Colored Particle Systems on the Ring: Stationarity from Yang-Baxter equation

Leonid Petrov
(University of Virginia)

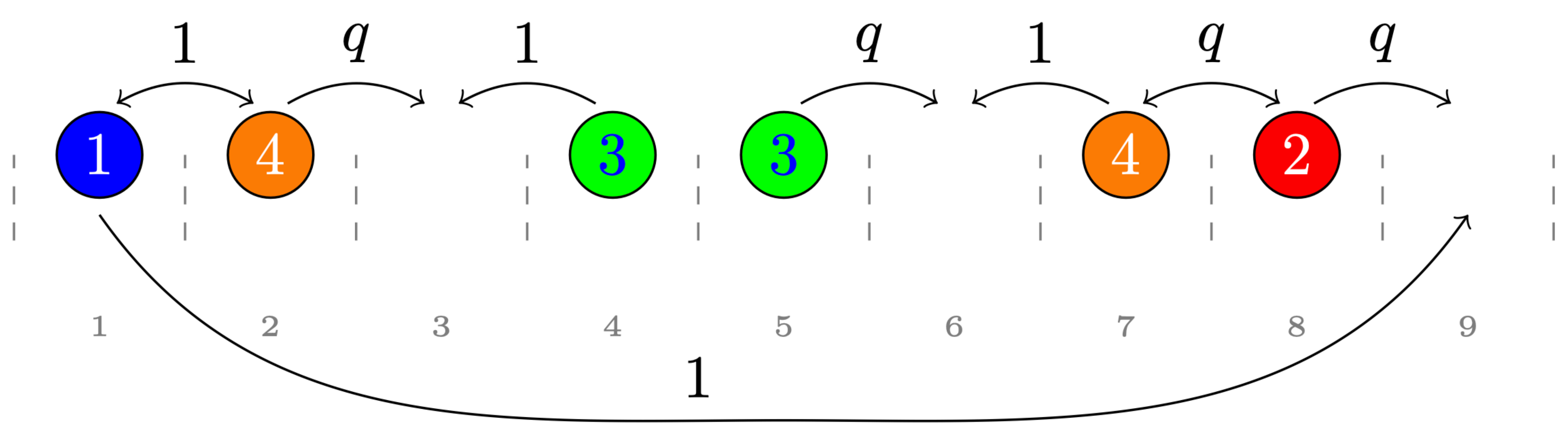
October 6, 2023
ASEP workshop at SCGP

Based on joint work with A.Aggarwal and M.Nicoletti arXiv:2309.11865

Multispecies ASEP and its stationary measure

(Results)

Colored ASEP (multispecies ASEP, mASEP)

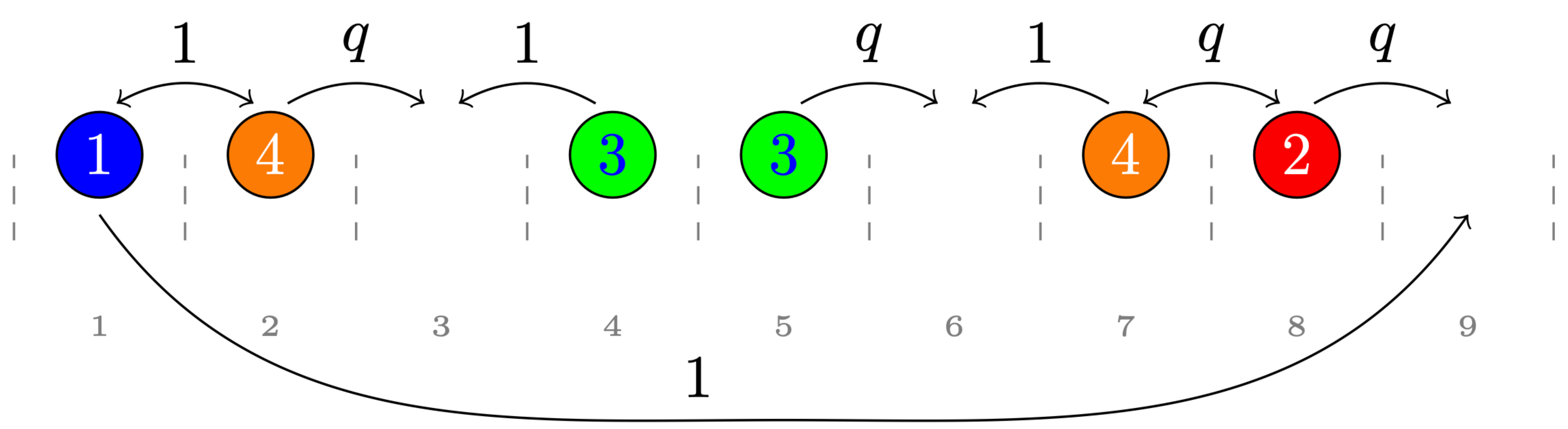


- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors (i_k, i_{k+1}) at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

Colored ASEP (multispecies ASEP, mASEP)



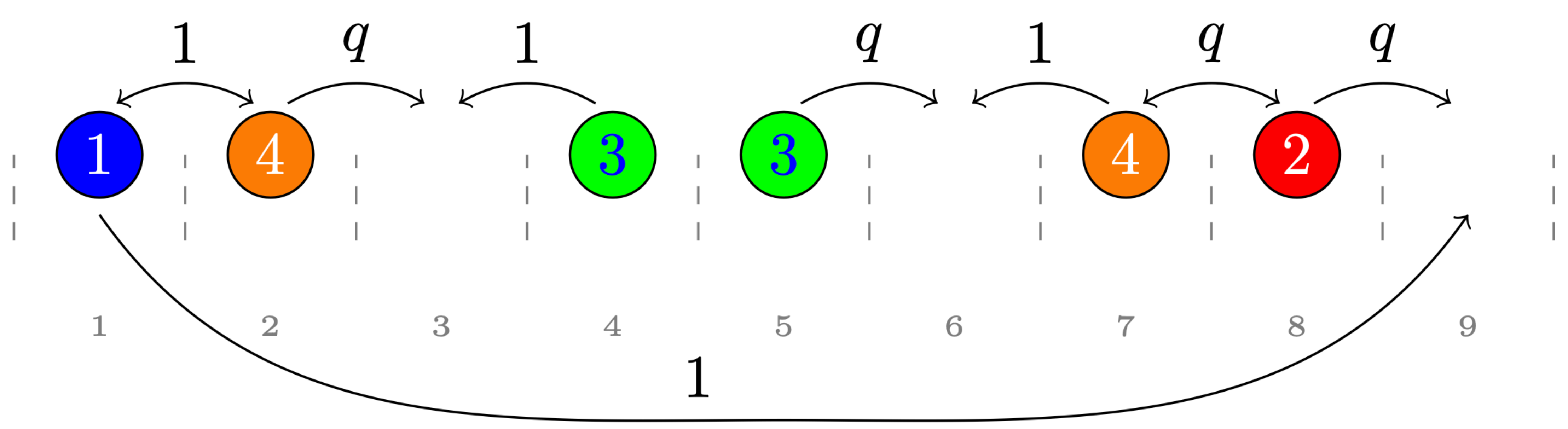
- Particles have colors (types) in $\{1, \ldots, n\}$.
- Particles of colors $\left(i_k, i_{k+1}\right)$ at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

  $$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ in each "sector" $\overrightarrow{N} = (N_1, \ldots, N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$

Colored ASEP (multispecies ASEP, mASEP)



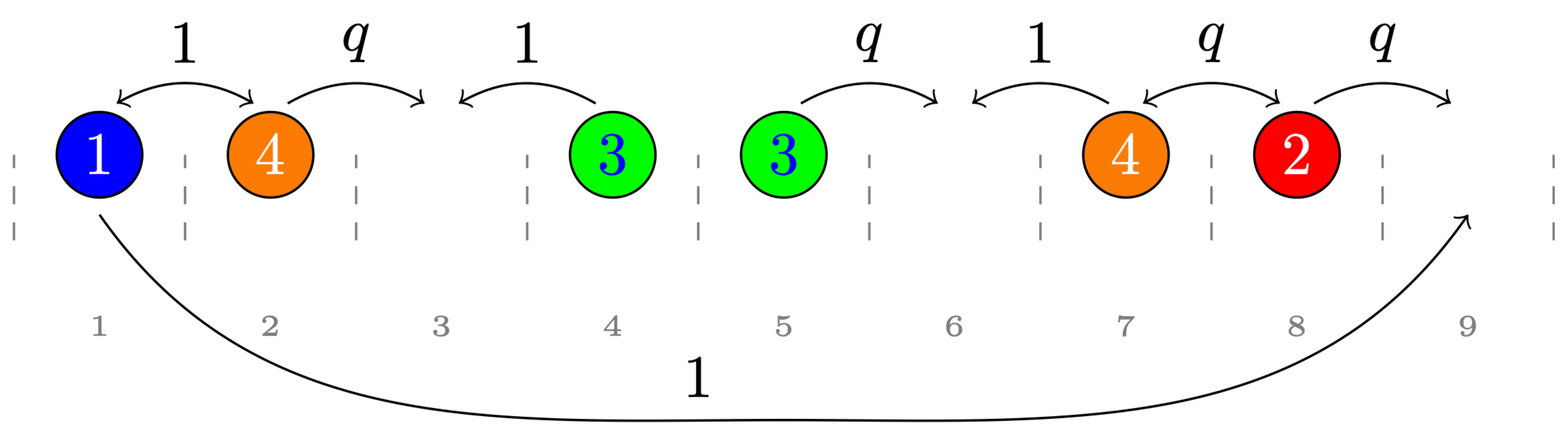
- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors $\left(i_k, i_{k+1}\right)$ at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

$$\text{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\text{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ in each "sector" $\vec{N} = (N_1, \ldots, N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations

Colored ASEP (multispecies ASEP, mASEP)

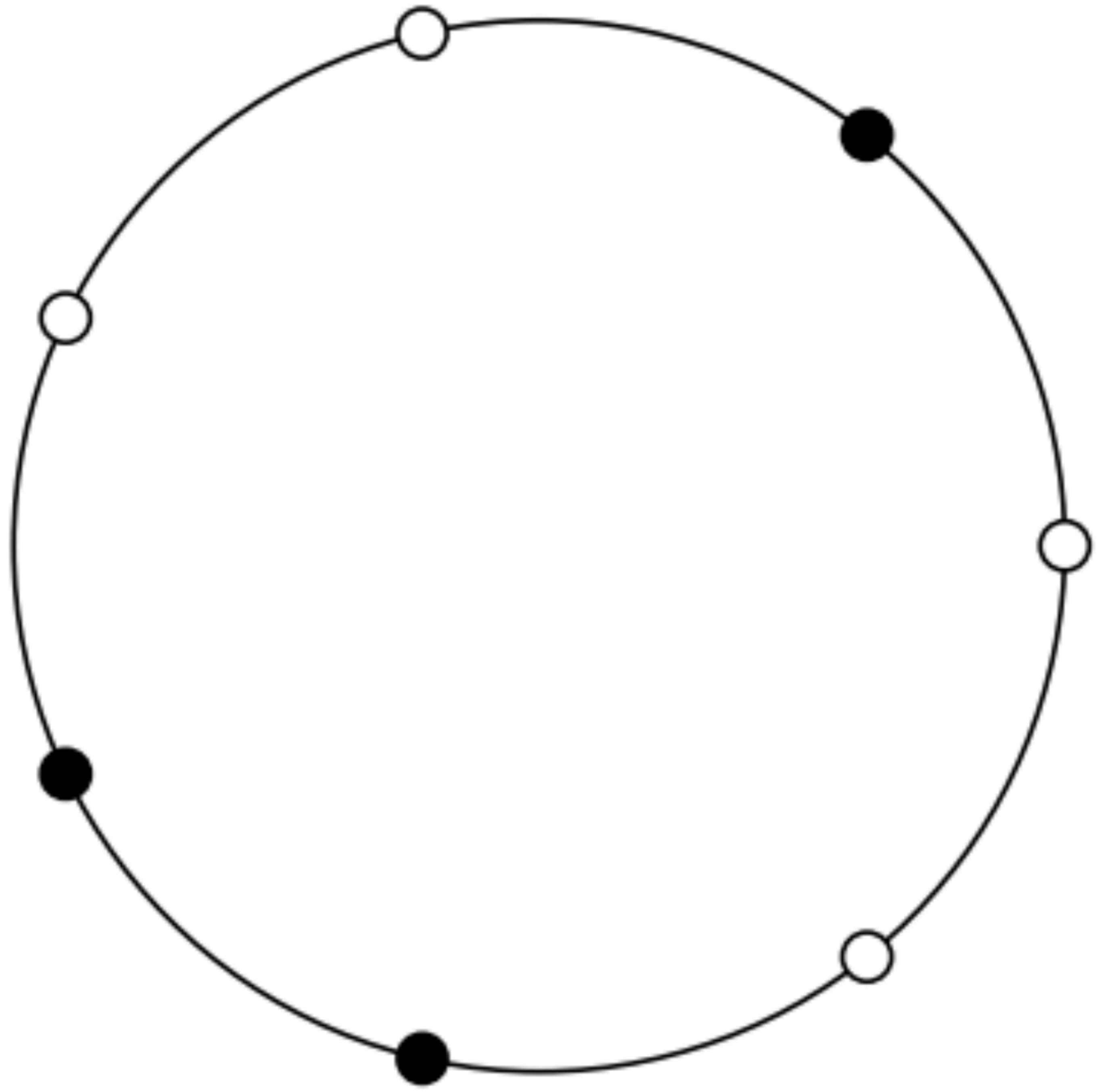


- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors (i_k, i_{k+1}) at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

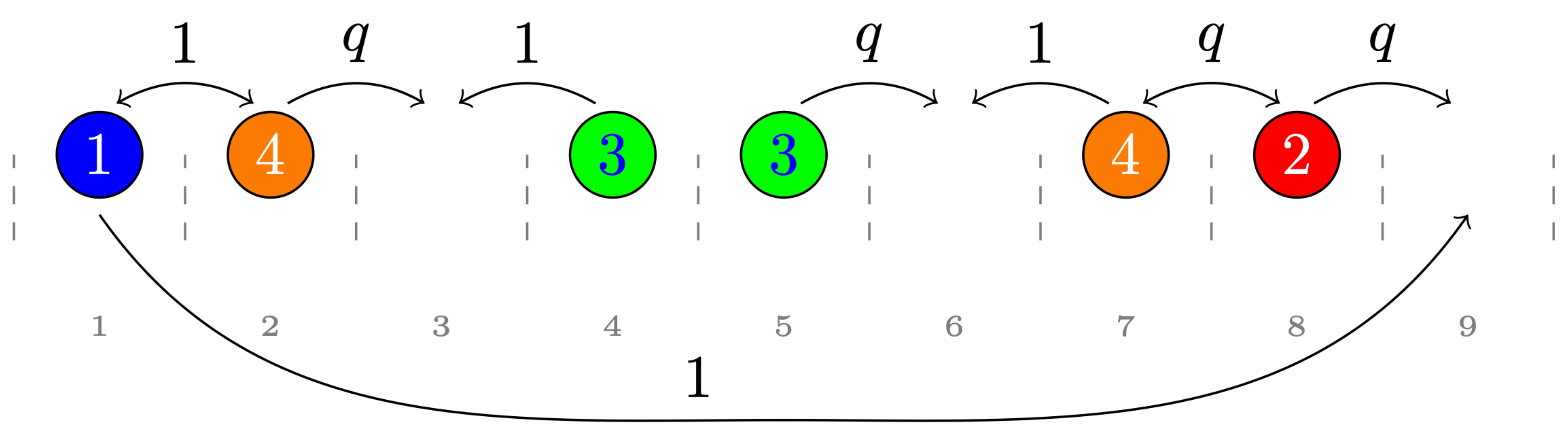
$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ in each "sector" $\vec{N} = (N_1, \ldots, N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations



Colored ASEP (multispecies ASEP, mASEP)



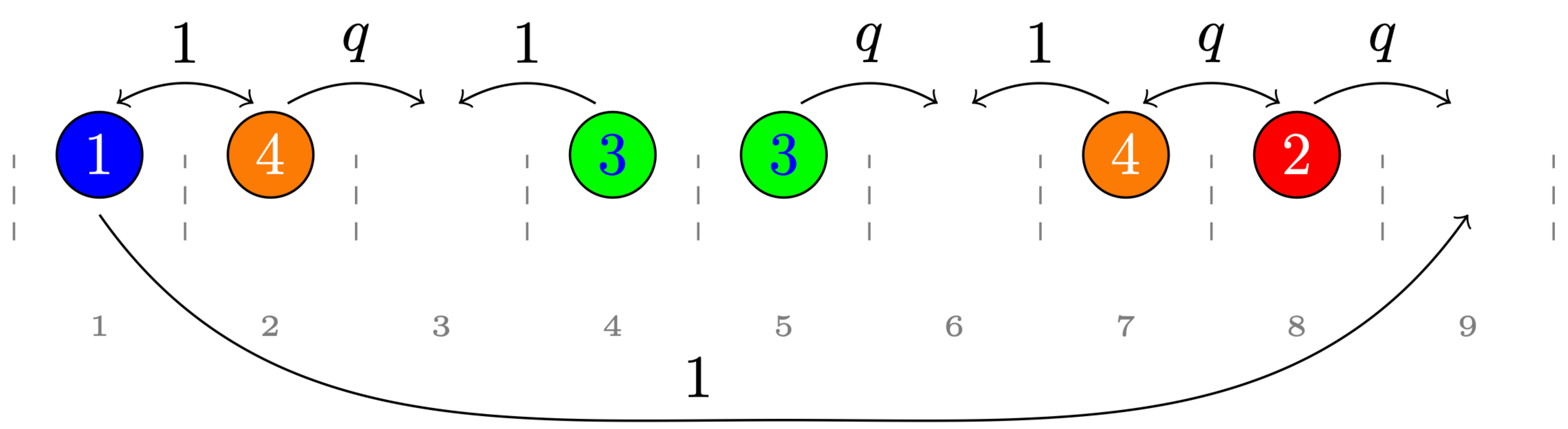
- Particles have colors (types) in $\{1, \ldots, n\}$.
- Particles of colors $\left(i_k, i_{k+1}\right)$ at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1, \ldots, N_n}(\eta_1, \ldots, \eta_N)$ in each "sector" $\overrightarrow{N} = (N_1, \ldots, N_n)$, where $\eta_j \in \{0, 1, \ldots, n\}$
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations

Colored ASEP (multispecies ASEP, mASEP)



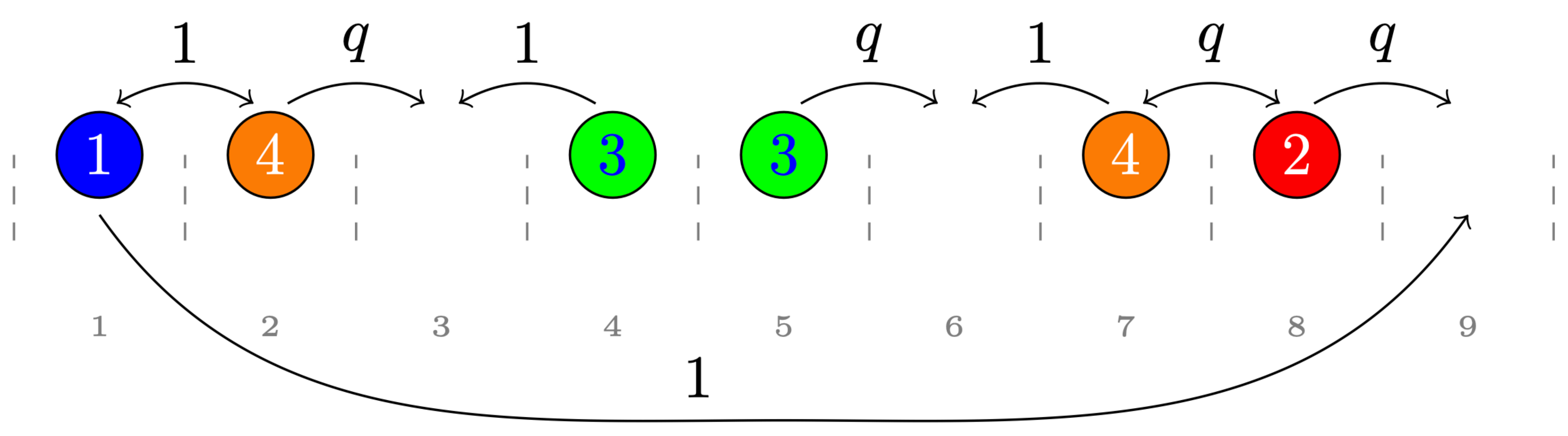
- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors (i_k, i_{k+1}) at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ in each "sector" $\overrightarrow{N} = (N_1, \ldots, N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations
- For many colors, nontrivial correlations

Colored ASEP (multispecies ASEP, mASEP)



- Particles have colors (types) in $\{1,\ldots,n\}$.
- Particles of colors (i_k, i_{k+1}) at adjacent sites $k, k+1$ swap at rate (color n: highest priority)

$$\mathrm{Rate}((i_k, i_{k+1}) \to (i_{k+1}, i_k)) = \begin{cases} q, & i_k > i_{k+1} \\ 1, & i_k < i_{k+1} \end{cases}$$

- $q \in [0,1)$ is the parameter
- Lives on a ring with N sites; there are N_i particles of color i (conserved quantities)

- There is a unique **stationary distribution** $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ in each "sector" $\vec{N} = (N_1,\ldots,N_n)$, where $\eta_j \in \{0,1,\ldots,n\}$
- For $n = 1$ (single color), it is uniform among all $\binom{N}{N_1}$ configurations
- For many colors, nontrivial correlations

- Multiline queues: **[Angel 2006], [Ferrari–Martin 2007]** (mTASEP, $q = 0$), **[Martin 2018]** (full mASEP)
- Matrix Ansatz: **[Prolhac–Evans–Mallick 2009]**
- Macdonald polynomials: **[Cantini–de Gier–Wheeler 2015], [Corteel–Mandelshtam–Williams 2018]**
- We use ***integrable vertex models***

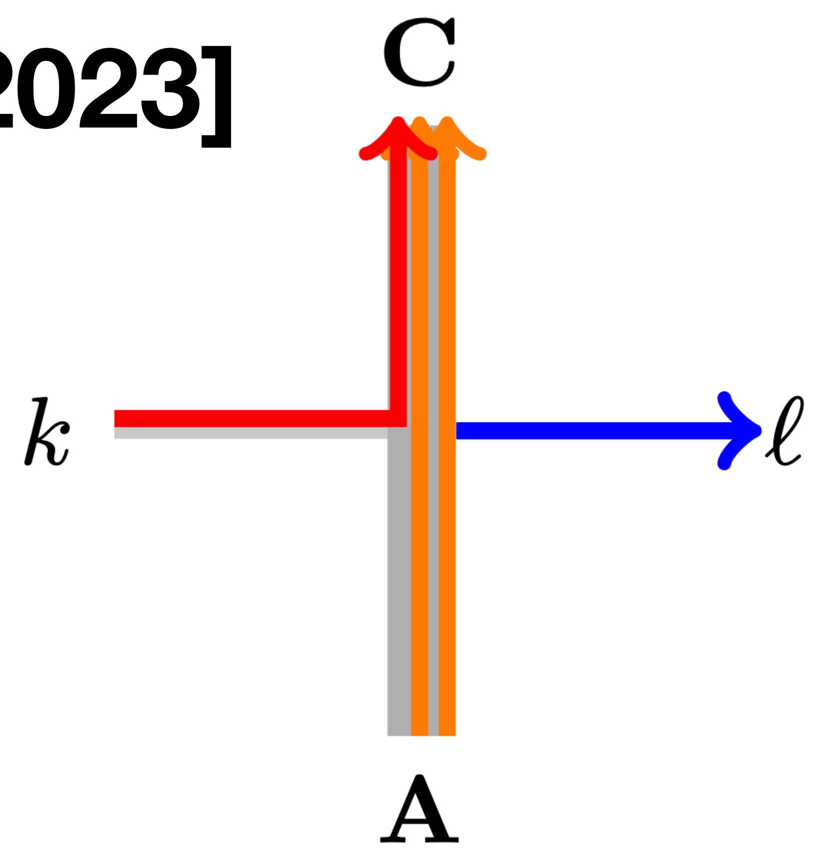
Main result for mASEP [Aggarwal-Nicoletti-P. 2023]

- We define a ***vertex model*** on the cylinder $\{-n, -n+1, \ldots, -2, -1\} \times (\mathbb{Z}/N\mathbb{Z})$

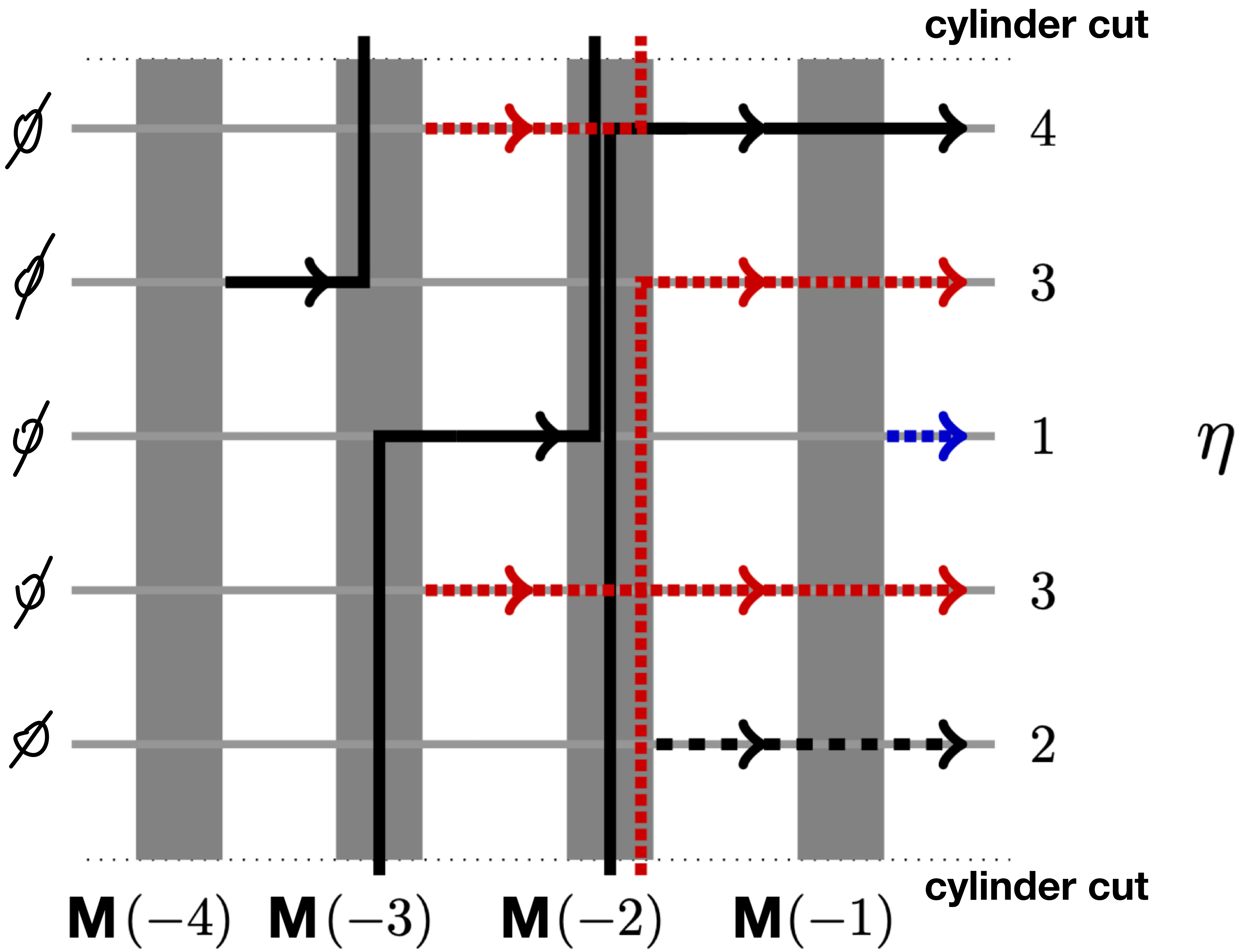
- The mASEP configuration $\eta = (\eta_1, \ldots, \eta_N)$ encodes the boundary condition.

- $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1, \ldots, \eta_N)$ is proportional to the ***partition function*** with the boundary η, which involves the summation over the wrappings $\mathbf{M}(-n), \ldots, \mathbf{M}(-1)$. There are infinitely many arrows of color m wrapping around column $(-m)$.

- Weights are denoted by $\mathbb{W}^{(-m)}_{s,x}(\mathbf{A}, k; \mathbf{C}, \ell)$, $\mathbf{A}, \mathbf{C} \in \mathbb{Z}^n_{\geq 0}$, $k, \ell \in \{0,1,\ldots,n\}$.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Main result for mASEP [Aggarwal-Nicoletti-P. 2023]

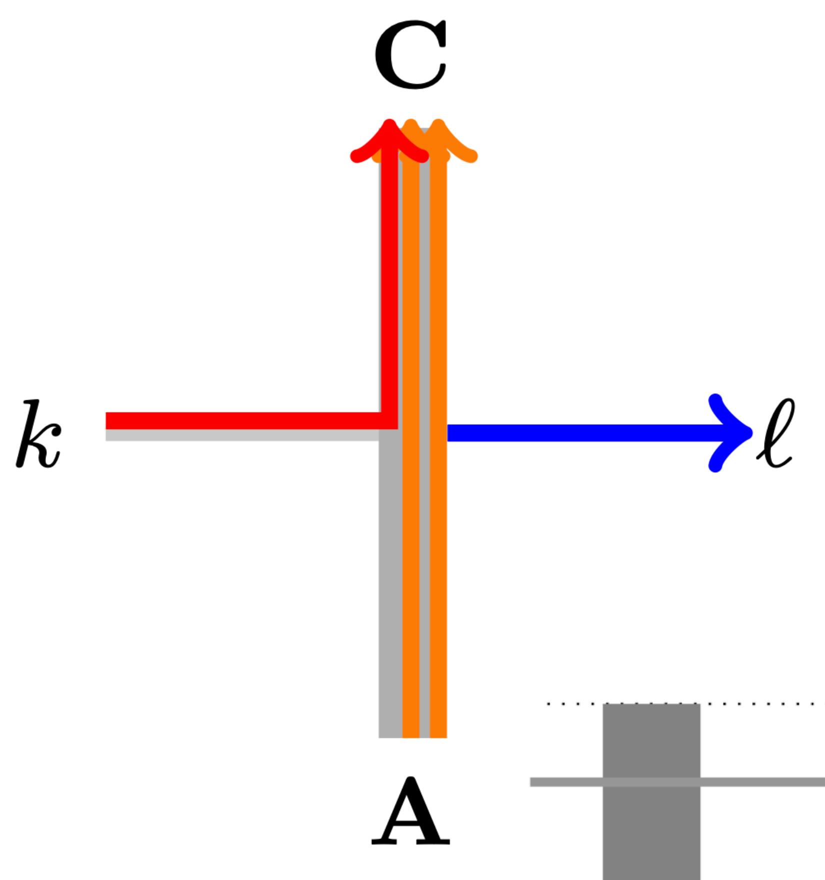
Theorem. $\mathrm{Prob}_{N_1,\dots,N_n}(\eta_1,\dots,\eta_N)$ is proportional to

$\sum_{\mathbf{M}(-n),\dots,\mathbf{M}(-1)} \sum_{path\ conf}$ of products of the weights

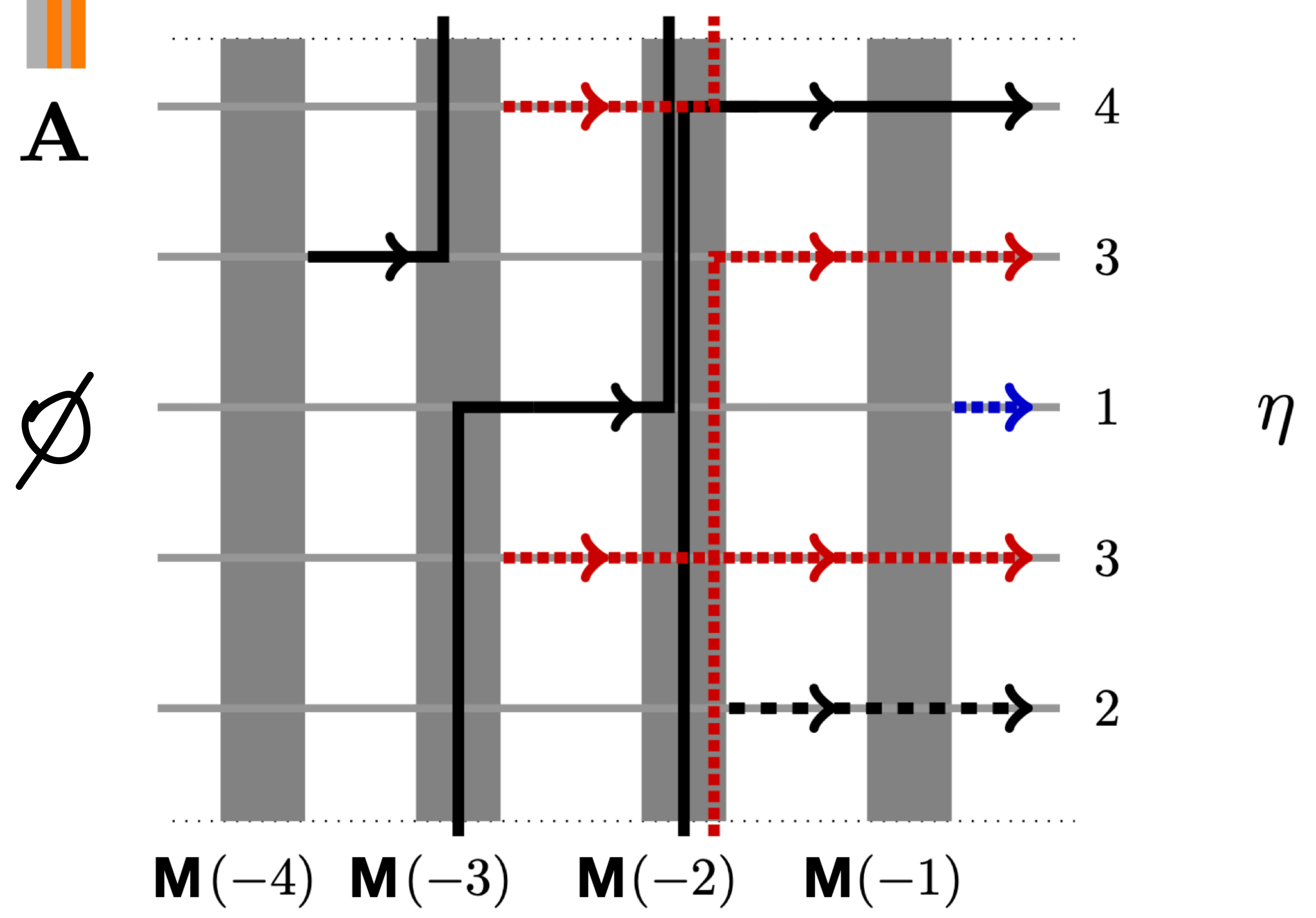
$\mathbb{W}^{(-m)}_{s_m,x_m}$ over all $n \times N$ vertices, with boundaries $\varnothing, \eta$.

top $\mathbf{A}$ left 0, right 0 bottom $\mathbf{A}$ 1	top $\mathbf{A}$ left k, right k bottom $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}^-_k$ left 0, right k bottom $\mathbf{A}$ $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$ left 0, right m bottom $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}^+_k$ left k, right 0 bottom $\mathbf{A}$ 1	top $\mathbf{A}^{+-}_{k\ell}$ left k, right ℓ bottom $\mathbf{A}$ $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}^{+-}_{\ell k}$ left ℓ, right k bottom $\mathbf{A}$ $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}^+_\ell$ left ℓ, right m bottom $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \le n$



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Main result for mASEP [Aggarwal-Nicoletti-P. 2023]

Theorem. $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ is proportional to

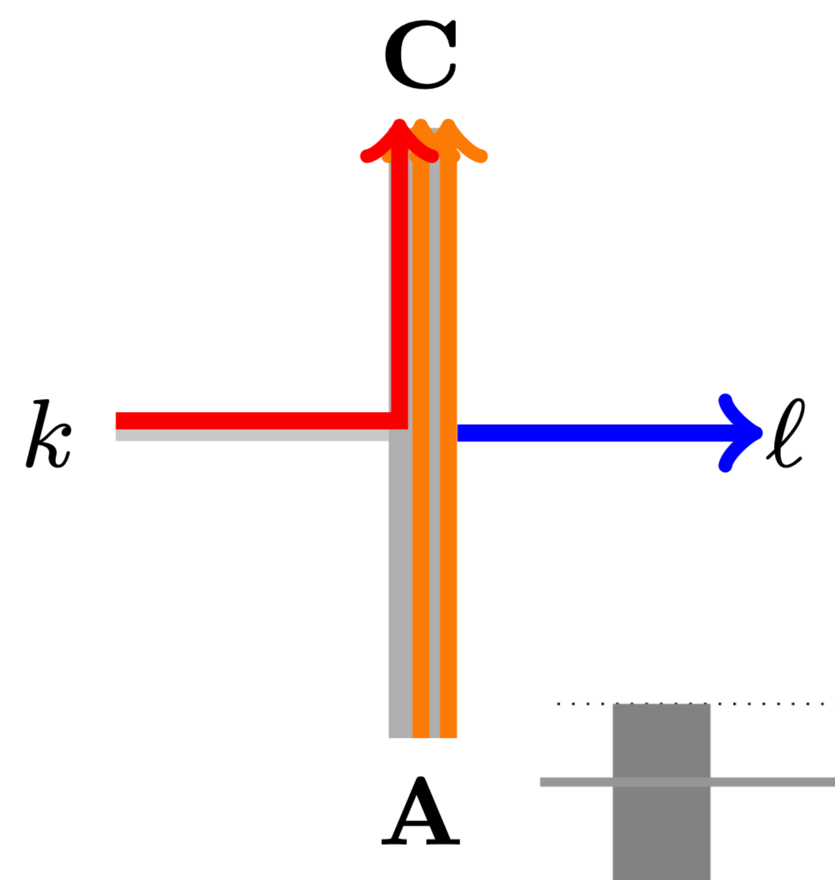
$\sum_{\mathbf{M}(-n),\ldots,\mathbf{M}(-1)} \sum_{path\ conf}$ of products of the weights

$\mathbb{W}^{(-m)}_{s_m,x_m}$ over all $n \times N$ vertices, with boundaries $\varnothing, \eta$.

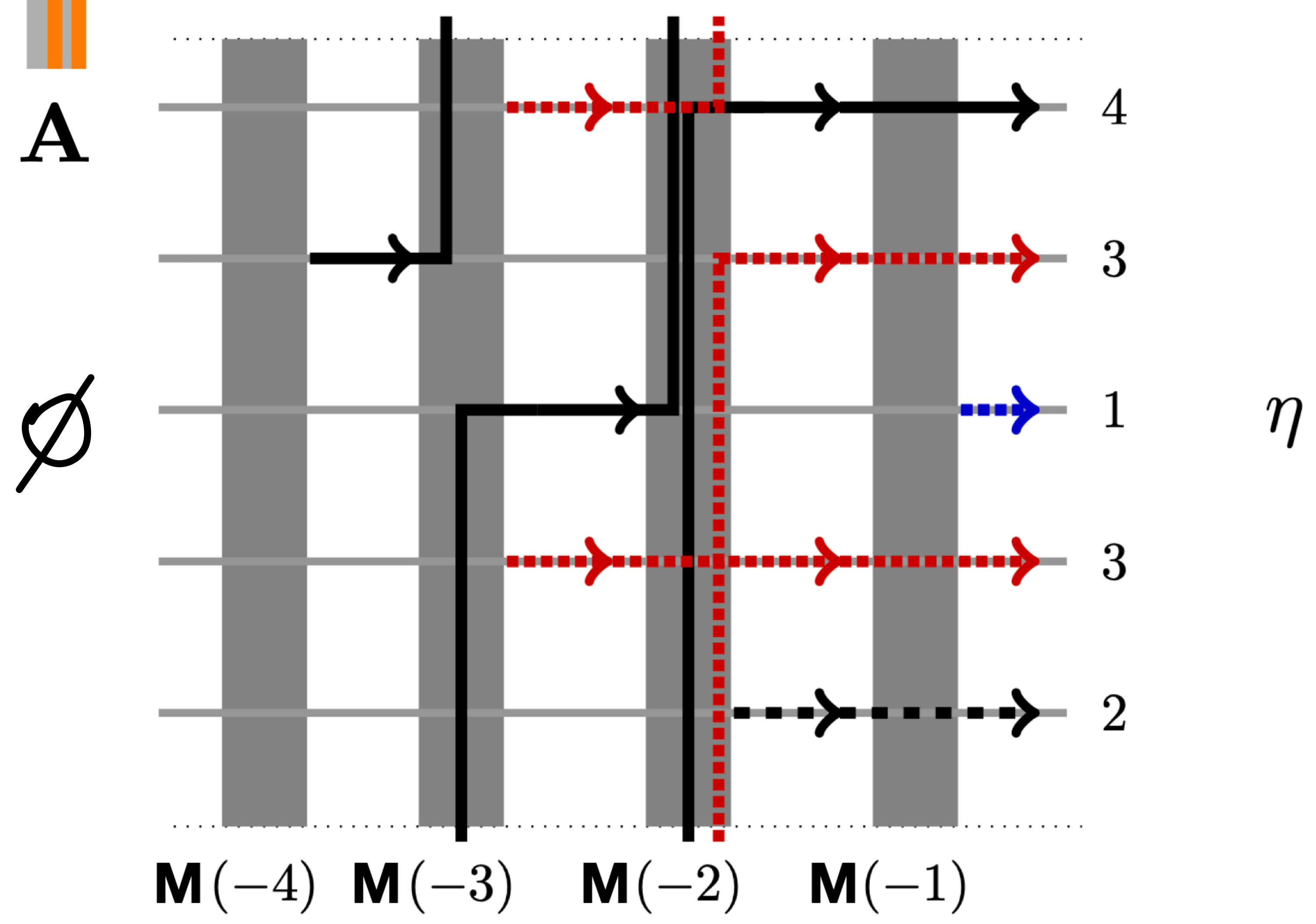
top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$: 1	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$: $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$, left 0, right k, bottom $\mathbf{A}$: $x\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$, left 0, right m, bottom $\mathbf{A}$: $x\,q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$, left k, right 0, bottom $\mathbf{A}$: 1	top $\mathbf{A}_{k\ell}^{+-}$, left k, right ℓ, bottom $\mathbf{A}$: $x\,(1-q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$, left ℓ, right k, bottom $\mathbf{A}$: $s\,(1-q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$, left ℓ, right m, bottom $\mathbf{A}$: $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \leq n$

- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Main result for mASEP [Aggarwal-Nicoletti-P. 2023]

Theorem. $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ is proportional to

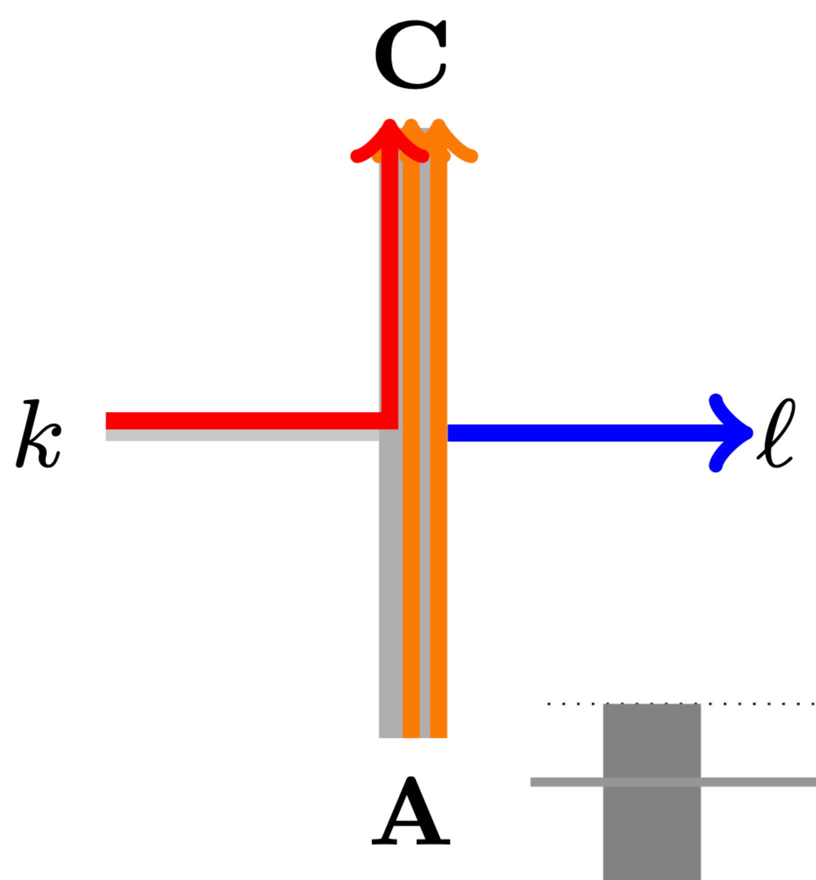
$$\sum_{\mathbf{M}(-n),\ldots,\mathbf{M}(-1)} \sum_{path\ conf}$$ of products of the weights

$\mathbb{W}^{(-m)}_{s_m,x_m}$ over all $n \times N$ vertices, with boundaries $\varnothing, \eta$.

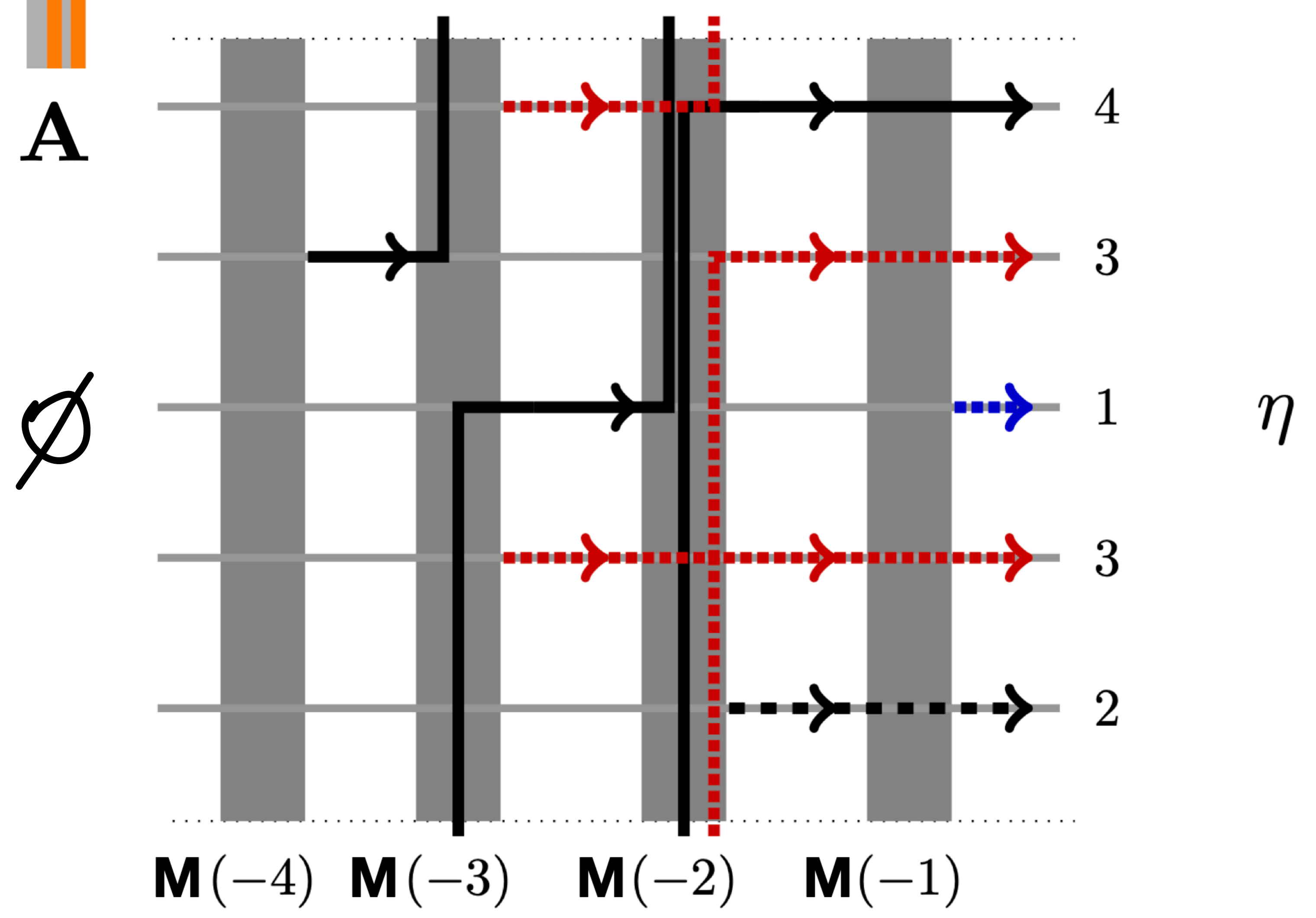
top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$: 1	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$: $(x - sq^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$, left 0, right k, bottom $\mathbf{A}$: $x\,(1-q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$, left 0, right m, bottom $\mathbf{A}$: $x\, q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$, left k, right 0, bottom $\mathbf{A}$: 1	top $\mathbf{A}_{k\ell}^{+-}$, left k, right ℓ, bottom $\mathbf{A}$: $x\,(1-q^{A_\ell})\, q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$, left ℓ, right k, bottom $\mathbf{A}$: $s\,(1-q^{A_k})\, q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$, left ℓ, right m, bottom $\mathbf{A}$: $s\, q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \le n$

- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.
- Parameters s_m, x_m do not affect $\mathrm{Prob}_{N_1,\ldots,N_n}(\eta_1,\ldots,\eta_N)$ on the ring. Positivity is automatic.



In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



Main result for mASEP [Aggarwal-Nicoletti-P. 2023]

Theorem. $\mathrm{Prob}_{N_1,\dots,N_n}(\eta_1,\dots,\eta_N)$ is proportional to

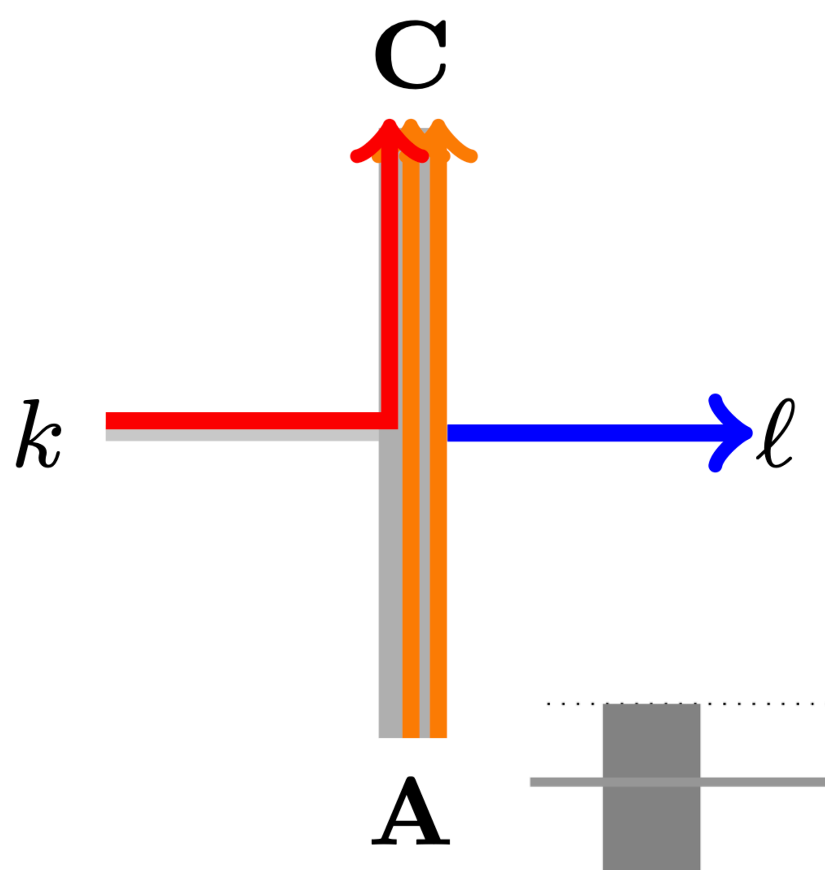
$\sum_{\mathbf{M}(-n),\dots,\mathbf{M}(-1)} \sum_{path\ conf}$ of products of the weights

$\mathbb{W}^{(-m)}_{s_m,x_m}$ over all $n \times N$ vertices, with boundaries $\varnothing, \eta$.

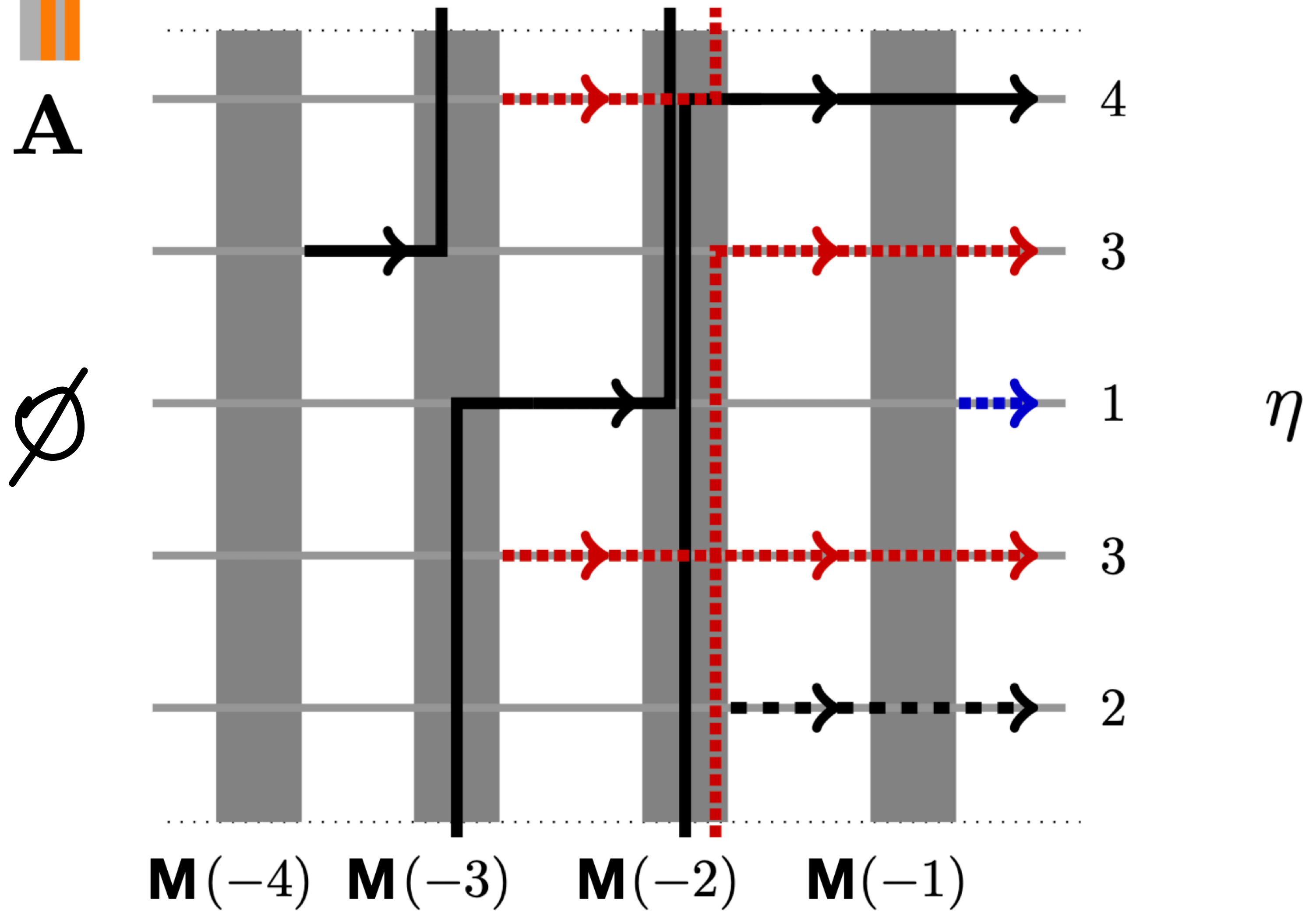
top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$ $x\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$; left 0, right m; bottom $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$ $x\,(1 - q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$ $s\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$; left ℓ, right m; bottom $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$

Infinitely many vertical arrows of color m; $m < k < \ell \le n$

- The sum $path\ conf$ is over path configurations in the $n \times N$ rectangle with boundaries $\varnothing, \mathbf{M}, \eta, \mathbf{M}$.
- Parameters s_m, x_m do not affect $\mathrm{Prob}_{N_1,\dots,N_n}(\eta_1,\dots,\eta_N)$ on the ring. Positivity is automatic.
- Similar result on the line (with fewer parameters for positivity). The remaining parameters are responsible for the color densities.

