Nonequilibrium particle systems in inhomogeneous space

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Outline

KPZ equation and TASEP

Inhomogeneous TASEP and slow bond problem

Integrable particle systems in inhomogeneous space

Schur measures

Continuous space TASEP

Conclusions

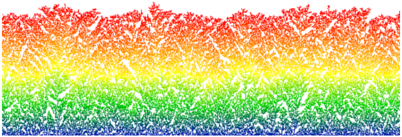
KPZ equation and TASEP

KPZ equation **[Kardar, Parisi, and Zhang, 1986]** — a stochastic PDE model for randomly growing interface $h(t,x)$, $t>0$, $x\in\mathbb{R}$:

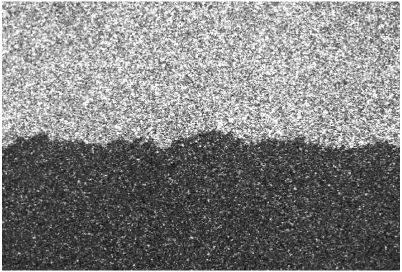
$$\frac{\partial h(t,x)}{\partial t} = \frac{\partial^2 h(t,x)}{\partial x^2} + \left(\frac{\partial h(t,x)}{\partial x}\right)^2 + \eta(t,x), \quad \mathbb{E}\eta(t,x)\eta(t',x') = \delta(t-t')\delta(x-x')$$

(the time evolution of the interface is governed by the smoothing and the slope-dependent growth terms, plus random noise)

- Existence and uniqueness of solutions **[Hairer, 2014]**, etc.
- Approximation of solutions of the KPZ equation by discrete-space interacting particle systems such as weakly ASEP **[Bertini and Giacomin, 1997]**, etc.
- Exact distributions and limits (e.g. $t\to+\infty$) of $h(t,x)$ for specific and (conjecturally) general initial data $h(0,x)$ **[Amir, Corwin, and Quastel, 2011, Matetski, Quastel, and Remenik, 2017]**, etc.

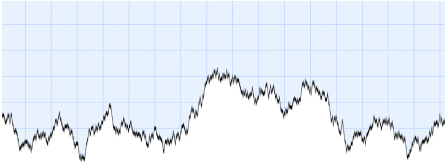


surface growth model



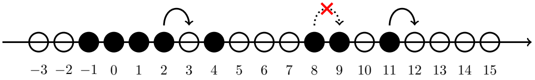
liquid crystal experiment

[Takeuchi and Sano, 2010]



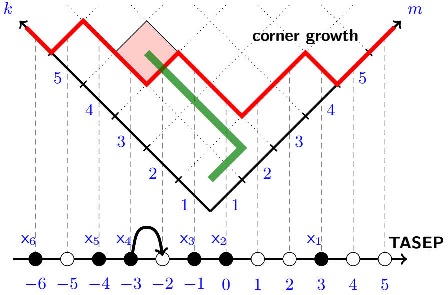
simulation by M. Hairer

TASEP (totally asymmetric simple exclusion process)



Each particle has an exponential clock with rate 1: $\mathbb{P}(\text{wait} > s) = e^{-s}$, $s > 0$, clocks are independent for each particle.

When the clock rings, the particle jumps to the right by one if the destination is not occupied.

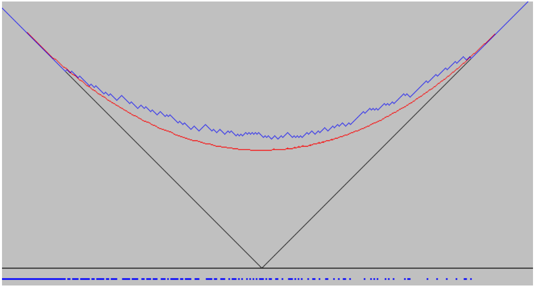


- TASEP on $\mathbb{Z}$
- directed last passage percolation
- corner growth
- longest increasing subsequences
- tandem queues

