



McInnes, C.R. (2003) Orbits in a generalized two-body problem. *Journal of Guidance, Control and Dynamics*, 26 (5). pp. 743-749. ISSN 0731-5090

<http://eprints.cdlr.strath.ac.uk/6246/>

Strathprints is designed to allow users to access the research output of the University of Strathclyde. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. You may not engage in further distribution of the material for any profitmaking activities or any commercial gain. You may freely distribute both the url (<http://eprints.cdlr.strath.ac.uk>) and the content of this paper for research or study, educational, or not-for-profit purposes without prior permission or charge. You may freely distribute the url (<http://eprints.cdlr.strath.ac.uk>) of the Strathprints website.

Any correspondence concerning this service should be sent to The Strathprints Administrator: eprints@cis.strath.ac.uk

Orbits in a Generalized Two-Body Problem

Colin R. McInnes*

University of Glasgow, Glasgow, Scotland G12 8QQ, United Kingdom

The two-body problem is a well-known case of the general central force problem with an attractive, inverse square force. However, there are forms of spacecraft propulsion, such as solar sails and minimagnetospheric plasma propulsion, which generate a repulsive, inverse square force. Because this force can be modulated, a more general central force problem is then formed. Such a problem is investigated and the families of orbits available using both forward integration and an inverse approach are explored. Both are used to explore various modes of transfer between circular coplanar orbits and to determine strategies for escape.

Nomenclature

a	= orbit semimajor axis
e	= orbit eccentricity
G	= gravitational constant
h	= specific orbital angular momentum
m	= mass
r	= orbit radius
T	= orbit period
t	= time
u	= inverse of orbit radius
W	= Wronskian
β	= lightness number
θ	= polar angle
λ	= spiral parameter
μ	= gravitational parameter, $G(m_1 + m_2)$
τ	= transfer duration
ω	= argument of perigee

Introduction

BOTH solar sail propulsion¹ and minimagnetospheric plasma propulsion² (M2P2) generate thrust-induced forces that vary as the inverse square of heliocentric distance (as do so-called magnetic sails³). For a solar sail, a fixed-sail area will intercept a flux of photons that diminishes as the inverse square of heliocentric distance, whereas for M2P2 propulsion, the power available to drive the system diminishes as the inverse square of heliocentric distance. (Although an M2P2 system using a nuclear power source would provide constant thrust.) The M2P2 system provides an essentially radial thrust as its magnetic field deflects the solar wind plasma, whereas a large, high-performance solar sail is well suited to delivering a radial thrust because sun pointing can be achieved passively if the sail has a slightly conical form with the apex directed sunward. This is a different mode of operation from articulated solar sails that can direct the sail thrust vector within 90 deg of the sun line. However, high performance sails may be unsuited to such articulation to minimize loads on their gossamer structure.

In addition to the inverse square form of these propulsive forces, the thrust magnitude can in principle be modulated between limits. For the M2P2 system, the thrust can be modulated by inflating or deflating its bubble of magnetic field, whereas a solar sail can in principle alter its effective area. This can be achieved by partly restowing the sail, or rotating panels of a segmented sail. More likely,

for a high-performance spin-stabilized disk sail, a variation of the coning angle of the angular velocity vector will lead to an averaged, modulated radial thrust. The main constraint on such systems is that the inverse square thrust is always directed along the sun line, radially away from the sun. In addition, there will be some maximum available thrust available, determined by the sail area or the M2P2 sizing.

The dynamics of low-thrust propulsion with constant radial thrust has previously been considered by a number of authors.⁴⁻⁸ Here, the related problem of a modulated, inverse square radial thrust is posed and solved with specific application to solar sail and M2P2 propulsion. As will be seen, large families of orbits can be investigated using analytical methods resulting in both closed and open orbits. The general central force problem with the force scaling as r^{-N} (integer N) has long been investigated, for example, see Ref. 9. Here, however, we present the thrust modulation as a function of polar angle and exploit such general force laws for specific applications such as orbit transfer. Although there is clear application of the open orbits to escape missions, such as fast trips to the heliopause,¹⁰ closed orbits may also find applications for space physics missions that monitor and explore the structure of the solar wind plasma.

Central Force Problem

The equations of motion for a spacecraft with a modulated, inverse square radial thrust may be written in plane polar coordinates (r, θ) as¹

$$\ddot{r}(t) - r(t)\dot{\theta}(t)^2 = -[1 - \beta(\theta)]\frac{\mu}{r(t)^2} \quad (1a)$$

$$\frac{1}{r(t)}\frac{d}{dt}[r(t)^2\dot{\theta}(t)] = 0 \quad (1b)$$

where $r(t)$ is the heliocentric distance of the spacecraft (m_2) from the sun (m_1) and θ is the polar angle of the spacecraft, measured anticlockwise from some reference direction, as defined in Fig. 1. Because both the spacecraft thrust and solar gravity have an inverse square variation, the induced thrust can be parameterized by the lightness number β , defined as the ratio of the thrust force to solar gravitational force acting on the spacecraft. For the case of a solar sail, a sail with a fixed surface area will have a constant lightness number, whereas for M2P2 propulsion, a fixed input power will also lead to a constant lightness number. However, the effective sail area and M2P2 thrust can be modulated, so that β can be a function of θ , with the constraint that $0 \leq \beta \leq \beta$ where β is the maximum lightness number attainable.

