Spin deformation of q-Whittaker polynomials and $\mathfrak{gl}_N$ Whittaker functions

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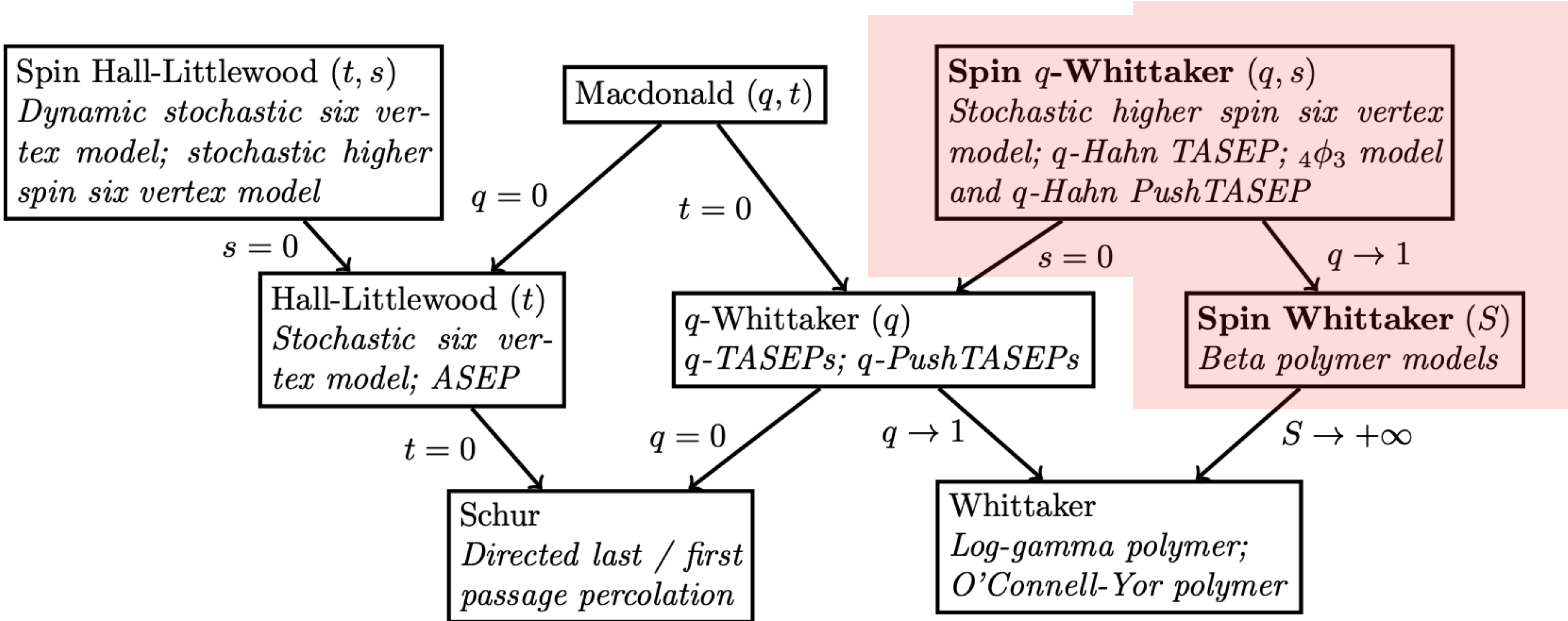
joint works with Alexey Bufetov, Matteo Mucciconi
https://arxiv.org/abs/2003.14260
https://arxiv.org/abs/1905.06815
https://arxiv.org/abs/1712.04584



"Abstract"

1. Spin q-Whittaker polynomials were originally introduced by Borodin and Wheeler (2017). We discovered a "better" version which satisfies the **usual properties**: *Cauchy identity, symmetry, combinatorial formula, plus certain eigenoperators.* They follow from **Yang-Baxter equation** and orthogonality relations for spin Hall-Littlewood polynomials.

2. In the limit $q \to 1$, we get new *spin Whittaker functions* indexed by interlacing real arrays. They also satisfy the limits of the **usual properties**. Plus, they are eigenfunctions of a deformed quantum Toda Hamiltonian.

Summary of results: Spin q-Whittaker polynomials

$$\mathbb{F}_\nu(x_1, \ldots, x_n)$$

Notation (q-Pocchammer)

$$(a;q)_k := (1-a)(1-aq)\ldots(1-aq^{k-1}), \qquad (a;q)_0 := 1.$$

Spin q-Whittaker polynomials - “combinatorial formula"

$$\mathbb{F}_{\lambda/\mu}(x) := x^{|\lambda|-|\mu|} \prod_{i=1}^{k} \frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}} \quad \textbf{(*)}$$

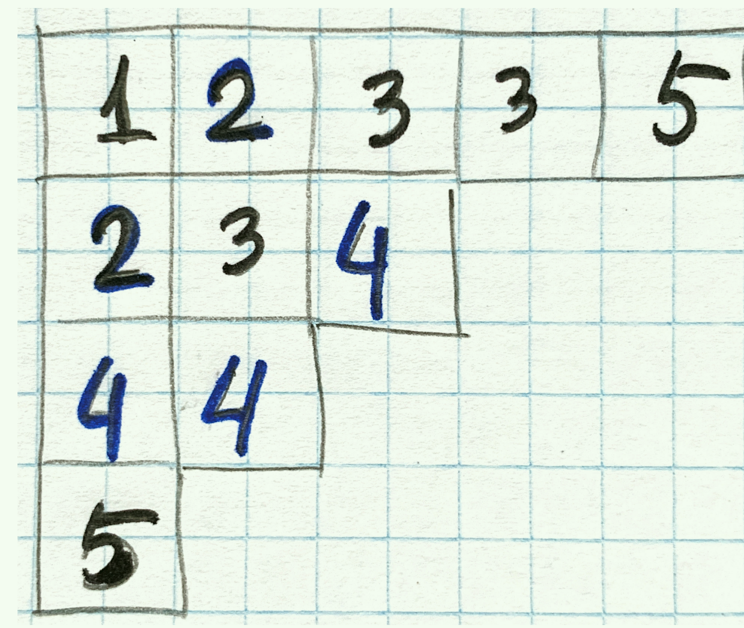
(this is a polynomial in x)

$$0 \leq \lambda_{k+1} \leq \mu_k \leq \lambda_k \leq \ldots \mu_1 \leq \lambda_1, \qquad \lambda_i, \mu_i \in \mathbb{Z}$$

$$\mathbb{F}_{\nu}(x_1,\ldots,x_n) = \sum_{\varkappa} \mathbb{F}_{\varkappa}(x_1,\ldots,x_{n-1})\, \mathbb{F}_{\nu/\varkappa}(x_n)$$

This is a symmetric polynomial in $x_1, \ldots, x_n$, which is a nontrivial fact

$$(1) \prec (2,1) \prec (4,2,0) \prec (4,3,2) \prec (5,3,2,1,0)$$



Borodin-Wheeler’s version (2017)

$$\mathbb{F}^{BW}_{\lambda/\mu}(x) = \frac{(-s/x;q)_{\lambda_{k+1}}}{(s^2;q)_{\lambda_{k+1}}}\, \mathbb{F}_{\lambda/\mu}(x).$$

$$\mathbb{F}_{\lambda}(0,x_2,\ldots,x_n) = \mathbb{F}^{BW}_{\lambda}(x_2,\ldots,x_n).$$

We made a typo implementing $\mathbb{F}^{BW}_{\lambda/\mu}$ and wrote **(*)** instead of the correct expression. But surprisingly **(*)** leads to symmetric polynomials satisfying nice properties - that is how our sqW polynomials were discovered. 😁

Brief history of spin deformations (omitting probabilistic applications & combinatorics)

The starting point is ***the spin Hall-Littlewood symmetric rational functions***

$$\mathsf{F}_\lambda(u_1,\ldots,u_n)=\frac{(1-t)^n}{(t;t)_{n-\ell(\lambda)}}\sum_{\sigma\in\mathfrak{S}_n}\sigma\Bigg\{\prod_{1\le i<j\le n}\frac{u_i-tu_j}{u_i-u_j}\prod_{i=1}^{n}\left(\frac{u_i-s}{1-su_i}\right)^{\lambda_i}\prod_{i=1}^{\ell(\lambda)}\frac{u_i}{u_i-s}\Bigg\}.$$

(1) $s=0$: the classical Hall-Littlewood polynomials. Yang-Baxter / vertex model picture appeared in **Tsilevich** (2005), further explored by **Wheeler** and **Zinn Justin** (2014+).

(2)$s=1/\sqrt{t}$: Bethe Ansatz eigenfunctions of ASEP (**Schutz** 1990s, **Tracy-Widom** 2007+).

(3) In a scaling $u=s\tilde{u}$, $s^2=0$, they are Bethe Ansatz eigenfunctions of the stochastic q-Boson dual to q-TASEP (**Sasamoto-Wadati** 1998; **Borodin-Corwin-Sasamoto** 2012, **Borodin-Corwin-P.-Sasamoto** 2013).

(4) q-TASEP / q-Boson generalize to q-Hahn TASEP / q-Hahn Boson (**Povolotsky** 2013; **Corwin** 2014).

(5) The fully general spin Hall-Littlewood polynomials are eigenfunctions of this system and of a yet more general *stochastic higher spin six vertex model* (**Borodin-Corwin-P.-Sasamoto** 2014, **Corwin-P.** 2015).

(6) Vertex model construction and connections to Yang-Baxter due to **Borodin** (2014).

(7) Spin q-Whittaker polynomials arise as *fusion* + change of variables from the spin Hall-Littlewood polynomials (**Borodin-Wheeler** 2017).

Properties

(1) $$\mathbb{F}_\lambda(x_1,\ldots,x_N) = x_1\cdots x_N\,\mathbb{F}_{\lambda-1^N}(x_1,\ldots,x_N)$$

(2) Cauchy identity

$$\mathbb{F}^*_{\lambda/\mu}(y) := y^{|\lambda|-|\mu|}\,\frac{(-s/y;q)_{\lambda_N-\mu_N}}{(q;q)_{\lambda_N-\mu_N}}\prod_{i=1}^{N-1}\frac{(-s/y;q)_{\lambda_i-\mu_i}(-sy;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\mu_i-\mu_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\mu_i-\mu_{i+i}}}$$

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\ldots,x_N)\,\mathbb{F}^*_\lambda(y_1,\ldots,y_k) = \prod_{j=1}^{k}\left(\frac{(-sy_j;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i=1}^{N}\prod_{j=1}^{k}\frac{(-sx_i;q)_\infty}{(x_iy_j;q)_\infty}.$$

Example $N=1,\ k=1$ - q-binomial theorem

$$\sum_{n=0}^{\infty}\underbrace{x^n}_{\mathbb{F}}\cdot\underbrace{y^n\frac{(-s/y;q)_n}{(q;q)_n}}_{\mathbb{F}^*} = \frac{(-sx;q)_\infty}{(xy;q)_\infty}$$

Compare with Macdonald case, also q-binomial theorem

$$\sum_{r\geq 0}\underbrace{x^r}_{P}\cdot\underbrace{y^r\frac{(t;q)_r}{(q;q)_r}}_{Q} = \frac{(txy;q)_\infty}{(xy;q)_\infty}$$

Any relation to Cauchy identity for interpolation Macdonald polynomials (**Olshanski** 2017)?

Compare with Borodin-Wheeler (2017): q-Gauss summation formula. In our setting we get a similar identity for $N=2, k=1$

$$\sum_{i=0}^{\infty}\frac{(-s/x;q)_i(-s/y;q)_i}{(s^2;q)_i(q;q)_i}(xy)^i = \frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty}.$$

q-difference operators ($T_{q,x}$ maps $f(x)$ to $f(qx)$)

$$\mathfrak{D}_1 := \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{(1+sx_i)}{1-x_i/x_j} T_{q,x_i}, \qquad \mathfrak{D}_1 \mathbb{F}_\lambda(x_1,\ldots,x_N) = q^{\lambda_N} \mathbb{F}_\lambda(x_1,\ldots,x_N).$$

$$\overline{\mathfrak{D}}_1 := \sum_{i=1}^{N} \prod_{\substack{j=1\\ j\neq i}}^{N} \frac{(1+s/x_i)}{1-x_j/x_i} T_{q^{-1},x_i}. \qquad \overline{\mathfrak{D}}_1 \mathbb{F}_\lambda(x_1,\ldots,x_N) = q^{-\lambda_1} \mathbb{F}_\lambda(x_1,\ldots,x_N).$$

Compare to q-Whittaker situation

At $s = 0$, the sqW polynomials become the q-Whittaker polynomials which possess higher order eigenoperators:

$$W_N^r := \sum_{|I|=r} \prod_{i\in I,\, j\notin I} \frac{1}{1-x_i/x_j} \prod_{i\in I} T_{q,x_i}, \qquad \tilde{W}_N^r := \sum_{|I|=r} \prod_{i\in I,\, j\notin I} \frac{1}{1-x_j/x_i} \prod_{i\in I} T_{q^{-1},x_i},$$

In the spin deformation the situation is more mysterious. First of all, $[\mathfrak{D}_1, \overline{\mathfrak{D}}_1] = 0$.
Next, both of them are conjugations of the first order q-Whittaker operators:

$$\mathbb{U}_N := \prod_{i=1}^{N} \frac{1}{(-sx_i;q)_\infty^{N-1}}, \qquad \mathbb{V}_N := \prod_{i=1}^{N} \frac{1}{(-s/x_i;q)_\infty^{N-1}}.$$

$$\mathfrak{D}_1 = \mathbb{U}_N^{-1} W_N^1 \mathbb{U}_N, \qquad \overline{\mathfrak{D}}_1 = \mathbb{V}_N^{-1} \tilde{W}_N^1 \mathbb{V}_N,$$

Same conjugations of the q-Whittaker operators are not diagonal in the spin q-Whittaker poly's.

