Integrable Probability: Random Polymers, Random Tilings, and Interacting Particle Systems

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1. Introduction
2. Random polymers and KPZ equation
3. Random tilings
4. Particle systems as zero temperature limits of random polymers
5. Positive temperature and q-deformed particle systems

"Integrable" ("exactly solvable") probability — study of stochastic systems which can be analyzed by essentially algebraic methods.

Historically: De Moivre–Laplace's explicit computation for the binomial distribution; then (after almost 100 years) — the general Central Limit Theorem

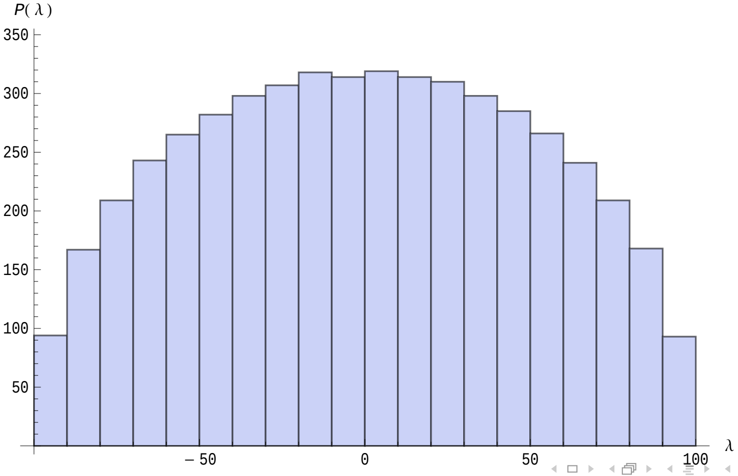
1. Identify new asymptotic phenomena by **explicit computations** for a particular integrable model

2. Understand the general class of (possibly **non-integrable**) stochastic systems which have the same asymptotic properties (**universality**)

Examples of integrable stochastic systems

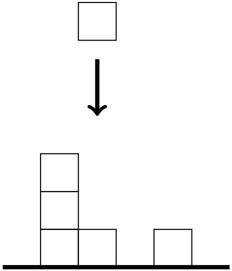
Random matrix ensembles [Wigner], [Dyson] (1950-60s). [T. Tao et al.], [H.-T. Yau et al.] — universality

(5000×5000 random symmetric matrix)



Examples of integrable stochastic systems

Random growth of interfaces



Examples of integrable stochastic systems

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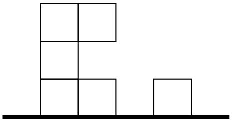


Examples of integrable stochastic systems

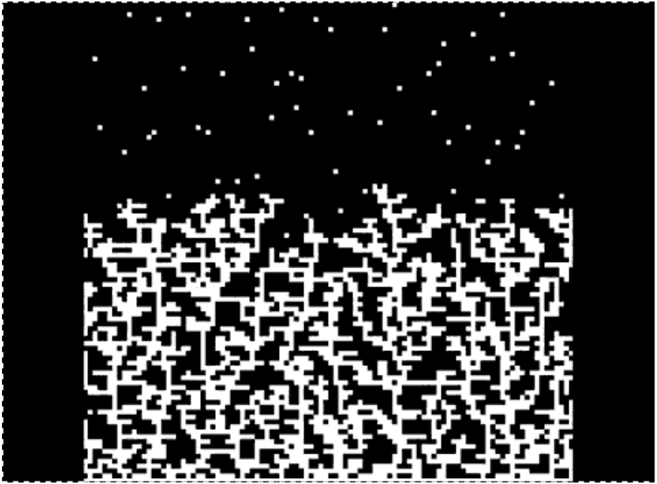
Random growth of interfaces

Simulation — integrable model?

http://www.wired.com/wiredscience/2013/03/
the-universal-laws-behind-growth-patterns-or-what-tetris-can-teach-us-about-co

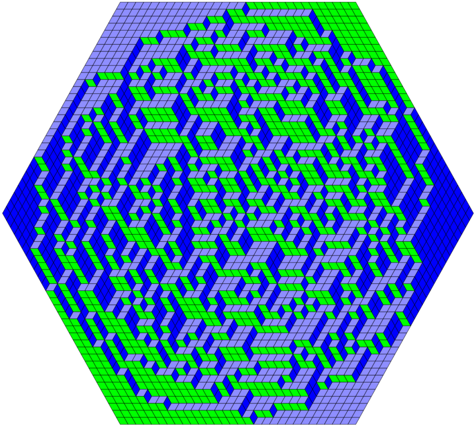
Examples of integrable stochastic systems

Random growth of interfaces



Examples of integrable stochastic systems

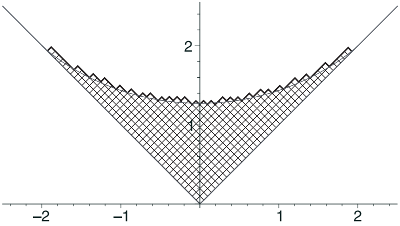
Random tilings/dimer models (two-dimensional interfaces)



Examples of integrable stochastic systems

Random systems motivated by representation theory

Example: Plancherel measure on Young diagrams
$\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$,
$P(\lambda) = (\dim \lambda)^2 / n!$
Vershik–Kerov–Logan–Shepp limit shape; longest increasing subsequence of random permutations



Also: infinite-dimensional diffusions (related to population dynamics and Poisson–Dirichlet distributions), combinatorics of Young diagrams, domino tilings, ...

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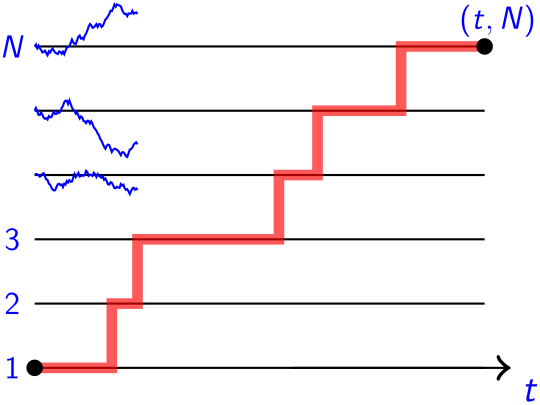
Semi-discrete directed Brownian polymer [O'Connell–Yor '01]

$$Z_N(t) := \int_{0<s_1<\ldots<s_{N-1}<t} e^{E(s_1,\ldots,s_{N-1})} ds_1 \ldots ds_{N-1}$$



Semi-discrete directed Brownian polymer [O'Connell–Yor '01]

$$Z_N(t) := \int\limits_{0<s_1<\ldots<s_{N-1}<t} e^{E(s_1,\ldots,s_{N-1})} ds_1 \ldots ds_{N-1}$$

where the energy is

$$E(s_1,\ldots,s_{N-1}) = B_1(s_1) + \big(B_2(s_2) - B_2(s_1)\big) + \ldots + \big(B_N(t) - B_N(s_{N-1})\big)$$

$B_1,\ldots,B_N$ — independent standard Brownian motions

Semi-discrete directed Brownian polymer: SDEs

$$Z_N(t) = \int_0^t e^{B_N(t)-B_N(s_{N-1})} Z_{N-1}(s_{N-1})\, ds_{N-1},$$

so $\frac{d}{dt} Z_N = Z_{N-1} + Z_N \dot{B}_N$

$$\begin{cases} dZ_N = Z_{N-1} dt + Z_N dB_N, & N = 1, 2, \ldots; \\ Z_N(0) = \mathbf{1}_{N=1}. \end{cases}$$

Semi-discrete directed Brownian polymer: SDEs

$$\begin{cases} dZ_N &= Z_{N-1}dt + Z_N dB_N, \qquad N = 1, 2, \ldots; \\ Z_N(0) &= \mathbf{1}_{N=1}. \end{cases}$$

Questions:

1. Distribution of $Z_N(t)$ for
   - $Z_N(0) = \mathbf{1}_{N=1}$
   - Any initial condition
2. Scaling limit of $Z_N(t)$ as $t, N \to \infty$

Semi-discrete polymer: scaling limit

[Borodin–Corwin–Ferrari '12]

For $Z_N(0) = \mathbf{1}_{N=1}$, one has

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\log Z_N(\varkappa N) - c_1(\varkappa)N}{c_2(\varkappa)N^{1/3}} \le u\right) = F_2(u)$$

F_2 — Tracy-Widom distribution (originated in random matrix theory '94)

$c_1(\varkappa), c_2(\varkappa) > 0$ — explicit constants

$c_1(\varkappa)$ established by [Moriarty–O'Connell '06], conjectured in [O'Connell–Yor '01]

Semi-discrete polymer: scaling limit

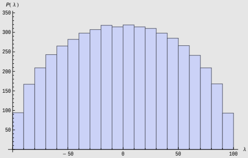
[Borodin–Corwin–Ferrari '12]

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{\log Z_N(\varkappa N) - c_1(\varkappa)N}{c_2(\varkappa)N^{1/3}} \le u\right) = F_2(u)$$

$c_1 N$ — Law of large numbers; $c_2 N^{1/3}$ — fluctuations (not $N^{1/2}$ as for the Gaussian)

Random matrices [TW '94]

$\lambda_{\max}$ — the rightmost eigenvalue,
Law of large numbers $\sim \sqrt{N}$;
fluctuations $\sim (\sqrt{N})^{1/3}$.