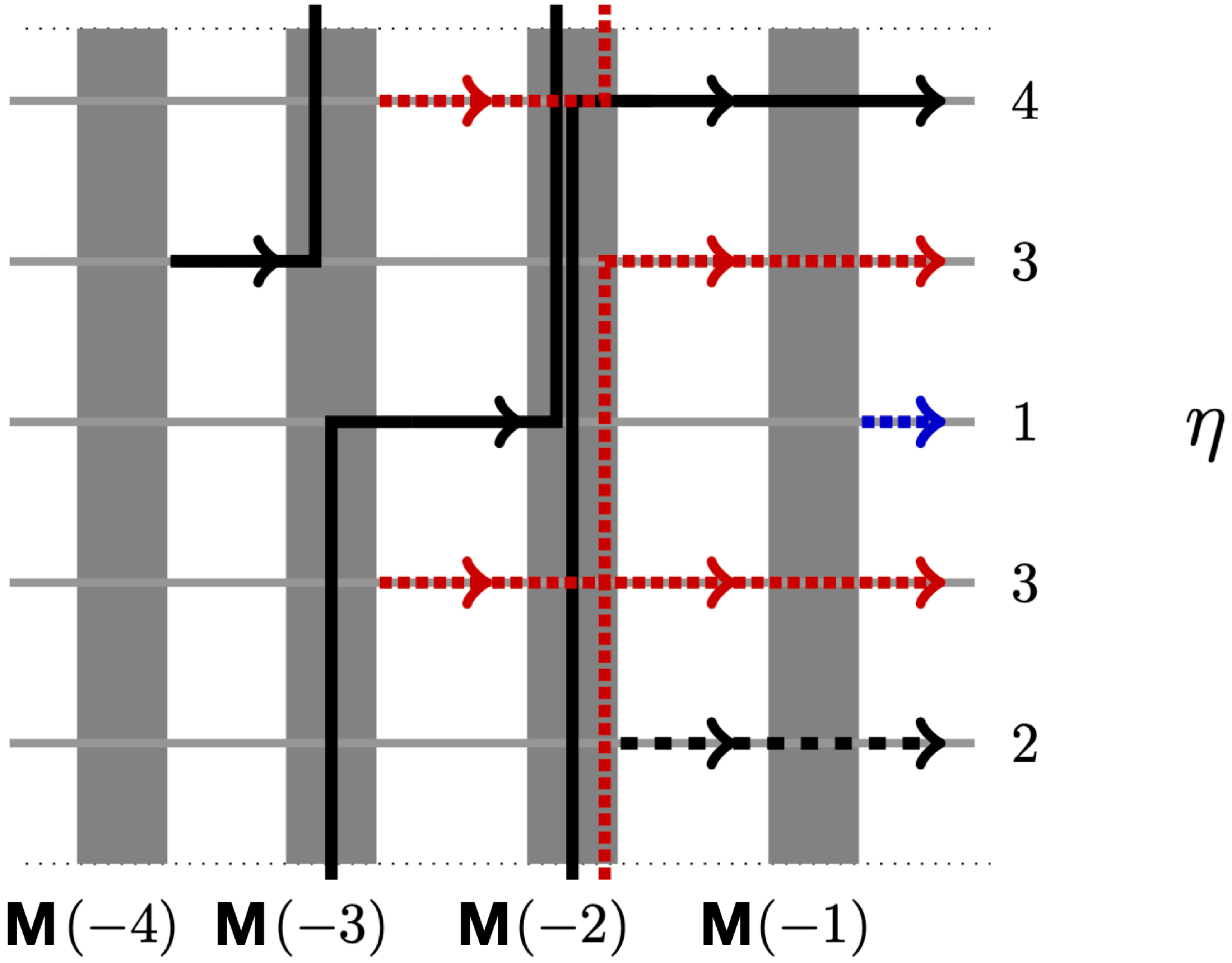


In column $(-m)$, use weight $\mathbb{W}^{(-m)}_{s_m,x_m}$



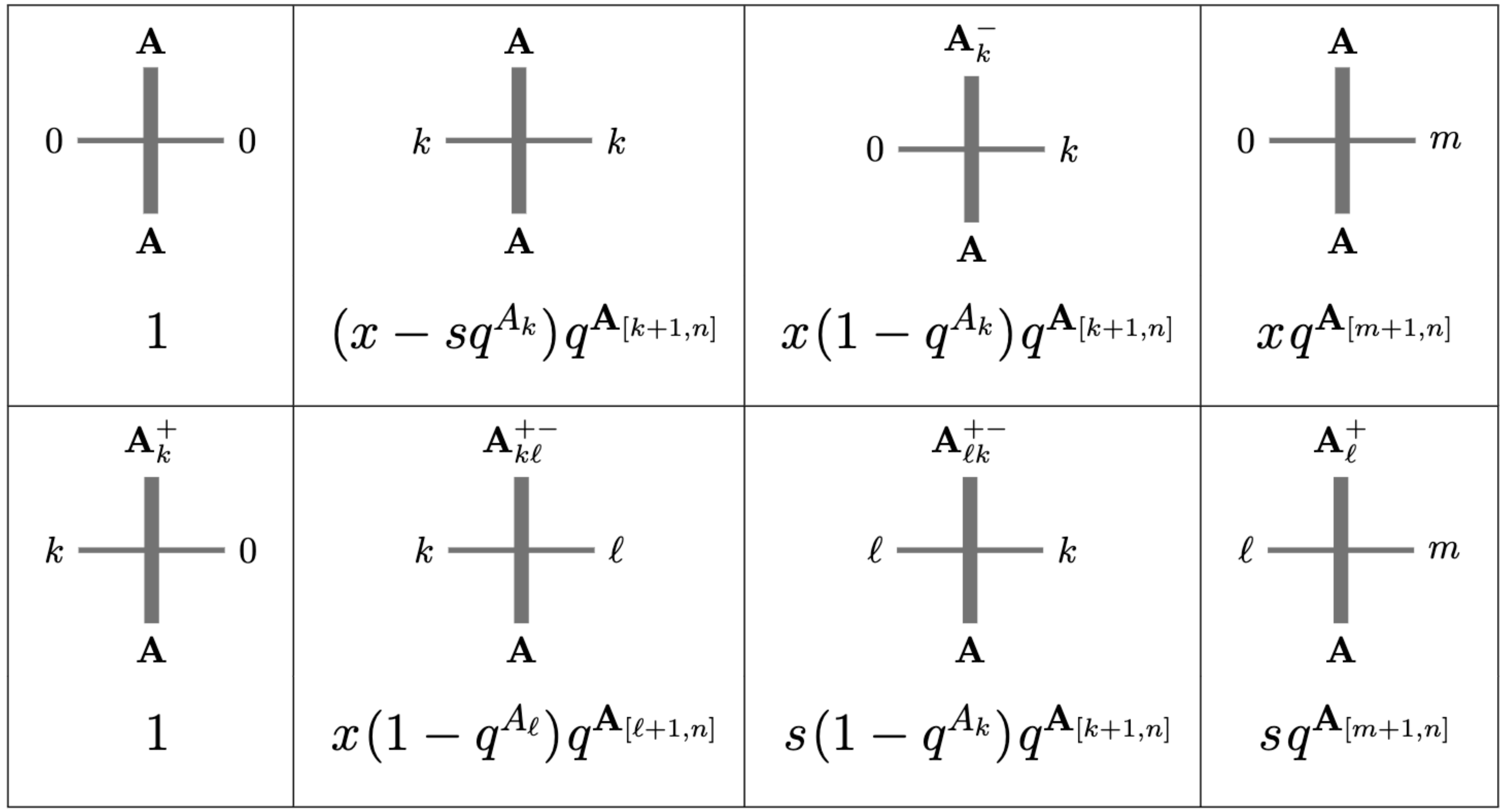
Matching to previous results

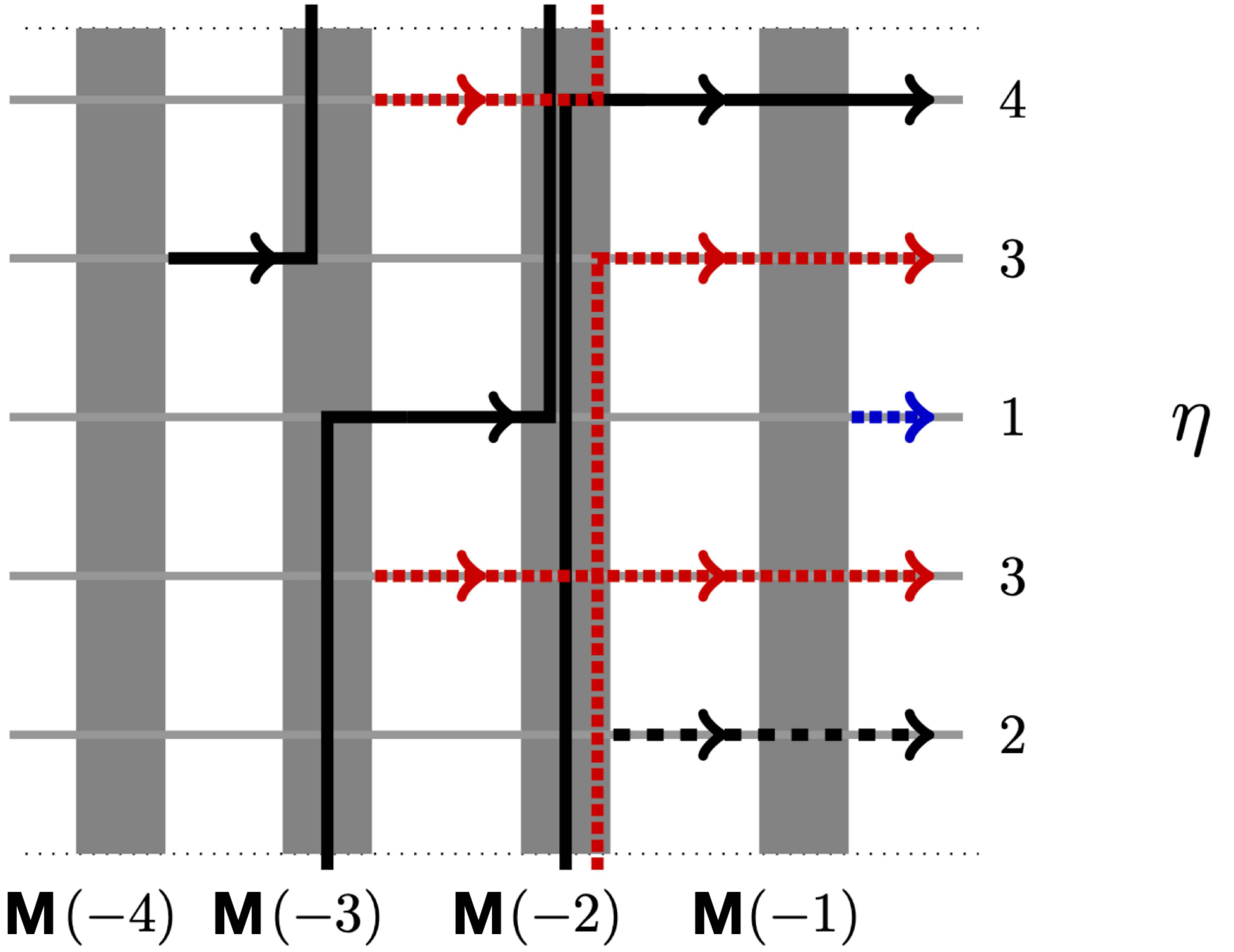
top: $\mathbf{A}$ left: 0, right: 0 bottom: $\mathbf{A}$ 1	top: $\mathbf{A}$ left: k, right: k bottom: $\mathbf{A}$ $(x - sq^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top: $\mathbf{A}_k^-$ left: 0, right: k bottom: $\mathbf{A}$ $x\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top: $\mathbf{A}$ left: 0, right: m bottom: $\mathbf{A}$ $x\,q^{\mathbf{A}_{[m+1,n]}}$
top: $\mathbf{A}_k^+$ left: k, right: 0 bottom: $\mathbf{A}$ 1	top: $\mathbf{A}_{k\ell}^{+-}$ left: k, right: ℓ bottom: $\mathbf{A}$ $x\,(1 - q^{A_\ell})\,q^{\mathbf{A}_{[\ell+1,n]}}$	top: $\mathbf{A}_{\ell k}^{+-}$ left: ℓ, right: k bottom: $\mathbf{A}$ $s\,(1 - q^{A_k})\,q^{\mathbf{A}_{[k+1,n]}}$	top: $\mathbf{A}_\ell^+$ left: ℓ, right: m bottom: $\mathbf{A}$ $s\,q^{\mathbf{A}_{[m+1,n]}}$



Matching to previous results

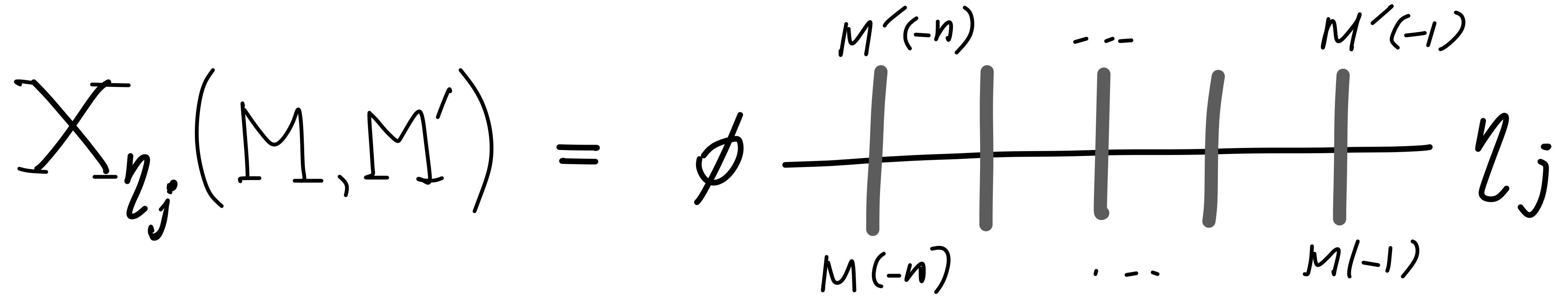
- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:

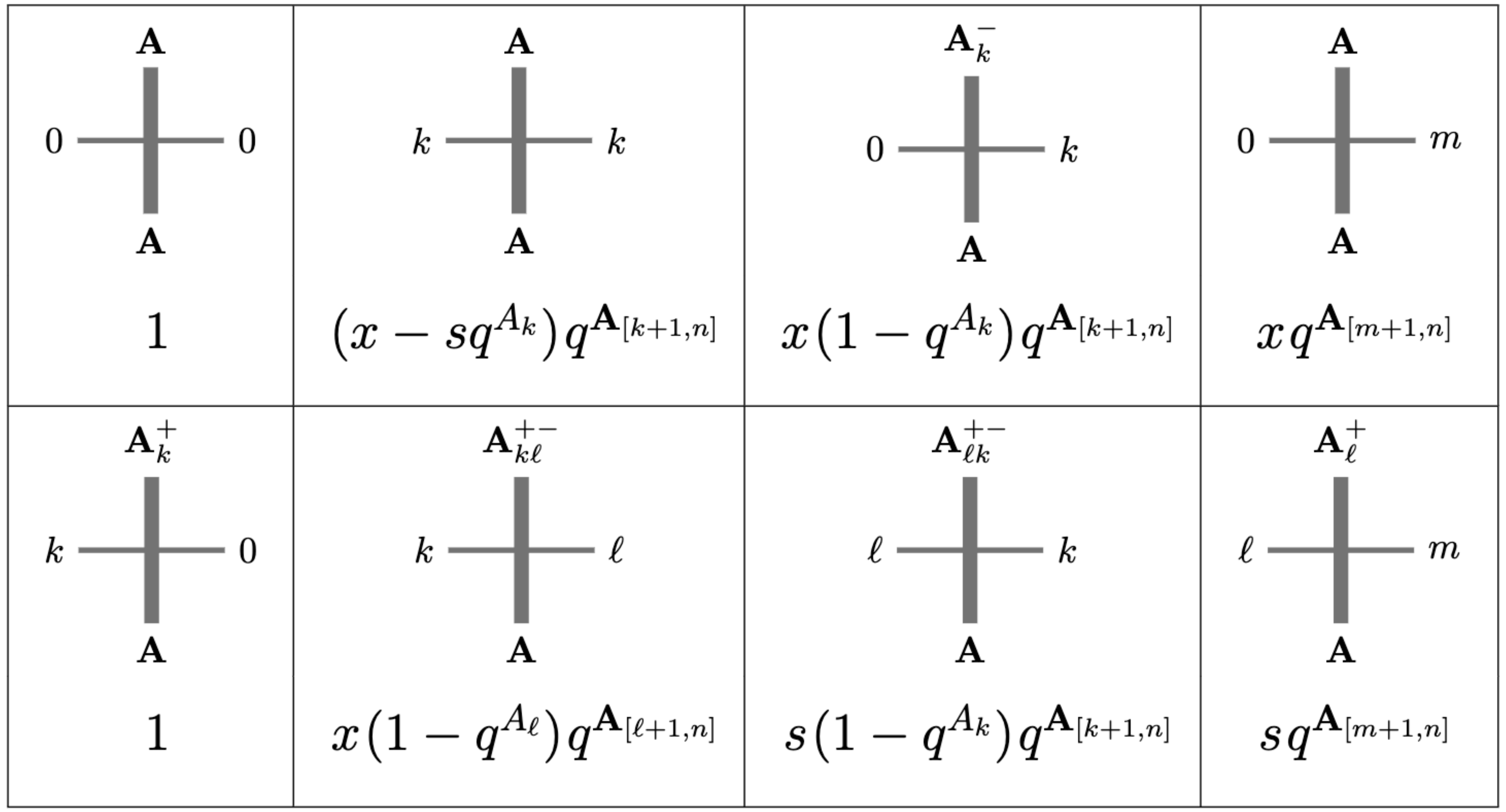


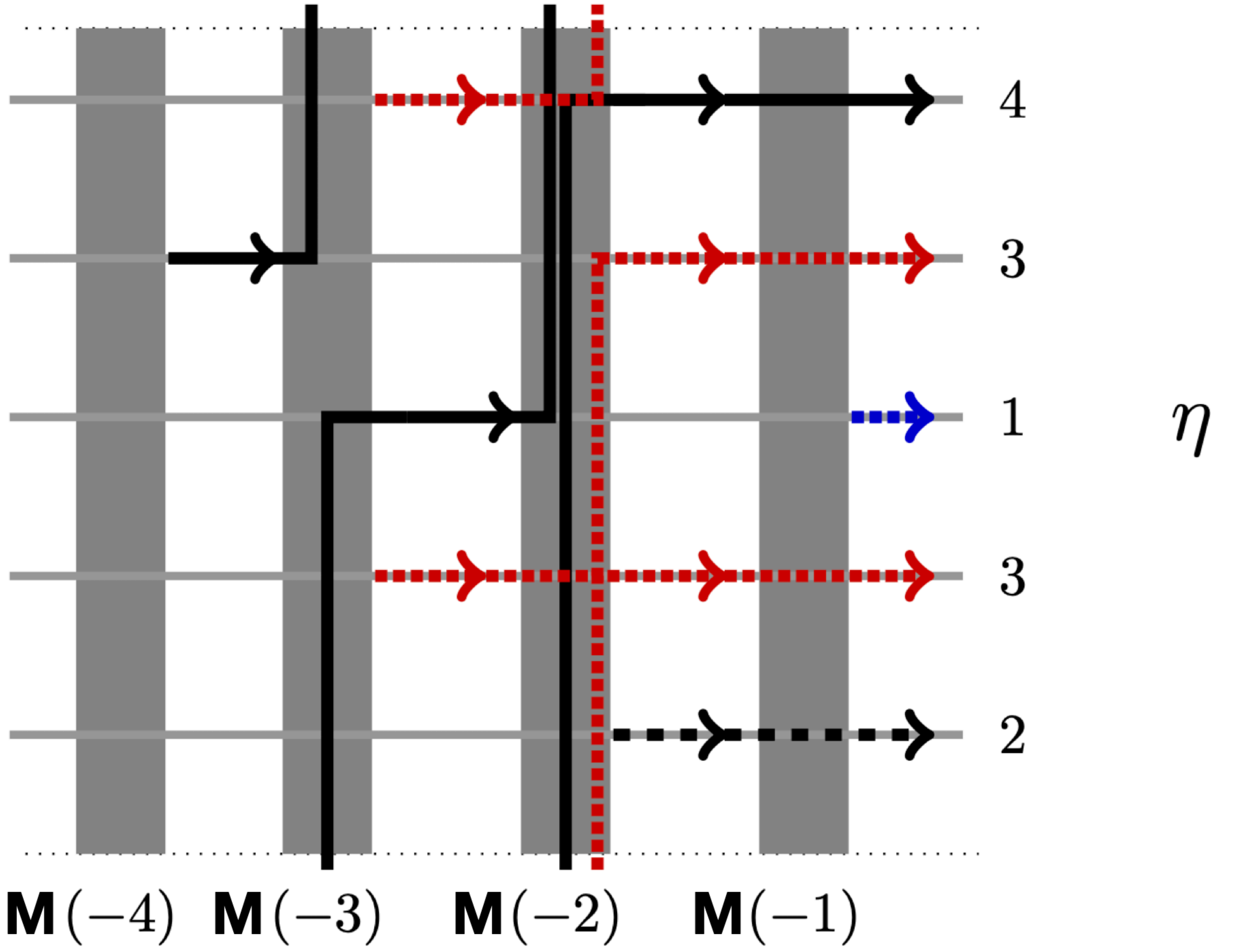


Matching to previous results

- The vertex model for $s=0, x=1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:

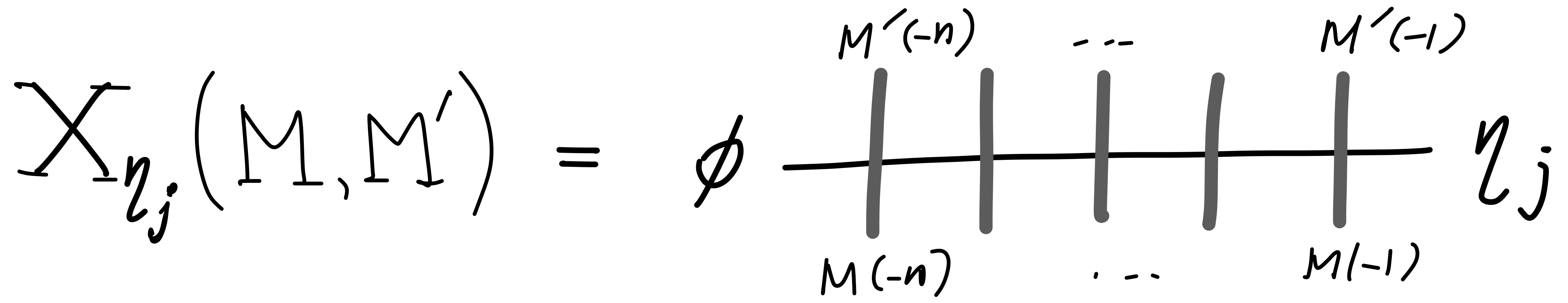


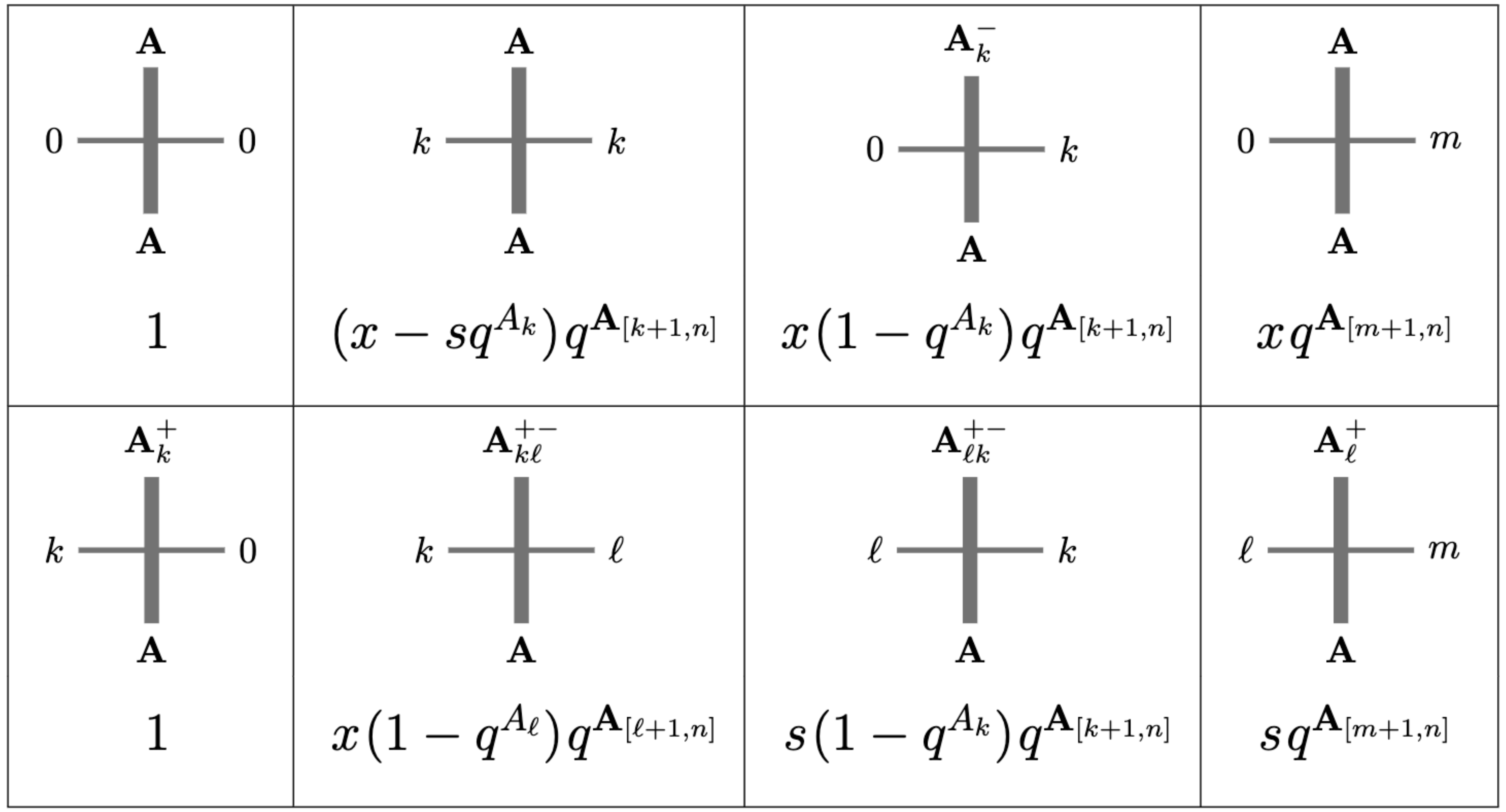


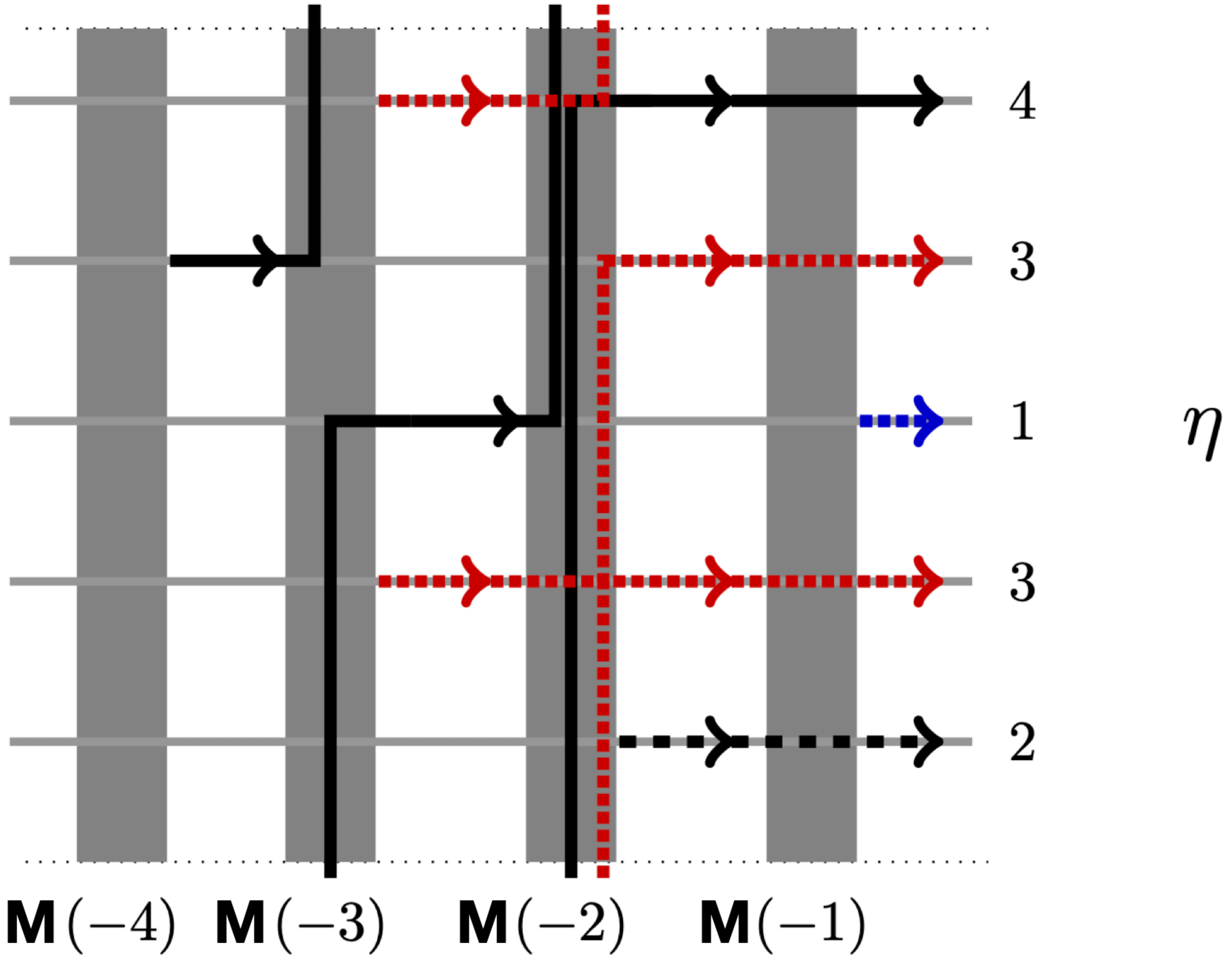


Matching to previous results

- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:

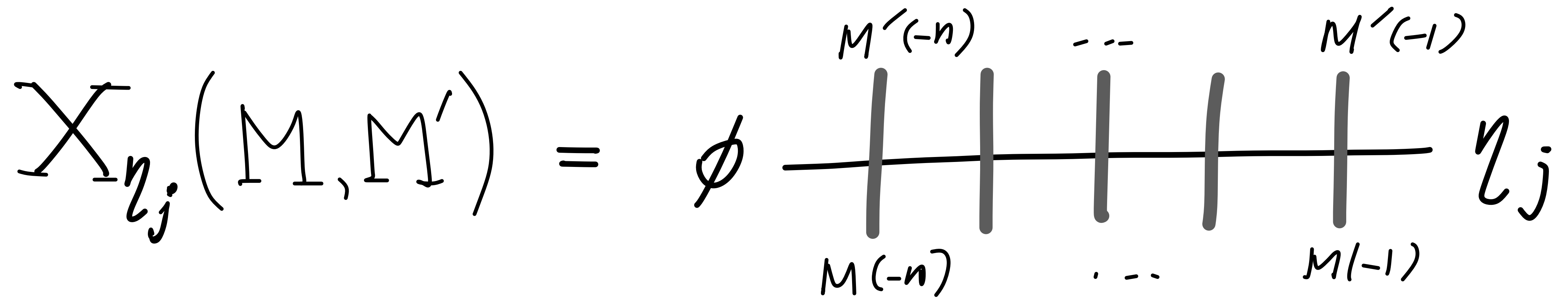


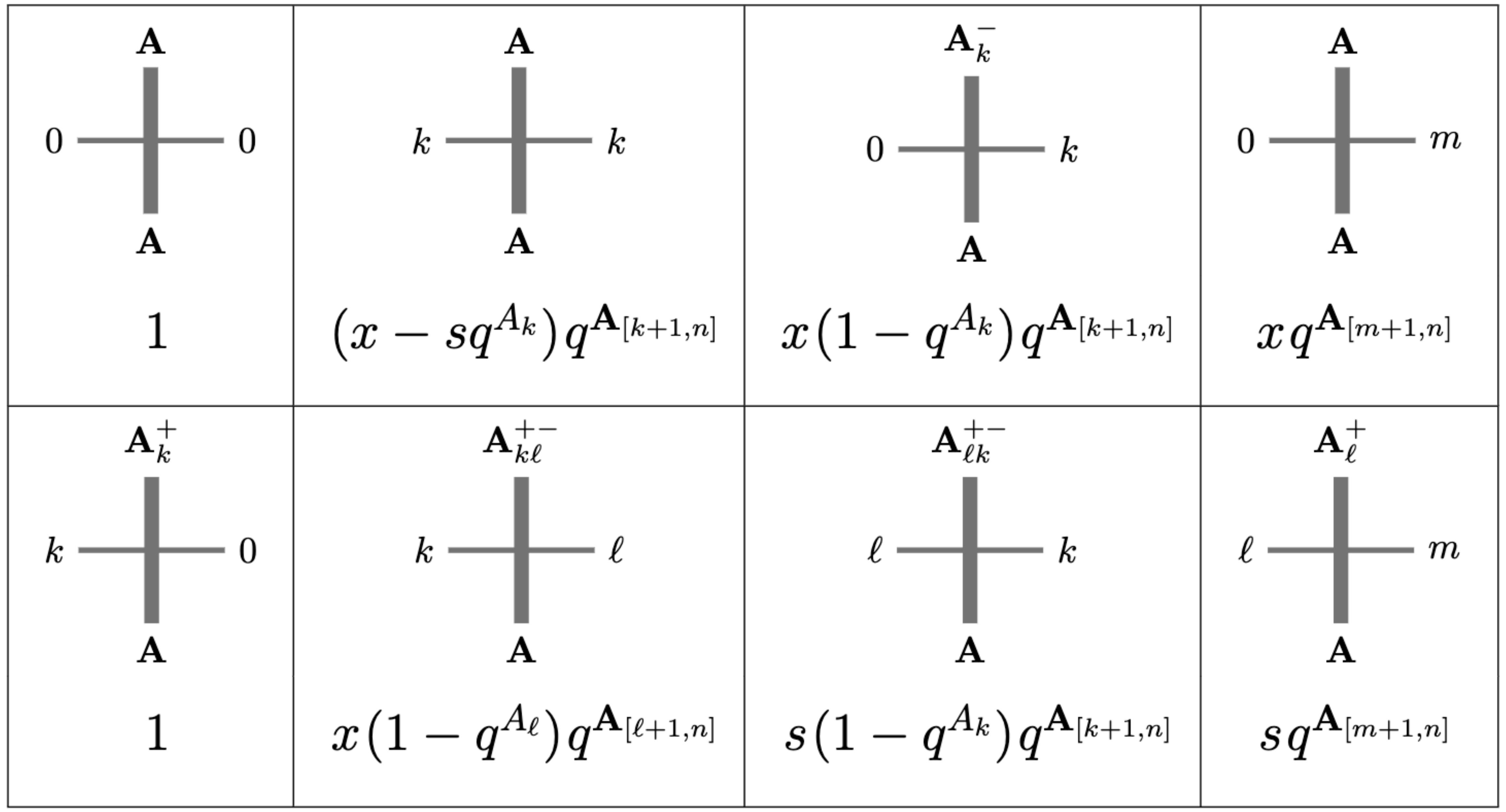


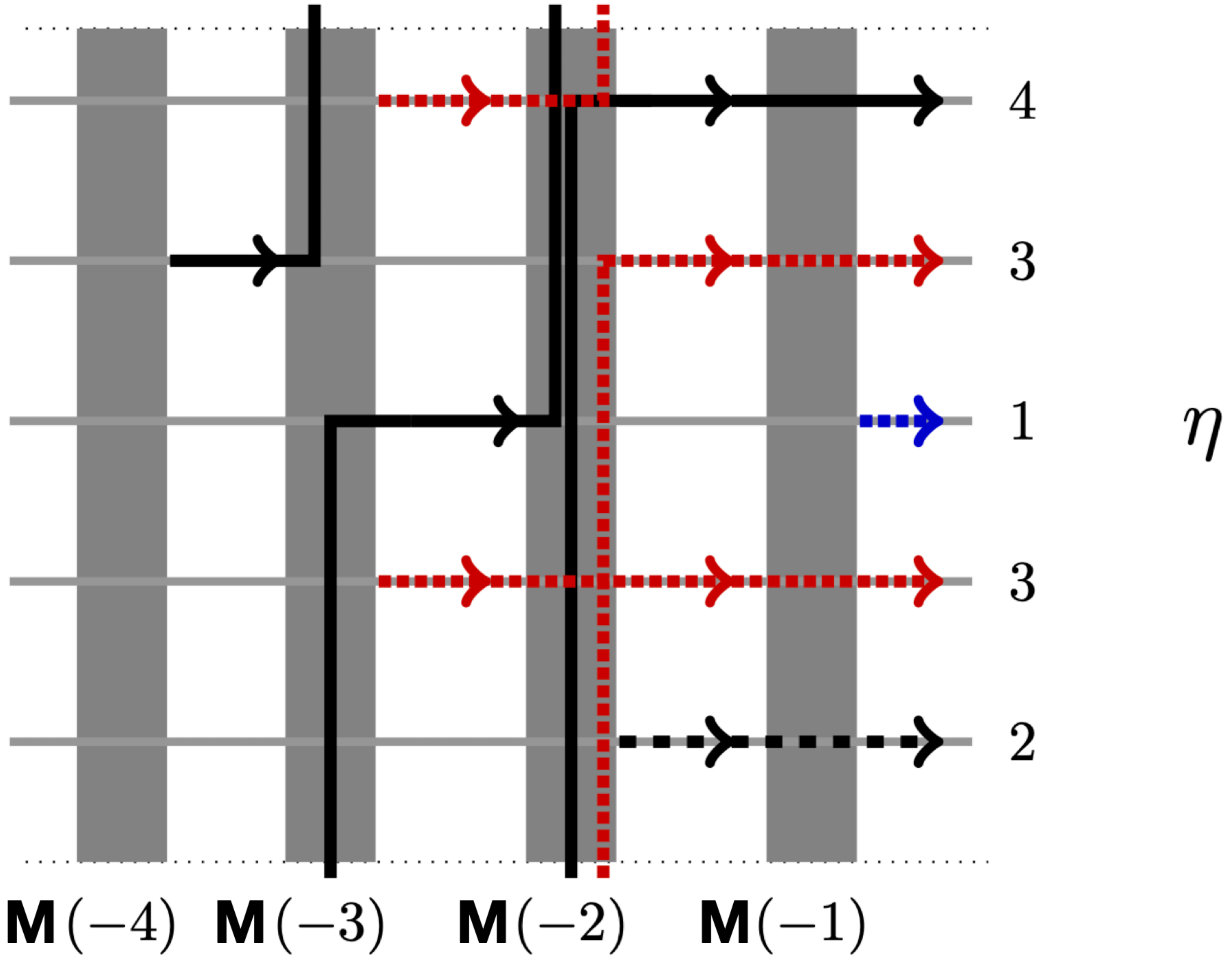


Matching to previous results

- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:

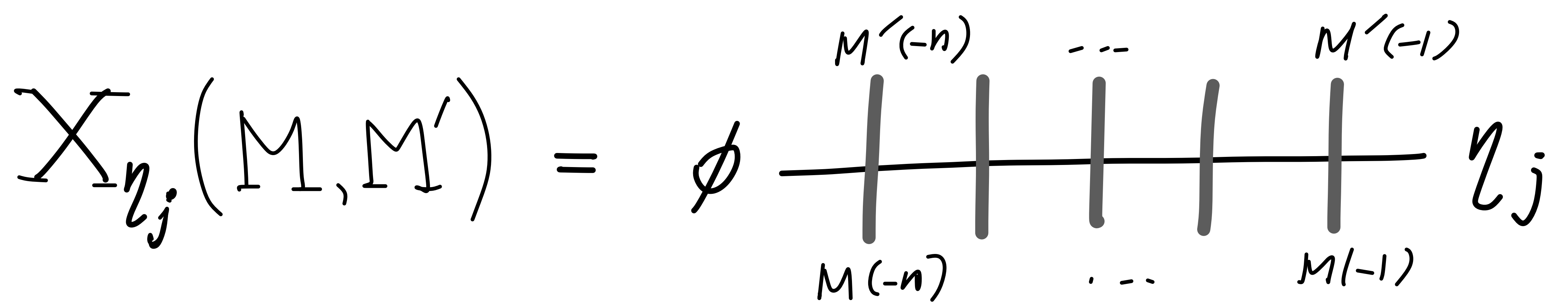




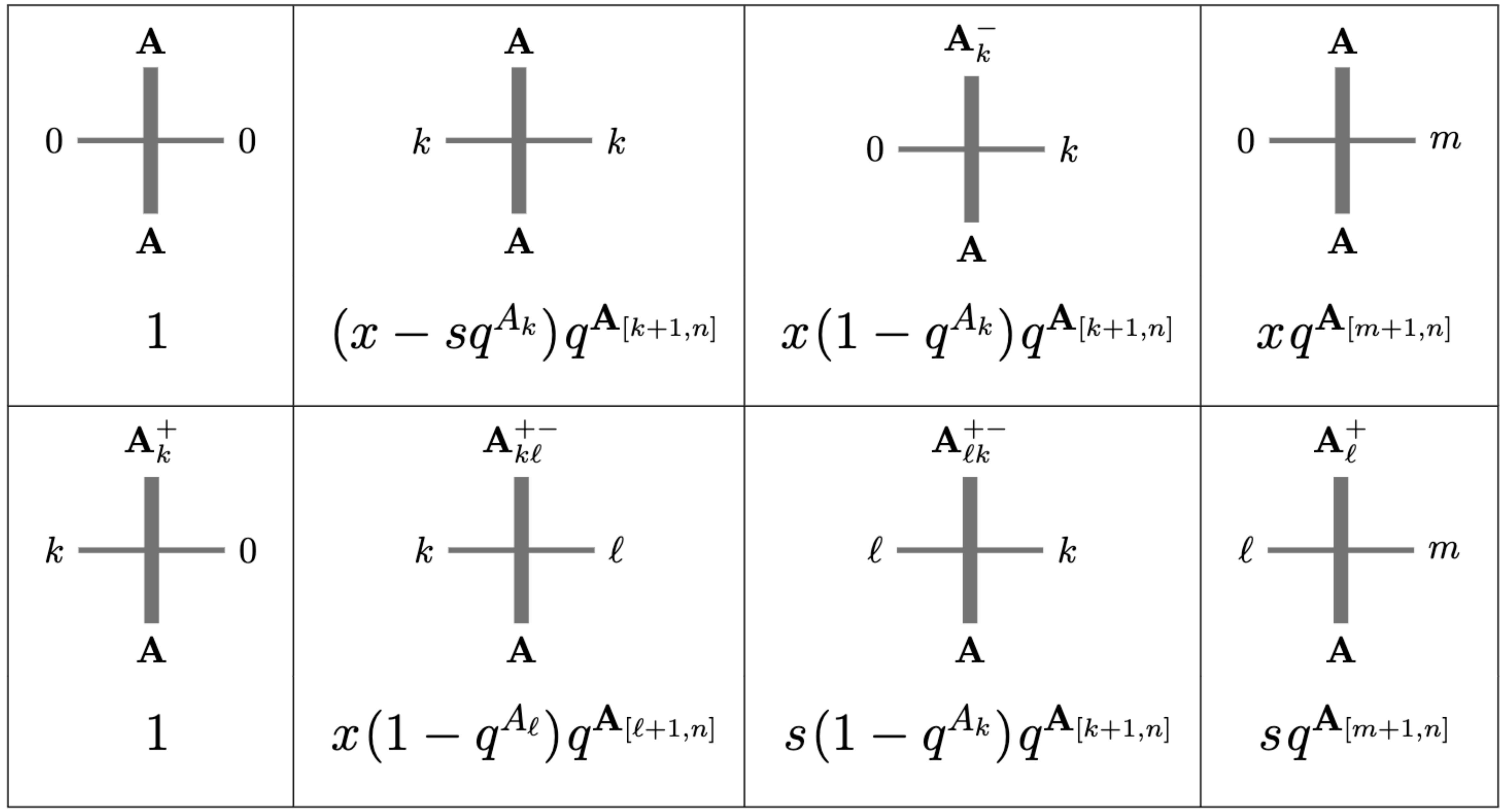


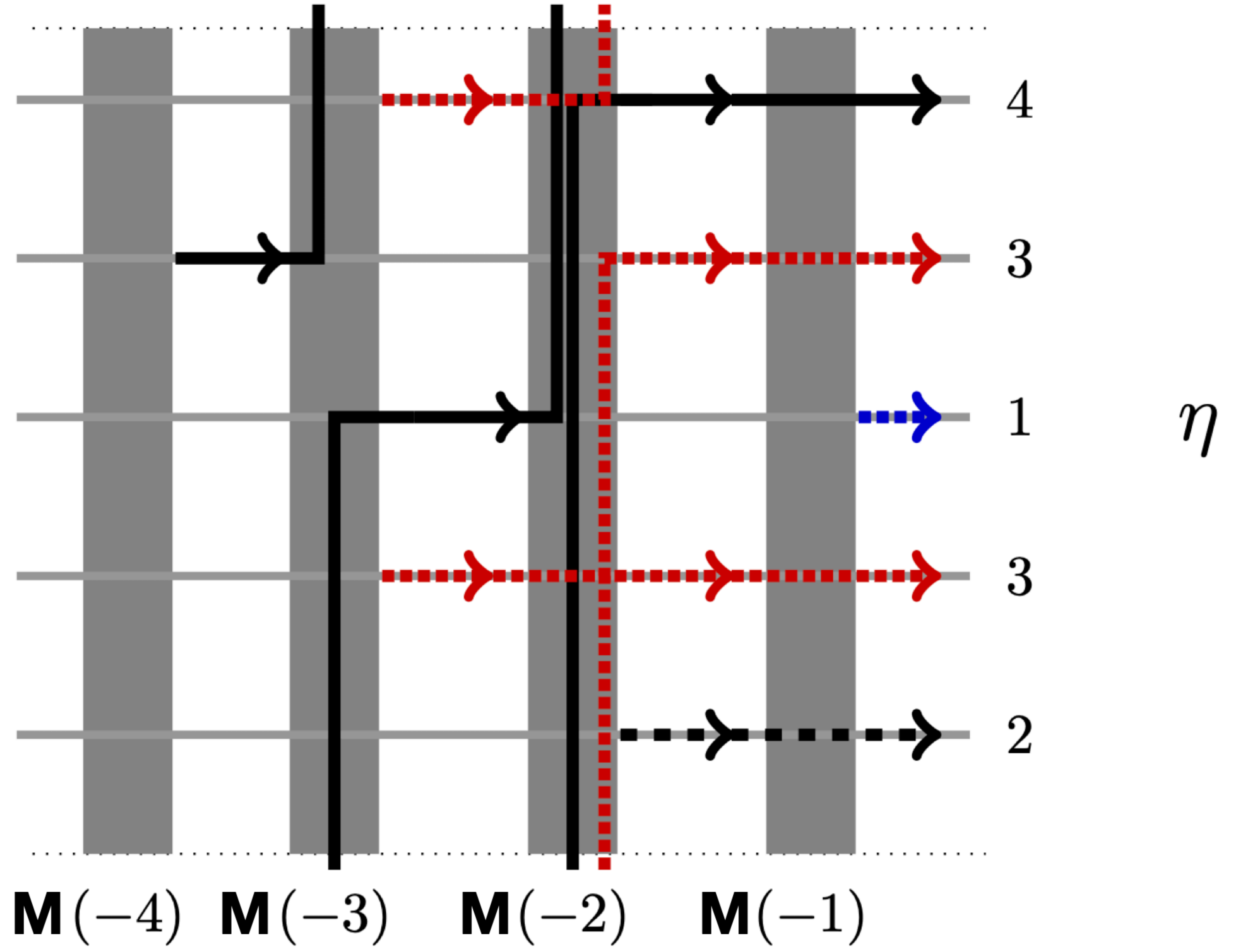
Matching to previous results

- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:



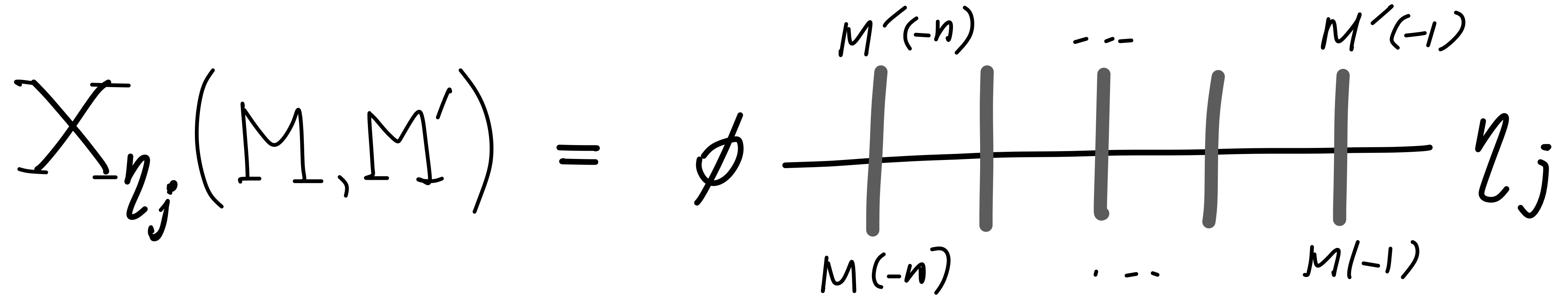
- **[Martin 2018]** found a multiline queue sampling algorithm that nontrivially corresponds to MPA



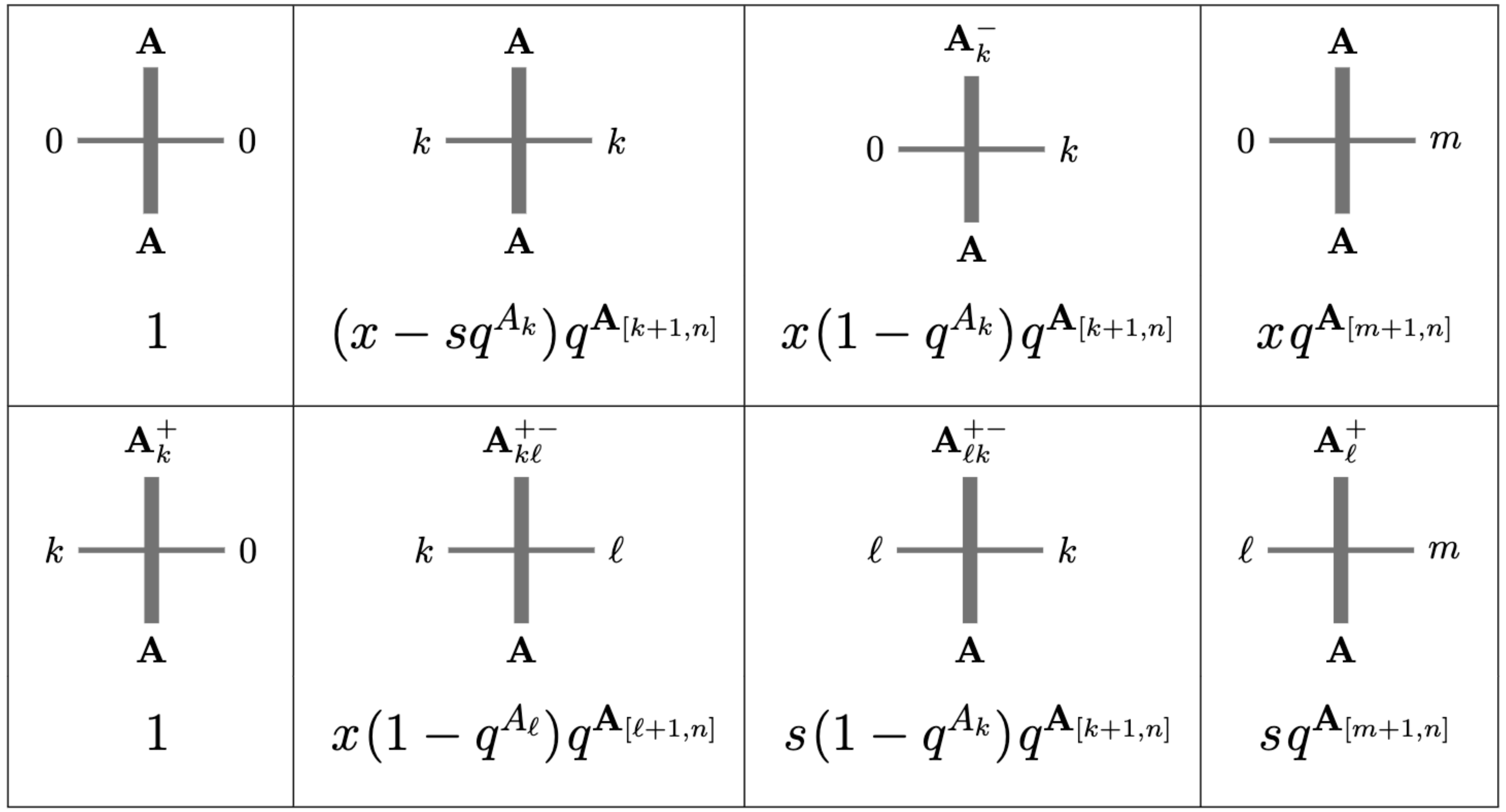


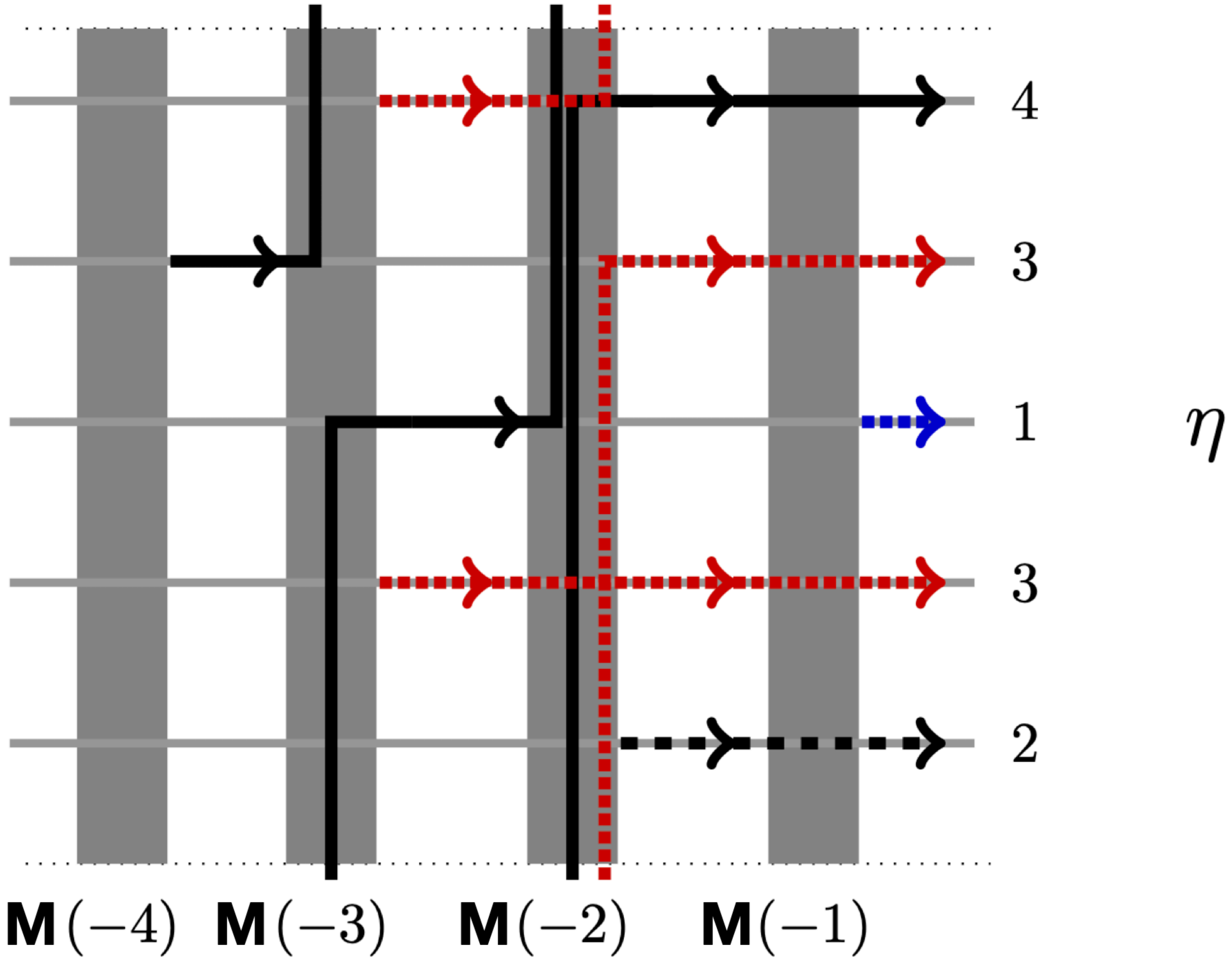
Matching to previous results

- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:



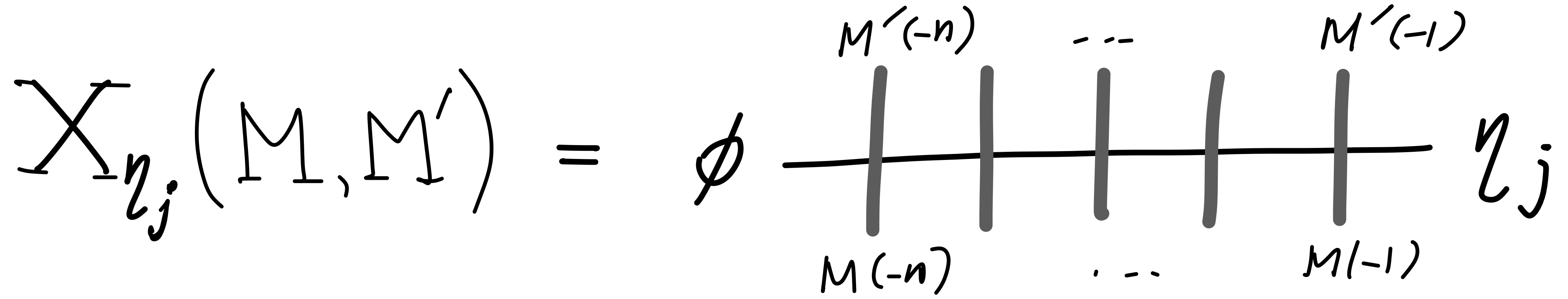
- **[Martin 2018]** found a multiline queue sampling algorithm that nontrivially corresponds to MPA
- $s = q, x = 1$ gives Martin's "alternative queues" (resolves conjecture); interpolation $s \in [0,1)$.