Conjectural torus orthogonality

$$m_{q,s}^{N}(z_1,\ldots,z_N) := \frac{1}{N!}\prod_{1\le i\ne j\le N}\frac{(s^2, z_i/z_j;q)_\infty}{(-sz_i,-s/z_i;q)_\infty}\prod_{i=1}^{N}\frac{1}{2\pi \mathrm{i} z_i}, \qquad (z_1,\ldots,z_N)\in\mathbb{T}^N.$$

$$\int_{\mathbb{T}^N}\mathbb{F}_\lambda(z_1,\ldots,z_N)\,\mathbb{F}_\mu(1/z_1,\ldots,1/z_N)\,m_{q,s}^{N}(z_1,\ldots,z_N)\,dz_1\cdots dz_N = c_\lambda \mathbf{1}_{\lambda=\mu}$$

At $s=0$, the q-Whittaker orthogonality is known, and it follows from the presence of many self-adjoint eigenoperators for Macdonald polynomials (whose eigenvalues separate the polynomials).

For $q=s=0$, this is the statement of orthogonality of characters of the unitary groups $U(N)$.

Both sqW eigenoperators $\mathfrak{D}_1$, $\overline{\mathfrak{D}}_1$ are self-adjoint with respect to $m_{q,s}$, but their eigenvalues do not separate labels λ.

Summary of results: Spin Whittaker functions

$$\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N)$$

Notation (Gauss hypergeometric function, Pochhammer symbol)

$$ {}_2F_1\left(\begin{matrix} a\,,\, b \\ c \end{matrix}\,\middle|\, z\right) = \sum_{k\geq 0} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

$$(r)_k = r(r+1)\cdots(r+k-1)$$

Limit transitions sqW → sW → W

$\mathbb{F}_\lambda(x_1, \dots, x_N \mid q, s)$ $\quad 0<q<1$, $-1<s<0$

$q \to 1$, $x_i = q^{X_i}$, $L_i = q^{-\lambda_i}$

$s = -q^{S}$

$S > 0$, $|X_i| < S$, $1 \le L_N \le L_{N-1} \le \dots \le L_1$

Theorem (Mucciconi-P. 2020)

$$\lim_{q \to 1} \frac{\mathbb{F}_\lambda(x_1, \dots, x_N)}{(-\log q)^{N(N-1)/2}} = \mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N)$$

$$1 \le L_N \le \dots \le L_1$$

Here $\mathfrak{f}_{X_1,\dots,X_N}$ is the *spin Whittaker* function, which is symmetric in X_i, depends on $\underline{L}_N$ and on a parameter S.

Reduction to the usual $\mathfrak{gl}_N$ Whittaker functions, $S \to +\infty$

$$L_i = S^{N+1-2i} e^{u_i}, \qquad X_k = -\mathrm{i}\lambda_k,$$

Whittaker symmetric functions
(Kostant, Givental, Bump, Stade, Gerasimov-Lebedev-Oblezin, Corwin-O'Connell-Seppalainen-Zygouras,…)

Conjecture (Mucciconi-P. 2020, holds modulo decay estimates)

$$\left(\frac{4\pi}{S16^S}\right)^{\frac{N(N-1)}{4}} \mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N) \xrightarrow{S\to\infty} \psi_{\lambda_1,\dots,\lambda_N}(u_1, \dots, u_N)$$

$$\psi_{\lambda_1,\dots,\lambda_N}(\underline{u}_N) = \int_{\mathbb{R}^{N-1}} \psi_{\lambda_1,\dots,\lambda_{N-1}}(\underline{u}_{N-1})\, Q^{N\to N-1}_{\lambda_N}(\underline{u}_N, \underline{u}_{N-1}) \prod_{k=1}^{N-1} du_{N-1,k},$$

where

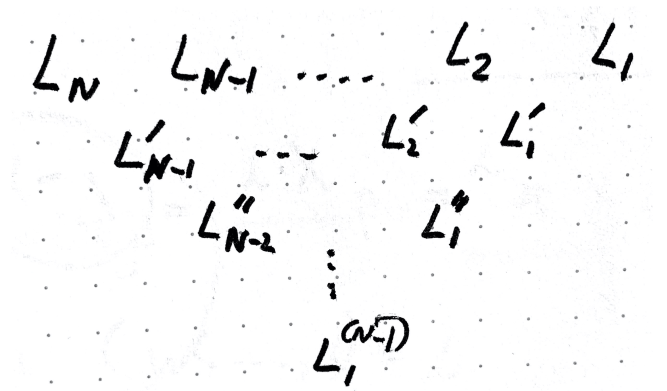
$$Q^{N\to N-1}_{\lambda}(\underline{u}_N, \underline{u}_{N-1}) = e^{\mathrm{i}\lambda\left(\sum_{i=1}^N u_{N,i} - \sum_{i=1}^{N-1} u_{N-1,i}\right)} \prod_{i=1}^{N-1} \exp\left\{-e^{u_{N-1,i}-u_{N,i}} - e^{u_{N,i+1}-u_{N-1,i}}\right\}$$

Definition of spin Whittaker functions via “combinatorial formula” (or “Givental integral”)

$$x^{|\lambda|-|\mu|}\prod_{i=1}^{k}\frac{(-s/x;q)_{\lambda_i-\mu_i}(-sx;q)_{\mu_i-\lambda_{i+1}}(q;q)_{\lambda_i-\lambda_{i+1}}}{(q;q)_{\lambda_i-\mu_i}(q;q)_{\mu_i-\lambda_{i+i}}(s^2;q)_{\lambda_i-\lambda_{i+i}}}.$$

$$\mathcal{A}_{S,X}(u,v,z):=\frac{1}{\mathrm{B}(S+X,S-X)}\left(1-\frac{v}{z}\right)^{S-X-1}\left(1-\frac{u}{v}\right)^{S+X-1}\left(1-\frac{u}{z}\right)^{1-2S}$$

$$\mathfrak{f}_X(\underline{L}_k;\underline{L}_{k+1}):=\mathbf{1}_{\underline{L}_k\prec\underline{L}_{k+1}}\left(\frac{L_{k+1,k+1}\cdots L_{k+1,1}}{L_{k,k}\cdots L_{k,1}}\right)^{-X}\prod_{i=1}^{k}\mathcal{A}_{S,X}(L_{k+1,i+1},L_{k,i},L_{k+1,i})$$



Interlacing $1\le L_{k,k}\le L_{k-1,k-1}\le L_{k,k-1}\le\ldots\le L_{k-1,1}\le L_{k,1}$

Definition. $$\mathfrak{f}_{X_1,\ldots,X_N}(\underline{L}_N):=\int_{\underline{L}_{N-1}\prec\underline{L}_N}\mathfrak{f}_{X_1,\ldots,X_{N-1}}(\underline{L}_{N-1})\,\mathfrak{f}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\frac{d\underline{L}_{N-1}}{\underline{L}_{N-1}}$$

Examples. $$\mathfrak{f}_{X_1}(L_{1,1})=L_{1,1}^{-X_1}$$

$$\mathfrak{f}_{X,Y}(u,z)=(z/u)^S u^{-X-Y}{}_2F_1\left(\begin{matrix}S+X\,,\,S+Y\\2S\end{matrix}\,\middle|\,1-\frac{z}{u}\right).$$

“Dual” functions. $$\mathfrak{g}_Y(\widetilde{\underline{L}}_k;\underline{L}_k)=\frac{L_{k,1}^{-Y}}{\Gamma(S-Y)}\left(1-\frac{\widetilde{L}_{k,1}}{L_{k,1}}\right)^{S-Y-1}\mathfrak{f}_{-Y}(\underline{\ell}_{k-1};\widetilde{\underline{L}}_k),$$

$$\mathfrak{g}_{Y_1,\ldots,Y_M}(\underline{L}_N)=\begin{cases}\displaystyle\int\mathfrak{g}_{Y_1,\ldots,Y_{M-1}}(\widetilde{\underline{L}}_N)\mathfrak{g}_{Y_M}(\widetilde{\underline{L}}_N;\underline{L}_N)\frac{d\widetilde{\underline{L}}_N}{\widetilde{\underline{L}}_N} & \text{if } N<M,\\ \displaystyle\int\mathfrak{g}_{Y_1,\ldots,Y_{N-1}}(\widetilde{\underline{L}}_{N-1})\mathfrak{g}_{Y_N}(\widetilde{\underline{L}}_{N-1};\underline{L}_N)\frac{d\widetilde{\underline{L}}_{N-1}}{\widetilde{\underline{L}}_{N-1}} & \text{if } N=M.\end{cases}$$

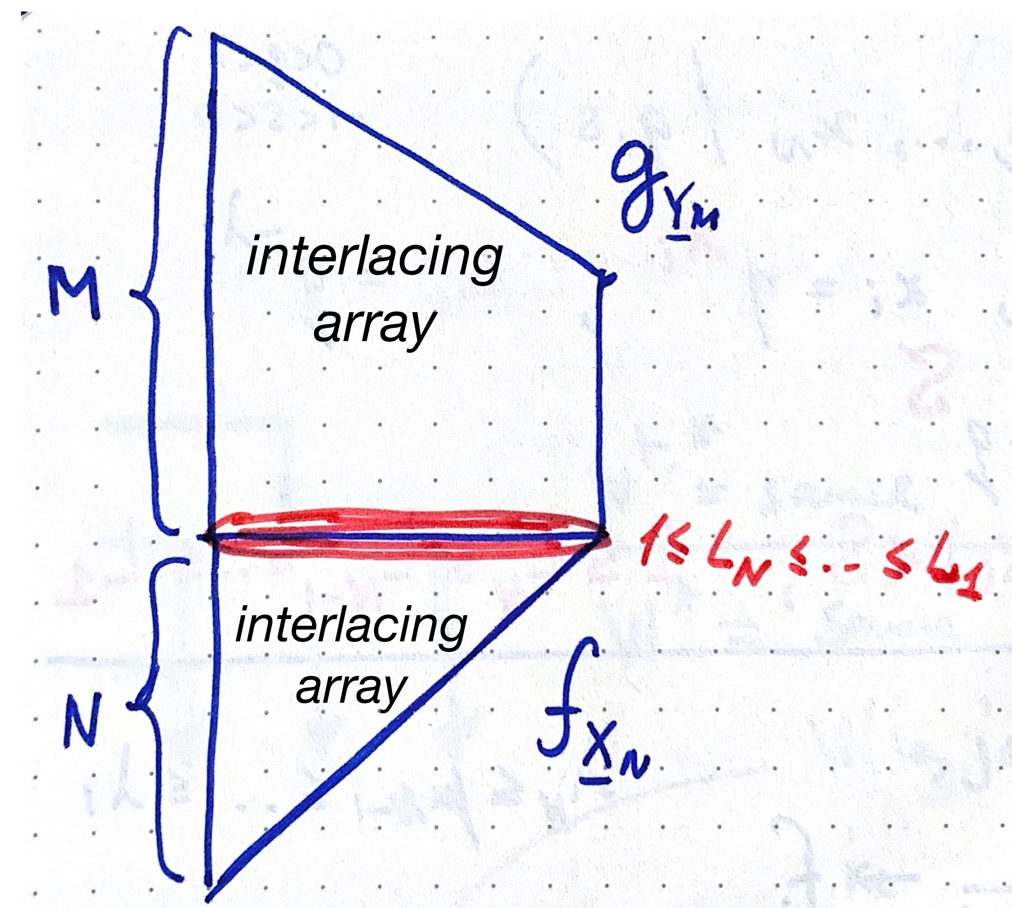
$$\mathfrak{g}_Y(L)=\mathfrak{g}_Y(1;L)=\frac{L^{-Y}(1-L^{-1})^{S-Y-1}}{\Gamma(S-Y)}.$$

Properties

(1) $$\mathfrak{f}_{X_1,\dots,X_N}(a\underline{L}_N) = a^{-X_1-\dots-X_N}\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N), \qquad a > 1.$$

(2) Cauchy identity, $M \geq N$

$$\int \mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N)\,\mathfrak{g}_{Y_1,\dots,Y_M}(\underline{L}_N)\,\frac{d\underline{L}_N}{\underline{L}_N} = \prod_{j=1}^{M}\frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)}\left(\prod_{i=2}^{N}\frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)}\right).$$



(3) Difference eigenoperators ($\mathcal{T}_X$ - shift by 1)

$$\mathcal{D}_1 := \sum_{i=1}^{N}\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{X_i+S}{X_i-X_j}\,\mathcal{T}_{X_i}, \qquad \overline{\mathcal{D}}_1 := \sum_{i=1}^{N}\prod_{\substack{j=1\\ j\neq i}}^{N}\frac{X_i-S}{X_i-X_j}\,\mathcal{T}_{X_i}^{-1}.$$

$$\mathcal{D}_1\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N) = L_{N,N}^{-1}\,\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N),$$

$$\overline{\mathcal{D}}_1\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N) = L_{N,1}\,\mathfrak{f}_{X_1,\dots,X_N}(\underline{L}_N).$$

(4) Deformed quantum Toda (scaling limit of Pieri rules, similar to [Gerasimov-Lebedev-Oblezin 2011-12])

$$\mathcal{H}_2 := -\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2 + \sum_{1\le i<j\le N} S^{-2(j-i)} e^{u_j-u_i}(S-\partial_{u_i})(S+\partial_{u_j}).$$

Additive variables u_i

$$L_i = S^{N+1-2i}e^{u_i}.$$

Theorem.

