Schur rational functions, vertex models, and random tilings

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j.w. Amol Aggarwal, Alexei Borodin, Michael Wheeler (https://arxiv.org/abs/2109.06718)

(119 pages...)

I. Symmetric functions as partition function of vertex models

Schur symmetric polynomials

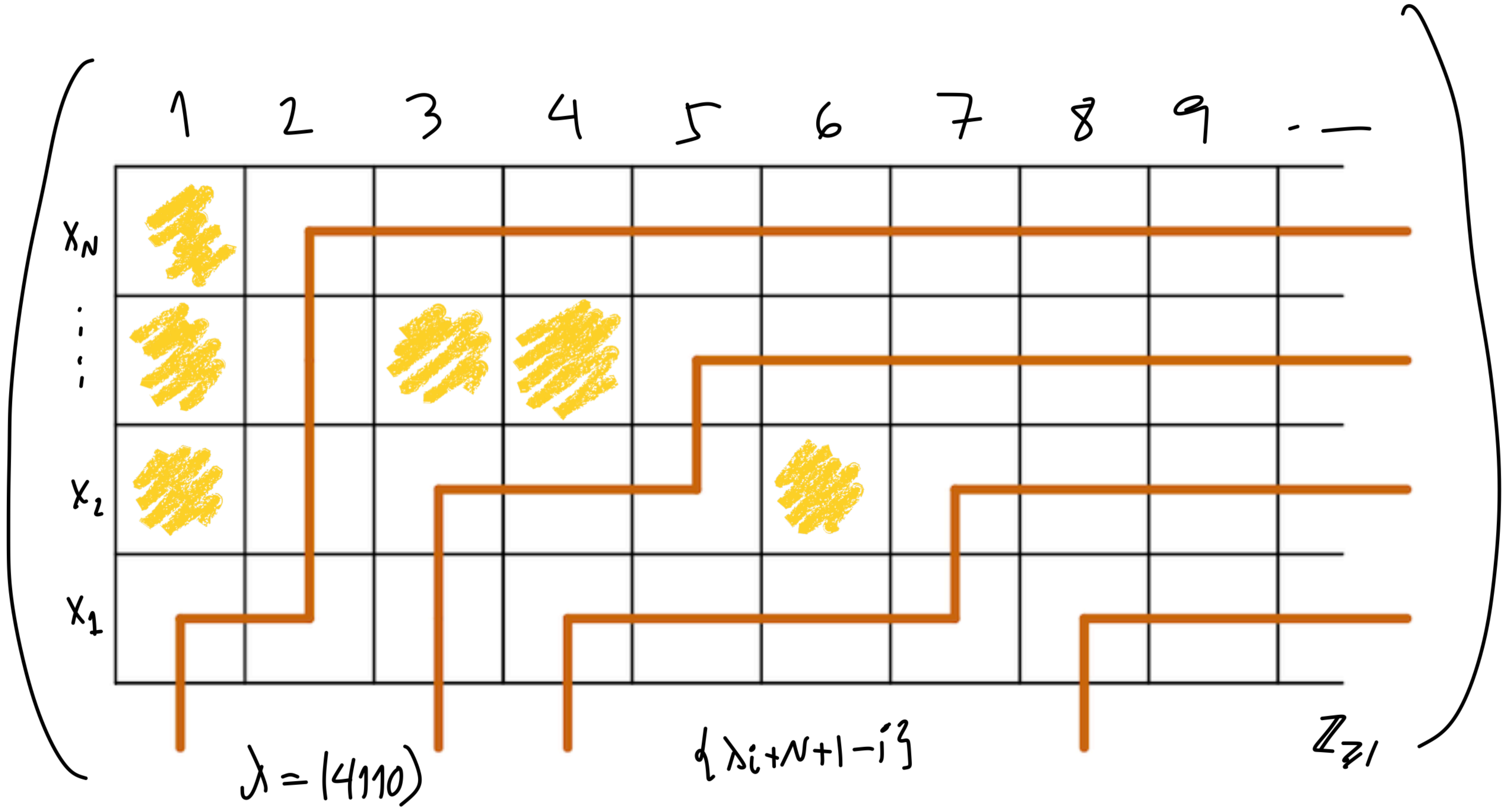
$\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$ - partition

$s_\lambda(x_1, \ldots, x_N) = \dfrac{\det[x_i^{\lambda_j + N - j}]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$ - Schur symmetric polynomial

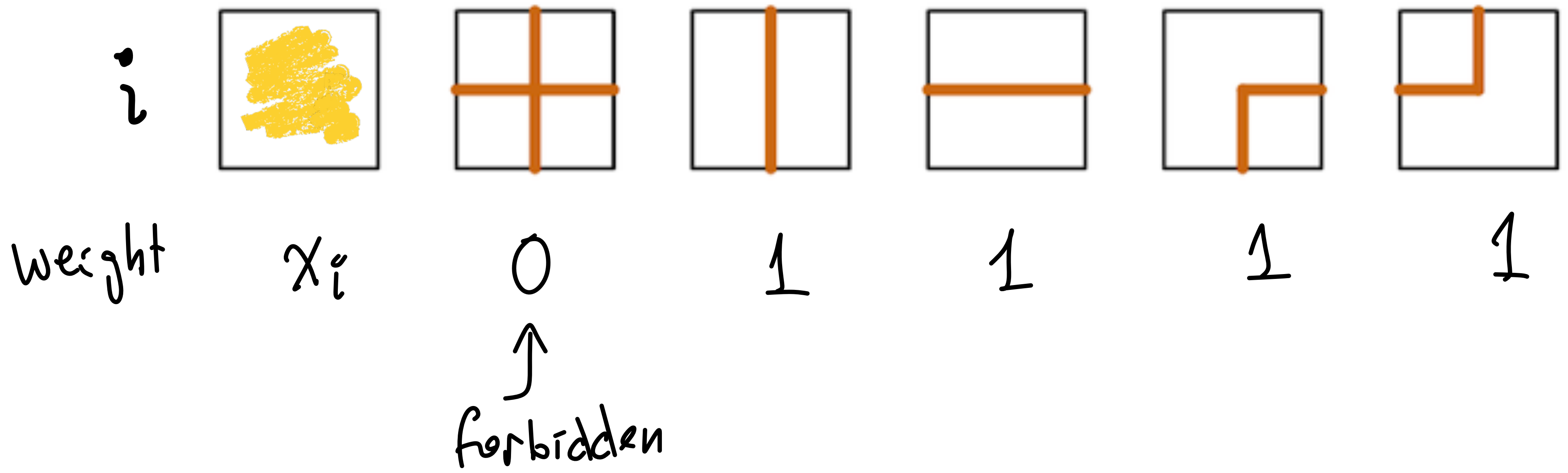
Exercise (Combinatorial formula).

The polynomial $s_\lambda(x_1, \ldots, x_N)$ is equal to the *partition function* - the sum of weights of all collections of up-right paths in $\mathbb{Z}_{\geq 1} \times \{1, \ldots, N\}$ with boundary conditions determined by λ, where the weight of a path collection is the product of the **vertex weights**

$S_\lambda(x_1, \ldots, x_N) = Z$

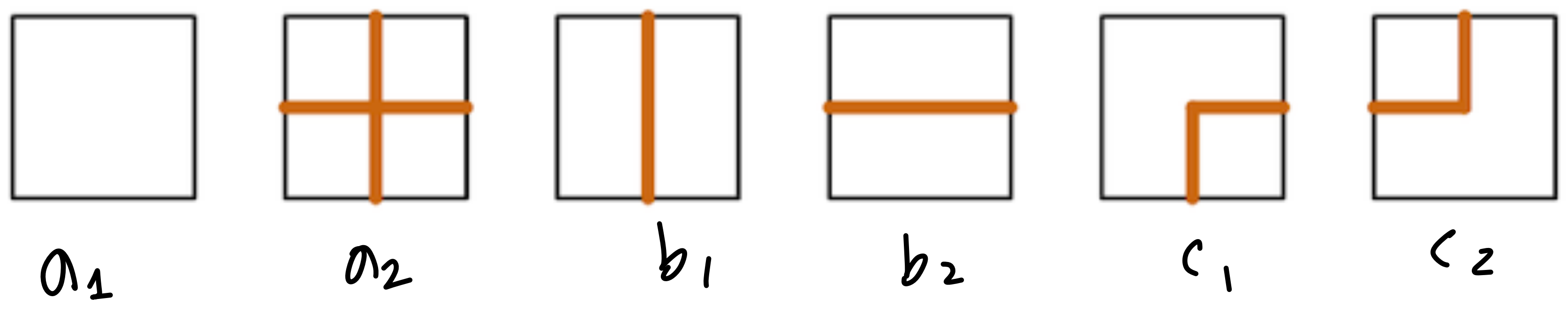


weight $= x_2^2 \, x_3^3 \, x_4$



one of possible five-vertex models

Vertex models



Moreover: $a_1(i,j)$, $i = 1 \ldots N$, $j \geq 1$

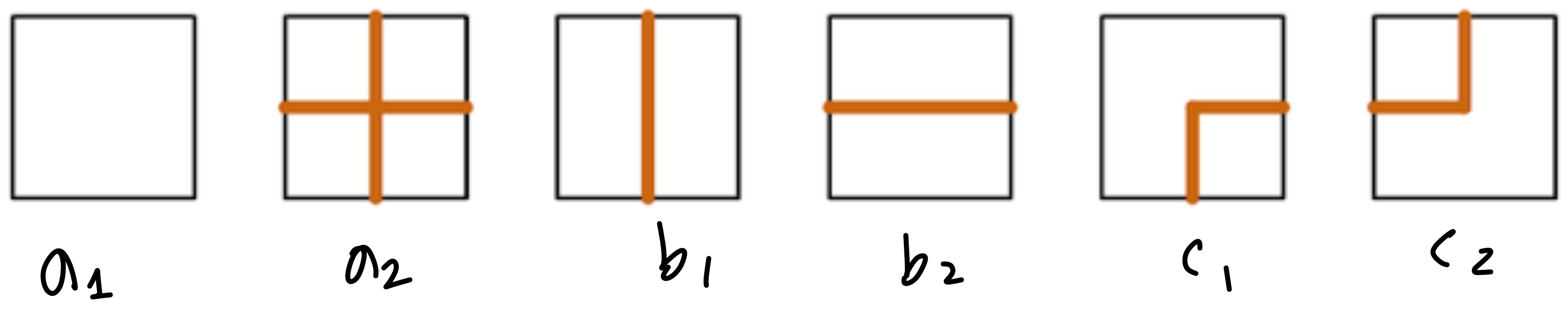
Previous vertex weights for s_λ were too simple! They can be much more general:

- We can allow paths to intersect
- There are 6 choices of weights, traditional notation $a_1, a_2, b_1, b_2, c_1, c_2$
- The weights $a_1, a_2, b_1, b_2, c_1, c_2$ may depend on the lattice site as $a_1(i, j)$, etc.

But, with the dependence on i, j it is much too general - we lose integrability.

At a minimum, $\Delta = \dfrac{a_1a_2 + b_1b_2 - c_1c_2}{2\sqrt{a_1a_2b_1b_2}}$ should stay the same throughout the lattice.

Vertex models



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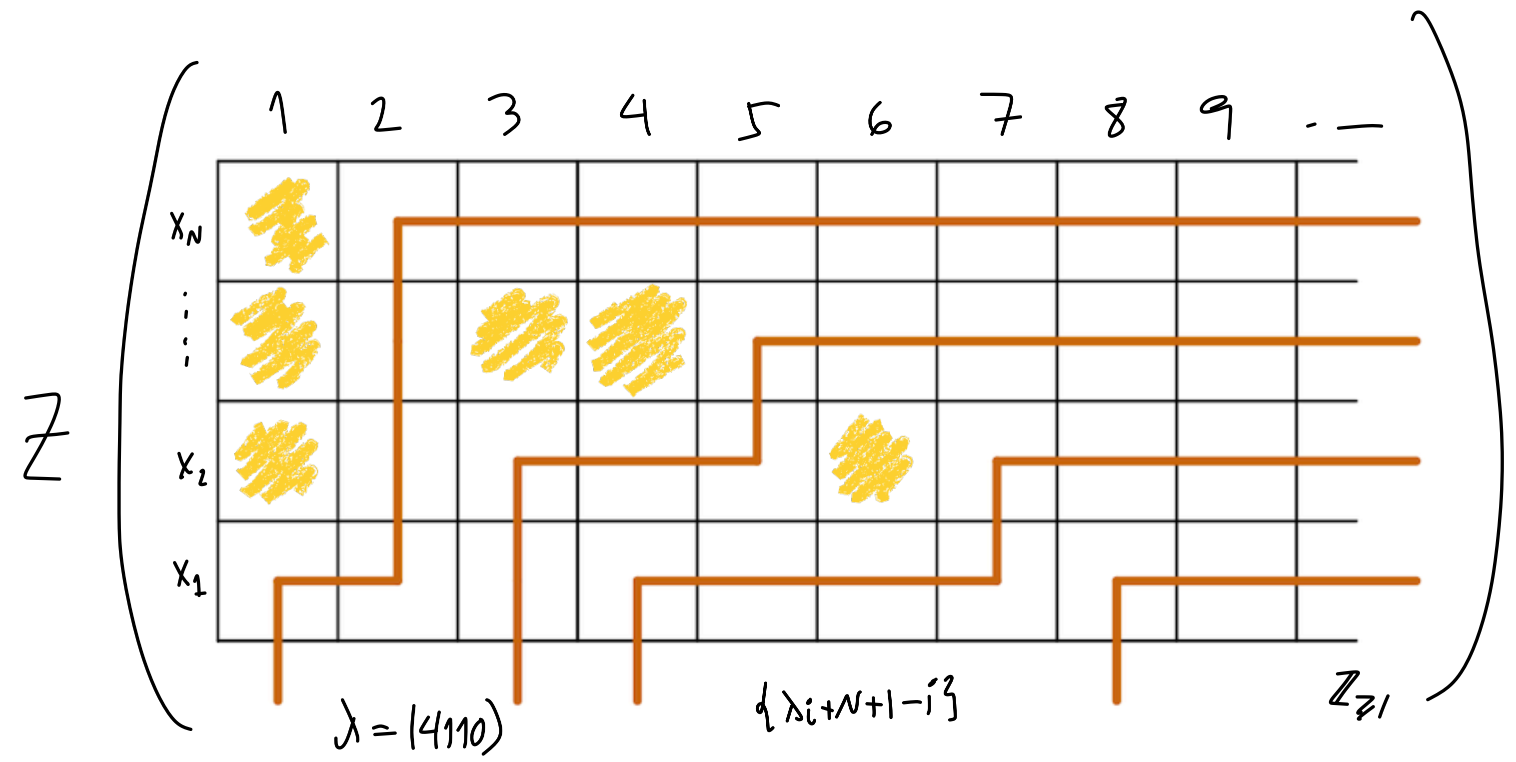
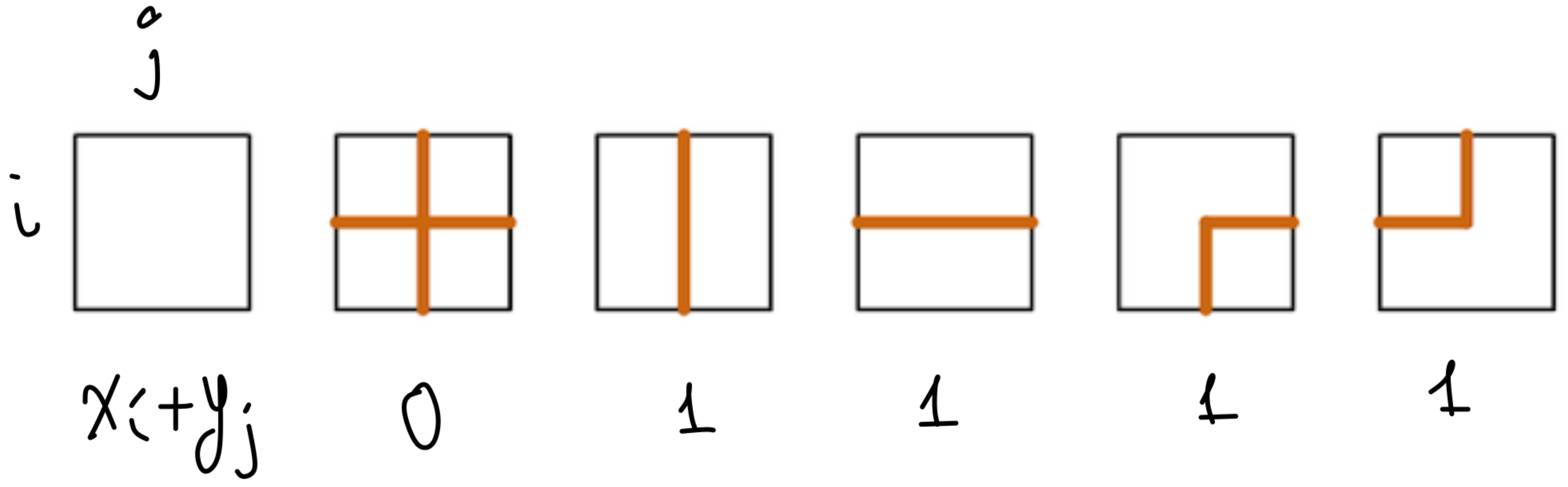


