$\mathfrak{sl}(2)$ Operators and Markov Dynamics on Branching Graphs

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Young diagrams

Partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0),$$
$$\lambda_i \in \mathbb{Z}$$

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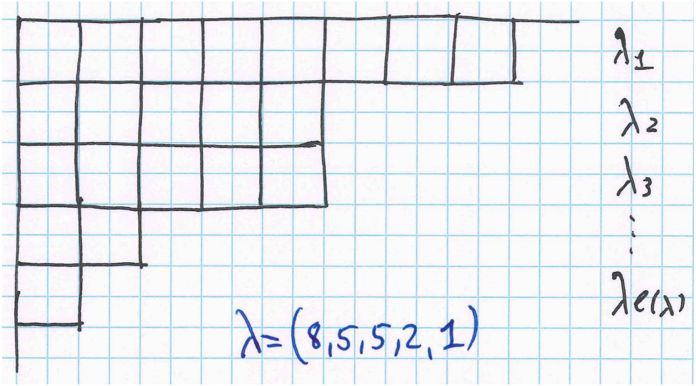
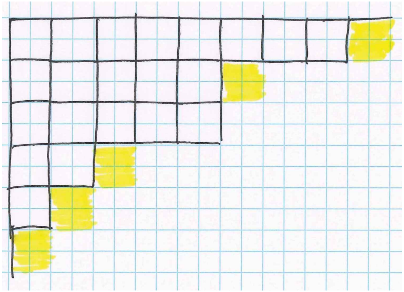
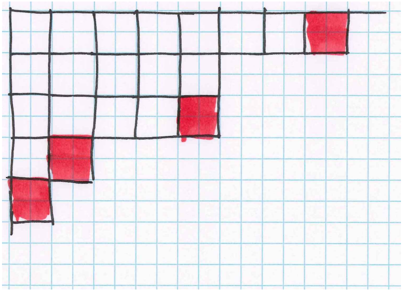


Young diagrams

$$\#\{\text{boxes that can be added to } \lambda\} = \#\{\text{boxes that can be deleted from } \lambda\} + 1.$$





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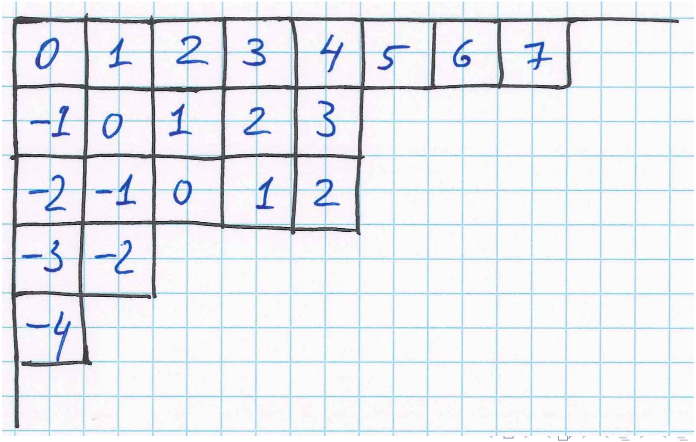

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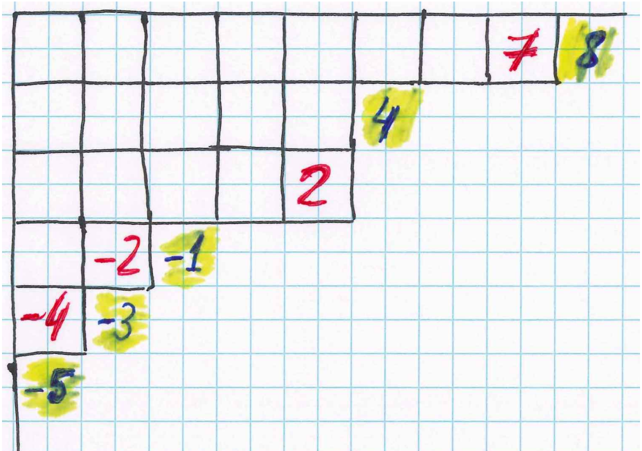

Kerov's identities ['90s]

$$\sum_{i=1}^{k} x_i - \sum_{j=1}^{k-1} y_j = 0$$

$$\sum_{i=1}^{k} x_i^2 - \sum_{j=1}^{k-1} y_j^2 = 2|\lambda|$$

($|\lambda|$ = number of boxes)