$$\mathcal{H}_2 \mathfrak{f}^{add}_{\underline{X}}(u_1,\ldots,u_N) = -\frac{1}{2}\left(X_1^2+\cdots+X_N^2\right)\mathfrak{f}^{add}_{\underline{X}}(u_1,\ldots,u_N).$$

Remark. For $S \to +\infty$ we get the usual $\mathfrak{gl}_N$ quantum Toda Hamiltonian

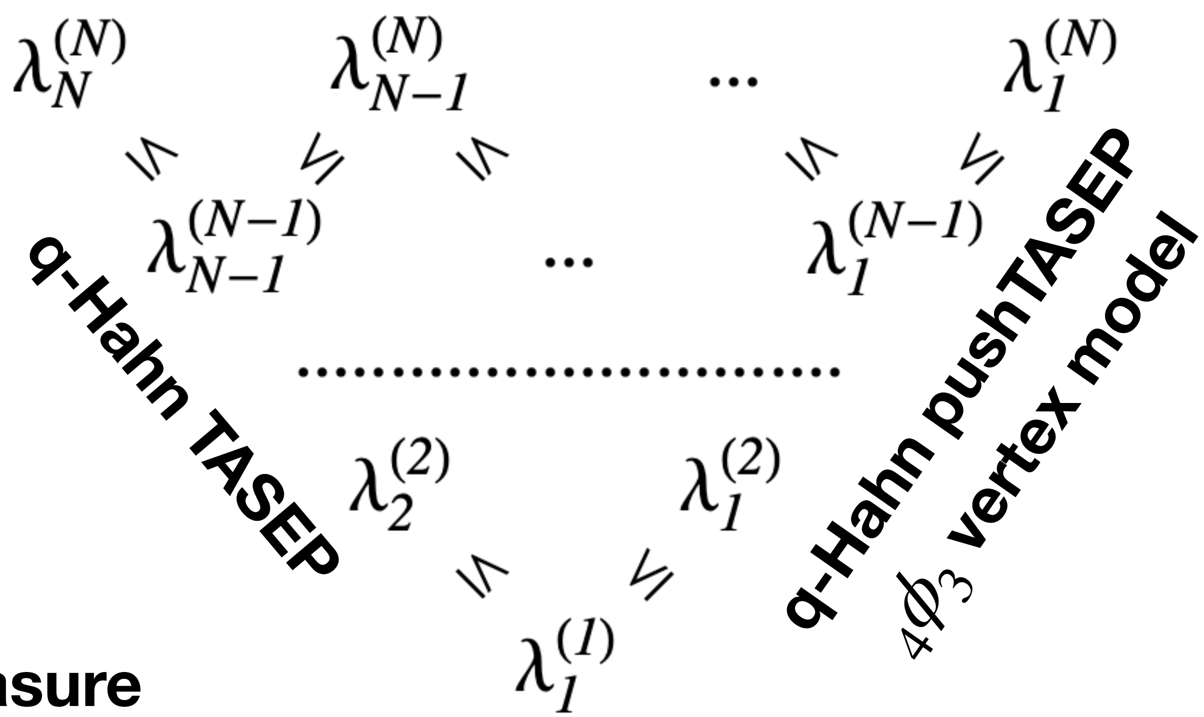
$$\mathcal{H}_2^{\text{Toda}} := -\frac{1}{2}\sum_{i=1}^{N}\partial_{u_i}^2 + \sum_{i=1}^{N-1} e^{u_{i+1}-u_i}.$$

(5) Conjectural "weak" orthogonality with deformed Sklyanin measure

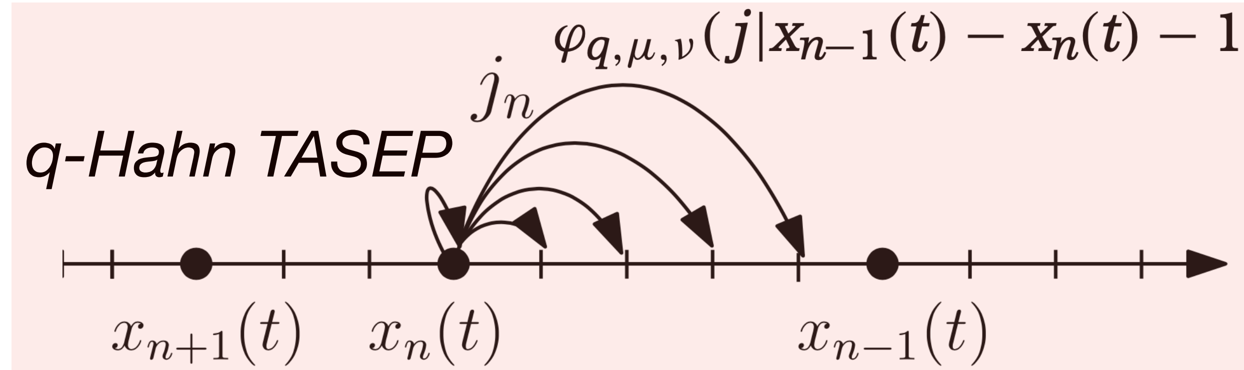
$$\int_{(\mathrm{i}\mathbb{R})^N} \mathfrak{f}_{\underline{Z}}(\underline{L}_N)\,\mathfrak{f}_{-\underline{Z}}(\underline{L}'_N)\,\mathfrak{M}_S^N(\underline{Z})\,dZ_1\ldots dZ_N = \prod_{i=1}^{N-1}\left(1-\frac{L_{N,i+1}}{L_{N,i}}\right)^{1-2S}\delta_{\underline{L}_N-\underline{L}'_N},$$

$$\mathfrak{M}_S^N(\underline{Z}) = \frac{1}{N!(2\pi\mathrm{i})^N}\prod_{1\le i\ne j\le N}\frac{\Gamma(S+Z_i)\Gamma(S-Z_i)}{\Gamma(2S)\Gamma(Z_i-Z_j)},$$

Probabilistic applications



Discrete case (spin q-Whittaker processes)



$$\varphi_{q,\mu,\nu}(j|m) := \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{m-j}}{(\nu;q)_m}\frac{(q;q)_m}{(q;q)_j(q;q)_{m-j}}$$

Measure

$$\frac{1}{Z}\mathbb{F}_{\lambda^{(1)}}(x_1)\mathbb{F}_{\lambda^{(2)}/\lambda^{(1)}}(x_2)\dots\mathbb{F}_{\lambda^{(N-1)}/\lambda^{(N-2)}}(x_{N-1})\mathbb{F}_{\lambda^{(N)}/\lambda^{(N-1)}}(x_N)\mathbb{F}^*_{\lambda^{(N)}}(y_1,y_2,\dots,y_t).$$

(Bijectivised) **Yang-Baxter equation** provides (two different) discrete time **Markov dynamics** on interlacing arrays increasing the parameter t.

- Under one of the dynamics, the leftmost coordinates $\lambda_k^{(k)} - k$ evolve as the **q-Hahn TASEP** [Povolotsky 2013, Corwin 2014, …]

- Under the other dynamics, the rightmost coordinates $\lambda_1^{(k)} + k$ evolve as the **q-Hahn PushTASEP** (or, very similarly, as the **stochastic $_4\phi_3$ vertex model**) [Corwin-Matveev-P. 2018, Bufetov-Mucciconi-P. 2019]

For both, difference eigenoperators on sqW polynomials give probabilistic information similarly to [Borodin-Corwin 2011].

Continuous case (spin Whittaker processes)

Define similarly a *process* based on the spin Whittaker functions

$$\mathfrak{P}_{\mathbf{X};\mathbf{Y}}(\underline{\underline{L}}_N) = \frac{\mathfrak{f}_{X_1}(\underline{L}_1)\mathfrak{f}_{X_2}(\underline{L}_1;\underline{L}_2)\cdots\mathfrak{f}_{X_N}(\underline{L}_{N-1};\underline{L}_N)\mathfrak{g}_{\mathbf{Y}}(\underline{L}_N)}{\Pi(\mathbf{X};\mathbf{Y})}.$$

$$\Pi(\mathbf{X};\mathbf{Y}) = \prod_{j=1}^{T} \frac{\Gamma(X_1+Y_j)}{\Gamma(S+X_1)} \left(\prod_{i=2}^{N} \frac{\Gamma(X_i+Y_j)\Gamma(2S)}{\Gamma(S+X_i)\Gamma(S+Y_j)} \right).$$

$$\mathbf{X} = (X_1, \ldots, X_N)$$

$$\mathbf{Y} = (Y_1, \ldots, Y_T)$$

We also have Markov dynamics on spin Whittaker processes which increase the parameter T.

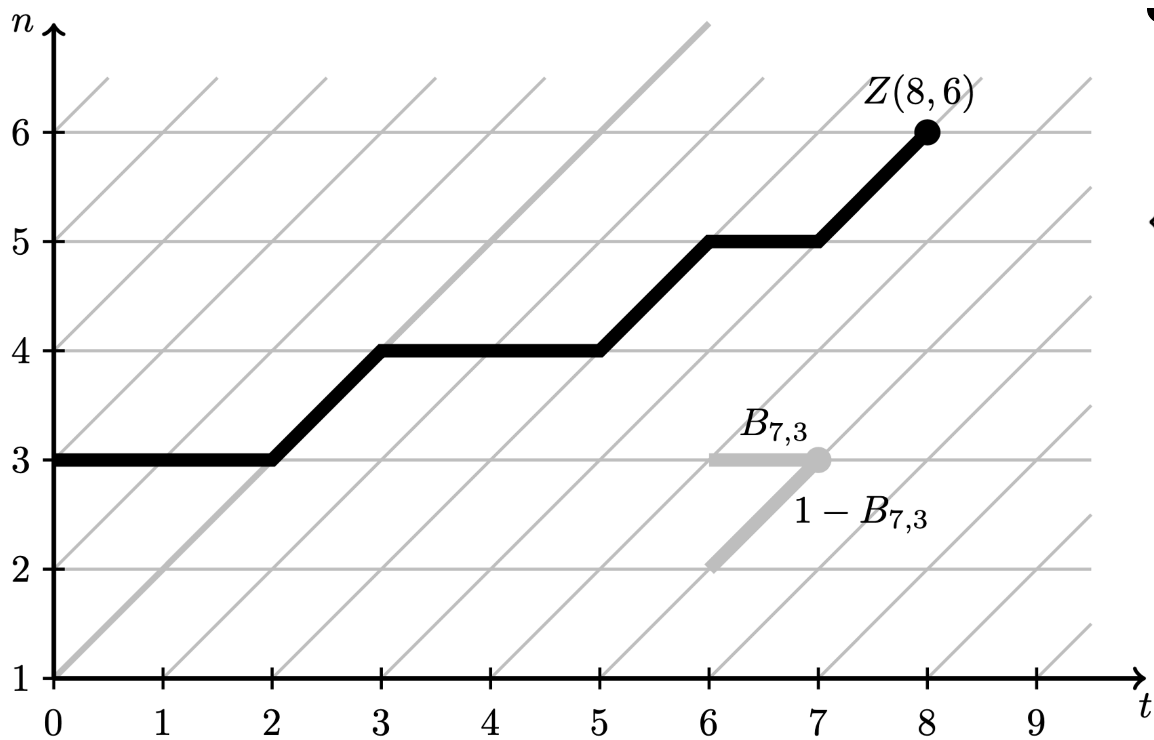
Theorem (Mucciconi-P. 2020).
The marginals $L_{k,k}(T)^{-1}$ have the same distribution as the strict-weak beta polymer model $Z(k, T)$ of [Barraquand-Corwin 2015].

The marginals $L_{k,1}(T)^{-1}$ have the same distribution as the “weird” beta polymer model $\tilde{Z}(k, T)$ of [Corwin-Matveev-P. 2018].

Both polymer models arise as $q \to 1$ limits of the q-Hahn particle systems.

The “weird” model as $S \to \infty$ also reduces to the more usual **log-gamma polymer model.**
The strict-weak beta polymer reduces to the strict-weak log-gamma polymer.