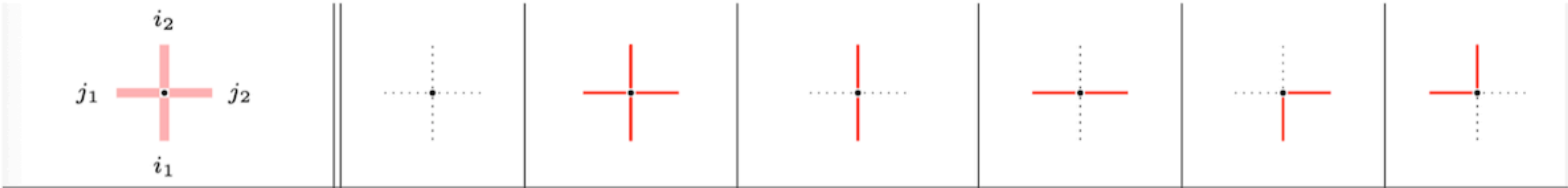
We focus on the **free fermion six vertex model** where $\Delta = 0$.
(≈ determinantal)

Simpler running example. *Factorial Schur functions:*



$$s_\lambda(\mathbf{x} \mid \mathbf{y}) = \frac{\det\left[(x_i \mid \mathbf{y})^{\lambda_j + N - j}\right]_{i,j=1}^{N}}{\prod_{1 \leq i < j \leq N}(x_i - x_j)}, \qquad (x \mid \mathbf{y})^k := (x + y_1)\dots(x + y_k)$$

Fully inhomogeneous free fermion six vertex model (our definition)



$w_{6\mathrm{V}}(i_1,j_1;i_2,j_2)$	a_1	a_2	b_1	b_2	c_1	c_2
$W(i_1,j_1;i_2,j_2)$	1	$\frac{\theta^{-2}w-y}{s^{-2}y-w}$	$\frac{s^{-2}y-\theta^{-2}w}{s^{-2}y-w}$	$\frac{y-w}{s^{-2}y-w}$	$\frac{w(\theta^{-2}-1)}{s^{-2}y-w}$	$\frac{y(s^{-2}-1)}{s^{-2}y-w}$

$\widehat{W}(i_1,j_1;i_2,j_2)$	$\frac{s^{-2}y-x}{y-x}$	$\frac{r^{-2}x-y}{y-x}$	$\frac{s^{-2}y-r^{-2}x}{y-x}$	1	$\frac{x(r^{-2}-1)}{y-x}$	$\frac{y(s^{-2}-1)}{y-x}$

- Renormalize so that the infinitely often vertex has weight 1
- Plus free fermion condition, leaves us with 4 parameters
- Organize the parameters by rows and columns in the interest of integrability (via **Yang-Baxter equation**)
- Row parameters are **variables**

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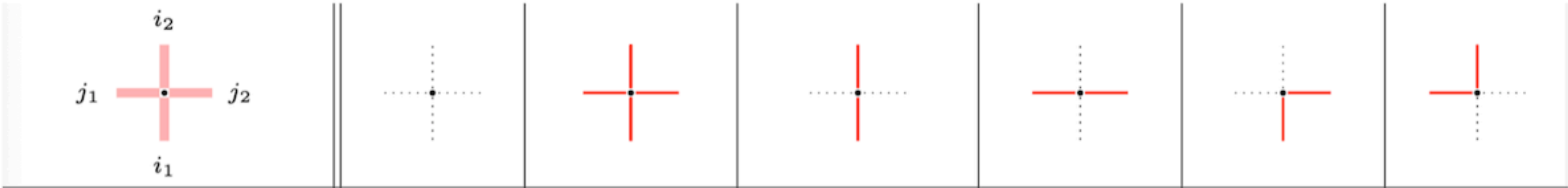

$w_{6V}(i_1, j_1; i_2, j_2)$	a_1	a_2	b_1	b_2	c_1	c_2
$W(i_1, j_1; i_2, j_2)$	1	$\frac{\theta^{-2}w - y}{s^{-2}y - w}$	$\frac{s^{-2}y - \theta^{-2}w}{s^{-2}y - w}$	$\frac{y - w}{s^{-2}y - w}$	$\frac{w(\theta^{-2} - 1)}{s^{-2}y - w}$	$\frac{y(s^{-2} - 1)}{s^{-2}y - w}$

$\widehat{W}(i_1, j_1; i_2, j_2)$	$\frac{s^{-2}y - x}{y - x}$	$\frac{r^{-2}x - y}{y - x}$	$\frac{s^{-2}y - r^{-2}x}{y - x}$	1	$\frac{x(r^{-2} - 1)}{y - x}$	$\frac{y(s^{-2} - 1)}{y - x}$

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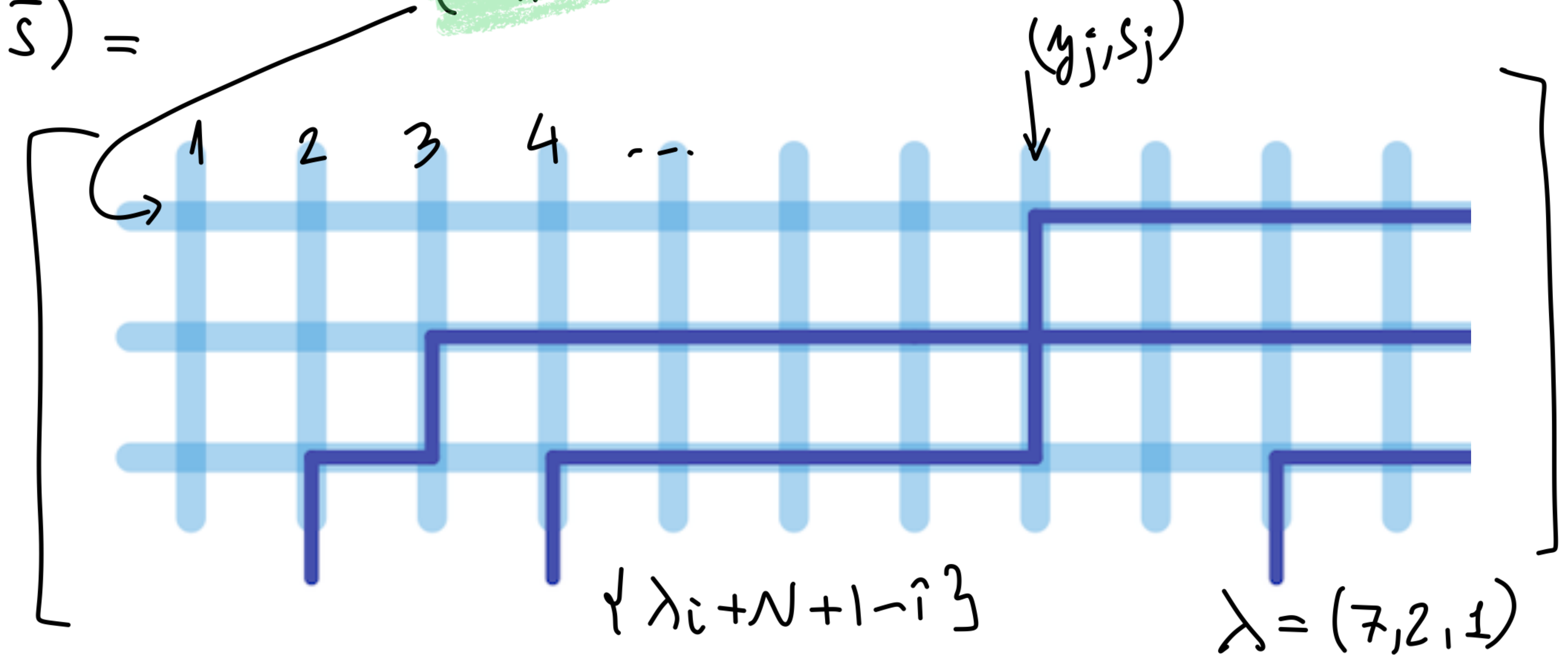
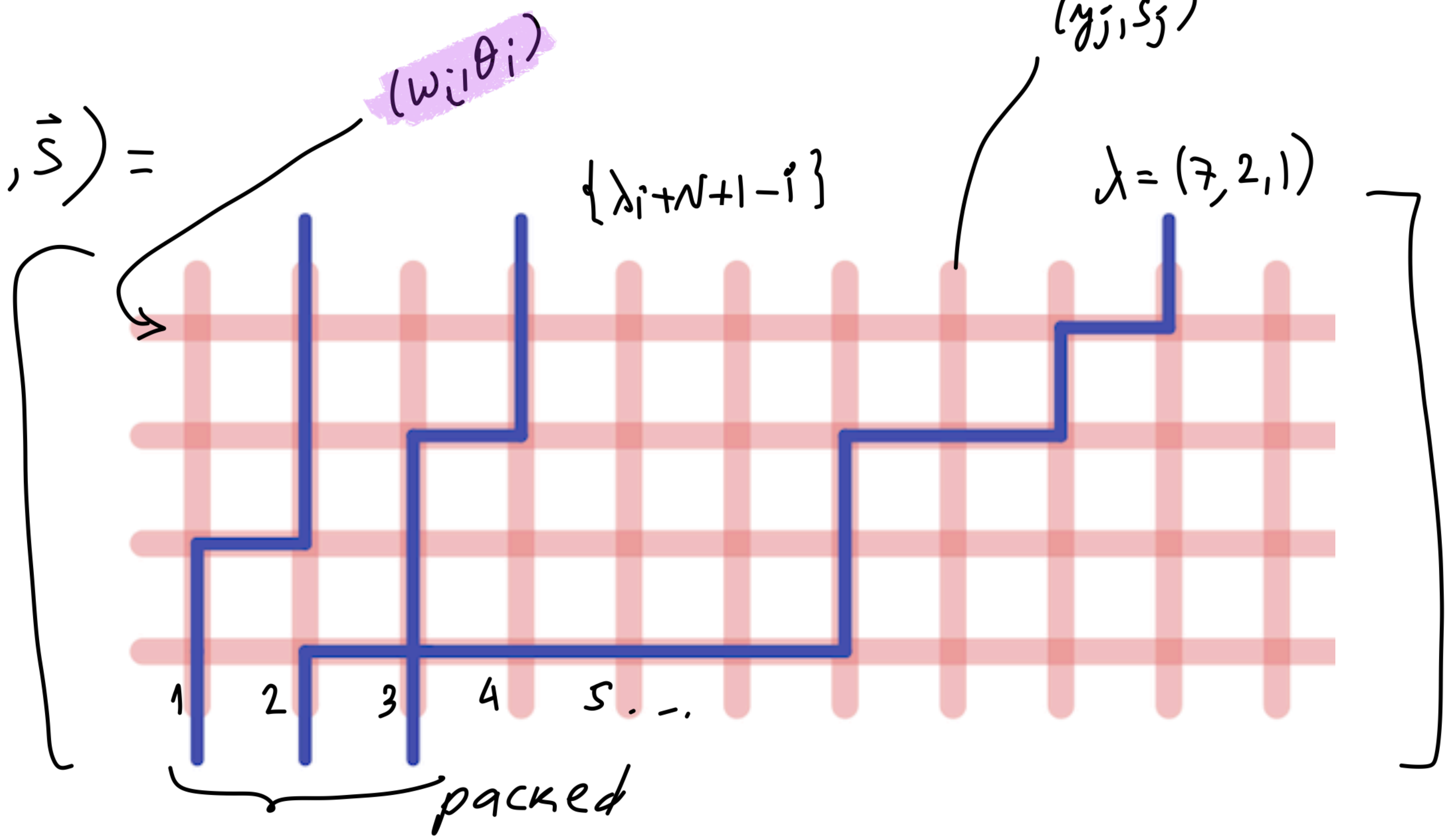
$F_\lambda(\vec{x}, \vec{y}, \vec{r}, \vec{s}) = Z\Big[$