The semi-discrete directed Brownian polymer (and random matrix ensembles) belongs to the **Kardar–Parisi–Zhang (KPZ) universality class**

Connection to the KPZ equation

Taking diffusive scaling limit in (t, N) (polymer goes from $(0, 1)$ to (t, N); look at fluctuations around), one arrives at the continuous **stochastic heat equation**:

$$\frac{\partial}{\partial t} Z(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z(t,x) + Z(t,x)\xi(t,x), \qquad (SHE)$$

where $\xi(t,x)$ is the space-time white noise, $\mathbb{E}\xi(t,x)\xi(s,y) = \delta(t-s)\delta(x-y)$.

$$Z(t,x) = \mathbb{E} : \exp : \int_0^t \xi\big(s, b(s)\big)\, ds$$

where $\mathbb{E}$ is with respect to the Brownian bridge $b(s)$ with $b(0) = 0$ and $b(t) = x$ (**continuum directed random polymer**).

Long-term behavior of $Z(t,x)$ (SHE) with a certain initial condition is described by F_2 — the Tracy-Widom distribution [Amir–Corwin–Quastel '10].

Connection to the KPZ equation

$$\frac{\partial}{\partial t} Z(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z(t,x) + Z(t,x)\xi(t,x), \qquad (SHE)$$

If $h(t,x) := \log Z(t,x)$, then formally h satisfies the **KPZ equation** [Kardar–Parisi–Zhang '86]

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi \qquad (KPZ)$$

The SHE is the Hopf-Cole transform of the KPZ. Rigorous meaning: [Hairer '11]

Connection to the KPZ equation

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$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi \qquad (KPZ)$$

$u := \partial_x h$ satisfies **stochastic Burgers equation**

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \xi \qquad (stochastic\ Burgers)$$

Connection to the KPZ equation

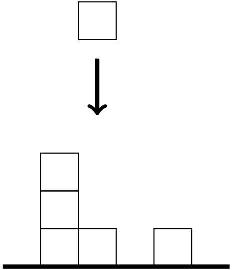
$$\frac{\partial}{\partial t} Z(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z(t,x) + Z(t,x)\xi(t,x), \qquad (SHE)$$

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi \qquad (KPZ)$$

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \frac{1}{2}\partial_x u^2 + \partial_x \xi \qquad (stochastic\ Burgers)$$

KPZ universality

- KPZ equation is a scaling limit of a number of systems (like the semi-discrete directed polymer). There are many open problems.



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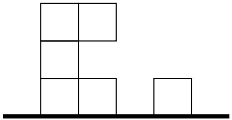


KPZ universality

- KPZ equation is a scaling limit of a number of systems (like the semi-discrete directed polymer). There are many open problems.

- Long-term behavior of $Z(t, x)$ (SHE) is described by F_2 — the Tracy-Widom distribution [Amir–Corwin–Quastel '10].

- Many more systems scale to F_2 or another Tracy-Widom distribution without scaling to KPZ equation; they belong to the wider **KPZ universality class**.
  Conjectural ingredients (already considered in [KPZ '86])
  - **Smoothing**
  - **Rotationally invariant, slope-dependent growth**
  - **Space-time uncorrelated noise**

  See [Corwin '11] for more detail.

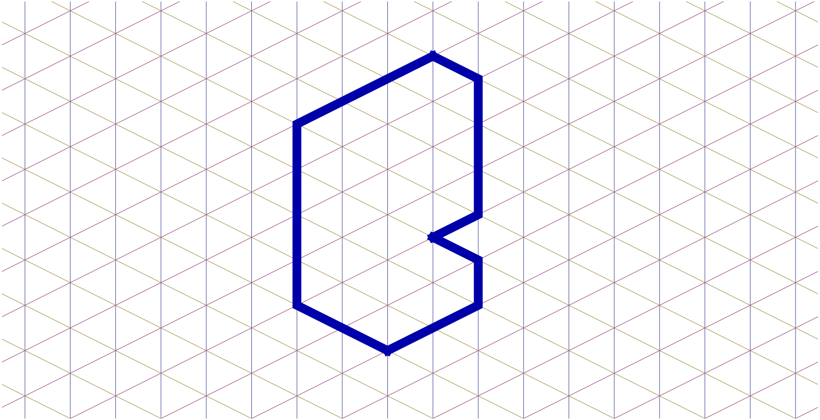
Integrable Probability

- Studying integrable members of the KPZ universality class help to understand many general (universal) properties.

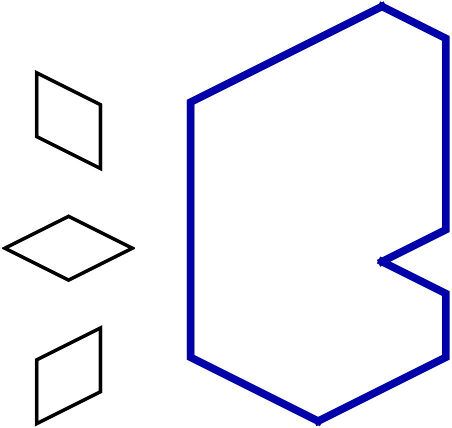
  "Small perturbations" of integrable models should not break the asymptotic results.

- This property of integrable models extends **beyond the KPZ universality class**.

Polygon on the triangular lattice



Lozenge tilings of a polygon



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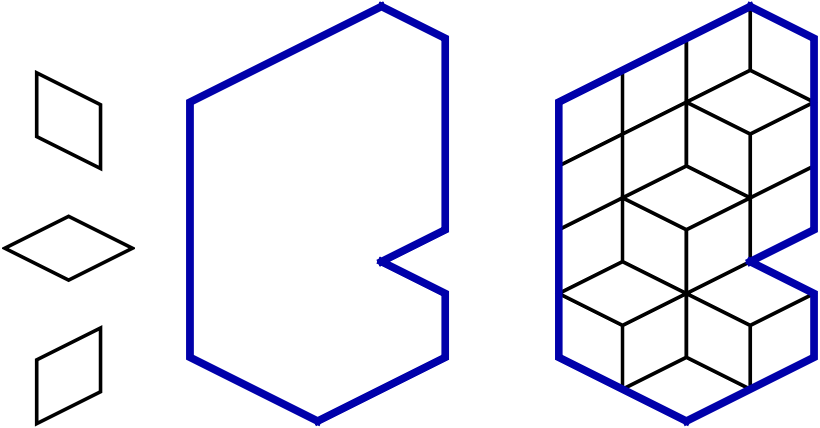
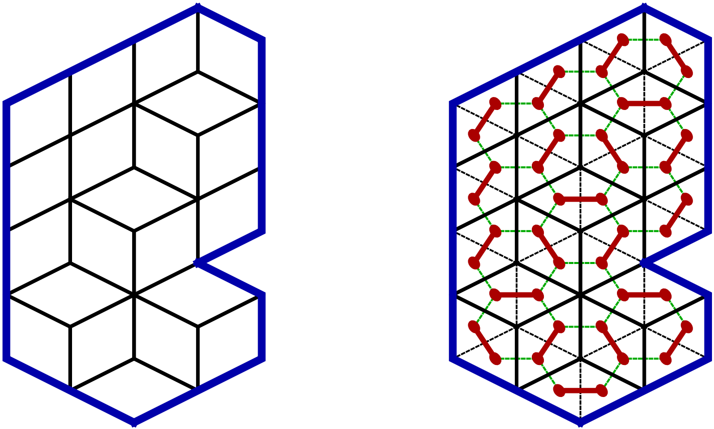
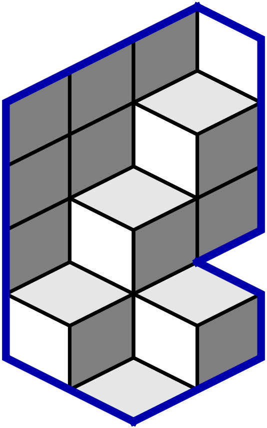
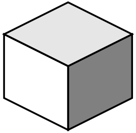


Lozenge tilings $\Longleftrightarrow$ Dimer Coverings



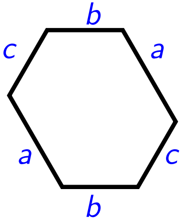
3D stepped surfaces with "polygonal" boundary conditions; **random interfaces** between two media in 3 dimensions ("melted crystal")



Unit cube = 

(polygon = **projection** of the boundary of 3D surfaces on the plane $x + y + z = 1$)

Tilings of the hexagon



Number of tilings:
P. MacMahon [1915–16]