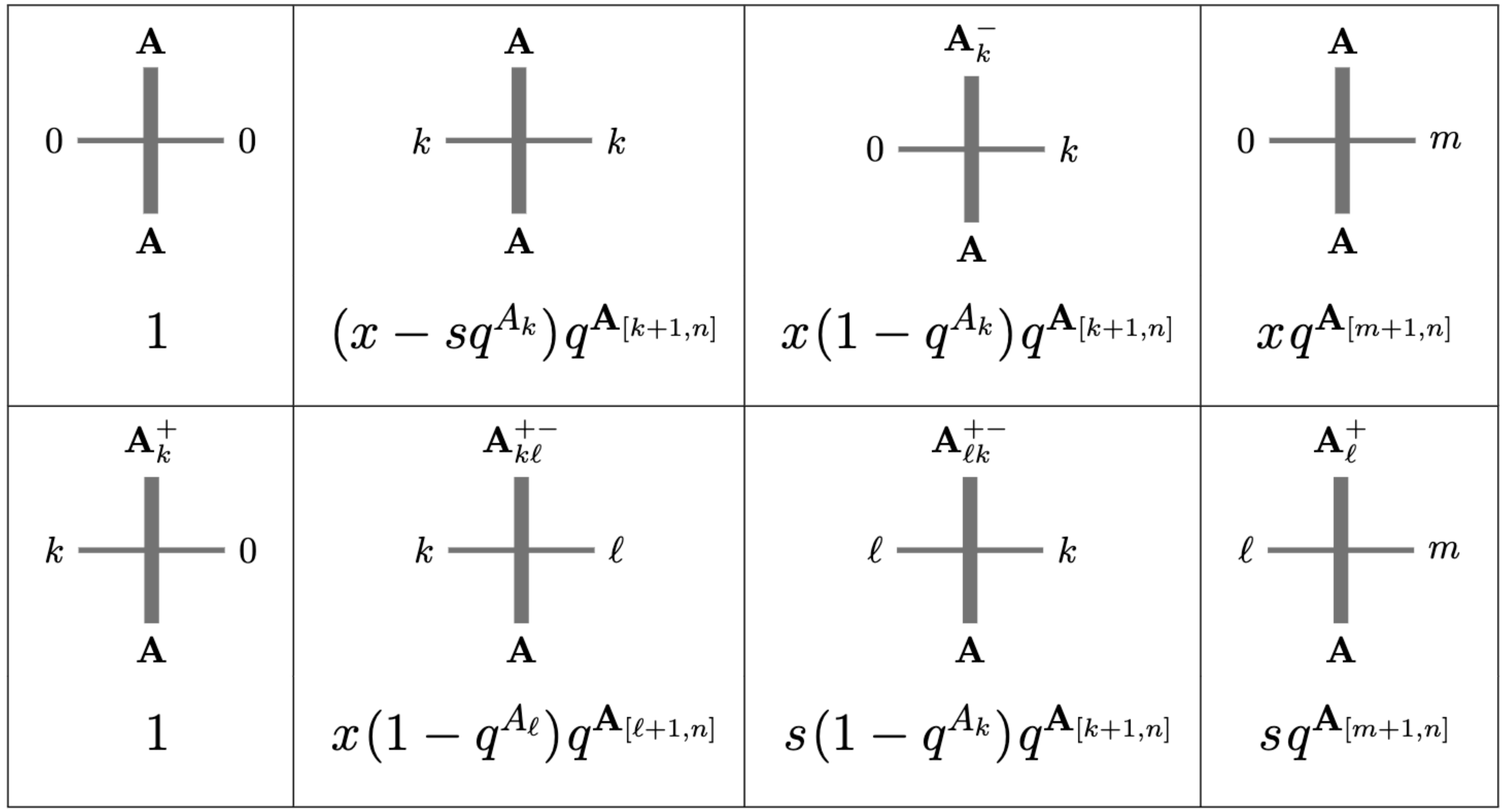


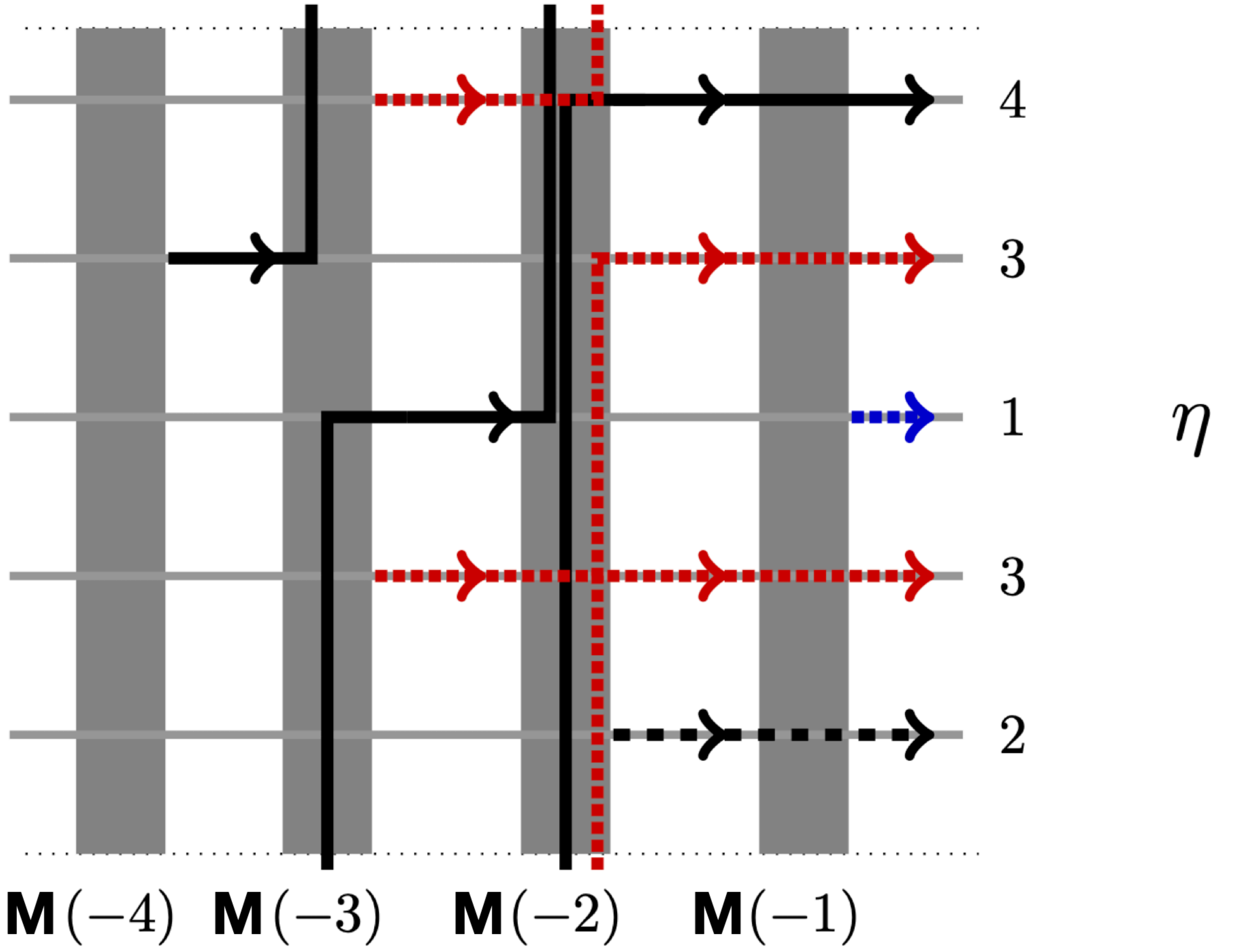
Matching to previous results

- The vertex model for $s = 0, x = 1$ is essentially the Matrix Product Ansatz (MPA) solution **[Prolhac–Evans–Mallick 2009]**. The matrices are row partition functions:

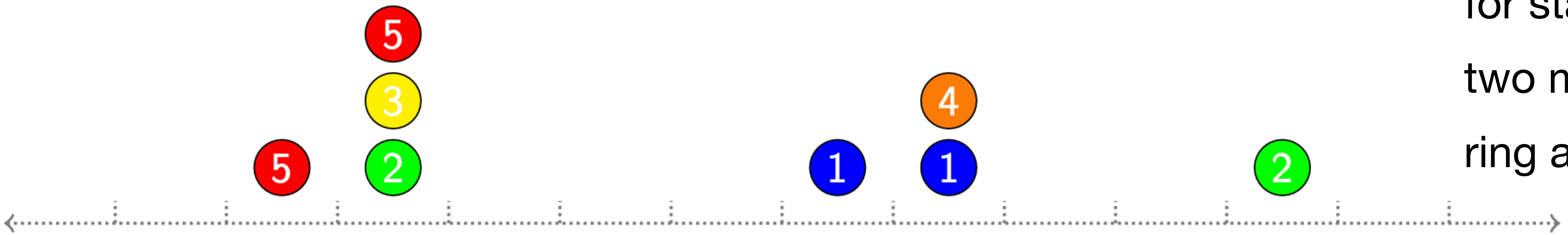


- **[Martin 2018]** found a multiline queue sampling algorithm that nontrivially corresponds to MPA
- $s = q, x = 1$ gives Martin's "alternative queues" (resolves conjecture); interpolation $s \in [0,1)$.
- We can use row-dependent x_j and weighted wrappings to produce nonsymmetric Macdonald polynomials like **[Cantini–de Gier–Wheeler 2015]**, **[Corteel–Mandelshtam–Williams 2018]**; apparently different from **[Borodin-Wheeler 2019]**





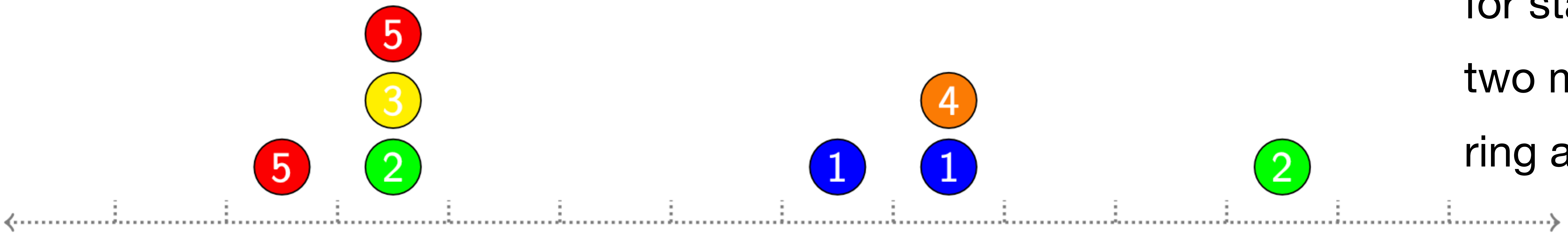
Two other particle systems [Aggarwal-Nicoletti-P. 2023]



We present vertex models for stationary measures of two more systems on the ring and the line

Two other particle systems [Aggarwal-Nicoletti-P. 2023]

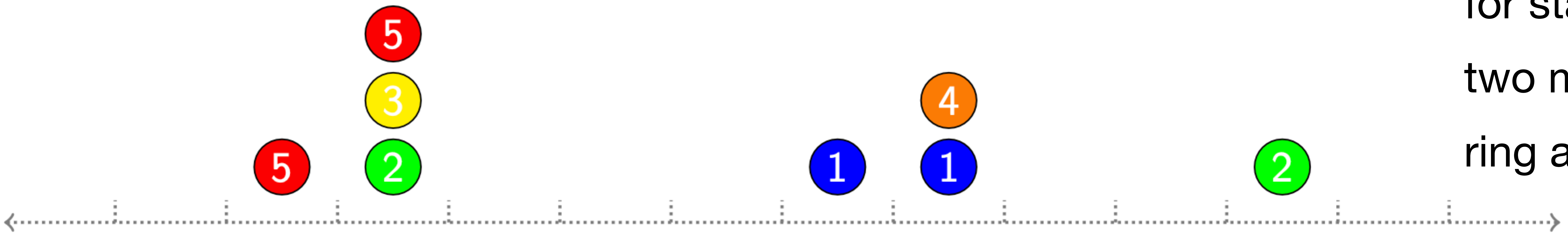
We present vertex models for stationary measures of two more systems on the ring and the line



- **Colored stochastic q-Boson** (**q-TAZRP**), introduced by **[Takeyama 2015]**
- A particle of color i hops from k to $k-1 \pmod N$ according to an independent exponential clock with rate $x_k^{-1}(1-q^{\mathbf{V}(k)_i})q^{\mathbf{V}(k)_{[i+1,n]}}$.
- Tableau/queue model for stationary distributions **[Ayyer–Mandelshtam–Martin 2022]**; we match to our vertex models

Two other particle systems [Aggarwal-Nicoletti-P. 2023]

We present vertex models for stationary measures of two more systems on the ring and the line



- **Colored stochastic q-Boson** (**q-TAZRP**), introduced by **[Takeyama 2015]**
- A particle of color i hops from k to $k-1 \,(\mathrm{mod}\, N)$ according to an independent exponential clock with rate $x_k^{-1}(1-q^{\mathbf{V}(k)_i})q^{\mathbf{V}(k)_{[i+1,n]}}$.
- Tableau/queue model for stationary distributions **[Ayyer–Mandelshtam–Martin 2022]**; we match to our vertex models

- **Colored q-PushTASEP** of capacity P
- **[Borodin-Wheeler 2018], [Bukh-Cox 2019], [Angel–Ayyer–Martin, in progress 2023]**
- A particle activates with rate $x_k^{-1}(q^{-A_j}-1)q^{\mathsf{P}-A_{[j+1,n]}}$
- Active particle hops from site to site, where it can either stop; stop activate another particle of lower color; or move through, with prob. $1-q^{\mathsf{P}-|\mathbf{B}|}$, $(q^{-B_d}-1)q^{\mathsf{P}-B_{[d+1,n]}}$, $q^{\mathsf{P}-B_{[c,n]}}$

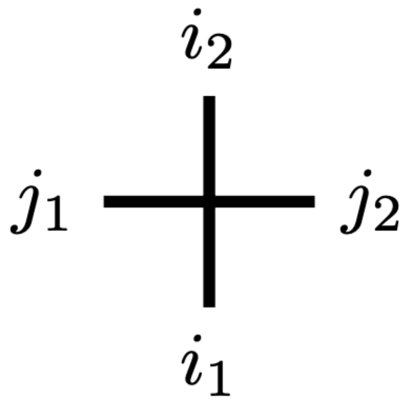
Stationarity from Yang-Baxter equation

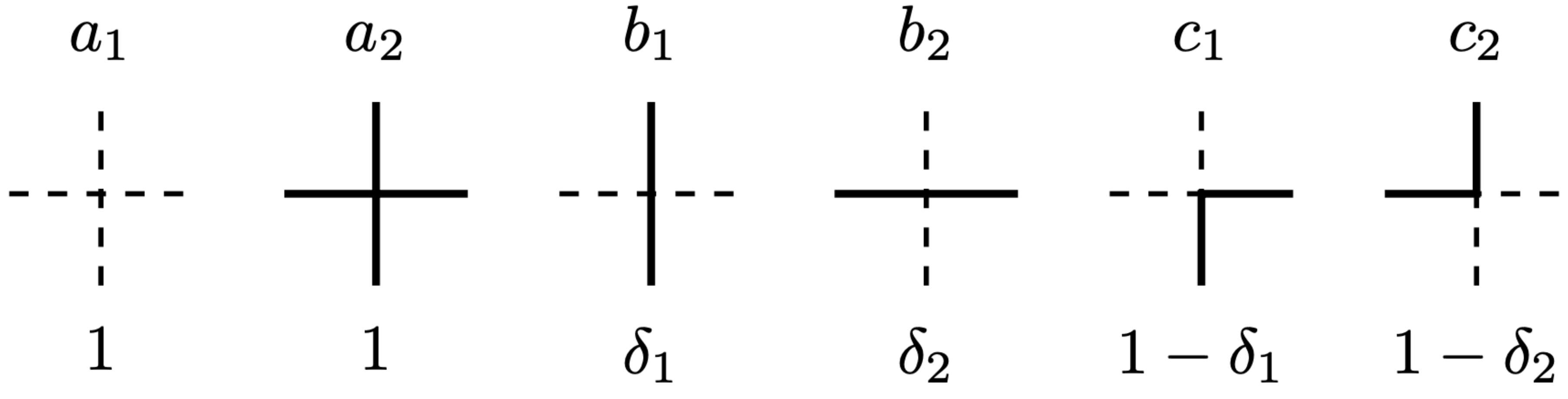
“Toy” example: stationarity for the single-color stochastic six-vertex model in the quadrant

(Explain the main idea in a simpler setting than the ring)

“Toy” example: stationarity for the single-color stochastic six-vertex model

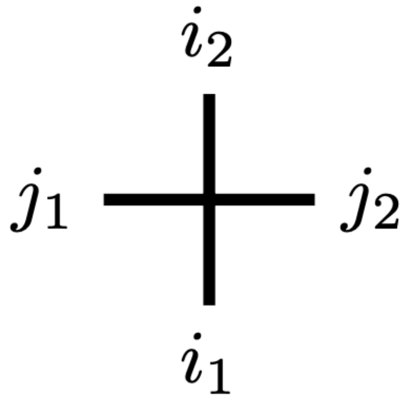
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

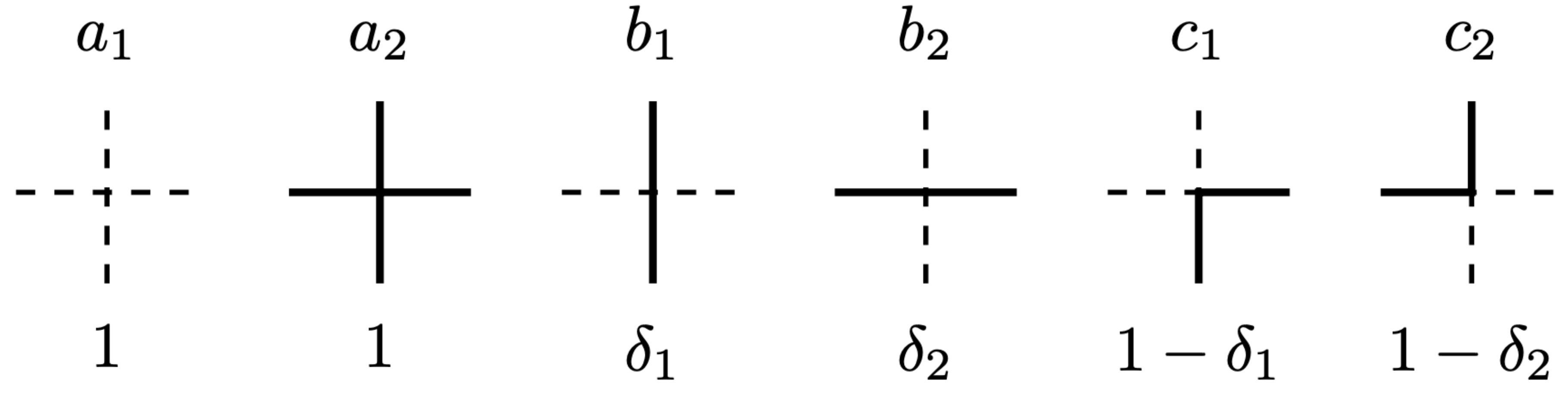




“Toy” example: stationarity for the single-color stochastic six-vertex model

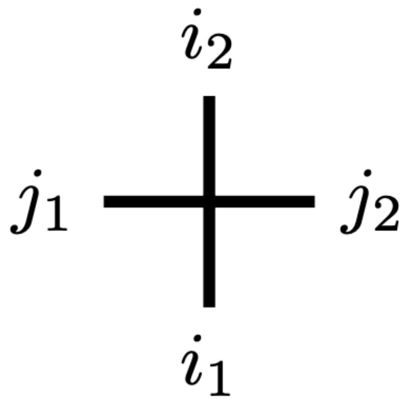
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

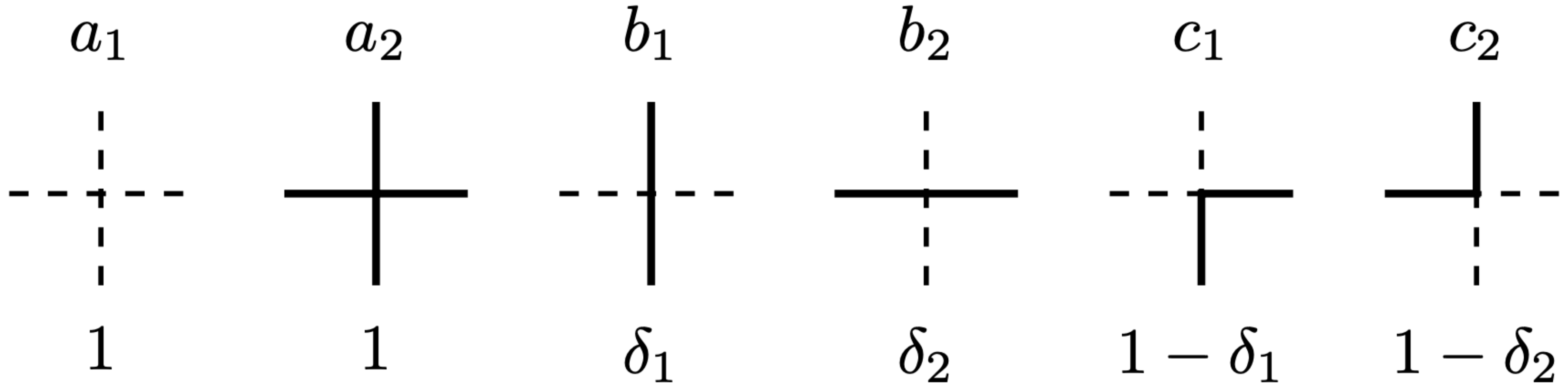




The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1 - \delta_1$, $b_2 = \delta_2$, $c_2 = 1 - \delta_2$ are *stochastic*: $\sum_{i_2, j_2} w(i_1, j_1; i_2, j_2) = 1$.

“Toy” example: stationarity for the single-color stochastic six-vertex model





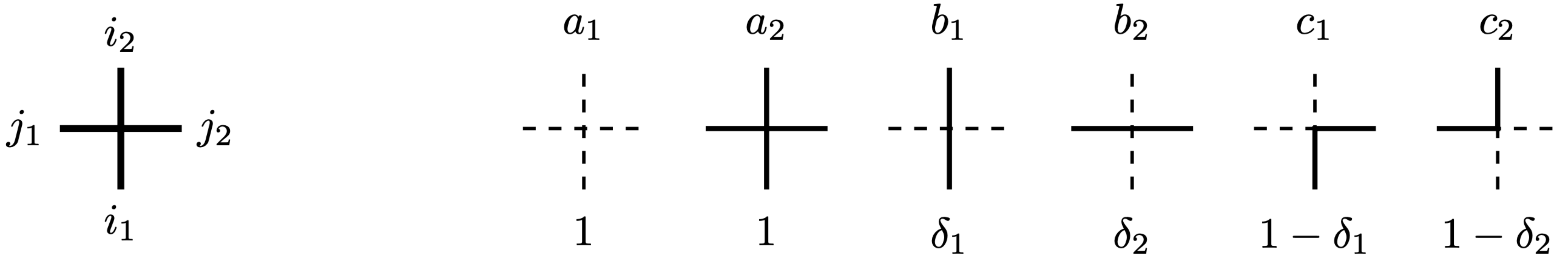
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1 - \delta_1$, $b_2 = \delta_2$, $c_2 = 1 - \delta_2$ are *stochastic*: $\sum_{i_2, j_2} w(i_1, j_1; i_2, j_2) = 1$.

“Toy” example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

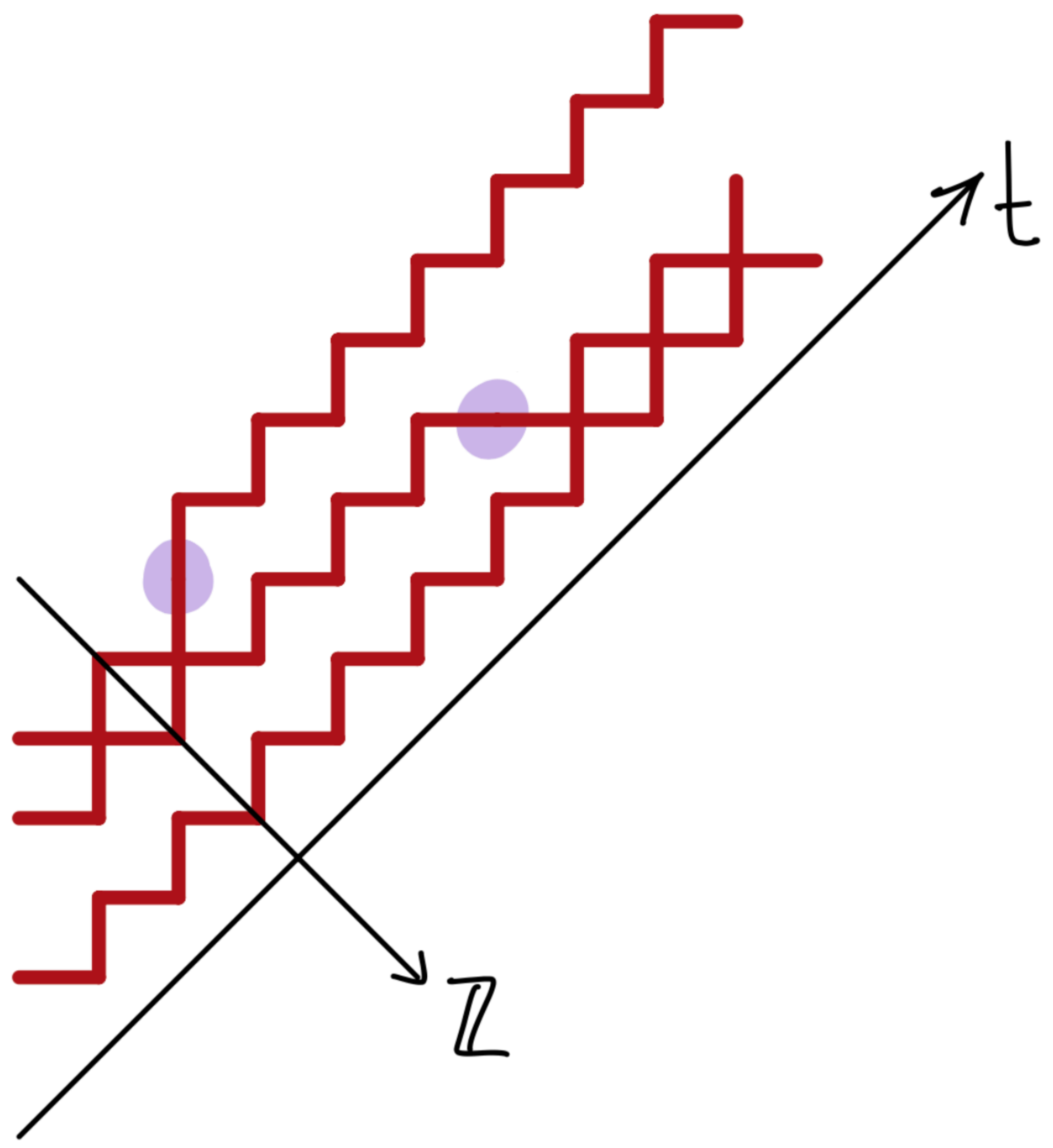


$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1 - \delta_1$, $b_2 = \delta_2$, $c_2 = 1 - \delta_2$ are *stochastic*: $\sum_{i_2, j_2} w(i_1, j_1; i_2, j_2) = 1$.

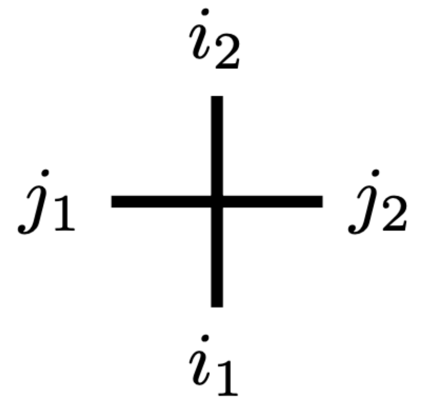
Converges to ASEP along the diagonal as

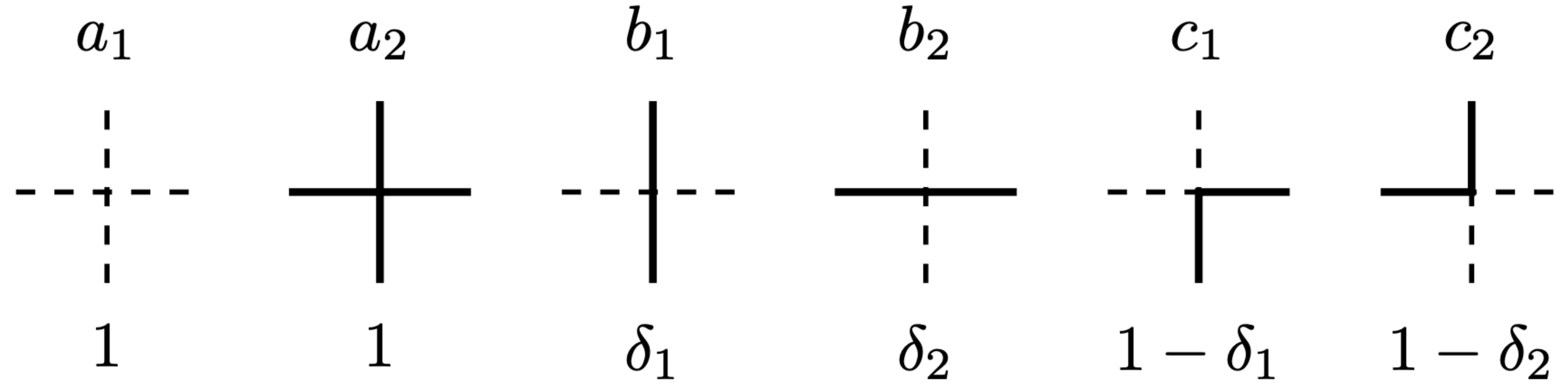
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)



"Toy" example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]



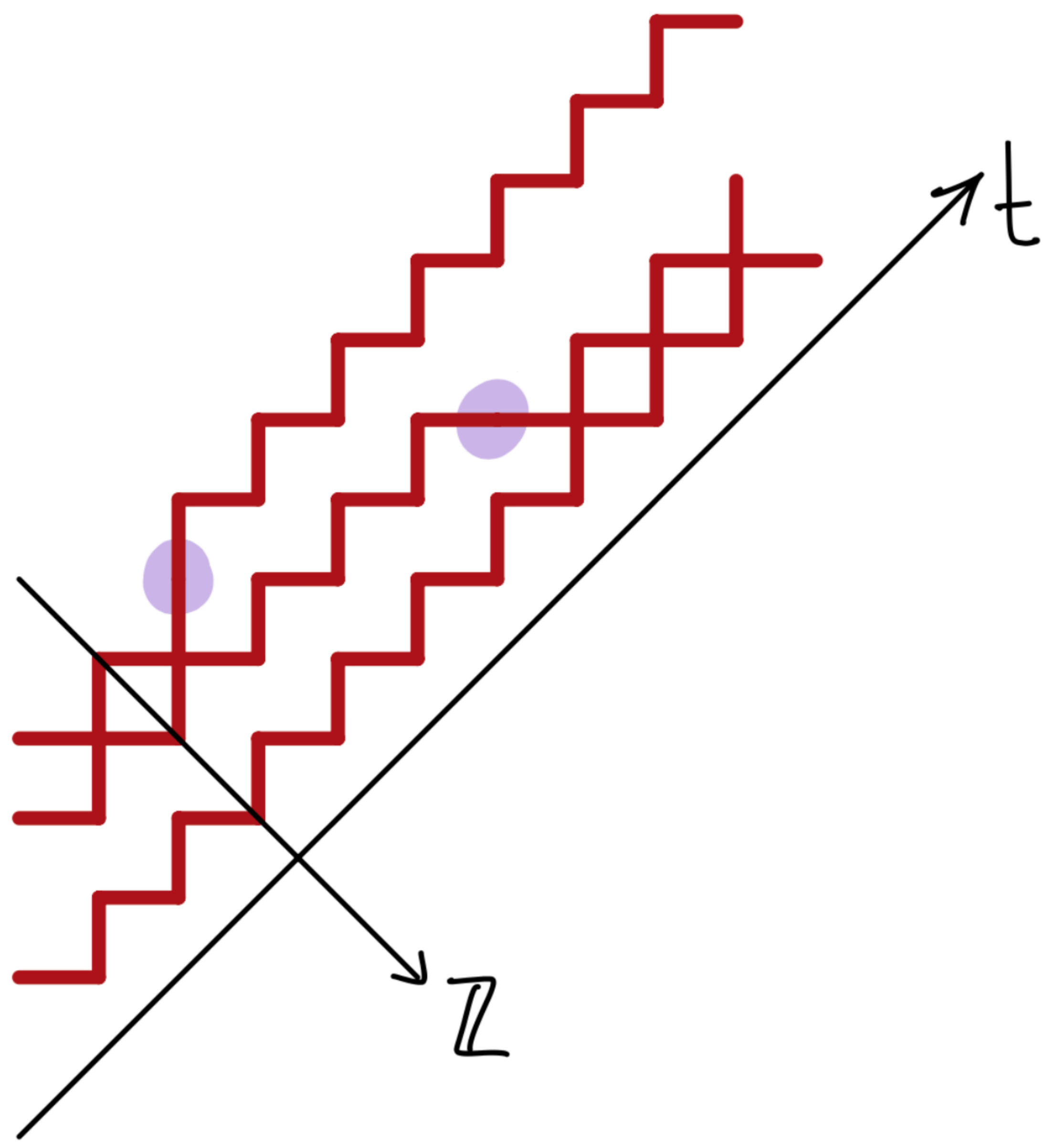


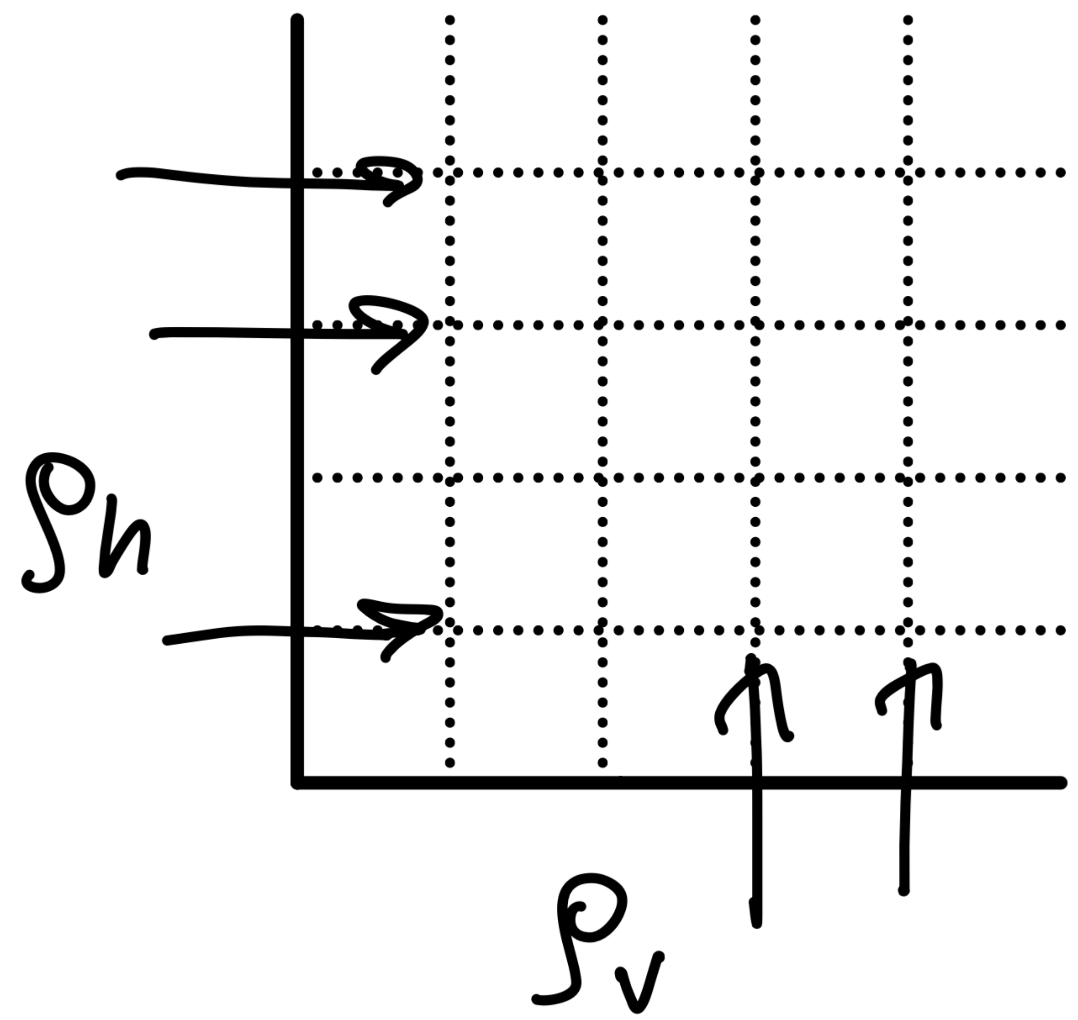
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

Converges to ASEP along the diagonal as

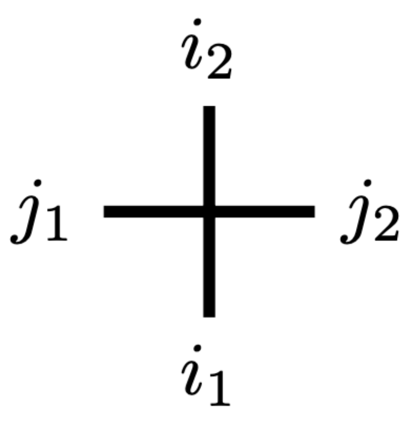
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)

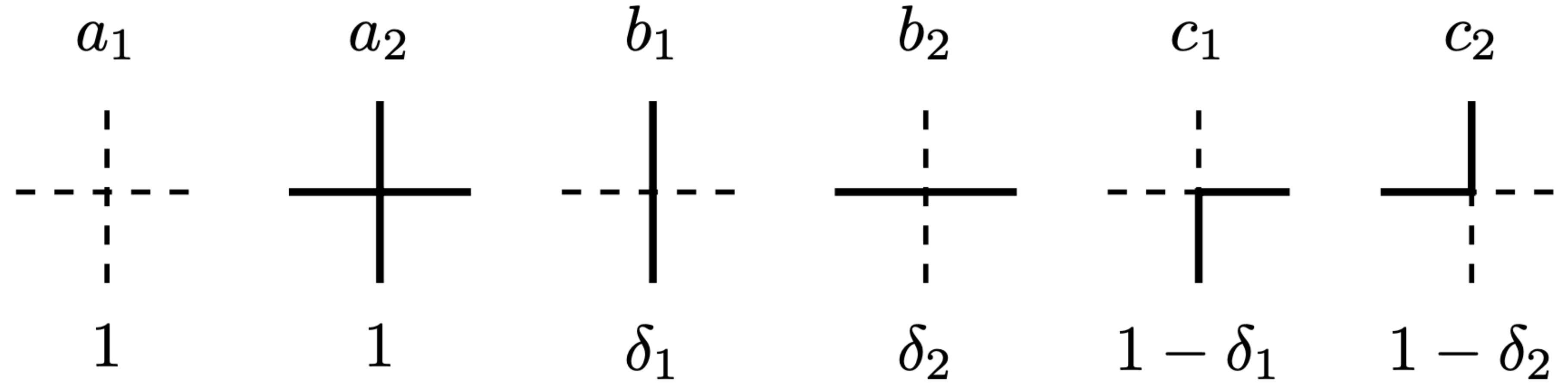




"Toy" example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]



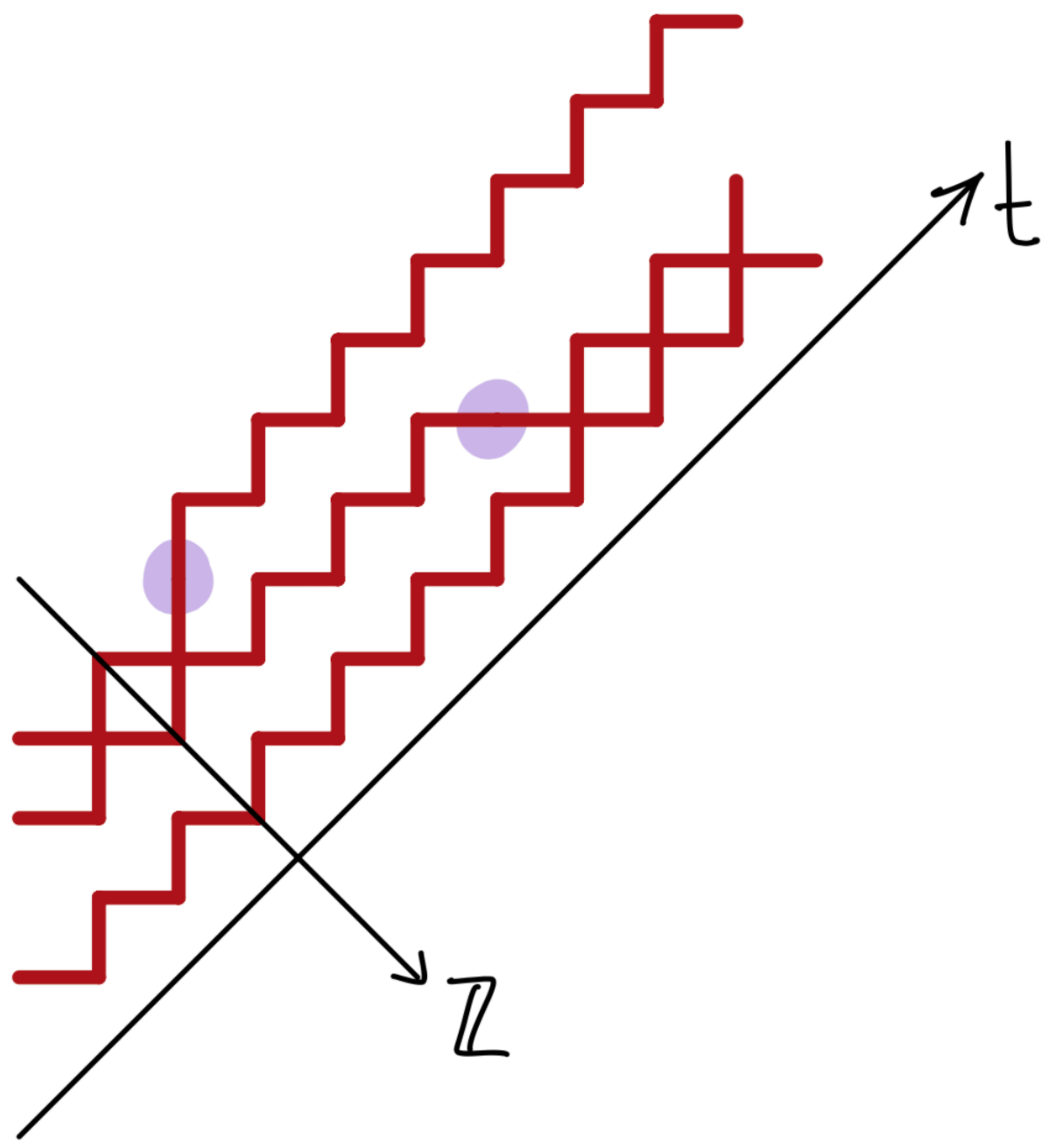


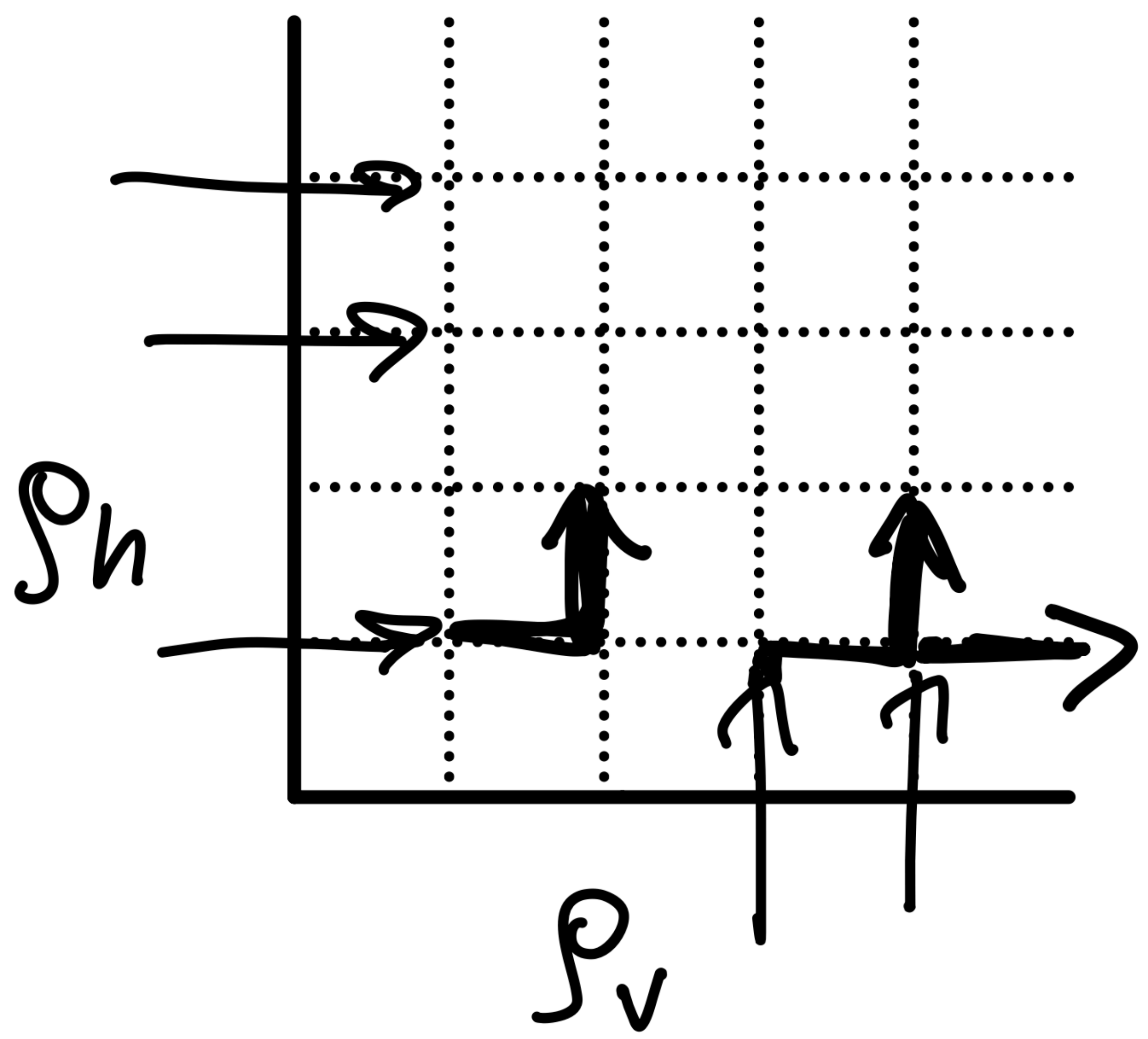
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1 - \delta_2$ are *stochastic*: $\sum_{i_2, j_2} w(i_1, j_1; i_2, j_2) = 1$.

Converges to ASEP along the diagonal as

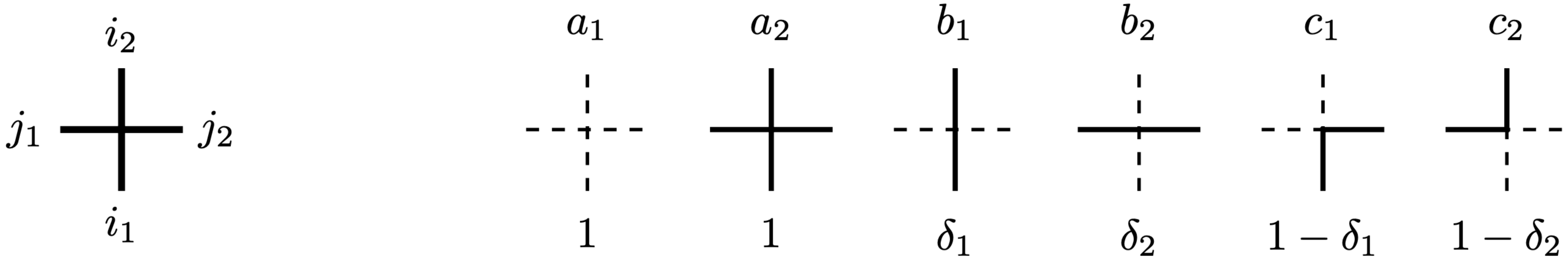
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)





“Toy” example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

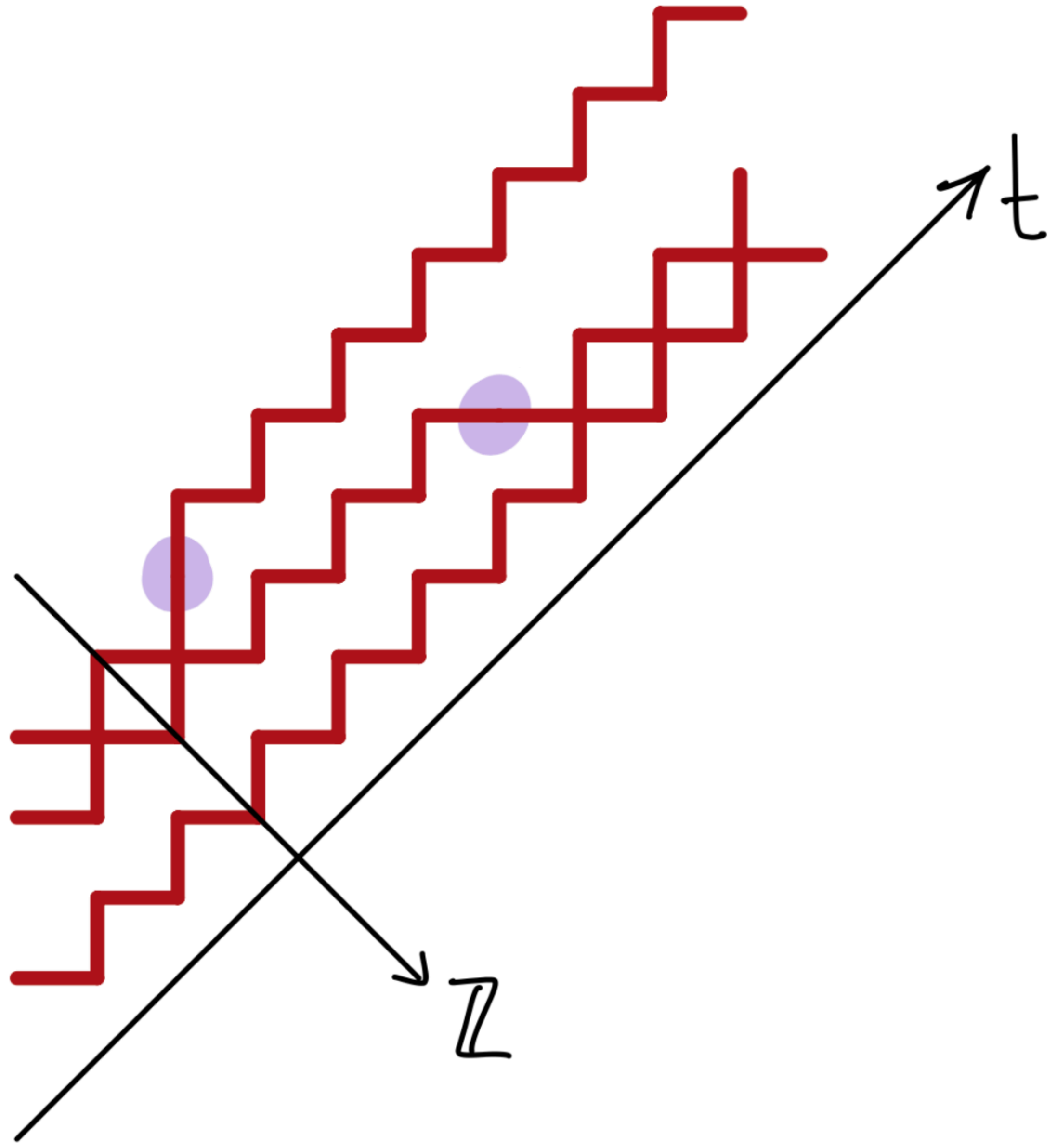


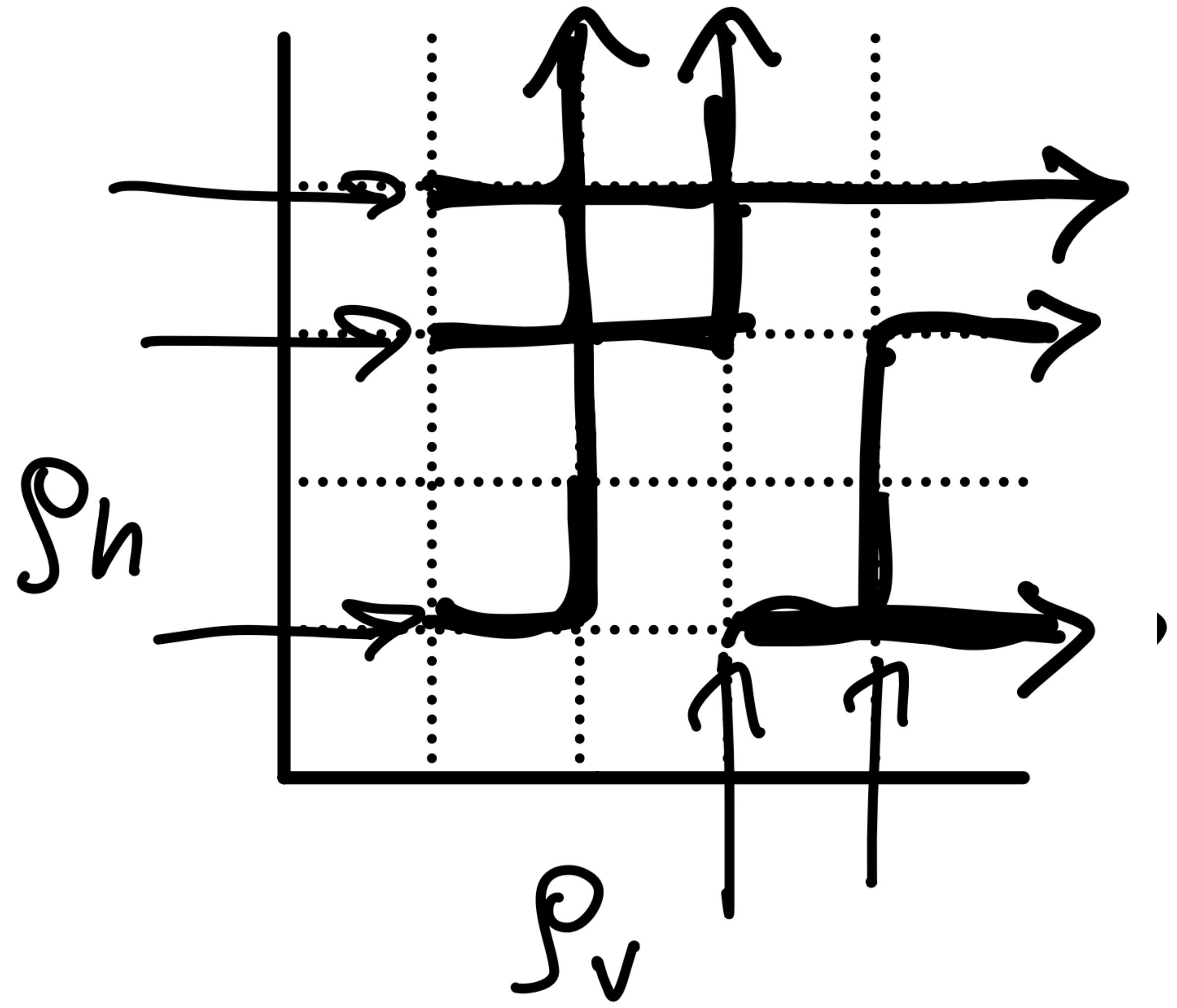
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

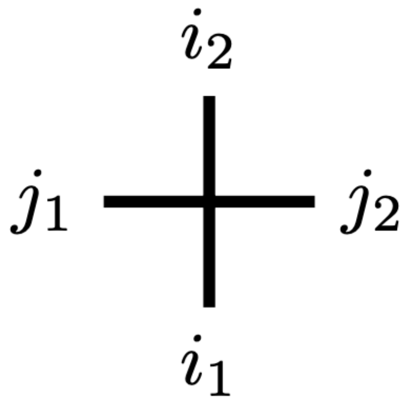
Converges to ASEP along the diagonal as

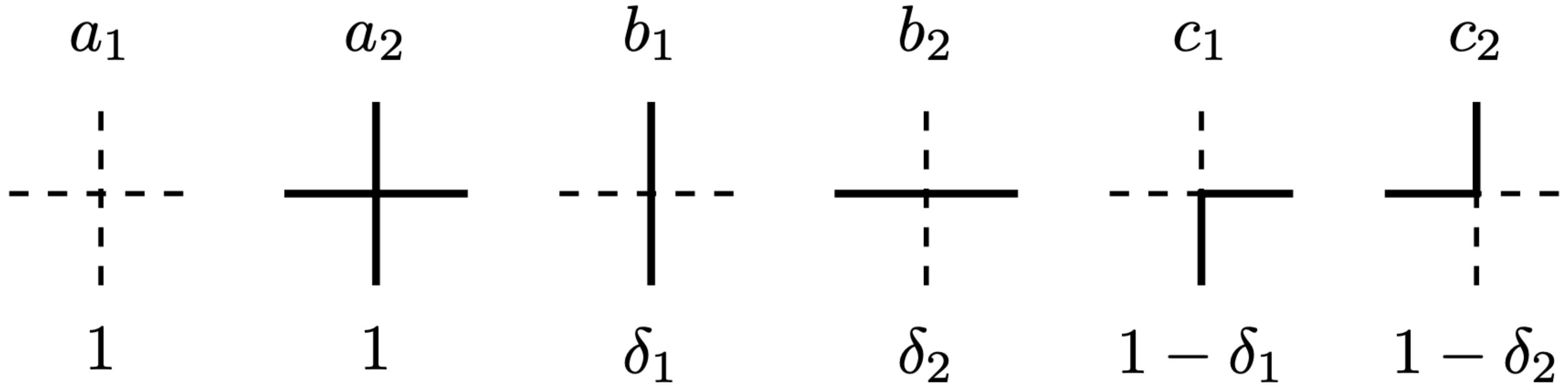
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)





“Toy” example: stationarity for the single-color stochastic six-vertex model





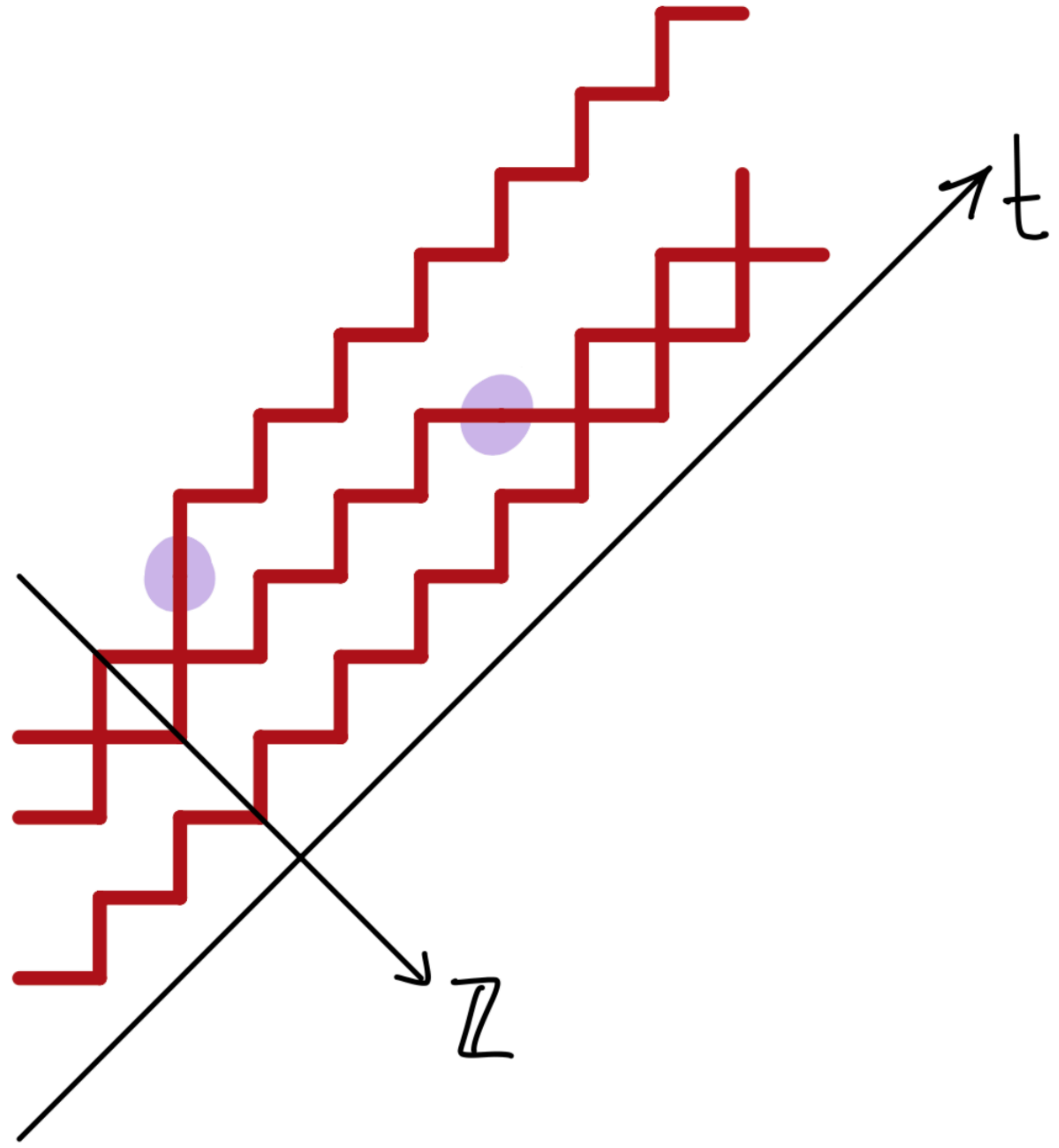
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

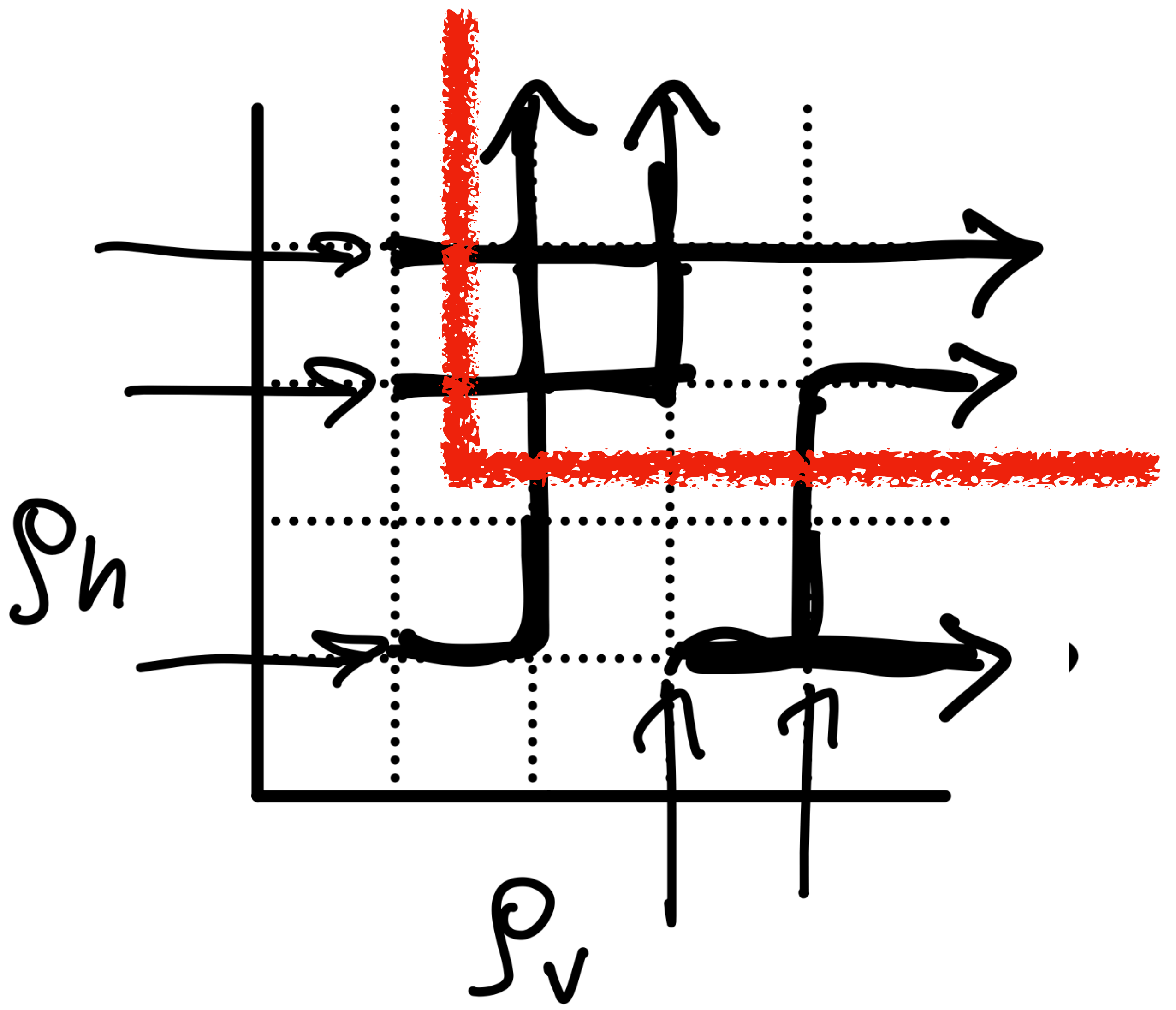
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