Beta polymers



Strict-weak beta polymer

$$\begin{cases} Z(i,j) = Z(i,j-1)B_{i,j} + Z(i-1,j-1)(1-B_{i,j}) & \text{for } 1 < i \le j; \\ Z(1,j) = Z(1,j-1)B_{1,j} & \text{for } j > 0; \\ Z(i,0) = 1 & \text{for } i > 0. \end{cases}$$

$$B_{i,j} \sim \mathcal{Beta}(X_i + Y_j, S - Y_j)$$

$$\mathcal{B}(m,n)[x] = \frac{x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)} \qquad \text{for } x \in (0,1).$$

Another beta polymer-type model - a random recursion

$$\widetilde{Z}(i,j) = \begin{cases} 1 & \text{for } j = 0, \\ \widetilde{Z}(1,j-1)\widetilde{B}_{1,j} & \text{for } i = 1, \\ W^{>}_{i,j}\widetilde{Z}(i,j-1) + (1 - W^{>}_{i,j})\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) > \widetilde{Z}(i-1,j), \\ (1 - W^{<}_{i,j})\widetilde{Z}(i,j-1) + W^{<}_{i,j}\widetilde{Z}(i-1,j) & \text{if } \widetilde{Z}(i,j-1) < \widetilde{Z}(i-1,j), \end{cases}$$

$$\widetilde{B}_{1,j} \sim \mathcal{Beta}^{-1}(X_1 + Y_j, S - Y_j)$$

$$W^{>}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i-1,j) - \widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i,j-1) - \widetilde{Z}(i-1,j-1)}, X_i + Y_j, S - Y_j\right),$$

$$W^{<}_{i,j} \sim \mathcal{NBB}^{-1}\left(2S-1, \frac{\widetilde{Z}(i,j-1) - \widetilde{Z}(i-1,j-1)}{\widetilde{Z}(i-1,j) - \widetilde{Z}(i-1,j-1)}, X_i + Y_j, S - X_i\right).$$

Where NBB is a random variable on [0,1] with density

$$\mathcal{NBB}(r,p,m,n)[x] = \frac{(1-p)^r x^{m-1}(1-x)^{n-1}}{\mathrm{B}(n,m)} \, {}_2F_1\left(\begin{matrix} r, n+m \\ n \end{matrix} \,\middle|\, p(1-x)\right),$$

The role of the Yang-Baxter equation

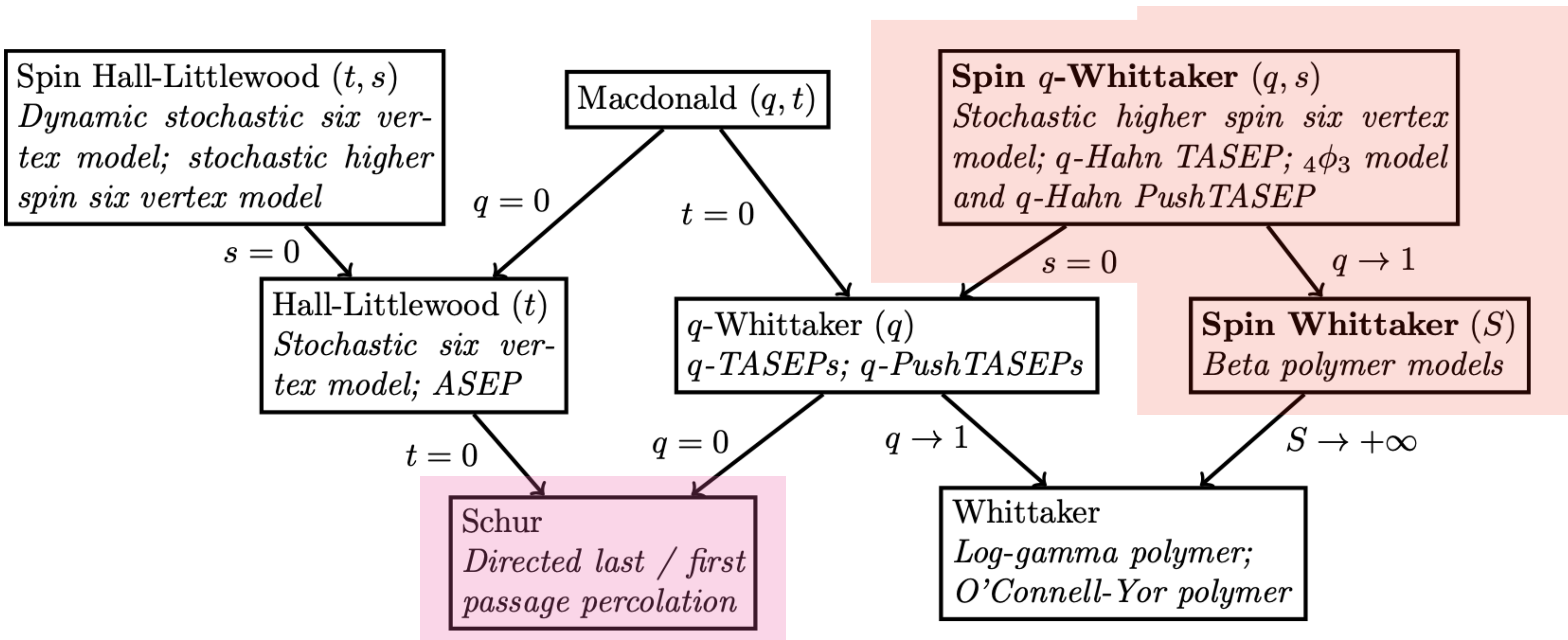
(mostly in the Schur $q = s = 0$ case)

Schur polynomials ($q = s = 0$ degenerations of spin q-Whittaker polynomials)

For simplicity, restrict to **Schur polynomials** which are most well-known symmetric functions in the scheme

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det[u_i^{\lambda_j+N-j}]_{i,j=1}^N}{\det[u_i^{N-j}]_{i,j=1}^n}, \qquad \textbf{(1)}$$

$$\lambda = (\lambda_1 \geq \ldots \geq \lambda_N), \;\; \lambda_i \in \mathbb{Z}_{\geq 0}.$$

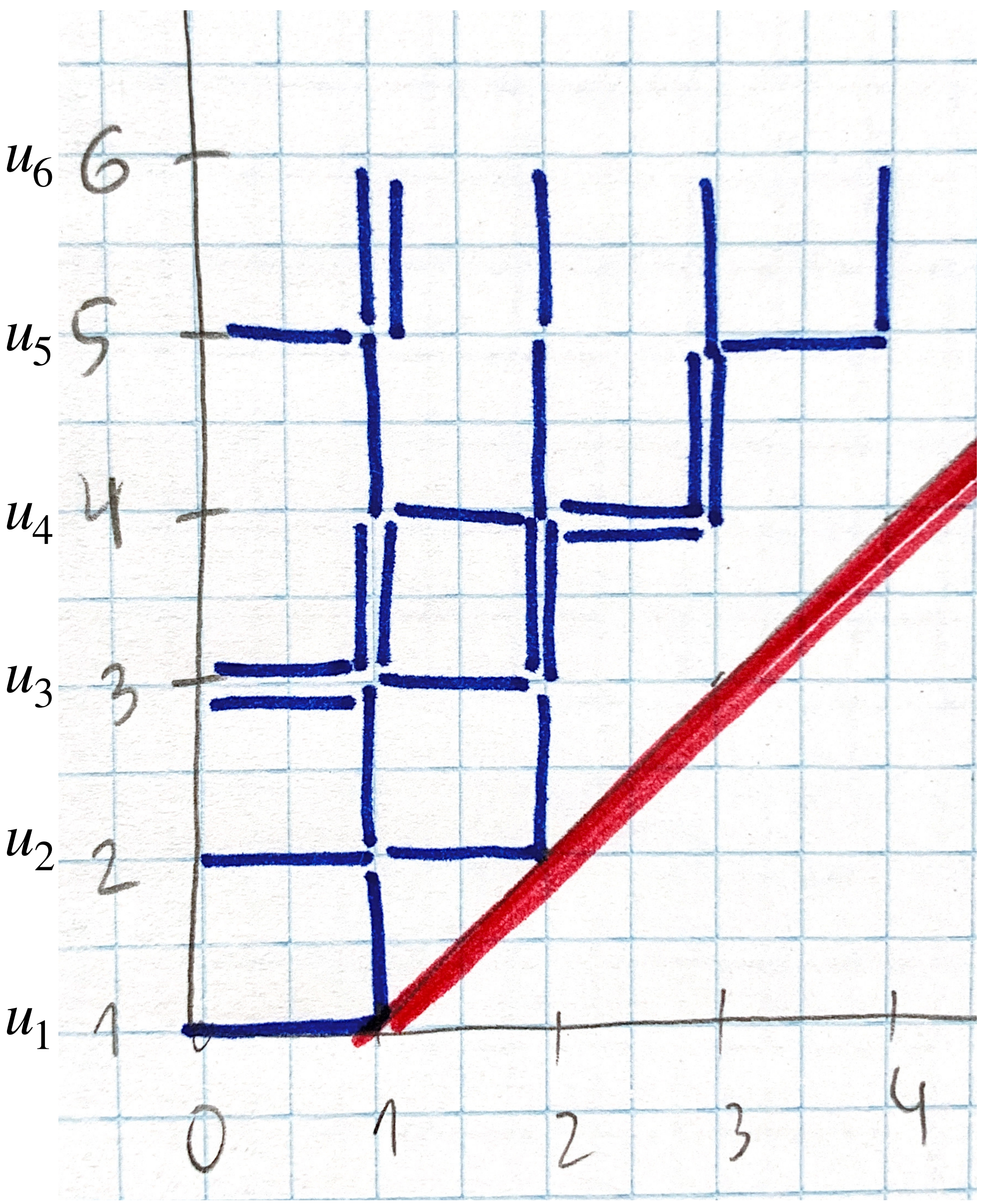


Goal.

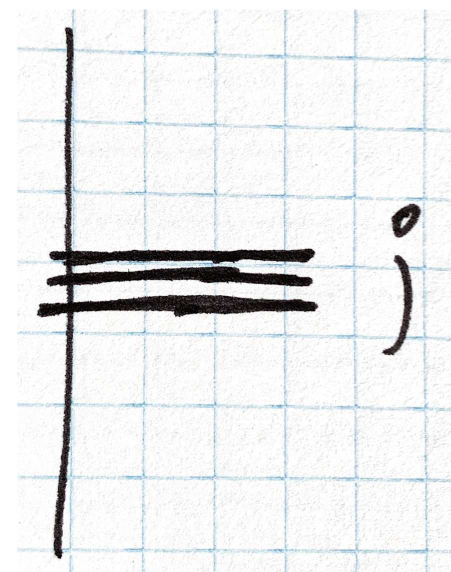
Get many properties (**symmetry, Cauchy identities**) of Schur polynomials via Yang-Baxter equation, starting from essentially the combinatorial formula **(2)**

$$s_\lambda(u_1, \ldots, u_N) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T \qquad \textbf{(2)}$$

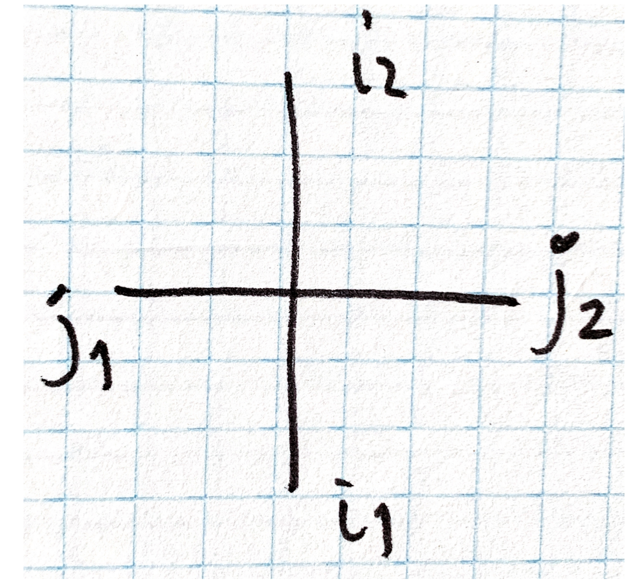
Combinatorial formula for Schur polynomials via a vertex model



Lattice paths in the half-quadrant with weights



u^j, where j paths start on the left

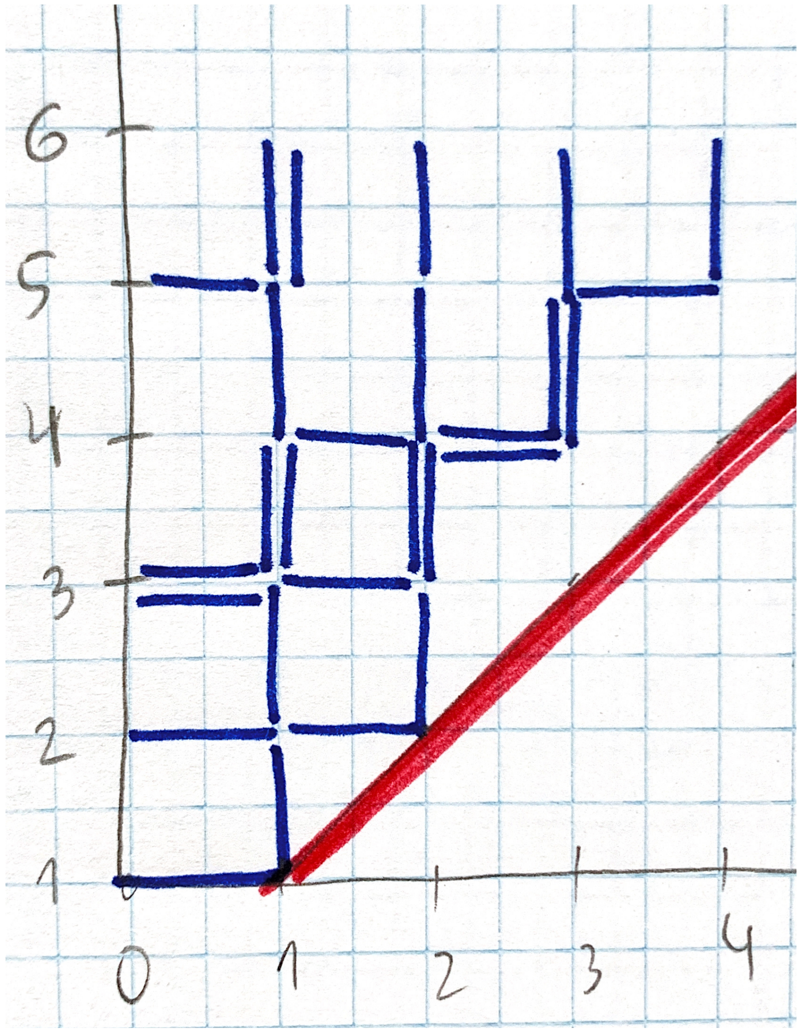


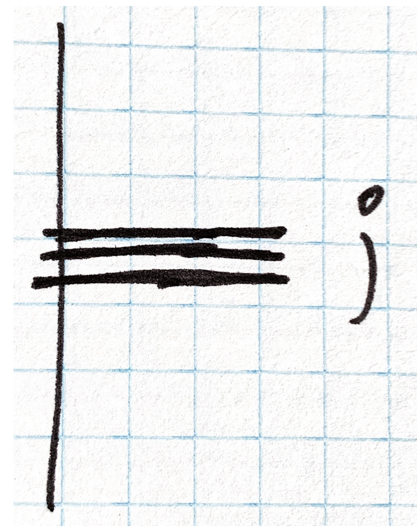
$\mathbf{1}_{i_1+j_1=i_2+j_2}\mathbf{1}_{i_1\geq j_2}u^{j_2}$

At the red diagonal, all paths must turn with weight **1**.