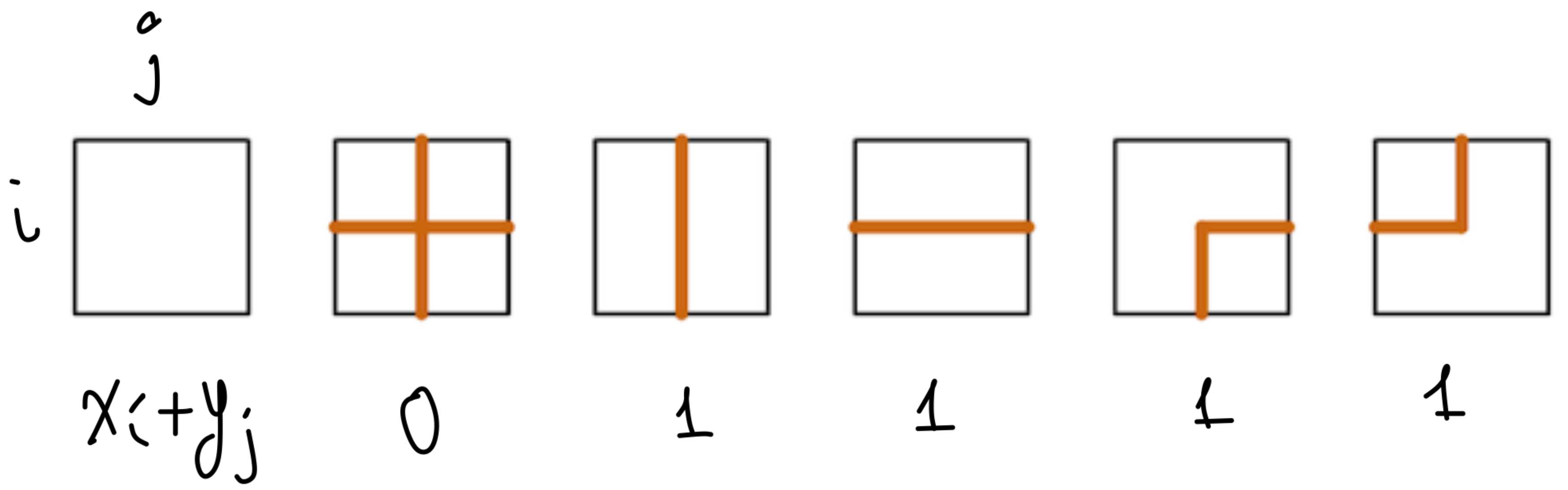
$\Big]$

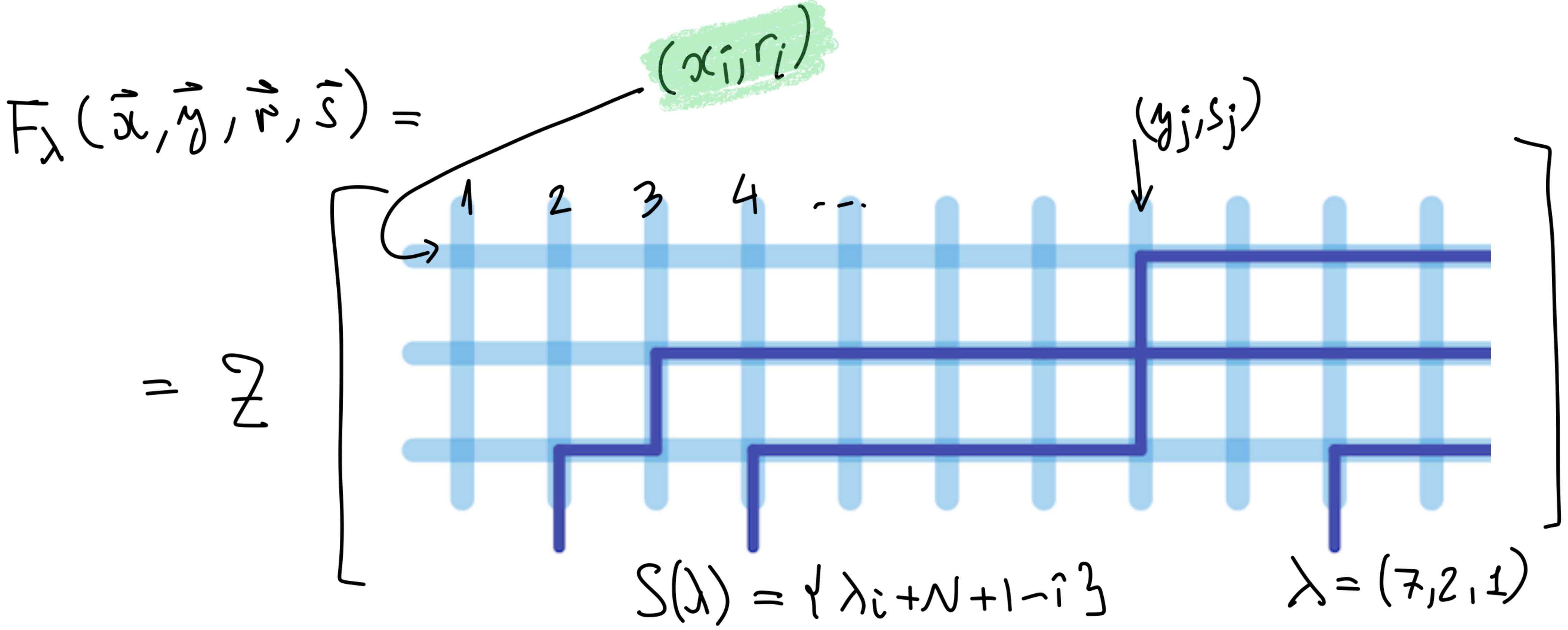
$G_\lambda(\vec{w}, \vec{y}, \vec{\theta}, \vec{s}) = Z\Big[$



$\Big]$

From F_λ to factorial Schur functions





$\widehat{W}(i_1,j_1;i_2,j_2)$	$\frac{s^{-2}y-x}{y-x}$	$\frac{r^{-2}x-y}{y-x}$	$\frac{s^{-2}y-r^{-2}x}{y-x}$	1	$\frac{x(r^{-2}-1)}{y-x}$	$\frac{y(s^{-2}-1)}{y-x}$

Lemma.

$$\lim_{s\to 0,\ r\to+\infty} F_\lambda(s^{-2}\mathbf{x}^{-1};-\mathbf{y}^{-1};\mathbf{r};\mathbf{s}) = \frac{x_1^{N-1}x_2^{N-2}\dots x_{N-1}}{\prod_{i\ge 1} y_i^{\#\{k\in\mathcal{S}(\lambda):\, k>i\}}}\, s_\lambda(\mathbf{x}\mid\mathbf{y}).$$

We will also see G_λ counterparts, which are symmetric rational functions $\check{s}_\lambda(\mathbf{x}\mid\mathbf{y})$ **[Morales-Pak-Panova 2017]**

Next: our favorite properties of the symmetric functions F_λ, G_λ

Cauchy identity

Theorem [ABPW '21].
$$\sum_{\lambda=(\lambda_1\geq\ldots\geq\lambda_N\geq 0)} G_\lambda(w_1,\ldots,w_T;\mathbf{y};\theta_1,\ldots,\theta_T;\mathbf{s})\,F_\lambda(x_1,\ldots,x_N;\mathbf{y};r_1,\ldots,r_N;\mathbf{s})$$
$$=\frac{\prod_{1\leq i\leq j\leq N}(r_i^{-2}x_i-x_j)\prod_{1\leq i<j\leq N}(s_i^{-2}y_i-y_j)}{\prod_{i,j=1}^N(y_i-x_j)}\prod_{i=1}^N\prod_{j=1}^T\frac{x_i-\theta_j^{-2}w_j}{x_i-w_j}$$

$$\left|\frac{s_p^{-2}y_p-x_i}{y_p-x_i}\frac{y_p-w_j}{s_p^{-2}y_p-w_j}\right|<1-\delta<1$$

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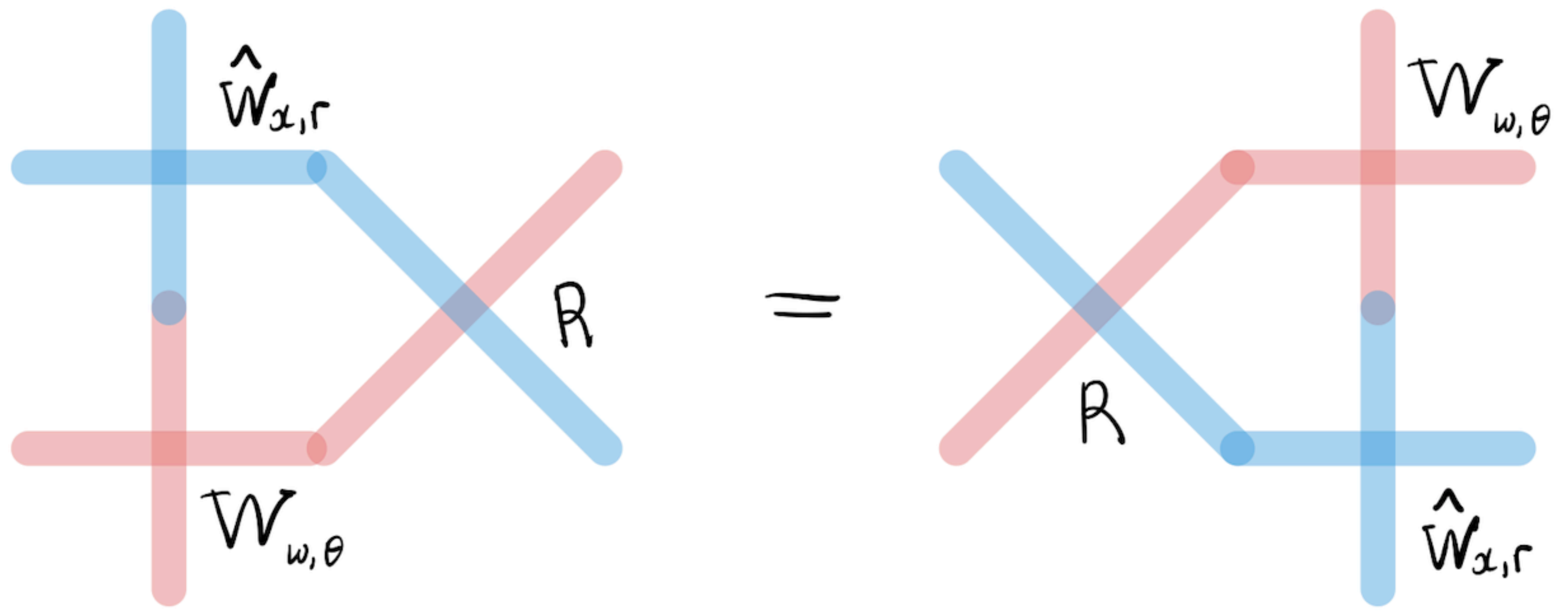
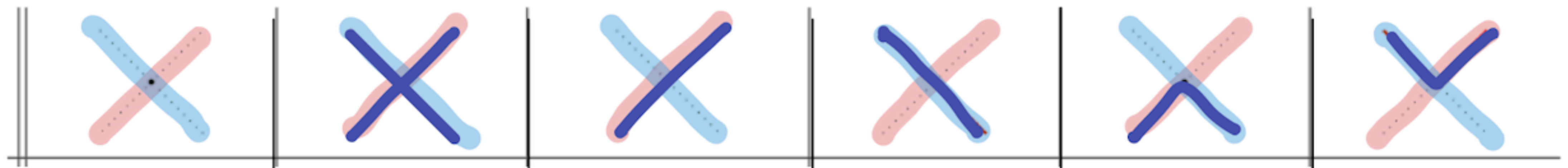
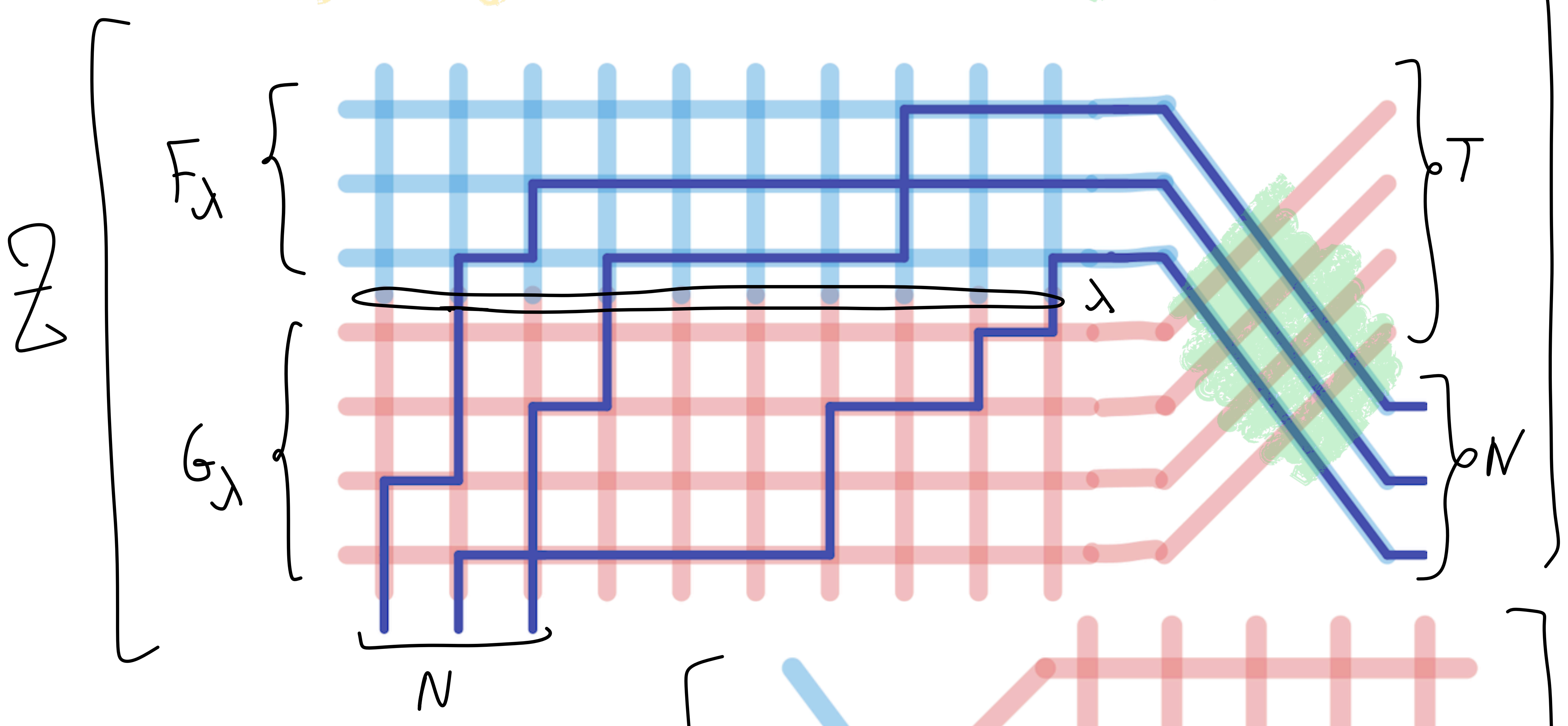
$$\left|\frac{s_p^{-2}y_p-x_i}{y_p-x_i}\,\frac{y_p-w_j}{s_p^{-2}y_p-w_j}\right|<1-\delta<1$$