The equations of motion may now be reduced because a central force problem is still being considered, and so orbital angular momentum is conserved. Integrating Eq. (1b) yields $r^2\dot{\theta} = h$, where h is the specific orbital angular momentum. Then, Eq. (1a) may be written as

$$\ddot{r}(t) - h^2/r(t)^3 = -[1 - \beta(\theta)][\mu/r(t)^2] \quad (2)$$

Received 10 October 2002; revision received 11 March 2003; accepted for publication 16 April 2003. Copyright © 2003 by Colin R. McInnes. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission. Copies of this paper may be made for personal or internal use, on condition that the copier pay the \$10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0731-5090/03 \$10.00 in correspondence with the CCC.

*Professor, Department of Aerospace Engineering; colinmc@aero.gla.ac.uk.

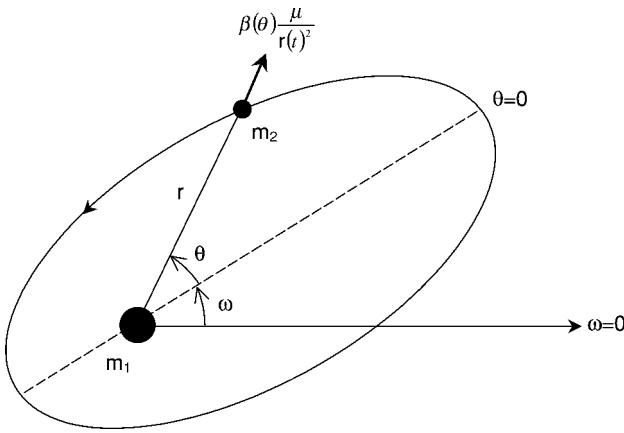


Fig. 1 Central force problem.

When the substitution $u(\theta) = 1/r(\theta)$ is made, and conservation of angular momentum is used to change the independent variable from time t to polar angle θ , it can be seen that Eq. (2) is transformed to

$$u''(\theta) + u(\theta) = (\mu/h^2)[1 - \beta(\theta)] \tag{3}$$

where the prime indicates a derivative with respect to polar angle θ . Because $\beta(\theta)$ can be specified a priori, with the constraint $0 \leq \beta \leq \tilde{\beta}$, Eq. (3) can in principle be solved in closed form to determine the resulting spacecraft orbit $r(\theta)$. To solve Eq. (3), the associated homogeneous equation, defined by

$$u''(\theta) + u(\theta) = 0 \tag{4}$$

must be solved. The general solution of this homogeneous equation $u_H(\theta)$ is then given by

$$u_H(\theta) = C_1 u_1(\theta) + C_2 u_2(\theta) \tag{5}$$

where C_1 and C_2 are arbitrary constants, determined from the initial conditions of the problem, and $u_1(\theta)$ and $u_2(\theta)$ are linearly independent solutions to Eq. (4). It is clear that two solutions to Eq. (4) are $u_1(\theta) = \cos \theta$ and $u_2(\theta) = \sin \theta$, the fundamental set of solutions to Eq. (3). The linear independence of these two solutions, although clear, can be verified by calculating the Wronskian $W(\theta)$ so that

$$W(\theta) = \begin{vmatrix} u_1(\theta) & u_2(\theta) \\ u_1'(\theta) & u_2'(\theta) \end{vmatrix} = 1 \tag{6}$$

Given that Eq. (3) has no dependence on $u'(\theta)$, it is expected that $W'(\theta) = 0$. Now that the general solution to the associated homogeneous equation has been determined, the general solution to Eq. (3) can be found by finding a second, particular solution $u_P(\theta)$. This particular solution can be found using the method of variation of parameters, by proposing a solution of the form $u_P(\theta) = v_1(\theta)u_1(\theta) + v_2(\theta)u_2(\theta)$. It can be shown that the unknown functions $v_1(\theta)$ and $v_2(\theta)$ must satisfy

$$v_1'(\theta)u_1(\theta) + v_2'(\theta)u_2(\theta) = 0 \tag{7a}$$

$$v_1'(\theta)u_1'(\theta) + v_2'(\theta)u_2'(\theta) = (\mu/h^2)[1 - \beta(\theta)] \tag{7b}$$

When it is noted that $W(\theta) = 1$, the solution to this set of simultaneous, first-order differential equations is then given by

$$v_1(\theta) = -\frac{\mu}{h^2} \int u_2(\theta)[1 - \beta(\theta)] d\theta \tag{8a}$$

$$v_2(\theta) = \frac{\mu}{h^2} \int u_1(\theta)[1 - \beta(\theta)] d\theta \tag{8b}$$

so that the particular solution to Eq. (3) may be written as

$$u_P(\theta) = -\frac{\mu}{h^2} u_1(\theta) \int u_2(\theta)[1 - \beta(\theta)] d\theta + \frac{\mu}{h^2} u_2(\theta) \int u_1(\theta)[1 - \beta(\theta)] d\theta \tag{9}$$

Finally, the general solution to Eq. (3) is given by $u(\theta) = u_H(\theta) + u_P(\theta)$ so that

$$u(\theta) = C_1 \cos \theta + C_2 \sin \theta - \frac{\mu}{h^2} \cos \theta \int \sin \theta [1 - \beta(\theta)] d\theta + \frac{\mu}{h^2} \sin \theta \int \cos \theta [1 - \beta(\theta)] d\theta \tag{10}$$

For closed periodic orbits $u(\theta)$, the orbit period T can now be obtained, defined here as the time between successive periapsis passages (at θ_o and θ_p). The orbit period can be obtained in closed form by a quadrature using $r^2 \dot{\theta} = h$ so that

$$T = \frac{1}{h} \int_{\theta_o}^{\theta_p} \frac{d\theta}{u^2(\theta)} \tag{11}$$

If $\beta(\theta)$ is now defined from a large class of elementary functions, the integrals in Eq. (10) can be performed and $u(\theta)$ obtained and, thus, so can the orbit $r(\theta)$.