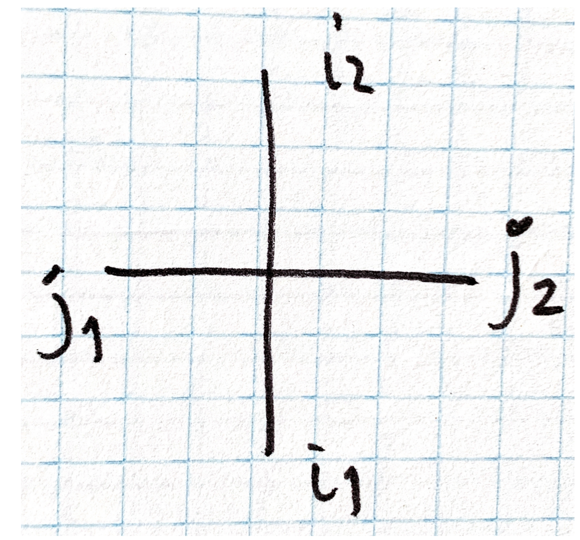
At each level we read off a partition by
$\lambda_i - \lambda_{i+1} = \#\{$paths through i-th vertical edge$\}$.
Top partition is $(5,3,2,1,0)$.
Path ensembles are in bijection with semistandard tableaux.

1	2	3	3	5
2	3	4		
4	4			
5				





u^j, where j paths start on the left



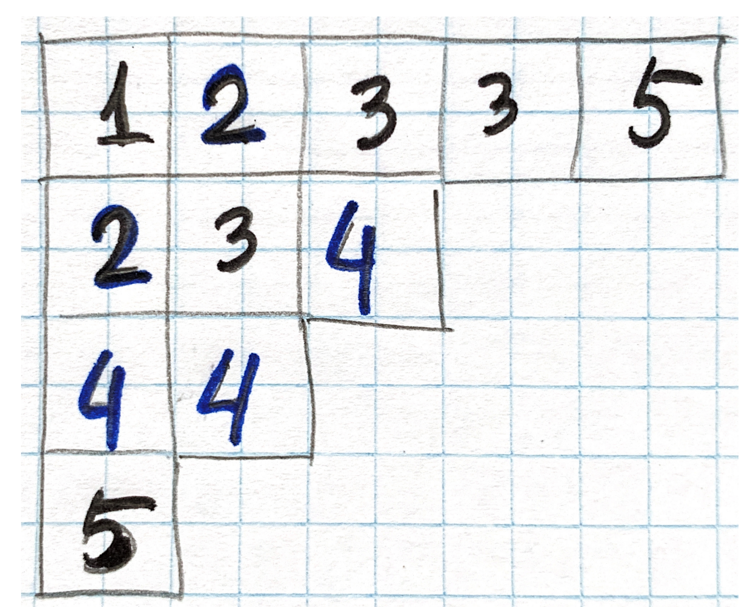
$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}\,u^{j_2}$

Exercise.

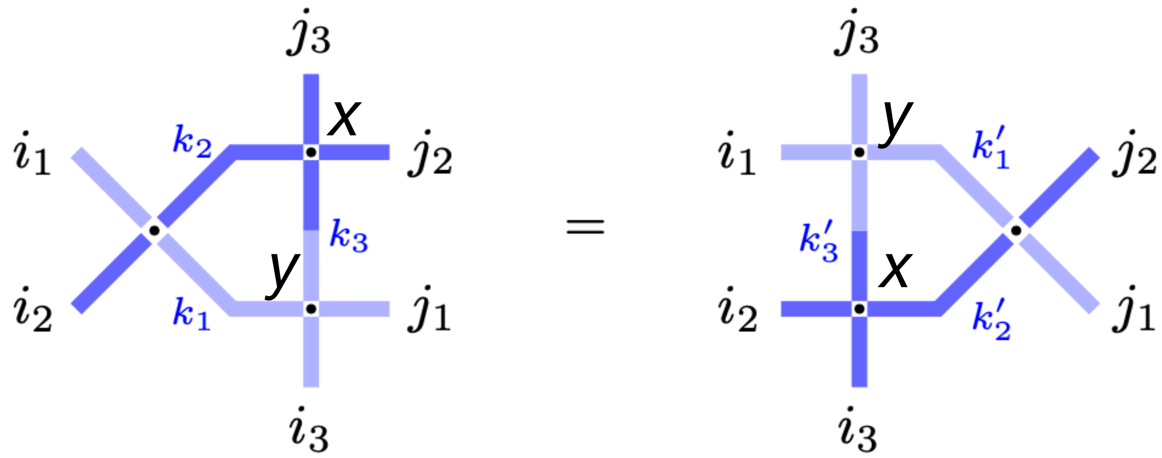
$$s_\lambda(u_1,\ldots,u_N) = \sum_{\text{path ensembles with top row } \lambda} \prod \text{vertex weights}$$

Example in the picture: $s_{(5,3,2,1,0)}(u_1,u_2,u_3,u_4,u_5)$

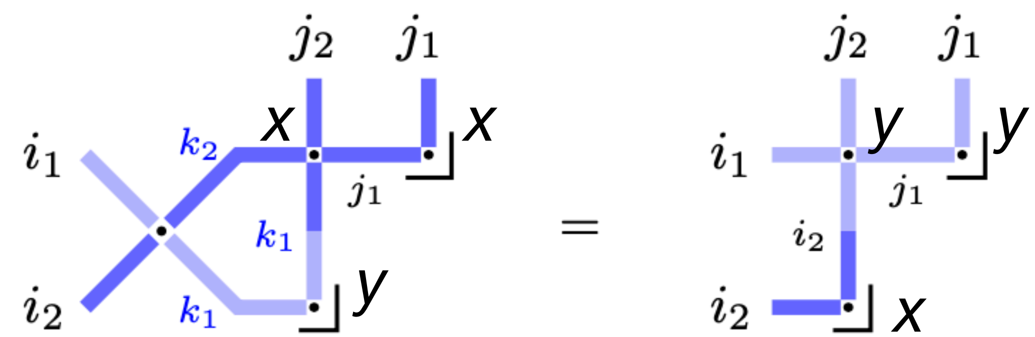
Weight of the picture: $u_1u_2^2u_3^3u_4^3u_5^2$



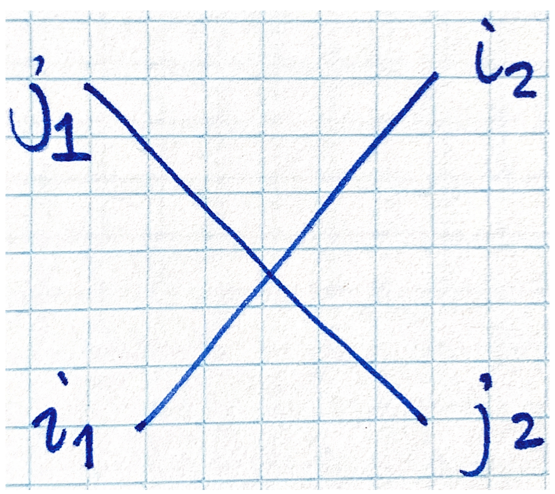
Yang-Baxter equations (all paths up-right)



(a)

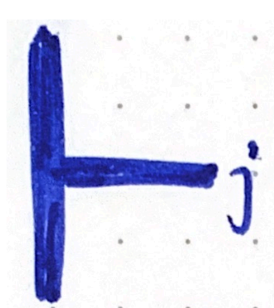


(b)



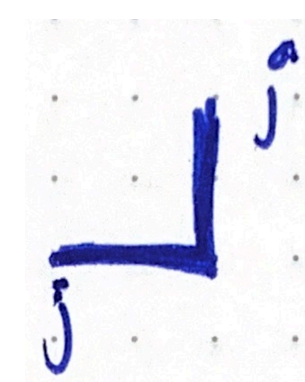
Cross vertex weights

$$R_{x,y}(i_1, j_1; i_2, j_2) = \mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}(y/x)^{j_2}\,(1 - y/x\,\mathbf{1}_{i_1>j_2})$$

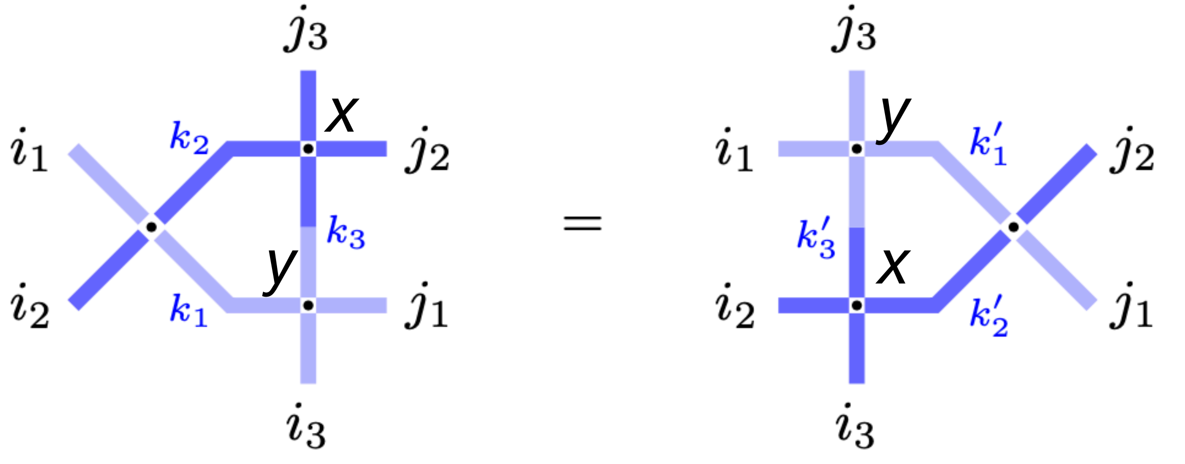


x^j

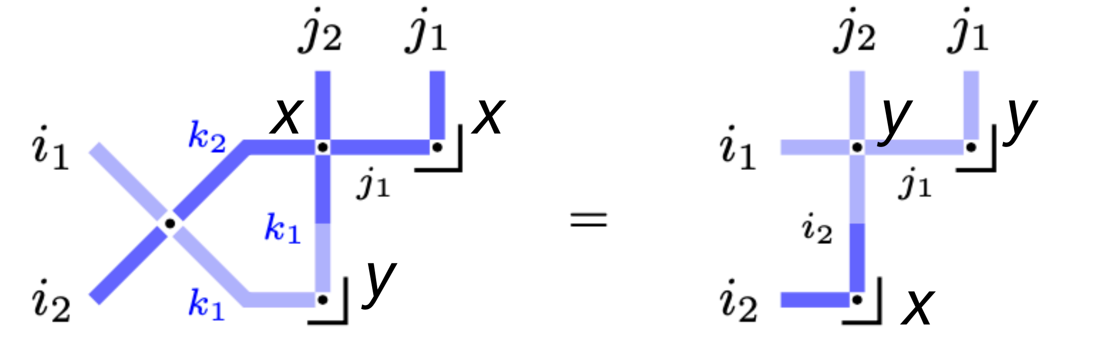
$$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}\,x^{j_2}$$



1

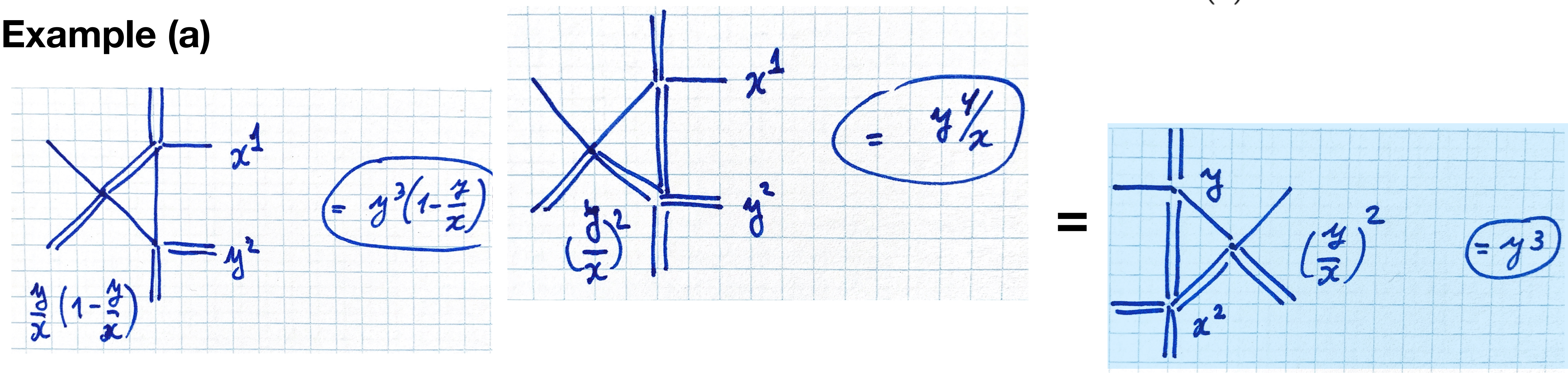


(a)