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Linear Transformations

$\mathbb{Y}$:= lattice of all Young diagrams ordered by inclusion

$\mathbb{C}\mathbb{Y}$:= linear space with basis $\{\underline{\lambda}\}_{\lambda \in \mathbb{Y}}$

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Operators in $\mathbb{C}\mathbb{Y}$

$$U^\circ \underline{\lambda} := \sum_{\nu=\lambda+\square} \underline{\nu}, \qquad D^\circ \underline{\lambda} := \sum_{\mu=\lambda-\square} \underline{\mu}$$

Then

$$[D^\circ, U^\circ] := D^\circ U^\circ - U^\circ D^\circ = Id.$$

Kerov's operators [Okounkov '00]

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$z, z' \in \mathbb{C}$ — parameters.

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$\mathfrak{sl}(2)$ commutation relations ($\Leftarrow$ Kerov's identities)

$$[D, U] = H, \qquad [H, U] = 2U, \qquad [H, D] = -2D$$

Differential posets [Stanley]
Dual graded graphs [Fomin], ‘80s

Generalize $[D^\circ, U^\circ] = Id$ for other objects.

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Branching graphs

$\mathbb{G} := \bigsqcup_{n=0}^{\infty} \mathbb{G}_n$, $\mathbb{G}_n$ — finite, $\mathbb{G}_0 := \{\varnothing\}$
$\boldsymbol{\kappa} > 0$ — edge multiplicity function

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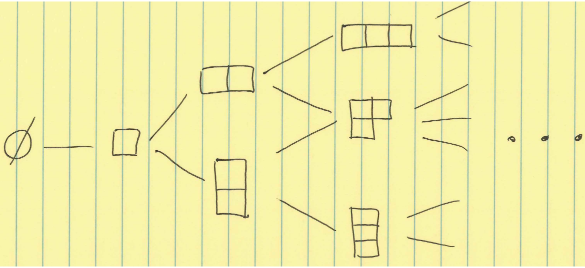

Differential posets, dual graded graphs

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Branching graph $\mathbb{G}$ is called r-self-dual ($r > 0$) iff

$$[D^\circ, U^\circ] = r \cdot Id.$$

(also more general **r**-duality is tractable)

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(Combinatorial) dimension

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$$\sum_{\lambda \in \mathbb{G}_n} (\dim \lambda)^2 = r^n n!.$$

Much more in [Stanley '88, '90], [Fomin '94 and other works].

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$M_n^{U^\circ D^\circ}(\lambda) := \dfrac{(\dim \lambda)^2}{r^n n!}$ — probability measure on $\mathbb{G}_n$ for all n.

Generalizing $\mathfrak{sl}(2)$ operators

For the Young graph:

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How to introduce dependence on the box in general?

Ideal branching graphs

$\mathbb{G}$ = *lattice of finite order ideals in some poset L*
+ an edge multiplicity function $\boldsymbol{\kappa} > 0$.

$\mu \nearrow \lambda$ (connected by an edge) iff $\mu \subset \lambda$ and $|\lambda| = |\mu| + 1$

For the Young graph $\mathbb{Y}$:

$$L = \mathbb{Z}^2_{\geq 0}, \qquad \boldsymbol{\kappa} \equiv 1.$$

Ideal branching graphs

Examples:

1. Chain
2. Pascal triangle
3. Young graph with edge multiplicities:
   - Young (simple edges)
   - Kingman (branching of set partitions)
   - Jack (β)
   - Macdonald (q, t)
4. Shifted shapes
5. Rim-hook and shifted rim-hook shapes (fixed $\#$ of boxes in a rim-hook)
6. 3D Young diagrams ($=$ plane partitions)

Kerov's operators

Definition

Operators U, D, H in $\mathbb{C}\mathbb{G}$ are called *Kerov's operators* if

1. $U\underline{\lambda} = \sum_{\nu:\ \nu \searrow \lambda} \boldsymbol{\kappa}(\lambda, \nu) q(\nu/\lambda)\underline{\nu}$,

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2. $H\underline{\lambda} = c_{|\lambda|} \cdot \underline{\lambda}$ for all $\lambda \in \mathbb{G}$
3. These operators satisfy $\mathfrak{sl}(2)$ relations

$$[D,U] = H, \qquad [H,U] = 2U, \qquad [H,D] = -2D$$

Kerov's operators

$(\cdot,\cdot)$ — standard inner product in $\mathbb{C}\mathbb{G}$: $(\underline{\lambda},\underline{\mu}) = \delta_{\lambda,\mu}$.