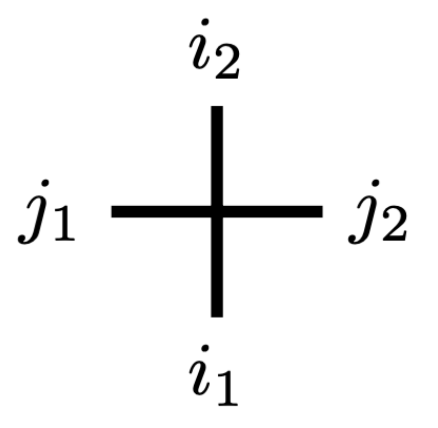
Converges to ASEP along the diagonal as

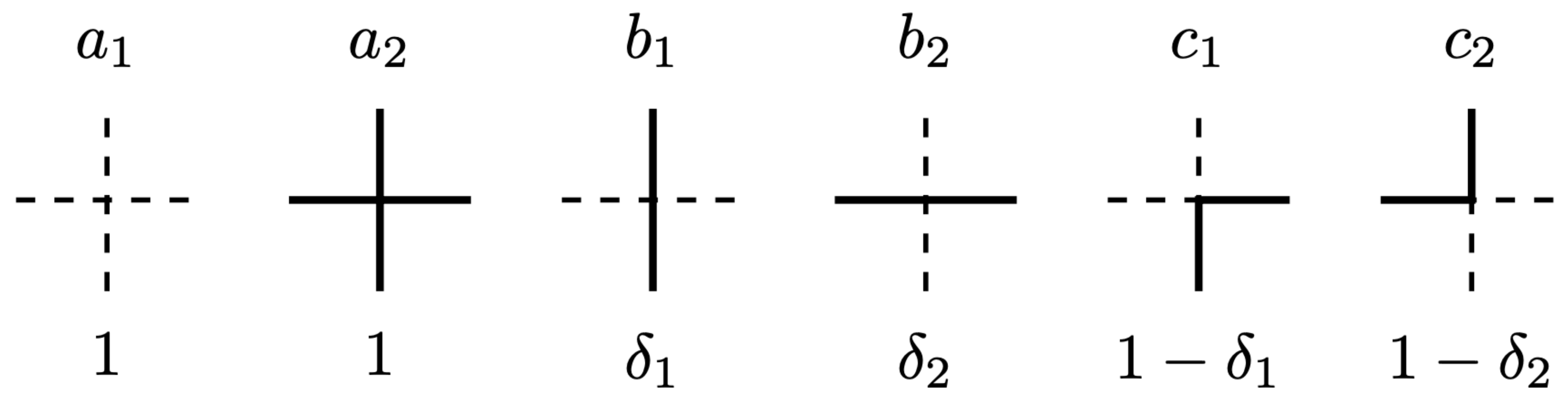
$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)





"Toy" example: stationarity for the single-color stochastic six-vertex model





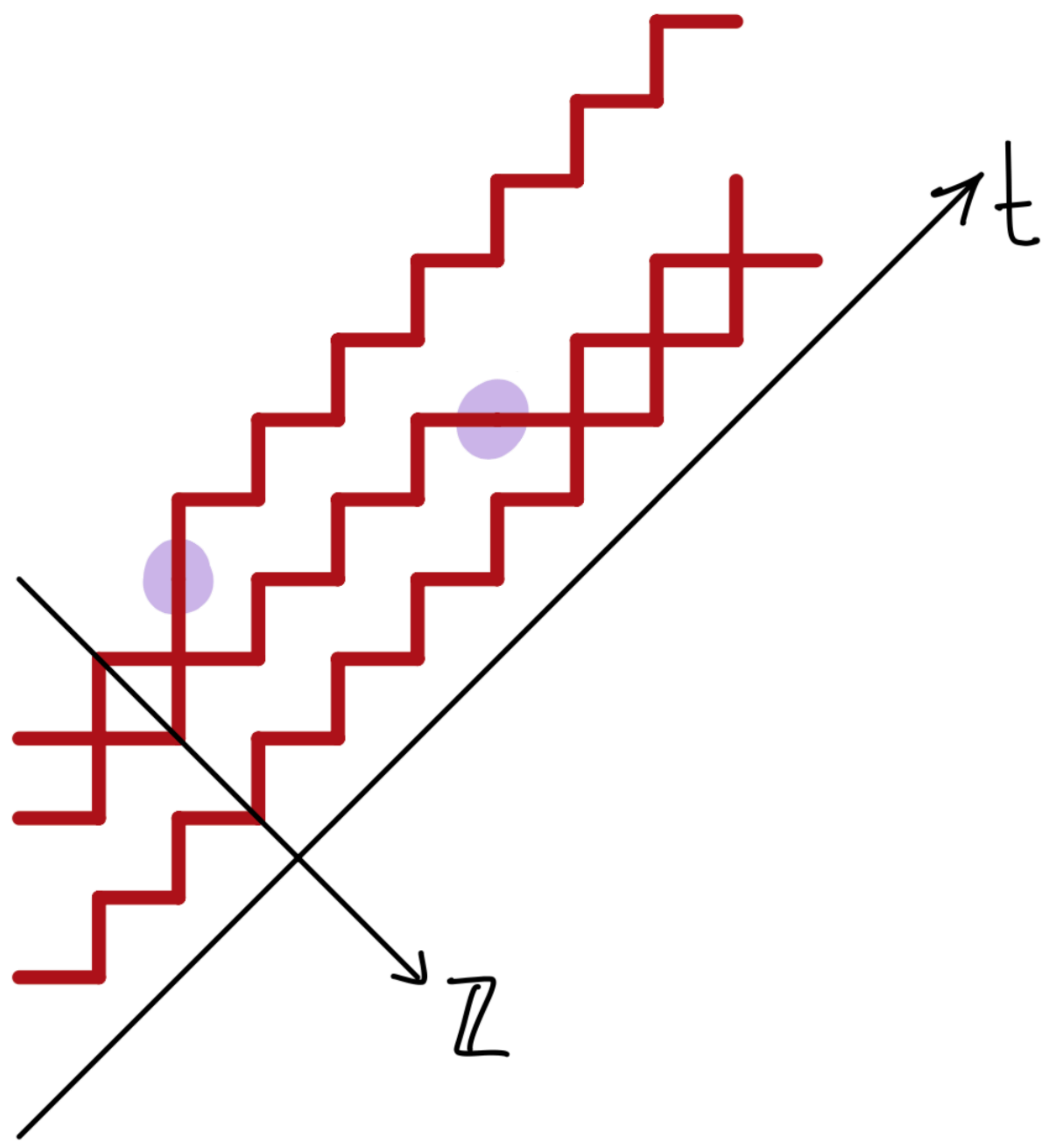
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

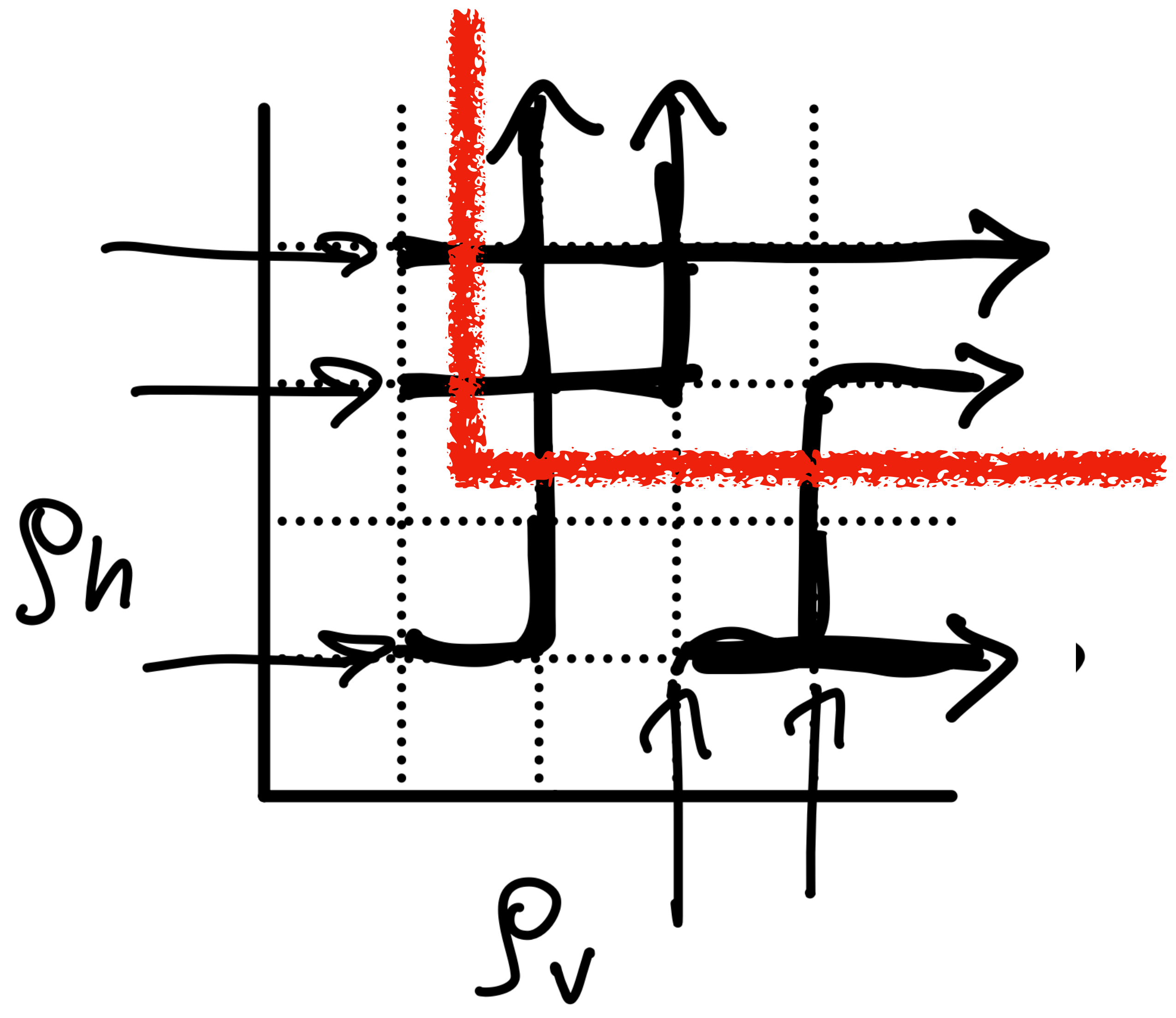
The weights with $a_1 = a_2 = 1,\ b_1 = \delta_1,\ c_1 = 1-\delta_1,\ b_2 = \delta_2,\ c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

Converges to ASEP along the diagonal as

$\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)

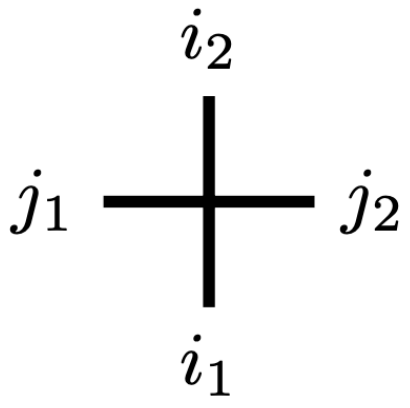


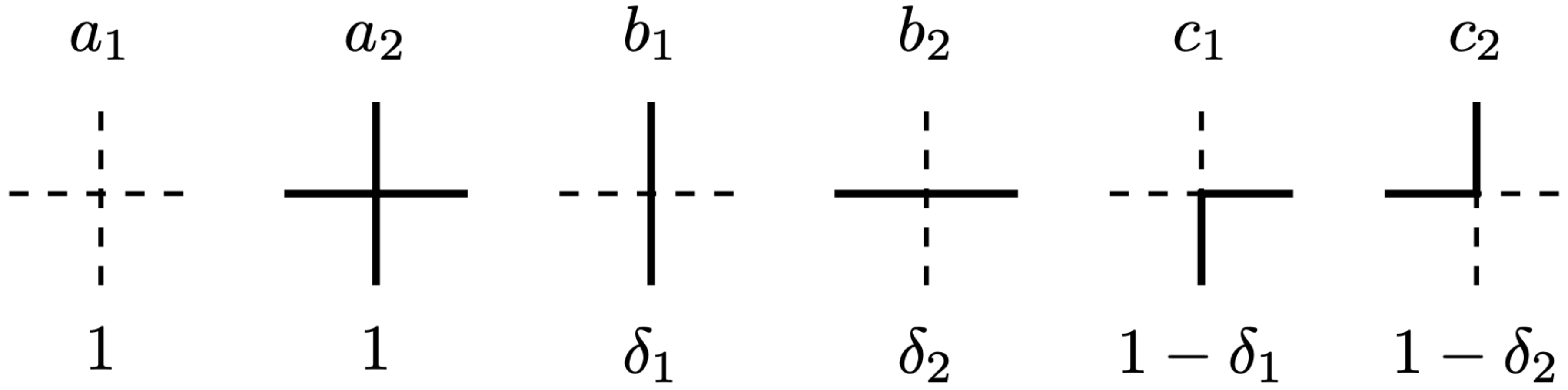
- **Stationarity.** Assume that the boundary conditions are Bernoulli with densities ρ_h, ρ_v. Then for $\rho_h = \dfrac{u\rho_v}{1-\rho_v+u\rho_v}$, the distribution is stationary in the quadrant.



“Toy” example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]



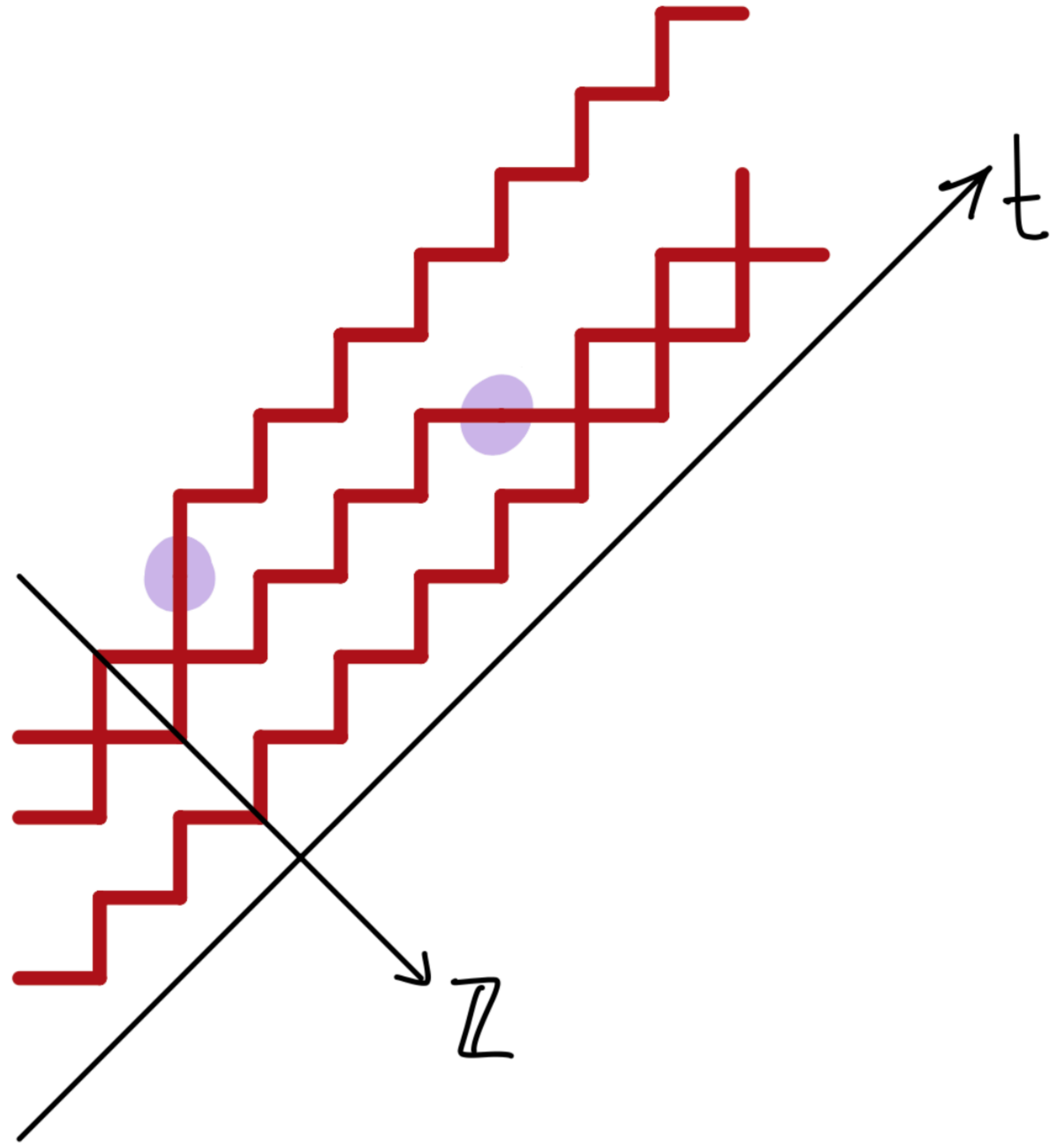


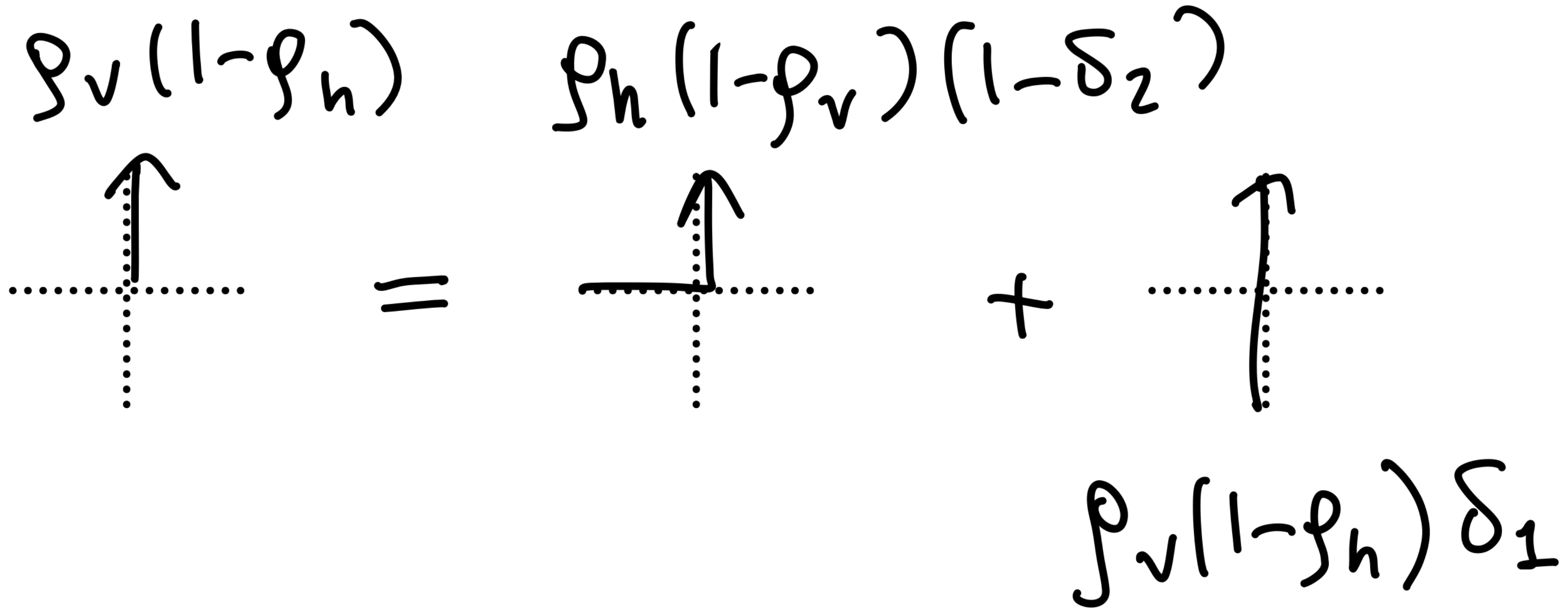
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

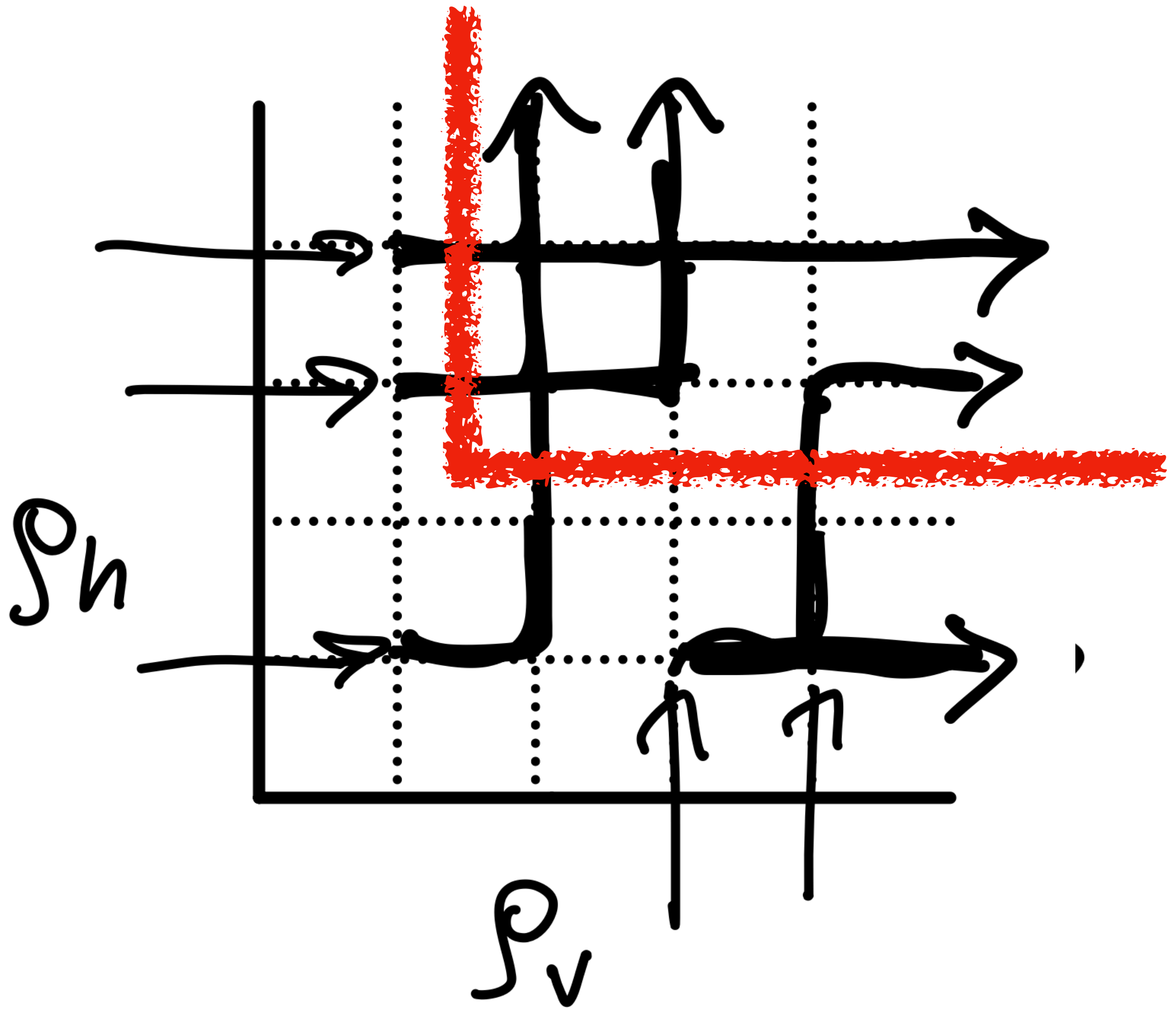
The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

Converges to ASEP along the diagonal as $\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)

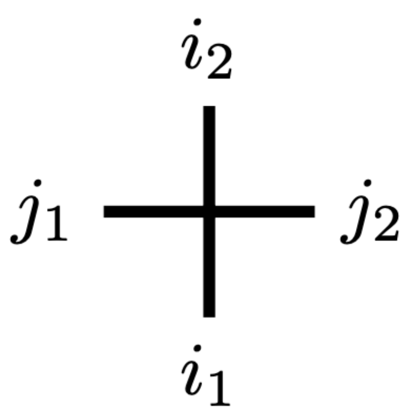
- **Stationarity.** Assume that the boundary conditions are Bernoulli with densities ρ_h, ρ_v. Then for $\rho_h = \dfrac{u\rho_v}{1-\rho_v+u\rho_v}$, the distribution is stationary in the quadrant.

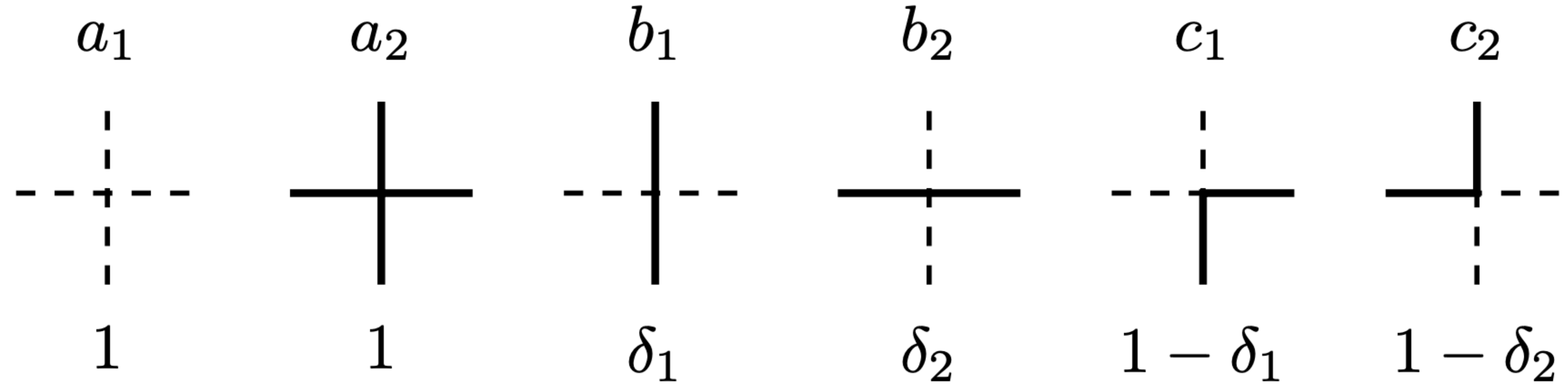






“Toy” example: stationarity for the single-color stochastic six-vertex model





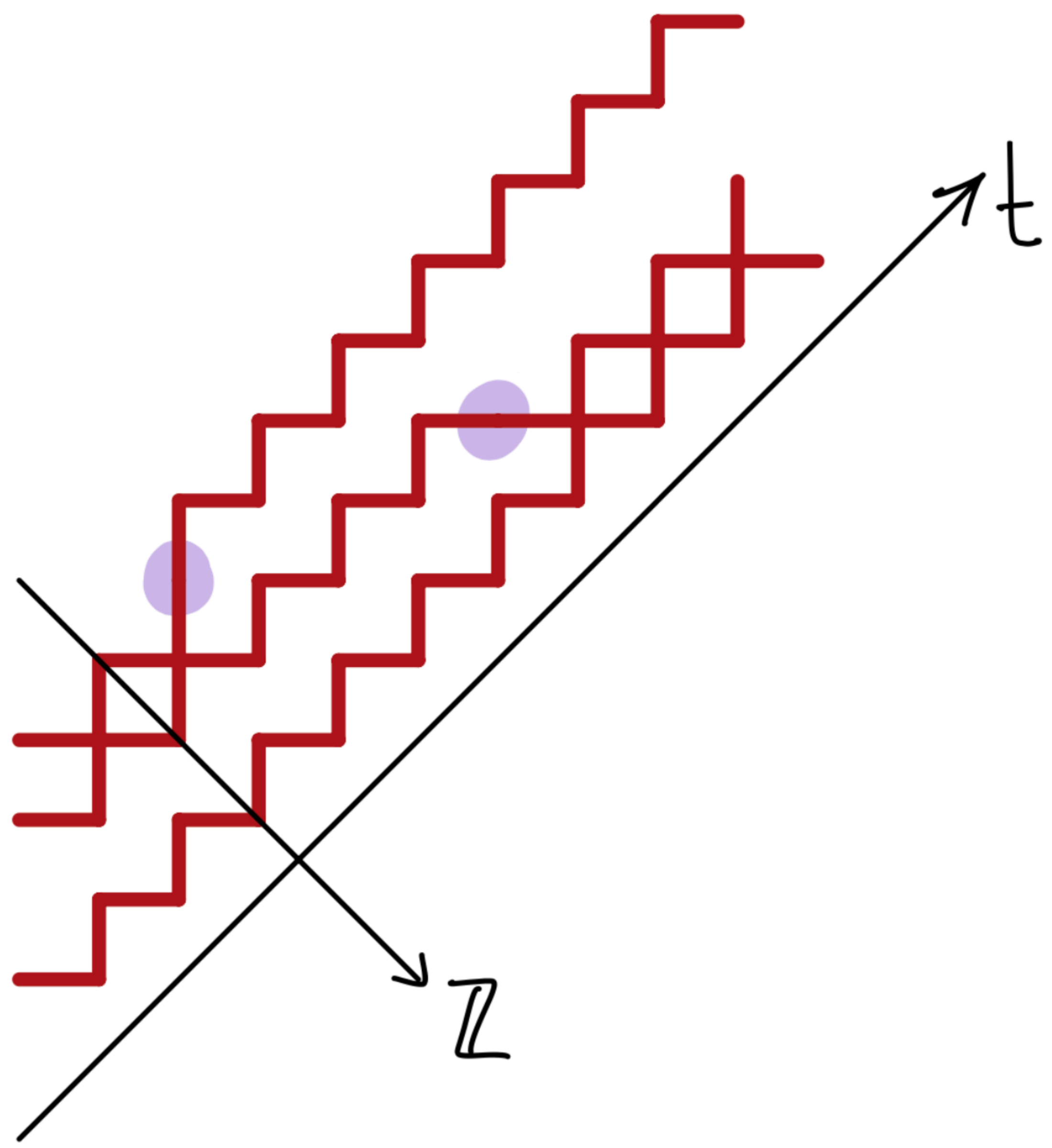
[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

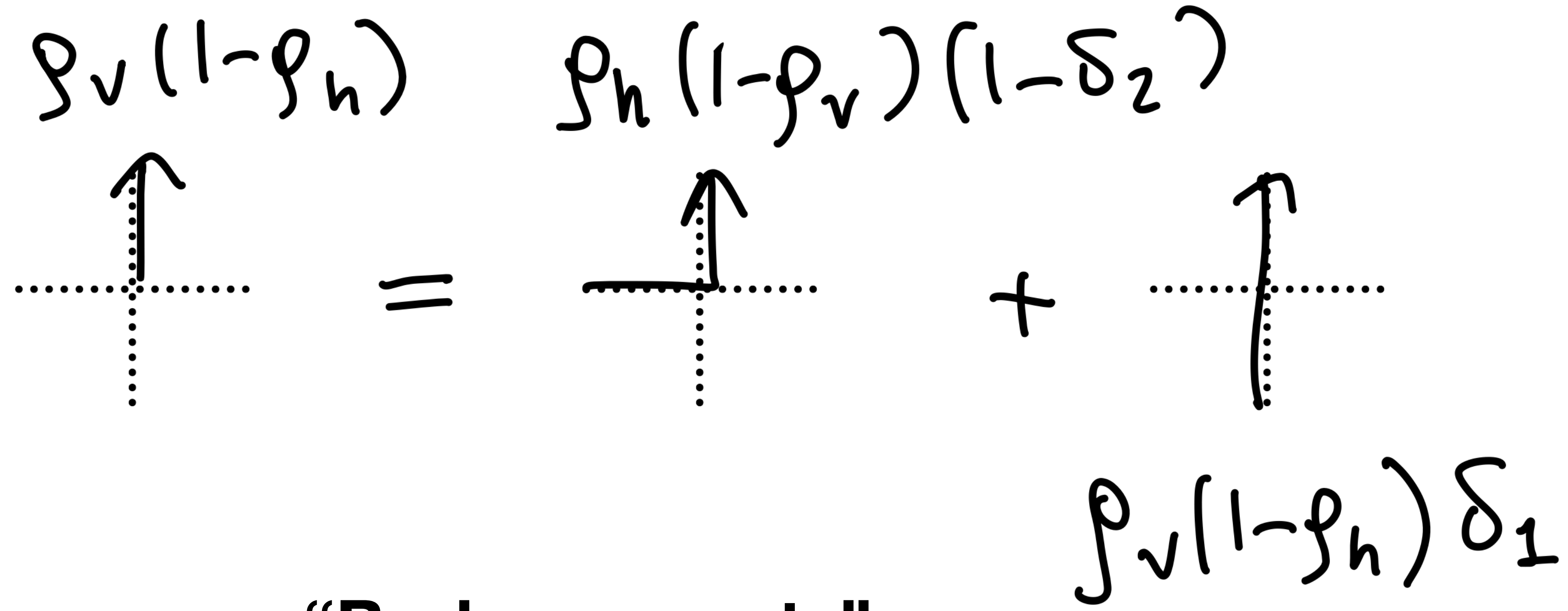
$$u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

The weights with $a_1 = a_2 = 1$, $b_1 = \delta_1$, $c_1 = 1-\delta_1$, $b_2 = \delta_2$, $c_2 = 1-\delta_2$ are *stochastic*: $\sum_{i_2,j_2} w(i_1,j_1;i_2,j_2) = 1$.

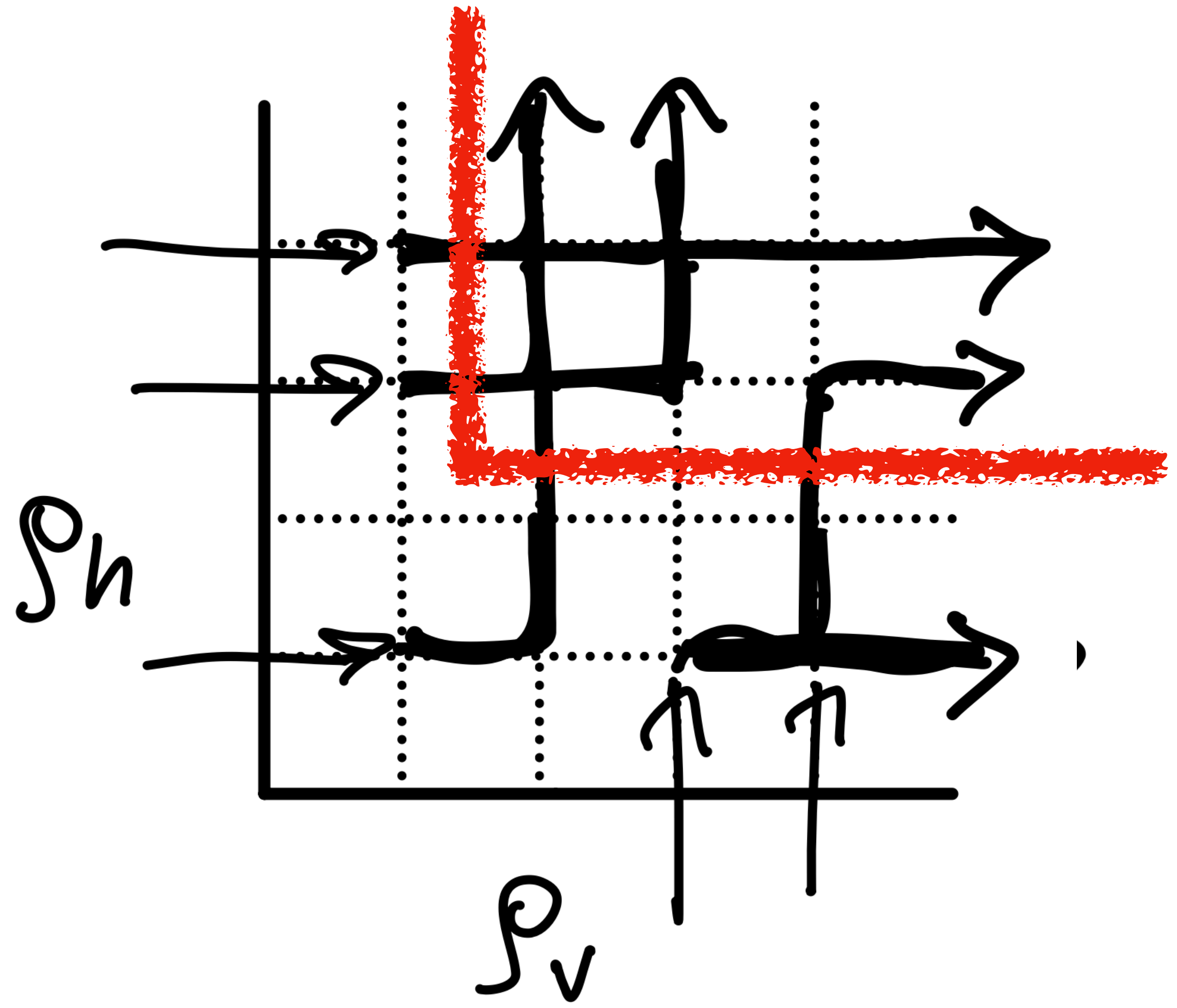
Converges to ASEP along the diagonal as $\delta_1, \delta_2 \to 0$ and q stays fixed (so, $u \to 1$)

- **Stationarity.** Assume that the boundary conditions are Bernoulli with densities ρ_h, ρ_v. Then for $\rho_h = \dfrac{u\rho_v}{1-\rho_v+u\rho_v}$, the distribution is stationary in the quadrant.



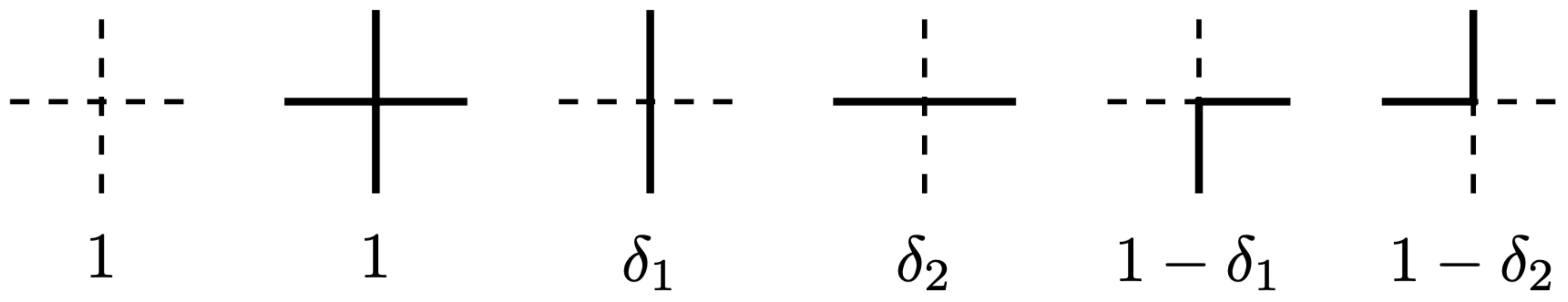


“Burke property”



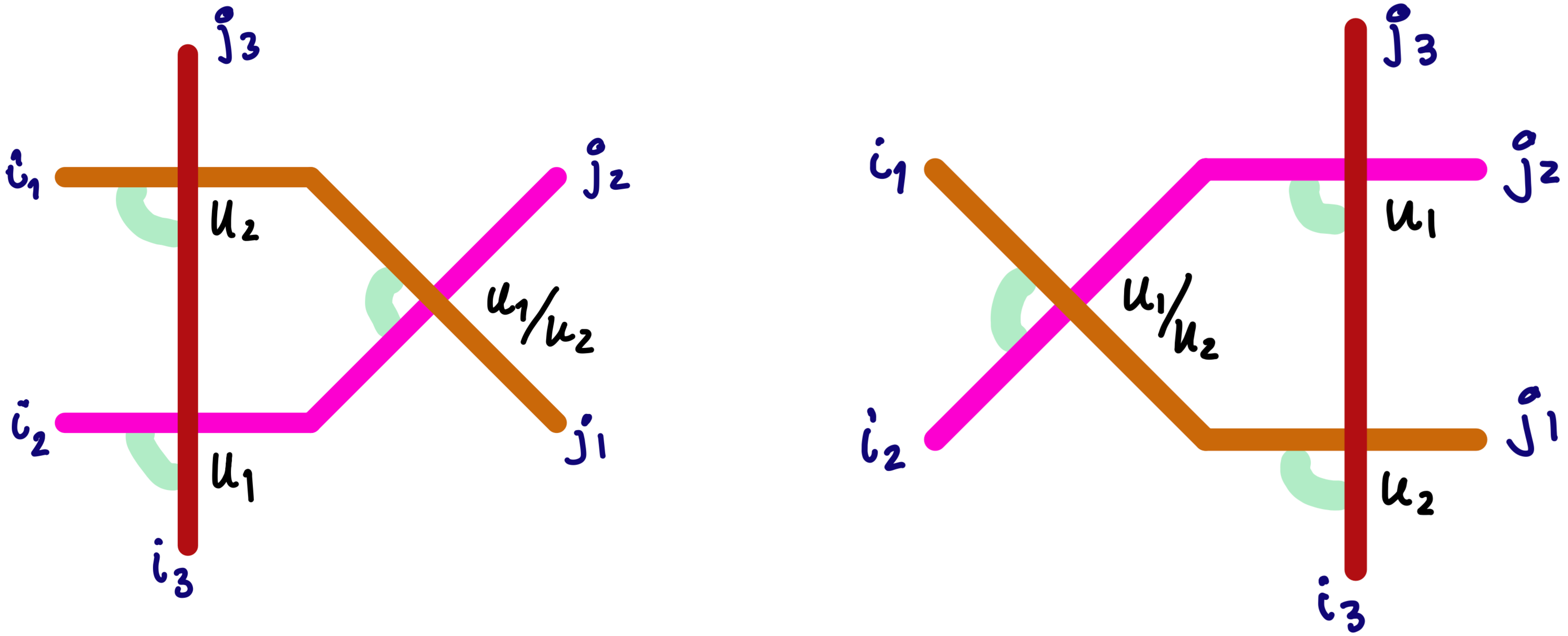
"Toy" example: stationarity for the single-color stochastic six-vertex model

[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

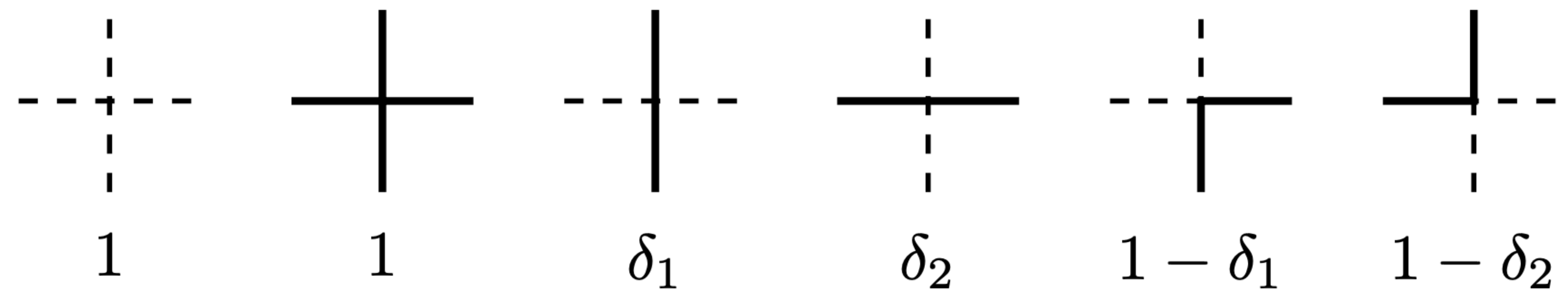


$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

- **Yang-Baxter equation.** For fixed q, and fixed $i_1, i_2, i_3 \in \{0,1\}$, the joint distribution of j_1, j_2, j_3 in two pictures is the same:



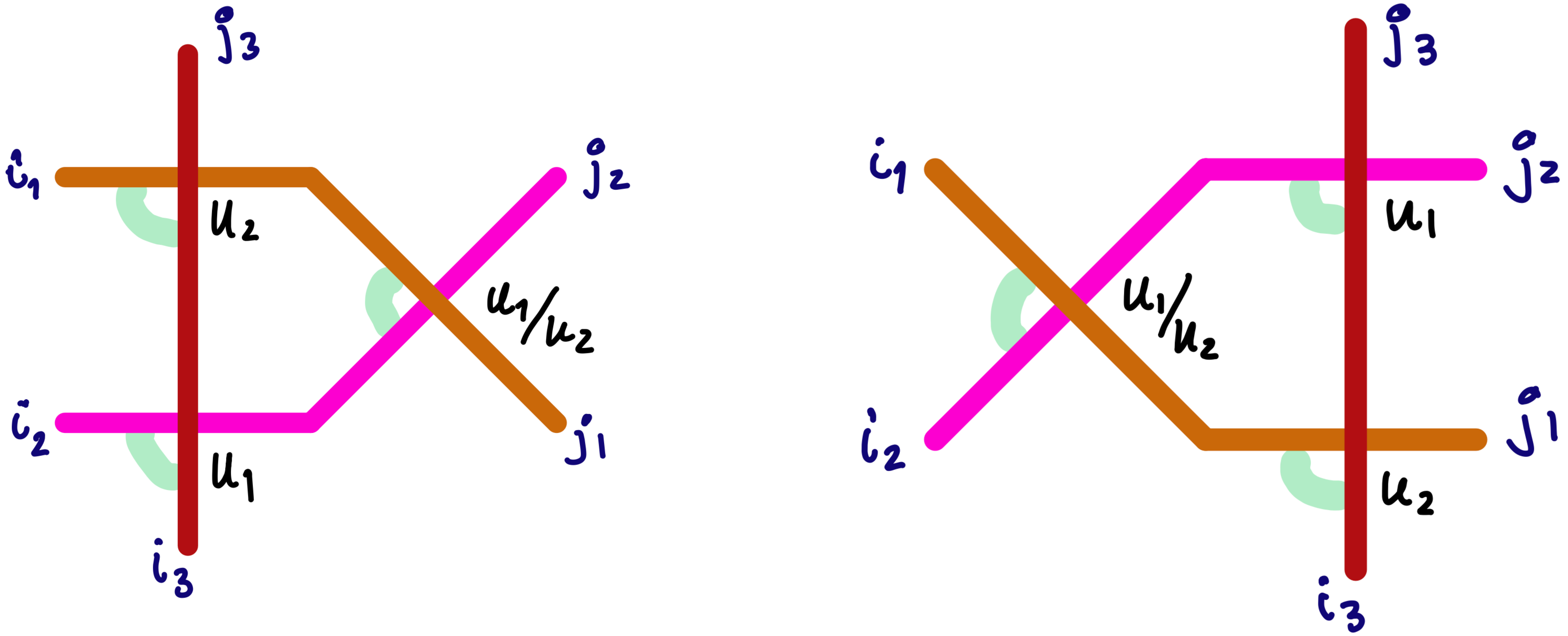
“Toy” example: stationarity for the single-color stochastic six-vertex model

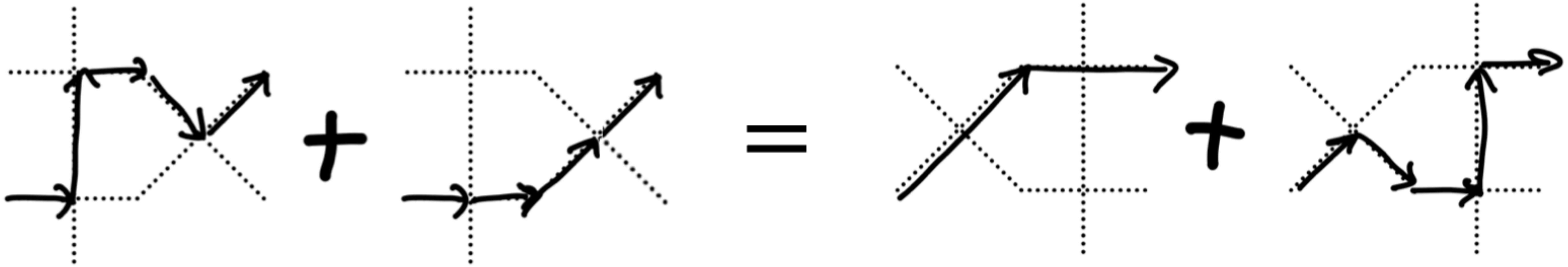


[Gwa-Spohn 1992], [Borodin-Corwin-Gorin 2014], [Aggarwal-Borodin 2016]

$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

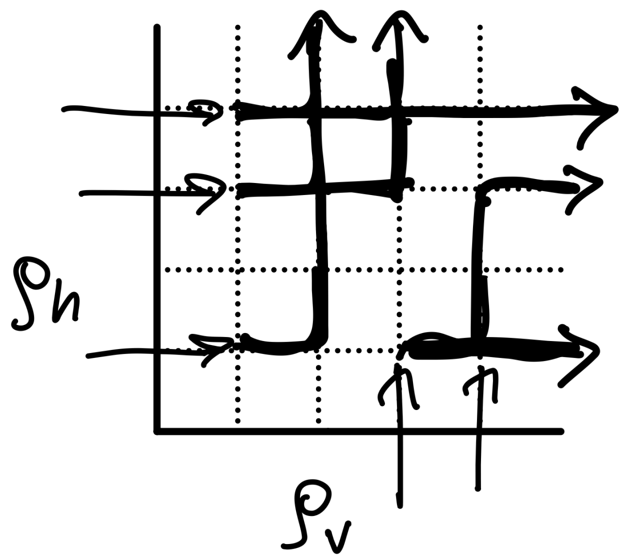
- **Yang-Baxter equation.** For fixed q, and fixed $i_1, i_2, i_3 \in \{0,1\}$, the joint distribution of j_1, j_2, j_3 in two pictures is the same:

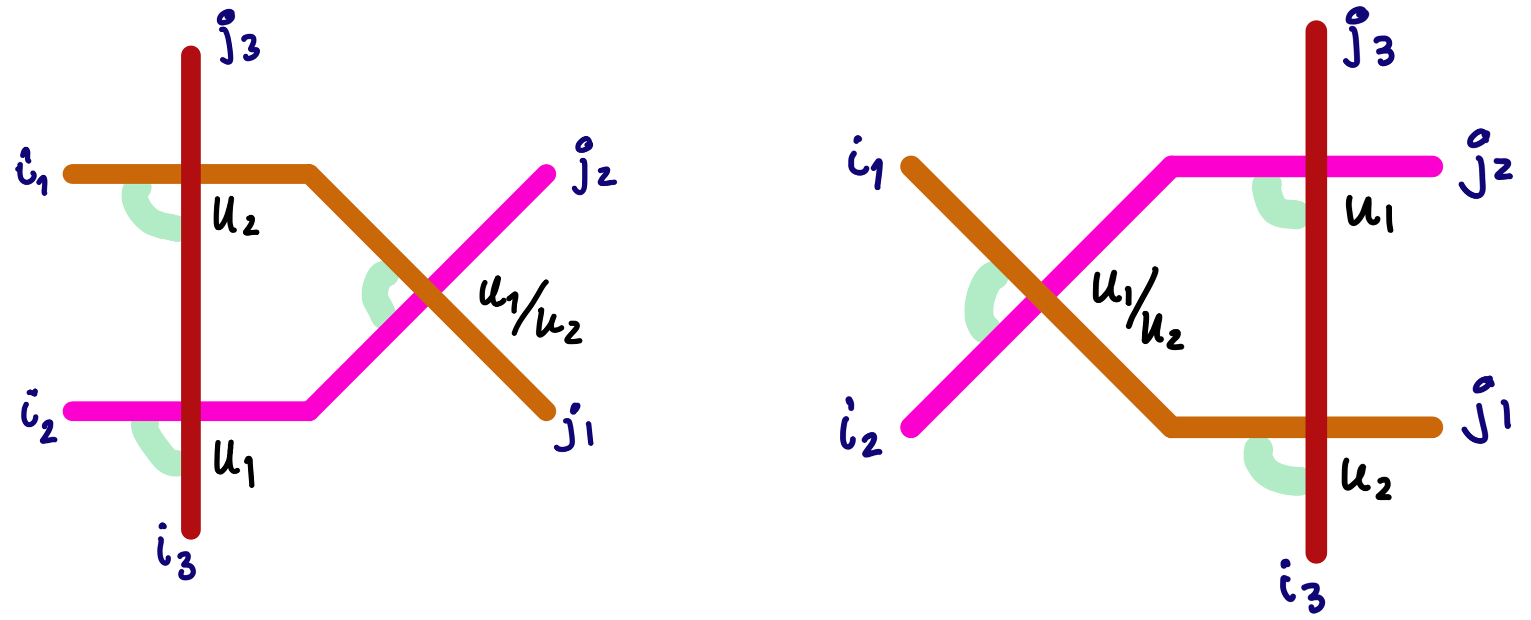




"Toy" example: stationarity for the single-color stochastic six-vertex model

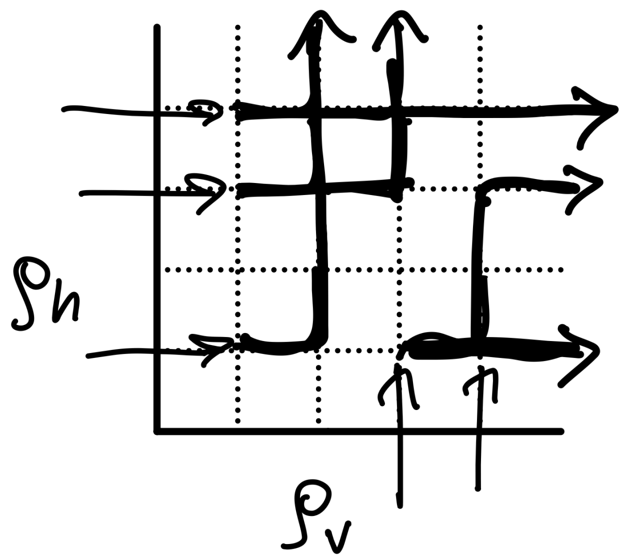
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

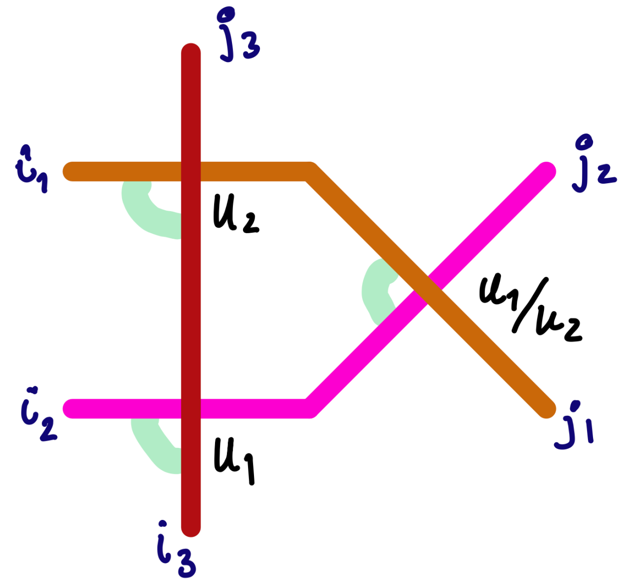


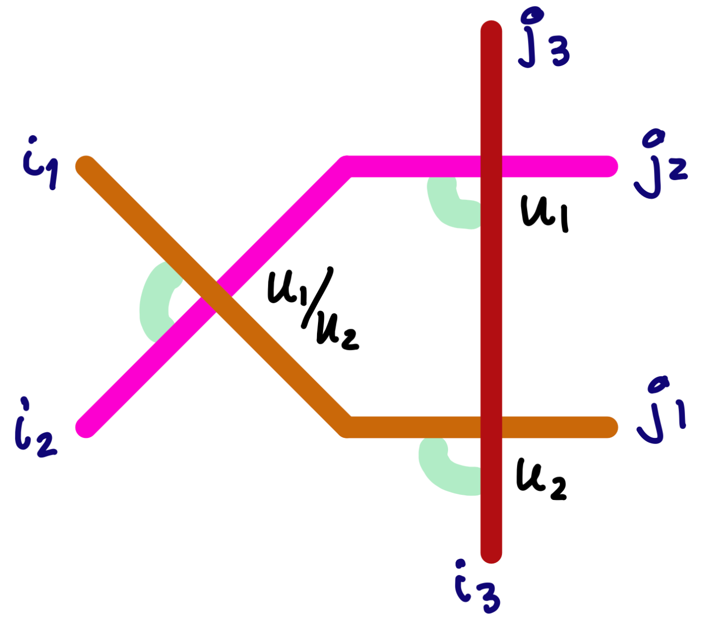


“Toy” example: stationarity for the single-color stochastic six-vertex model

$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



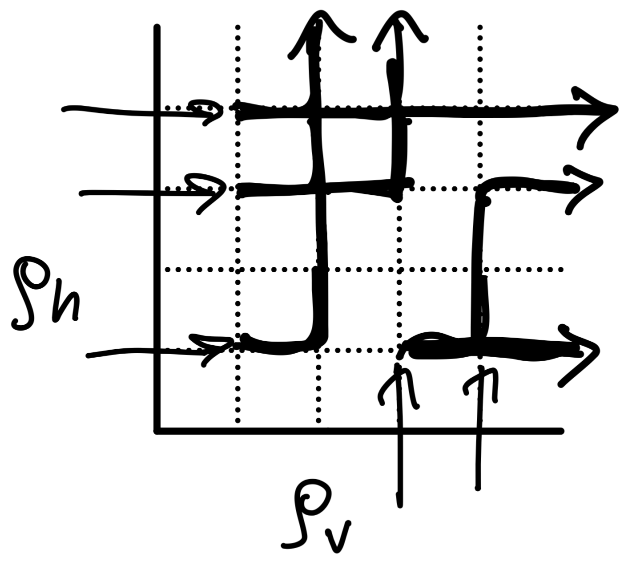


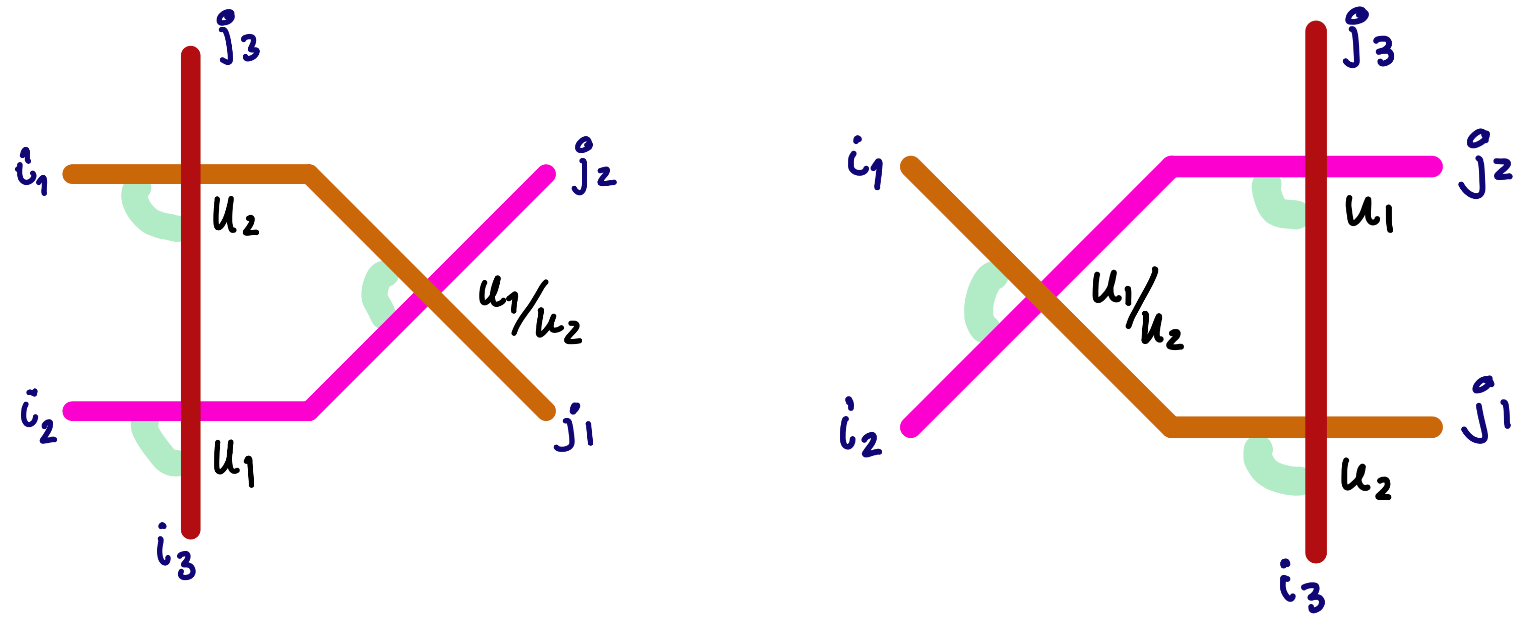


- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones.

