:**

$$f_{x_1 \dots x_N}(L_N, \dots, L_1) \text{ is symmetric in } x_j$$

2) **Conj:** $s \to \infty$ scaling limit of f leads to gl_n Whittaker functions

[Kostant, Givental, Bump, Stade, Gerasimov-Lebedev-Oblezin, Corwin-O'Connell-Seppalainen-Zygouras,…]

3) Two eigenoperators

$$D_1 F = \sum_i \prod_{j \neq i} \frac{x_i + s}{x_i - x_j} F \Big/_{x_i \to x_i + 1}$$

$$\bar{D}_1 F = \sum_i \prod_{j \neq i} \frac{x_i - s}{x_i - x_j} F \Big/_{x_i \to x_i - 1}$$

Prop:

$$\boxed{\begin{aligned} D_1 f_{\vec{x}}(\vec{L}) &= L_N^{-1} f_{\vec{x}}(\vec{L}) \\ \bar{D}_1 f_{\vec{x}}(\vec{L}) &= L_1 f_{\vec{x}}(\vec{L}) \end{aligned}}$$

4) „Deformed Quantum Toda"

$$\mathcal{H} = -\frac{1}{2}\sum_i \left(\frac{\partial}{\partial u_i}\right)^2 + \sum_{i<j} s^{-2(i-j)} e^{u_j-u_i}\left(s-\frac{\partial}{\partial u_i}\right)\left(s+\frac{\partial}{\partial u_j}\right)$$

$$\left[\, s\to\infty: \quad -\frac{1}{2}\sum_i \left(\frac{\partial}{\partial u_i}\right)^2 + \sum_{\substack{i<j \\ j=i+1}} e^{u_j-u_i} \,\right]$$

Prop:

$$\boxed{\begin{aligned} &\mathcal{H} f_{\vec{x}}\left(L_i = s^{N+1-2i} e^{u_i}\right) = \\ &\qquad = -\frac{1}{2}\left(x_1^2 + \dots + x_N^2\right) f_{\vec{x}}\left(L_i = s^{N+1-2i} e^{u_i}\right) \end{aligned}}$$

5) Conjectural weak orthogonality with „spin Sklyanin measure"

Conj.

$$\frac{1}{N!\,(2\pi i)^N}\int_{(i\mathbb{R})^N} f_{\vec{x}}(\vec{L})\, f_{-\vec{x}}(\vec{L}') \overbrace{\prod_{i\neq j} \frac{\Gamma(s+x_i)\,\Gamma(s-x_j)}{\Gamma(2s)\,\Gamma(x_i-x_j)}}\, d\vec{x}$$

$$= \prod_{i=1}^{N-1}\left(1 - \frac{L_{i+1}}{L_i}\right)^{1-2s} \delta_{\vec{L},\vec{L}'}$$

Example. $N=2$

$$\frac{1}{2(2\pi i)^2}\int_{(i\mathbb{R})^2}\left(\frac{L_1L_1'}{L_2L_2'}\right)^s\left(\frac{L_2}{L_2'}\right)^{-x-y}{}_2F_1\left(\left.\begin{matrix}s+x,\ s+y\\ 2s\end{matrix}\right|1-\frac{L_1}{L_2}\right)$$

$$\cdot\,{}_2F_1\left(\left.\begin{matrix}s-x,\ s-y\\ 2s\end{matrix}\right|1-\frac{L_1'}{L_2'}\right)$$

$$\cdot\,\frac{\Gamma(s-x)\Gamma(s+x)\Gamma(s-y)\Gamma(s+y)}{\Gamma(x-y)\Gamma(y-x)\Gamma(2s)^2}\cdot dx\,dy$$

$$\overset{?}{=}\left(1-L_2/L_1\right)^{1-2s}\delta_{\vec{L},\vec{L}'}.$$

VI From spin Whittaker to polymers.

1. At q-level, there are couplings between spin q-Whittaker measures and q-Hahn TASEP/PushTASEP.
   Then $q \to 1$ scaling gives results

2. Compare $\oint$ moments of polymers to spin Whittaker measure observables which we get using $D_1, \overline{D_1}$ and Cauchy identity.

$\mathbb{E}(L_N^{-k}) = \oint_k$

$$D_1 F = \sum_i \prod_{j \neq i} \frac{x_i + s}{x_i - x_j} F_{x_i \to x_i + 1}$$

$$\overline{D_1} F = \sum_i \prod_{j \neq i} \frac{x_i - s}{x_i - x_j} F_{x_i \to x_i - 1}$$

3. Is there gRSK?

$\boxed{\sum_\lambda s_\lambda(x) s_\lambda(y) = \prod \frac{1}{1 - x_i y_j}}$

sW measures. (and processes)

Prop. (Cauchy Identity)

$$\int_{\mathbb{R}^N} f_{x_1, \dots x_N}(\vec{L}) \, g_{y_1, \dots y_M}(\vec{L}) \frac{d\vec{L}}{L_1 \dots L_N}$$