Theorem ([Johansson, 2000])
Start TASEP from the **step initial configuration** $x_i(0) = -i$, $i = 1, 2, \ldots$.
Let $h(t, x)$ be the height of the interface over x at time t. Then

$$\lim_{L \to +\infty} \mathbb{P}\left(\frac{h(\tau L, \chi L) - L\mathfrak{h}(\tau, \chi)}{c_{\tau,\chi} L^{\mathbf{1/3}}} \geq -s \right) = F_{GUE}(s),$$

where F_{GUE} is the **GUE** (Gaussian Unitary Ensemble) **Tracy–Widom distribution** originated in random matrix theory **[Tracy and Widom, 1993]**



Limit shape **[Rost, 1981]**
Simulation **[Ferrari, 2008]**

KPZ universality principle / conjecture: models in KPZ class (including the KPZ equation) *at large times and scales* behave as TASEP at *large times and scales*

Starting from Johansson's theorem, there is a very good understanding of TASEP asymptotics:

- multipoint distributions
- particle-dependent speeds
- other initial conditions, including general
- extensions to other models such as ASEP

[Okounkov, 2001, Its, Tracy, and Widom, 2001, Gravner, Tracy, and Widom, 2002, Prähofer and Spohn, 2002, Borodin, Ferrari, Prähofer, and Sasamoto, 2007, Matetski, Quastel, and Remenik, 2017, Borodin, Ferrari, and Sasamoto, 2009, Duits, 2013, Tracy and Widom, 2009]

- Some important aspects are **missing:**

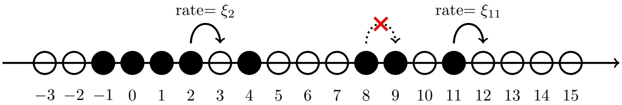
The focus today is on asymptotics of particle systems in **inhomogeneous space**

Inhomogeneous TASEP and slow bond problem

— one of the most complicated aspects of TASEP asymptotics, still not fully understood

Inhomogeneous TASEP

Particles are identical, but *jump rates depend on the location*



Slow bond problem. Let all ξ_i be 1 except that the jump to zero is with lower rate $\xi_{-1} = 1 - \varepsilon \in [0, 1]$.

Question: For step IC, does the flux of particles through zero **decrease** (from $\frac{1}{4}$ for $\varepsilon = 0$) for any $\varepsilon > 0$? Or is there a critical value $\varepsilon_c \neq 0$?

This question received competing predictions from various groups of physicists **[Janowsky and Lebowitz, 1992, Costin, Lebowitz, Speer, and Troiani, 2013]**.

This is a *hard analytic problem*: exact solutions **break down**

Warm up: hydrodynamics (why current through 0 is $\frac{1}{4}$)

Law of large numbers for regular (locally constant) behavior $\xi_i = \xi(i/t)$ and $t \to \infty$.

Understanding translation invariant stationary distributions in a *homogeneous* system, write down a PDE for the limiting density in the inhomogeneous case

Theorem (Liggett)
Bernoulli measures are all non-trivial extremal stationary measures of TASEP.

Let ρ be the density of the Bernoulli measure, then the flux (current) is $j(\rho) = \rho(1-\rho)$.

The continuity equation for the limiting density $\rho(t,x)$ (if it exists) is

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\Big(\xi(x)\rho(t,x)\big(1-\rho(t,x)\big)\Big) = 0, \qquad \rho(0,x) = \rho_0(x).$$

For $\xi \equiv 1$, $\rho_0(x) = \mathbf{1}_{x<0}$, we have $\rho(t,x) = \frac{1}{2}(1 - x/t)$, $|x| < t$. So $\rho(t,0) = \frac{1}{2}$.

[GKS10]: formulas when ξ takes 2 values. Conjecturally, Tracy–Widom fluctuations

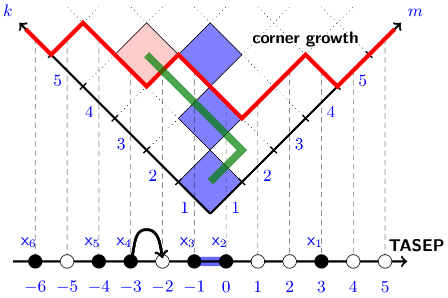
[Liggett, 1976, 1999, Andjel, 1982, Andjel and Kipnis, 1984, Rolla and Teixeira, 2008, Georgiou, Kumar, and Seppäläinen, 2010, Calder, 2015],

Slow bond problem and last passage percolation

For slow bond problem, ξ_i is not locally constant at 0 and Bernoulli measures are not invariant — so no hydrodynamics. The slow bond problem was solved in **[Basu, Sidoravicius, and Sly, 2014]**; invariant measures described in **[Basu, Sarkar, and Sly, 2017]**:

For every $\varepsilon > 0$, the asymptotic current through zero is strictly less than $\frac{1}{4}$.