Z = total # of tilings

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{i+j+c-1}{i+j-1}$$

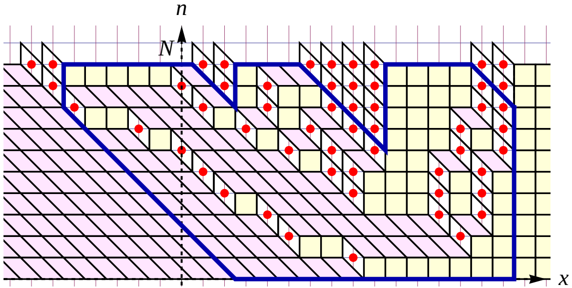
Partition functions (generalizing MacMahon's formulas)

Fixed N-th row of the particle array: $\mathbf{x}_N^N < \ldots < \mathbf{x}_1^N$

$Z =$ total # of tilings

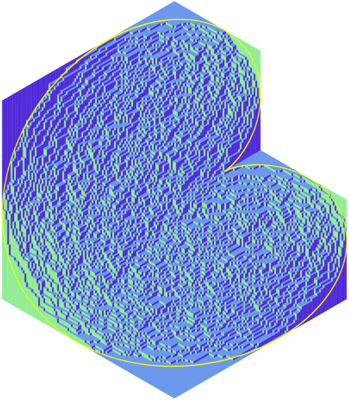
$$= \prod_{1 \le i < j \le N} \frac{\mathbf{x}_i^N - \mathbf{x}_j^N}{j - i} = s_\nu(\underbrace{1, \ldots, 1}_{N}) \text{ — Schur function,}$$

dimension of an irreducible representation of $U(N)$ indexed by the highest weight $\nu = (\mathbf{x}_1^N + 1, \mathbf{x}_2^N + 2, \ldots, \mathbf{x}_N^N + N)$ (**Weyl dimension formula**)

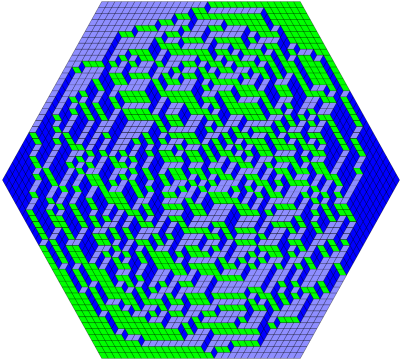


How very “large” uniformly random tilings look like?

Fix a polygon $\mathcal{P}$ and let the mesh $= N^{-1} = \varepsilon \to 0$ (**hydrodynamic scaling**).



[Kenyon-Okounkov '07]



Algorithm of [Borodin-Gorin '09]

Limit shape and frozen boundary for general polygonal domains

[Cohn–Larsen–Propp '98], [Cohn–Kenyon–Propp '01], [Kenyon-Okounkov '07]

- (LLN) As the mesh goes to zero, random 3D stepped surfaces concentrate around a **deterministic limit shape surface** (solution to a **variational problem**)
- The limit shape develops **frozen facets**
- There is a connected **liquid (disordered) region** where all three types of lozenges are present
- The limit shape surface and the separating **frozen boundary curve** are algebraic
- The frozen boundary is **tangent** to all sides of the polygon

Variational problem

$\mathbf{h}(\chi, \eta)$ — height of the limit shape at a point (χ, η) inside the polygon.

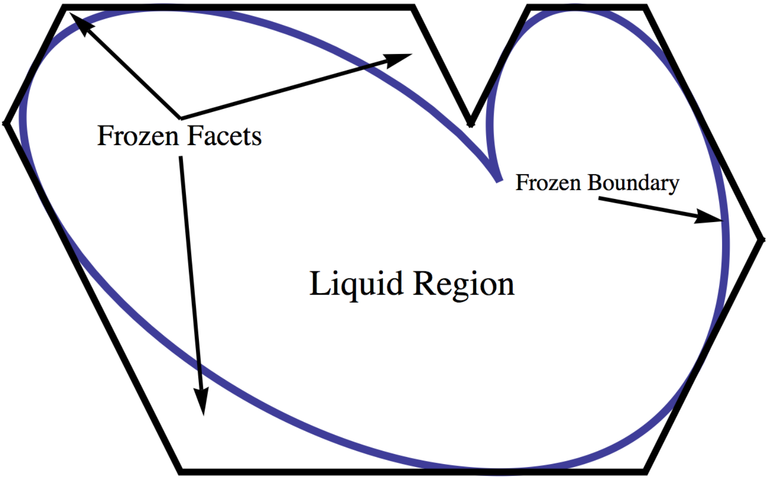
The height $\mathbf{h}$ is the *unique minimizer* of the functional

$$\int_{polygon} \sigma(\nabla \mathbf{h}(\chi, \eta)) d\chi d\eta,$$

where σ is the *surface tension*.

σ is the Legendre dual $(f^{\vee}(p^*) = \sup_p(\langle p, p^* \rangle - f(p)))$ of the Ronkin function of $z + w = 1$,

$$R(x, y) = \frac{1}{(2\pi i)^2} \int\int_{|z|=e^x,\ |w|=e^y} \log|z + w - 1| \frac{dz}{z} \frac{dw}{w}$$

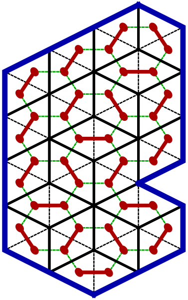


Frozen Facets
Frozen Boundary
Liquid Region

“Integrability” of random tilings

Thm. [Temperley–Fisher, Kasteleyn, 1960s]

The total number of dimer coverings of a hexagonal graph is the (absolute value of) the determinant of the incidence matrix $K(u, v)$



$$Prob(\text{dimers occupy } (u_1, v_1), \dots, (u_\ell, v_\ell))$$

$$= \frac{\det[K]_{\text{graph without } (u_1,v_1),\dots,(u_\ell,v_\ell)}}{\det[K]_{\text{all graph}}}$$

$$= \det[K^{-1}(u_i, v_j)]_{i,j=1}^{\ell}$$

K^{-1} can be written as a double contour integral **[P. ‘12]**, thus giving access to asymptotics

Asymptotic analysis of K^{-1}

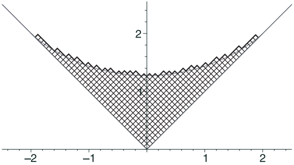
$K^{-1}(u; v) \sim$ additional summand

$$+ \oint \oint f(w,z) \frac{e^{N[S(w;u)-S(z;v)]}}{w-z} dwdz$$

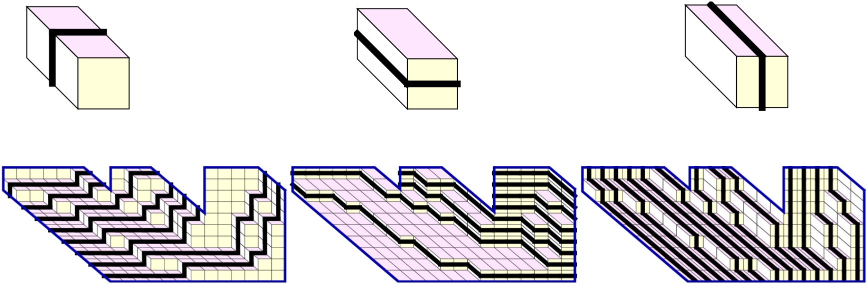
$f(w,z)$ — some "regular" part having a limit, $S(w;u)$ is an explicit function depending on the point u inside the polygon.

Then investigate critical points of the *action* $S(w;\chi,\eta)$ and transform the contours of integration so that the double contour integral goes to zero: $\Re S(w) < 0$, $\Re S(z) > 0$.

[Okounkov '02] — first application of double contour integrals to get asymptotics

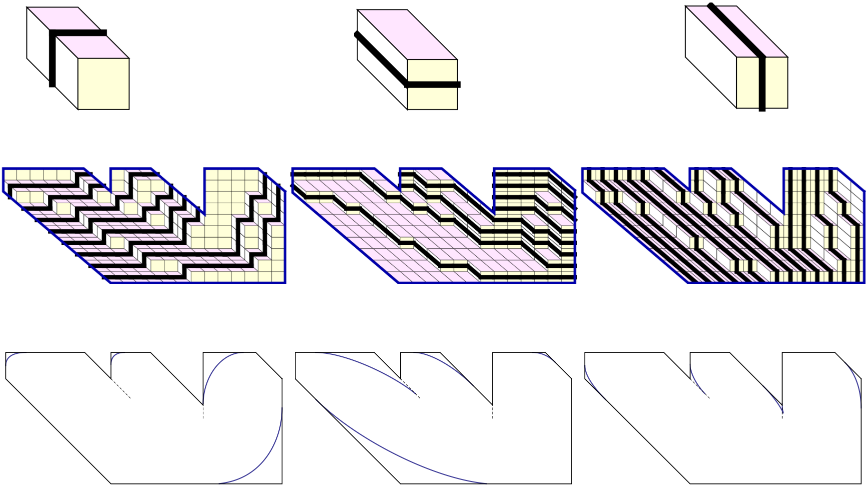


Local behavior at the edge: 3 directions of nonintersecting paths

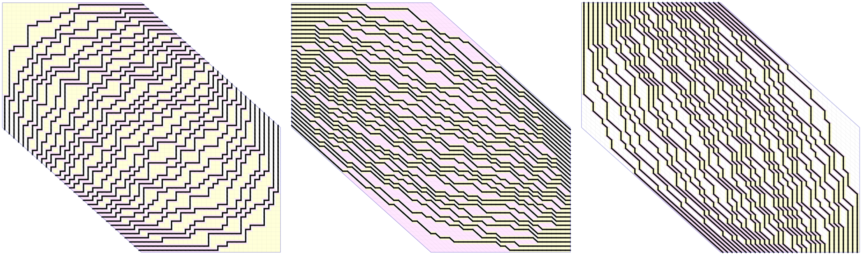


Counting nontintersecting paths with the help of determinants dates back to [Karlin–McGregor '59], [Lindstrom '73], [Gessel–Viennot '89]