For example, if the lightness number is modulated according to $\beta(\theta) = \cos^2 \theta$, so that $0 \leq \beta \leq 1$, the described methodology yields

$$u(\theta) = (u_o/6)[3 + (6C_1/u_o) \cos \theta + \cos 2\theta + (6C_2/u_o) \sin \theta] \tag{12}$$

Then, if the initial conditions of the problem are chosen such that the spacecraft begins on a circular orbit with $u(0) = u_o$ and $u'(0) = 0$, so that $\mu/h^2 = u_o$, the constants are found to be $C_1 = u_o/3$ and $C_2 = 0$. Therefore, because $u_o = 1/r_o$, where r_o is the initial circular orbit radius, and $r(\theta) = u(\theta)^{-1}$, it can be seen that the resulting orbit is defined by

$$r(\theta) = \frac{6r_o}{3 + 2 \cos \theta + \cos 2\theta} \tag{13}$$

where $r_o \leq r(\theta) \leq 4r_o$, as shown in Fig. 2. The orbit period is then obtained from Eq. (11) as

$$T = (r_o^{3/2} / \sqrt{\mu}) [3^{3/4} \sqrt{2} (3 + \sqrt{3}) \pi] \tag{14}$$

It has been shown then that the classical two-body problem can be extended to a more generalized central force problem applicable to

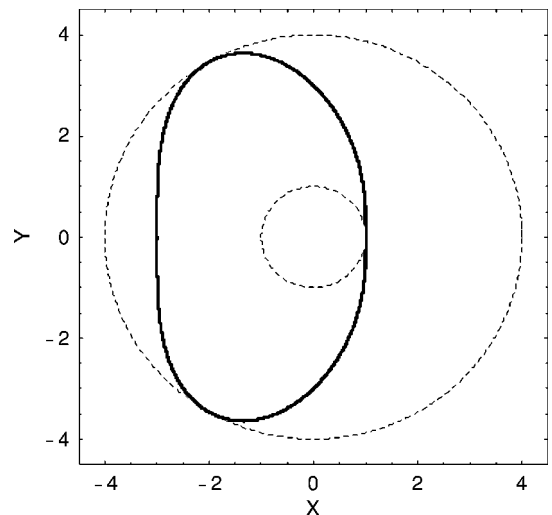


Fig. 2 Closed orbit for $\beta = \cos^2 \theta$.

spacecraft that are able to generate an inverse square radial, modulated thrust. If the spacecraft lightness number can be defined as a function of polar angle, the resulting orbit and orbit period can in general be determined in closed form. Applications to both solar sail and M2P2 propulsion will be explored later.

Inverse Problem

Now that the forward integration problem has been investigated and solved, the inverse problem will be considered. From Eq. (3) it is clear that, if $u(\theta)$ is sufficiently smooth, then the required functional form of the lightness number can be determined in closed form using

$$\beta(\theta) = 1 - (h^2/\mu)[u''(\theta) + u(\theta)] \quad (15)$$

Therefore, with the constraint $0 \leq \beta \leq \tilde{\beta}$ imposed, large families of orbits can be defined a priori and the required lightness number modulation determined using Eq. (15). The key constraint on Eq. (15) is that $\beta \geq 0$. Again, if it is assumed that $u(0) = u_o$ and $u'(0) = 0$, then $\mu/h^2 = u_o$, which implies that

$$u''(\theta) + u(\theta) \leq u_o \quad (16)$$

This constraint will be used later to determine bounds on admissible orbits that possess the property that $\beta \geq 0$. Now that the inverse problem has been defined, several individual cases will be considered.

Circular Orbit

For a closed circular orbit of radius R , the orbit equation is simply $r(\theta) = R$ so that Eq. (15) yields

$$\beta = 1 - h^2/\mu R \quad (17)$$

It can be seen that the minimum distance at which a circular orbit can be sustained is constrained by the orbit angular momentum h . Therefore, in order that $\beta \geq 0$, it is clear that there is a minimum orbit radius such that $R \geq h^2/\mu$. The orbit period can also be obtained from Eq. (11) as

$$T = (2\pi/h)R^2 \quad (18)$$

so that there is a minimum orbit period such that $T \geq 2\pi h^4/\mu^2$. Transfer between circular orbits will be considered later when these constraints will be of some importance. Note that such orbits are non-Keplerian and that the orbit period is decoupled from the orbit radius, as can be seen by solving Eq. (17) for h and substituting in Eq. (18).⁷

Rectilinear Orbit

For an open rectilinear orbit, the orbit equation can be defined in plane polar coordinates using

$$r(\theta) = r_o \sec \theta \quad (19)$$

where r_o is the minimum orbit radius at $\theta = 0$. Then, transforming to $u(\theta) = 1/r(\theta)$ and using Eq. (15) yields $\beta = 1$, which is expected because there can be no net force acting in this case. Similarly, when Eq. (11) is used, it is clear that $T \rightarrow \infty$, again as expected.