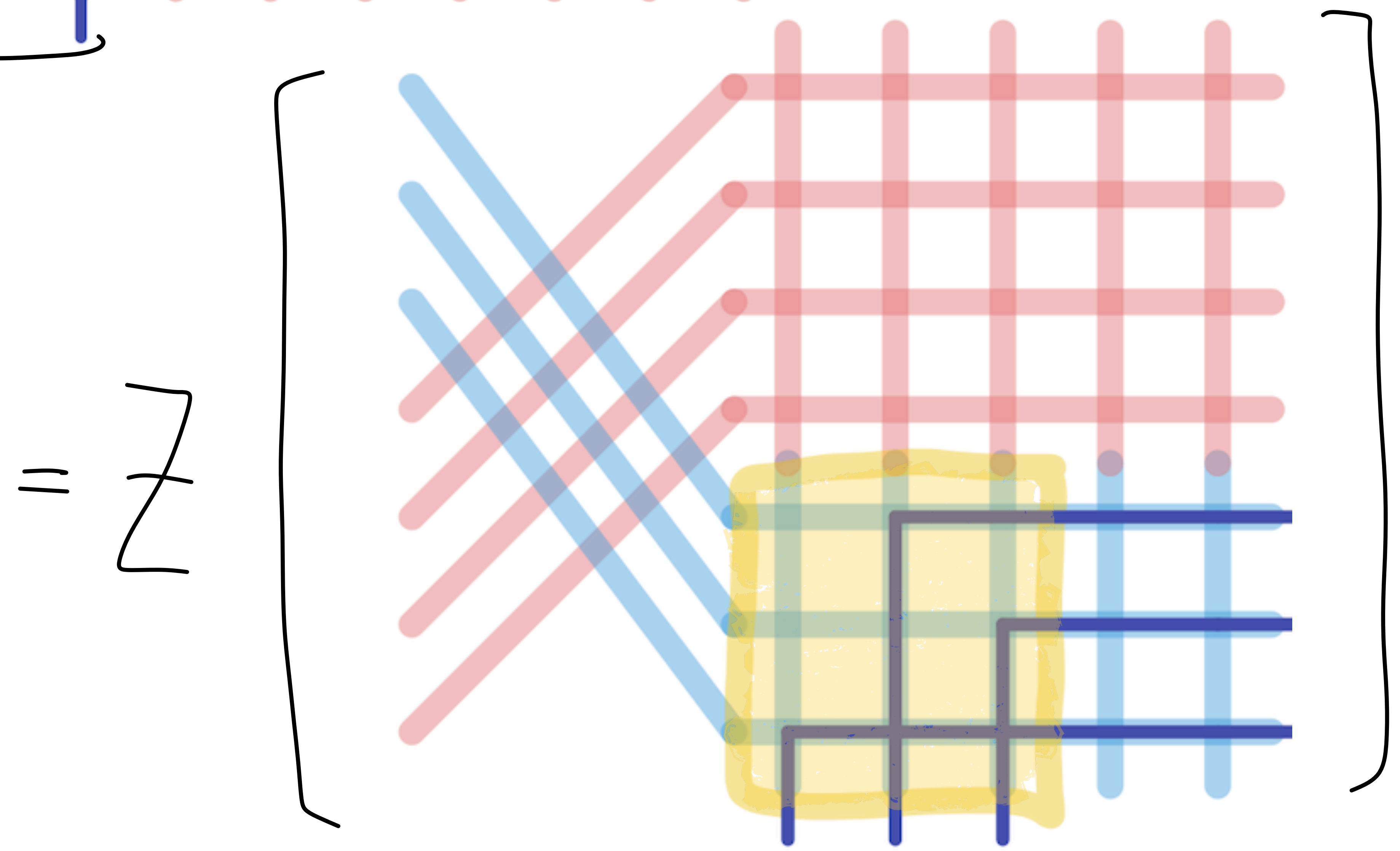
Proof. Yang-Baxter equation and rewriting of partition function





1	$\frac{w-r^{-2}x}{x-\theta^{-2}w}$	$\frac{r^{-2}x-\theta^{-2}w}{x-\theta^{-2}w}$	$\frac{x-w}{x-\theta^{-2}w}$	$\frac{x(1-r^{-2})}{x-\theta^{-2}w}$	$\frac{w(1-\theta^{-2})}{x-\theta^{-2}w}$





Add extra cross vertices, move them to the left,

then remains only the domain-wall partition function which is

an explicit product in the *free fermion case*

Explicit determinantal formulas

1. Double alternant formula for F_λ

$$F_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) = \frac{\prod_{1\le i\le j\le N}(r_i^{-2}x_i - x_j)}{\prod_{1\le i<j\le N}(x_i - x_j)} \ \det\left[\varphi_{\lambda_j+N-j}(x_i \mid \mathbf{y};\mathbf{s})\right]_{i,j=1}^{N}$$

(Note: F_λ is symmetric up to a simple prefactor)

$$\varphi_k(x \mid \mathbf{y};\mathbf{s}) := \frac{1}{y_{k+1}-x}\prod_{j=1}^{k}\frac{x - s_j^{-2}y_j}{x-y_j}$$

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2. Biorthogonality

$$\psi_k(x \mid \mathbf{y};\mathbf{s}) := \frac{y_{k+1}(s_{k+1}^{-2}-1)}{x - s_{k+1}^{-2}y_{k+1}}\prod_{j=1}^{k}\frac{x-y_j}{x-s_j^{-2}y_j}$$

$$\frac{1}{2\pi i}\oint_{y_i}\varphi_k(z)\,\psi_\ell(z) = \delta_{k=\ell}, \qquad k,\ell \geq 0$$

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3. Jacobi-Trudi type formula for G_λ (Macdonald's "variation")

$$\mathsf{h}_{k,m}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s}) = \frac{1}{2\pi\mathbf{i}}\oint_{\gamma'} dz\,\frac{\psi_k(z \mid \mathbf{y};\mathbf{s})}{y_m - z}\prod_{j=1}^{M}\frac{z-\theta_j^{-2}w_j}{z-w_j}$$

y_i, w_j

$$G_\lambda(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s}) = \prod_{1\le i<j\le N}\frac{s_i^{-2}y_i - y_j}{y_j - y_i}\det\left[\mathsf{h}_{\lambda_i+N-i,\,j}(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s})\right]_{i,j=1}^N$$

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4. We also prove a Sergeev-Pragacz type formula for G_λ

like for supersymmetric Schur polynomials (long...)

special case $y_j \equiv y$, $s_j \equiv s$

$$G_\lambda(x_1,\ldots,x_M;\mathbf{y};\mathbf{r};\mathbf{s}) = s_\lambda\left(\left\{\frac{s^2(1-x_j)}{1-s^2x_j}\right\}_{j=1}^M \Big/ \left\{\frac{s^2(x_jr_j^{-2}-1)}{1-s^2r_j^{-2}x_j}\right\}_{j=1}^M\right)\prod_{i=1}^M\left(\frac{1-s^2r_i^{-2}x_i}{1-s^2x_i}\right)^N$$

Factorial Schur case

1. Double alternant formula $$s_\lambda(\mathbf{x} \mid \mathbf{y}) = \frac{\det\left[(x_i \mid \mathbf{y})^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{1\le i<j\le N}(x_i - x_j)}, \qquad (x \mid \mathbf{y})^k := (x+y_1)\dots(x+y_k)$$

2. Biorthogonality $$\check{\psi}_k(x \mid y) = +\frac{1}{x+y_{k+1}}\prod_{i=1}^k \frac{1}{x+y_i}, \qquad k \ge 0.$$

$$\frac{1}{2\pi\mathbf{i}}\oint (z \mid y)^l \psi_k(z \mid y)dz = \mathbf{1}_{k=l}$$

around $-y_i$

Similar to some det formulas from **[Morales-Pak-Panova 2017]**

3. Jacobi-Trudi type formula for $\check{s}_\lambda$

$$\check{s}_\lambda(w_1,\dots,w_M \mid y) = \det\left[\frac{1}{2\pi\mathbf{i}}\oint \frac{z^{i-1}\check{\psi}_{\lambda_j+j-1}(z)}{\prod_{k=1}^M(w_k - z)}\,dz\right]$$

around $-y_i$

4. Cauchy identity (new? at least different from [Molev 2009]**)**

$$\sum_\lambda s_\lambda(x \mid y)\check{s}_\lambda(w \mid y) = \prod_{i,j}\frac{1}{w_i - x_j}, \qquad |x_i| < |w_j|.$$

II. From symmetric functions to probability distributions

Cauchy identity + positivity ⇒ probability

$$\sum_{\lambda=(\lambda_1\geq\ldots\geq\lambda_N\geq 0)} G_\lambda(w_1,\ldots,w_T;\mathbf{y};\theta_1,\ldots,\theta_T;\mathbf{s})\, F_\lambda(x_1,\ldots,x_N;\mathbf{y};r_1,\ldots,r_N;\mathbf{s})$$

$$= \underbrace{\frac{\prod_{1\leq i\leq j\leq N}(r_i^{-2}x_i - x_j)\prod_{1\leq i<j\leq N}(s_i^{-2}y_i - y_j)}{\prod_{i,j=1}^{N}(y_i - x_j)} \prod_{i=1}^{N}\prod_{j=1}^{T}\frac{x_i - \theta_j^{-2}w_j}{x_i - w_j}}_{Z}$$

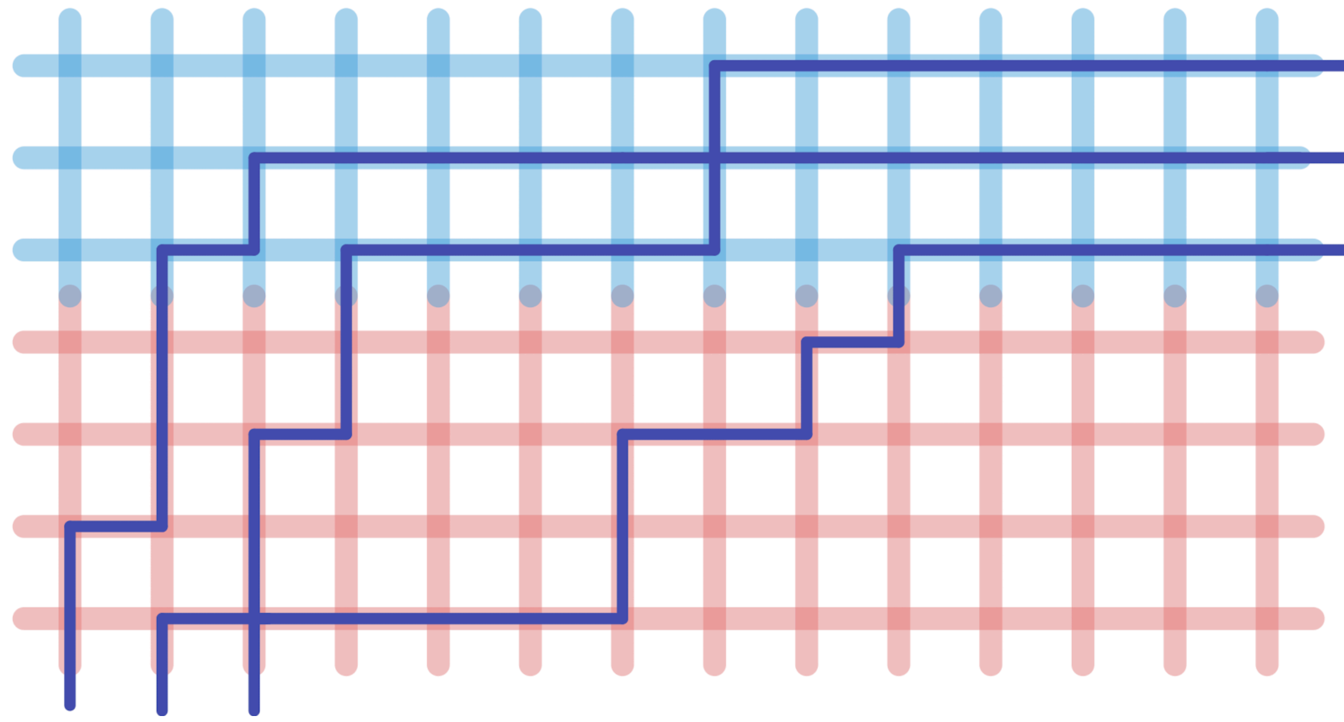
Lemma. $F_\lambda, G_\lambda \geq 0$ (and also all vertex weights) under conditions

$$w_i < y_j < w_i\theta_i^{-2} < y_j s_j^{-2} \text{ and } x_i < y_j < x_i r_i^{-2} < y_j s_j^{-2} \text{ for all } i, j$$