The proof uses mapping to *last passage percolation* with $\exp(1-\varepsilon)$ random variables on the diagonal, and multiscale analysis.



Integrable particle systems in inhomogeneous space

— let us “repair” the inhomogeneous TASEP by taking a “similar” system which is exactly solvable

Relatives of TASEP which are integrable (= exactly solvable) in inhomogeneous space:

1. **PushTASEP**
   A known old relative of TASEP, also called *long-range TASEP*, a special case of the *Toom's model* **[Derrida, Lebowitz, Speer, and Spohn, 1991]**, known since 1970s **[Spitzer, 1970]**.

   (**[Petrov, 2018]** in preparation)

2. ⟶ A new **continuous space** version of **TASEP**
   Brings a whole new class of relative models: queuing systems, a deformation of last passage percolation, etc.

   (a harder model **[Knizel, Petrov, and Saenz, 2018]**, `arXiv:1808.09855 [math.PR]`)

Ultimate triggers of integrability: **Schur measures and the Yang-Baxter equation**

Schur measures

— main integrability tool

Schur measures **[Okounkov, 2001]** — a powerful tool in integrable probability. We use it to solve inhomogeneous PushTASEP and continuous space TASEP

Definition (Schur polynomials)

$\lambda = (\lambda_1 \ge \dots \lambda_N \ge 0)$, $$s_\lambda(\xi_1, \dots, \xi_N) := \frac{\det[\xi_i^{\lambda_j + N - j}]_{i,j=1}^N}{\det[\xi_i^{N-j}]_{i,j=1}^N}$$

$s_\lambda(\xi_1, \dots, \xi_N) \ge 0$ if all $\xi_i \ge 0$.

Definition (Schur measure)

$$\mathbb{P}(\lambda) := \frac{1}{Z} s_\lambda(\xi_1, \dots, \xi_N) s_\lambda(\eta_1, \dots, \eta_N), \qquad Z = \prod_{i,j=1}^N \frac{1}{1 - \xi_i \eta_j}.$$

Many parameters ξ_i, η_j

Explicit normalization follows from the *Cauchy identity*.

Working example

Of particular interest is the Schur measure $\propto s_\lambda(\xi_1, \ldots, \xi_N) s_\lambda(\Gamma_t)$

When $\xi_i \equiv 1$, it is sometimes called the *Schur-Weyl measure*, and appears in dimension counting in Schur-Weyl duality.

Schur-Weyl measure is closely related to the *Plancherel measure on partitions* and *longest increasing subsequences*
[Baik, Deift, and Johansson, 1999], **[Okounkov, 2000]**, **[Borodin, Okounkov, and Olshanski, 2000]**, **[Biane, 2001]**, **[Romik, 2015]**, etc.

Determinantal structure

The half-infinite random point configuration $\{\lambda_j - j\}_{j\ge 0}$ is a *determinantal point process* on $\mathbb{Z}$, that is,

$$\mathbb{P}(\text{random configuration } \{\lambda_j - j\} \text{ contains } a_1, \ldots, a_r) = \det_{i,j=1}^{r} \left[K(a_i, a_j)\right].$$

The kernel for $\propto s_\lambda(\xi_1,\ldots,\xi_N)s_\lambda(\Gamma_t)$ is

$$K(x,y)=\frac{1}{(2\pi\mathbf{i})^2}\oint\oint\frac{dwdz}{z-w}\frac{w^y}{z^{x+1}}e^{t(z-w)}\prod_{i=1}^{N}\frac{1-\xi_i/z}{1-\xi_i/w}$$

(contours are around 0 and $\{\xi_i\}$, and $|z|>|w|$)

- Asymptotic questions about Schur measures can in principle be answered via asymptotic analysis of contour integrals (*saddle point methods*)
- Extension to Schur processes **[Okounkov and Reshetikhin, 2003]**
- Markov dynamics on Schur processes (changing their specializations) are a rich source of integrable particle systems in 1 and 2 dimensions.