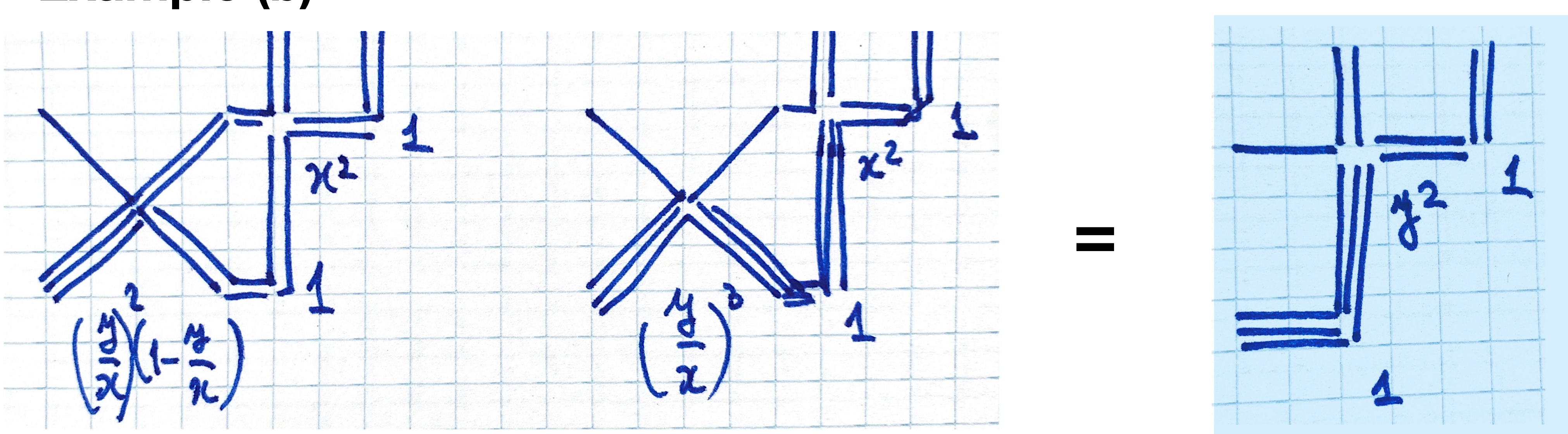


(b)

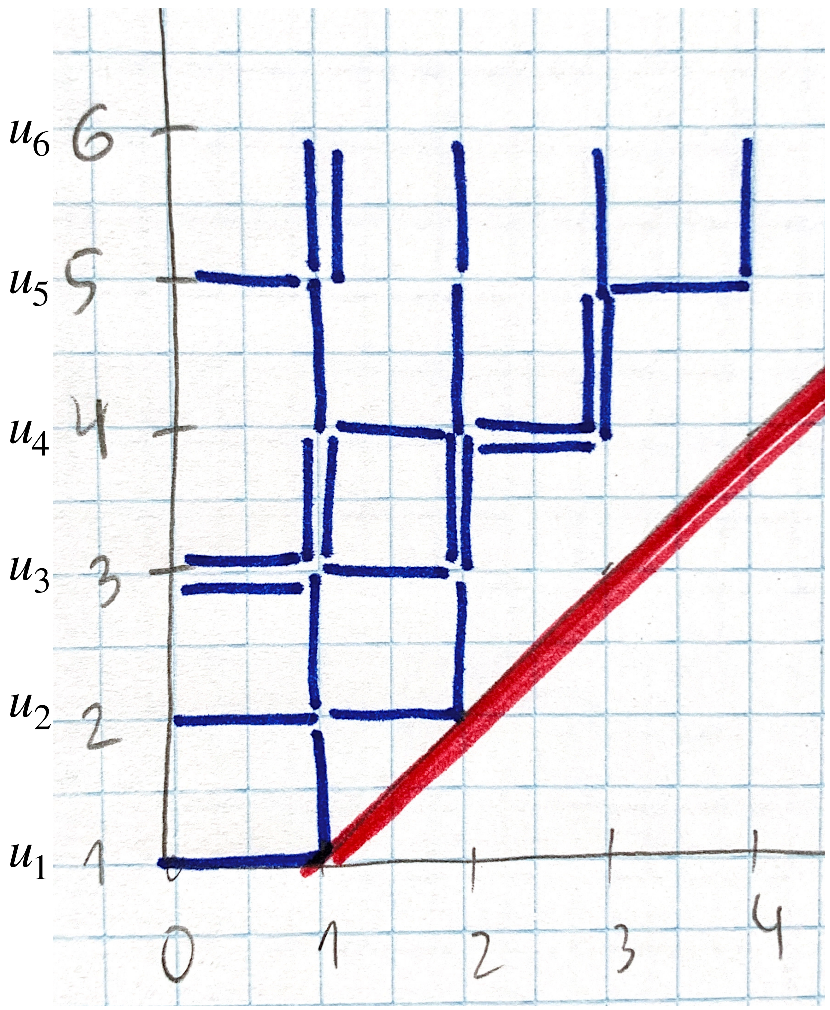
Example (a)



Example (b)

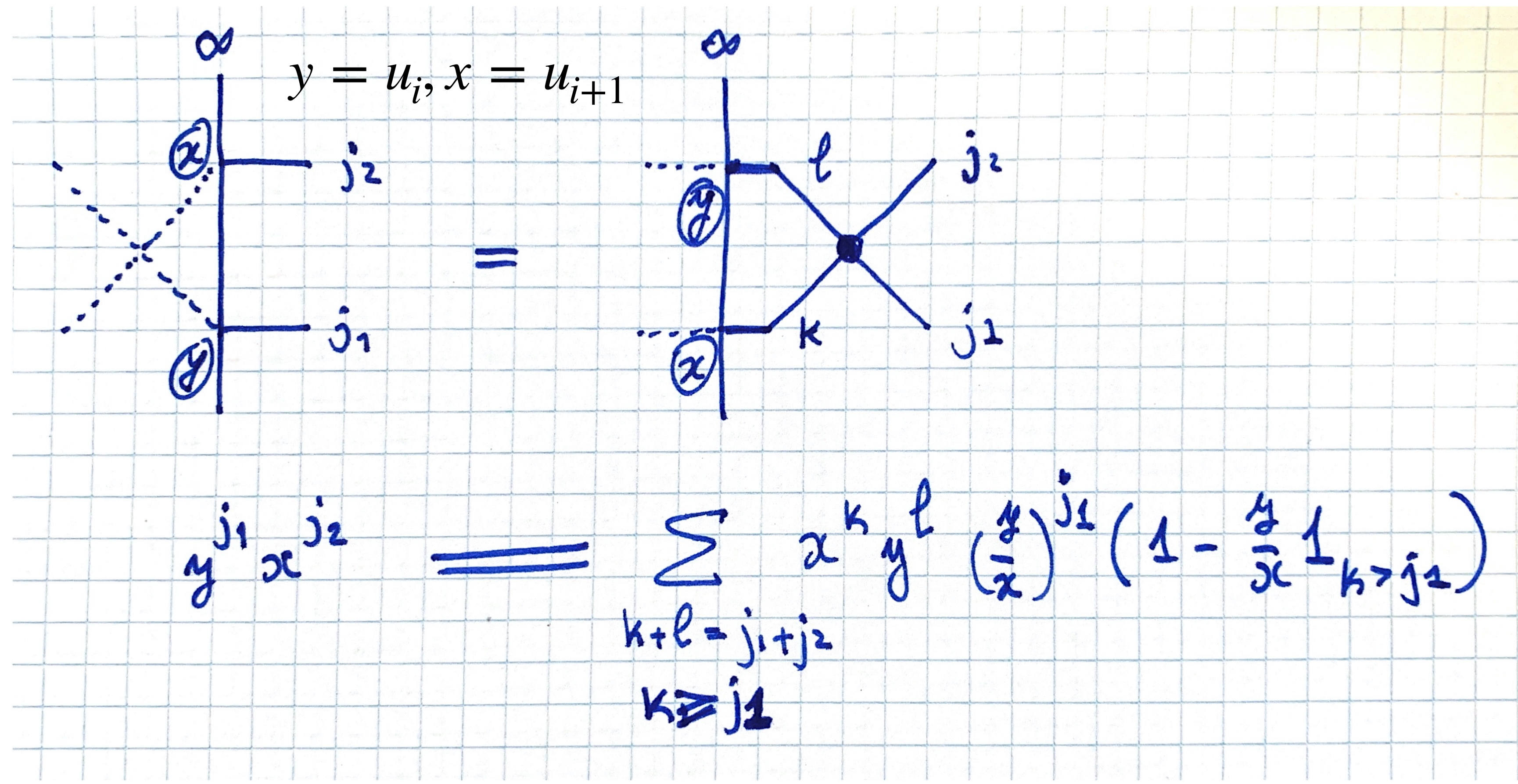


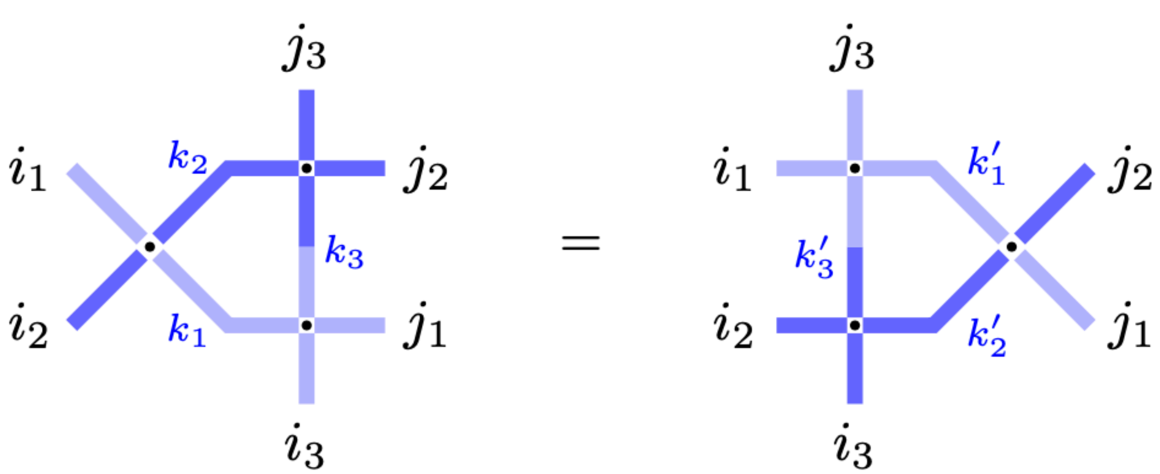
Symmetry of polynomials



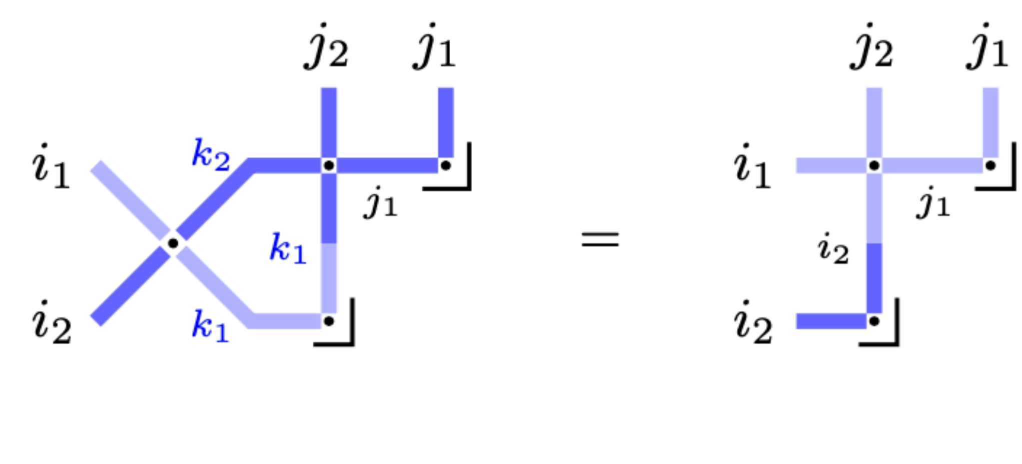
To swap any pair $u_i \leftrightarrow u_{i+1}$, first apply YBE (b) on the right. This produces a cross which we then move to the left using YBE (a) many times.