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"Enumerative" consequences

$$\sum_{\lambda \in \mathbb{G}_n} (U^n\underline{\varnothing},\underline{\lambda})(D^n\underline{\lambda},\underline{\varnothing}) = \theta(\theta+1)\dots(\theta+n-1)n! =: (\theta)_n n!$$

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$r^n \longrightarrow (\theta)_n$ deformation

Kerov's operators and probability measures

Probability measure on $\mathbb{G}_n$ for all n

$$M_n^{UD}(\lambda) = \frac{1}{(\theta)_n n!}(U^n \underline{\varnothing}, \underline{\lambda})(D^n \underline{\lambda}, \underline{\varnothing}) = \frac{(\dim \lambda)^2}{(\theta)_n n!} \prod_{b \in \lambda} q(b)^2$$

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UD-self-dual graph $\mathbb{G}$

For $\mathbb{G}$ to have Kerov's operators,

$$[D^\circ, U^\circ] \text{ must be a diagonal operator.}$$

(more general than Stanley–Fomin's differentiality/duality).

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Unified characterization of many interesting measures

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Remark about Jack (β) z-measures

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The measures on partitions arising from the Young graph with Jack edge multiplicities are natural discrete analogues of:

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β random matrix ensembles

N-particle random point configurations on $\mathbb{R}$ with joint density

$$\text{const} \cdot \prod_{i=1}^{N} \mu(dx_i) \cdot \prod_{1 \le i < j \le N} |x_i - x_j|^{\beta}.$$

Young graph corresponds to $\beta = 2$.

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4. On an abstract level — diagonalize the generator of dynamics
5. In concrete examples — go much further (use Fock space structure):
   - Young graph — *determinantal dynamics*
   - Schur graph of shifted shapes — *determinantal random point fiels + Pfaffian dynamics*

Down and up Markov transition kernels on $\mathbb{G}$

Down Markov transition kernels

As a branching graph, $\mathbb{G}$ comes with a natural family of *down Markov transition kernels* $p^{\downarrow}_{n,n-1}$ from $\mathbb{G}_n$ to $\mathbb{G}_{n-1}$:

$$p^{\downarrow}_{n,n-1}(\lambda, \mu) := \frac{\boldsymbol{\kappa}(\mu, \lambda) \dim \mu}{\dim \lambda},$$

where $|\mu| = n - 1$, $|\lambda| = n$.

$$\sum_{\mu \colon |\mu| = n-1} p^{\downarrow}_{n,n-1}(\lambda, \mu) = 1.$$

(randomly remove one element from λ)

Down and up Markov transition kernels on $\mathbb{G}$

Fact ($\Leftarrow \mathfrak{sl}(2)$ commutation relations)

The measures $\{M_n^{UD}\}$ are compatible with the down transition kernel $p^{\downarrow}_{n,n-1}$:

$$M_n^{UD} \circ p^{\downarrow}_{n,n-1} = M_{n-1}^{UD},$$

i.e.,

$$\sum_{\lambda \in \mathbb{G}_n} M_n^{UD}(\lambda) p^{\downarrow}_{n,n-1}(\lambda, \mu) = M_{n-1}^{UD}(\mu).$$

(random removal preserves measures M_n^{UD})

Down and up Markov transition kernels on $\mathbb{G}$

Up Markov transition kernels

There are *up Markov transition kernels* $p^{\uparrow}_{n,n+1}$ from $\mathbb{G}_n$ to $\mathbb{G}_{n+1}$:

$$p^{\uparrow}_{n,n+1}(\lambda,\nu) := \frac{M^{UD}_{n+1}(\nu)}{M^{UD}_{n}(\lambda)} p^{\downarrow}_{n+1,n}(\nu,\lambda),$$

where $|\lambda| = n$, $|\nu| = n+1$.

Down and up Markov transition kernels on $\mathbb{G}$

Up Markov transition kernels

There are *up Markov transition kernels* $p^{\uparrow}_{n,n+1}$ from $\mathbb{G}_n$ to $\mathbb{G}_{n+1}$:

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where $|\lambda| = n$, $|\nu| = n+1$.