Local behavior at the edge: 3 directions of nonintersecting paths



Limit shape $\Rightarrow$ outer paths of every type concentrate around the corresponding direction of the frozen boundary:



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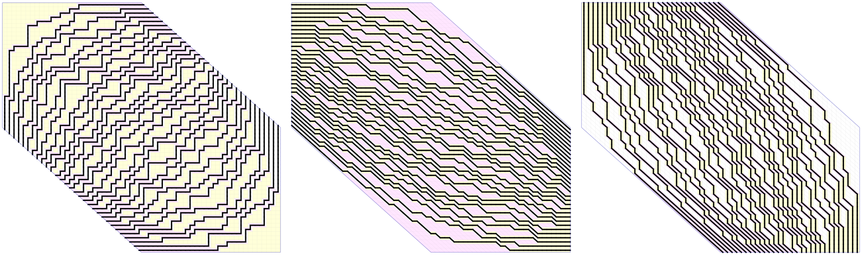


Theorem [P. '12]. Edge behavior: Tracy-Widom

Fluctuations $O(N^{2/3})$ in tangent and $O(N^{1/3})$ in normal direction

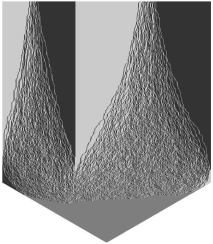
Thus scaled fluctuations are governed by the (space-time) $Airy_2$ process (its marginal is Tracy-Widom F_2) at **not tangent nor turning** point $(\chi, \eta) \in$ **boundary**

- First appearances:

random matrices (in part., Tracy-Widom distribution F_2), *random partitions* (in part., the longest increasing subsequence)

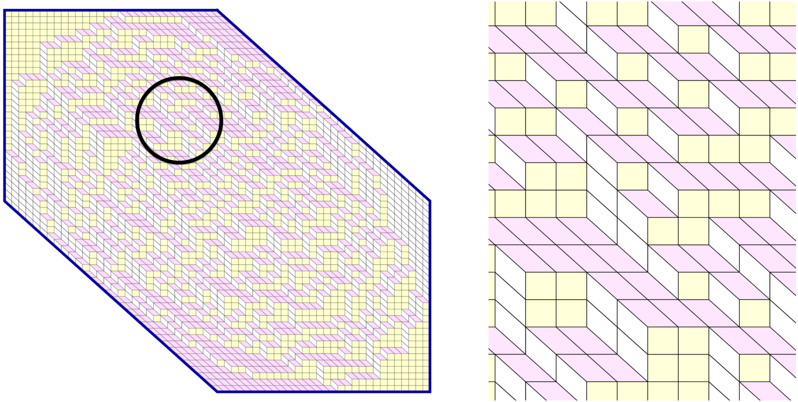
- Space-time Airy process: [Prähofer–Spohn '02]

- Random tilings of infinite polygons, same results: [Okounkov-Reshetikhin '07], [Borodin-Ferrari '08]



- K^{-1} computed by [Johansson '05] in terms of orthogonal polynomials (only for the hexagon), used in [Baik-Kriecherbauer-McLaughlin-Miller '07] to prove Tracy-Widom fluctuations

Studying asymptotics of K^{-1} also allows to obtain local lattice behavior. From it: understand geometry of the limit shape surface and of the frozen boundary [BKMM '07], [Gorin '07], [Borodin-Gorin-Rains '09], [P. '12].



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Theorem [P. '12]. Gaussian Free Field

Random field of fluctuations
$h_N([\chi N],[\eta N]) - \mathbb{E}(h_N([\chi N],[\eta N]))$, where h_N is the random (discrete) height function,
converges to a **Gaussian Free Field** on the liquid region with zero boundary conditions

Note that limit shape result is $h_N([\chi N],[\eta N])/N \to \mathbf{h}(\chi,\eta)$, where $\mathbf{h}$ is the deterministic continuous height function.

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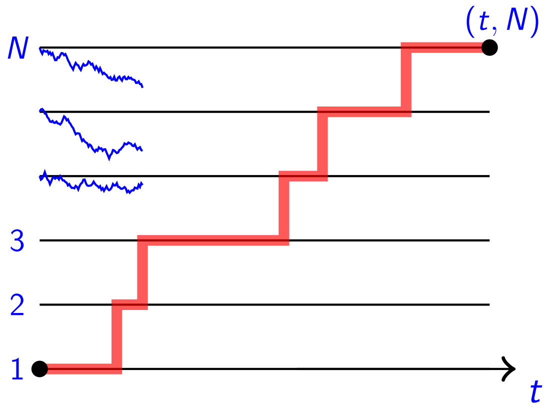
- Same result about fluctuations was obtained by Kenyon (preprint '04) for boundary conditions not allowing frozen parts of the limit shape, by analytic tools. He also conjectured the above theorem.

Zero temperature limit $\beta \to +\infty$

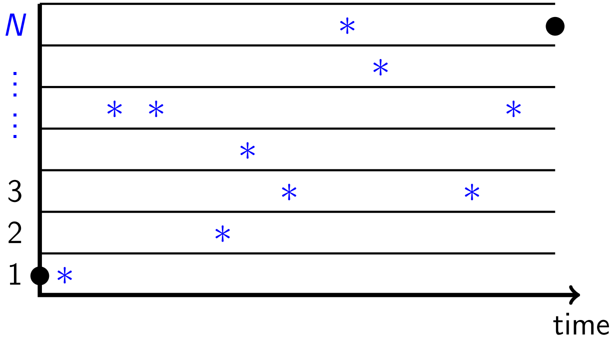
$$Z_N(t) := \int_{0<s_1<\ldots<s_{N-1}<t} e^{\boxed{\beta} E(s_1,\ldots,s_{N-1})} ds_1 \ldots ds_{N-1}$$

converges to a trajectory (depending on the environment in a **deterministic way**) which maximizes the energy



Let us also **discretize**, replacing Brownian motions by Poisson processes, then

$$Z_N(t) \longrightarrow L_N(t) := \begin{Bmatrix} \text{maximal number of points collected by} \\ \text{an up-right path from } (0,1) \text{ to } (t,N) \end{Bmatrix}$$



$$L_1 \leq L_2 \leq \ldots \leq L_{N-1} \leq L_N$$

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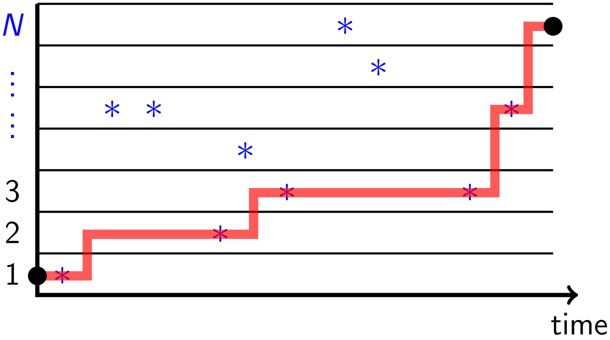


$L_1 \leq L_2 \leq \ldots \leq L_{N-1} \leq L_N$

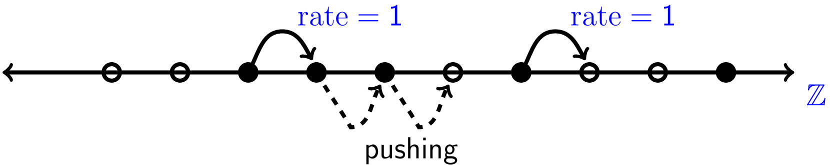
PushTASEP

(Pushing Totally Asymmetric Simple Exclusion Process)

Time evolution of

$$x_n(t) := L_n(t) + n, \qquad n = 1, 2, 3, \ldots$$

is Markov:

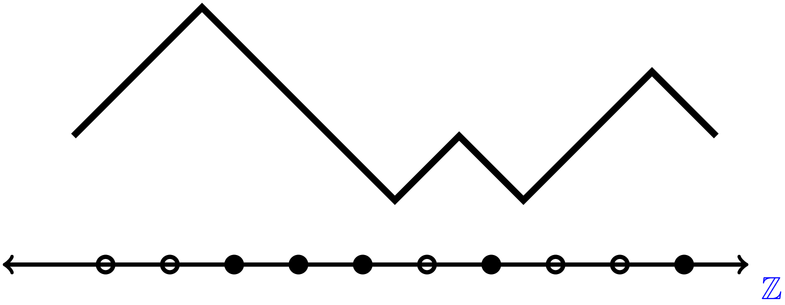


This is a **discrete, zero temperature** version of the stochastic heat equation

“Long-range TASEP” [Spitzer ’70]

PushTASEP as a growth model

(slope $+1$ over a hole, slope -1 over a particle)



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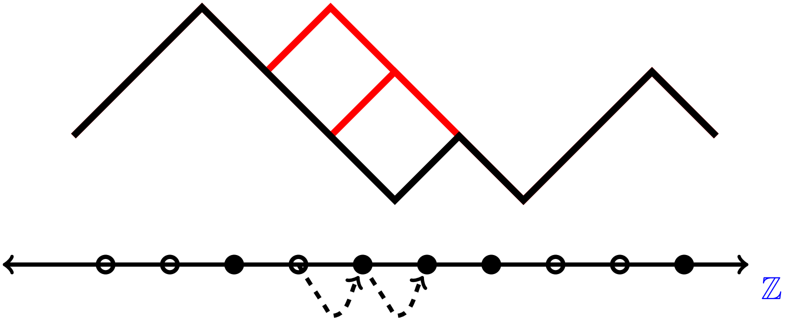
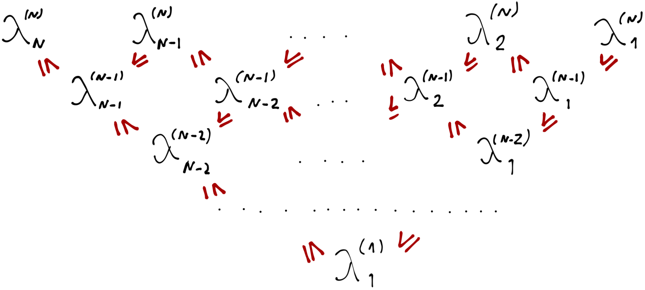


(growth speed depends on the "macroscopic" slope)

A two-dimensional extension of PushTASEP [Borodin–Ferrari '08]

Interlacing integer arrays (= Gelfand-Tsetlin schemes)

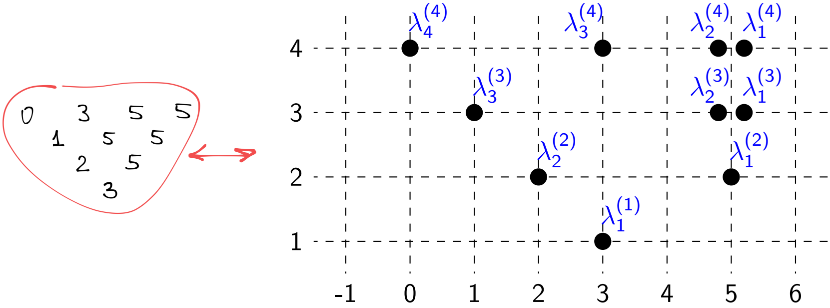


Each row $\lambda^{(k)} = (\lambda_k^{(k)} \leq \lambda_{k-1}^{(k)} \leq \ldots \leq \lambda_1^{(k)})$ is the *highest weight* of an irreducible representation of $GL(k)$.

Interlacing arrays parametrize vectors in the Gelfand-Tsetlin basis in the representation of $GL(N)$ defined by $\lambda^{(N)}$.

A two-dimensional extension of PushTASEP [Borodin–Ferrari '08]

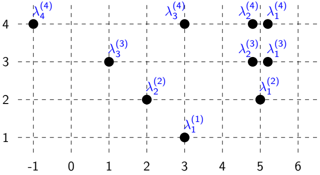
interlacing integer arrays $\longleftrightarrow$ particles in 2 dimensions



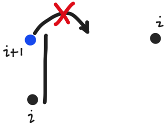
1 particle at level 1,
2 particles at level 2, etc.

A two-dimensional extension of PushTASEP [Borodin–Ferrari ’08]

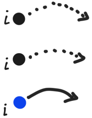
1. Each particle $\lambda_j^{(k)}$ jumps to the right by one according to an independent exponential clock of rate 1.



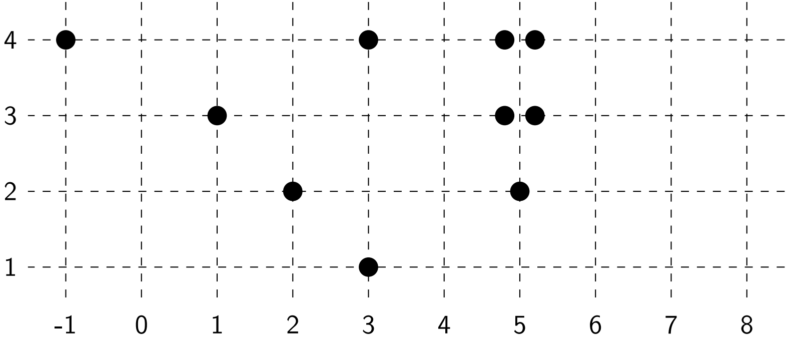
2. If it is blocked from below, there is no jump



3. If violates interlacing with above, it pushes the above particles



A two-dimensional extension of PushTASEP [Borodin–Ferrari '08]



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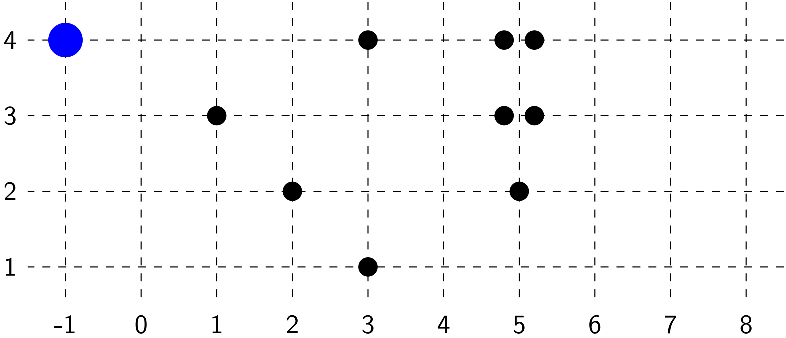

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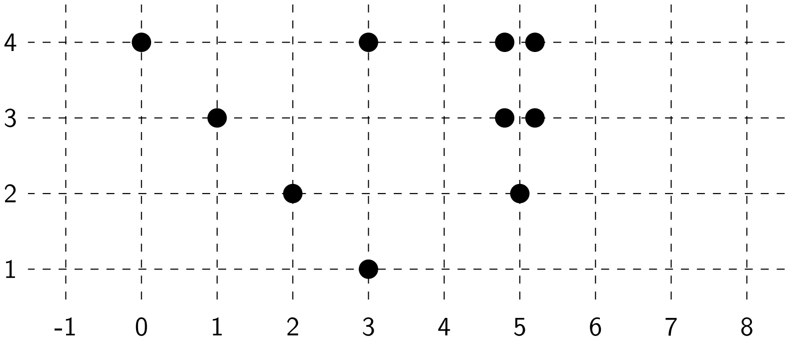

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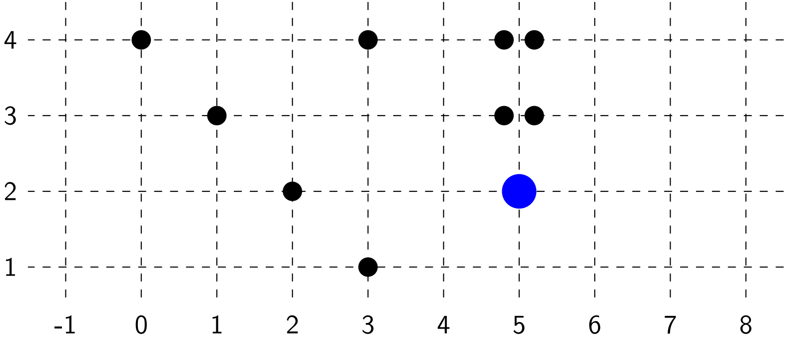

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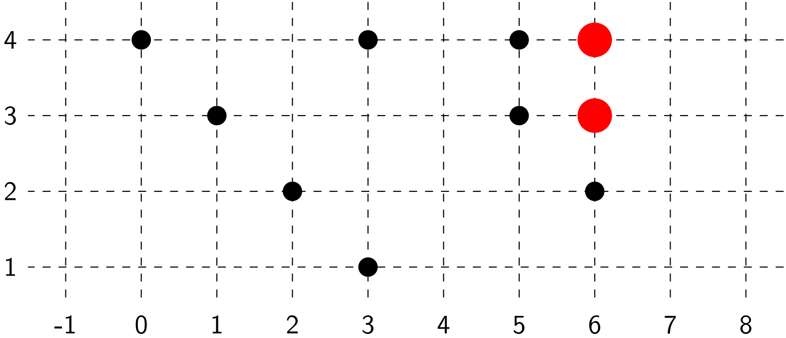