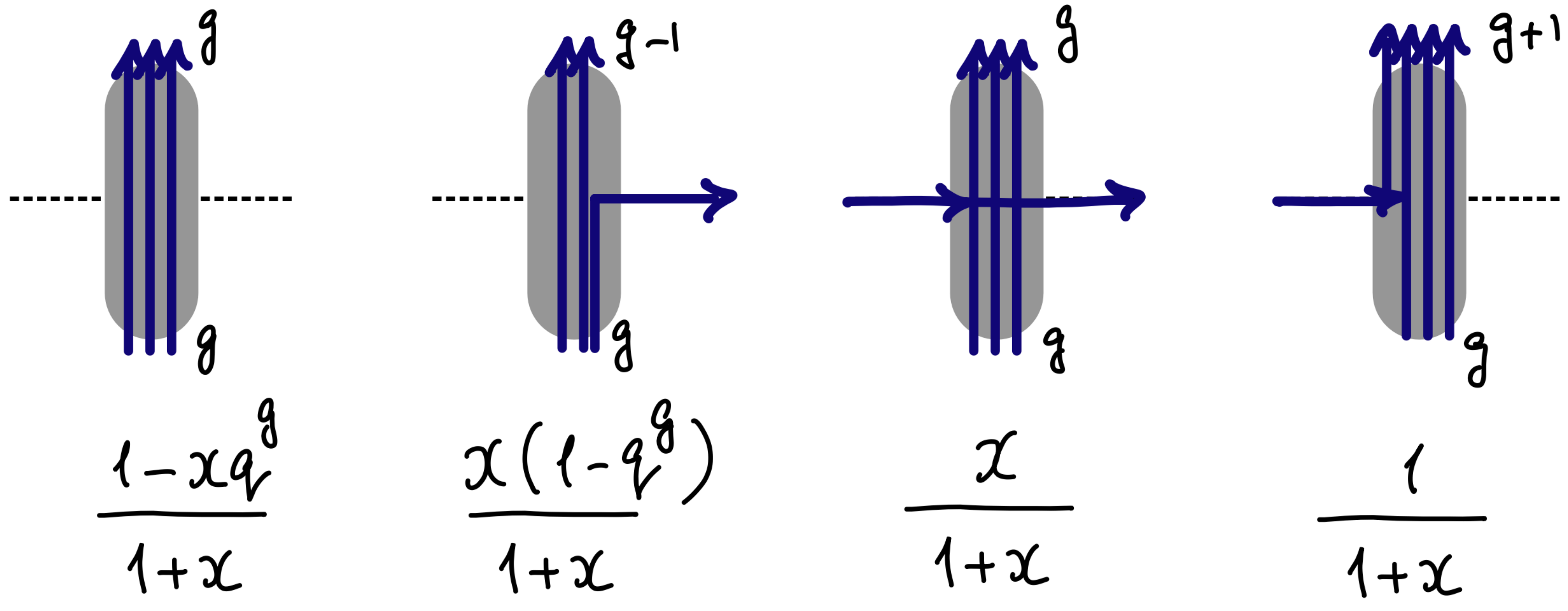
“Toy” example: stationarity for the single-color stochastic six-vertex model

$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



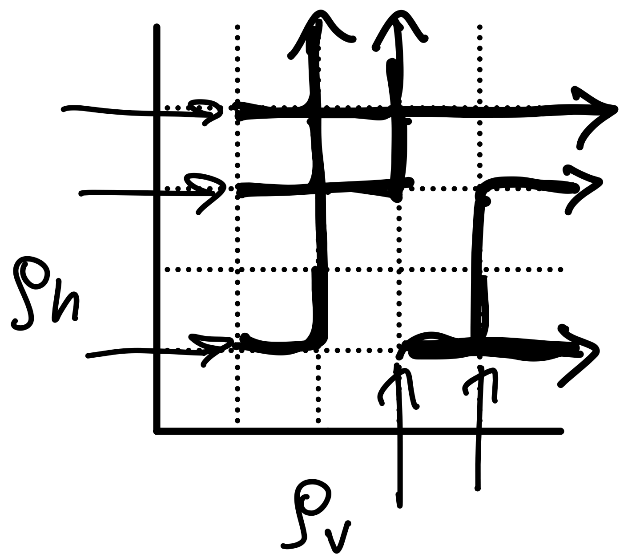


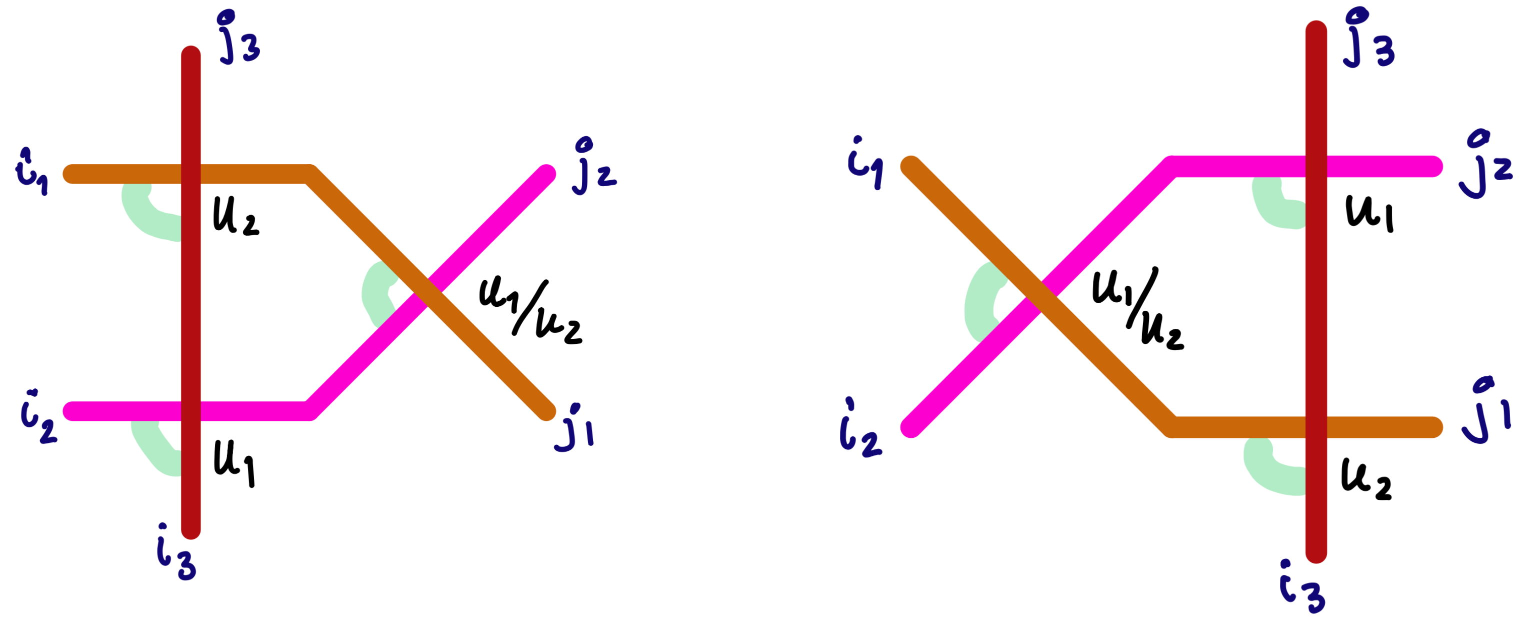
- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones.



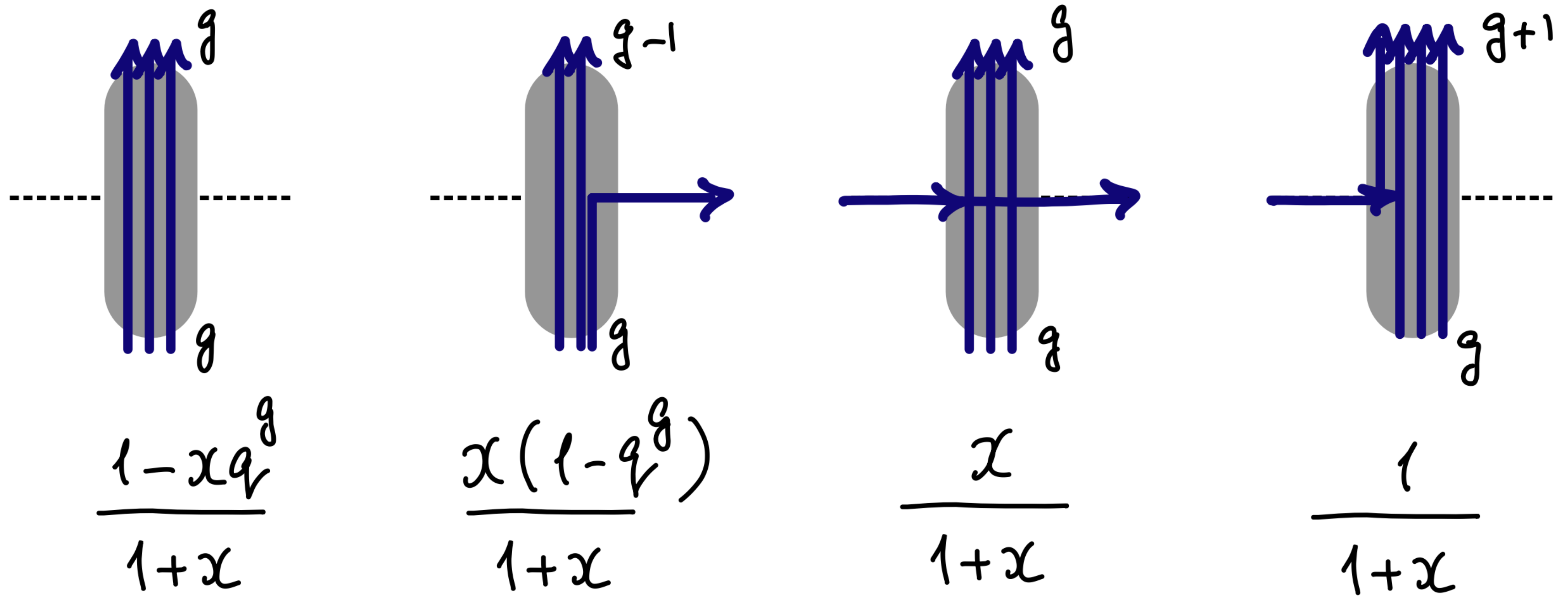
“Toy” example: stationarity for the single-color stochastic six-vertex model

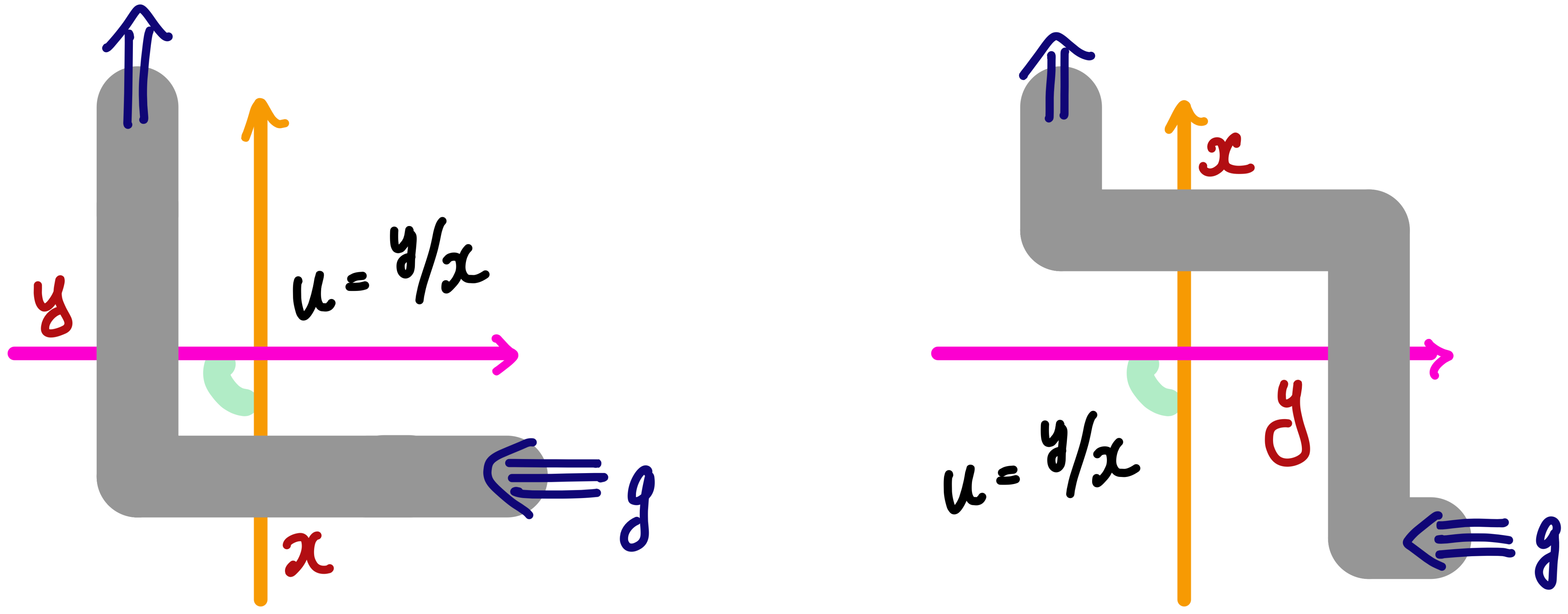
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$





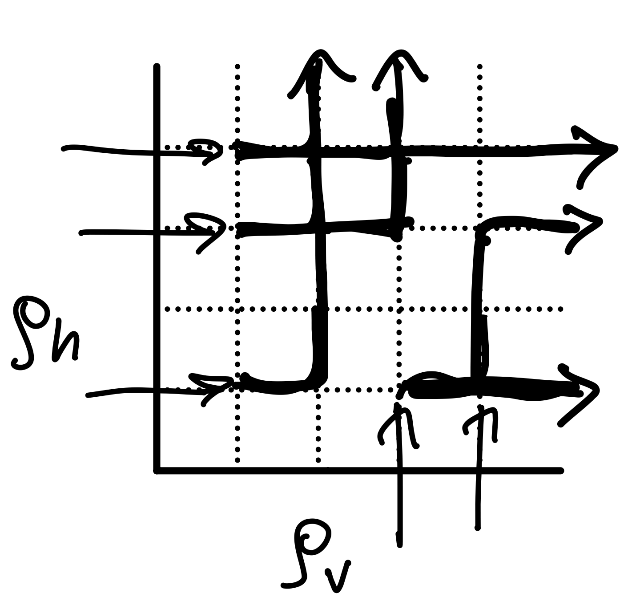
- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones.

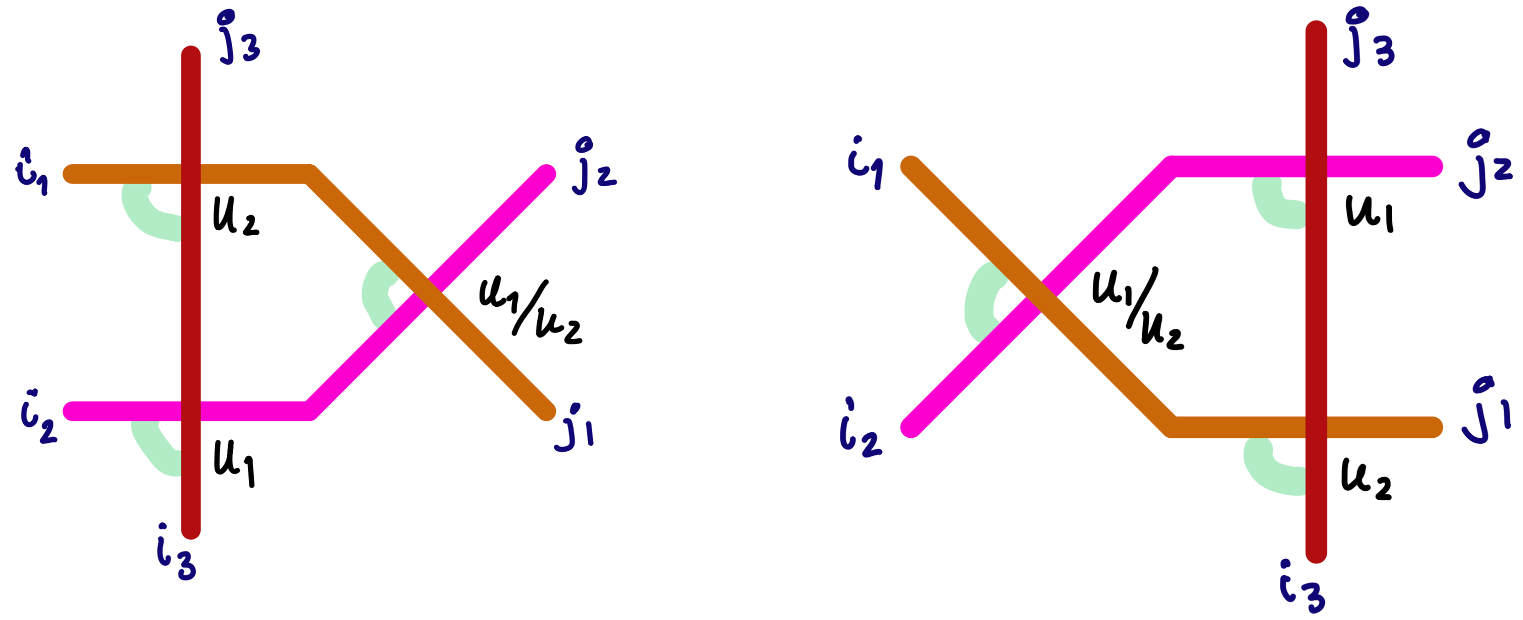




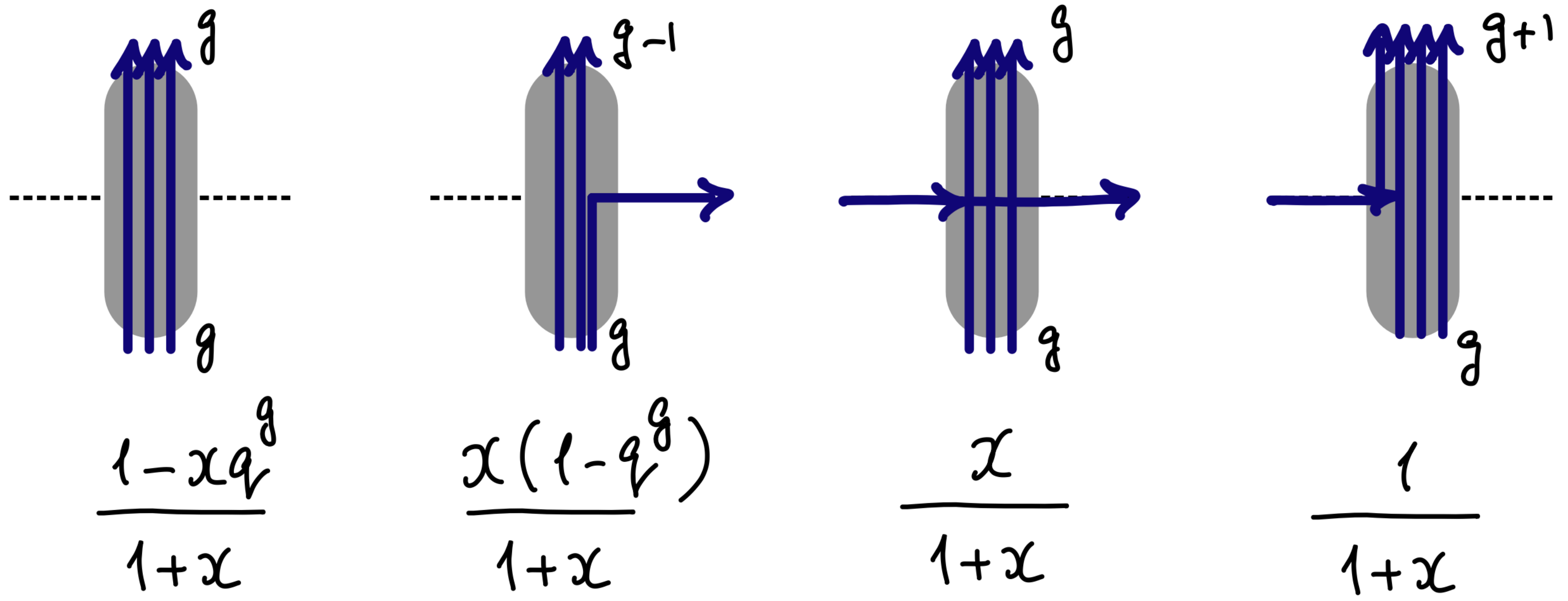
“Toy” example: stationarity for the single-color stochastic six-vertex model

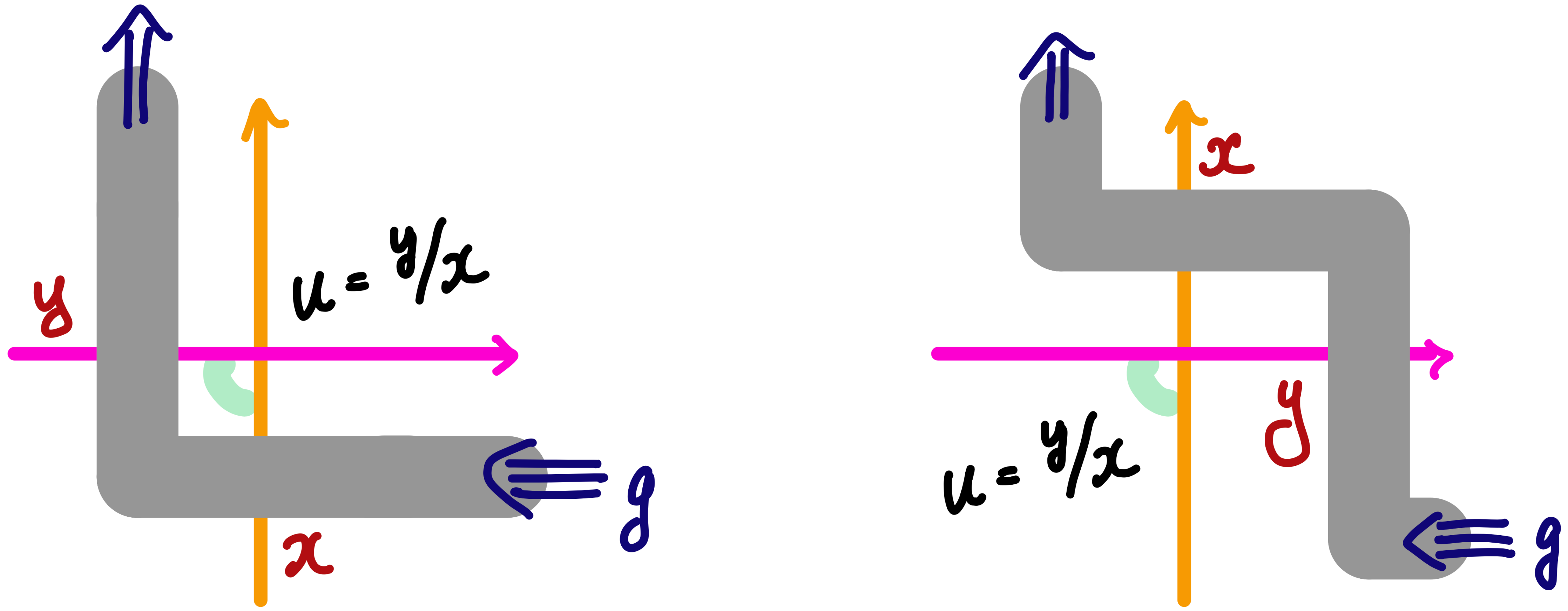
$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$





- **Fusion [Kulish-Reshetikhin-Sklyanin 1983], [Corwin-P. 2015]** - a way to construct new YBE solutions from existing ones.



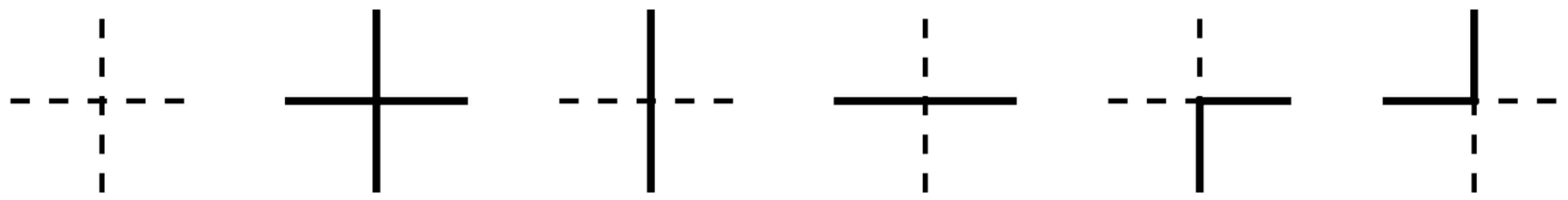


Stationarity via Yang-Baxter

- For $g = +\infty$, the right output of the fat vertex is $Bernoulli(\frac{x}{x+1})$, independent of the bottom and the left inputs.
- The Yang-Baxter equation is equivalent to the previous “Burke” computation: $\rho_v = \dfrac{x}{x+1}$,

$$\rho_h = \frac{ux}{ux+1} \;\Rightarrow\; \rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v}.$$

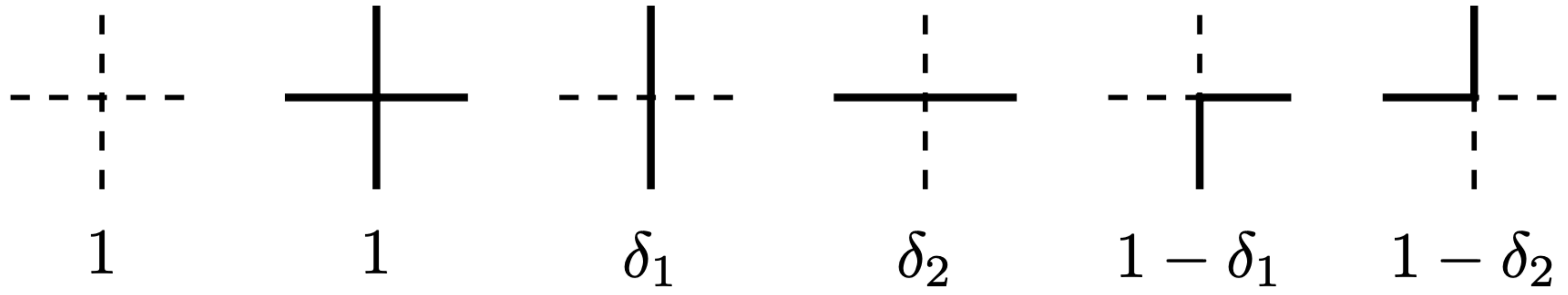
“Toy” example: stationarity for the single-color stochastic six-vertex model



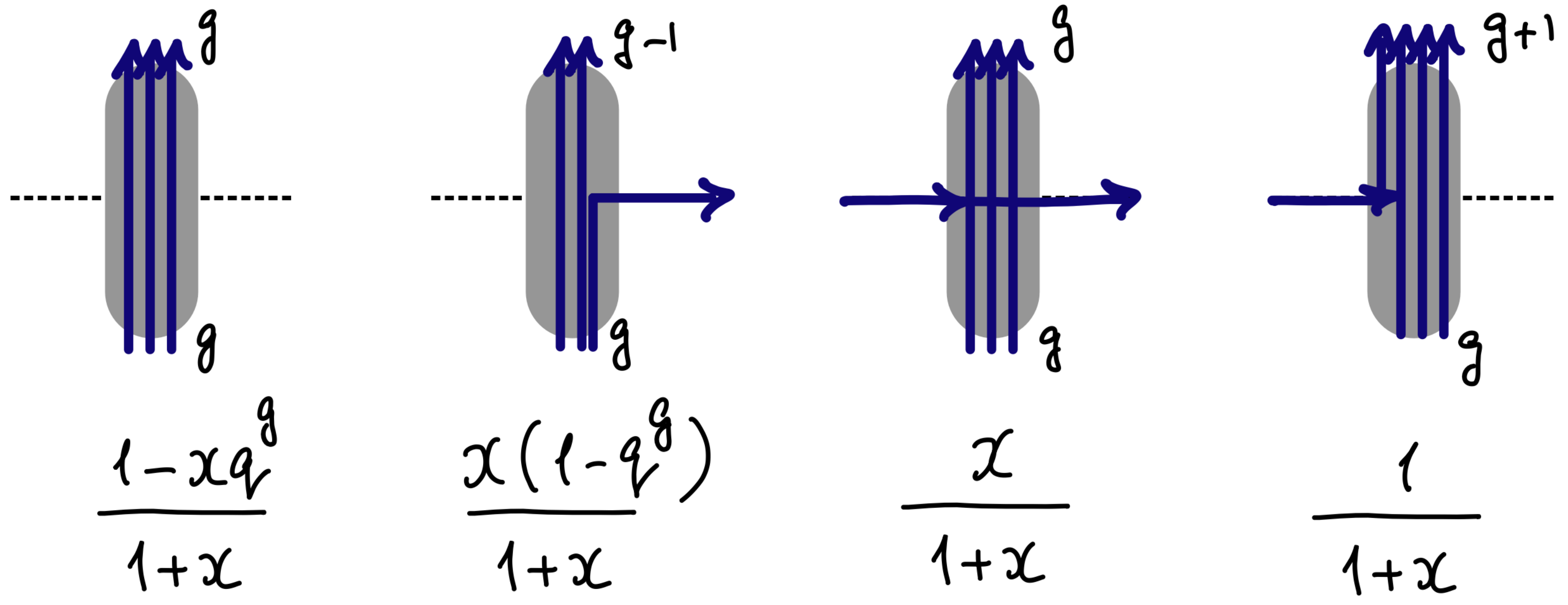
$1 \quad 1 \quad \delta_1 \quad \delta_2 \quad 1-\delta_1 \quad 1-\delta_2$

$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

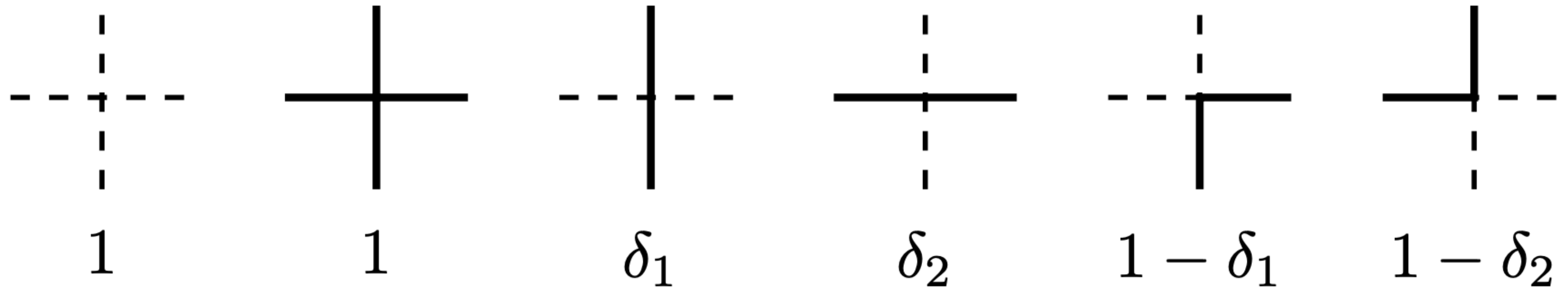
“Toy” example: stationarity for the single-color stochastic six-vertex model



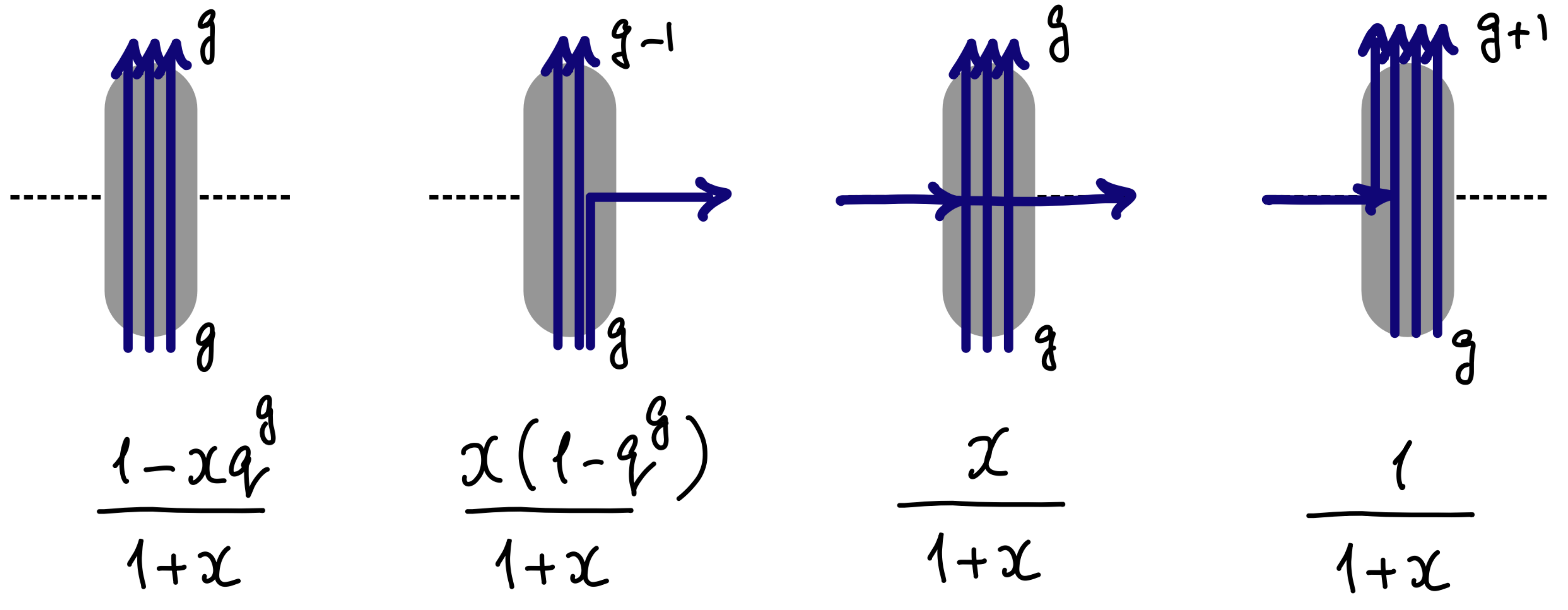
$$\rho_h = \frac{u\rho_v}{1 - \rho_v + u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

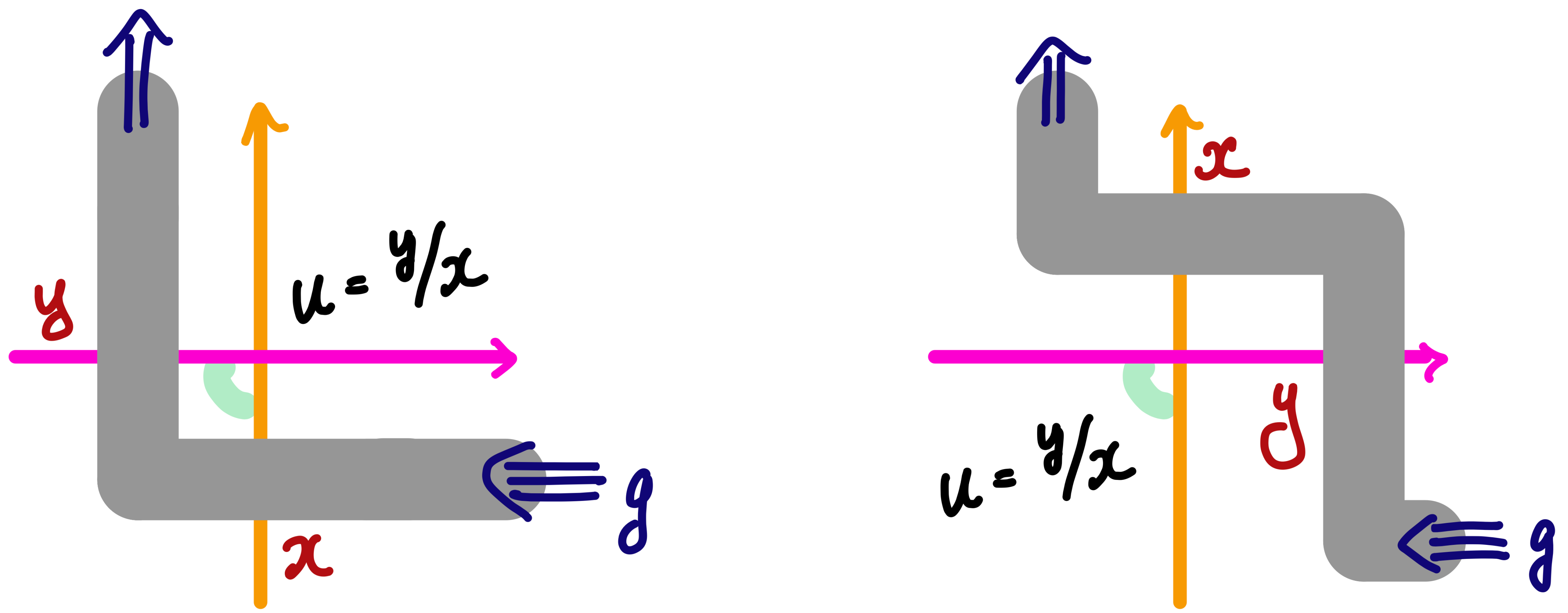


"Toy" example: stationarity for the single-color stochastic six-vertex model

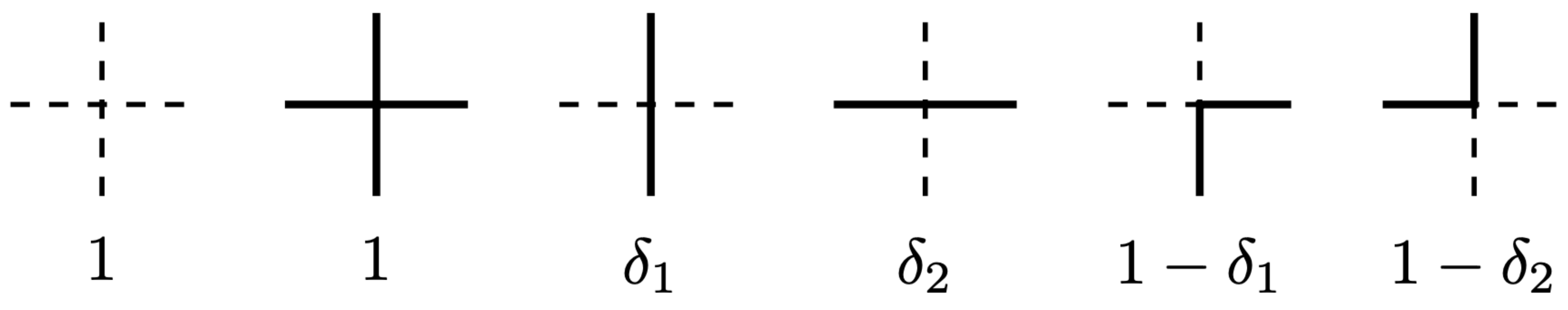


$$\rho_h = \frac{u\rho_v}{1 - \rho_v + u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$

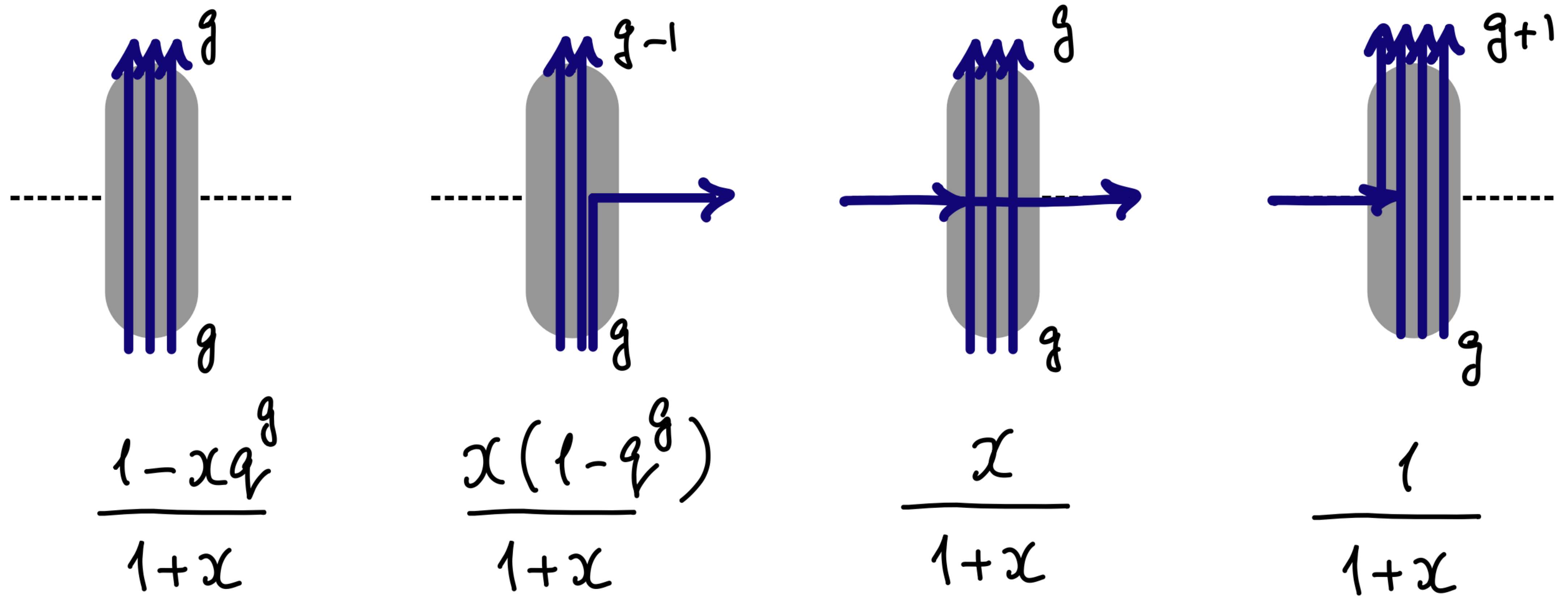


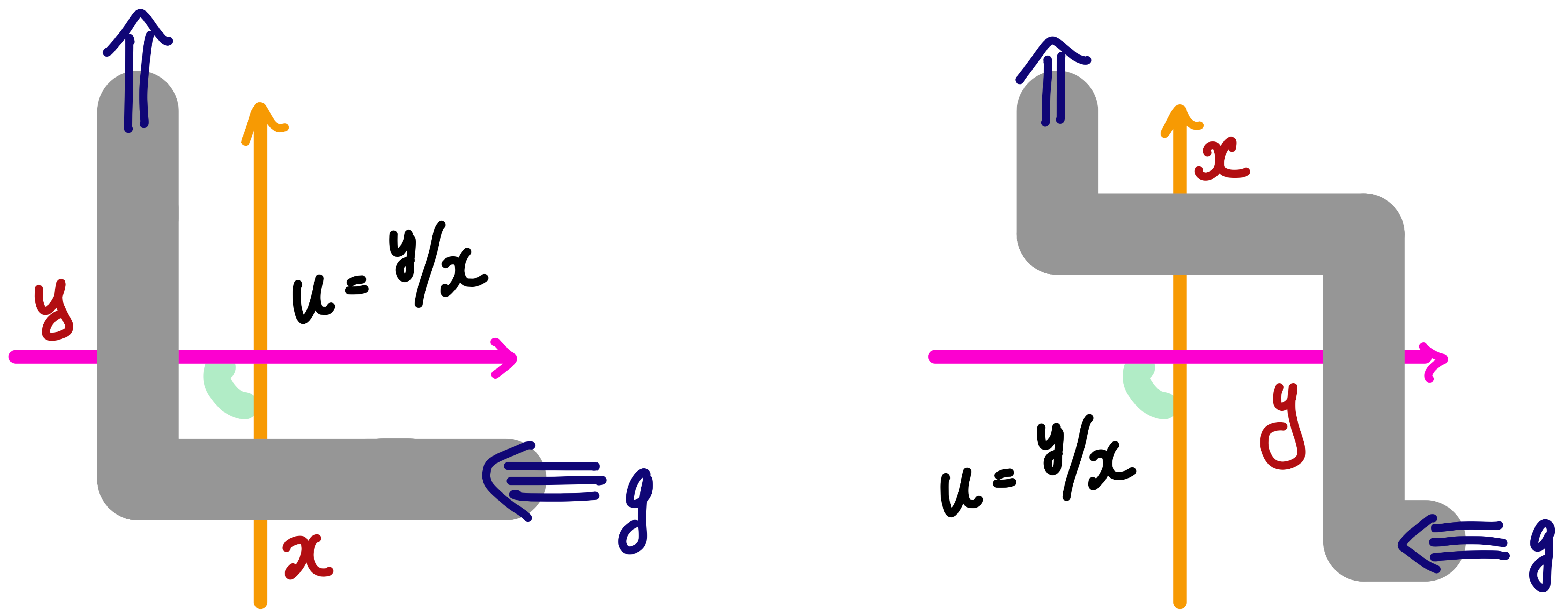


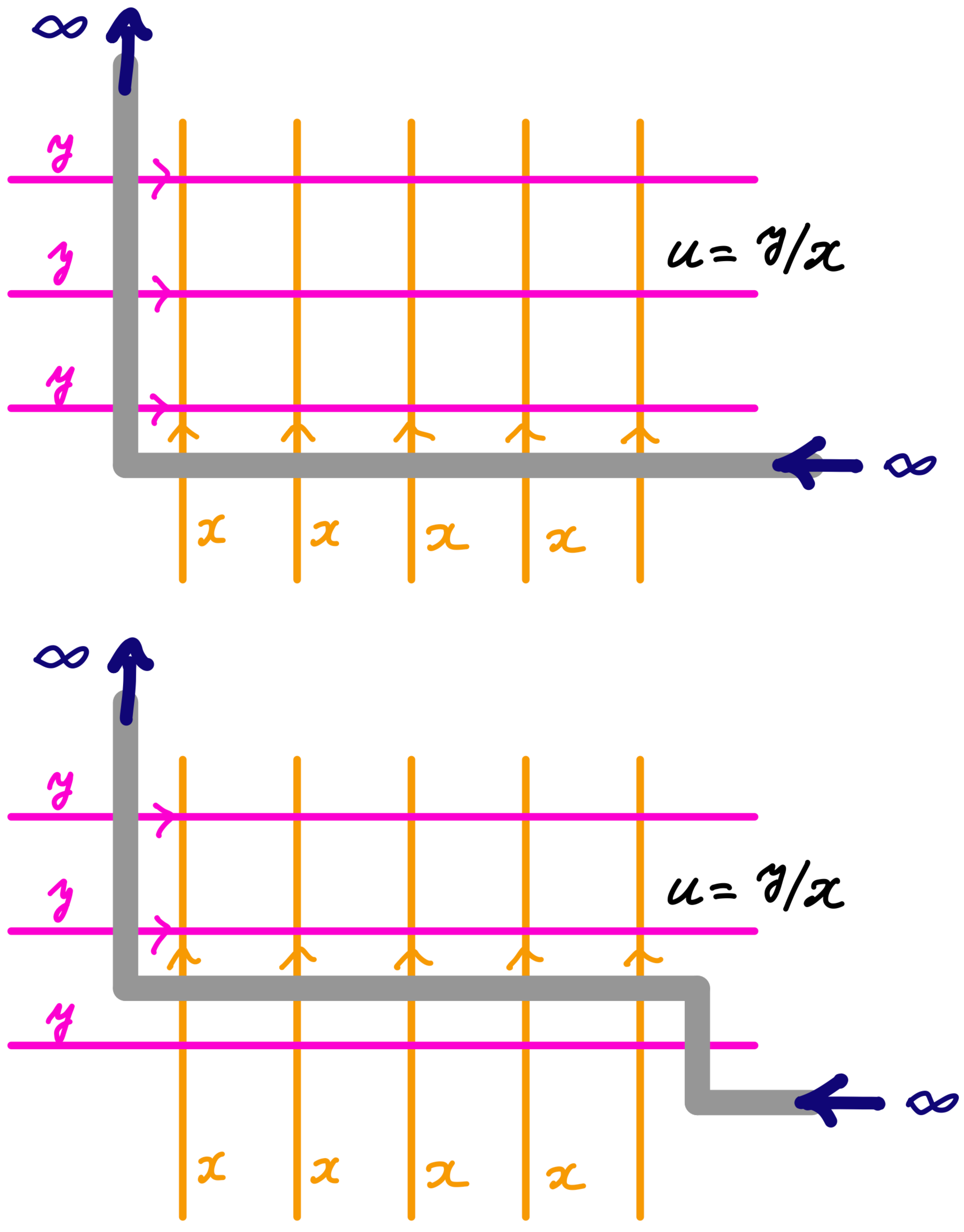
"Toy" example: stationarity for the single-color stochastic six-vertex model



$$\rho_h = \frac{u\rho_v}{1-\rho_v+u\rho_v} \qquad u := \frac{1-\delta_1}{1-\delta_2}, \qquad q := \delta_1/\delta_2$$



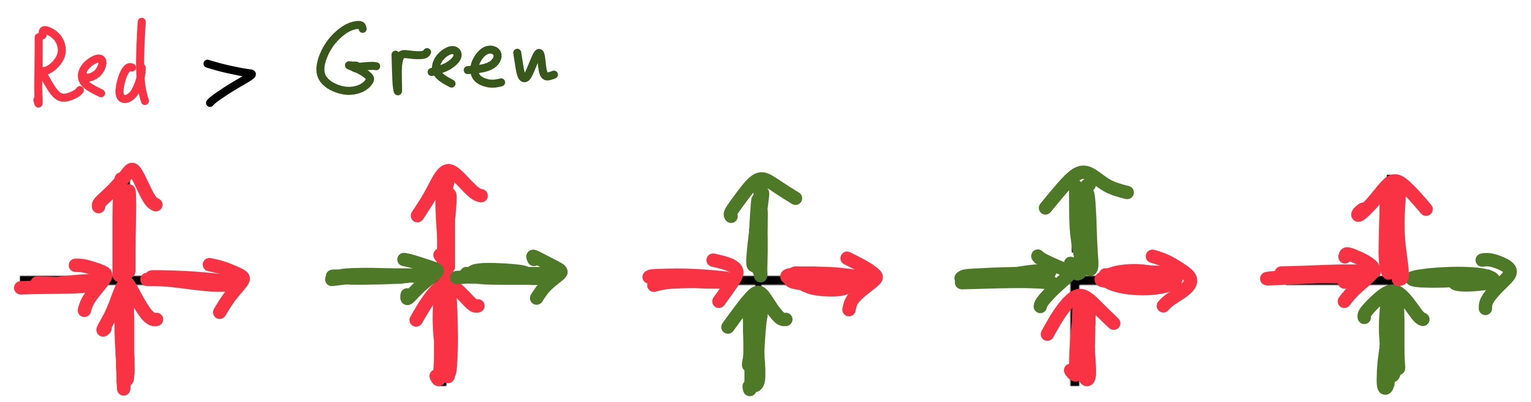




Stationarity from Yang-Baxter equation

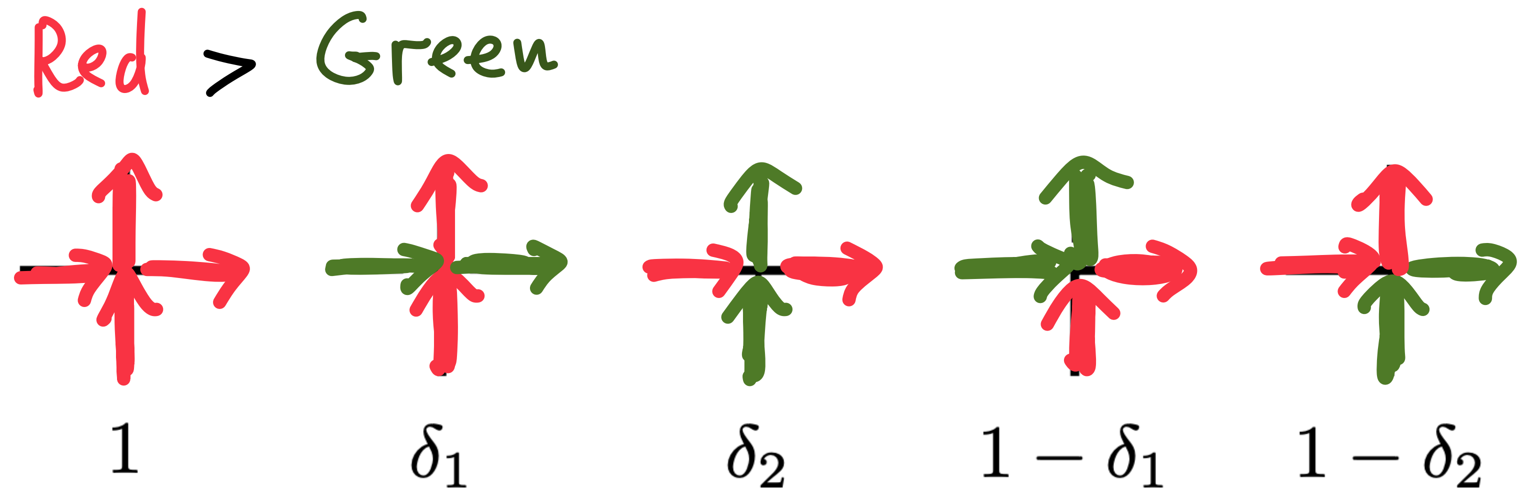
Colored stochastic six-vertex model in the quarter plane

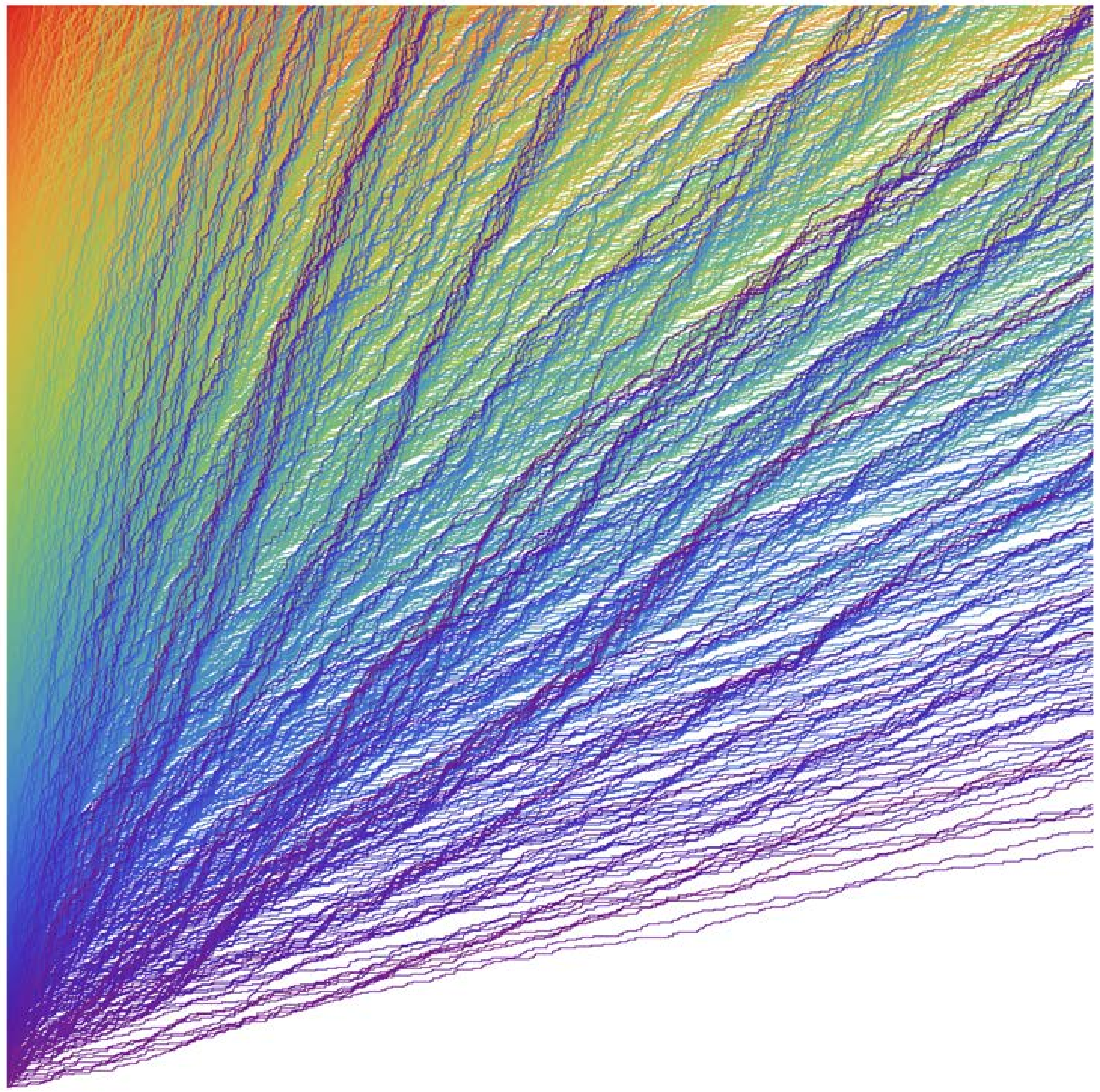
Colored stochastic six-vertex model. Many colors ⇒ many fat lines



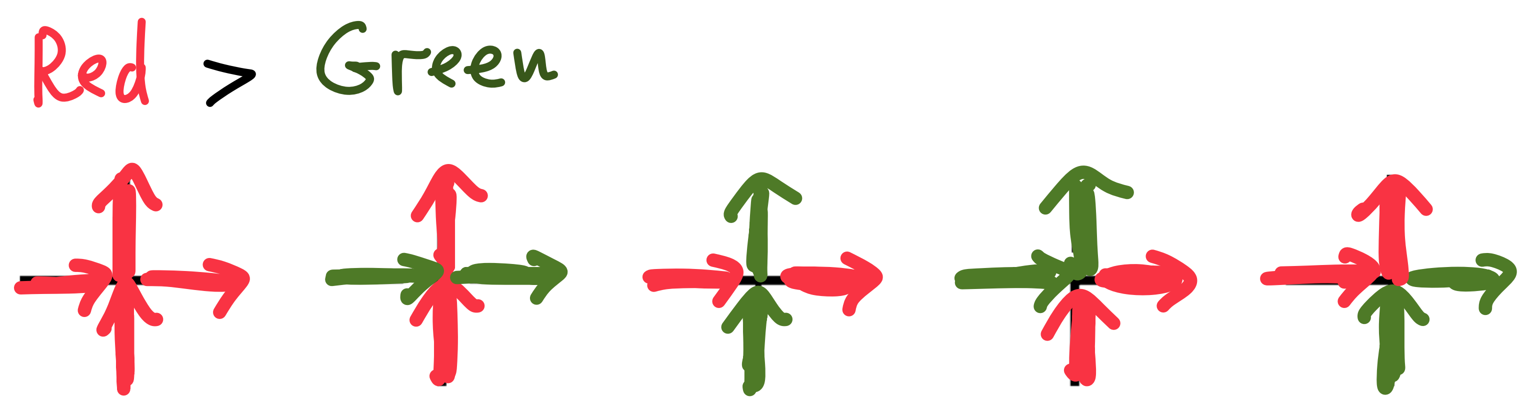
$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

Colored stochastic six-vertex model. Many colors ⇒ many fat lines



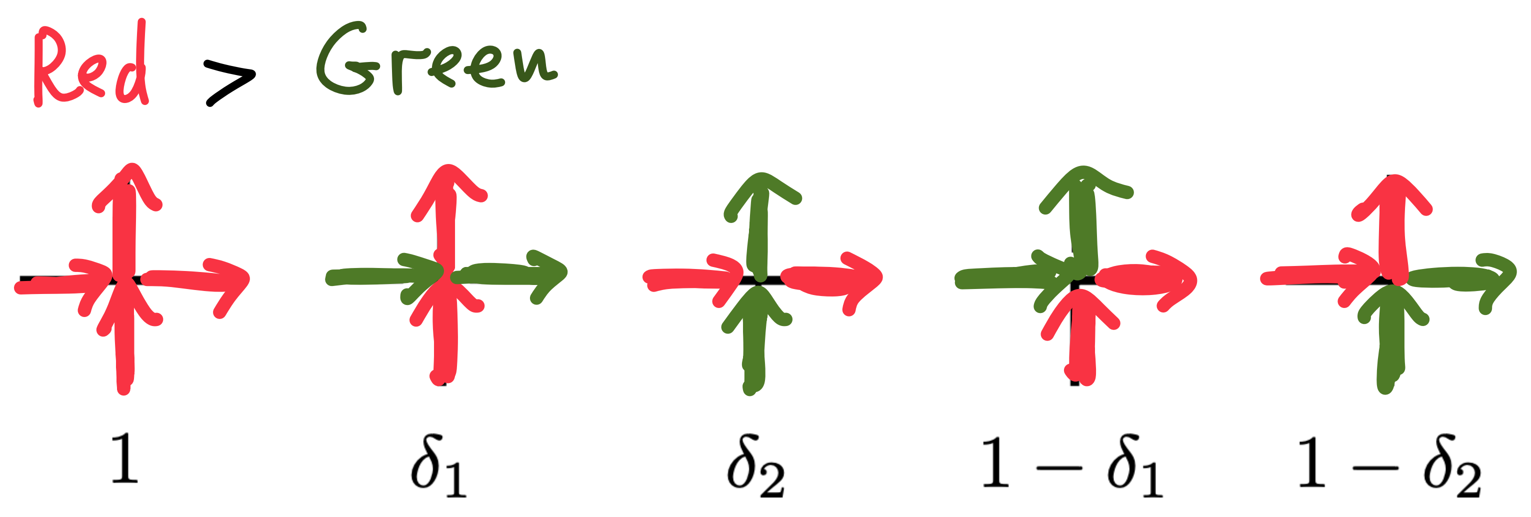


Colored stochastic six-vertex model. Many colors ⇒ many fat lines



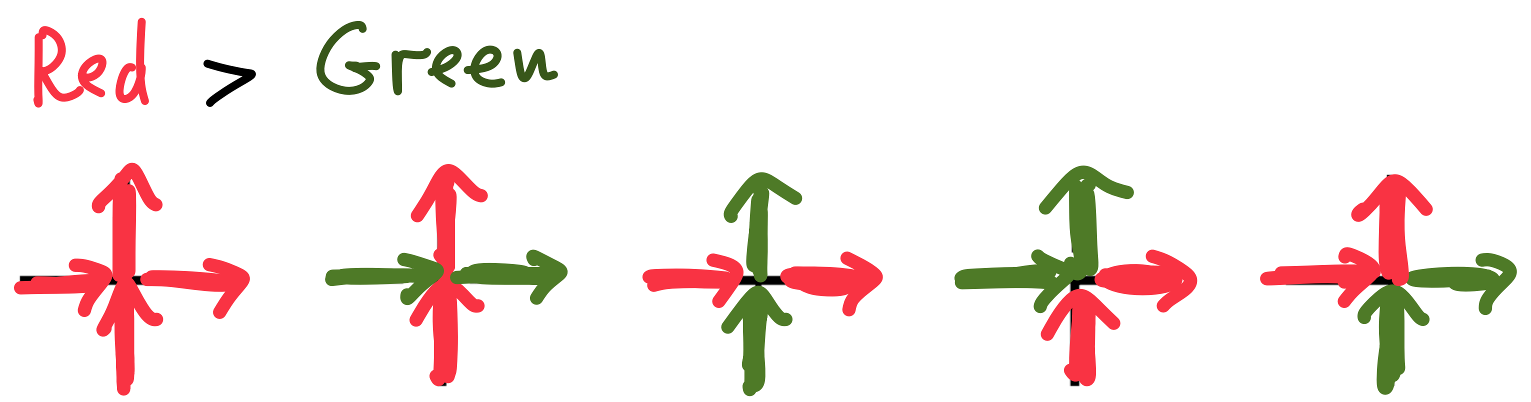
1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

Colored stochastic six-vertex model. Many colors ⇒ many fat lines



- **Fusion and Yang-Baxter equation.**

Colored stochastic six-vertex model. Many colors ⇒ many fat lines

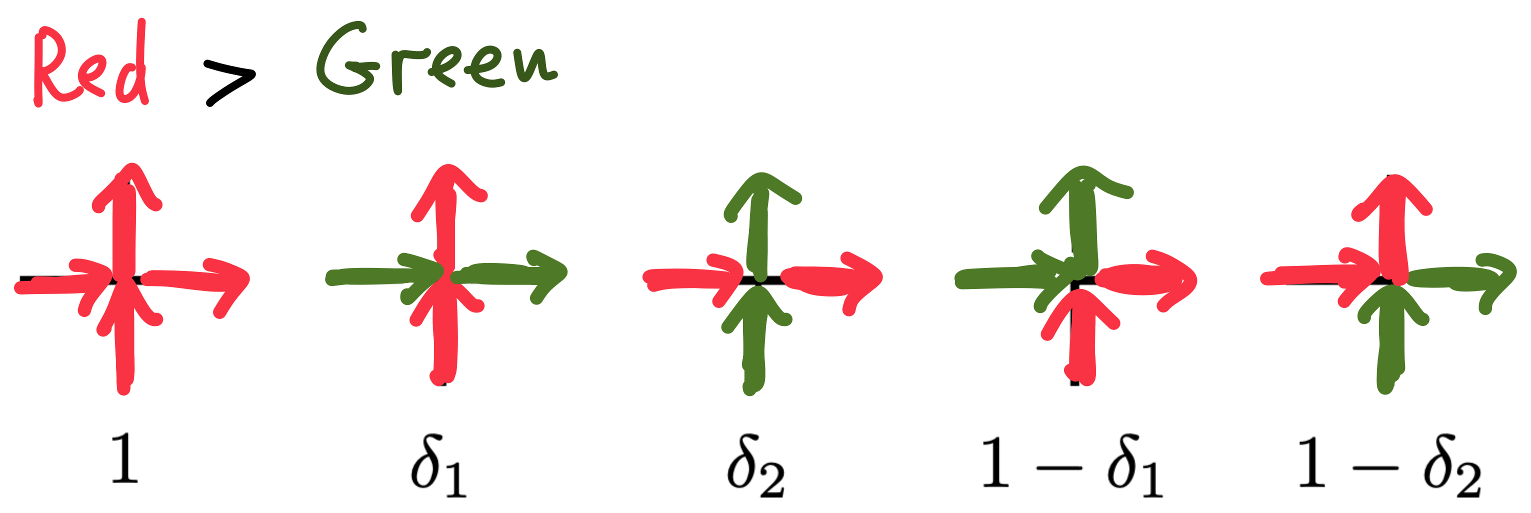


1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

- **Fusion and Yang-Baxter equation.**

top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$; $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$; $\dfrac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$; $\dfrac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$; $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$; $\dfrac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$; $\dfrac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

Colored stochastic six-vertex model. Many colors ⇒ many fat lines



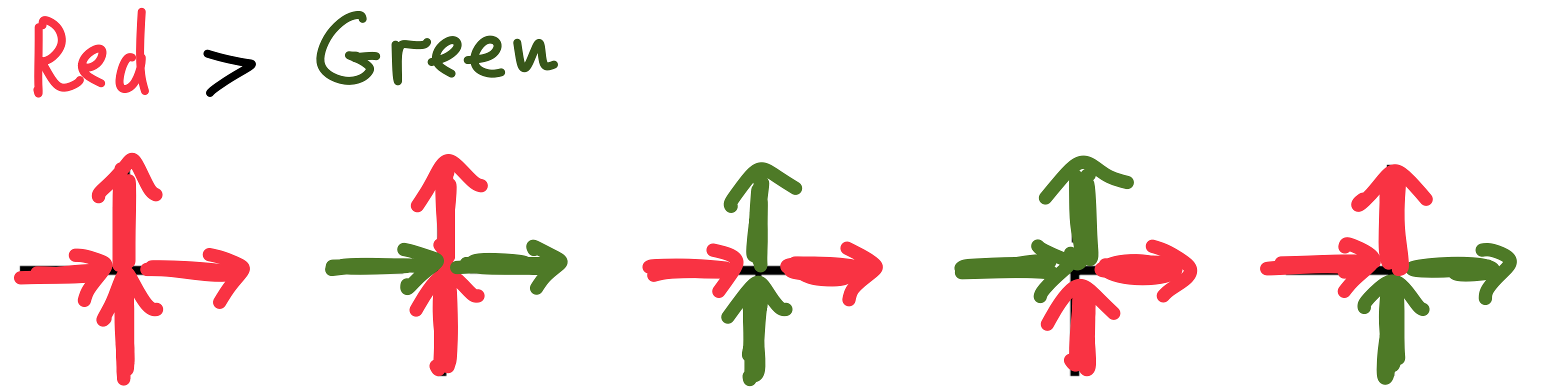
$\mathbf{A}$; 0 — 0; $\mathbf{A}$ $\frac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	$\mathbf{A}$; k — k; $\mathbf{A}$ $\frac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	$\mathbf{A}_k^-$; 0 — k; $\mathbf{A}$ $\frac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
$\mathbf{A}_k^+$; k — 0; $\mathbf{A}$ $\frac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	$\mathbf{A}_{k\ell}^{+-}$; k — ℓ; $\mathbf{A}$ $\frac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	$\mathbf{A}_{\ell k}^{+-}$; ℓ — k; $\mathbf{A}$ $\frac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.

  Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$

Colored stochastic six-vertex model. Many colors ⇒ many fat lines



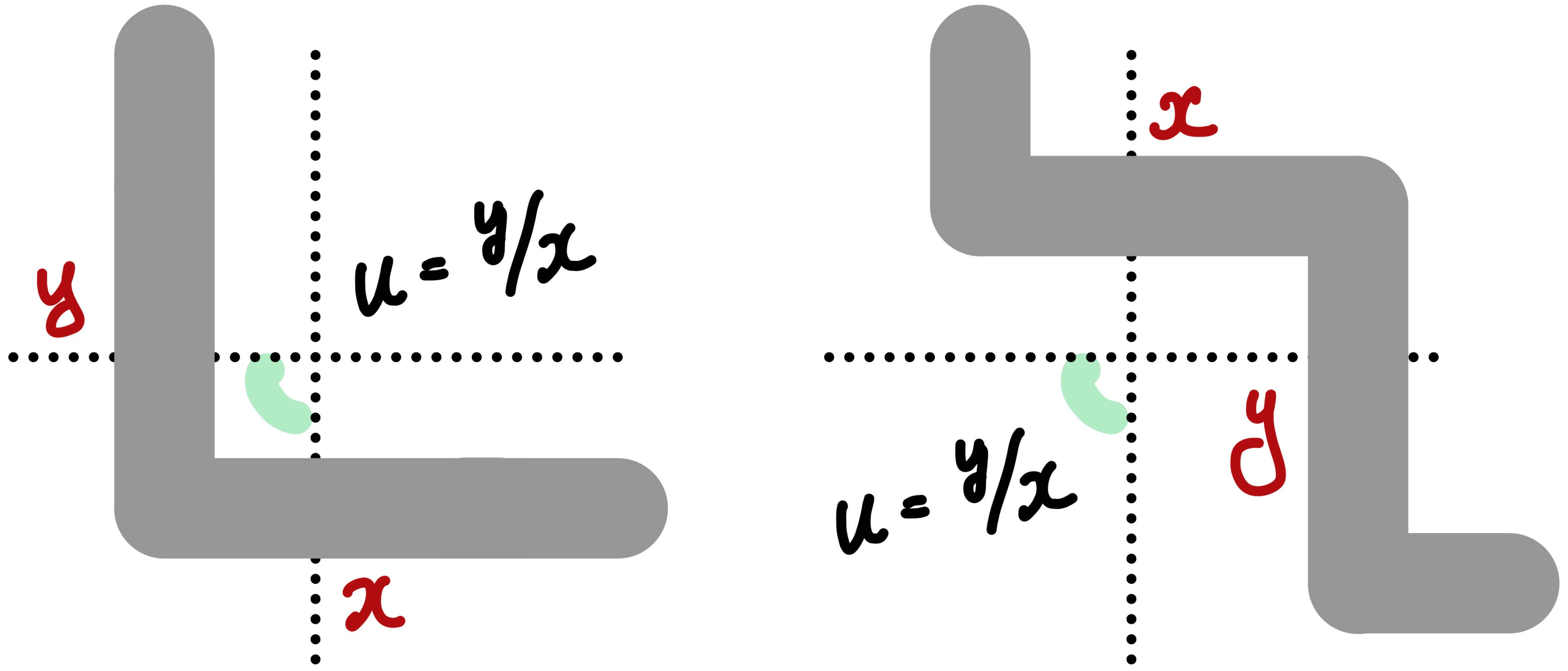
1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$: $\frac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$: $\frac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$: $\frac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$: $\frac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$: $\frac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$: $\frac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

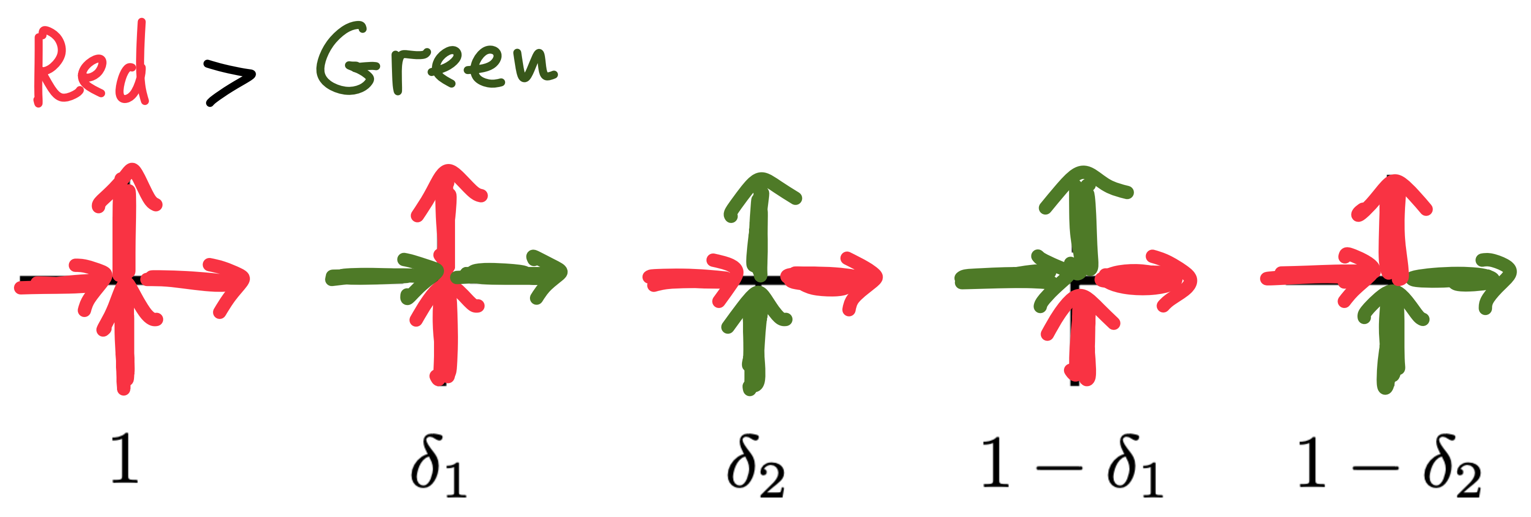
- **Fusion and Yang-Baxter equation.**

Higher spin, higher rank stochastic weights.

Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$



Colored stochastic six-vertex model. Many colors ⇒ many fat lines

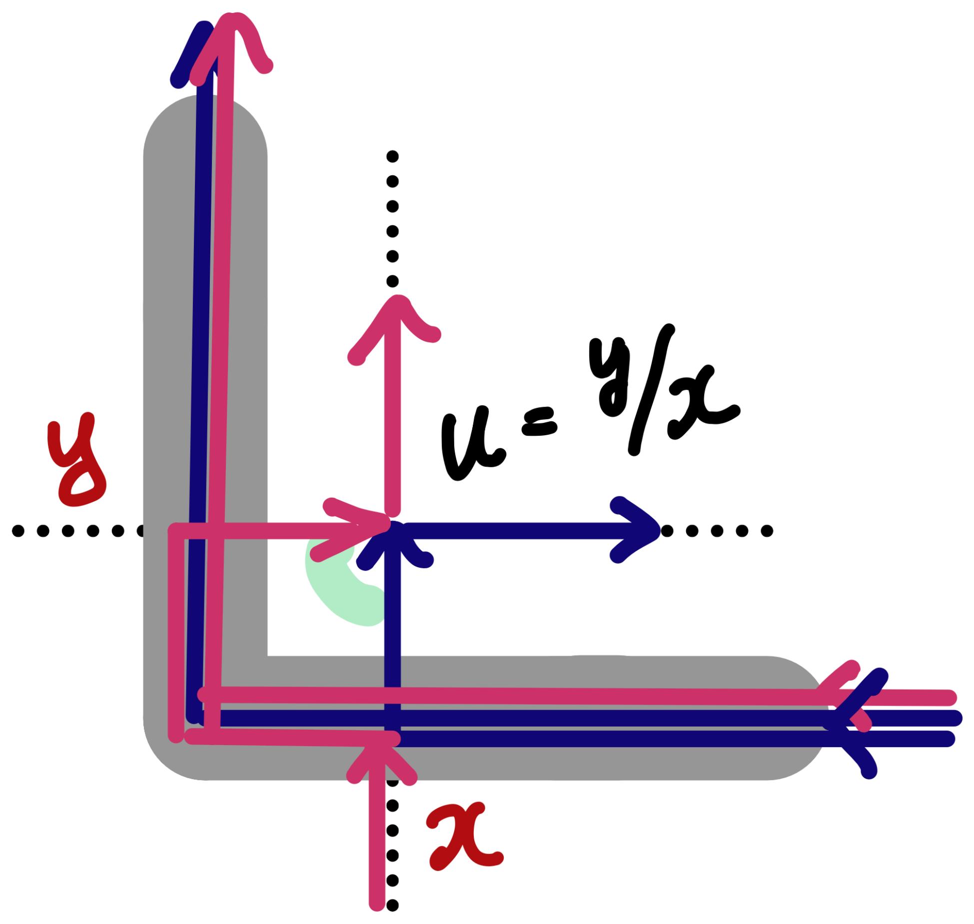


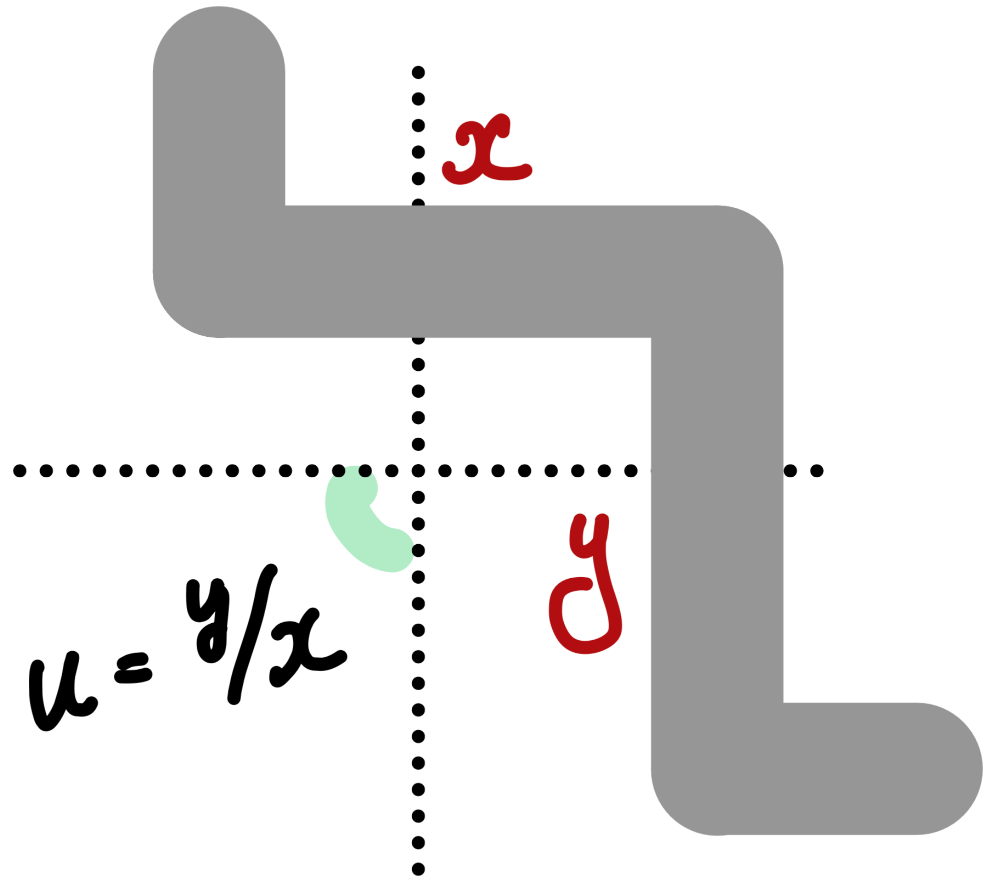
- **Fusion and Yang-Baxter equation.**

Higher spin, higher rank stochastic weights.

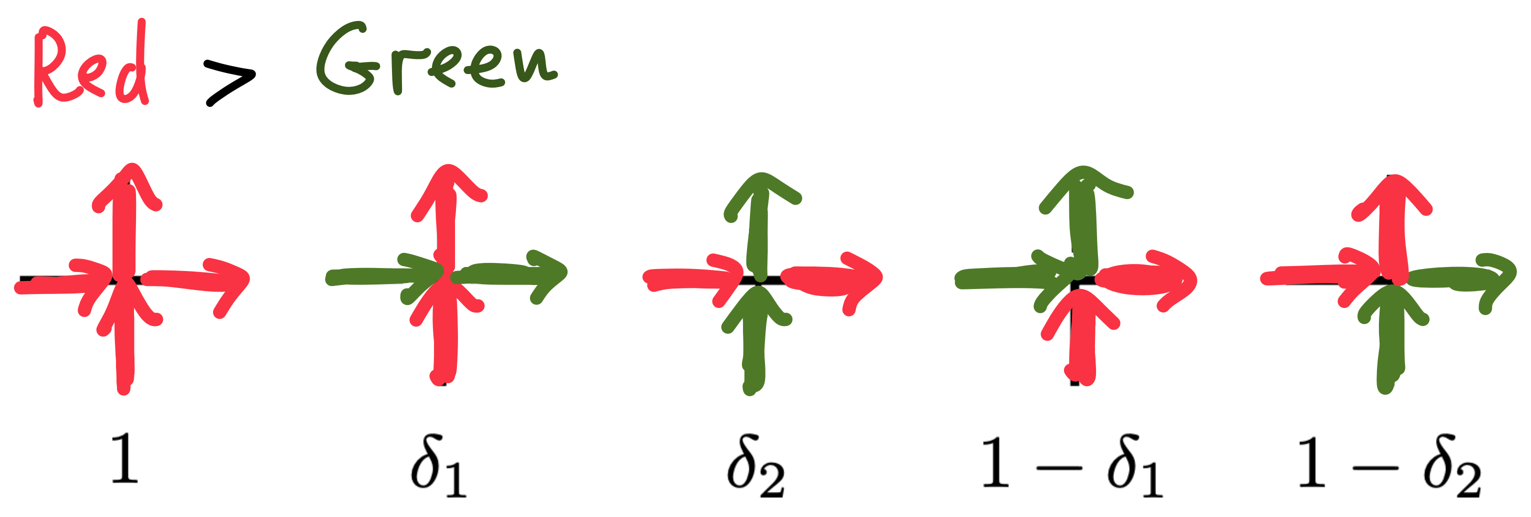
Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$

top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$: $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$: $\dfrac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$: $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$: $\dfrac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$





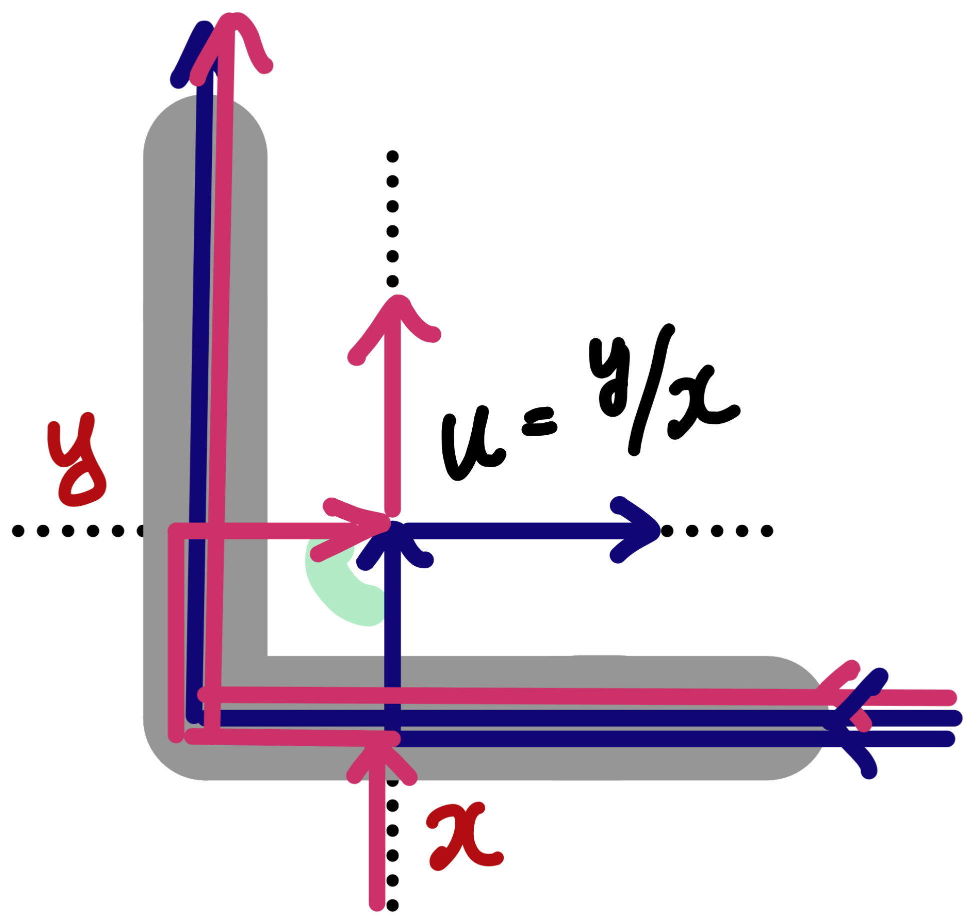
Colored stochastic six-vertex model. Many colors ⇒ many fat lines

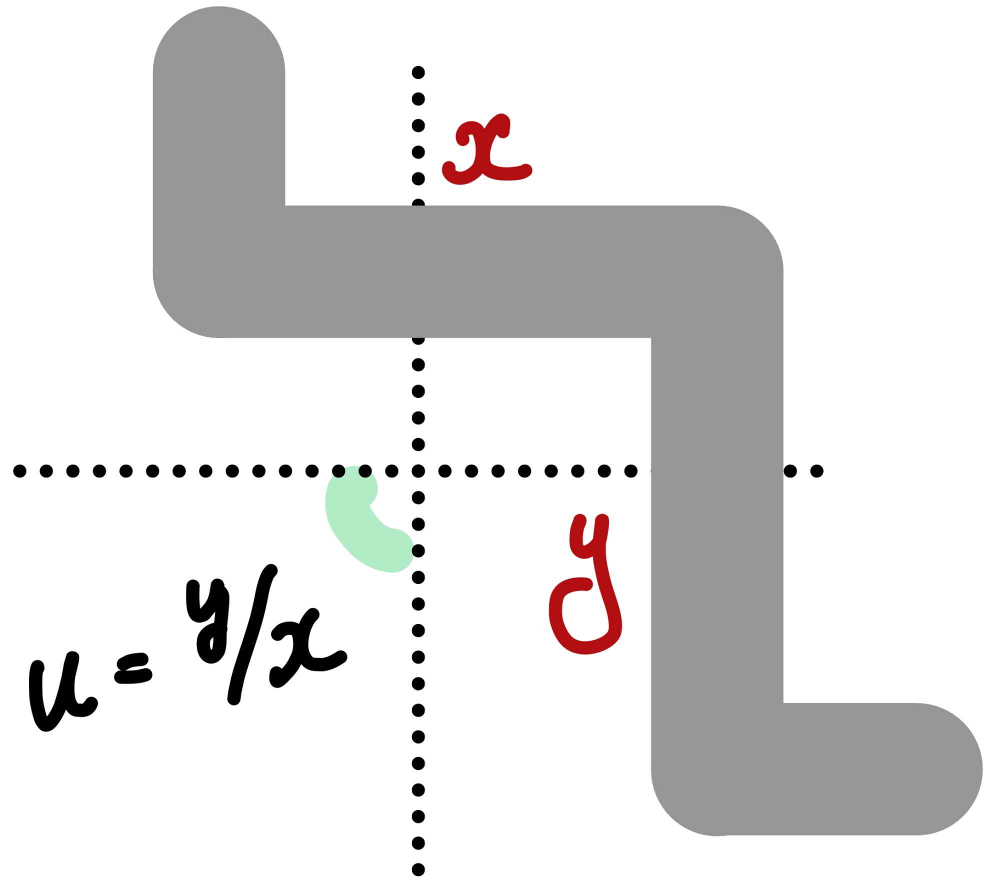


top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$: $\dfrac{1+xq^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$: $\dfrac{(x+\nu q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$: $\dfrac{1-\nu q^{\lvert\mathbf{A}\rvert}}{1+x}$	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$: $\dfrac{x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}}{1+x}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$: $\dfrac{\nu(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}}{1+x}$

- **Fusion and Yang-Baxter equation.**

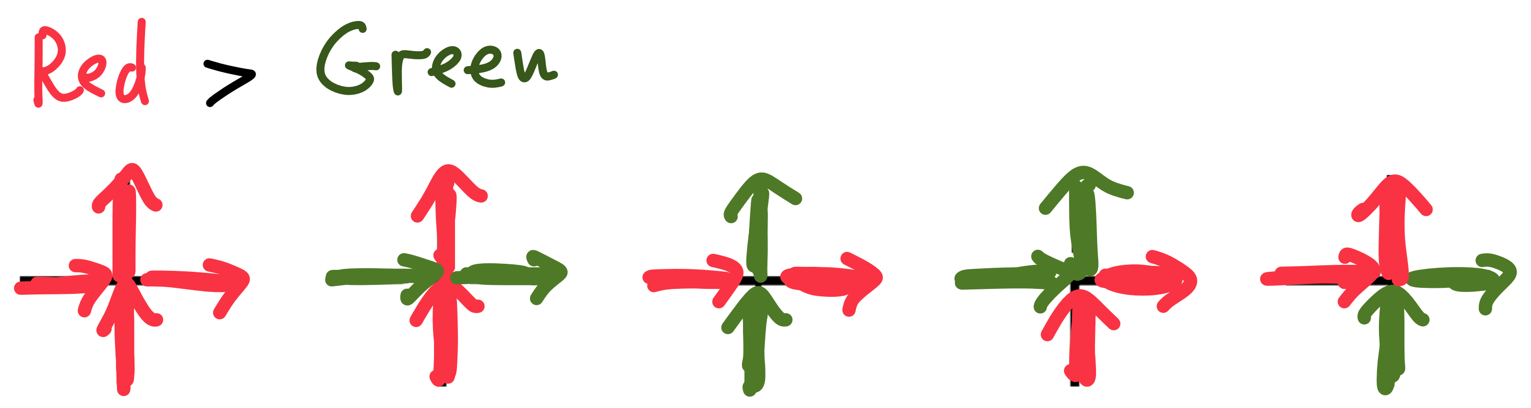
  Higher spin, higher rank stochastic weights.
  Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$





- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

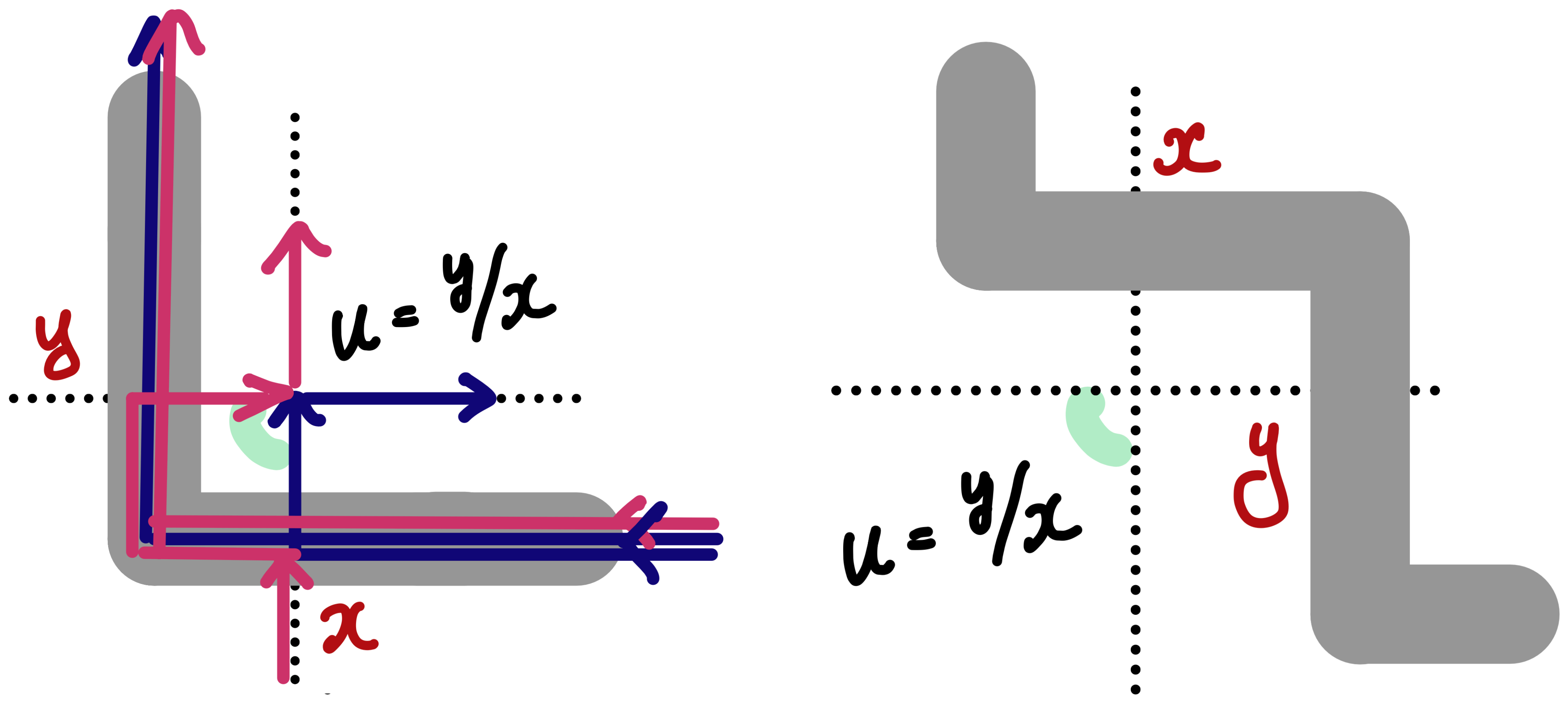
Colored stochastic six-vertex model. Many colors ⇒ many fat lines



$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

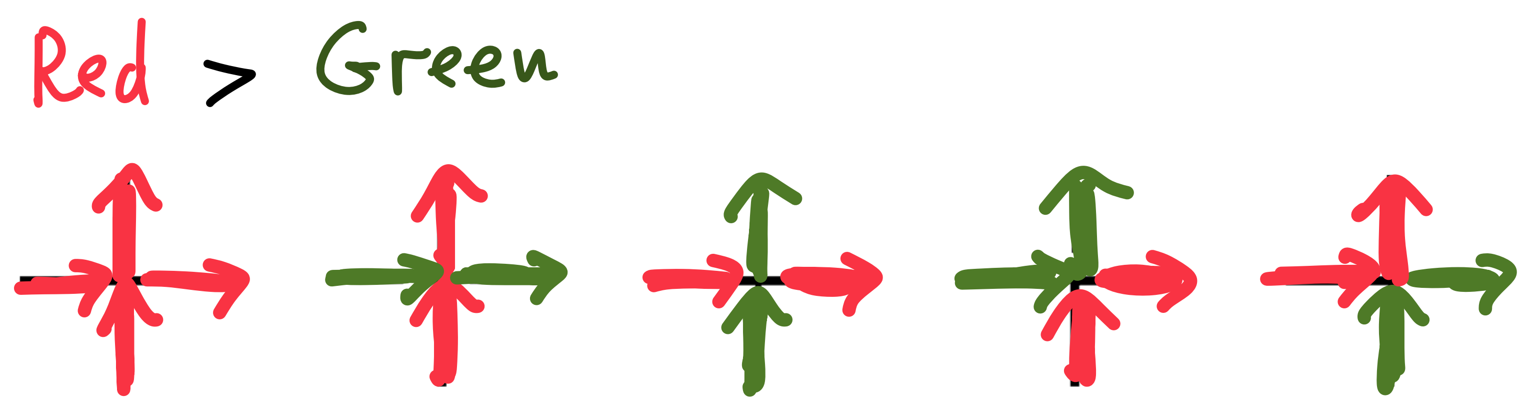
- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.
  Related to $U_q(\widehat{sl}_{n+1})$**;** $1 \leq k < \ell \leq n$



- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m, x_m}$ from the beginning (up to simple factors).

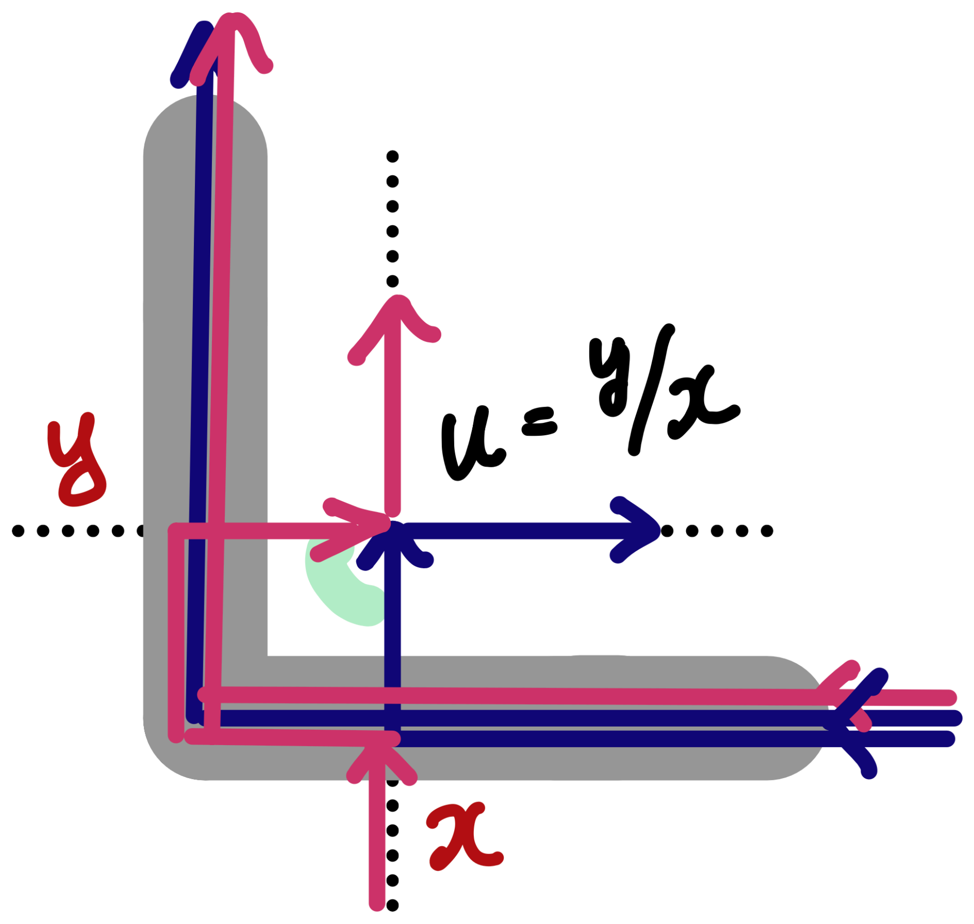
Colored stochastic six-vertex model. Many colors ⇒ many fat lines

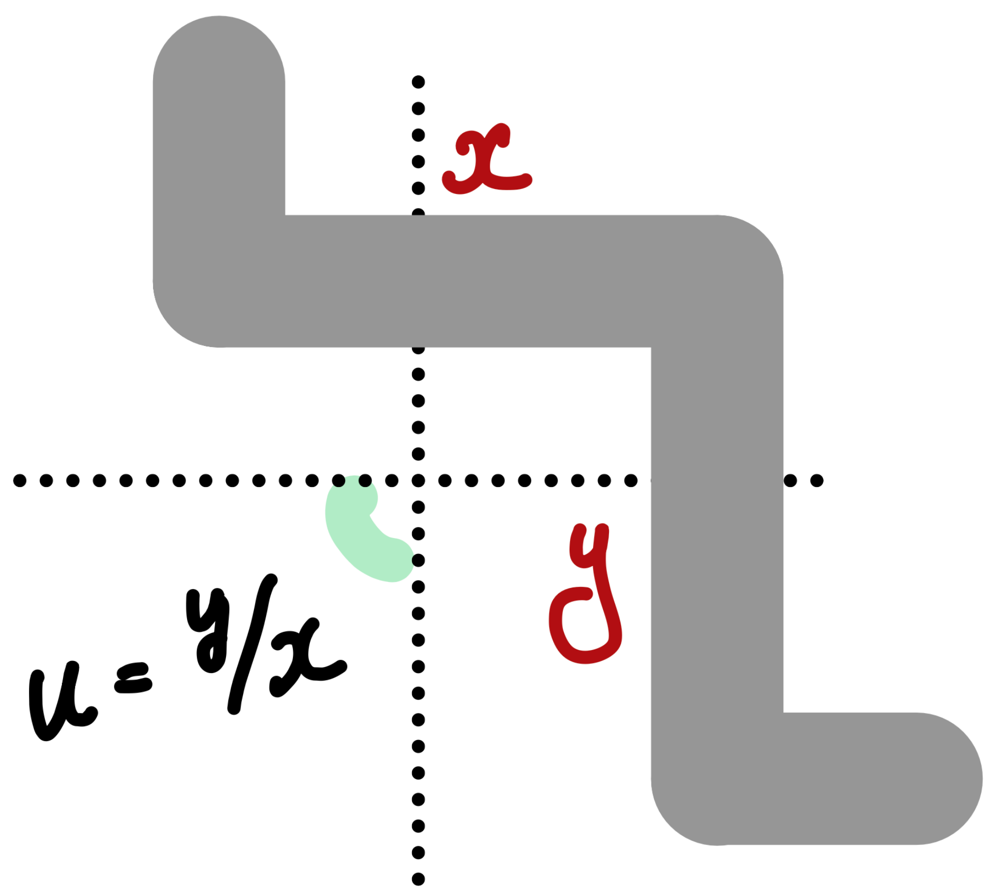


1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

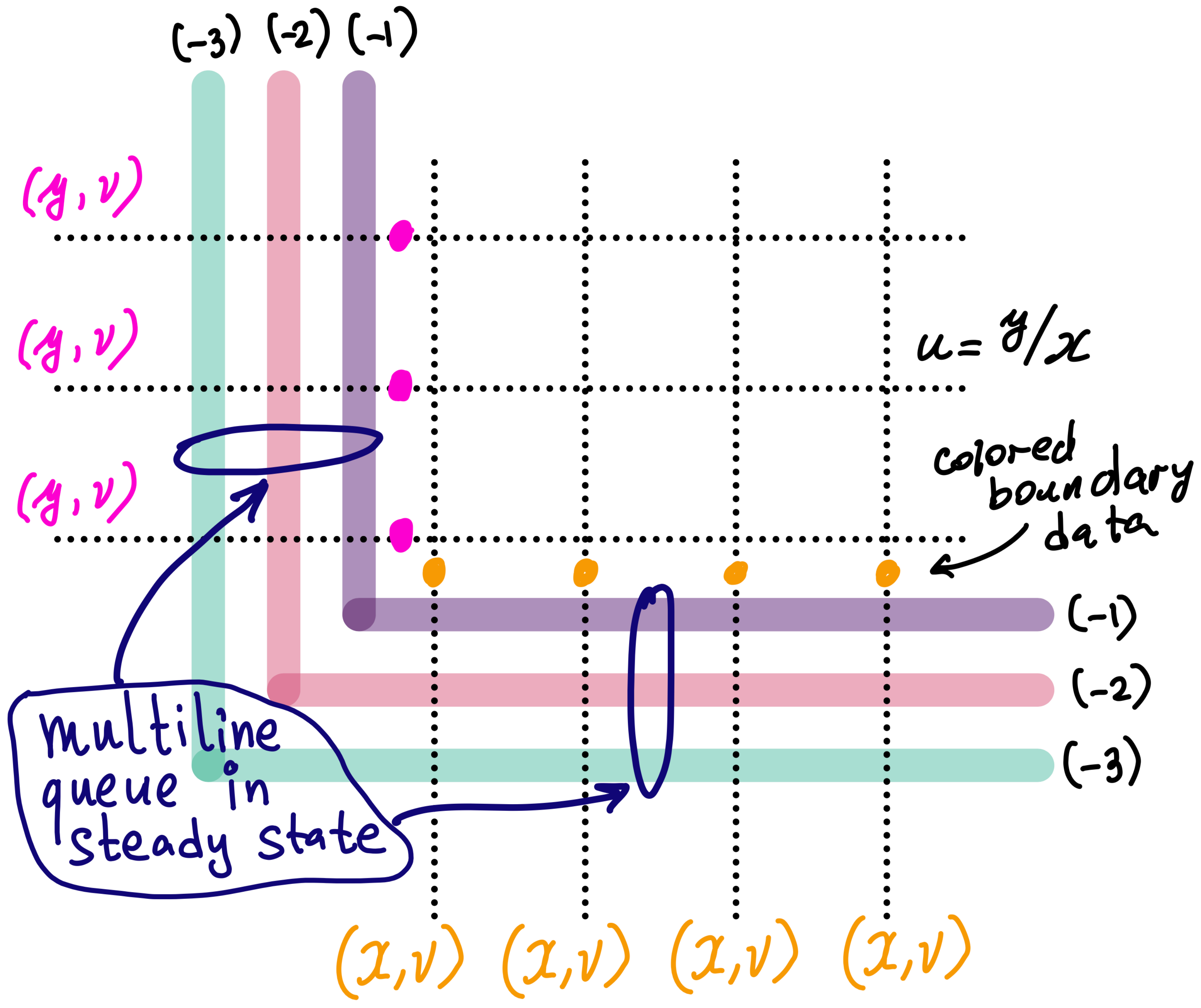
- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.
  Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$

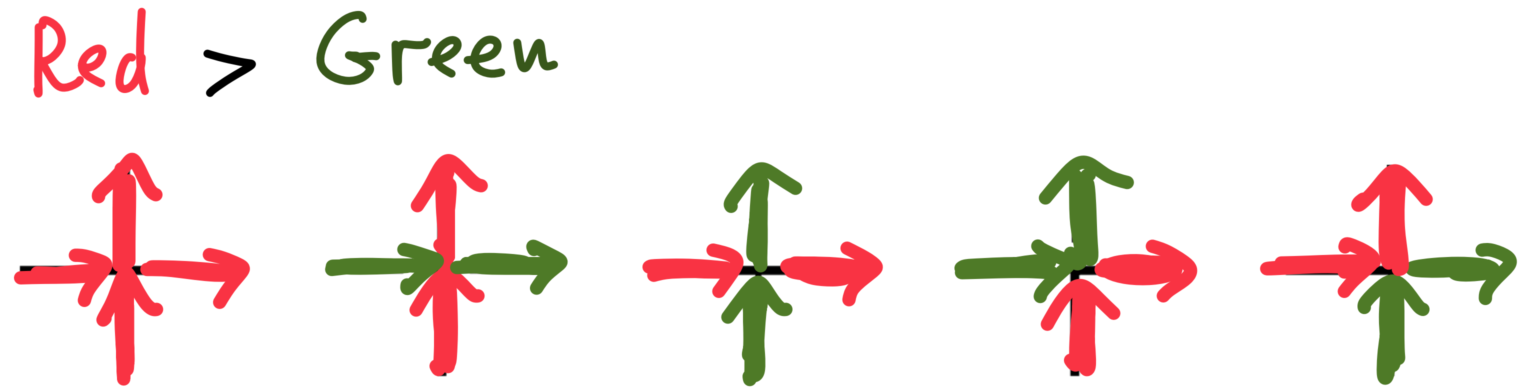




- **Set the number of arrows of a given color** m **to** $+\infty$. We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).



Colored stochastic six-vertex model. Many colors ⇒ many fat lines

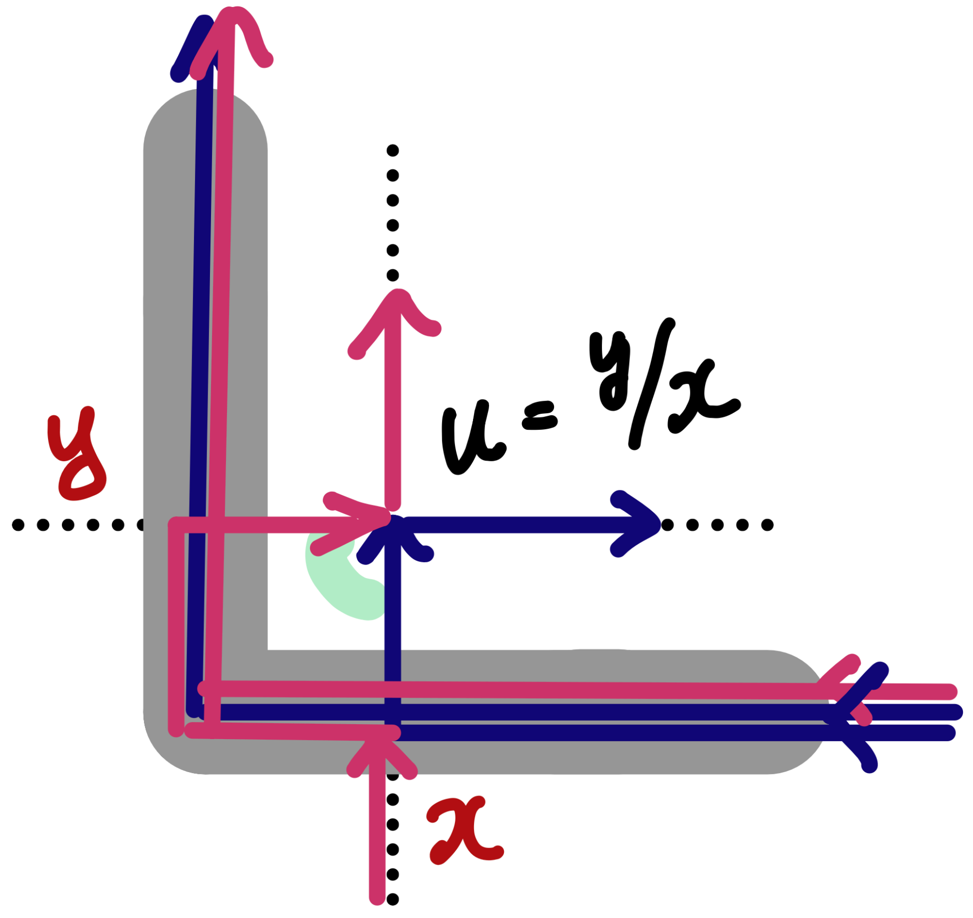


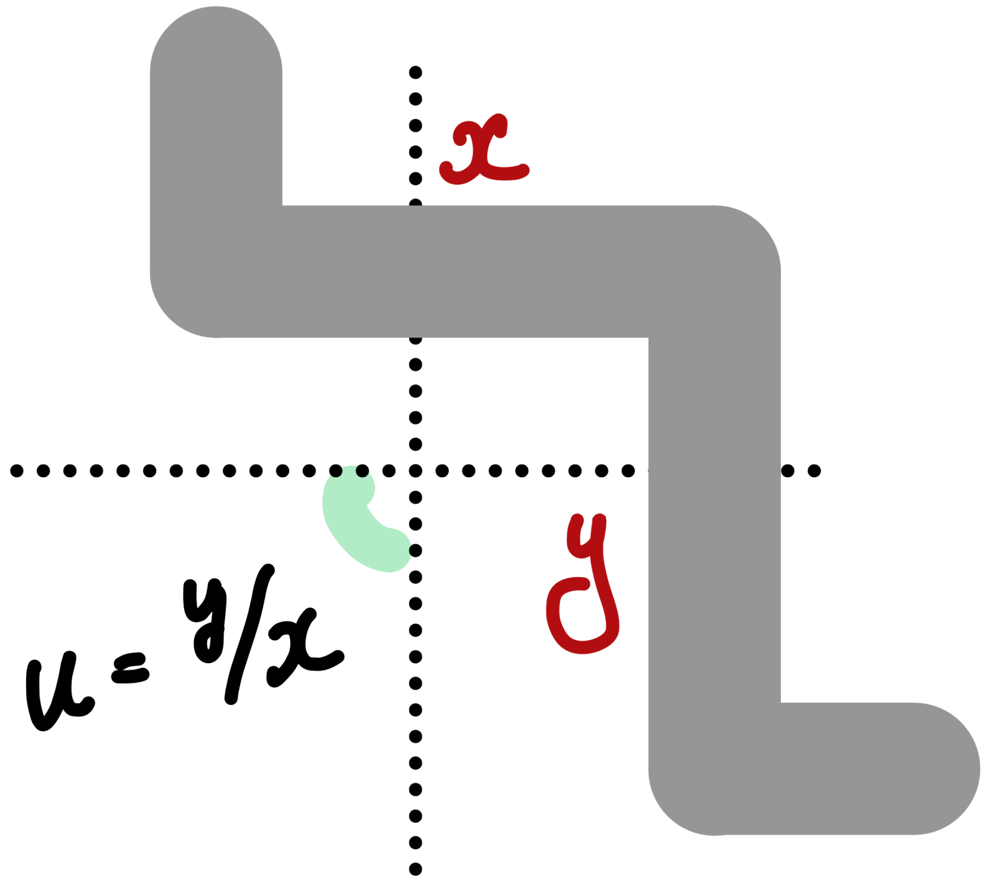
1 δ_1 δ_2 $1-\delta_1$ $1-\delta_2$

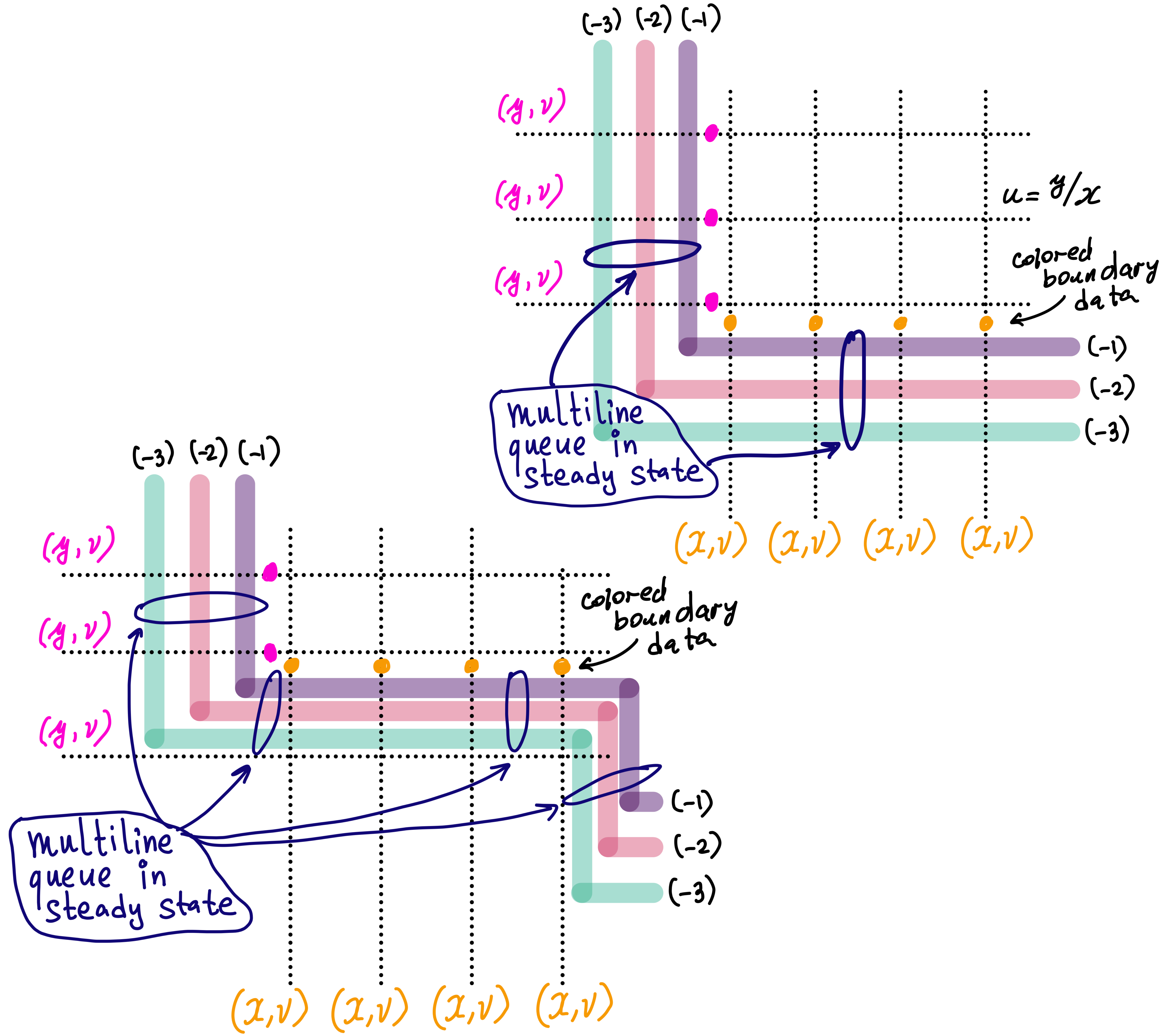
- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.

  Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$

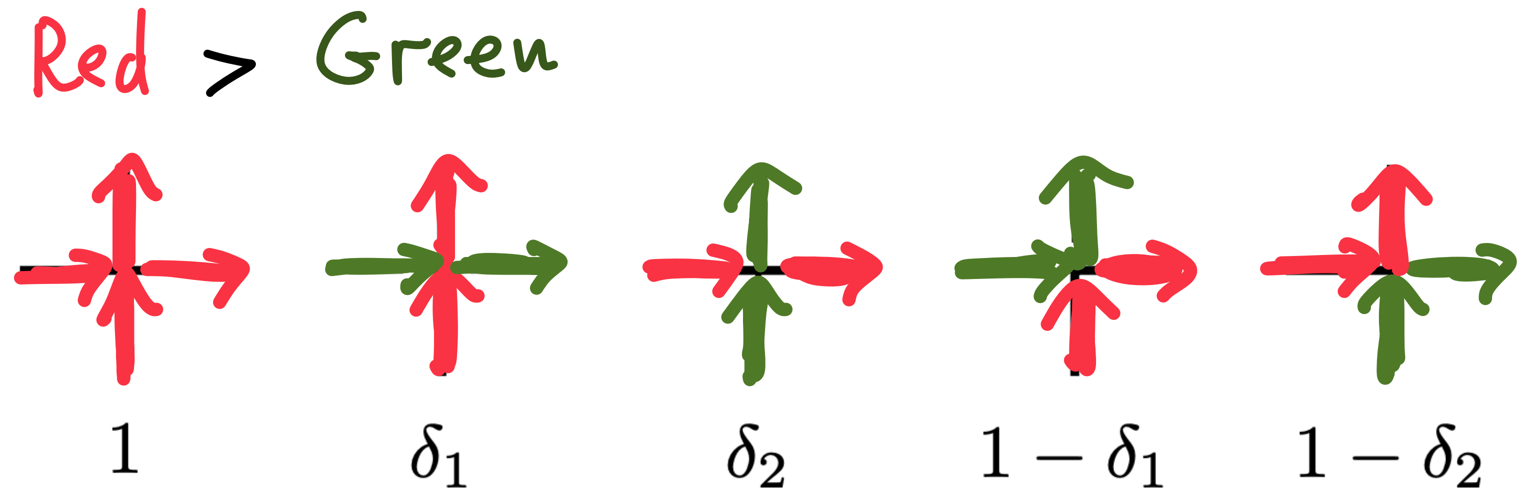






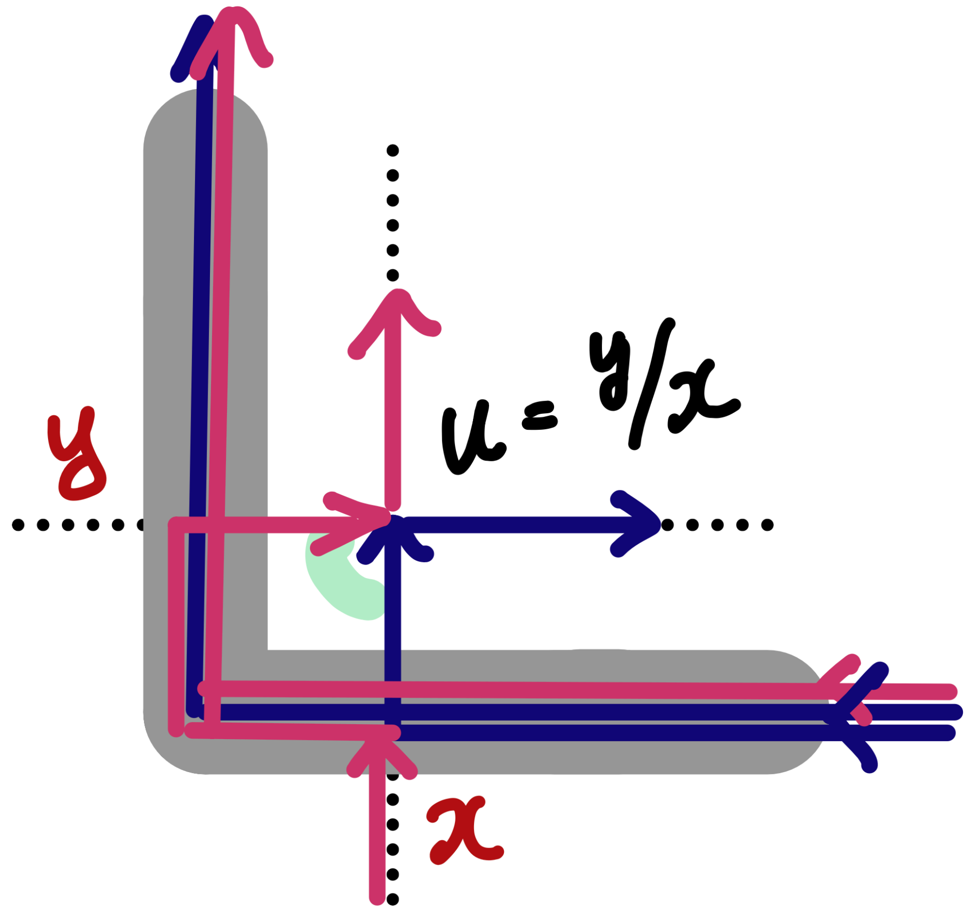
- **Set the number of arrows of a given color m to $+\infty$.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

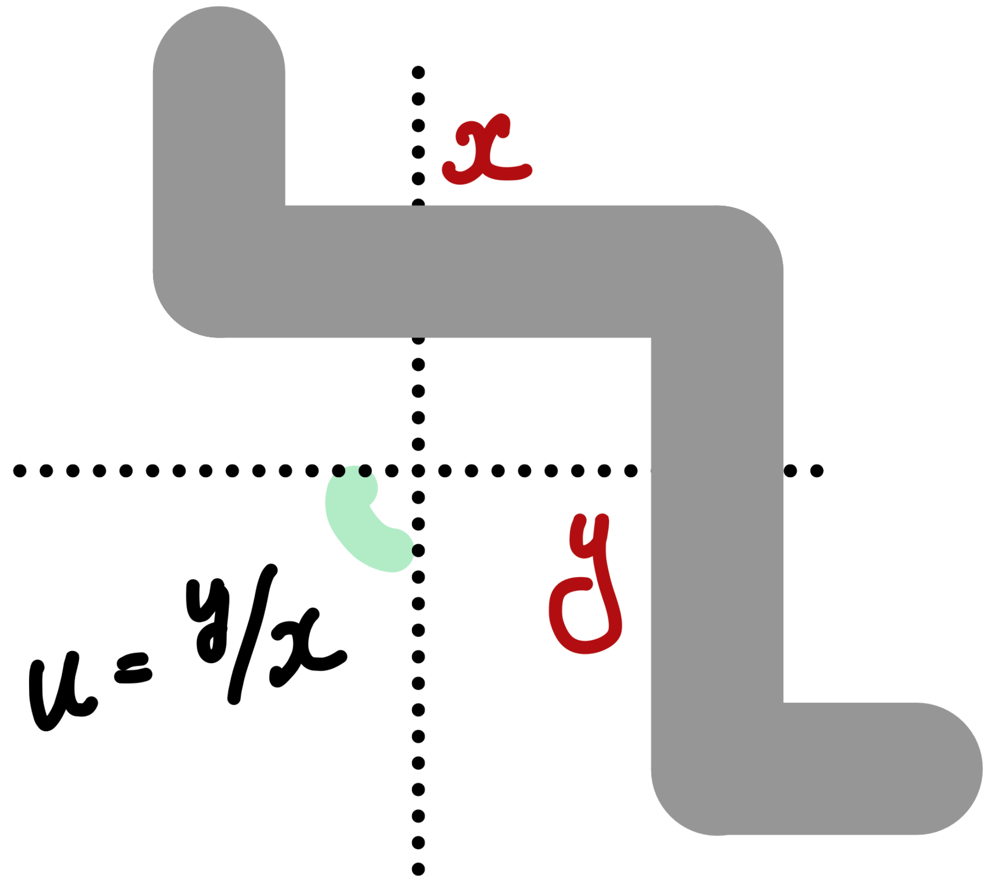
Colored stochastic six-vertex model. Many colors ⇒ many fat lines

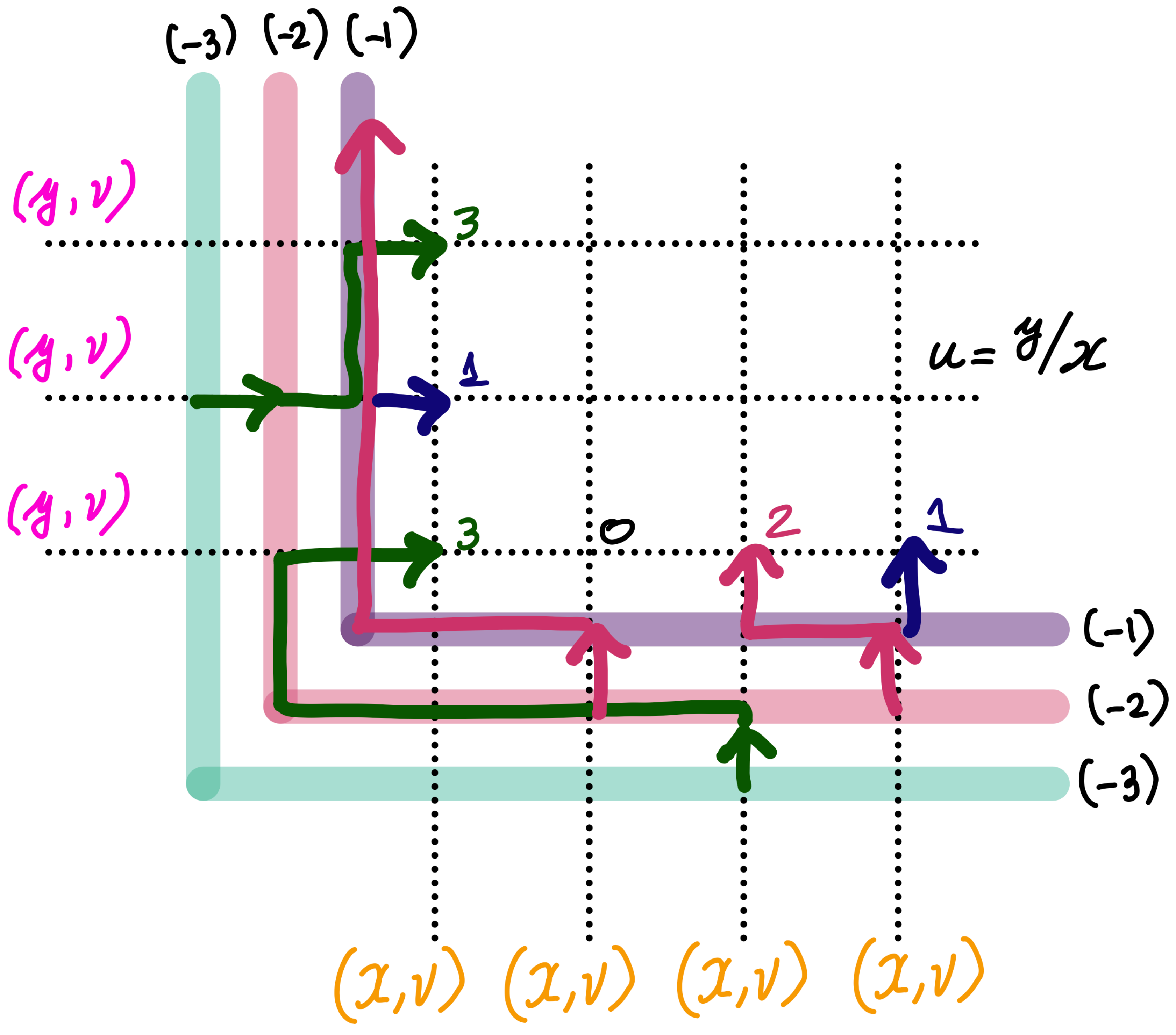


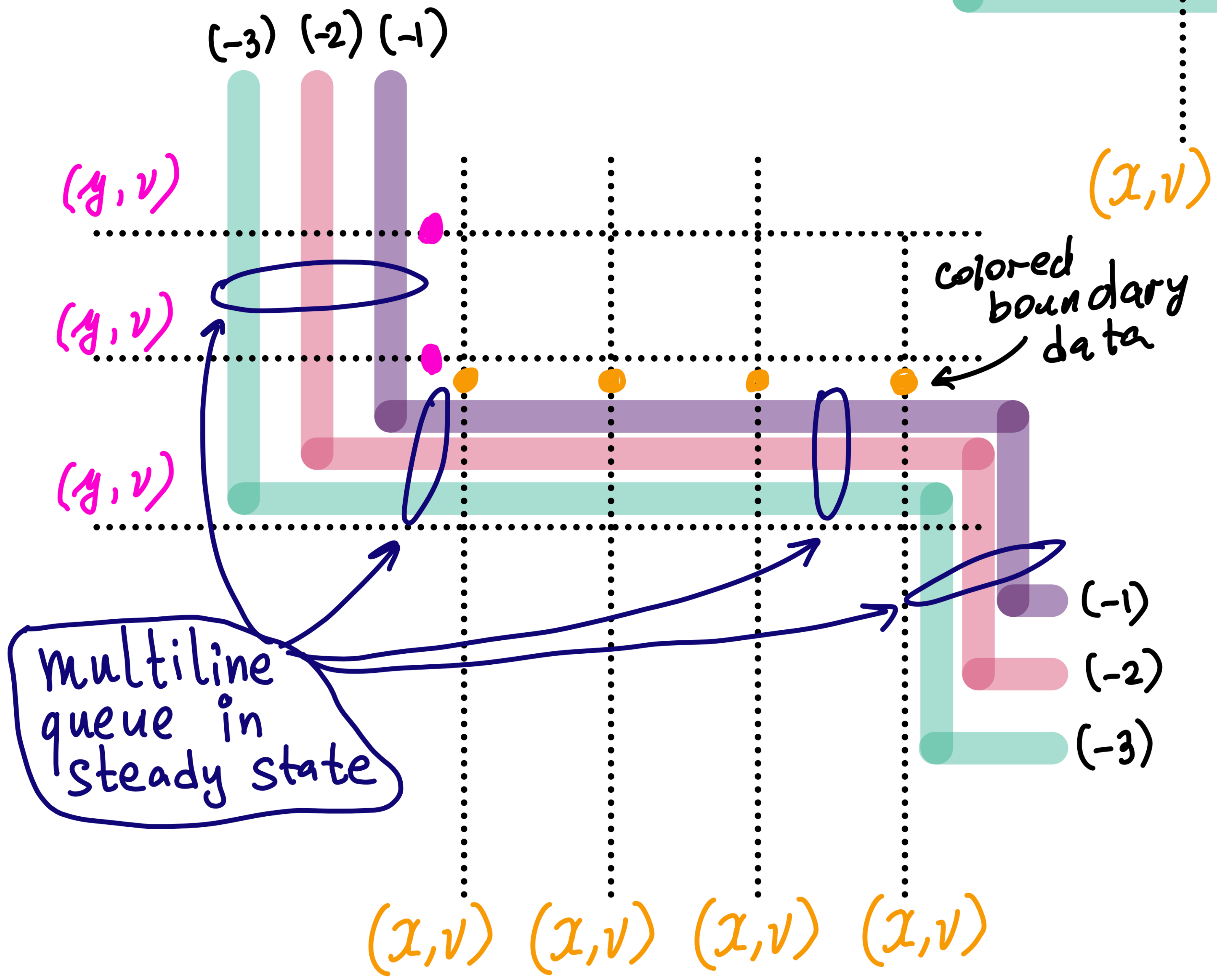
- **Fusion and Yang-Baxter equation.**

Higher spin, higher rank stochastic weights.
Related to $U_q(\widehat{sl}_{n+1})$; $1 \le k < \ell \le n$



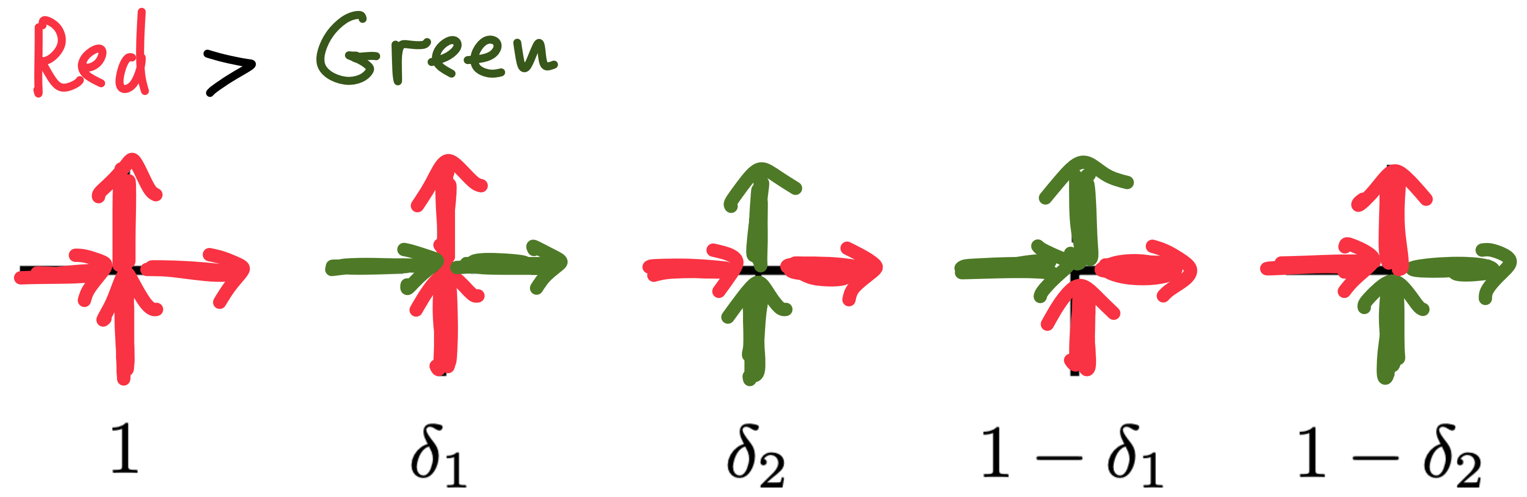


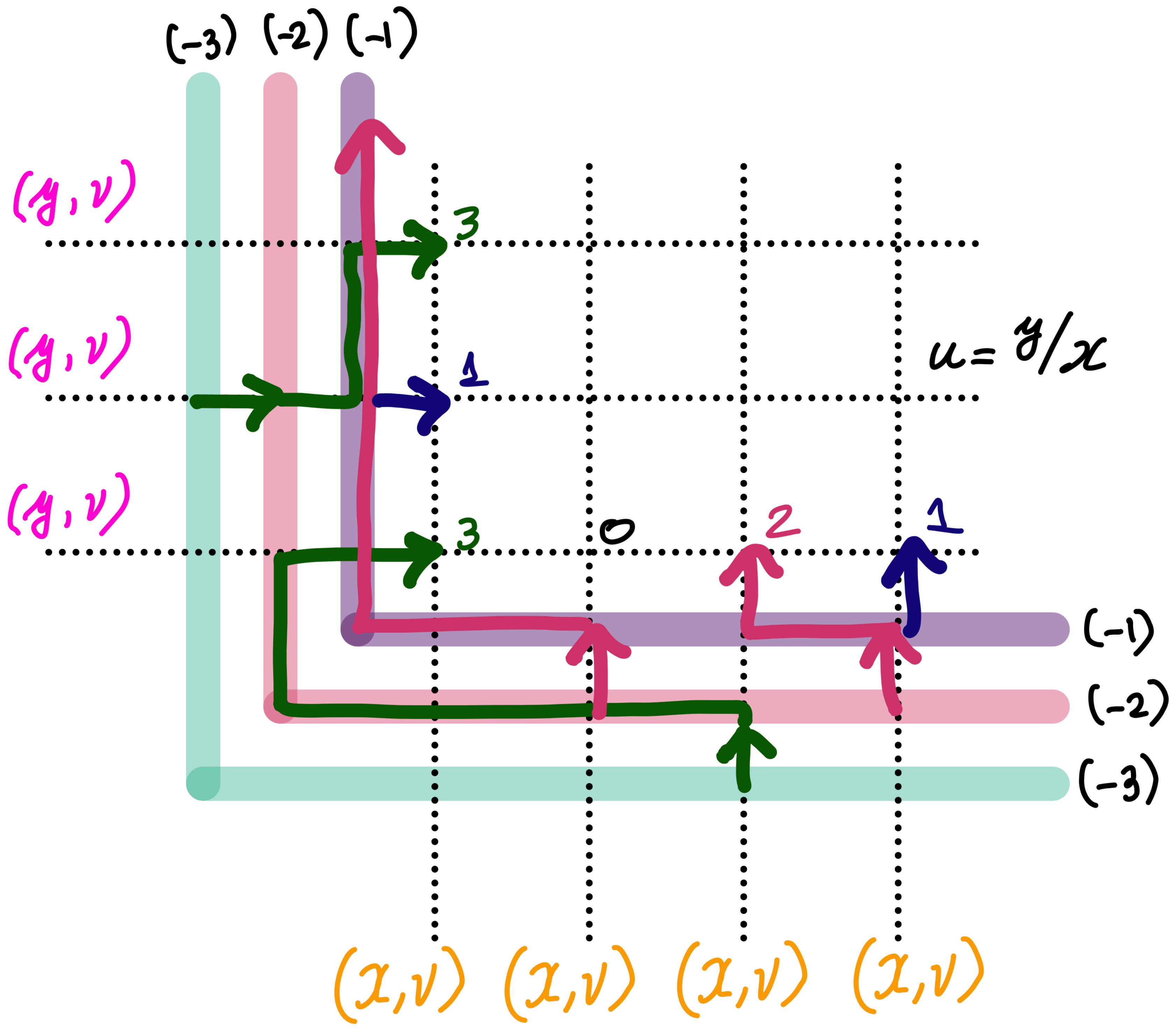




- **Set the number of arrows of a given color** m **to** $+\infty$. We get $\mathbb{W}^{(-m)}_{s_m, x_m}$ from the beginning (up to simple factors).

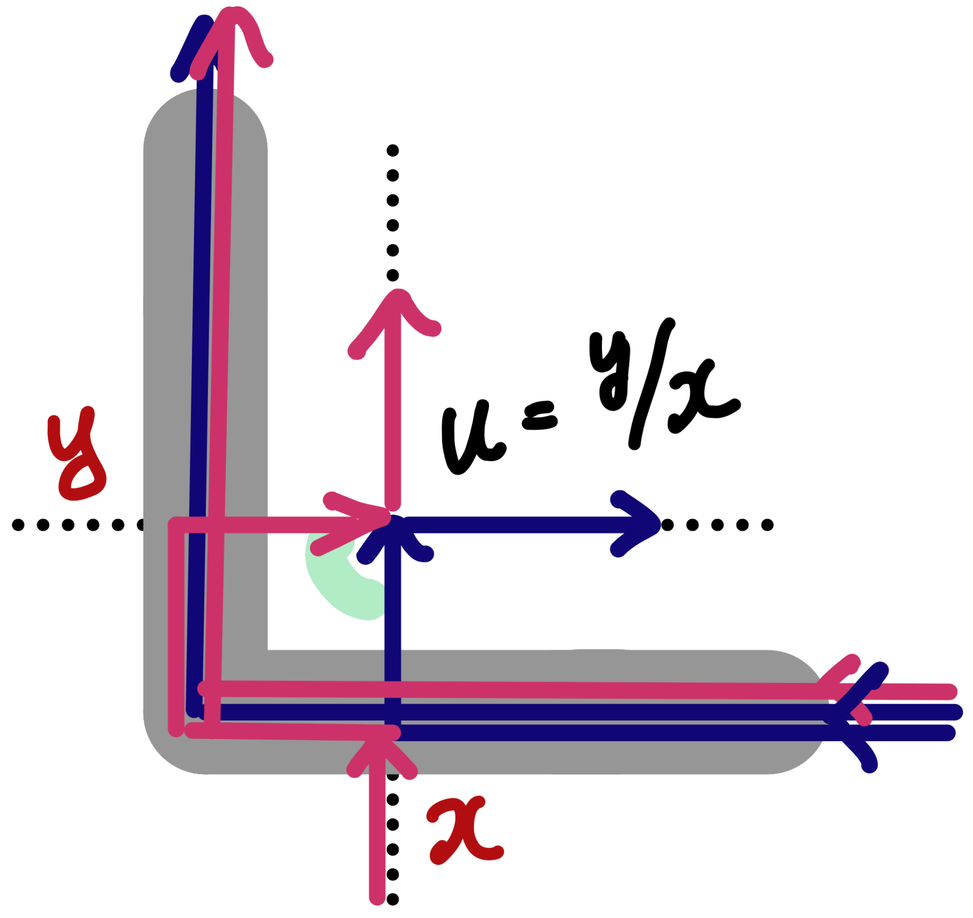
Colored stochastic six-vertex model. Many colors ⇒ many fat lines

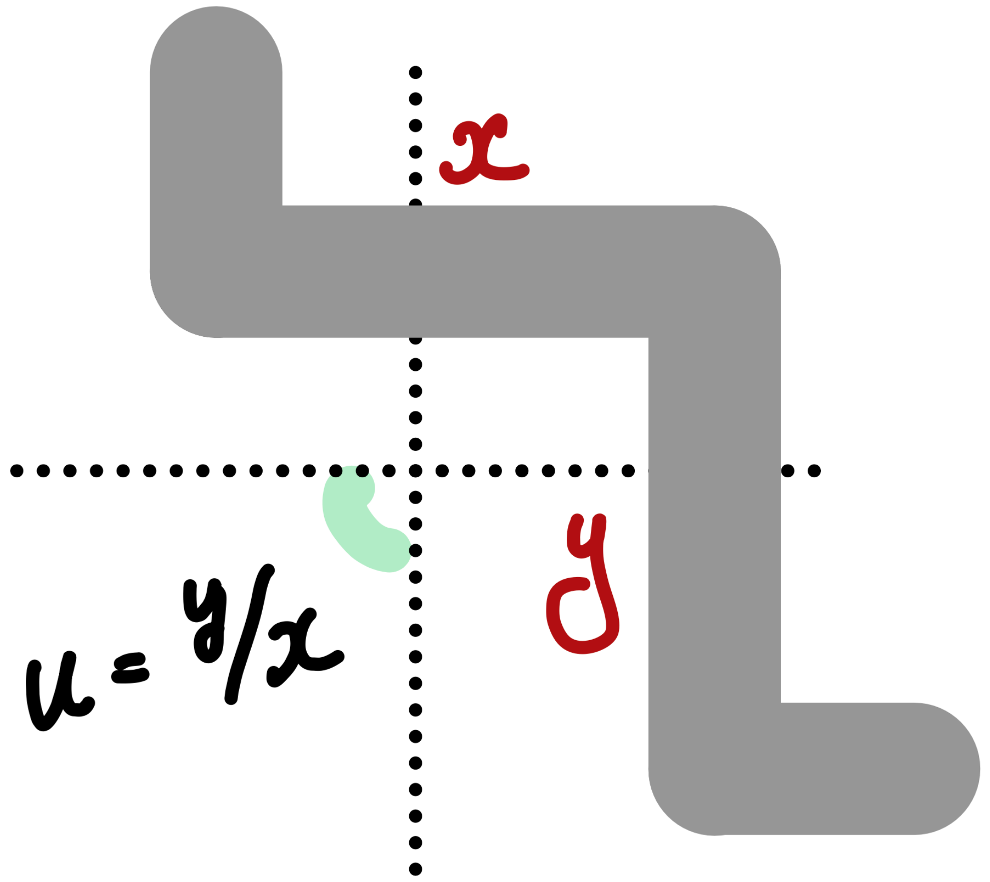


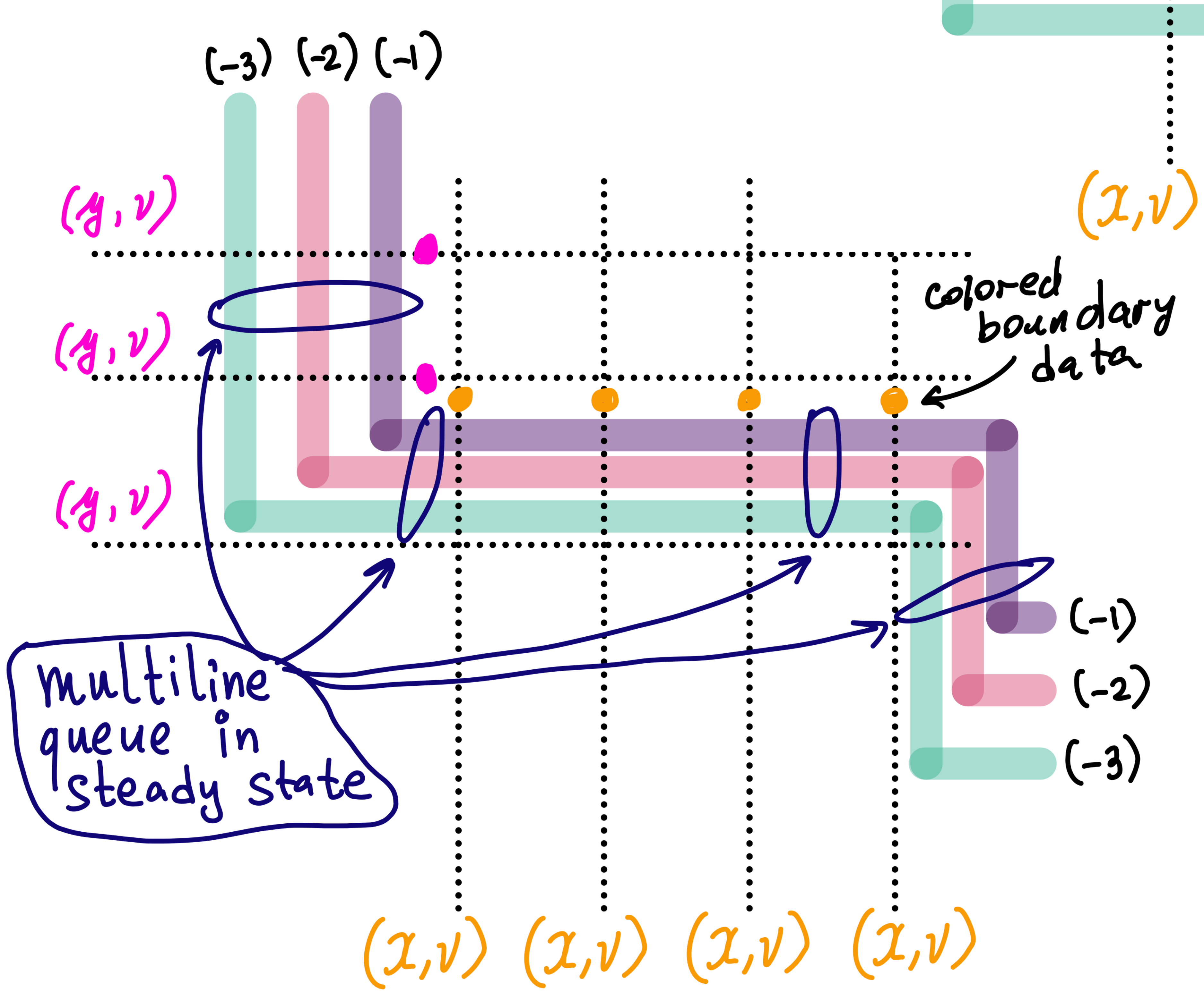


- **Fusion and Yang-Baxter equation.**

  Higher spin, higher rank stochastic weights.
  Related to $U_q(\widehat{sl}_{n+1})$**;** $1 \leq k < \ell \leq n$







- **Set the number of arrows of a given color** m **to** $+\infty$**.** We get $\mathbb{W}^{(-m)}_{s_m,x_m}$ from the beginning (up to simple factors).

(Implies mASEP stationary measure results on the line)

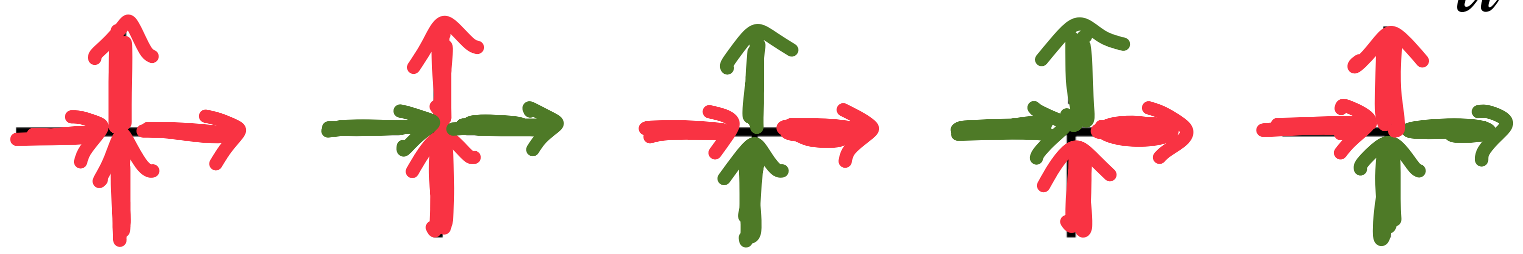
Stationarity from Yang-Baxter equation

mASEP on the ring

Yang-Baxter equation on the $n \times N$ cylinder

Red > Green $\qquad R_u$

$$u = \frac{1-\delta_1}{1-\delta_2}$$

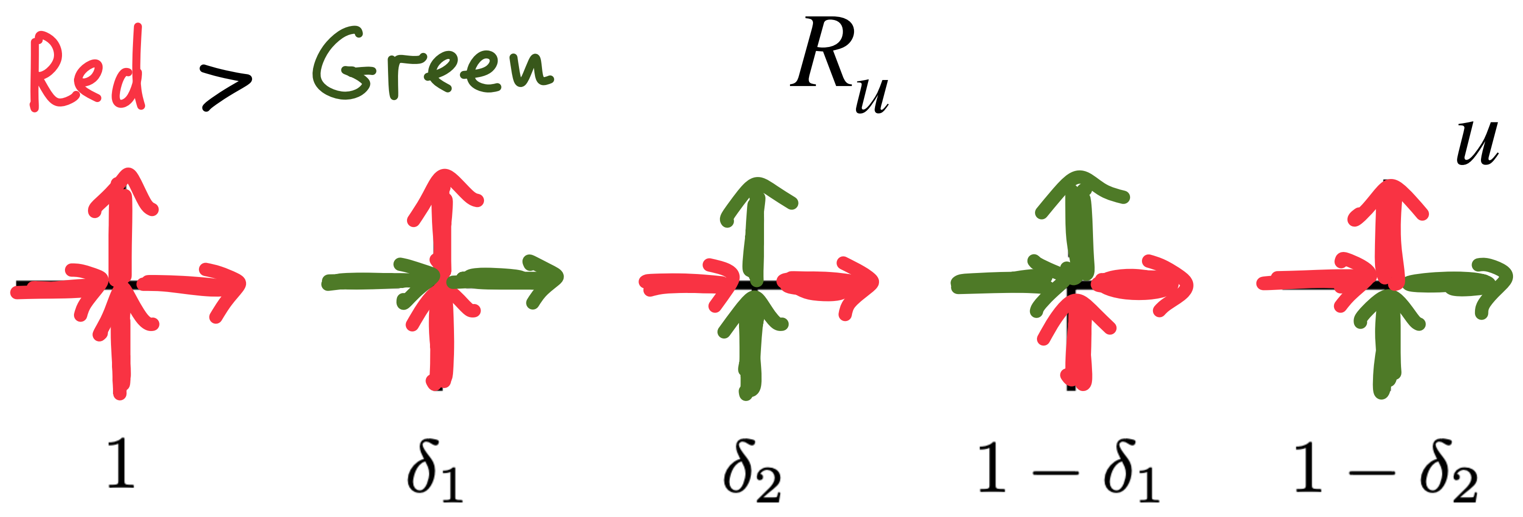


$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

$\mathbf{A}$ 0 — 0 $\mathbf{A}$ 1	$\mathbf{A}$ k — k $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 — k $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 — m $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k — 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k — ℓ $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ — k $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ — m $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

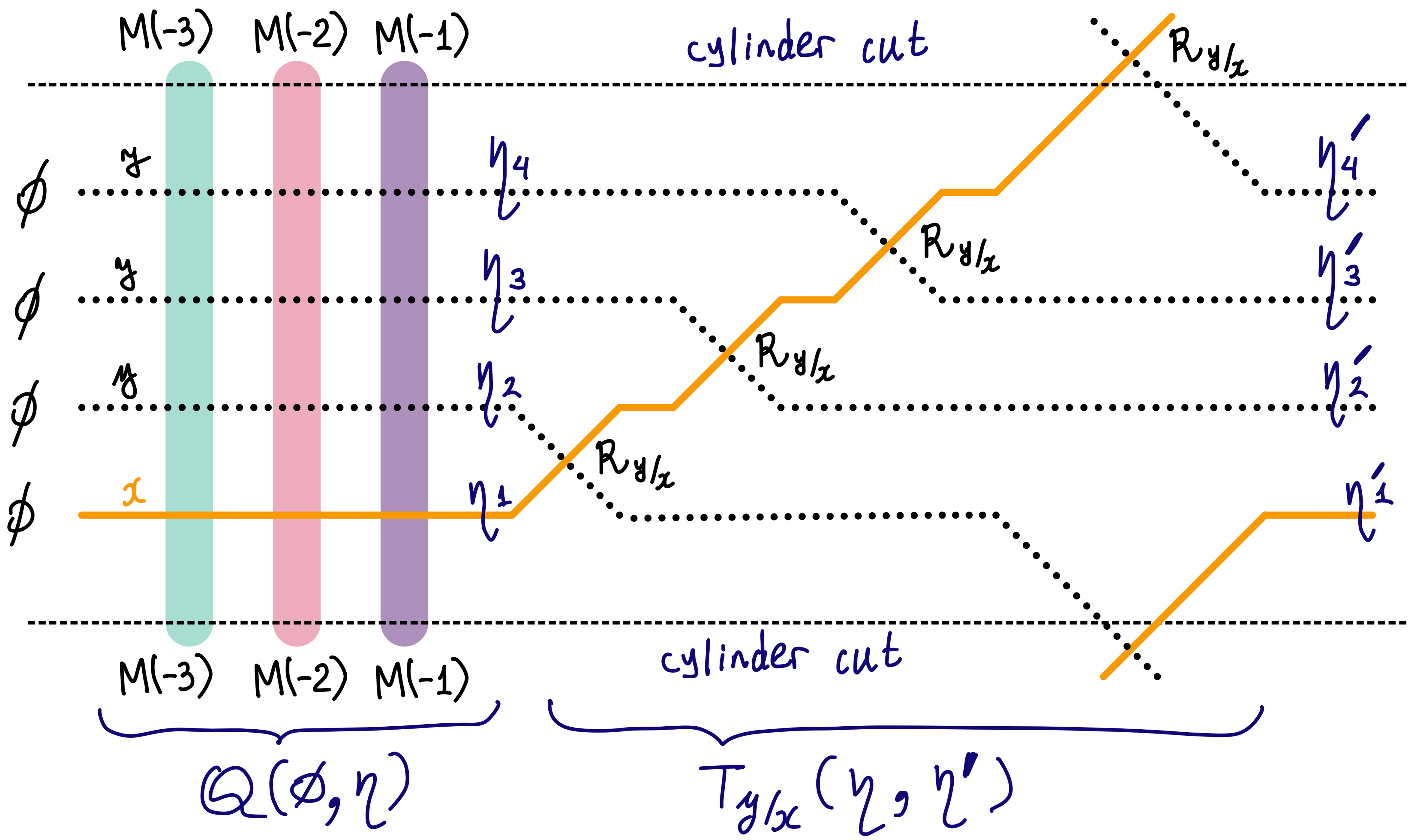
Yang-Baxter equation on the $n \times N$ cylinder



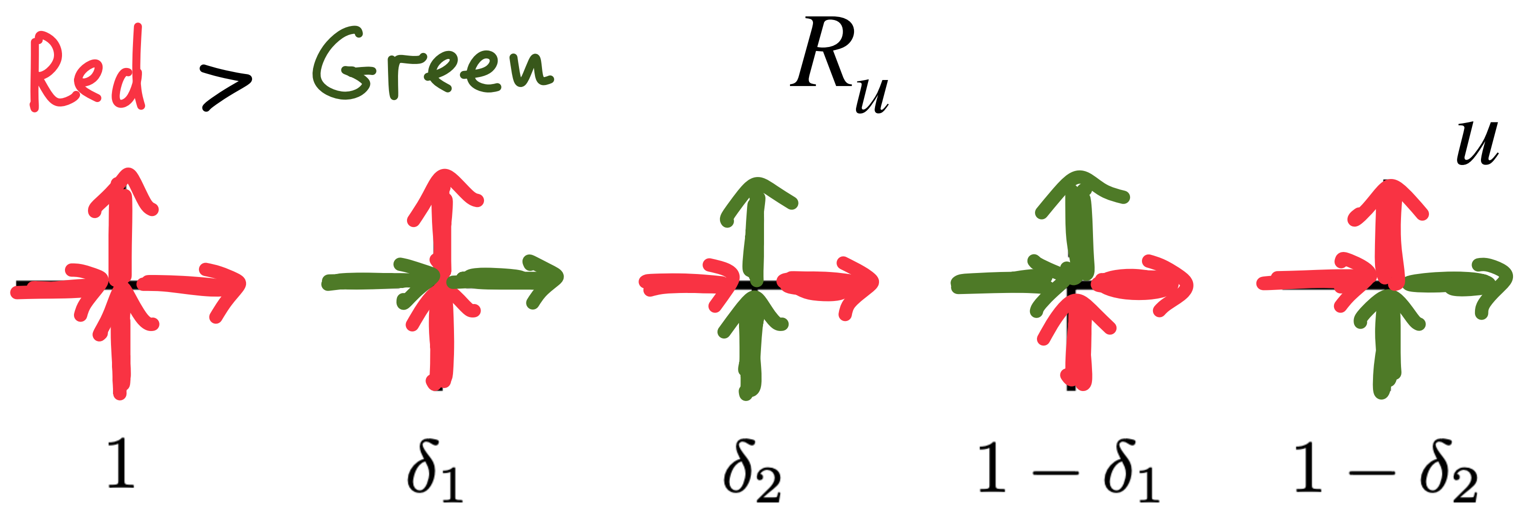
$$u = \frac{1-\delta_1}{1-\delta_2}$$

$\mathbf{A}$ 0 — 0 $\mathbf{A}$ 1	$\mathbf{A}$ k — k $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 — k $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 — m $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k — 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k — ℓ $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ — k $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ — m $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)



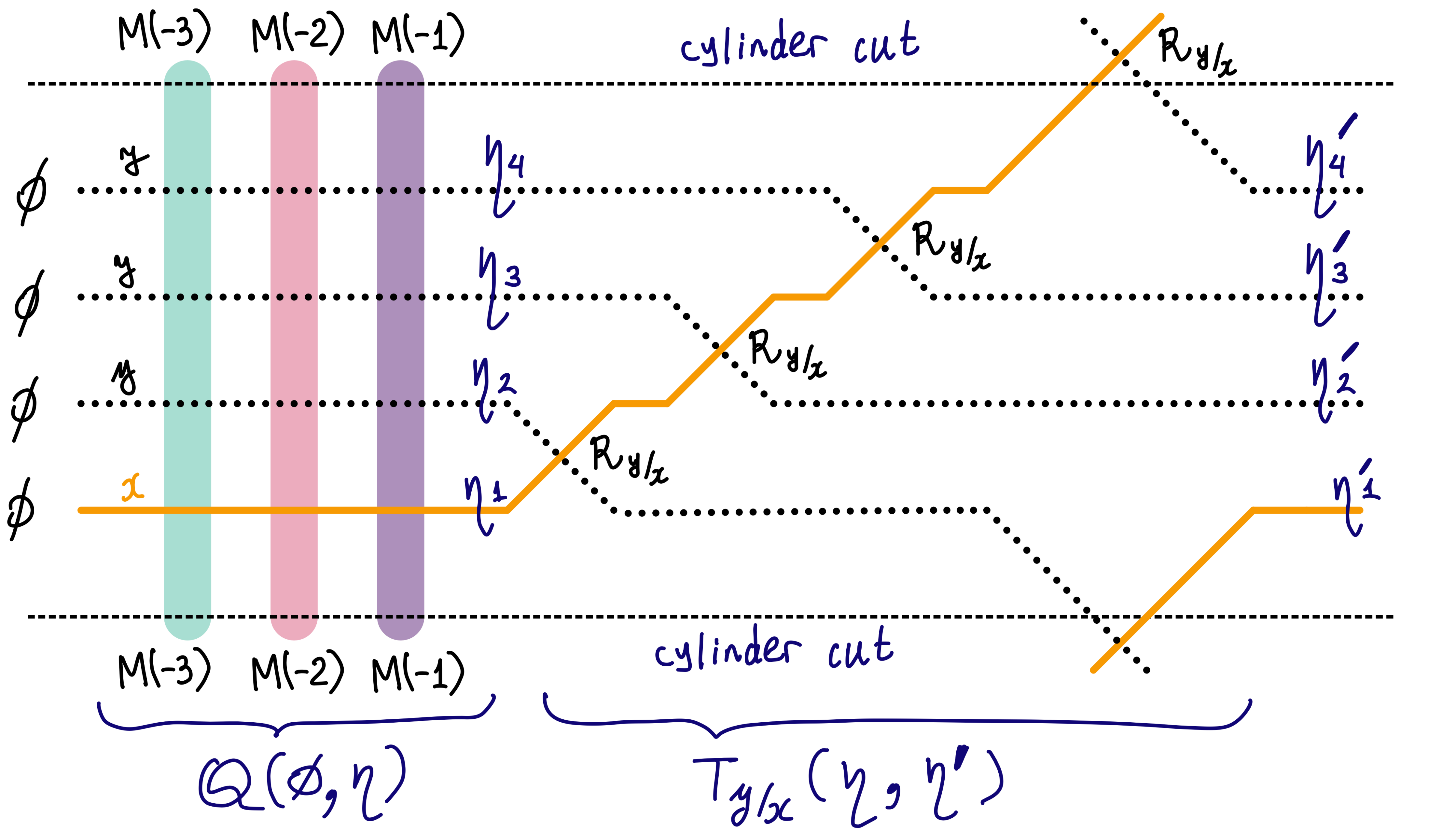
Yang-Baxter equation on the $n \times N$ cylinder

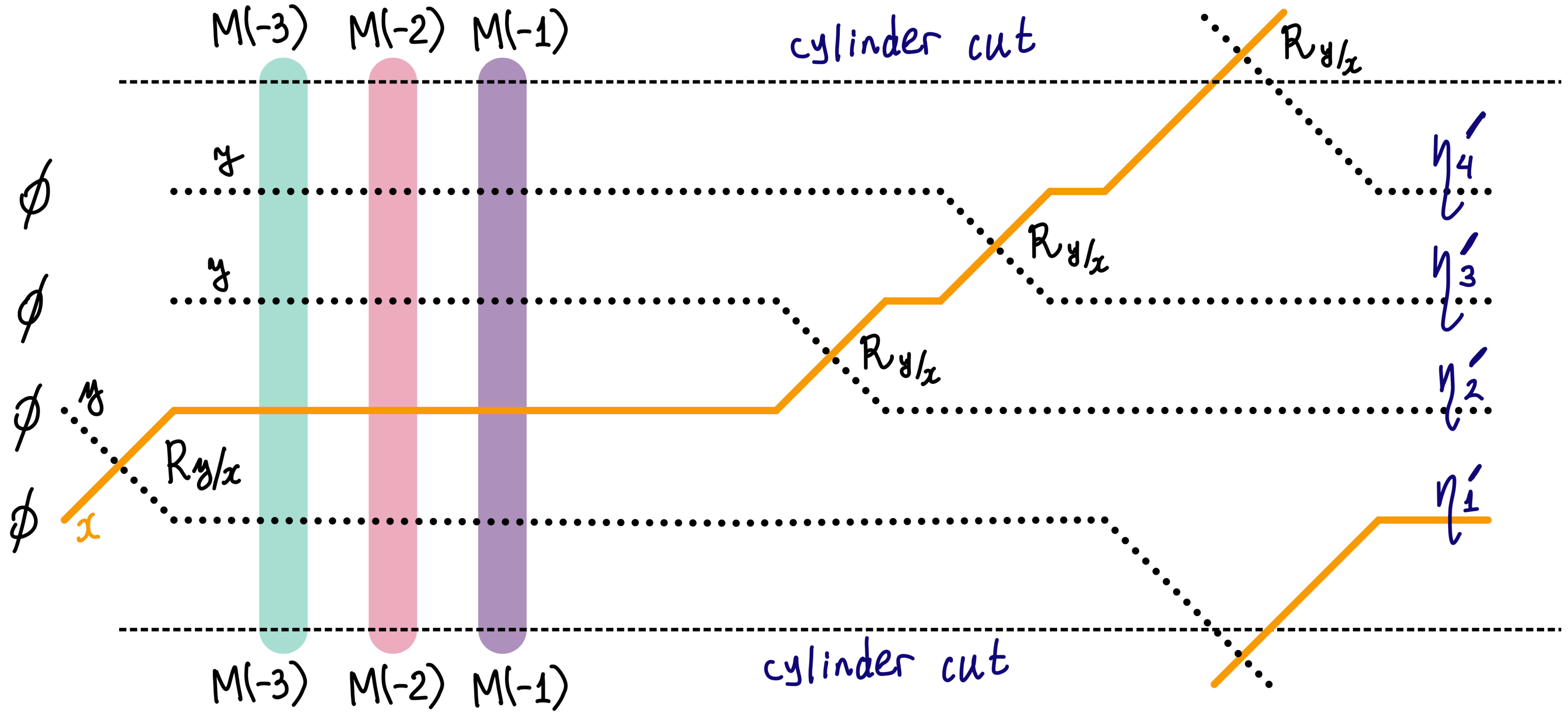


$$u = \frac{1-\delta_1}{1-\delta_2}$$

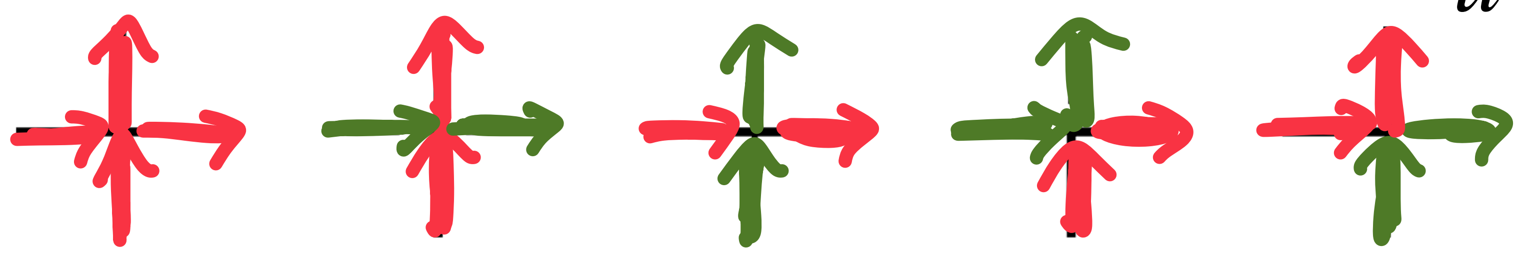
top $\mathbf{A}$, left 0, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}$, left k, right k, bottom $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$, left 0, right k, bottom $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$, left 0, right m, bottom $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$, left k, right 0, bottom $\mathbf{A}$ 1	top $\mathbf{A}_{k\ell}^{+-}$, left k, right ℓ, bottom $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$, left ℓ, right k, bottom $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$, left ℓ, right m, bottom $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)





Yang-Baxter equation on the $n \times N$ cylinder

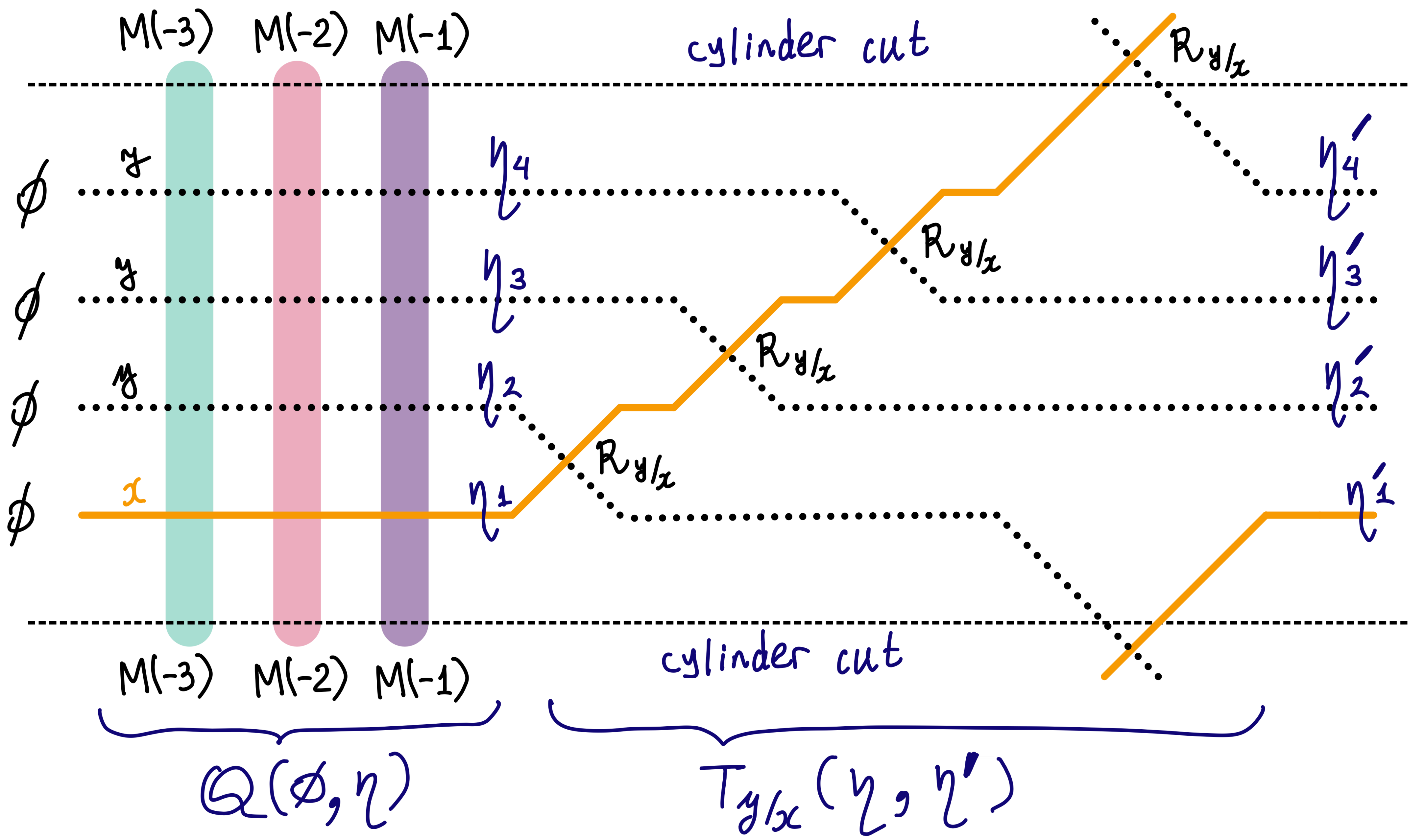


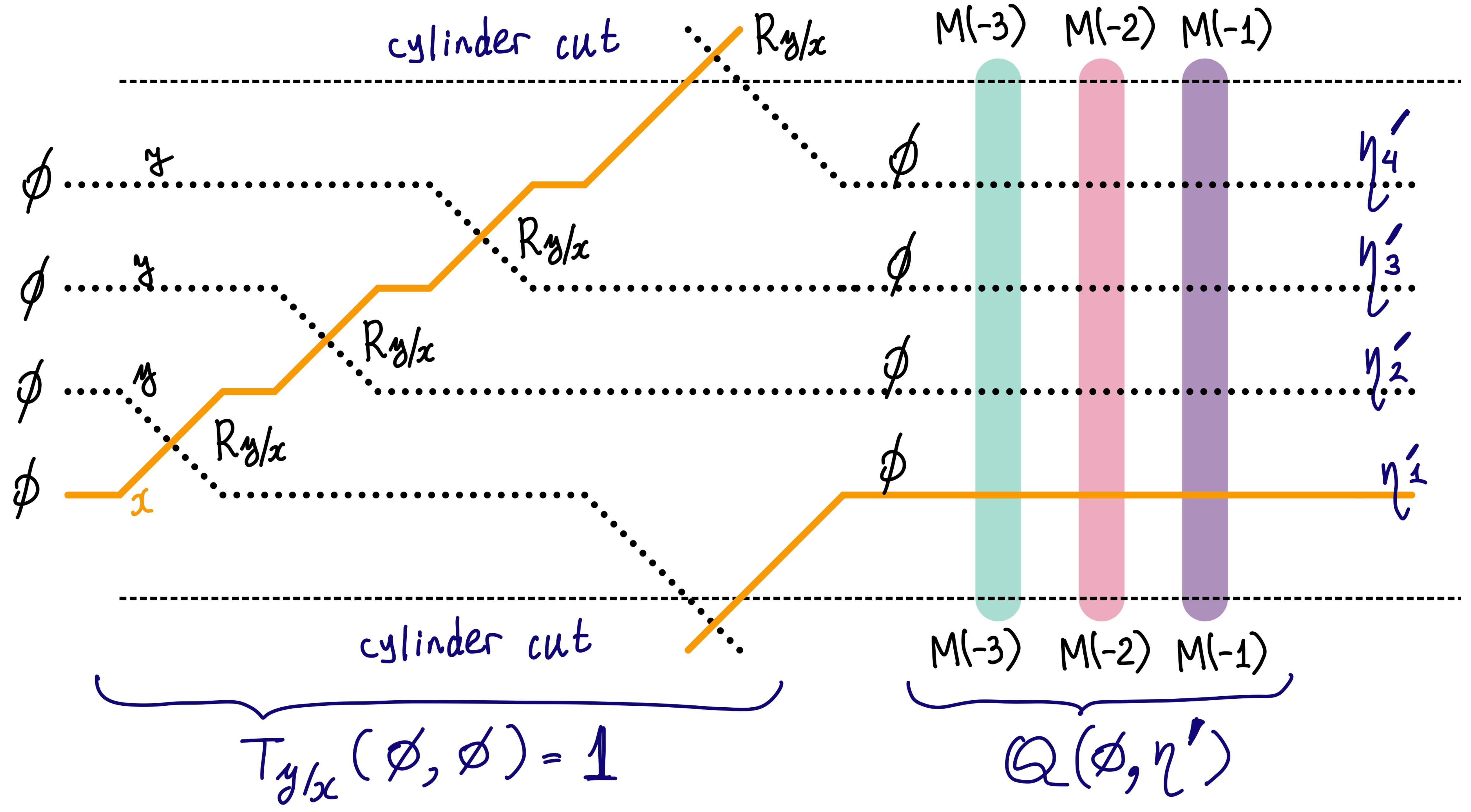
$1 \quad \delta_1 \quad \delta_2 \quad 1-\delta_1 \quad 1-\delta_2$

$$u = \frac{1-\delta_1}{1-\delta_2}$$

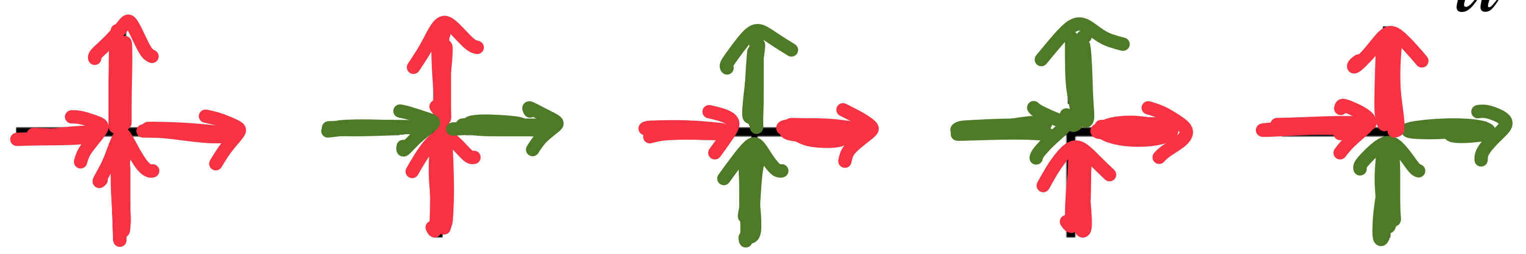
top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$; left 0, right m; bottom $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$; left ℓ, right m; bottom $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)





Yang-Baxter equation on the $n \times N$ cylinder



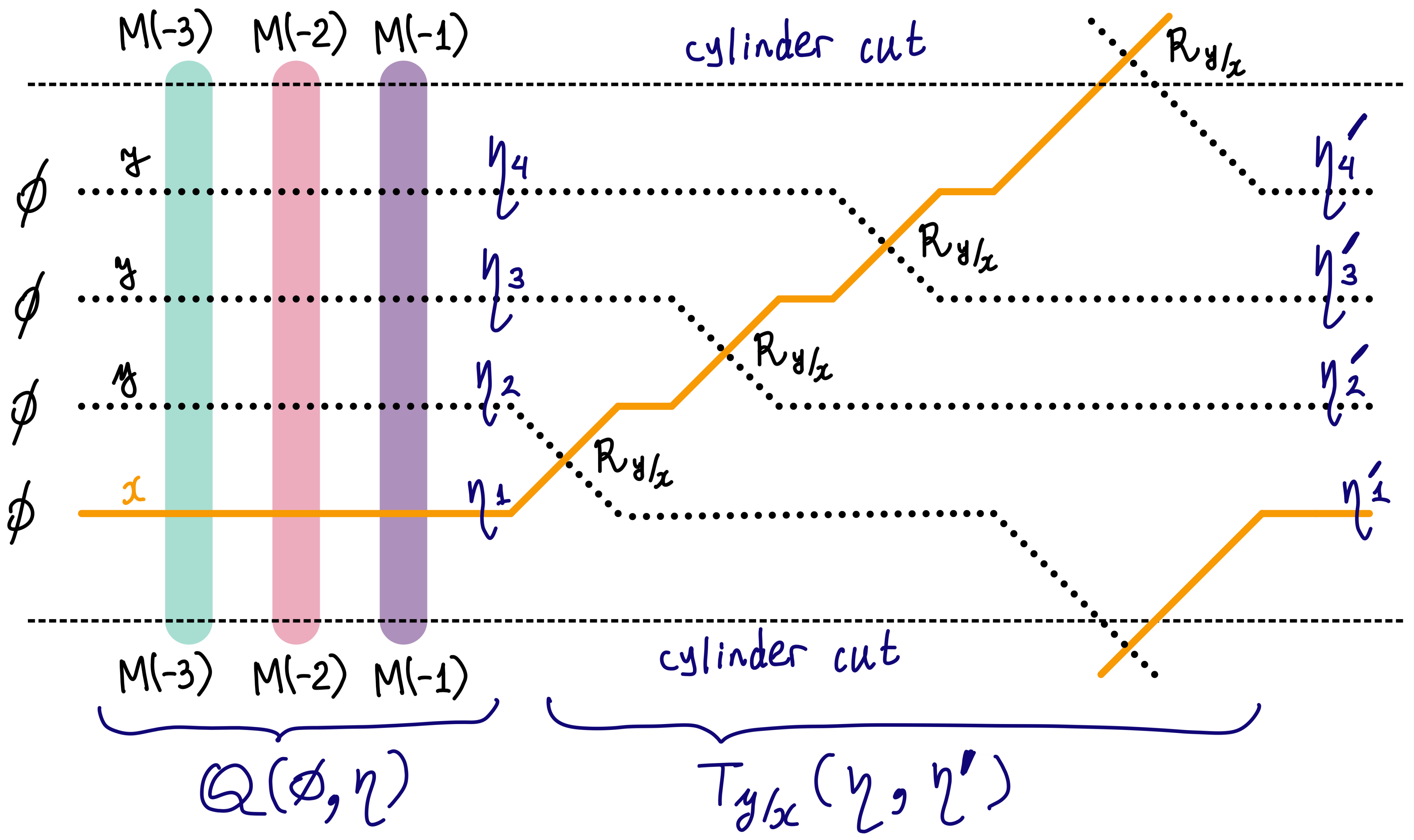
$$u = \frac{1-\delta_1}{1-\delta_2}$$

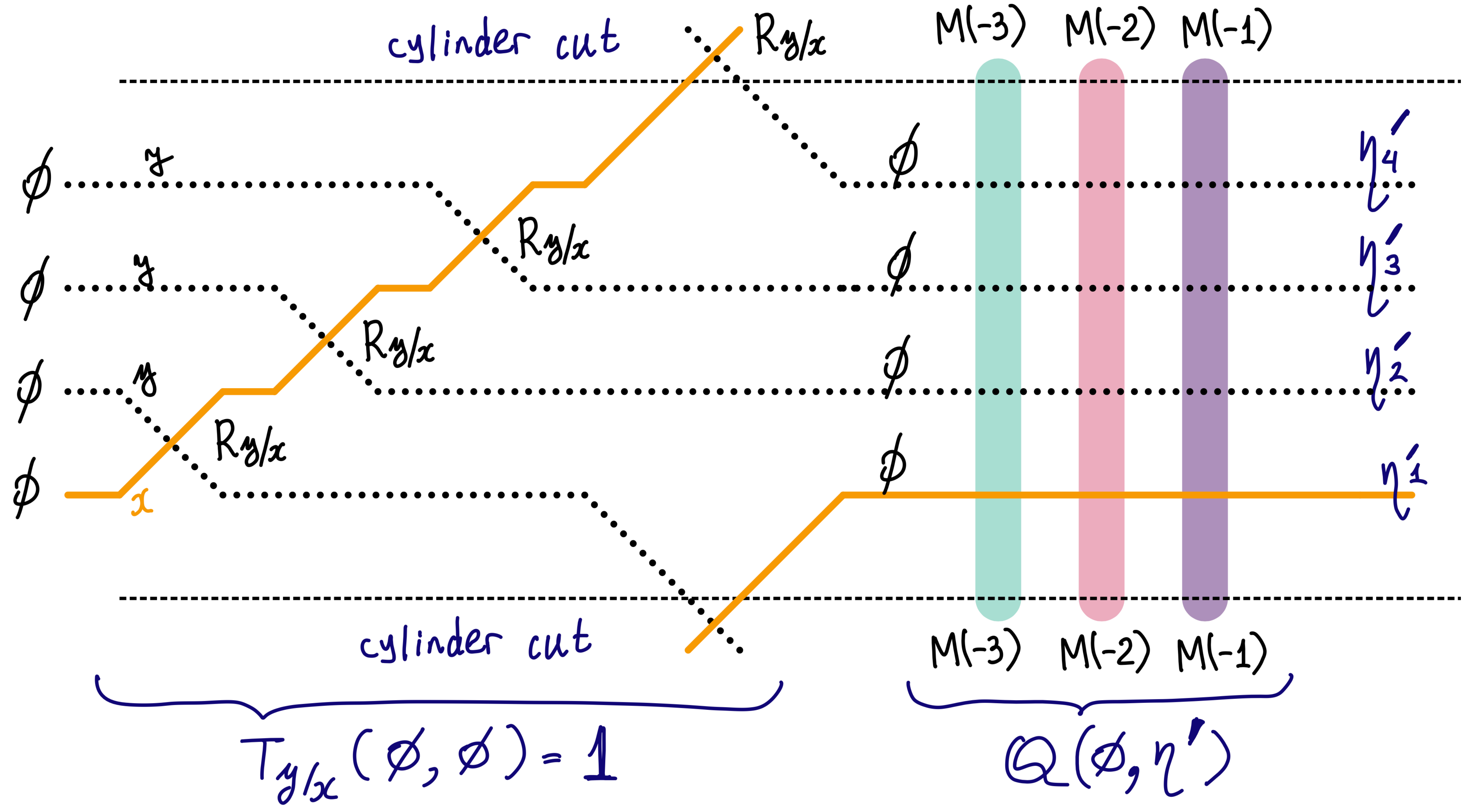
top $\mathbf{A}$; left 0, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}$; left k, right k; bottom $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_k^-$; left 0, right k; bottom $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}$; left 0, right m; bottom $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
top $\mathbf{A}_k^+$; left k, right 0; bottom $\mathbf{A}$ 1	top $\mathbf{A}_{k\ell}^{+-}$; left k, right ℓ; bottom $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	top $\mathbf{A}_{\ell k}^{+-}$; left ℓ, right k; bottom $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	top $\mathbf{A}_\ell^+$; left ℓ, right m; bottom $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)