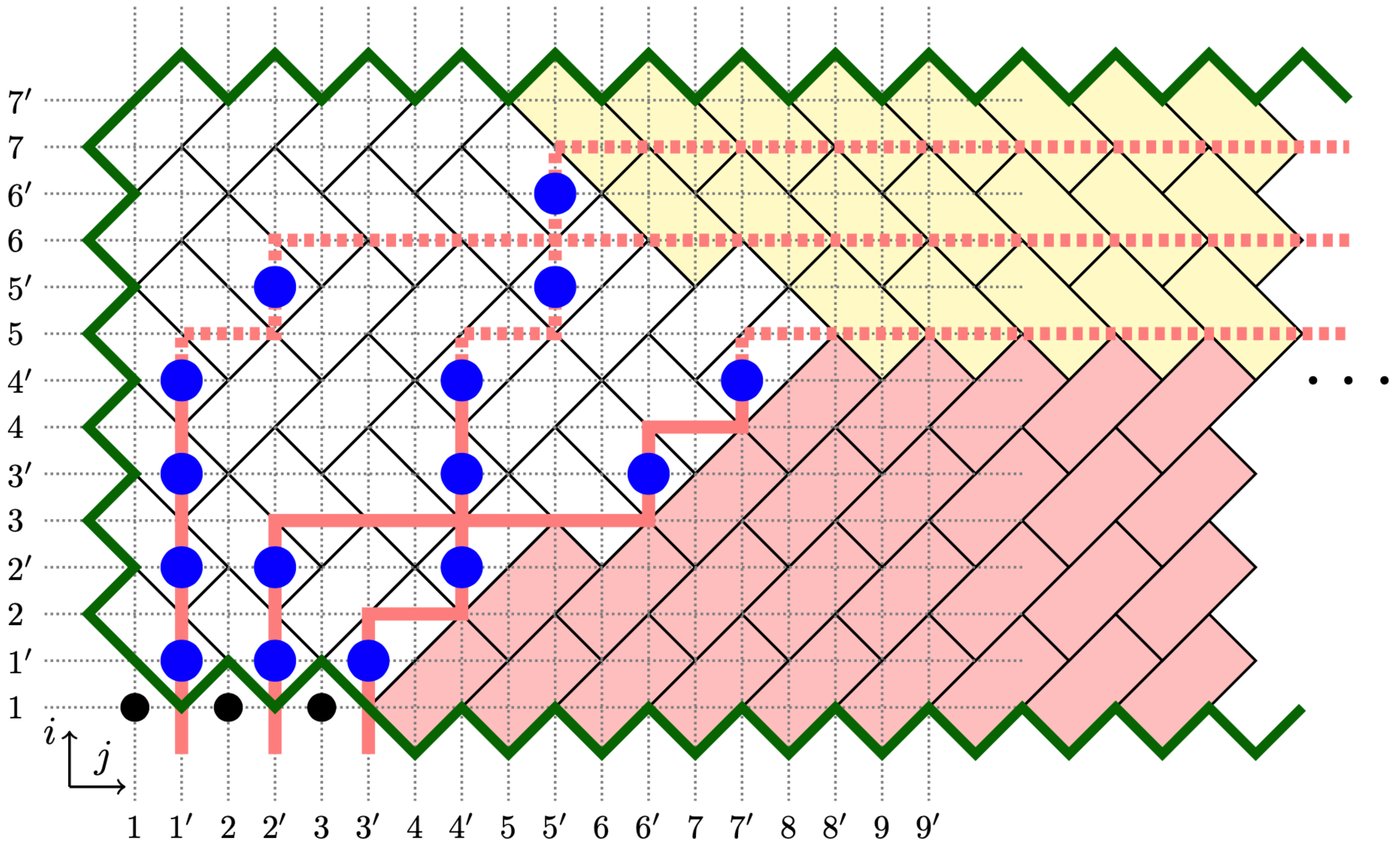
Define probability distribution on $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$ by normalizing the Cauchy identity:

$$\mathbb{P}(\lambda) = \frac{1}{Z} F_\lambda(\mathbf{x};\mathbf{y};\mathbf{r};\mathbf{s}) G_\lambda(\mathbf{w};\mathbf{y};\boldsymbol{\theta};\mathbf{s})$$

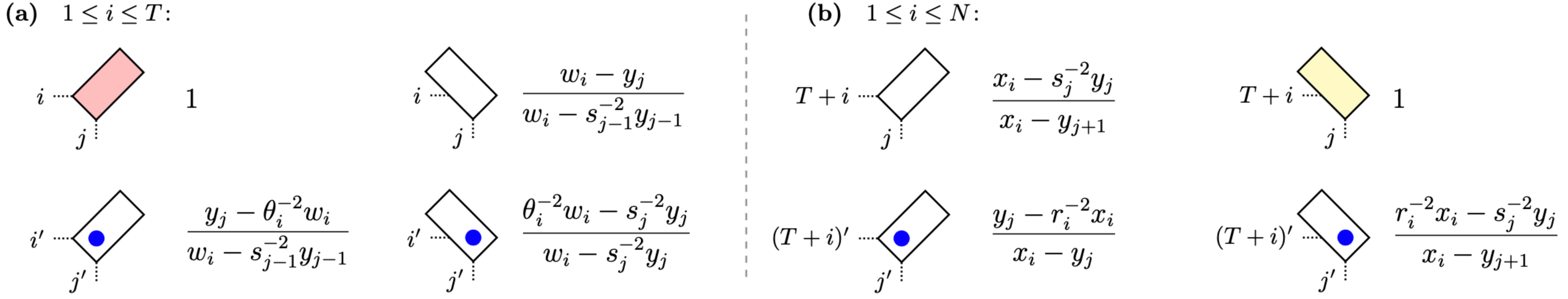
More generally, we can consider random path ensemble whose probability weights are proportional to products of vertex weights ("Random tableaux")



An equivalent model of domino tilings, for which we get bulk asymptotics



(thanks to free fermion structure, it's a dimer model!)

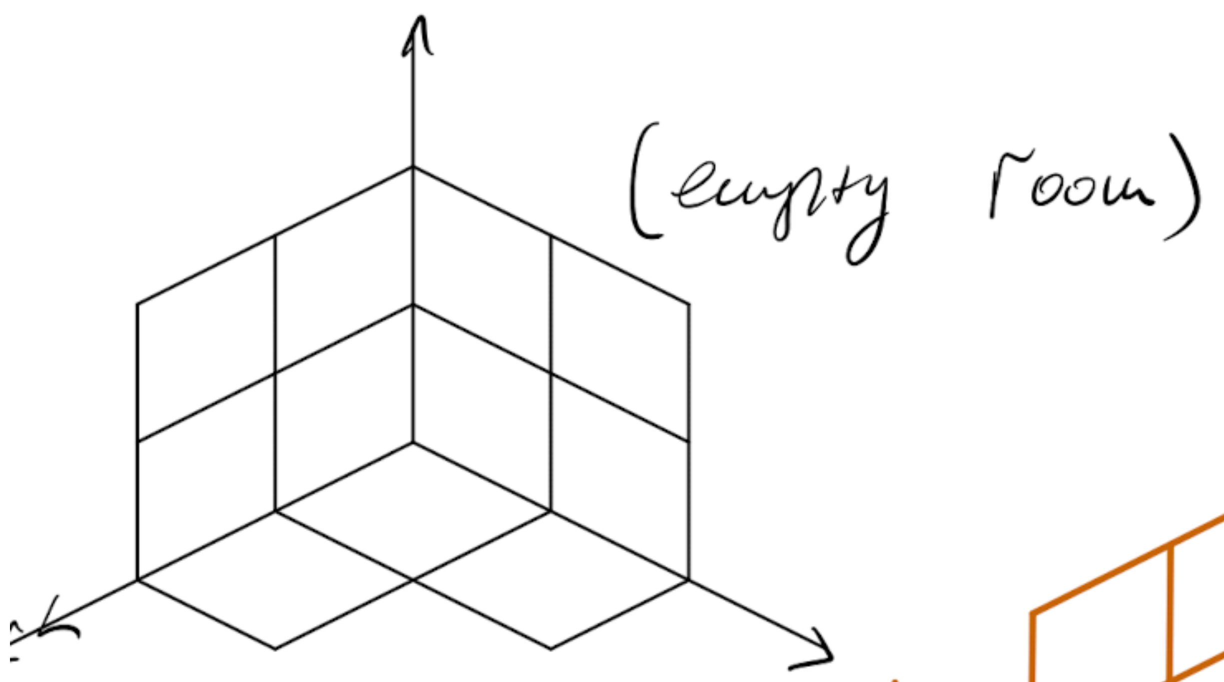


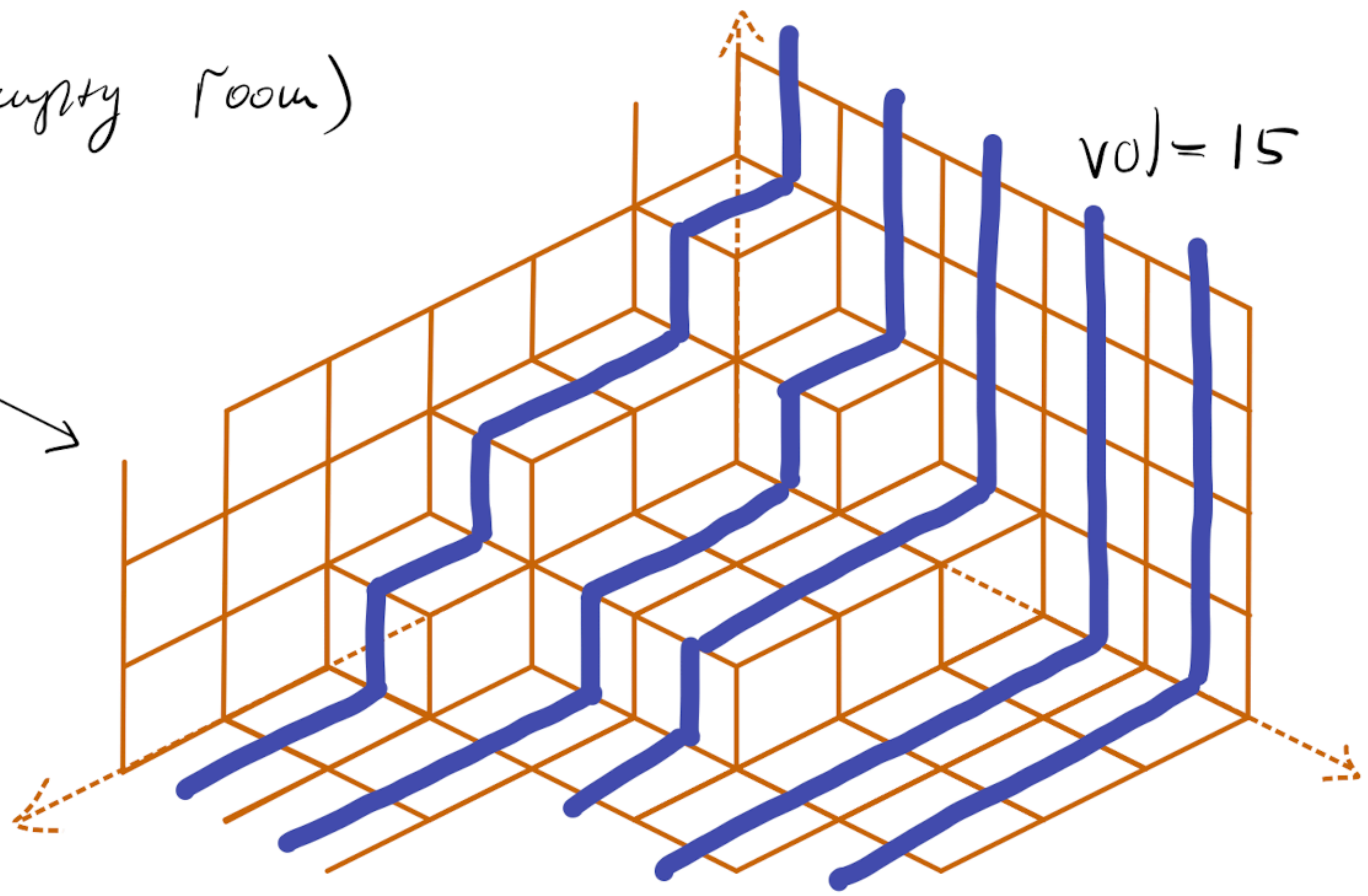
Analogy with plane partitions

$$\mathbb{P}(\text{plane partition } P) = \frac{q^{\text{volume}(P)}}{Z}, 0 < q < 1$$

$$Z = \sum_P q^{\text{volume}(P)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$$ **[MacMahon, 1900s]**

(measure is uniform, conditioned on the number of boxes)