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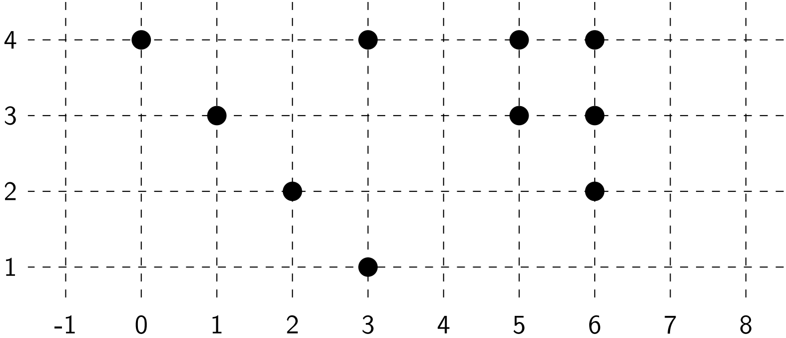

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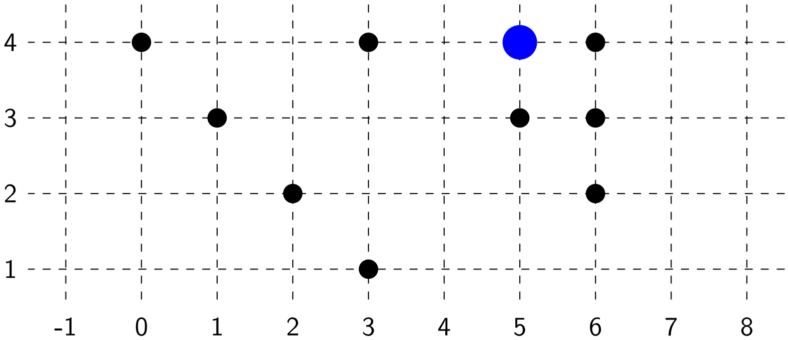

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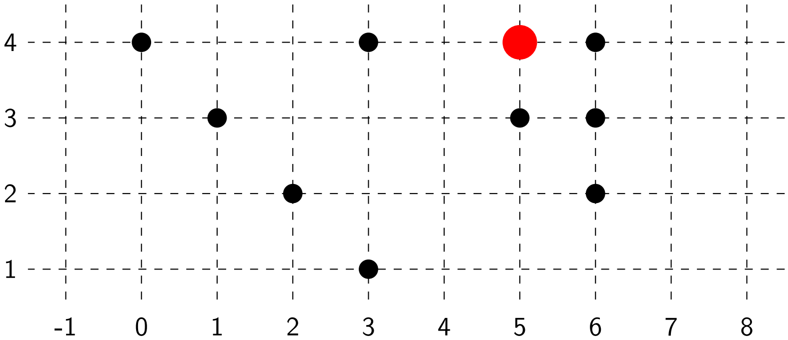

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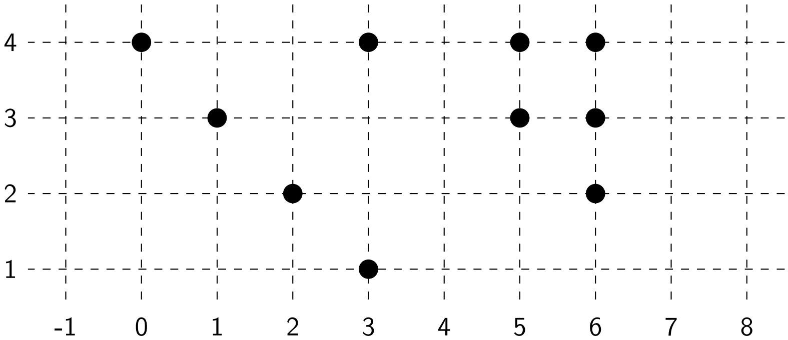

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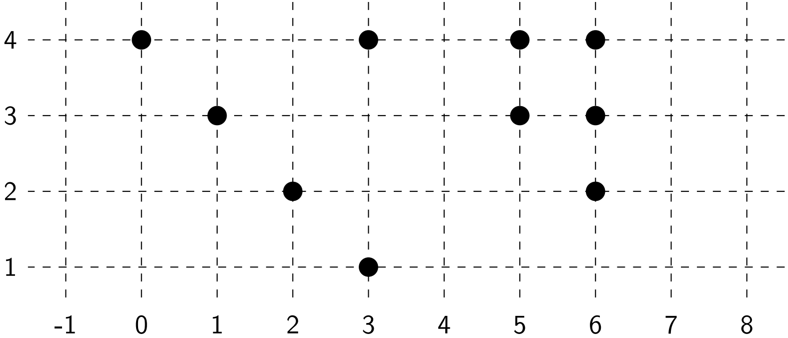

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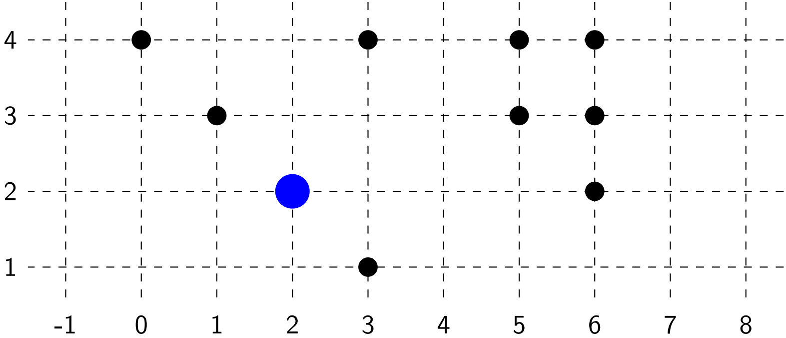

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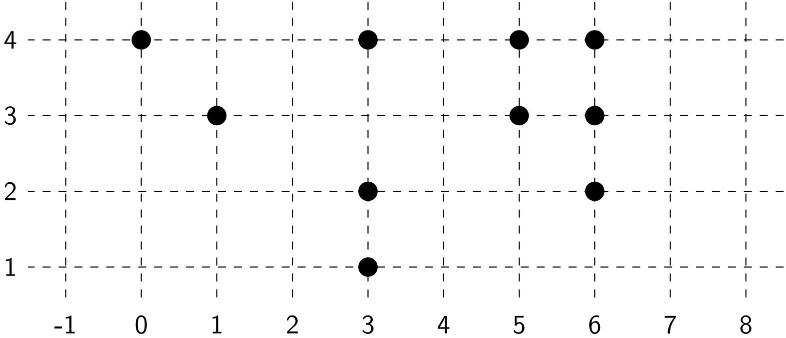
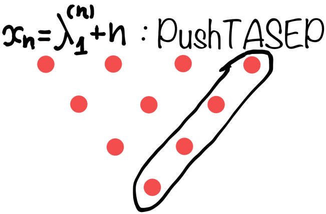


TASEP and PushTASEP

Markovian projection to the right-most particles — PushTASEP



TASEP and PushTASEP

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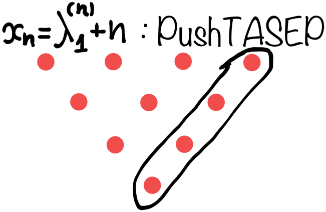
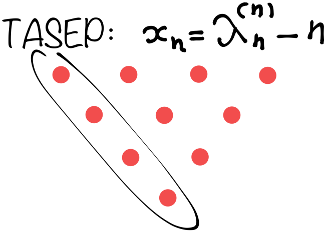


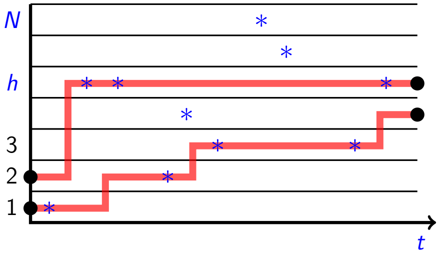


Markovian projection to the leftmost particles — TASEP (another discrete, zero temperature version of the stochastic heat equation)