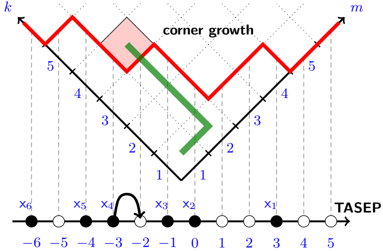
Remark. Markov dynamics on Schur processes are related / extend to

(1) *shuffling* for lozenge or domino tilings;

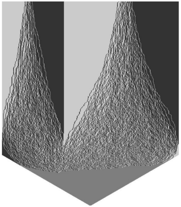
(2) Robinson-Schensted-Knuth insertion;

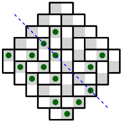
(3) integrable models of random polymers in random media; etc.

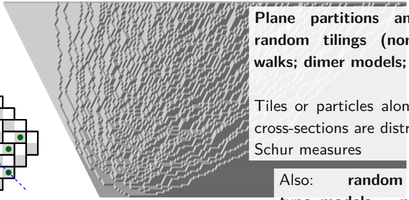
(A sample of) models solvable by Schur measures



- **Homogeneous** or **particle-inhomogeneous** TASEP on $\mathbb{Z}$
- directed last passage percolation
- corner growth
- longest increasing subsequences
- tandem queues







Plane partitions and other random tilings (noncolliding walks; dimer models; etc.)

Tiles or particles along certain cross-sections are distributed as Schur measures

Also: **random matrix type models, z-measures, polynuclear growth, . . .**

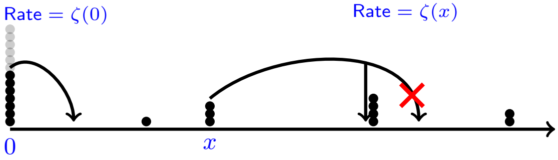
lozenge tilings pictures: [Okounkov and Reshetikhin, 2003, Borodin and Ferrari, 2014]

Continuous space TASEP

A continuous time particle system $\mathbf{X}(t)$ on ordered particles $x_1 \geq x_2 \geq \dots$ in $\mathbb{R}_{\geq 0}$.
Step IC: initially infinitely many particles at 0, the rest of the space is empty.

- one particle can leave a stack at location x at rate $\zeta(x)$, where ζ is an arbitrary positive piecewise continuous **speed function**;
- the jumping particle wants to jump an **exponential distance** with mean $1/L$;
- particles **preserve order** — an overflying particle joins the first stack to the right

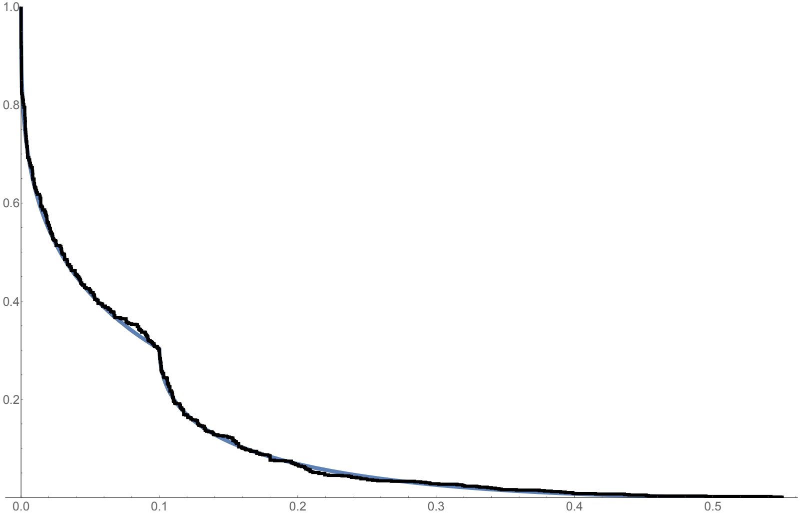
Height function $h_{cont}(t, x) := \#\{\text{number of particles} \geq x \text{ at time } t\}$.
Limit regime: $L \to +\infty$, $t = L\tau$ — more particles, long time, short jumps; the speed function $\zeta(\cdot)$ and location x are not scaled