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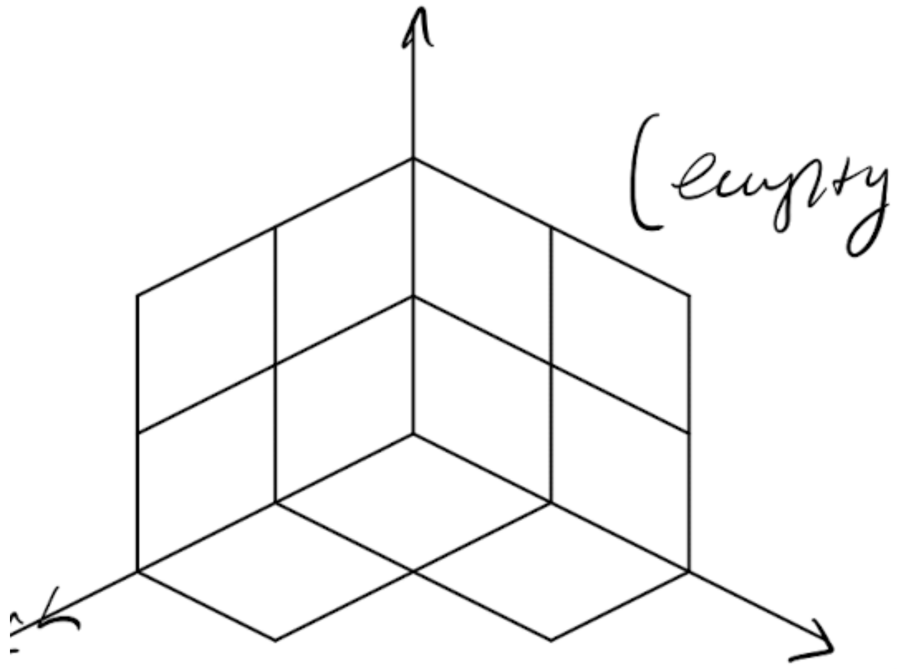
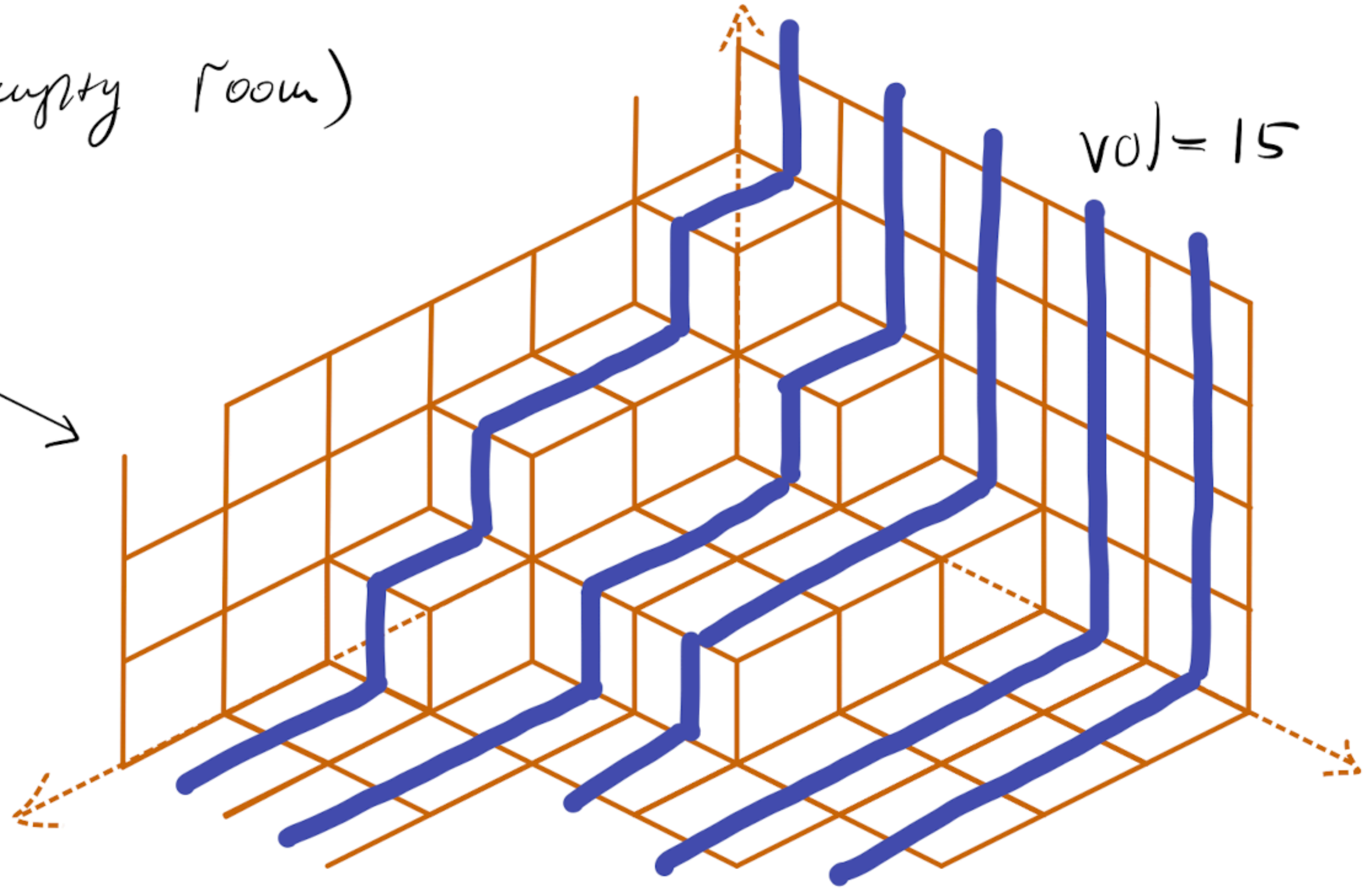
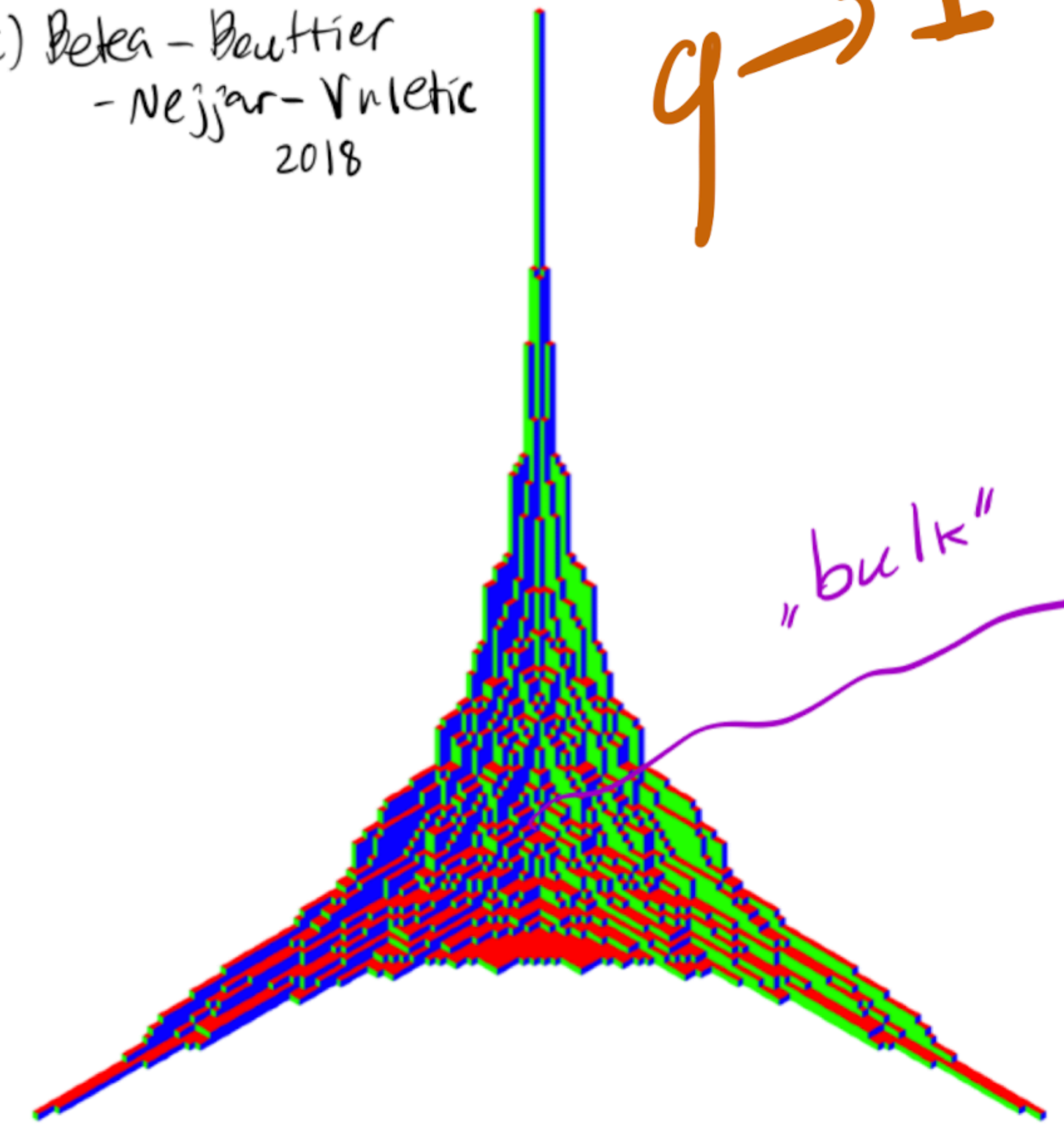


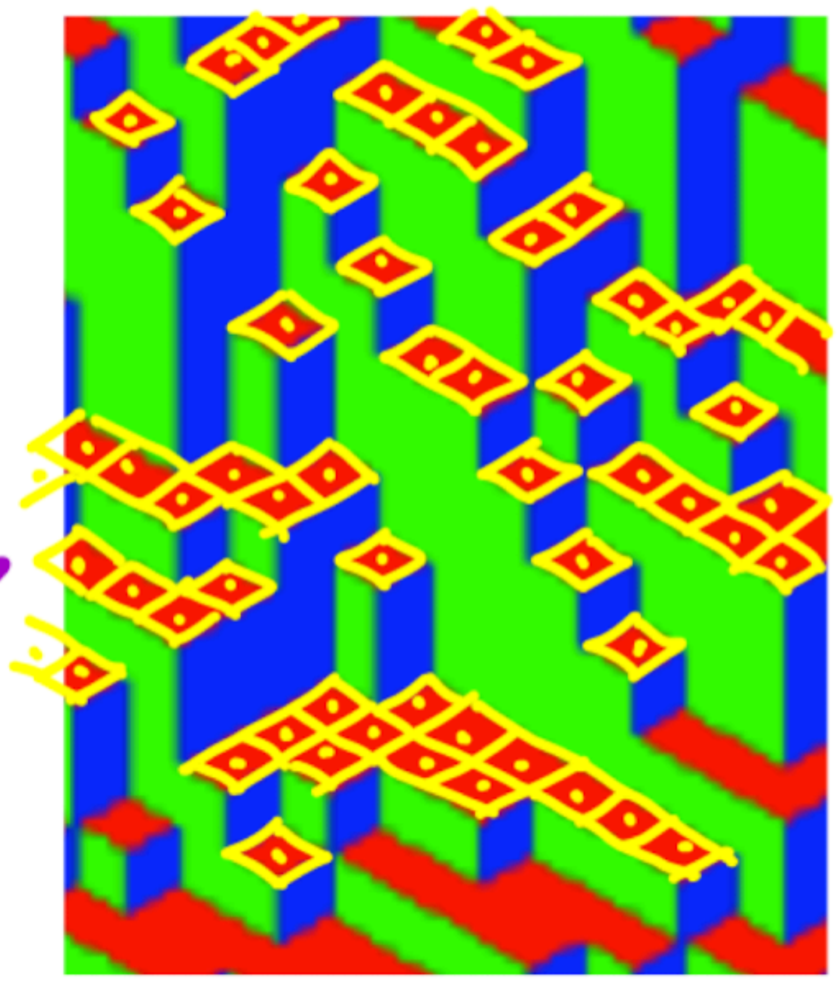


$q \to 1$

(c) Betea - Boutillier - Nejjar - Vuletic 2018



„bulk"



determinantal process on $\mathbb{Z}^2$

[Okounkov-Reshetikhin 2001]

- Determinantal point process structure, based on Schur polynomials
- Bulk (lattice) asymptotic behavior. Pure Gibbs states classified by Sheffield (2003)

III. Free fermions and correlations

Bethe Ansatz operators A,B,C,D

$$\begin{array}{c|cc} & 1 & 0 \\ \hline 1 & A & B \\ 0 & C & D \end{array}$$

- Act in $V = \mathbb{C}^2 = \mathrm{span}\{e_0, e_1\}$

$$Ae_0 = W(\;)\,e_0 \qquad Ae_1 = W(\;)\,e_1 \qquad Be_1 = W(\;)\,e_0$$

$$De_0 = W(\;)\,e_0 \qquad De_1 = W(\;)\,e_1 \qquad Ce_0 = W(\;)\,e_1$$

- Know how to act in (finite) tensor powers of V

$$B\, v_1 \otimes v_2 = Bv_1 \otimes Av_2 + Dv_1 \otimes Bv_2$$

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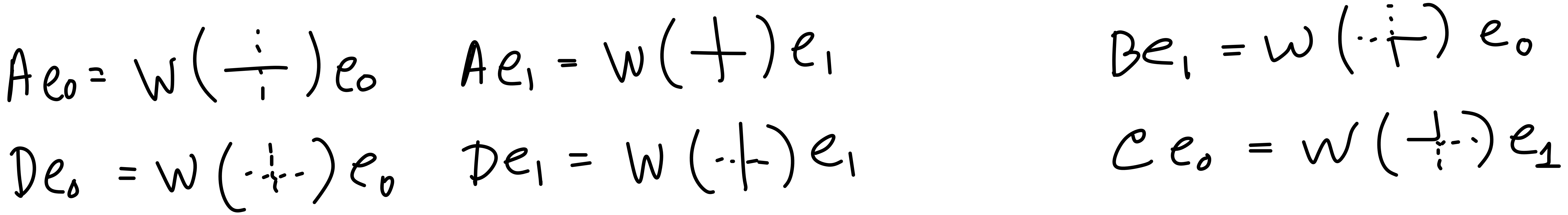


- Know how to act in (finite) tensor powers of V

$$B\, v_1 \otimes v_2 = Bv_1 \otimes Av_2 + Dv_1 \otimes Bv_2$$

- Thanks to the Yang-Baxter equation, satisfy certain quadratic relations, for example:

$$B(x_2, r_2)D(x_1, r_1) = \frac{r_1^{-2}x_1 - x_2}{x_1 - x_2} D(x_1, r_1)B(x_2, r_2) + \frac{(1 - r_2^{-2})x_2}{x_1 - x_2} D(x_2, r_2)B(x_1, r_1);$$

- Functions F_λ, G_λ are matrix elements of products of these operators. We use relations to compute explicit formulas

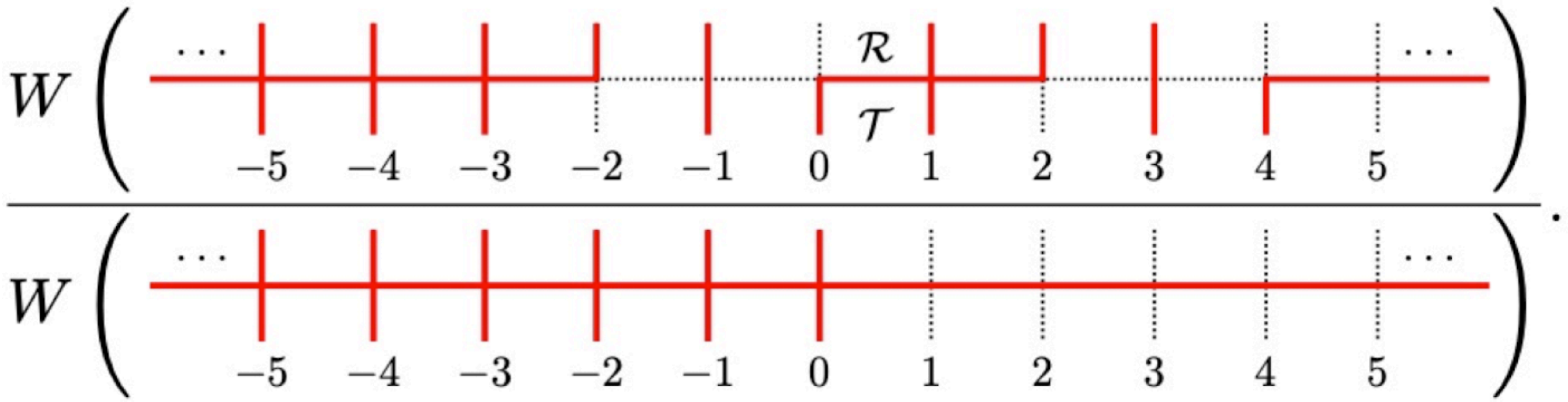
[Korepin-Bogoliubov-Izergin 1993; Borodin-P. 2016]

$$F_\lambda = \left\langle e_0 \otimes e_0 \otimes \ldots, \;\; B(r_N, x_N) \ldots B(r_1, x_1) e_\lambda \right\rangle$$