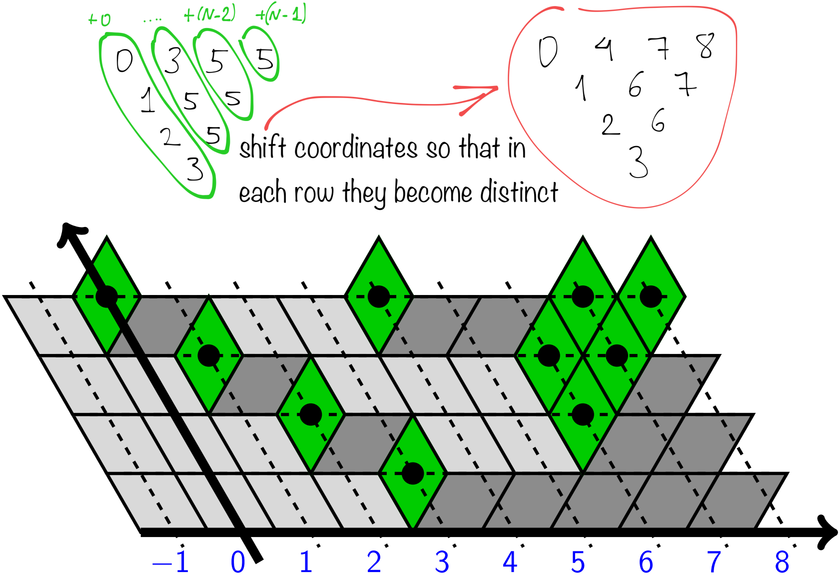
PushTASEP has another extension related to nonintersecting up-right paths and the Robinson–Schensted–Knuth correspondence

$\lambda_1^{(h)} + \lambda_2^{(h)} + \ldots + \lambda_j^{(h)} =$ the maximal number of $(*)$ one can collect along j **nonintersecting** up-right paths that connect points $(1, 2, \ldots, j)$ on the left border $(\mathrm{time} = 0)$, and $(h-j+1, h-j+2, \ldots, h)$ on the right border $(\mathrm{time} = t > 0)$.

[Borodin–P. '13]: common axiomatics for 2-dimensional dynamics with nice properties & their complete classification



Interlacing integer arrays ⟷ lozenge tilings

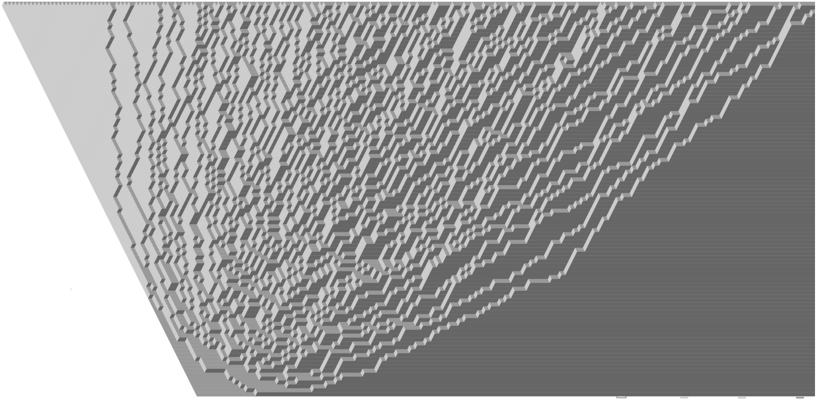


Growing 2-dimensional random interface $h(\eta, \nu)$ (with frozen parts), models the following continuous random growth:

$$\partial_t h = \Delta h + Q(\partial_\eta h, \partial_\nu h) + \xi(\eta, \nu)$$

(Q quadratic form of signature $(-1, 1)$; **anisotropic KPZ growth**)

fluctuations: $\sim L^{1/3}$ with time (L — large parameter)



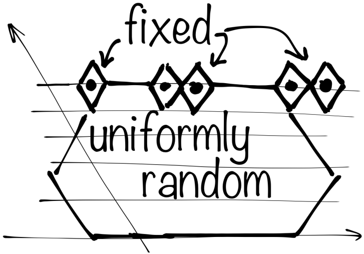
Gibbs property of the dynamics on interlacing arrays

Definition. Gibbs probability measures on interlacing arrays

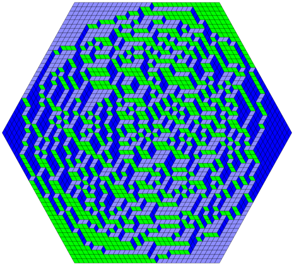
A measure M is called *Gibbs* if for each $h = 1, \ldots, N$:

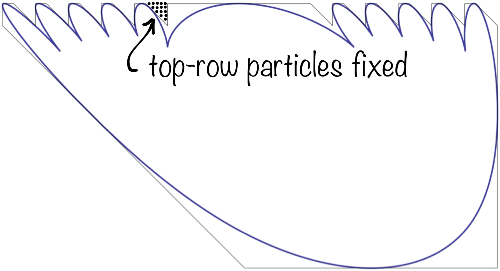
> *Given (fixed) $\lambda_h^{(h)} \le \ldots \le \lambda_1^{(h)}$, the conditional distribution of all the lower levels $\lambda^{(1)}, \ldots, \lambda^{(h-1)}$ is uniform (among configurations satisfying the interlacing constraints).*

The dynamics on arrays *preserves the class of Gibbs measures*: it maps one Gibbs measure into another.

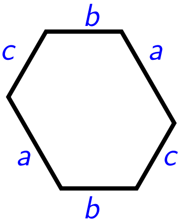


"Simplest" Gibbs measures — uniformly random tilings





(uniformly random configuration with fixed top row)



As $a, b, c \to +\infty$ such that $ab/c \to t$, uniformly random tilings of the hexagon converge to the distribution of the 2-dimensional dynamics at time $t > 0$.

1. Introduction
2. Random polymers and KPZ equation
3. Random tilings
4. Particle systems as zero temperature limits of random polymers
5. Positive temperature and q-deformed particle systems

Macdonald polynomials

$P_\lambda(x_1,\ldots,x_N) \in \mathbb{Q}(q,t)[x_1,\ldots,x_N]^{S(N)}$ labeled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\mathcal{D}^{(1)} = \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i}, \qquad (T_q f)(z) := f(zq),$$

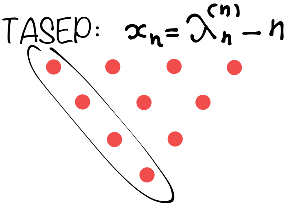
with (generically) pairwise different eigenvalues

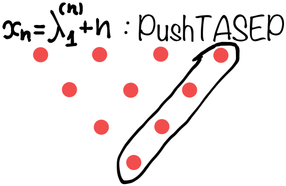
$$\mathcal{D}^{(1)} P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \ldots + q^{\lambda_N}) P_\lambda.$$

Macdonald polynomials have many remarkable properties (similar to those of Schur polynomials corresponding to $q = t$) including orthogonality, simple reproducing kernel (Cauchy identity), Pieri and branching rules, index/variable duality, etc. There are also simple higher order Macdonald difference operators commuting with $\mathcal{D}^{(1)}$.

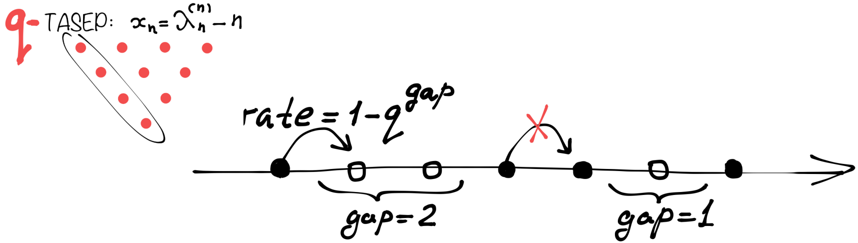
q-deformed particle systems

2-dimensional dynamics on interlacing arrays can be constructed using Macdonald polynomials (with $t = 0$) as well [Borodin–Corwin '11], [Borodin–P. '13]. They lead to **q-deformations of TASEP and PushTASEP.**





q-TASEP [Sasamoto–Wadati ’98], [Borodin–Corwin ‘11]



q-TASEP

1. Exact contour integral formulas for q-moments $\mathbb{E}\left(\prod_{j=1}^{k} q^{x_{N_j}(t)+N_j}\right)$ (where $N_1 \geq N_2 \geq \ldots \geq N_k > 0$), with a special initial condition [BC '11], [BC–Sasamoto '12]. Exact formulas for arbitrary initial condition, and a related Plancherel isomorphism theorem [BC–P.–Sasamoto '13]

q-TASEP

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2. q-TASEP locations $x_n(t)$ converge (under rescaling, as $q = e^{-\varepsilon}$, $t = \tau\varepsilon^{-2}$) to the semi-discrete directed polymer partition functions $Z_n(\tau)$.

q-TASEP

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3. Moments of q-TASEP particles are all bounded, and thus determine the distribution. This is not true for the polymer case (replica trick in physics literature).

q-TASEP

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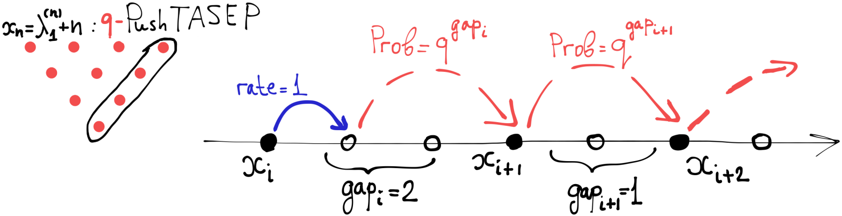
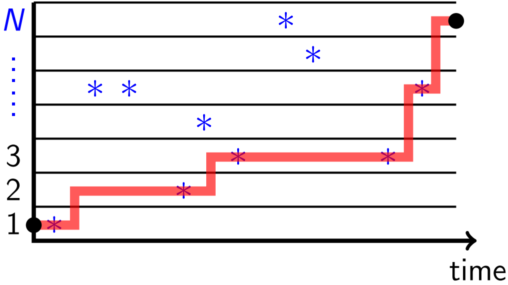
3. Moments of q-TASEP particles are all bounded, and thus determine the distribution. This is not true for the polymer case (replica trick in physics literature).
4. Tracy-Widom asymptotics: [Ferrari-Veto '13].