Logarithmic Spiral Orbit

As a further example of the inverse problem, an open logarithmic spiral orbit can be defined using

$$r(\theta) = r_o \exp(\lambda\theta) \quad (20)$$

for some constant $\lambda > 0$; thus, transforming to $u(\theta) = 1/r(\theta)$ and using Eq. (15) yields

$$\beta(\theta) = 1 - (h^2/\mu)u_o(1 + \lambda^2) \exp(-\lambda\theta) \quad (21)$$

where again $u_o = 1/r_o$. It can be seen that $\beta(\theta) \rightarrow 1$ as $\theta \rightarrow \infty$, as shown in Fig. 3 (with $r_o = 1$), where the orbit number is defined as $\theta/2\pi$. In addition, it can be seen that for $\beta(0) = 0$, and so

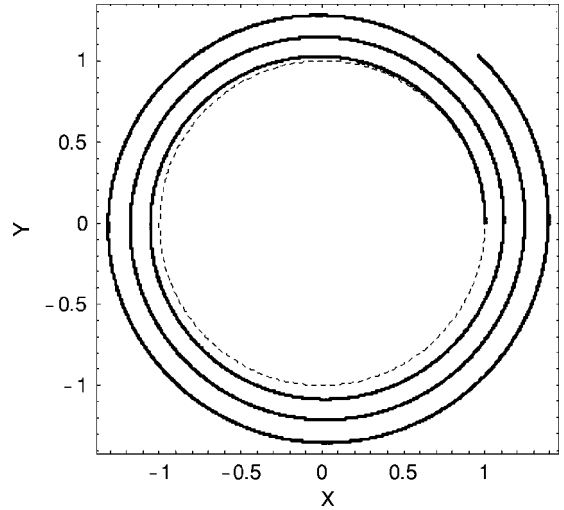


Fig. 3a Logarithmic spiral orbit.

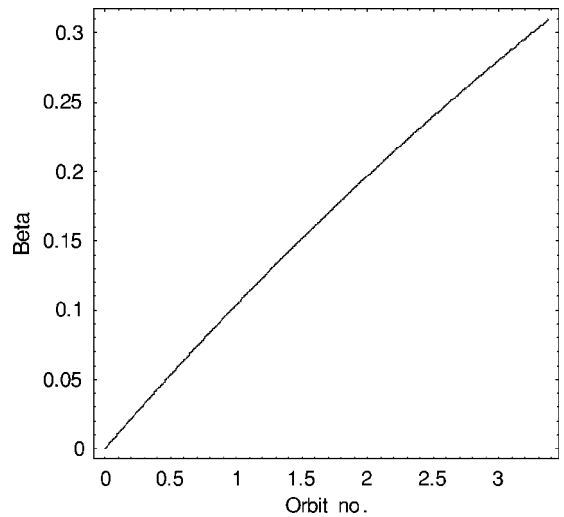


Fig. 3b Required lightness number for a logarithmic spiral orbit.

$0 \leq \beta(\theta) \leq 1$, the orbital angular momentum $h^2 = \mu r_o(1 + \lambda^2)^{-1}$. Therefore, the orbital angular momentum $h^2 < \mu r_o$ and so is not equal to the orbital angular momentum of a circular Keplerian orbit at $r = r_o$. This is also the case for solar sail logarithmic spirals with a fixed, nonradial, sail pitch angle.¹ The constraint defined by Eq. (16) can now be used to obtain

$$\lambda^2 \leq u_o/u(\theta) - 1 \quad (22)$$

Therefore, if $u(\theta) \geq u_o \forall \theta [r(\theta) \leq r_o]$, then $\lambda^2 \leq 0$ and so such orbits are forbidden, whereas if $u(\theta) \leq u_o \forall \theta [r(\theta) \geq r_o]$, then $\lambda^2 \geq 0$ and so logarithmic spiral orbits exist in this case. It is, therefore, concluded from Eq. (16) that inward spirals are always forbidden whereas outward spirals are allowed.

Last, conservation of angular momentum can be used to obtain the trajectory along the logarithmic spiral orbit. Since $r(\theta)^2 \dot{\theta} = h$, where $r(\theta)$ is defined by Eq. (20), the time t at polar angle θ (with $\theta = 0$ at $t = 0$) can be obtained by integration as

$$t(\theta) = (r_o^2/2\lambda h)[\exp(2\lambda\theta) - 1] \quad (23)$$

so that the trajectory along the logarithmic spiral orbit is defined as a parametric curve of the form

$$r(t) = r_o \sqrt{1 + 2h\lambda t/r_o^2} \quad (24a)$$

$$\theta(t) = (1/2\lambda) \log[1 + (2h\lambda/r_o^2)t] \quad (24b)$$

In addition, note that, because $\beta(\theta)$ scales as the inverse of the radial distance from the sun, the total radial force acting on the spacecraft scales as the inverse cube of the radial distance from the sun. It is known that an inverse cube force law will generate spiral trajectories.^{9,11}

Doubly Periodic Orbit

In addition to standard families of closed and open orbits, more complex, doubly periodic orbits can also be considered such as

$$r(\theta) = r_o[1 + p \sin(\theta/4)^q] \tag{25}$$

where p and q are constants that parameterize the orbit. The required lightness number can then be obtained from Eq. (15), as shown in Fig. 4 (with $r_o = 1$, $p = 0.5$ and $q = 4$), although it is not listed here for brevity.