At the leftmost boundary we also use YBE (a):



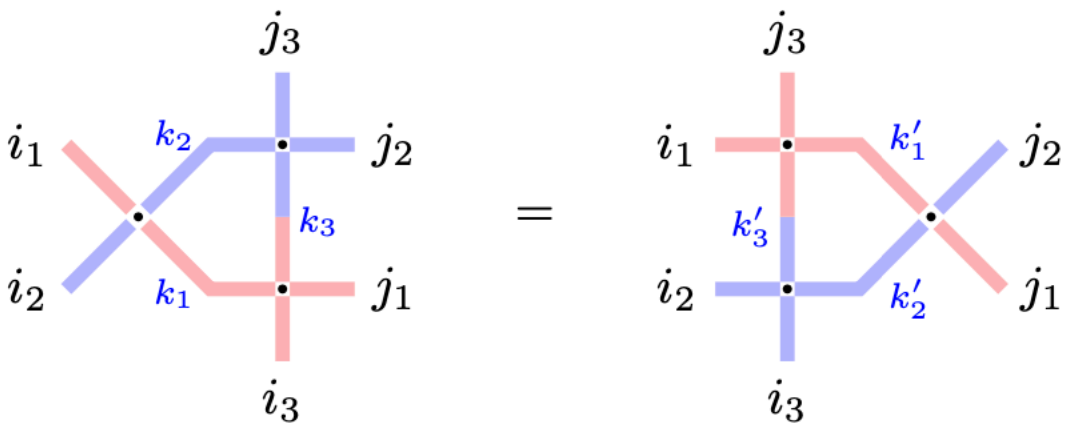


(a)

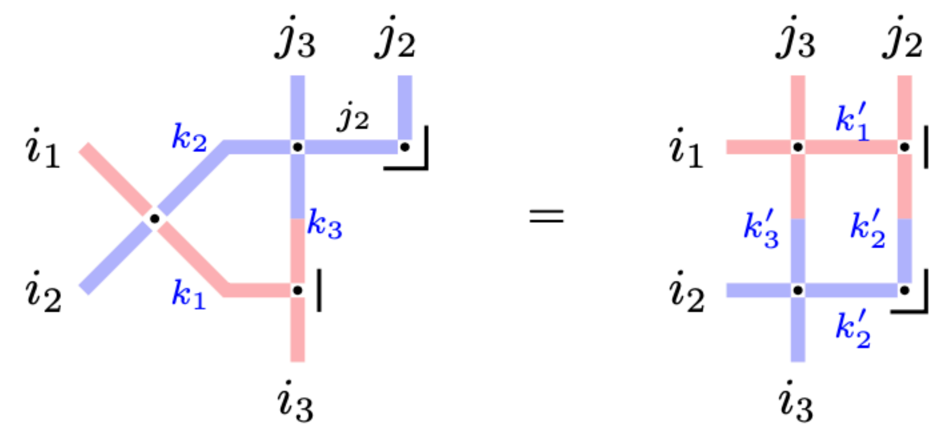


(b)

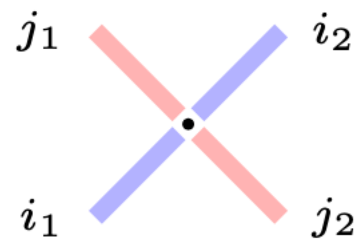
More Yang-Baxter equations for Cauchy identity (blue - up-right, red - down-right)



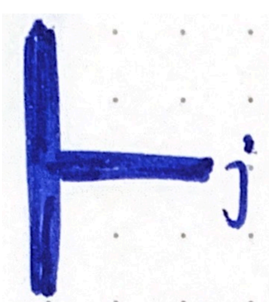
(a)



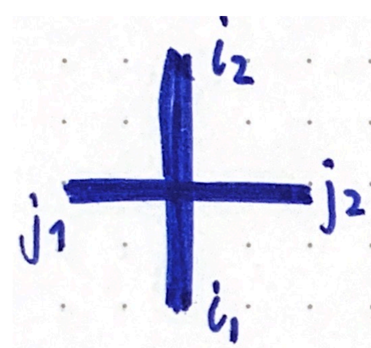
(b)



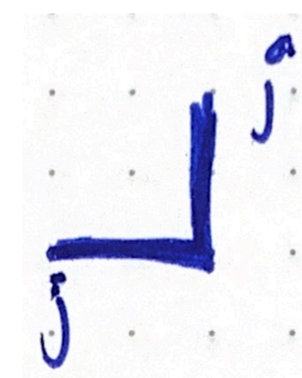
$$\mathbb{R}_{x,y} = \mathbf{1}_{i_1+j_2=j_1+i_2}(xy)^{\min(i_2,j_2)}$$



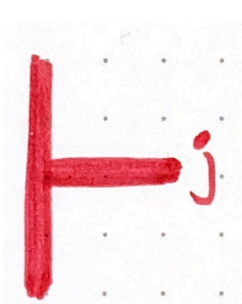
x^j



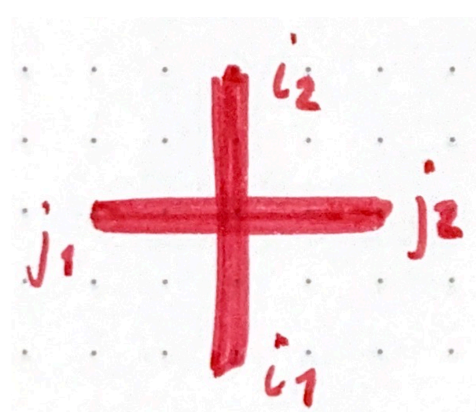
$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\geq j_2}\,x^{j_2}$



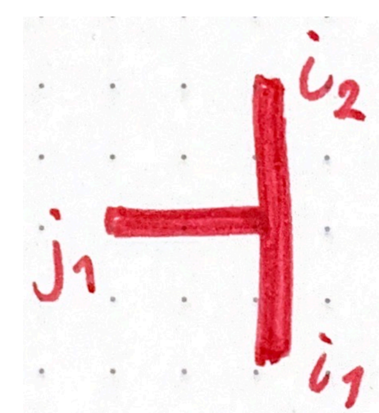
1



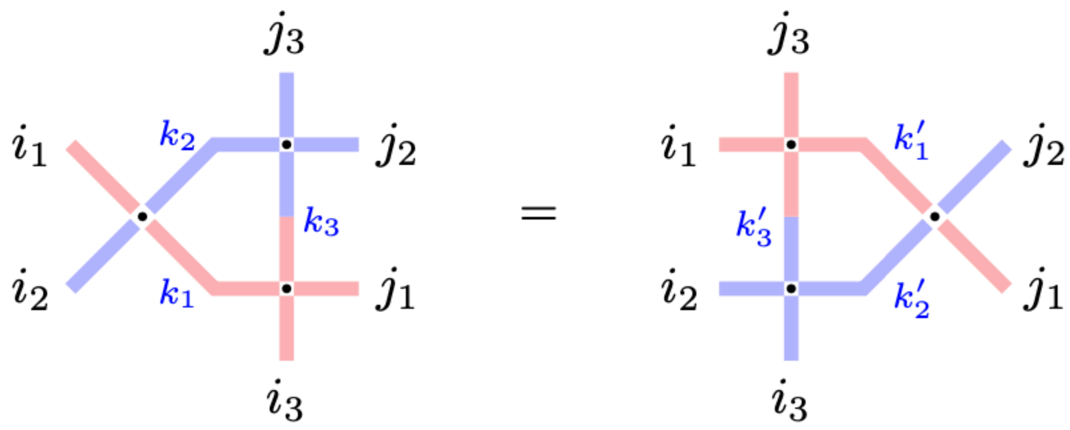
x^j



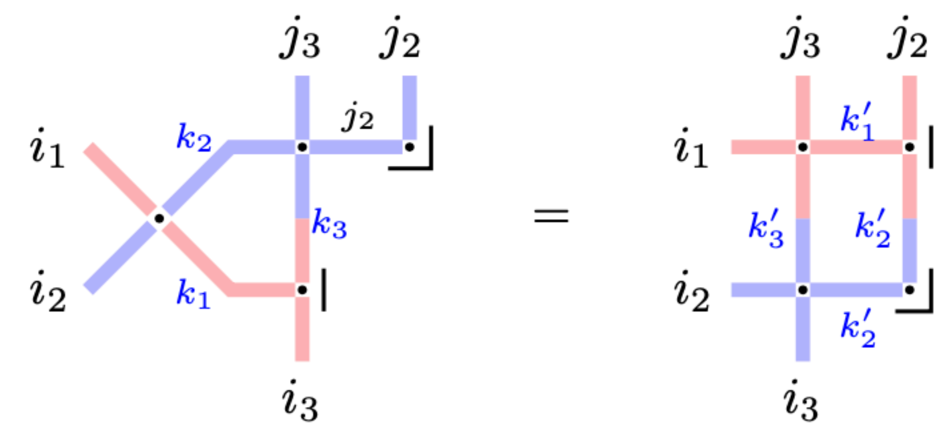
$\mathbf{1}_{i_2+j_1=i_1+j_2}\,\mathbf{1}_{i_2\geq j_2}\,x^{j_2}$



$\mathbf{1}_{i_1=j_1+i_2}$

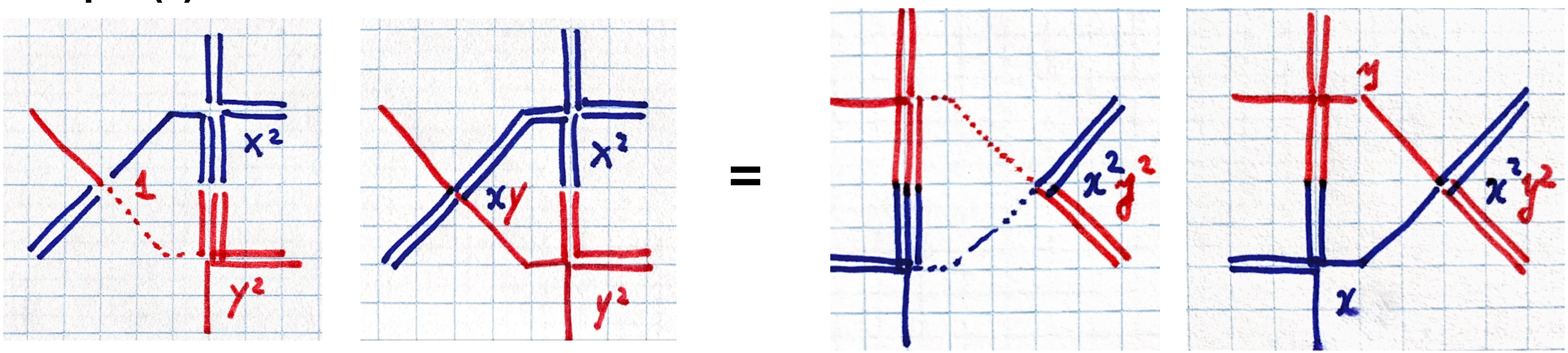


(a)

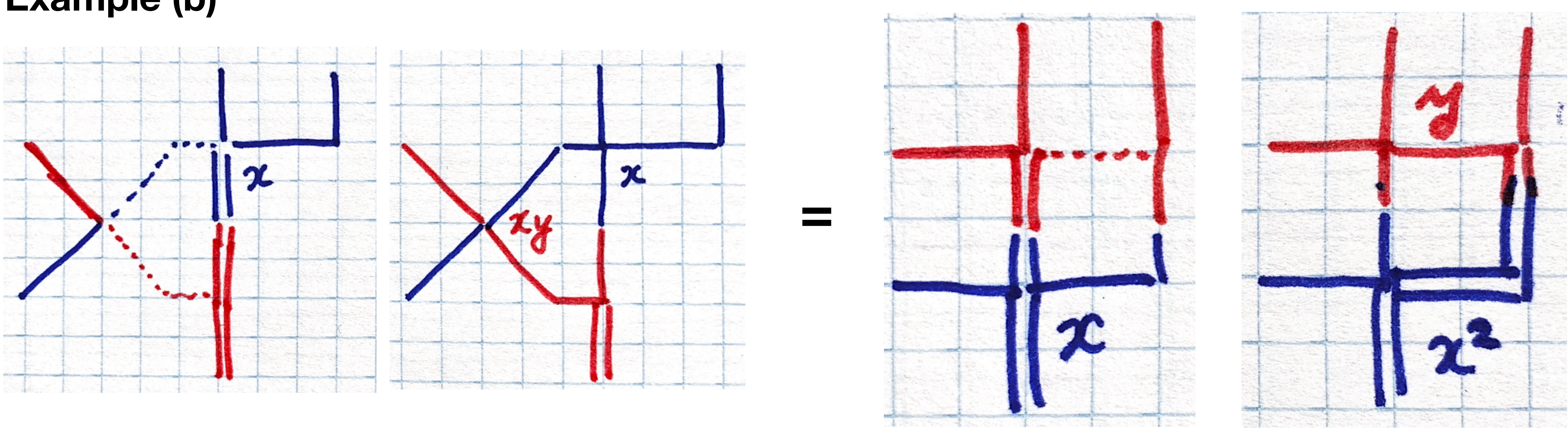


(b)

Example (a)



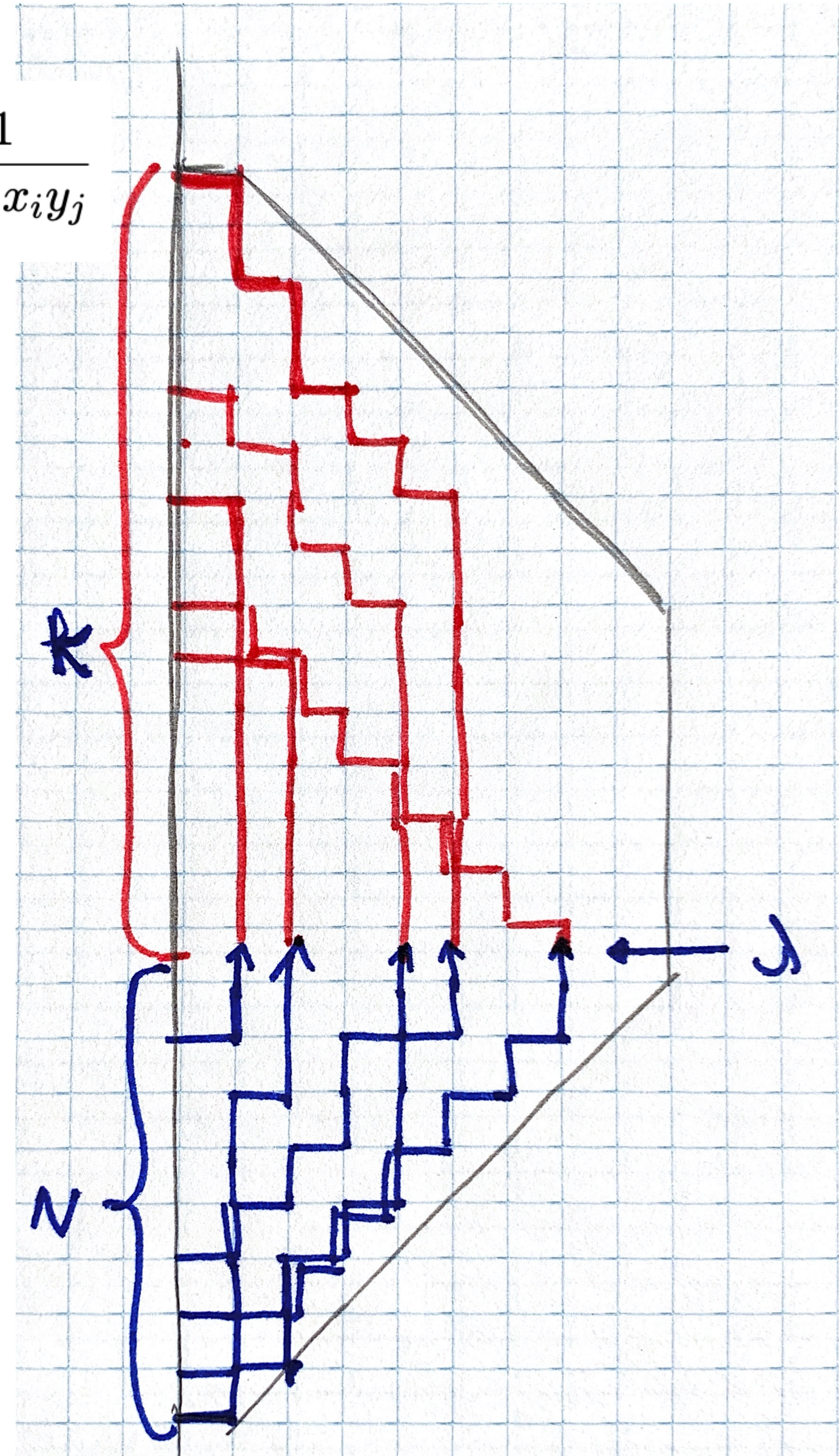
Example (b)