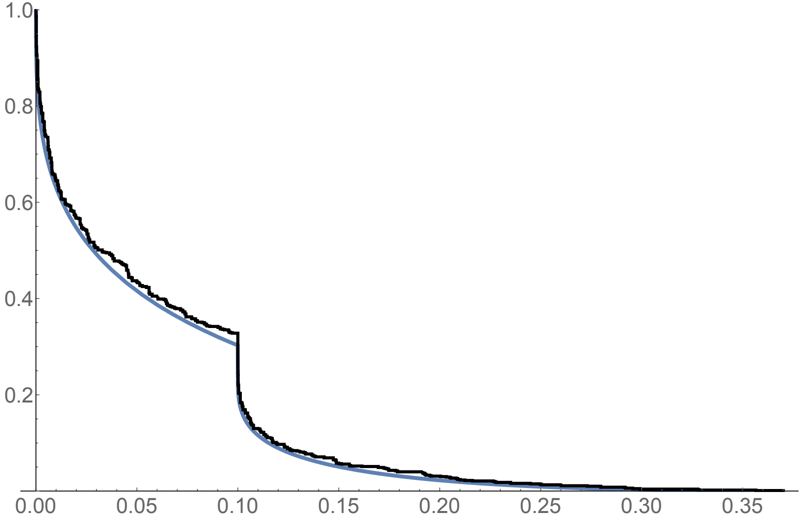
- This is a "natural" definition of TASEP in continuous space
- Also a continuous space limit of the discrete space *generalized TASEP* of **[Derbyshev, Povolotsky, and Priezzhev, 2015]**
- Resembles *queuing systems*, with Poisson service
- Can incorporate *roadblocks* (= *slow bonds*) in the space catching particles with fixed probability; for simplicity let's leave this for now
- A q-*deformation* was first studied in **[Borodin and Petrov, 2018]**. Continuous space TASEP is the $q = 0$ limit, and methods of [BP17] *break* for $q = 0$

Theorem ([Orr and Petrov, 2017], via stochastic vertex models)
Take λ_M under $\propto s_\lambda\left(\zeta(\frac{x}{M}), \zeta(\frac{2x}{M}), \ldots, \zeta(x)\right) s_\lambda\left(\frac{e^{-L/M}}{\zeta(x/M)}, \frac{e^{-L/M}}{\zeta(2x/M)}, \ldots, \frac{e^{-L/M}}{\zeta(x)}; \Gamma_t\right)$.
Then $h_{cont}(t, x) \stackrel{d}{=} \lim_{M\to\infty} \lambda_M$.
(recall: L^{-1} is the mean jump distance in the continuous TASEP)

Simulations of the random height function



Simulations of the random height function



Remark. Invariance under permutations / exchangeability

The height function's distribution in **both** PushTASEP and continuous space TASEP is **invariant under permutations of space inhomogeneity**:

$$h_{push}(t,x) \sim s_\lambda(\xi_1,\ldots,\xi_x)s_\lambda(\Gamma_t),$$
$$h_{cont}(t,x) \sim s_\lambda\left(\text{"}\zeta\big([0,x)\big)\text{"}\right) s_\lambda(\text{"}\frac{1}{\zeta\big([0,x)\big)}\text{"};\Gamma_t)$$

(Schur polynomials are symmetric)

These posterior facts can be traced to the Yang-Baxter equation for the $\mathfrak{sl}_2$ higher spin six vertex model

This invariance under permutations of the environment can be viewed as a good indicator towards solvability of space-inhomogeneous models
(does not naively work for TASEP with a slow bond)

Hydrodynamics in continuous space TASEP

Examples of translation invariant stationary distributions are Poisson processes on $\mathbb{R}$ with random geometric number of particles at points of Poisson process.