The extremal values of the required lightness number can also be obtained from Eq. (15) by calculating $\beta'(\theta) = 0$, where

$$\beta'(\theta) = -(h^2/\mu)[u'''(\theta) + u'(\theta)] \tag{26}$$

From Eq. (25) it can then be shown that extremal values of $\beta(\theta)$ occur when $\theta = 2\pi K$, for some integer K . The minimum lightness number β^- and maximum lightness number β^+ required are then

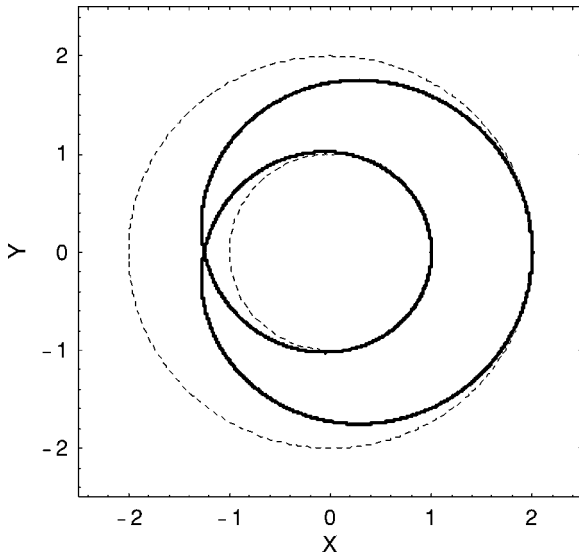


Fig. 4a Doubly periodic orbit; $p = 0.5$ and $q = 4$.

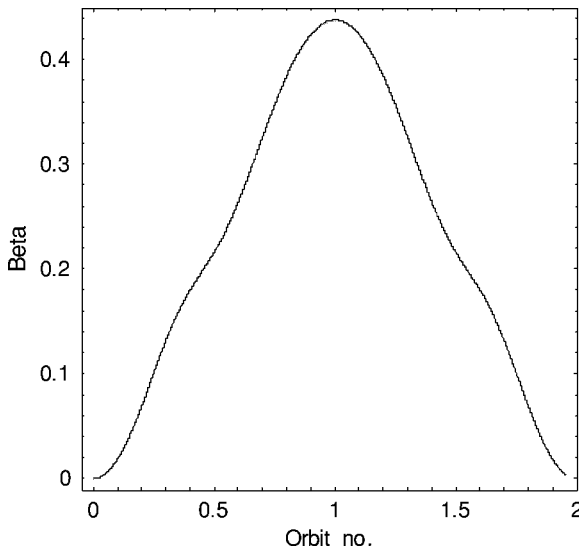


Fig. 4b Required lightness number for a doubly periodic orbit.

found to be

$$\beta^- = 1 - \frac{h^2}{\mu r_o} \tag{27a}$$

$$\beta^+ = 1 - \frac{h^2}{\mu r_o} \left[\frac{pq}{16(1+q)^2} + \frac{1}{1+q} \right] \tag{27b}$$

so that $\beta^- = 0$ if the initial conditions are representative of a Keplerian circular orbit of radius r_o with $h^2 = \mu r_o$. Last, the orbit period can be obtained from Eq. (11), where the limits of integration for the orbit apsides must be set to $[0, 4\pi)$, so that

$$T = \frac{4\sqrt{\pi}r_o^2}{h} \left\{ \sqrt{\pi} + \frac{2q\Gamma[(1+p)/2]}{\Gamma(1+p/2)} + \frac{q^2\Gamma(\frac{1}{2}+p)}{\Gamma(1+p)} \right\} \tag{28}$$

where Γ is the Euler gamma function. Now that a range of open- and closed-, periodic orbits have been presented by way of example, transfer between circular coplanar orbits will be investigated.

Transfer Orbits

For a spacecraft with a maximum attainable lightness number $\tilde{\beta}$, a circular orbit can be sustained over a range of orbit radii, but with a non-Keplerian orbit period. If the initial conditions are representative of a circular orbit of radius r_o , the orbital angular momentum is given by $h^2 = \mu r_o$, as discussed earlier. Therefore, the maximum orbit radius \tilde{r} at which a circular orbit can be sustained is defined by Eq. (17) as

$$\tilde{r} = r_o / (1 - \tilde{\beta}) \tag{29}$$

so that the reachable domain for circular orbit transfer is $r_o \leq r \leq \tilde{r}$ with $0 \leq \beta \leq \tilde{\beta}$. It can be seen that the constraint $\beta \geq 0$ implies that $r \geq r_o$, whereas $\tilde{r} \rightarrow \infty$ as $\tilde{\beta} \rightarrow 1$. Whereas these circular orbits are possible over a range of orbit radii, again note that the orbit period will be non-Keplerian, as demonstrated by Eq. (18).