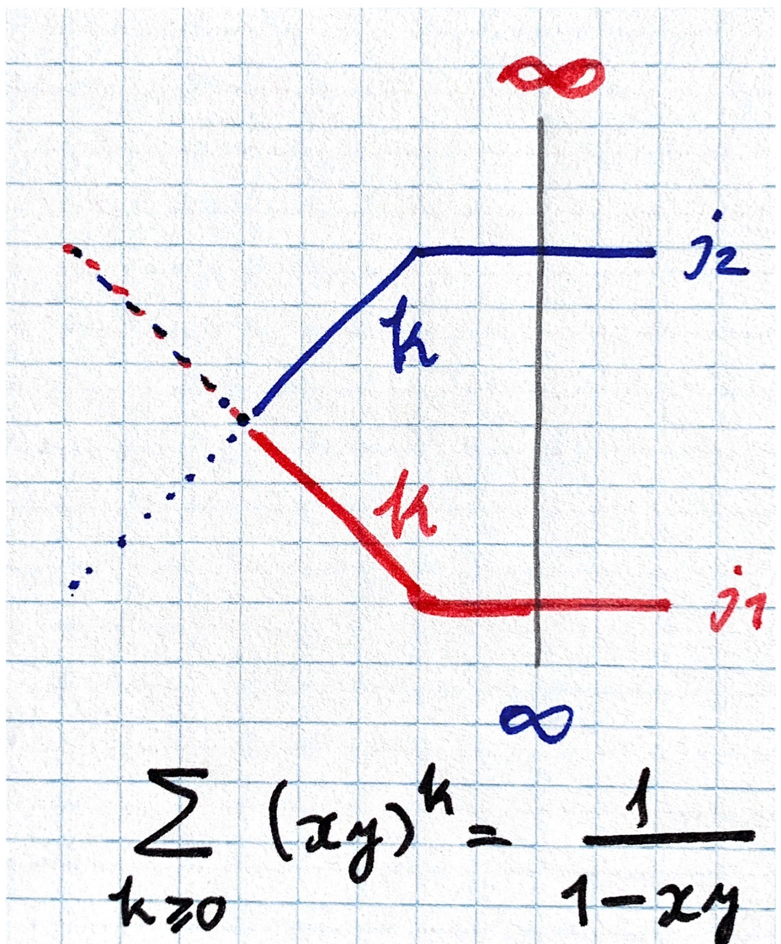
Proof of the Cauchy identity

$$\sum_{\lambda_1 \geq \ldots \geq \lambda_N} s_\lambda(x_1, \ldots, x_N) s_\lambda(y_1, \ldots, y_k) = \prod_{i=1}^{N} \prod_{j=1}^{k} \frac{1}{1 - x_i y_j}$$

The left-hand side is a partition function of configurations like these.



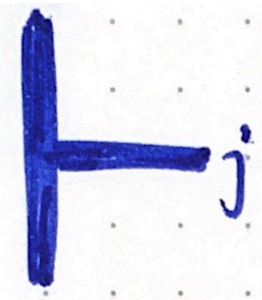
Do the following Nk times:
Start from the right, apply YBE (b) to produce a new cross vertex, then YBE (a) to drag it the left.
This exchanges one x_i with one y_j. On the left boundary we have



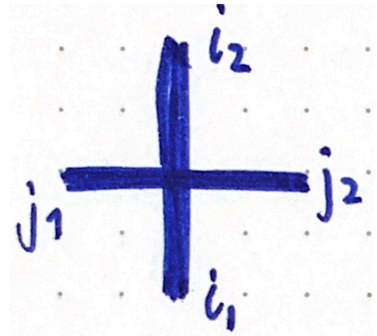
$$\sum_{k \geq 0} (xy)^k = \frac{1}{1-xy}$$

After Nk exchanges, the resutling partition function becomes 1, and we get the Cauchy identity. ■

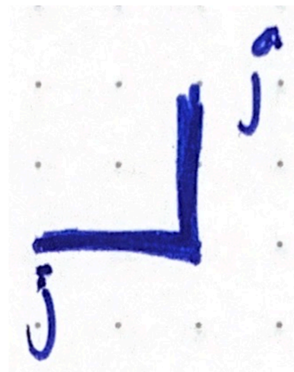
Spin q-Whittaker weights (for the second YBE)



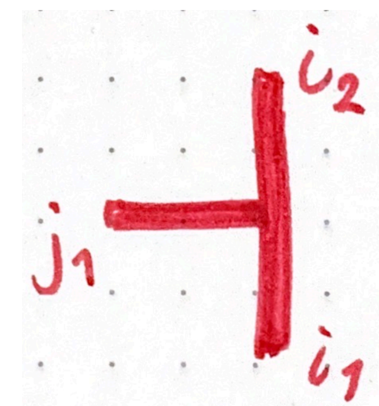
$$x^j \frac{(-s/x;q)_j}{(q;q)_j}$$



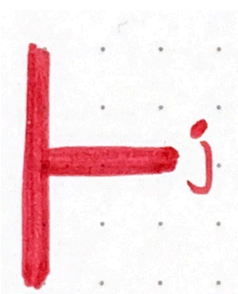
$$\mathbf{1}_{i_1+j_1=i_2+j_2}\,\mathbf{1}_{i_1\ge j_2}\, x^{j_2}\frac{(-s/x;q)_{j_2}(-sx;q)_{i_1-j_2}(q;q)_{i_2}}{(q;q)_{j_2}(q;q)_{i_1-j_2}(s^2;q)_{i_2}}$$



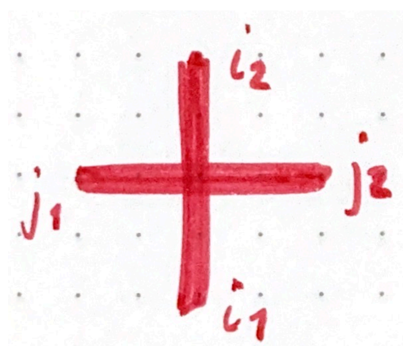
$$\frac{(q;q)_j}{(-s/x;q)_j}.$$



$$\mathbf{1}_{i_1=j_1+i_2}$$

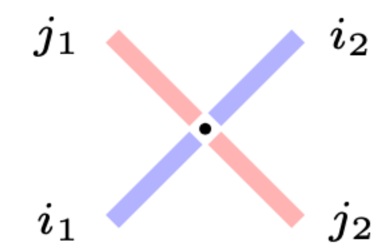


$$x^j \frac{(-s/x;q)_j}{(q;q)_j}$$



$$\mathbf{1}_{i_2+j_1=i_1+j_2}\mathbf{1}_{i_2\ge j_2}\frac{y^{j_2}(q;q)_{i_2}(-s/y;q)_{j_2}(-sy;q)_{i_2-j_2}}{(q;q)_{i_2-j_2}(q;q)_{j_2}(s^2;q)_{i_2}}$$

$$\mathbb{R}_{x,y,s}(i_1,j_1;i_2,j_2) := \mathbf{1}_{i_2+j_1=i_1+j_2}\frac{q^{i_2 i_1+\frac{1}{2}j_2(j_2-1)}(sx)^{j_2}(q;q)_{j_1}}{(s^2;q)_{j_1+i_2}(q;q)_{j_2}(q;q)_{i_2}(-q/(sx);q)_{i_1-j_1}}$$



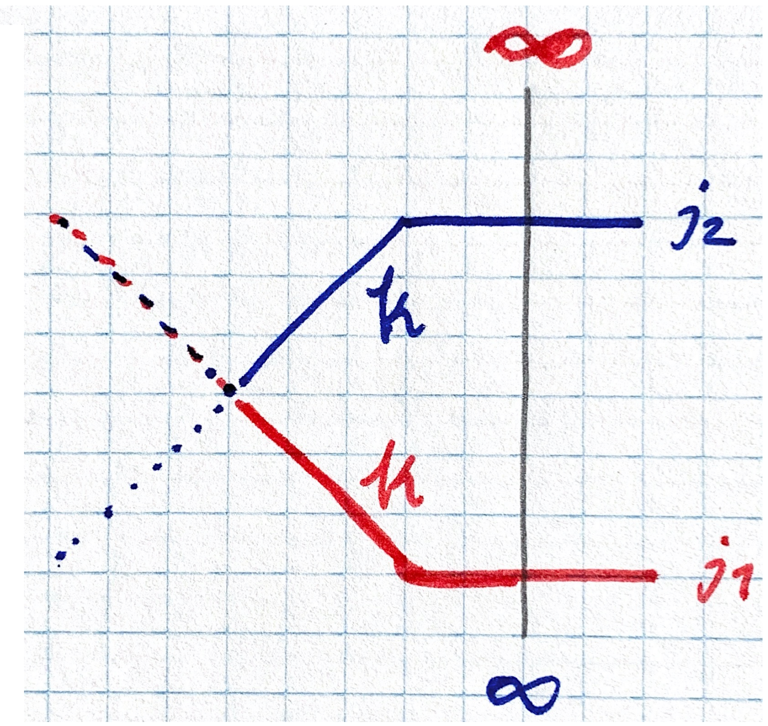
$${}_4\overline{\phi}_3\left(\begin{matrix} q^{-n} & a_1 & a_2 & a_3 \\ & b_1 & b_2 & b_3 \end{matrix}; q, z\right) = \sum_{j=0}^{n} z^j \frac{(q^{-n};q)_j}{(q;q)_j}(a_1,a_2,a_3;q)_j\,(q^j b_1, q^j b_2, q^j b_3;q)_{n-j}.$$

$$\times\, {}_4\overline{\phi}_3\left(\begin{matrix} q^{-i_2}; q^{-i_1}, -sy, -q/(sx) \\ -s/x, q^{1+j_2-i_2}, -yq^{1-i_1-j_2}/s \end{matrix}\,\middle|\, q,q\right).$$

Cauchy identity for the spin q-Whittaker polynomials

$$\sum_{\lambda\in\mathrm{Sign}_N}\mathbb{F}_\lambda(x_1,\ldots,x_N)\,\mathbb{F}^*_\lambda(y_1,\ldots,y_k)=\prod_{j=1}^{k}\left(\frac{(-sy_j;q)_\infty}{(s^2;q)_\infty}\right)^{N-1}\prod_{i=1}^{N}\prod_{j=1}^{k}\frac{(-sx_i;q)_\infty}{(x_iy_j;q)_\infty}.$$

$$\begin{aligned}\sum_{i,j\ge 0}\mathbb{R}_{x,y,s}(0,0;i,j)&=\sum_{k\ge 0}\mathbb{R}_{x,y,s}(0,0;k,k)\\&=\sum_{k=0}^{\infty}(xy)^k\,\frac{(-s/x;q)_k(-s/y;q)_k}{(s^2;q)_k(q;q)_k}\\&=\frac{(-sx;q)_\infty(-sy;q)_\infty}{(s^2;q)_\infty(xy;q)_\infty}.\end{aligned}$$



At the last step (which is basically the q-binomial theorem), we would get a factor of $\frac{(-sx;q)_\infty}{(xy;q)_\infty}$ instead of the full product, so this explains the powers in the right-hand side of the Cauchy identity. ■

Note. For probabilistic applications, we repeat the Yang-Baxter moves in the Cauchy identity, and at each step make a stochastic choice ("*bijectivisation*"). We can start the YBE from left or right, and this produces both couplings, for "TASEP" and "PushTASEP" sides.

Conclusions and further problems

- Two new families of symmetric functions - "better" *spin q-Whittaker polynomials*, and (possibly more fundamental) *spin Whittaker functions*.
- **Is there an integrable vertex model explanation of the "corner vertices"?**
- Satisfy the "usual" properties of symmetric functions, including Cauchy identity, some eigenoperators, etc.
- **How to prove the conjectural orthogonality relations for these symmetric functions?**
- **Are there higher order eigenoperators for sqW or sW?**
- There are probabilistic applications (our initial motivation), and we have added an extra parameter to q-Whittaker / Whittaker setup ([COSZ], [BC] 2010+). We also get connections to random polymers, etc.
- **Is there a good polymer interpretation of multilayer distributions coming from spin Whittaker processes? (multilayer beta polymers)**
- **RSK like constructions associated with beta polymers and spin Whittaker processes?**
- **Is there any representation theory behind spin Whittaker functions?**