They depend on $\{M^{UD}_n\}$ and

$$M^{UD}_n \circ p^{\uparrow}_{n,n+1} = M^{UD}_{n+1}.$$

(randomly add an element to λ *in a way preserving* M_n)

Mixed measures

From $\{M_n^{UD}\}$ to measures on the whole graph $\mathbb{G}$

$$M_\xi^{UD}(\lambda) := (1-\xi)^\theta \xi^{|\lambda|} \frac{(\theta)_{|\lambda|}}{|\lambda|!} \cdot M_{|\lambda|}^{UD}(\lambda)$$

$$= (1-\xi)^\theta \xi^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!} \right)^2 \prod_{b \in \lambda} q(b)^2$$

$$\text{"} = \text{"} (1-\xi)^\theta \left(e^{\sqrt{\xi} U} \underline{\varnothing}, \underline{\lambda} \right) \left(e^{\sqrt{\xi} D} \underline{\lambda}, \underline{\varnothing} \right)$$

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Example: Chain $\mathbb{G} = \mathbb{Z}_{\geq 0}$

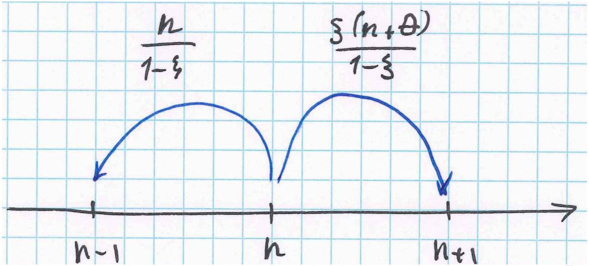
$$M_\xi^{UD}(n) = (1-\xi)^\theta \xi^n \frac{(\theta)_n}{n!} := \pi_{\theta,\xi}(n).$$

Example: Chain $\mathbb{G} = \mathbb{Z}_{\geq 0}$

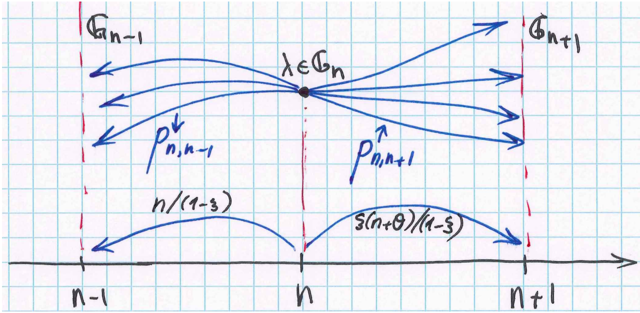
Birth and death process $\mathbf{n}_{\theta,\xi}$ preserving $\pi_{\theta,\xi}$ on $\mathbb{Z}_{\geq 0}$

$$\text{Prob}\Big(\mathbf{n}_{\theta,\xi}(t+dt) = n-1 \mid \mathbf{n}_{\theta,\xi}(t) = n\Big) = \frac{n}{1-\xi} + o(t);$$

$$\text{Prob}\Big(\mathbf{n}_{\theta,\xi}(t+dt) = n+1 \mid \mathbf{n}_{\theta,\xi}(t) = n\Big) = \frac{\xi(n+\theta)}{1-\xi} + o(t)$$



Markov process $\boldsymbol{\lambda}_\xi$ preserving M_ξ^{UD} (for general $\mathbb{G}$)



1. $|\boldsymbol{\lambda}_\xi(t)| \equiv \mathbf{n}_{\theta,\xi}(t)$
2. boxes are added/deleted to/from $\boldsymbol{\lambda}_\xi$ according to $p^{\uparrow}_{n,n+1}$ and $p^{\downarrow}_{n,n-1}$

Averages w.r.t. M_{ξ}^{UD}

Operator G_{ξ}

Let

$$G_{\xi} := e^{\sqrt{\xi} U} (1 - \xi)^{\frac{H}{2}} e^{-\sqrt{\xi} D}.$$

It is a *unitary* operator in $\ell^2(\mathbb{G})$ ($:= \mathbb{C}\mathbb{G}$ with standard inner product)

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Proposition

$$\langle f\rangle_{M_\xi^{UD}} := \sum_{\lambda\in\mathbb{G}} f(\lambda)M_\xi^{UD}(\lambda) = (G_\xi^{-1}fG_\xi\underline{\varnothing}, \underline{\varnothing}).$$

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Remark: *Fock space* structure of Young and Schur graphs allow to study M_ξ^{UD} and dynamics $\boldsymbol{\lambda}_\xi$ in great detail

Generator of dynamics $\boldsymbol{\lambda}_\xi$

Generator acting in $\ell^2(\mathbb{G}, M_\xi^{UD})$

$$(Af)(\lambda) := \sum_{\rho \in \mathbb{G}} Q_{\lambda,\rho} f(\rho),$$

$Q_{\lambda,\rho}$ — jump rates of $\boldsymbol{\lambda}_\xi$.