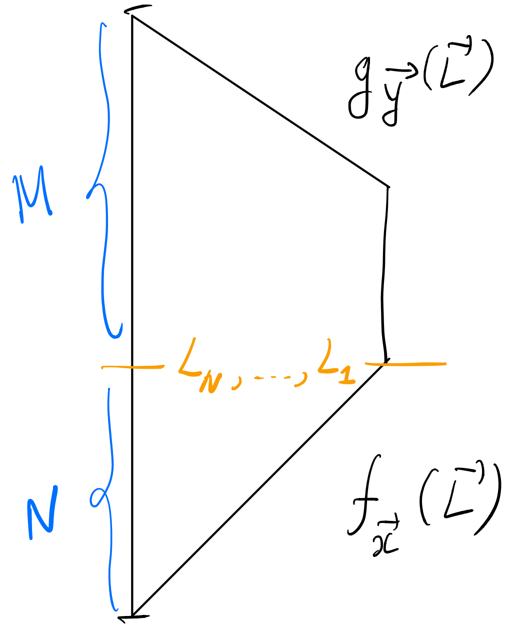
$$= \prod_{j=1}^{M} \left[\frac{\Gamma(x_1 + y_j)}{\Gamma(s + x_1)} \prod_{i=2}^{N} \frac{\Gamma(x_i + y_j) \Gamma(2s)}{\Gamma(s + x_i) \Gamma(s + y_j)} \right].$$

where $g_{y_1 \dots y_M}(L_N, \dots, L_1)$, $1 \le L_N \le \dots \le L_2 \le L_1$

is the „dual" sW function, symmetric in y_j.

sW measure

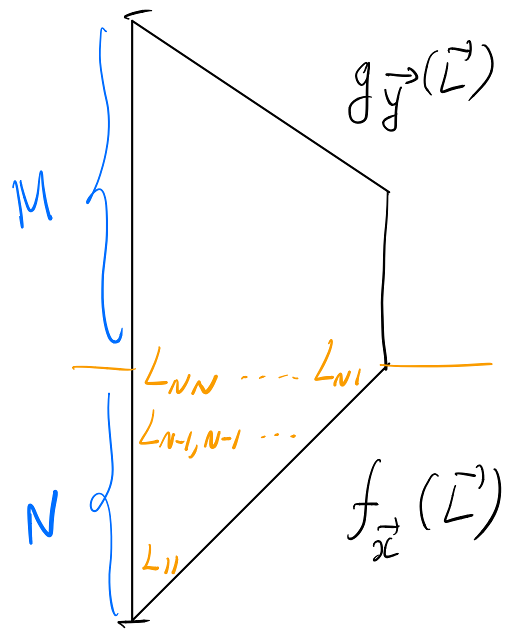
$$\mathrm{Prob}(\vec{L}) = \frac{1}{Z} f_{x_1 \dots x_N}(\vec{L})\, g_{y_1 \dots y_M}(\vec{L})$$



sW process

$$\mathrm{Prob}(\vec{L}_1 < \dots < \vec{L}_N) = \frac{1}{Z} g_{y_1 \dots y_M}(\vec{L}_N) \cdot$$

$$\cdot f_{x_1}(\vec{L}_1) f_{x_2}(\vec{L}_1, \vec{L}_2) \dots f_{x_N}(\vec{L}_{N-1}, \vec{L}_N)$$



Theorem (sW and Beta polymers)

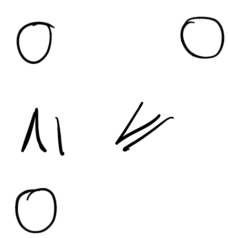
① $\{L_{N,N}^{-1}\}_{N\geq 1} \overset{\mathcal{D}}{=} \{Z^{\text{Beta strict weak}}(N, M)\}_{N \geq 1}$

② $\{L_{N,1}^{-1}\}_{N\geq 1} \overset{\mathcal{D}}{=} \{\tilde{Z}^{\text{"weird" Beta}}(N, M)\}_{N \geq 1}$

M — in $g_{y_1 \dots y_M}(\vec{L})$

$f_{x_1 \dots x_N}(\vec{L})\, g_{y_1 \dots y_M}(\vec{L})$

$$\begin{array}{cccc}
L_{NN} & L_{N-1,N-1} & \cdots & L_{N2} \quad L_{N1} \\
L_{N-1,N-1} & L_{N-1,N-2} & \cdots & L_{N-1,1} \\
\cdots & & & \\
L_{22} & L_{21} & & \\
L_{11} & & &
\end{array}$$



"weird" Beta

strict-weak Beta

VII Conclusions / Questions

- Motivated by probability, we built symmetric functions behind two **Beta polymer** models – **spin Whittaker functions.** They are full members of the hierarchy of symmetric functions

- Is the inverse Beta polymer a part of the same family?

- How to prove the conjectural orthogonality?

- Are there higher order eigenoperators like for the Macdonald / q-Whittaker polynomials and Whittaker functions?

- Polymer interpretation of multilayer distributions? Multilayer beta polymers? “Geometric RSK” for beta polymers?

- Representation theory / number theory behind spin Whittaker functions?

- Other symmetry types?