Also: [O'Connell–Pei '12], [Povolotsky '13], [van Diejen et al. '03], ...

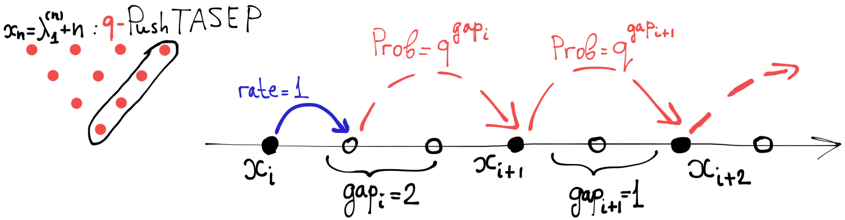
q-PushTASEP [Borodin–P. '13],



Describes the time evolution in a "positive temperature" version: random up-right paths in random environment



q-PushTASEP [Borodin–P. '13],



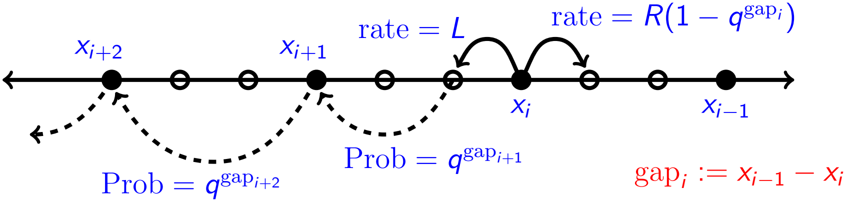
Particle locations converge to the polymer partition functions;

$$q = e^{-\varepsilon}, \qquad t = \tau\varepsilon^{-2},$$
$$x_n(\tau) = \tau\varepsilon^{-2} + (n-1)\varepsilon^{-1}\log(\varepsilon^{-1}) + \tilde{Z}_n(\tau)\varepsilon^{-1}$$

then $\tilde{Z}_n(\tau) \to Z_n(\tau)$, where

$$Z_n(\tau) = \int\limits_{0<s_1<\ldots<s_{n-1}<\tau} e^{B_1(s_1)+\ldots+\left(B_n(\tau)-B_n(s_{n-1})\right)} ds_1 \ldots ds_{n-1}$$

q-PushASEP [Corwin–P. '13]



$R * (q\text{-TASEP, to the right}) + L * (q\text{-PushTASEP, to the left})$

Traffic model (relative to a time frame moving to the right)

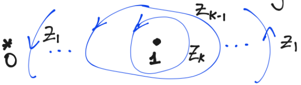
- Right jump = a car *accelerates*. Chance $1 - q^{\mathrm{gap}}$ is lower if another car is in front.
- Left jump = a car *slows down*. The car behind sees the brake lights, and may also quickly slow down, with probability q^{gap} (chance is higher if the car behind is closer).

q-PushASEP integrability

Theorem [Corwin–P. ‘13]. q-moment formulas for the q-PushASEP with the step initial condition $x_i(0) = -i$, $i = 1, \ldots, N$.

$$\mathbb{E}\left(\prod_{i=1}^{k} q^{x_{N_i}(t)+N_i}\right) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} (1-z_j)^{-N_j} \frac{G_t(q z_j)}{G_t(z_j)} \cdot \frac{dz_j}{z_j}$$

$N_1 \geq N_2 \geq \ldots \geq N_k > 0$



$$G_t(z) = e^{t(Rz + Lz^{-1})}$$

Obtained via a quantum integrable (many body) systems approach dating back to [H. Bethe ‘31]

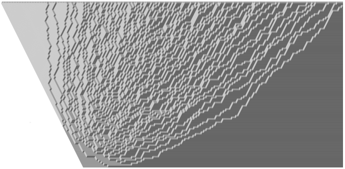
Conclusions

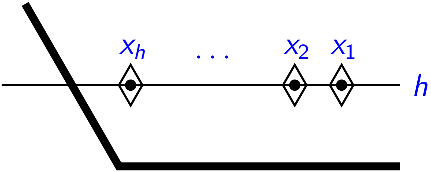
- Integrable probabilistic models help to understand general, universal behavior of stochastic systems. Algebraic tools are often the only ones available.
- Integrable properties of probabilistic models reveal connections with other areas (representation theory, combinatorics, integrable systems). This equips probabilistic computations and results with a richer structure.
- Algebraic structures provide deformations (regularisations) which **eliminate analytic issues** (replica trick for polymers/SHE/KPZ vs q-TASEP [BC '11]).

Surveys/lecture notes:

- Corwin arXiv:1106.1596 [math.PR]
- Borodin–Gorin arXiv:1212.3351 [math.PR]
- Borodin–P. arXiv:1310.8007 [math.PR]

Bonus: Back to zero-temperature dynamics





Distribution of vertical lozenges $(x_1 - h + 1, x_2 - h + 2, \ldots, x_h) = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_h)$ at height h is determined from the generating series

$$\prod_{i=1}^{h} e^{t(z_i - 1)} = \sum_{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_h} \mathrm{Prob}(\mu) \cdot \frac{s_\mu(z_1, \ldots, z_h)}{s_\mu(1, \ldots, 1)},$$

where s_μ — Schur symmetric polynomials.

Connection with irreducible characters of unitary groups $U(N)$, and of the infinite-dimensional unitary group $U(\infty)$ [Edrei, Schoenberg '50s, Voiculescu '76, Boyer, Vershik, Kerov '80s]

Apply Macdonald difference operators, $t = q$

$$\mathcal{D}^{(1)} = \sum_{i=1}^{h} \prod_{j \neq i} \frac{qz_i - z_j}{z_i - z_j} T_{q,z_i}, \qquad T_q f(z) := f(qz),$$

these operators are diagonalized by Schur polynomials (representation-theoretic meaning: operators which are scalar in each irreducible representation):

$$(\mathcal{D}^{(1)} s_\mu)(x_1, \ldots, x_h) = \left(\sum_{i=1}^{h} q^{\mu_i + h - i} \right) s_\mu(x_1, \ldots, x_h).$$

Then (idea first applied in [Borodin–Corwin '11], see also [Borodin–P. '13: Lecture notes])

$$\mathcal{D}^{(1)} \prod_{i=1}^{h} e^{t(z_i - 1)} = \sum_{\mu} \mathrm{Prob}(\mu) \left(\sum_{i=1}^{h} q^{\mu_i + h - i} \right) \frac{s_\mu(z_1, \ldots, z_h)}{s_\mu(1, \ldots, 1)}$$

We want to put $z_1 = \ldots = z_h$, which is best done with contour integrals.

Apply Macdonald difference operators, $t = q$

$$\mathcal{D}^{(1)} \prod_{i=1}^{h} e^{t(z_i-1)} \Big|_{z_1=\ldots=z_h=1}$$

$$= \frac{1}{2\pi\mathrm{i}} \oint_{|w-1|=\varepsilon} \prod_{j=1}^{h} \frac{qw - z_j}{w - z_j} \frac{1}{(q-1)w} e^{t(q-1)w} dw \Big|_{z_1=\ldots=z_h=1}$$

$$= \sum_{\mu_1 \geq \ldots \geq \mu_h} \left(\sum_{r=1}^{h} q^{\mu_r + h - r} \right) \mathrm{Prob}(\mu)$$

Now, q is arbitrary, so can take contour integral over q to compare powers of q. Get the density of vertical lozenges:

$$\mathrm{Prob}\left\{ n \in \{\mu_i + h - i\}_{i=1}^{h} \right\}$$

$$= \frac{1}{(2\pi\mathrm{i})^2} \oint_{|q|=\varepsilon} \frac{dq}{q^{n+1}} \oint_{|w-1|=\varepsilon} \left(\frac{qw-1}{w-1} \right)^h \frac{e^{t(q-1)w}}{(q-1)w} dw.$$

Asymptotics [Borodin–Ferrari '08]

Look at critical points of the integrand (L — large)

$$\mathrm{Prob}\{n \in \{\mu_i + h - i\}_{i=1}^{h}\} = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \frac{dv}{v} \oint_{\Gamma_1} dw \frac{e^{L(F(v)-F(w))}}{v(v-w)},$$

$$F(z) := \tau z + \eta \ln(z-1) - \nu \ln z.$$