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Isometry $\ell^2(\mathbb{G}, M_\xi^{UD}) \longleftrightarrow \ell^2(\mathbb{G})$

$$\ell^2(\mathbb{G}, M_\xi^{UD}) \ni f \longleftrightarrow f \cdot (M_\xi^{UD})^{\frac{1}{2}} \in \ell^2(\mathbb{G})$$

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operator A in $\ell^2(\mathbb{G}, M_\xi^{UD}) \longleftrightarrow$ operator B in $\ell^2(\mathbb{G})$

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Eigenfunctions of B in $\ell^2(\mathbb{G})$

Let $\mathfrak{F}_\lambda := G_\xi \underline{\lambda}$ (for all $\lambda \in \mathbb{G}$), then

$$B\mathfrak{F}_\lambda = -|\lambda|\mathfrak{F}_\lambda, \qquad \lambda \in \mathbb{G}.$$

Diagonalization of the generator

Isometry $\ell^2(\mathbb{G}) \longleftrightarrow \ell^2(\mathbb{G}, M_\xi^{UD})$

functions $\mathfrak{F}_\lambda$ in $\ell^2(\mathbb{G})$

$$\updownarrow$$

functions

$$\mathfrak{M}_\lambda := \left(\frac{\sqrt{\xi}}{1-\xi}\right)^{|\lambda|} \left(\prod_{b\in\lambda} q(b)\right) \cdot \mathfrak{F}_\lambda \cdot (M_\xi^{UD})^{-\frac{1}{2}}$$

in $\ell^2(\mathbb{G}, M_\xi^{UD})$

Explicit formula/definition of $\mathfrak{M}_\lambda$

(does not require the existence of G_ξ)

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$$\mathfrak{M}_\lambda(\rho) := \sum_{\mu\subseteq\lambda} \left(\frac{\xi}{\xi-1}\right)^{|\lambda|-|\mu|} \left(\prod_{b\in\lambda/\mu} q(b)^2\right) \times \\ \times \frac{|\rho|!}{(|\lambda|-|\mu|)!(|\rho|-|\mu|)!} \frac{\dim(\mu,\lambda)\dim(\mu,\rho)}{\dim \rho}$$

where

$\dim(\mu,\lambda) :=$ *the number of paths (with weights) from μ to λ.*

Functions $\mathfrak{M}_\lambda$

1. Diagonalize the generator of the Markov dynamics $\boldsymbol{\lambda}_\xi$:
$A\mathfrak{M}_\lambda = -|\lambda|\mathfrak{M}_\lambda, \;\; \lambda \in \mathbb{G}$

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2. Form a (Hilbert space) basis in $\ell^2(\mathbb{G}, M_\xi^{UD})$

3. Form an orthogonal basis:

$$(\mathfrak{M}_\lambda, \mathfrak{M}_\mu)_{M_\xi^{UD}} = \delta_{\lambda,\mu}\frac{\xi^{|\lambda|}}{(1-\xi)^{2|\lambda|}}\prod_{b\in\lambda} q(b)^2.$$

Example: Chain $\mathbb{G} = \mathbb{Z}_{\geq 0}$. Meixner polynomials

$$\mathfrak{M}_n(x) =$$
$$= \sum_{k=0}^{n} \left(\frac{\xi}{\xi - 1}\right)^{n-k} \binom{n}{k} \frac{\Gamma(\theta + n)}{\Gamma(\theta + k)} \cdot x(x-1)\dots(x-k+1).$$

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$\mathbb{G} = \mathbb{Y}$ — Meixner symmetric functions [Olshanski '10,'11]

Characterization of Meixner polynomials $\mathfrak{M}_n$

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1. $\mathfrak{M}_n = x^n +$ lower degree terms
2. These polynomials are eigenfunctions of our generator:

$$A\mathfrak{M}_n = -n \cdot \mathfrak{M}_n, \qquad n = 0, 1, \ldots$$

Final remarks

The general-case functions $\mathfrak{M}_\lambda$ on $\mathbb{G}$ can be characterized in a similar manner.

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Operators U°, D° (in particular, on the (q,t)-Young graph) gives rise to similar dynamics. There is explicit diagonalization. For the chain $\mathbb{G} = \mathbb{Z}_{\geq 0}$ — *monic Charlier orthogonal polynomials* (w.r.t. Poisson weight).