Commutation relation on the cylinder

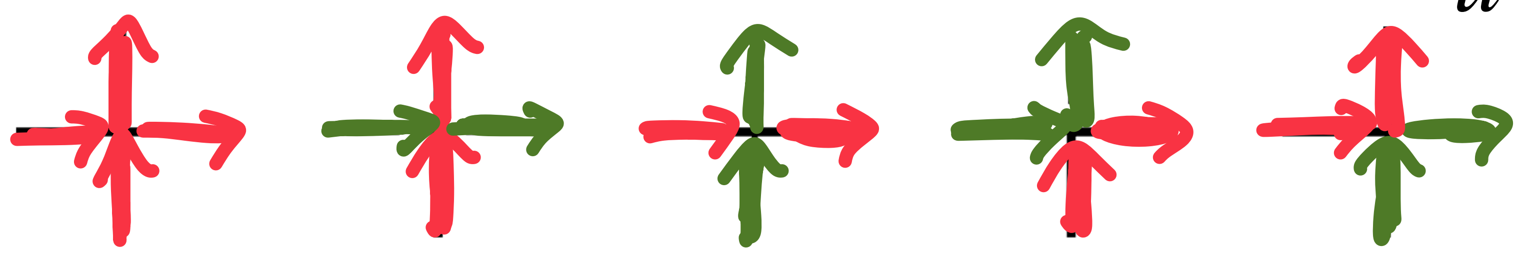
$$\sum_{\eta} \mathfrak{Q}(\varnothing,\eta)T_{y/x}(\eta,\eta') = T_{y/x}(\varnothing,\varnothing)\mathfrak{Q}(\varnothing,\eta') = \mathfrak{Q}(\varnothing,\eta')$$





Yang-Baxter equation on the $n \times N$ cylinder

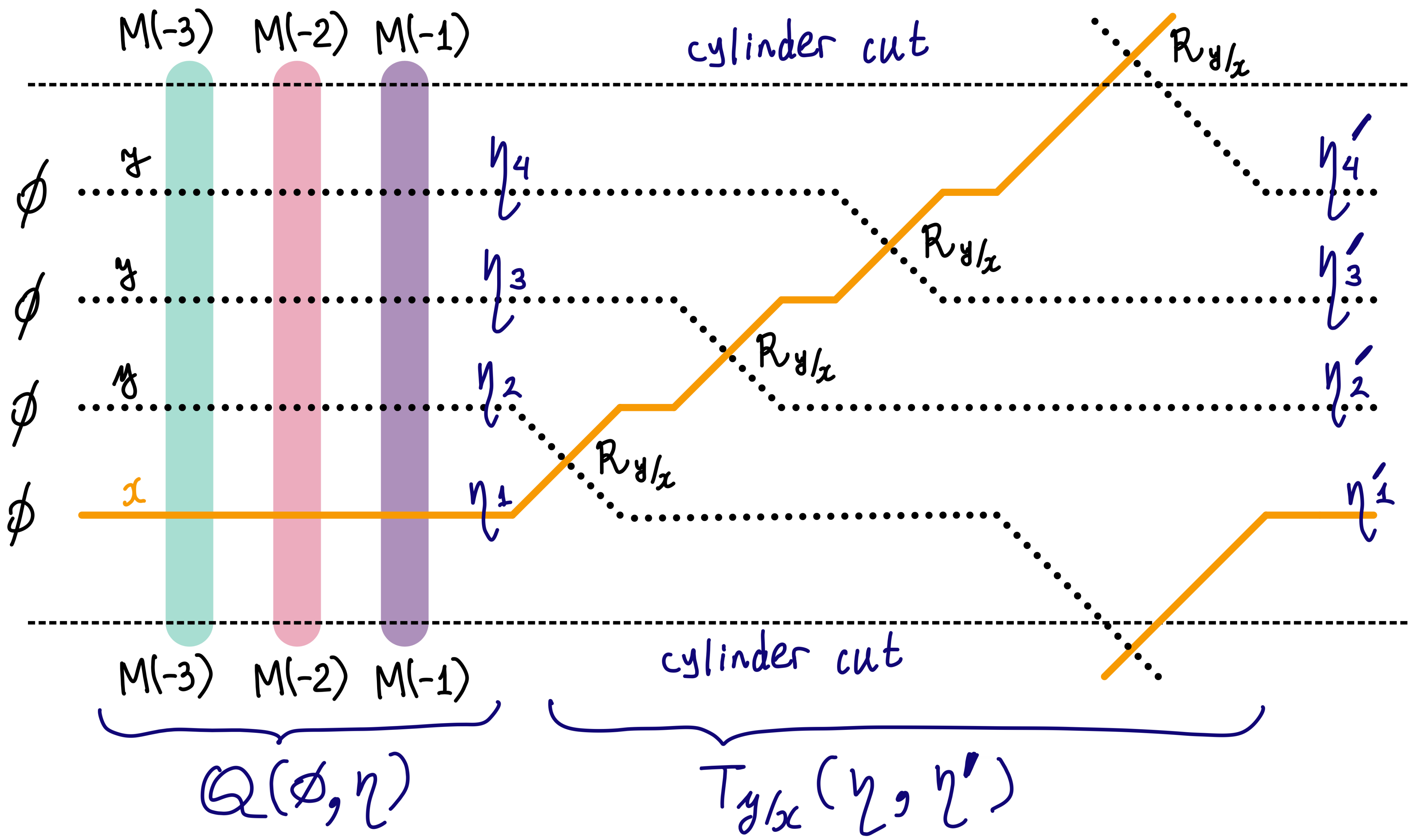
Red > Green $\quad R_u \quad u = \dfrac{1-\delta_1}{1-\delta_2}$



$1 \qquad \delta_1 \qquad \delta_2 \qquad 1-\delta_1 \qquad 1-\delta_2$

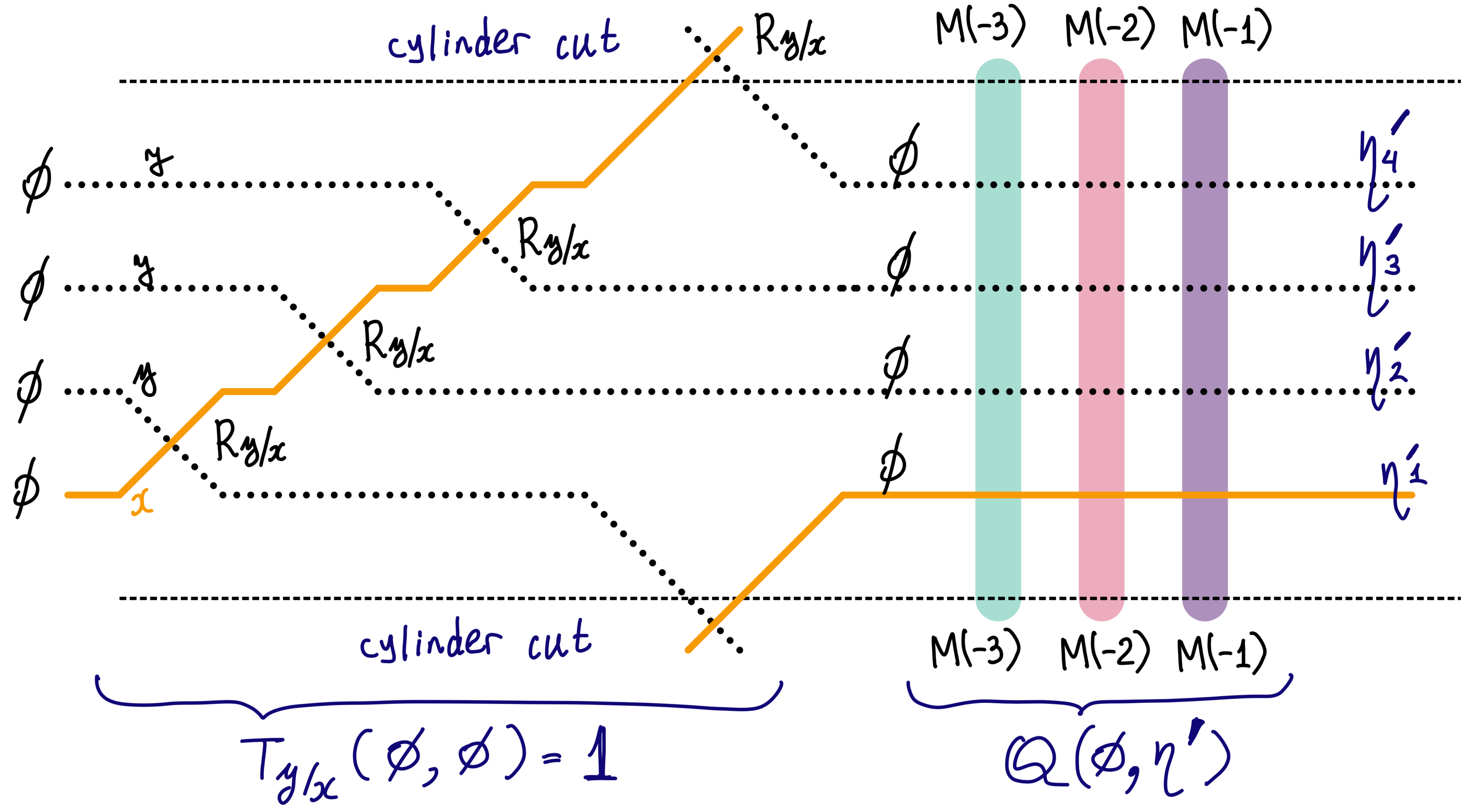
$\mathbf{A}$ 0 — 0 $\mathbf{A}$ 1	$\mathbf{A}$ k — k $\mathbf{A}$ $(x - sq^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_k^-$ 0 — k $\mathbf{A}$ $x(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}$ 0 — m $\mathbf{A}$ $xq^{\mathbf{A}_{[m+1,n]}}$
$\mathbf{A}_k^+$ k — 0 $\mathbf{A}$ 1	$\mathbf{A}_{k\ell}^{+-}$ k — ℓ $\mathbf{A}$ $x(1-q^{A_\ell})q^{\mathbf{A}_{[\ell+1,n]}}$	$\mathbf{A}_{\ell k}^{+-}$ ℓ — k $\mathbf{A}$ $s(1-q^{A_k})q^{\mathbf{A}_{[k+1,n]}}$	$\mathbf{A}_\ell^+$ ℓ — m $\mathbf{A}$ $sq^{\mathbf{A}_{[m+1,n]}}$

(These weights are not stochastic and have more parameters than on the line; all of this is okay on the ring)



Commutation relation on the cylinder

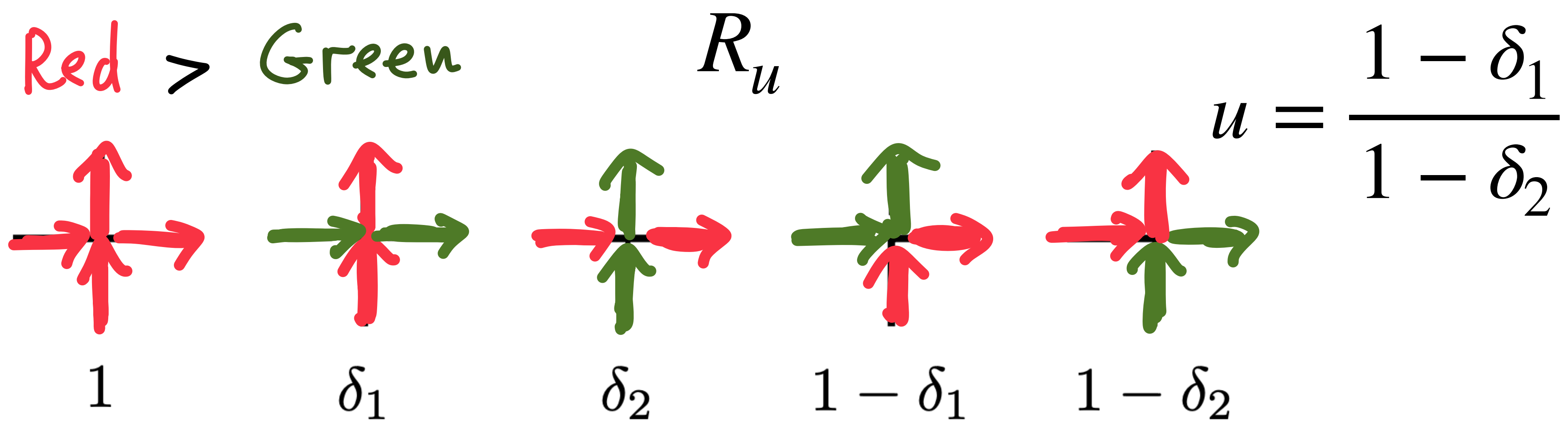
$$\sum_{\eta} \mathfrak{Q}(\varnothing, \eta) T_{y/x}(\eta, \eta') = T_{y/x}(\varnothing, \varnothing)\mathfrak{Q}(\varnothing, \eta') = \mathfrak{Q}(\varnothing, \eta')$$

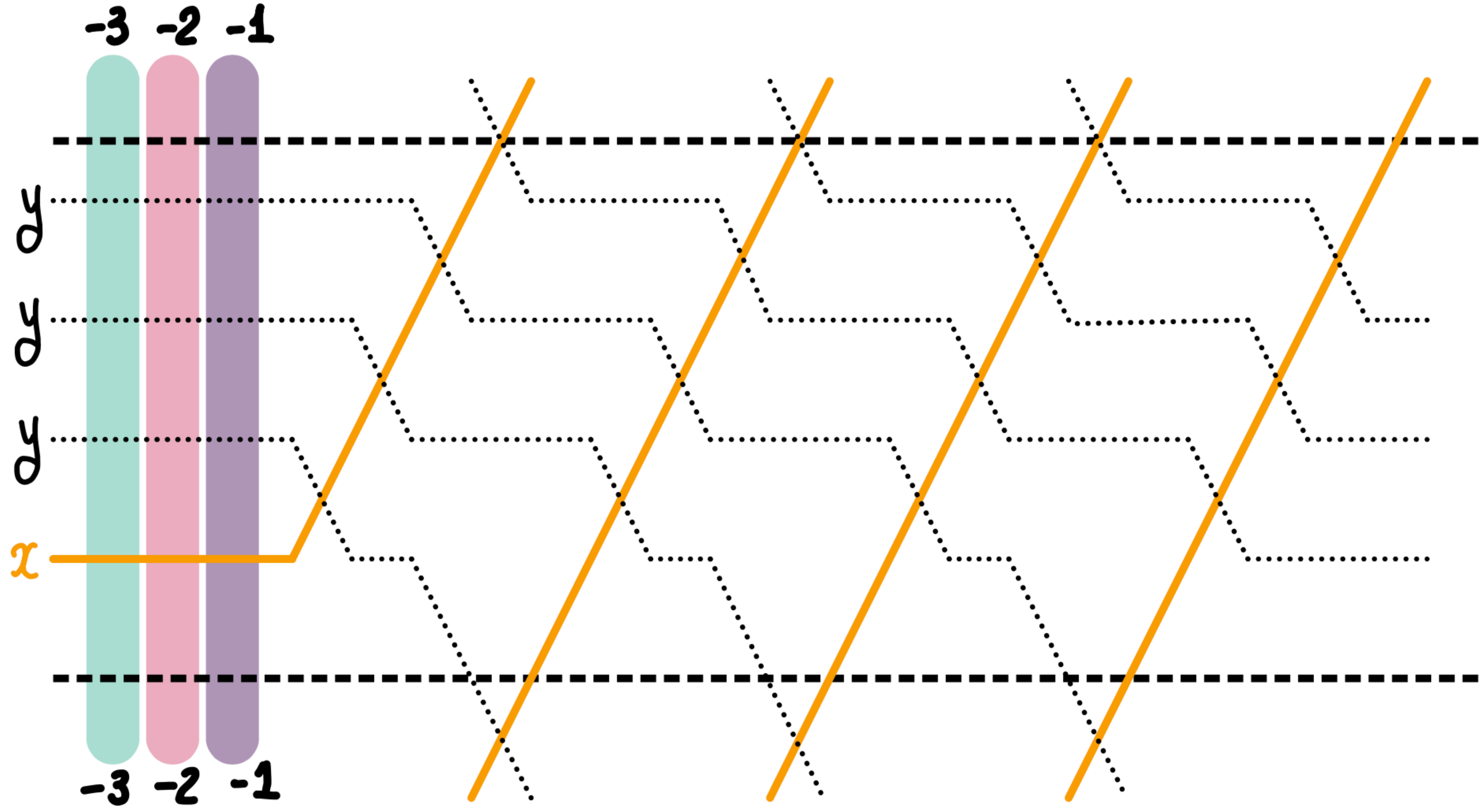


(Bethe Ansatz: construct eigenvalue of T as a partition function)

Limit to the mASEP, $y/x = 1 - \epsilon$, continuous time

mASEP limit $\delta_1, \delta_2 \to 0$, $q = \delta_1/\delta_2$

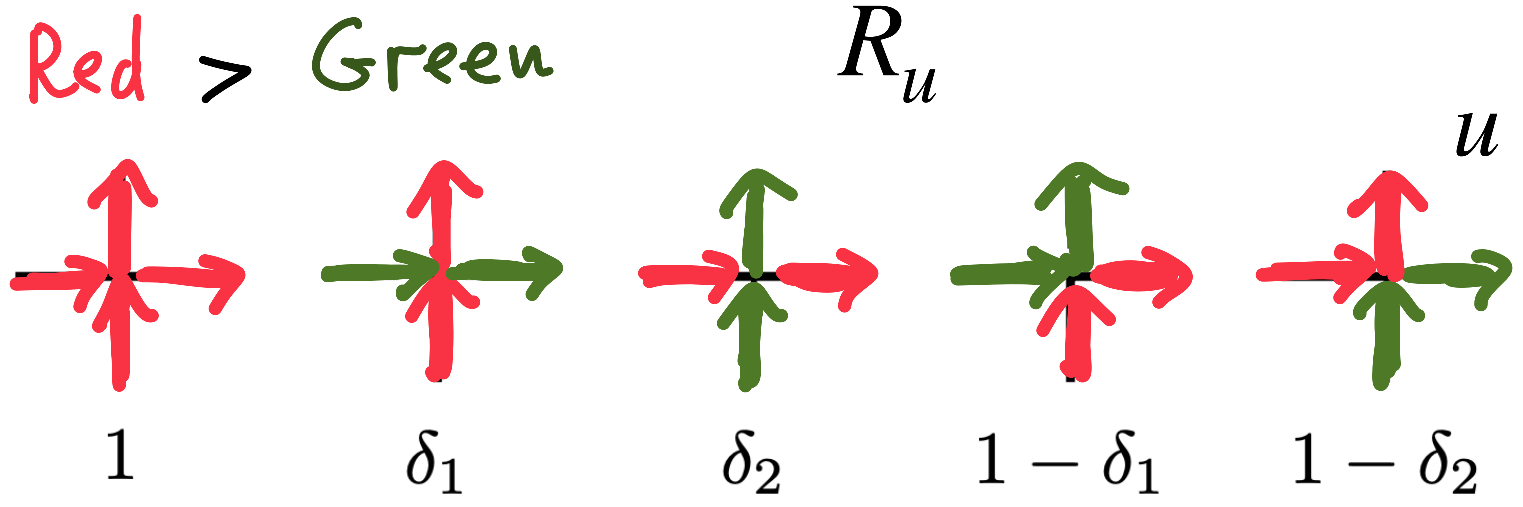




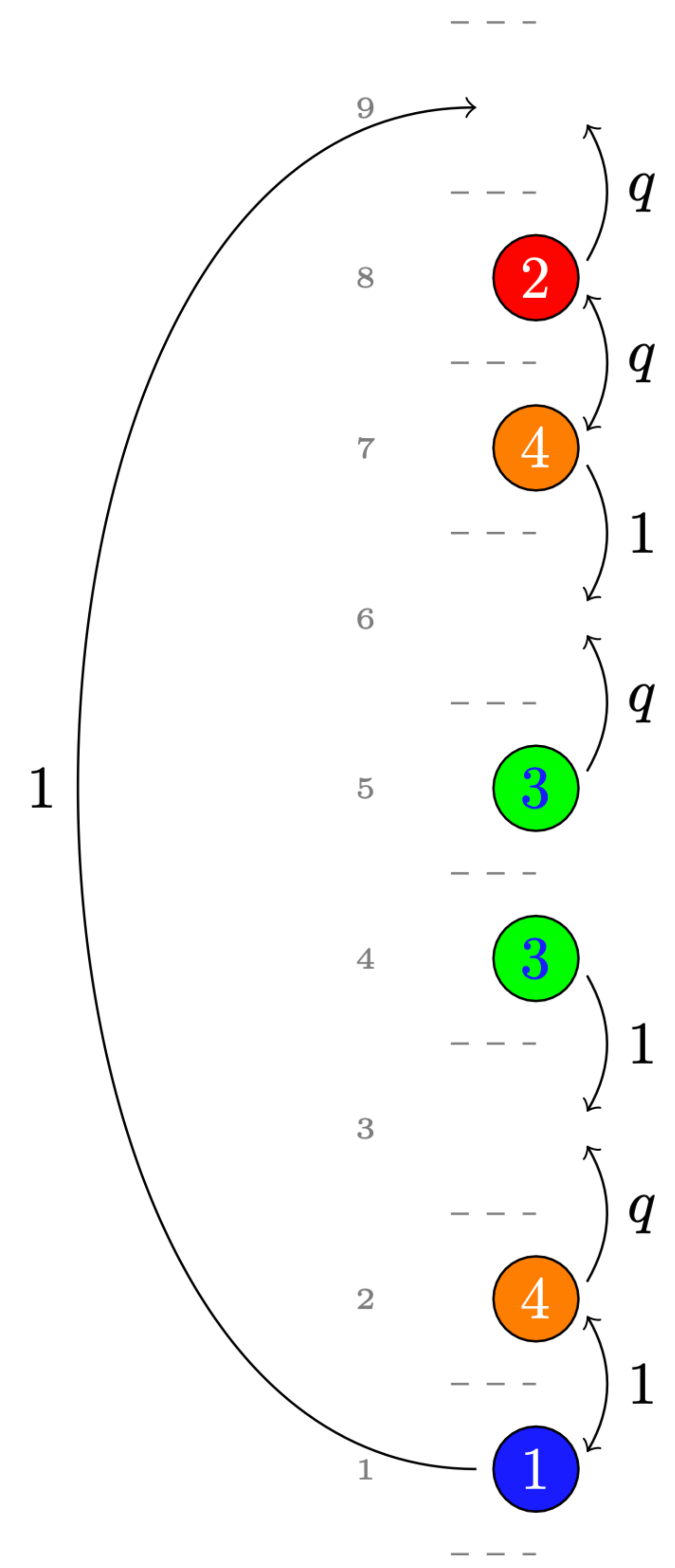
Time $\sim \tau/\epsilon$

Limit to the mASEP, $y/x = 1 - \epsilon$, continuous time

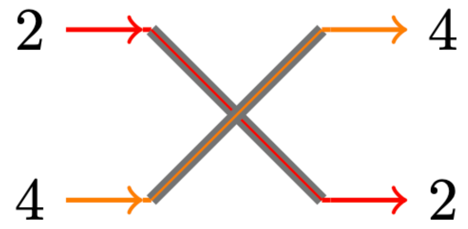
mASEP limit $\delta_1, \delta_2 \to 0$, $q = \delta_1/\delta_2$



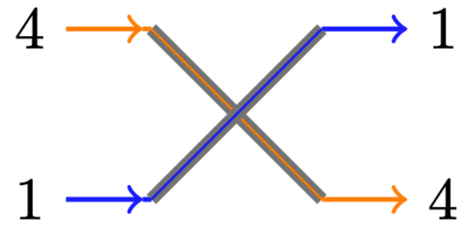
$$u = \frac{1-\delta_1}{1-\delta_2}$$

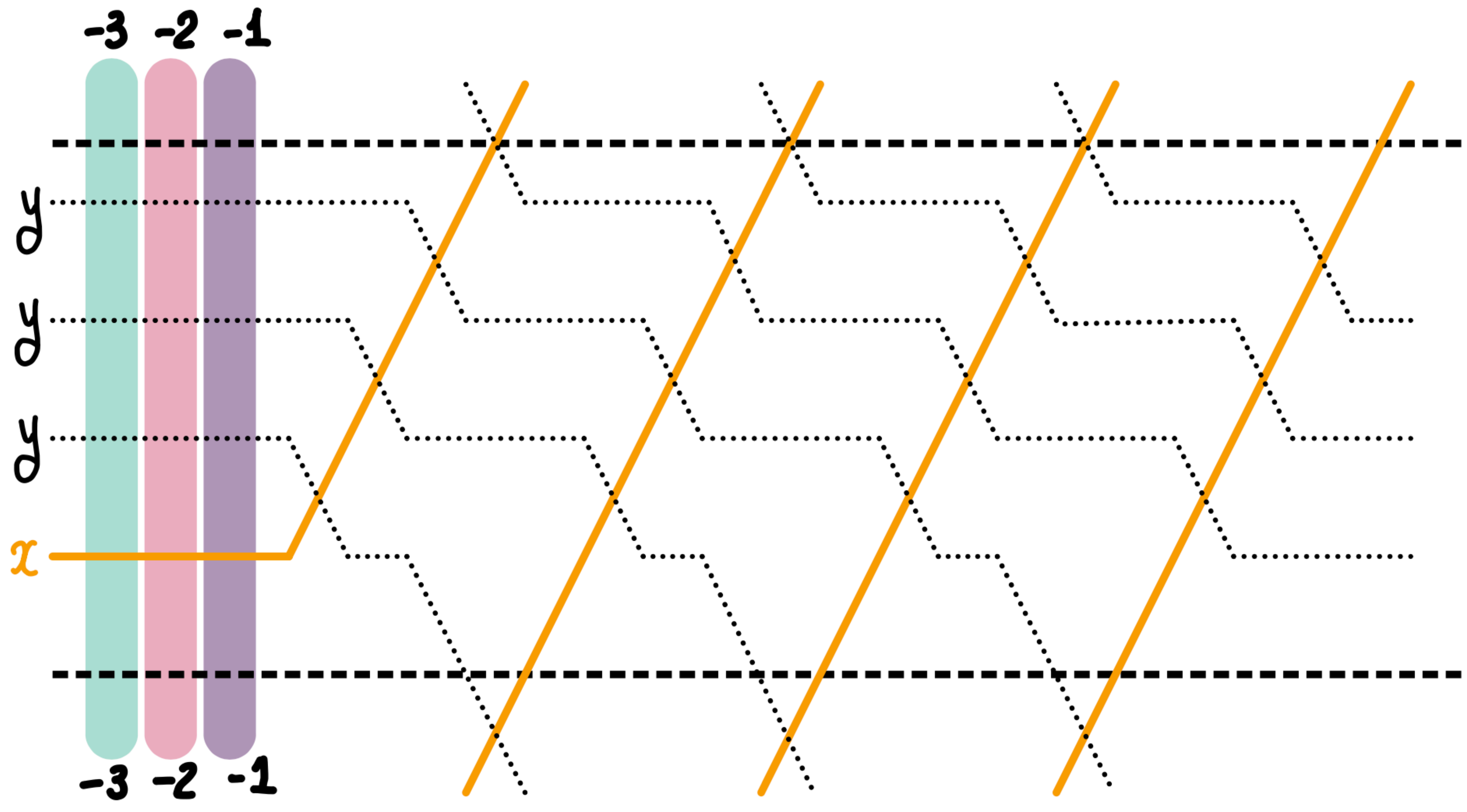


$$\langle 4,2|\check{\mathfrak{R}}_{1-\epsilon}|2,4\rangle = R_{1-\epsilon}(4,2,4,2) = \frac{q\epsilon}{1-q} + O(\epsilon^2)$$



$$\langle 1,4|\check{\mathfrak{R}}_{1-\epsilon}|4,1\rangle = R_{1-\epsilon}(1,4;1,4) = \frac{\epsilon}{1-q} + O(\epsilon^2)$$



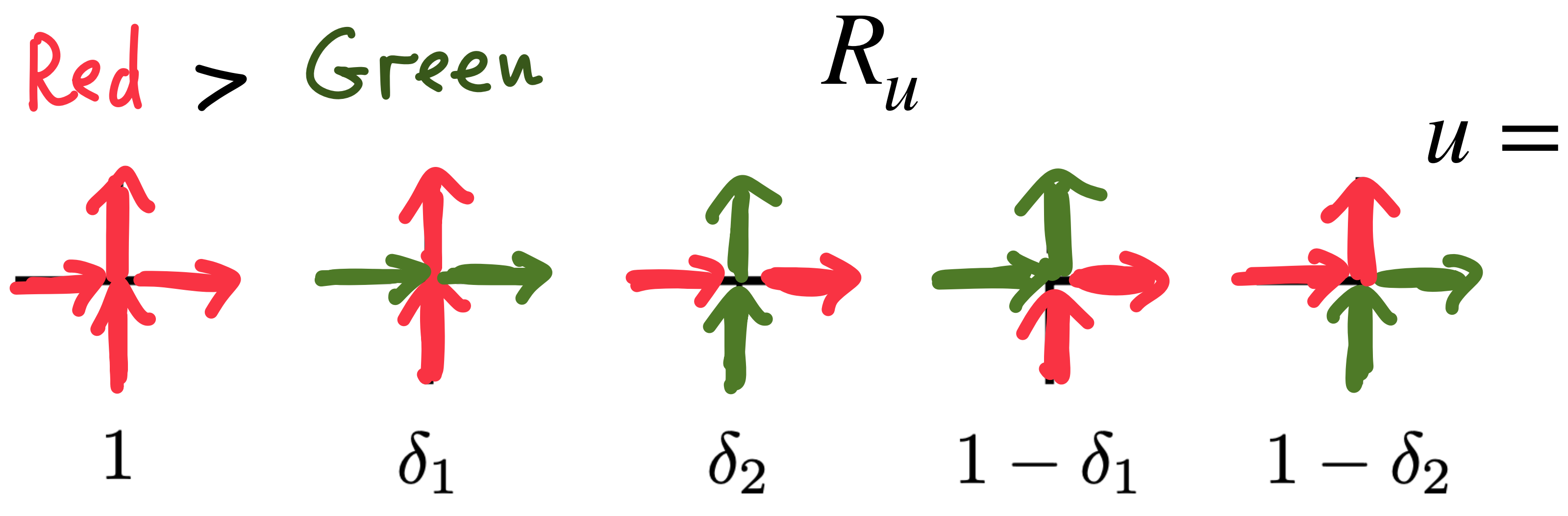


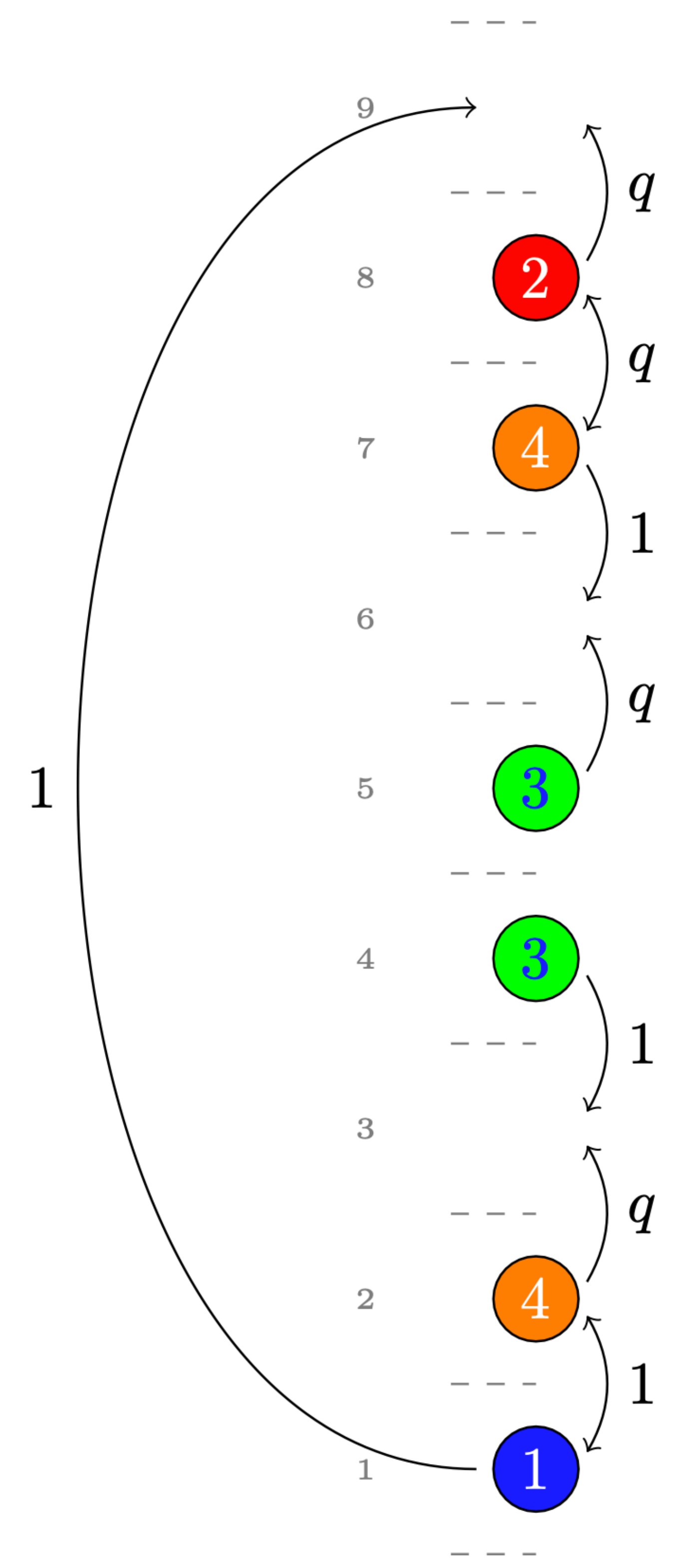
Time $\sim \tau/\epsilon$

Limit to the mASEP, $y/x = 1 - \epsilon$, continuous time

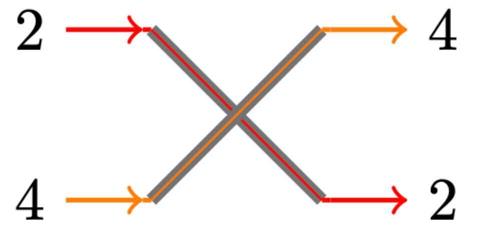
mASEP limit $\delta_1, \delta_2 \to 0$, $q = \delta_1/\delta_2$

R_u $\qquad u = \dfrac{1-\delta_1}{1-\delta_2}$

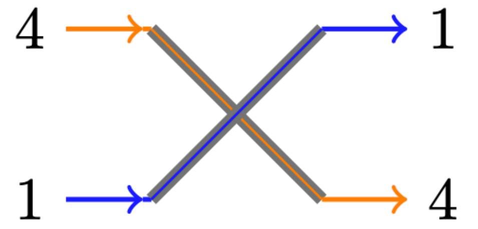


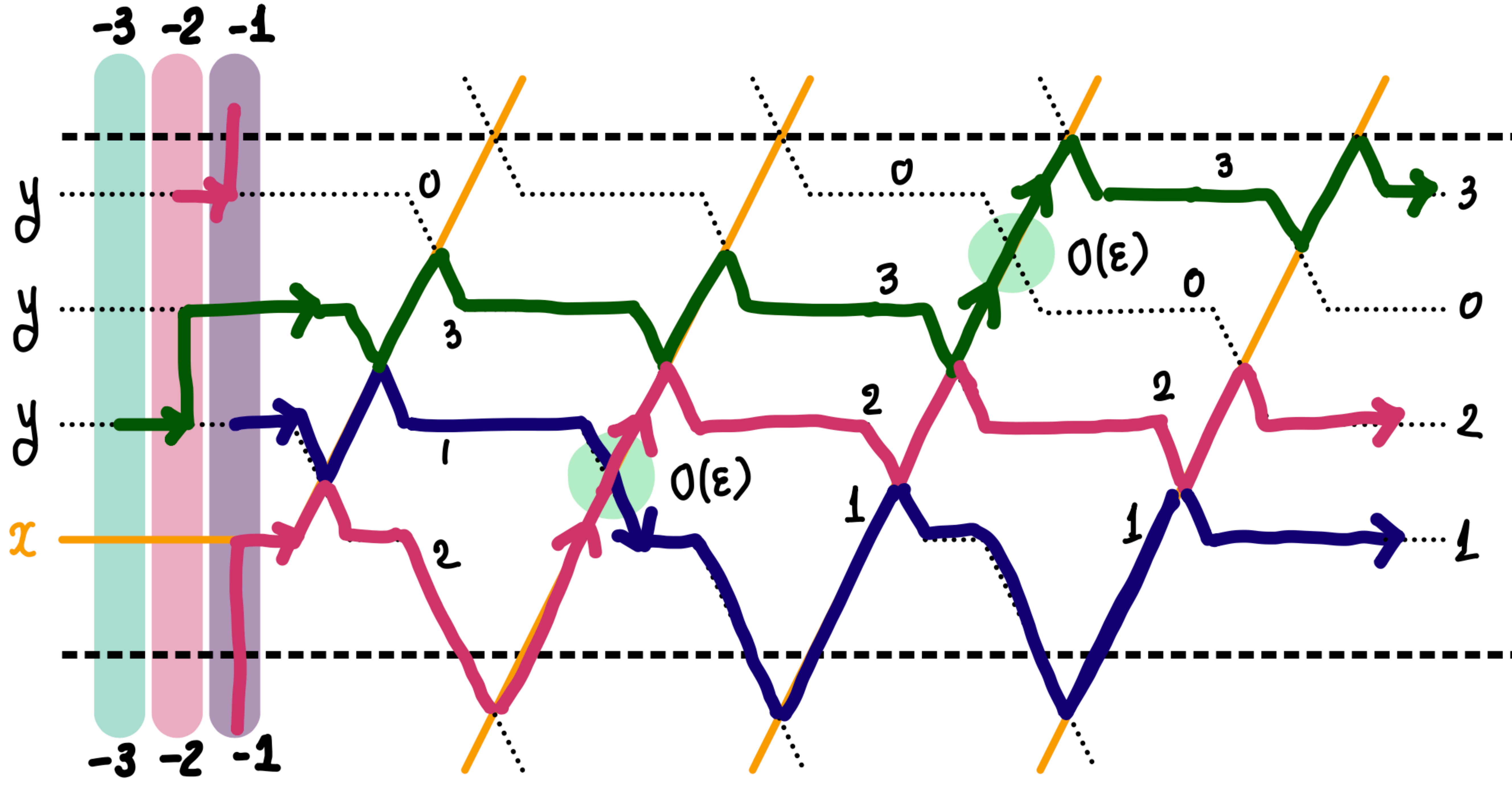


$$\langle 4,2|\check{\mathcal{R}}_{1-\epsilon}|2,4\rangle = R_{1-\epsilon}(4,2,4,2) = \frac{q\epsilon}{1-q} + O(\epsilon^2)$$



$$\langle 1,4|\check{\mathcal{R}}_{1-\epsilon}|4,1\rangle = R_{1-\epsilon}(1,4;1,4) = \frac{\epsilon}{1-q} + O(\epsilon^2)$$

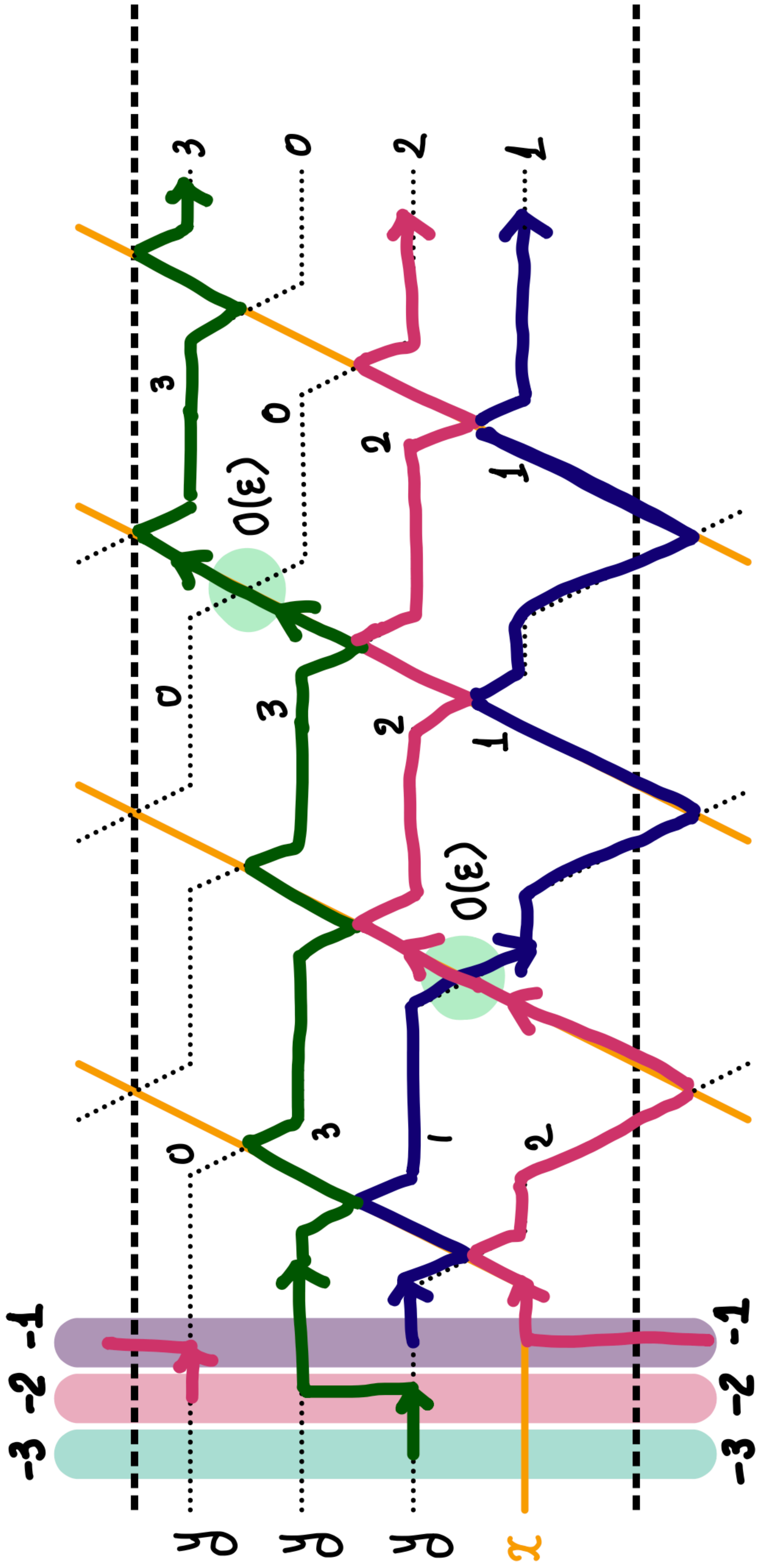




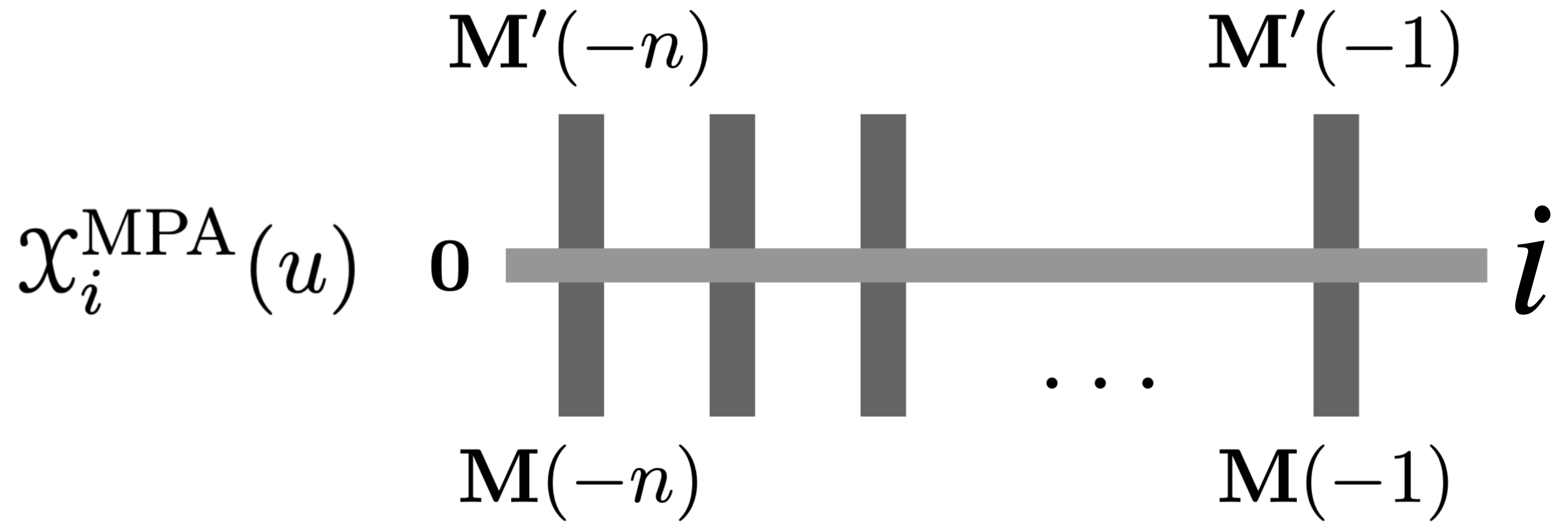
Time $\sim \tau/\epsilon$

Conclusions

- A lot of recent activity around stationary measures for colored (also called multi-species or multi-type) and monochrome interacting particle systems in different geometries (line, ring, half-space, segment).
  - Motivated by asymptotic phenomena (microscopic characteristics, stationary measures for KPZ equation)
  - Rich algebraic and combinatorial structure (e.g. nonsymmetric Macdonald polynomials)
- We show that the ring, line, and quadrant stationarity follow directly from the Yang-Baxter equation.
- Other geomeries?
- Box ball systems?
- Stationary horizons / speed processes?



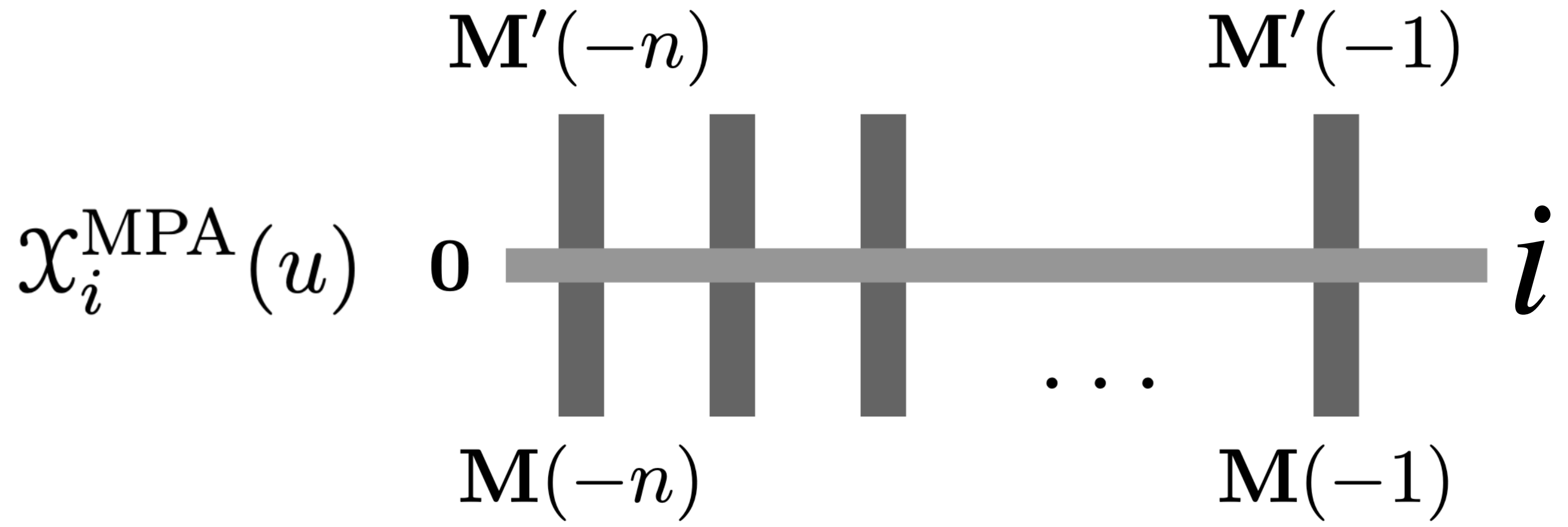
Bonus: Matrix Product Ansatz from Yang-Baxter equation



Matrix Product Ansatz expression
for the mASEP stationary measure

$$\mathrm{Prob}^{\mathrm{mASEP}}_{N_1,\ldots,N_n}(\eta) = \frac{\mathrm{Trace}\left(\mathcal{X}^{\mathrm{MPA}}_{\eta_1}\cdots\mathcal{X}^{\mathrm{MPA}}_{\eta_N}\right)}{Z^{\mathrm{MPA}}_{N_1,\ldots,N_n}}$$

Bonus: Matrix Product Ansatz from Yang-Baxter equation



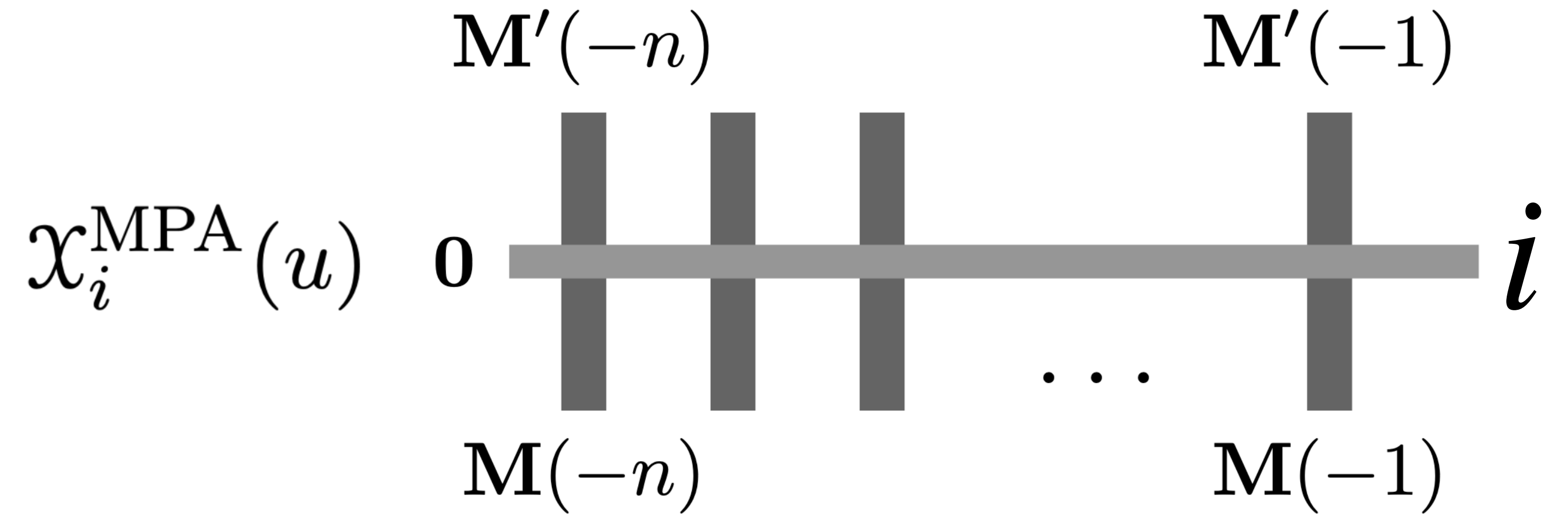
Matrix Product Ansatz expression for the mASEP stationary measure

$$\mathrm{Prob}^{\mathrm{mASEP}}_{N_1,\ldots,N_n}(\eta) = \frac{\mathrm{Trace}\left(\mathcal{X}^{\mathrm{MPA}}_{\eta_1}\cdots\mathcal{X}^{\mathrm{MPA}}_{\eta_N}\right)}{Z^{\mathrm{MPA}}_{N_1,\ldots,N_n}}$$

Key identity in the stationarity proof: existence of auxiliary matrices in **[Prolhac–Evans–Mallick 2009]**

$$\sum_{i,i'=0}^{n} \mathcal{X}^{\mathrm{MPA}}_{i}\mathcal{X}^{\mathrm{MPA}}_{i'}\left(\mathcal{M}_{loc}\right)_{ii',jj'} = \mathcal{X}^{\mathrm{MPA}}_{j}\widehat{\mathcal{X}}^{\mathrm{MPA}}_{j'} - \widehat{\mathcal{X}}^{\mathrm{MPA}}_{j}\mathcal{X}^{\mathrm{MPA}}_{j'}$$

Bonus: Matrix Product Ansatz from Yang-Baxter equation



Matrix Product Ansatz expression
for the mASEP stationary measure

$$\mathrm{Prob}^{\mathrm{mASEP}}_{N_1,\ldots,N_n}(\eta) = \frac{\mathrm{Trace}\left(\mathcal{X}^{\mathrm{MPA}}_{\eta_1}\cdots\mathcal{X}^{\mathrm{MPA}}_{\eta_N}\right)}{Z^{\mathrm{MPA}}_{N_1,\ldots,N_n}}$$

Key identity in the stationarity proof:
existence of auxiliary matrices in
[Prolhac–Evans–Mallick 2009]

$$\sum_{i,i'=0}^{n} \mathcal{X}^{\mathrm{MPA}}_{i}\mathcal{X}^{\mathrm{MPA}}_{i'}\left(\mathcal{M}_{loc}\right)_{ii',jj'} = \mathcal{X}^{\mathrm{MPA}}_{j}\widehat{\mathcal{X}}^{\mathrm{MPA}}_{j'} - \widehat{\mathcal{X}}^{\mathrm{MPA}}_{j}\mathcal{X}^{\mathrm{MPA}}_{j'}$$

Yang-Baxter equation

$$\sum_{i,i'=0}^{n} \mathcal{X}^{\mathrm{MPA}}_{i}(u)\,\mathcal{X}^{\mathrm{MPA}}_{i'}(u(1-\epsilon))\cdot R_{1-\epsilon}(i,i';j',j) = \mathcal{X}^{\mathrm{MPA}}_{j}(u(1-\epsilon))\,\mathcal{X}^{\mathrm{MPA}}_{j'}(u).$$

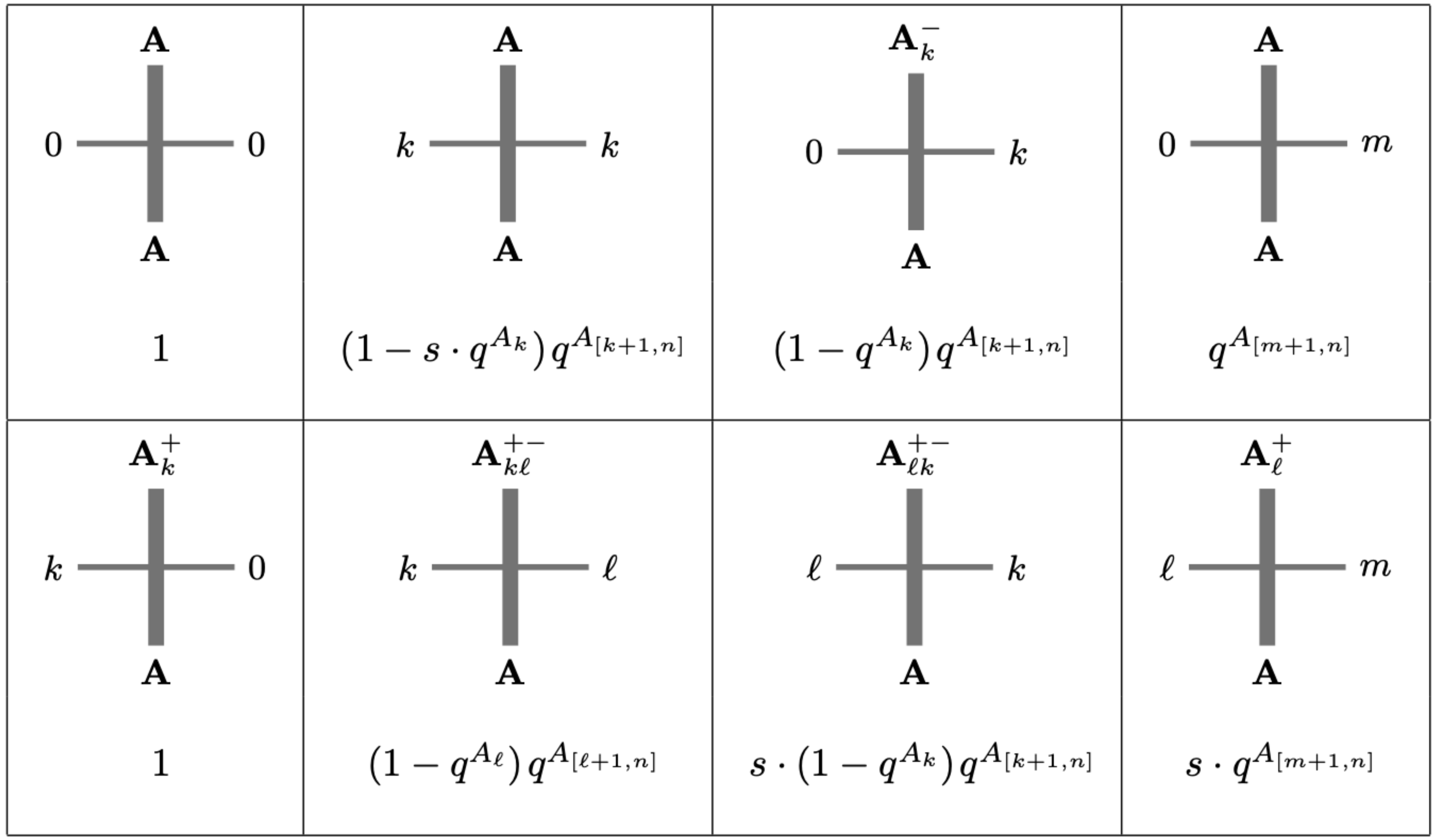
$$\widehat{\mathcal{X}}^{\mathrm{MPA}}_{j}(u) \coloneqq (1-q)\,u\,\frac{\partial}{\partial u}\,\mathcal{X}^{\mathrm{MPA}}_{j}(u)$$

Bonus: Matrix Product Ansatz from Yang-Baxter equation

$$AD - qDA = EA - qAE = (1-q)A, \qquad ED - qDE = (1-q)(E+D).$$

$$A \coloneqq \begin{pmatrix} 1 & s & 0 & \dots \\ 0 & q & qs & \dots \\ 0 & 0 & q^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad D \coloneqq u^{-1} \begin{pmatrix} u-s & 0 & 0 & \dots \\ 1-q & u-sq & 0 & \dots \\ 0 & 1-q^2 & u-sq^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$E \coloneqq \begin{pmatrix} 1 & u & 0 & \dots \\ 0 & 1 & u & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$



$s = 0$: **[Prolhac–Evans–Mallick 2009]**

$s = q$: conjectured alternative queues **[Martin 2018]**

General s: interpolation

Thank you for attention!

Special thanks to the organizers of the conference