The flux (current) is $j(\rho) = \dfrac{2\rho + 1 - \sqrt{4\rho+1}}{2\rho}$ (*why generating function for Catalan numbers?*), and the PDEs for the limiting density and the limiting height function $\mathfrak{h}(\tau, x)$ are

$$\frac{\partial}{\partial \tau}\rho(\tau,x) + \frac{\partial}{\partial x}\big(\zeta(x) j(\rho(\tau,x))\big) = 0 \qquad \Rightarrow \qquad \mathfrak{h}_x(\tau,x) = -\frac{\zeta(x)\mathfrak{h}_\tau(\tau,x)}{\left(\zeta(x) - \mathfrak{h}_\tau(\tau,x)\right)^2}$$

Fix piecewise continuous $\zeta(\cdot)$, scale time $t = \tau L$, send $L \to +\infty$

Theorem ([Knizel, Petrov, and Saenz, 2018])
There exists almost sure limit shape $\mathfrak{h}(\tau, x) = \lim_{L\to\infty} \frac{1}{L} h_{cont}(\tau, x)$.

Edge: $\mathfrak{h}(\tau, x) \equiv 0$, $x \geq x_e(\tau)$, where $\tau = \displaystyle\int_0^{x_e(\tau)} \frac{dy}{\zeta(y)}$. For $x \in (0, x_e(\tau))$:

- Let w_x be the unique solution of $\tau w = \displaystyle\int_0^x \frac{w\zeta(y)(w+\zeta(y))}{(\zeta(y)-w)^3} dy$ on the interval $0 < w_x < \min_{0\leq y<x} \zeta(y)$. If ζ jumps down, the interval shrinks and w_x is discontinuous in x.
- The limit shape is $\mathfrak{h}(\tau, x) = \tau w_x - \displaystyle\int_0^x \frac{\zeta(y) w_x}{\zeta(y) - w_x} dy$. One can check that it satisfies the continuity equation.

Example: If $\zeta(x) = \mathbf{1}_{x \leq x_0} + b\mathbf{1}_{x > x_0}$, the limit shape is piecewise of degrees 3 and 6

Fluctuations for continuous TASEP

Let $\zeta(\cdot)$ be piecewise continuous, and continuous at 0.

Theorem ([Knizel, Petrov, and Saenz, 2018])

When $x \in (0, x_e(\tau))$, define $\sigma_x := 2^{-\frac{1}{3}} \left[\int_0^x \frac{2(w_x)^2 \, \zeta(y) \, (w_x + 2\zeta(y))}{(\zeta(y) - w_x)^4} \, dy \right]^{\frac{1}{3}} > 0$.

Then

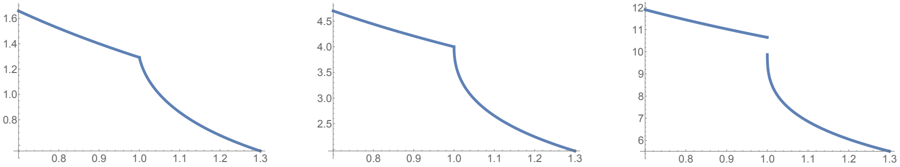
$$\lim_{L \to \infty} \mathbb{P}\left(\frac{h_{cont}(\tau L, x) - L\mathfrak{h}(\tau, x)}{L^{1/3}\sigma_x} \geq -r \right) = F_{GUE}(r), \qquad r \in \mathbb{R}.$$

$+$ multitime fluctuations are described by the Airy$_2$ kernel

Note: F_{GUE} fluctuations hold even at the points of infinite traffic jams

- There is a phase transition and a critical value $\zeta_c(\tau)$ of the slowdown speed ζ_c: for speed $\zeta > \zeta_c$ the height function is continuous, and for $\zeta < \zeta_c$ the height function becomes discontinuous.
- The density is discontinuous at such phase transitions. In infinite traffic jam the density is infinite.

Fluctuations at a traffic jam [Knizel, Petrov, and Saenz, 2018]



Let $\xi(y) = \mathbf{1}_{y\le 1} + \frac{1}{2}\cdot \mathbf{1}_{y>1}$. The traffic jam appears after $t = 12L$. Let $x = 1 + 10\varepsilon(L)$.

- If $\varepsilon(L) \ll L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - 4L}{2^{-2/3} c L^{1/3}} \to F_{GUE}$;
- If $\varepsilon(L) \gg L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - \mathfrak{h}(12, x) L}{c L^{1/3}} \to F_{GUE}$;
- If $\varepsilon(L) = 10^{-4/3}\delta L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - 4L}{2^{-2/3} c L^{1/3}} \to F_{GUE}^{(\delta)}$ (next slide).