Bethe Ansatz and correlations

- Fock space is spanned by e_J, where J are semi-infinite subspaces of $\mathbb{Z}$ - packed to the left, empty to the right
- We define normalized operators $A^{\mathbb{Z}}, B^{\mathbb{Z}}, C^{\mathbb{Z}}, D^{\mathbb{Z}}$ - infinite volume limits of A, B, C, D

$\langle A^{\mathbb{Z}}(x, r) e_{\mathcal{T}}, e_{\mathcal{R}} \rangle =$



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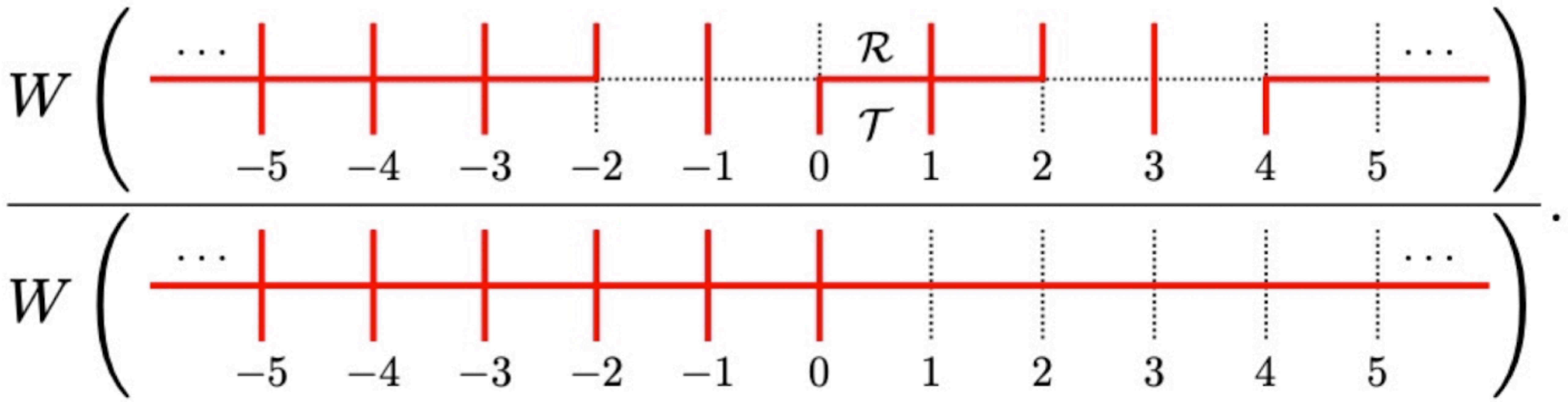

etc

$$F_\lambda(\vec{x},\vec{y},\vec{r},\vec{s}) = \langle e_{J(\lambda)}, B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)\, e_{\mathbb{Z}_{\le 0}}\rangle$$

$$G_\lambda(\vec{w},\vec{y},\vec{\theta},\vec{s}) = \langle e_{J(\lambda)}, D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)\, e_{\mathbb{Z}_{\le N}}\rangle$$

$$J(\lambda) = \{\lambda_i + N + 1 - i\} \cup \mathbb{Z}_{\le 0}$$

projection onto $e_{J(\lambda)}$

The measure has the form

$$\mathbb{P}(\lambda) = \frac{1}{Z}\langle e_{\mathbb{Z}_{\le 0}}, B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)\, I_\lambda D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1) e_{\mathbb{Z}_{\le N}}\rangle$$

Bethe Ansatz and correlations

- **Definition.** For a subset $A = \{a_1, \ldots, a_m\}$, the **correlation function** is the probability $\mathbb{P}[J(\lambda) \supset A]$.

  It has the form $\det[K(a_i, a_j)]_{i,j=1}^{m}$ because it's a **dimer model**. But we want K explicitly for asymptotics

- The correlation function is computed by replacing I_λ by the product of annihilation-creation pairs:

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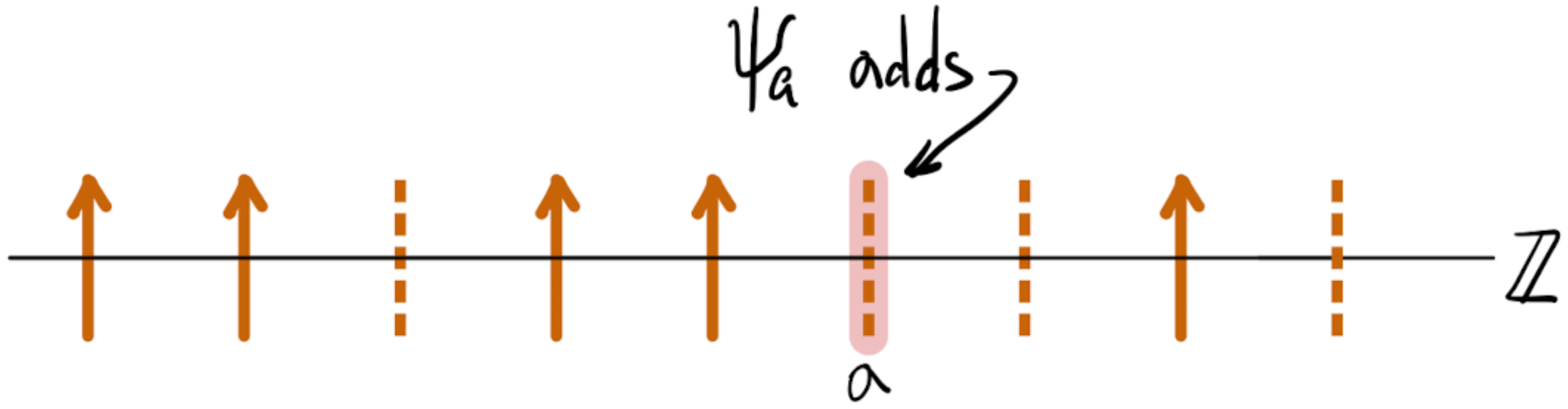
$$\psi_{a_m}\psi^*_{a_m} \cdots \psi_{a_1}\psi^*_{a_1}$$

- ψ_a creates a new arrow at a (if it's there, maps vector to 0)
- ψ_a^* annihilates an arrow at a (if it's not there, maps vector to 0)



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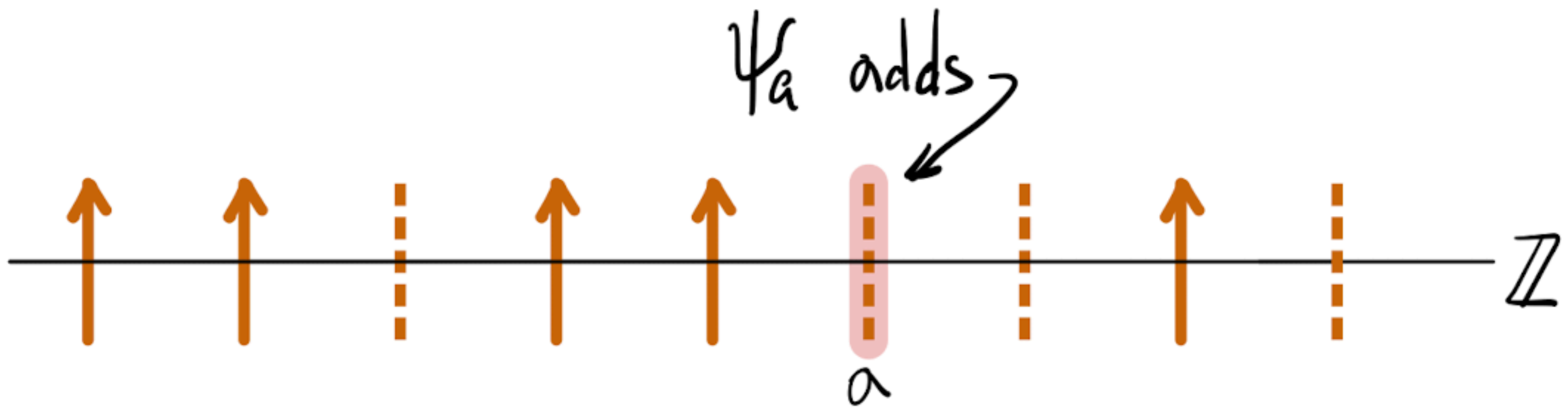

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- As opposed to Schur measures **[Okounkov 1999],** here we **don't know** how the operators ψ_a, ψ_a^* commute with $B^{\mathbb{Z}}, D^{\mathbb{Z}}$.
- Instead, we insert certain generating functions, built from the **same** Bethe Ansatz operators. Relations follow from YBE

Generating functions and correlations

Special parameters eliminate terms from YBE

Define

$$\Psi(u,\xi)\, e_{\mathcal{T}} := D^{\mathbb{Z}}(u, \sqrt{u/\xi})C^{\mathbb{Z}}(\xi, \sqrt{\xi/u})(-1)^{c(\mathcal{T})} e_{\mathcal{T}},$$

$$\Psi^*(\zeta,v)\, e_{\mathcal{T}} := D^{\mathbb{Z}}(\zeta, \sqrt{\zeta/v})B^{\mathbb{Z}}(v, \sqrt{v/\zeta})\, e_{\mathcal{T}}.$$

Theorem [ABPW] (Inhomogeneous Boson-Fermion correspondence).

$$\Psi(u,\xi) = \sum_{j\in\mathbb{Z}} \frac{y_j(1-s_j^{-2})}{u - s_j^{-2} y_j} \mathcal{P}_{0,j-1}(u \mid \mathbf{y}, \mathbf{s}^{-2}\mathbf{y})\, \psi_j,$$

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$$\mathcal{P}_{n,n'}(u \mid \mathbf{b};\mathbf{c}) := \begin{cases} \displaystyle\prod_{j=n+1}^{n'} \frac{u-b_j}{u-c_j}, & n < n'; \\ 1, & n = n'; \\ \displaystyle\prod_{j=n'+1}^{n} \frac{u-c_j}{u-b_j}, & n > n', \end{cases}$$

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Theorem.
$$\frac{1}{Z}\langle e_{\mathbb{Z}_{\le 0}},B^{\mathbb{Z}}(x_N,r_N)\dots B^{\mathbb{Z}}(x_1,r_1)\overbrace{\Psi(u_m)\Psi^*(v_m)\dots\Psi(u_1)\Psi^*(v_1)}^{\text{generating function of correlations}}$$
$$\times D^{\mathbb{Z}}(w_M,\theta_M)\dots D^{\mathbb{Z}}(w_1,\theta_1)e_{\mathbb{Z}_{\le N}}\rangle$$
$$=\prod_{i=1}^{M}\prod_{\alpha=1}^{m}\frac{(v_\alpha-\theta_i^{-2}w_i)(u_\alpha-w_i)}{(v_\alpha-w_i)(u_\alpha-\theta_i^{-2}w_i)}\prod_{\alpha=1}^{m}\prod_{j=1}^{N}\frac{u_\alpha-y_j}{v_\alpha-y_j}\frac{v_\alpha-x_j}{u_\alpha-x_j}\det\left[\frac{v_\alpha}{u_{\alpha'}-v_\alpha}\right]_{\alpha,\alpha'=1}^{m}.$$

follows from commutation relations / YBE + Wick's determinant

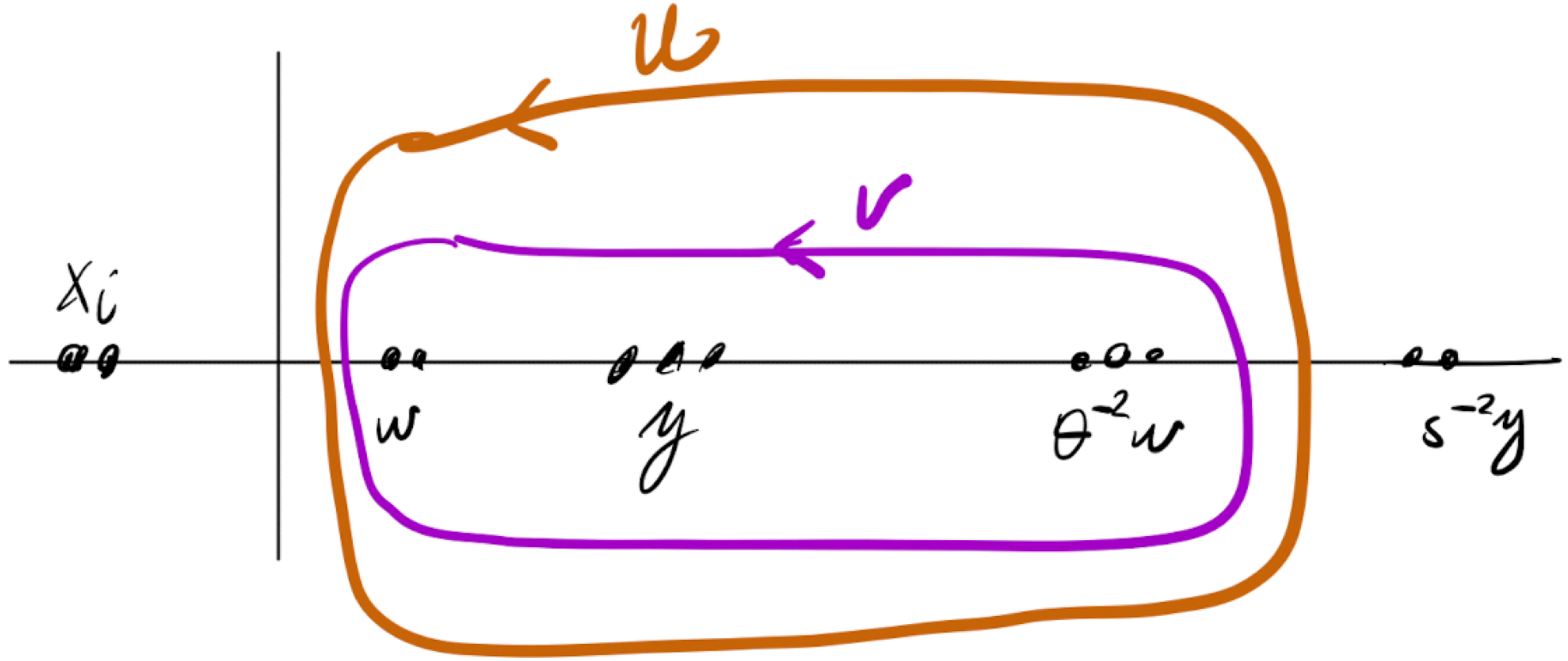
$$\langle e_{\mathbb{Z}_{\le 0}},\Psi(u_1)\Psi^*(v_1)\dots\Psi(u_m)\Psi^*(v_m)\,e_{\mathbb{Z}_{\le 0}}\rangle=\det\left[\frac{v_\alpha}{u_{\alpha'}-v_\alpha}\right]_{\alpha,\alpha'=1}^{m}.$$