Inverse Transfer Orbit

Because it has been shown that orbits can be defined a priori, with the constraint $0 \leq \beta \leq \tilde{\beta}$, the problem of transfer between circular orbits can be considered as an inverse problem. Perhaps the simplest functional form for $u(\theta)$ that satisfies the boundary conditions for a transfer between circular orbits is the cubic polynomial

$$u(\theta) = u_o + 3(u_f - u_o)(\theta/\theta_f)^2 - 2(u_f - u_o)(\theta/\theta_f)^3 \tag{30}$$

where θ_f is the transfer angle. Clearly, any choice of orbit $u(\theta)$ that satisfies the boundary conditions of the problem must also satisfy Eq. (16). It can be shown that $u(0) = u_o$ and $u(\theta_f) = u_f$, whereas $u'(0) = u'(\theta_f) = 0$. The required lightness number can be determined using the procedure detailed earlier. In addition, the extremal values of the required lightness number can be obtained from Eq. (15) by calculating $\beta'(\theta) = 0$. This then results in a quadratic equation of the form

$$\theta^2 - \theta\theta_f + 2 = 0 \tag{31}$$

which yields the polar angles at which the minimum and maximum lightness numbers occur. When the quadratic is solved, the minimum and maximum lightness numbers are found to occur at

$$\theta^- = \theta_f/2 - \sqrt{\theta_f^2/4 - 2} \tag{32a}$$

$$\theta^+ = \theta_f/2 + \sqrt{\theta_f^2/4 - 2} \tag{32b}$$

Then, substituting for θ^- and θ^+ , and assuming a 360-deg transfer ($\theta_f = 2\pi$) yields the minimum and maximum required lightness

numbers, which are found to be

$$\beta^- = \left(\frac{1}{2} - \sqrt{\pi^2 - 2}/2\pi + \sqrt{\pi^2 - 2}/\pi^3\right) \times (1 - u_f/u_o) \sim 0.144(1 - u_f/u_o) \quad (33a)$$

$$\beta^+ = \left(\frac{1}{2} + \sqrt{\pi^2 - 2}/2\pi - \sqrt{\pi^2 - 2}/\pi^3\right) \times (1 - u_f/u_o) \sim 0.856(1 - u_f/u_o) \quad (33b)$$

so that $\beta^- \geq 0$ if $r_f \geq r_o$. Therefore, as with the logarithmic spiral, inward transfers are forbidden whereas outward transfers are allowed. In addition, as $u_f \rightarrow 0$ ($r_f \rightarrow \infty$), the maximum required lightness number is of order 0.856. An example circle-to-circle transfer with $\theta_f = 2\pi$ is shown in Fig. 5 (with $r_o = 1$ and $r_f = 2$), where the extremal values of β , defined by Eqs. (33), can be seen.

Elliptical Transfer Orbit

Whereas the use of inverse methods allows the boundary conditions for circle-to-circle transfer to be satisfied by defining a transfer orbit a priori, other approaches can also be considered. For example, an elliptical orbit can be sought that connects the initial and final circular orbits in a quasi-Hohmann fashion. For initial and final circular orbit radii r_o and r_f , the required semimajor axis of the transfer ellipse a is $(r_o + r_f)/2$. Then, the speed v on the transfer

ellipse at orbit radius r is given by

$$v^2 = \mu(1 - \beta)[2/r - 2/(r_o + r_f)] \quad (34)$$

Matching the speed on the transfer ellipse at orbit radius r_o with the speed on the initial circular orbit $\sqrt{(\mu/r_o)}$ yields the required lightness number for the transfer as

$$\beta = \frac{1}{2}(1 - r_o/r_f) \quad (35)$$

As can be seen from Eq. (17), this is one-half of the lightness number required to sustain a circular orbit at orbit radius r_f . The transfer duration τ can also be found from Eq. (11) as one-half of the orbit period of the transfer ellipse so that

$$\tau = \pi\sqrt{a^3/\mu(1 - \beta)}, \quad a = \frac{1}{2}(r_o + r_f) \quad (36)$$

where β is defined by Eq. (35). In summary, the transfer begins with $\beta = 0$ at the initial circular orbit of radius r_o . An intermediate lightness number of $(1 - r_o/r_f)/2$ is then required for one-half of the transfer ellipse, to transfer to the final circular orbit at radius r_f in duration τ . Last, the lightness number is increased to $(1 - r_o/r_f)$ to inject the spacecraft into the final circular orbit, with a non-Keplerian orbit period.

Bielliptic Transfer Orbit

An alternative mode of transfer is to construct a transfer composed of two ellipses, an initial arc with the maximum lightness number $(1 - r_o/r_f)$ (as required for a circular orbit at orbit radius r_f) followed by a coast arc. The osculating semimajor axis and eccentricity at the end of the powered arc must correspond to an apocenter radius on the coast arc equal to the final circular orbit radius. The length of the initial powered arc must, therefore, be chosen to satisfy this condition. To investigate this requirement, the variational equations for the problem must be formed. Because the effect of the spacecraft thrust is to perturb an osculating two-body ellipse with a radial, inverse square force, the variational equations may be written as¹

$$\frac{da}{d\theta} = \frac{2ae}{1 - e^2}\beta(\theta) \sin \theta \quad (37a)$$

$$\frac{de}{d\theta} = \beta(\theta) \sin \theta \quad (37b)$$

$$\frac{d\omega}{d\theta} = -\frac{1}{e}\beta(\theta) \cos \theta, \quad e \neq 0 \quad (37c)$$