$+$ multitime fluctuations are described by the Airy$_2$ kernel or its δ-deformation

The deformation $F_{GUE}^{(\delta)}$ is a Fredholm determinant

$$F_{GUE}^{(\delta)}(r) = \det\left(1 - K^{(\delta)}\right)_{(r,+\infty)}$$

of a deformation of the Airy kernel:

$$K^{(\delta)}(r,r') = \frac{1}{(2\pi\mathbf{i})^2} \int_{e^{-2\pi\mathbf{i}/3}\infty}^{e^{2\pi\mathbf{i}/3}\infty} dw \int_{e^{-\pi\mathbf{i}/3}\infty}^{e^{\pi\mathbf{i}/3}\infty} dz \, \frac{1}{z-w} \times \exp\left\{ \frac{z^3}{3} - \frac{w^3}{3} - zr + wr' - \frac{\delta}{z} + \frac{\delta}{w} \right\}$$

Kernel like this appears in **[Borodin and Peche, 2008]** in fluctuations for a certain last-passage percolation model and a certain random matrix model.

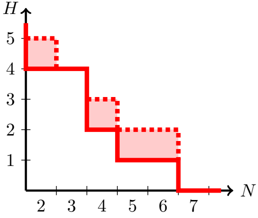
Why on both sides we recover the usual Tracy–Widom fluctuations:

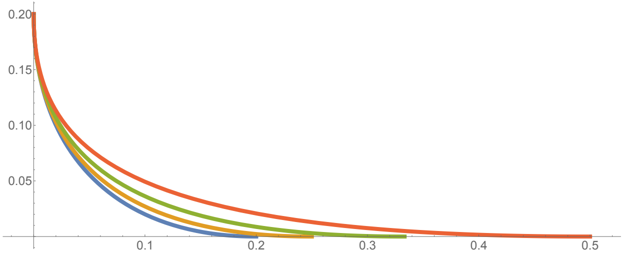
- If $\delta = 0$, we have $F_{GUE}^{(\delta)} = F_{GUE}$;
- As $\delta \to +\infty$, $F_{GUE}^{(\delta)}(r + 2\delta^{\frac{1}{2}}) \to F_{GUE}(2^{-\frac{2}{3}}r)$.

Remark. A discrete extension

The continuous space TASEP is in a **whole new class of Schur-solvable models**, as rich as the original family of models related to the usual TASEP.

Example: one-parameter extension of the last-passage percolation with geometric weights, where we add more than one box, but only in one direction.

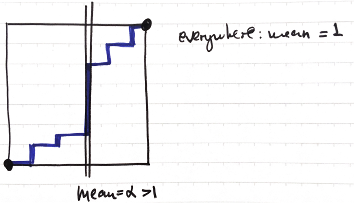


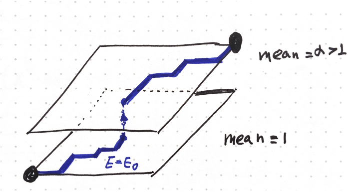


Limit shapes: a one-parameter extension of geometric corner growth (blue curve)

Remark. Last passage percolation view of phase transitions

- There is another type of phase transition first observed in **[Baik, Ben Arous, and Péché, 2005]** in random matrices;
- In directed last passage percolation setting this corresponds to having weights with slightly higher mean in a small region of the space. One can also organize a BBP transition in our continuous space TASEP
- Our phase transition can informally be related to a **layered directed last passage percolation**, when paths having energy $E > E_0$ (with E_0 fixed) move to a new layer with higher mean.





Conclusions

- The original slow bond problem for TASEP so far resists integrable tools
- There are two similarly looking models in inhomogeneous space, exactly solvable through Schur measures; both allow to model slow bond type behavior
- Common feature — invariance under permutations of the environment, can be traced to Yang–Baxter equation and Schur measures
- Explicit limit shape formulas and fluctuation results
- Continuous space TASEP allows for phase transitions of higher order (infinite density); this also nicely deforms fluctuation behavior
- On exactly solvable side, through the continuous space TASEP we discovered a new class of particle systems related to Schur measures.
  Many of the constructions lift to Macdonald processes and stochastic vertex models

Thank you!

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