Correlation kernel - extracted using inhomogeneous orthogonality

Theorem. We have $\mathbb{P}[J(\lambda) \supset \{a_1, \ldots, a_m\}] = \det[K(a_i, a_j)]_{i,j=1}^m$ where the kernel is given by

$$K_{\mathcal{M}}(a, a') = \frac{1}{(2\pi\mathbf{i})^2} \oint_{\Gamma_{y,\theta^{-2}w}} du \oint_{\Gamma_{y,w}} dv \prod_{k=1}^{N} \frac{(u - y_k)(v - x_k)}{(u - x_k)(v - y_k)} \prod_{i=1}^{M} \frac{(u - w_i)(v - \theta_i^{-2} w_i)}{(v - w_i)(u - \theta_i^{-2} w_i)}$$

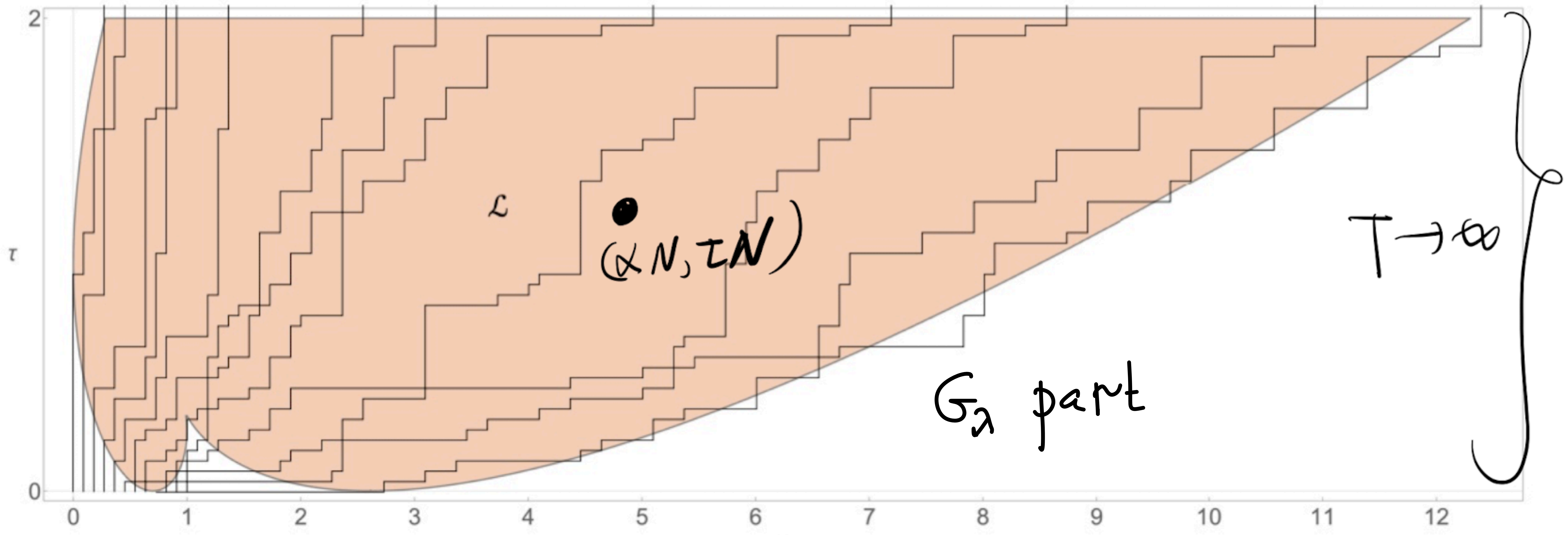
$$\times \frac{1}{u - v} \frac{y_a(1 - s_a^{-2})}{v - s_a^{-2} y_a} \frac{1}{u - y_{a'}} \prod_{j=1}^{a-1} \frac{v - y_j}{v - s_j^{-2} y_j} \prod_{j=1}^{a'-1} \frac{u - s_j^{-2} y_j}{u - y_j},$$



IV. Asymptotics

Asymptotics in the bulk

- For each point (α, τ) in the "liquid region" there exists a *complex slope* z parametrizing the slope of paths
- Around (α, τ), at the lattice level we see a determinantal point process depending on sequences $w_i, \theta_i; y_j, s_j, i, j \in \mathbb{Z}$



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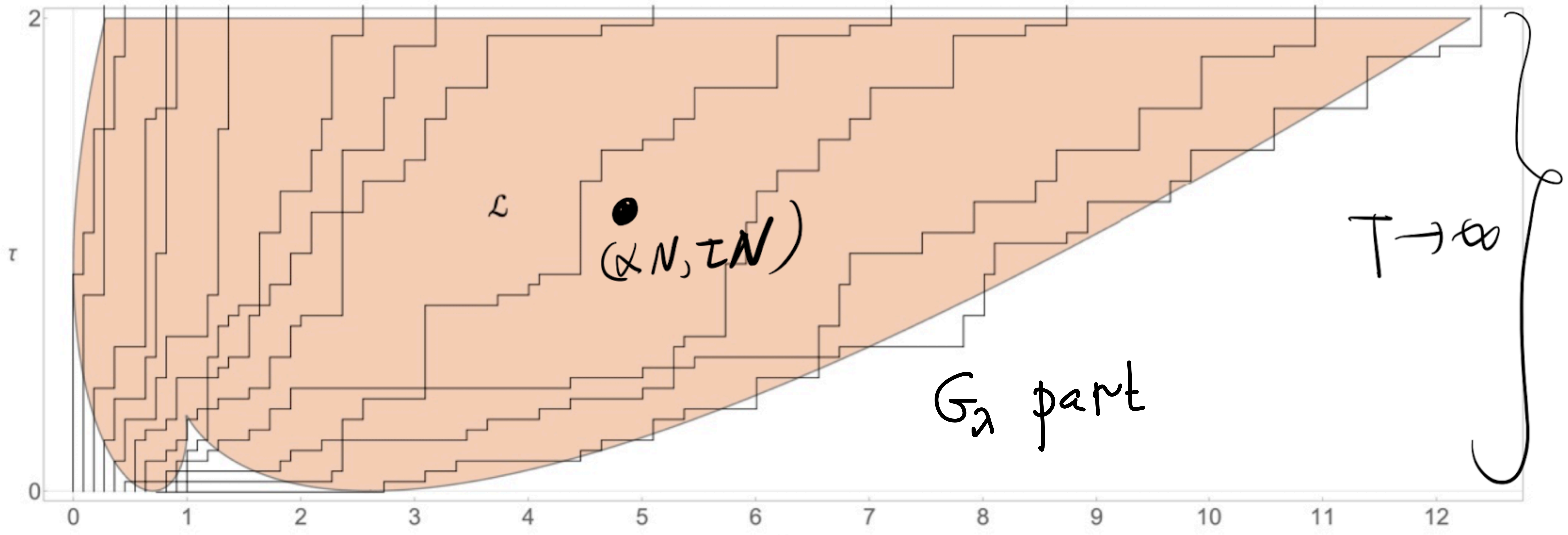

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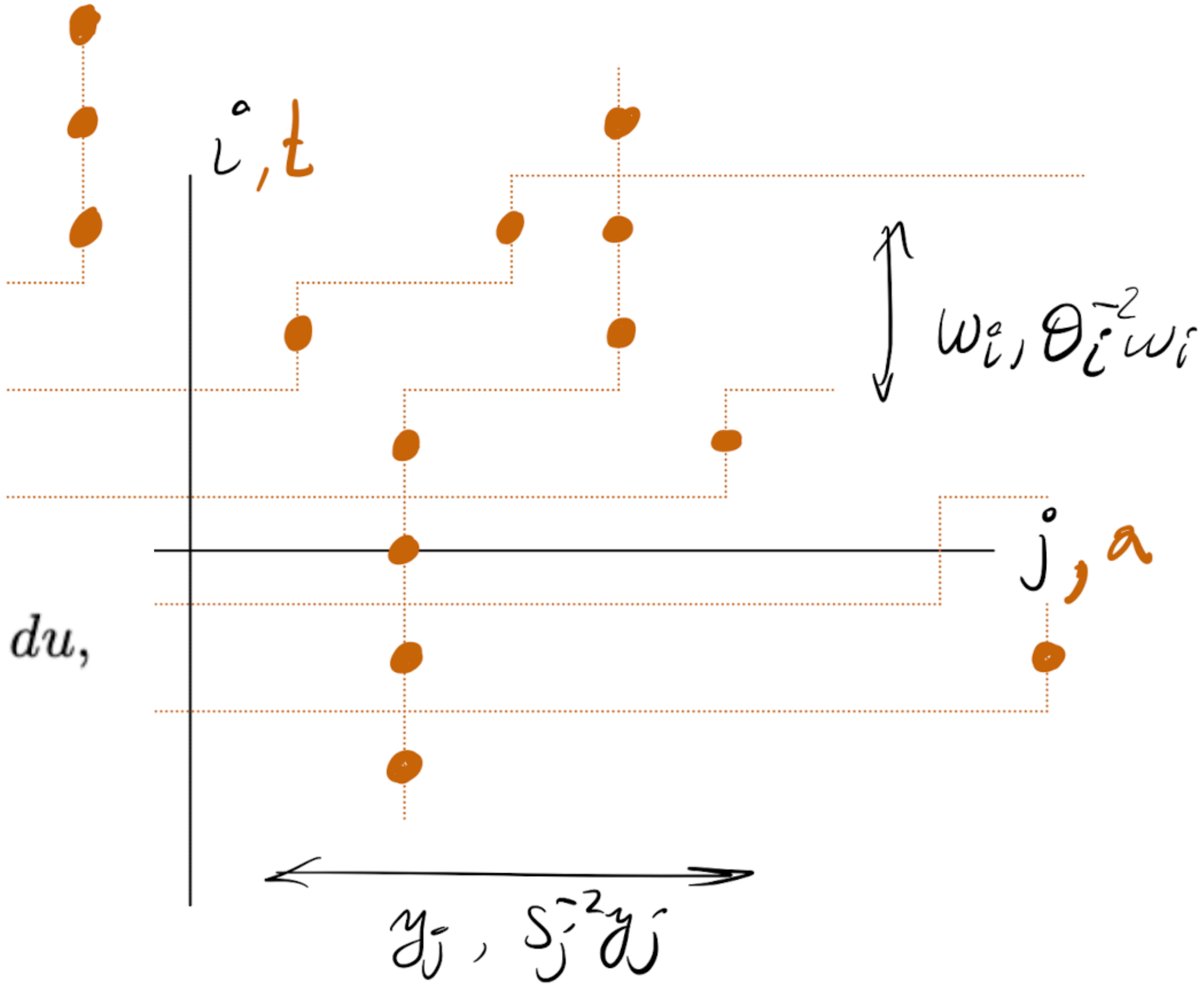
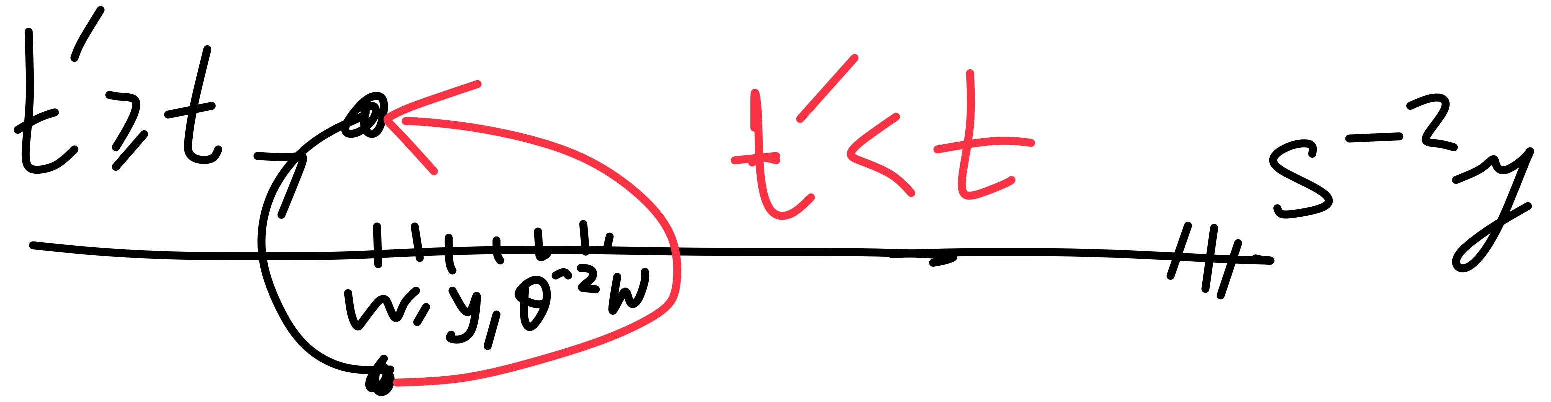
- **Inhomogeneous 2d sine kernel**:

$$\det\left[K^{z}_{2d}(t_i, a_i ; t_j, a_j)\right]_{i,j=1}^{m} = \mathrm{Prob}\left[\begin{array}{c}\text{there is } \bullet \\ \text{at each} \\ (t_k, a_k)\end{array}\right]$$



$$K^{z}_{2\mathrm{d}}(t, a; t', a') = -\frac{1}{2\pi\mathbf{i}} \int_{\bar{z}}^{z} \frac{y_a(1 - s_a^{-2})}{(u - y_a)(u - s_{a'}^{-2} y_{a'})} \mathcal{P}_{a,a'}(u \mid \mathbf{s}^{-2}\mathbf{y}; \mathbf{y})\, \mathcal{P}_{t,t'}(u \mid \mathbf{w}; \boldsymbol{\theta}^{-2}\mathbf{w})\, du,$$



Summary

- Free fermion six vertex model provides multiparameter (inhomogeneous) generalizations (4 families of parameters, 2 per each coordinate direction) of:
  - Schur polynomials, and also factorial / supersymmetric Schur polynomials
  - Schur measures and processes, with double contour integral determinantal structure
  - Translation invariant ergodic Gibbs measures ("pure Gibbs states") governed by the extended discrete sine kernel

- Technical features:
  - Fermionic operators naturally come from the Bethe ansatz operators A,B,C,D
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Thank you!