where ω is the osculating argument of perigee. To proceed, the spacecraft will begin on a circular orbit with $e = 0$ and $a = a_o$ ($a_o = r_o$, the initial circular orbit radius). Then, when Eqs. (37a) and (37b) are integrated over some arc length $\Delta\theta$, with a fixed lightness number β , the resulting elements are given by

$$e(\Delta\theta) = \beta(1 - \cos \Delta\theta) \quad (38a)$$

$$a(\Delta\theta) = a_o / [1 - \beta^2(1 - \cos \Delta\theta)^2] \quad (38b)$$

where Eq. (38b) may be obtained by integrating Eq. (37a), or more easily by using Eq. (38a) in combination with conservation of angular momentum $h^2 = \mu a(1 - e^2)$. The osculating apocenter radius $r_a = a(1 + e)$ can then be formed from Eqs. (38) as

$$r_a(\Delta\theta) = a_o \frac{1 + \beta(1 - \cos \Delta\theta)}{1 - \beta^2(1 - \cos \Delta\theta)^2} \quad (39)$$

Then, when $r_p = r_f$, the condition for orbit transfer becomes

$$\cos^2 \Delta\theta - \left(\frac{2\rho\beta^2 - \beta}{\rho\beta^2}\right) \cos \Delta\theta + \left(\frac{1 + \beta - \rho + \rho\beta^2}{\rho\beta^2}\right) = 0 \quad (40)$$

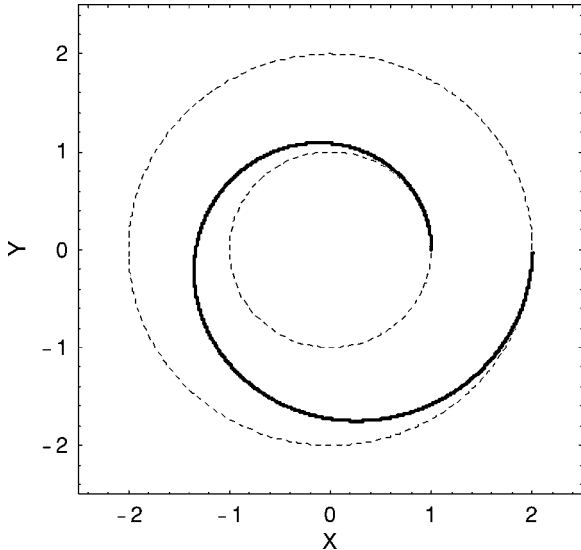


Fig. 5a Cubic transfer orbit.

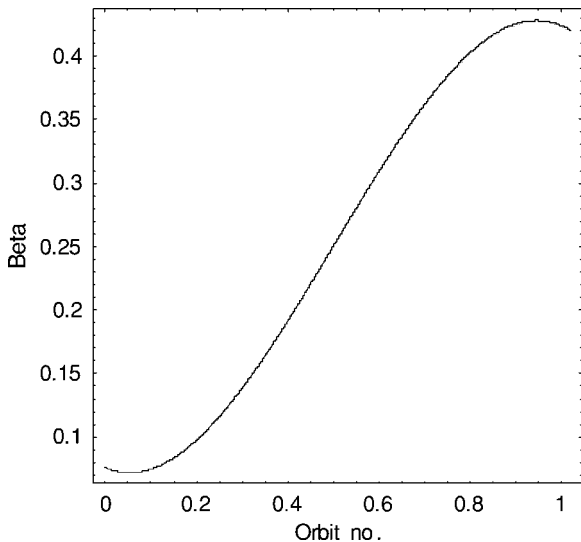


Fig. 5b Required lightness number for a cubic transfer orbit.

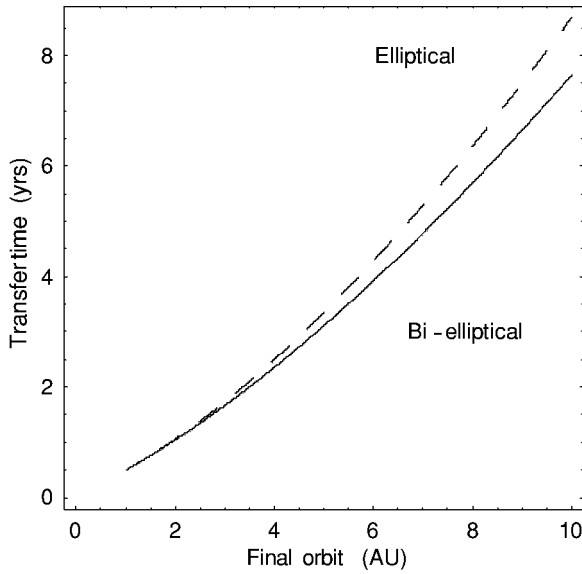


Fig. 6 Transfer duration for elliptic and bielliptic transfer modes.

where $\rho = r_f/r_o$. The admissible solution to this quadratic is then found to be

$$\cos \Delta\theta = 1 + (1 - \rho)/\rho\beta \tag{41}$$

If Eq. (35) is used to substitute for β as $(1 - r_o/r_f)/2$, it is then found that $\Delta\theta = \pi$, as expected. However, if a maximum lightness number of $(1 - r_o/r_f)$ is substituted, it is found that $\Delta\theta = \pi/2$. Therefore, the transfer is composed of two ellipses, each of arc length $\pi/2$.

In summary, the transfer begins with $\beta = 0$ at the initial circular orbit of radius r_o . A lightness number of $(1 - r_o/r_f)$ is then required for $\Delta\theta = \pi/2$ to attain the osculating elements required for a coast arc to the final circular orbit at radius r_f . Last, a lightness number of $(1 - r_o/r_f)$ is again required, to inject the spacecraft into the final circular orbit. The transfer duration can then be obtained by integrating Eq. (11). Although this can be carried out analytically, the full result is not listed here for brevity. A comparison of the transfer duration for the elliptic and bielliptic transfer modes is shown in Fig. 6. Note that in the limit $r_f \rightarrow r_o$ it is found that $\tau \rightarrow \pi\sqrt{(r_o^3/\mu)}$. This is a limiting process that results from the formulation of the problem, as can be seen from Eqs. (35) and (36).

Escape Orbits

To reach escape, a switching strategy is required to increase the orbit energy, while the orbit angular momentum is conserved. Because angular momentum is conserved, there will be a curve within the $a-e$ plane along which the transfer to escape will occur. From Eqs. (37a) and (37b), it can be seen that

$$\frac{da}{de} = \frac{2ae}{1 - e^2} \tag{42}$$

which integrates to $a(1 - e^2) = a_o(1 - e_o^2)$, which is of course a statement of conservation of angular momentum. The curve in the $a-e$ plane for a spacecraft starting from a circular heliocentric orbit with $a_o = 1$ is shown in Fig. 7. As the orbit energy is increased, the orbit semimajor axis is also increased, leading to an increase in orbit eccentricity until escape at $e = 1$. If $e_o = 0$, then, from conservation of angular momentum, the orbit pericenter and apocenter radii can be obtained as

$$r_a = a_o/(1 - e) \tag{43a}$$

$$r_p = a_o/(1 + e) \tag{43b}$$

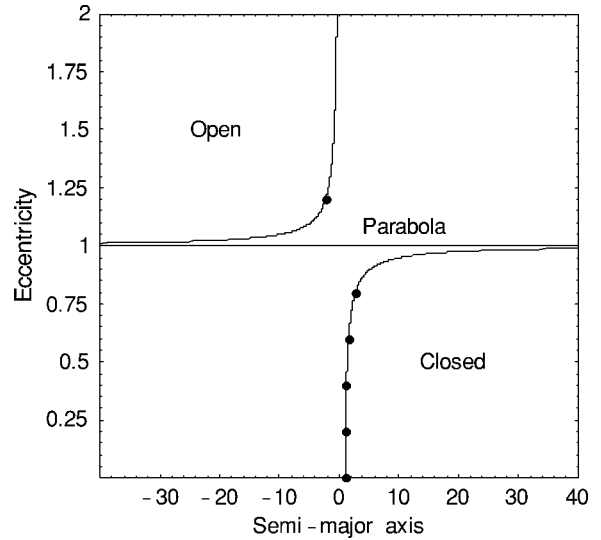


Fig. 7 Curve in $a-e$ space to escape: ●, steps along escape ladder with $\tilde{\beta} = 0.1$.

so that in the limit as $e \rightarrow 1$ the pericenter radius $r_p \rightarrow a_o/2$, whereas $r_a \rightarrow \infty$. The spacecraft is, therefore, limited to pericenter radii $r_p > a_o/2$.

It can be seen from Eq. (37a) that a strategy to increase the orbit energy is provided by the following switching law:

$$\beta(\theta) = \begin{cases} \tilde{\beta} & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \pi \leq \theta \leq 2\pi \end{cases} \tag{44}$$

where $\tilde{\beta}$ is again the maximum attainable lightness number. Maximum thrust is, therefore, applied on the outward arc, whereas null thrust is required on the inward arc. From Eqs. (38), this results in the following change in orbital elements in the $a-e$ plane

$$\Delta e = 2\tilde{\beta} \tag{45a}$$

$$\Delta a = a_o/(1 - 4\tilde{\beta}^2) \tag{45b}$$

Immediately, it can be seen that escape can be reached on the first-half orbit if $\tilde{\beta} = \frac{1}{2}$, since $\Delta e = 1$. If $\tilde{\beta} < \frac{1}{2}$, then at least one complete orbit is required before escape can be reached. Similarly, if $\tilde{\beta} > \frac{1}{2}$, then on the first arc $\Delta e > 1$ and so the orbit semimajor axis $a < 0$ as the orbit becomes hyperbolic because the orbit angular momentum $h^2 = \mu a(1 - e^2)$ is conserved. When Eq. (45) is used, an escape ladder can then be formed along the curve of constant angular momentum in the $a-e$ plane such that

$$e_j = 2j\tilde{\beta} \tag{46a}$$

$$a_j = a_o/[1 - (2j\tilde{\beta})^2] \tag{46b}$$

where $j = 0 - M$ is the number of orbits completed. The steps along the escape ladder from a circular orbit with $a_o = 1$ and $\tilde{\beta} = 0.1$ are shown in Fig. 7, whereas the resulting escape trajectory is shown in Fig. 8 with $r_p > \frac{1}{2}$.

The use of multiple loops for spacecraft with $\tilde{\beta} < \frac{1}{2}$ can lead to lengthy escape trajectories. However, future solar sail and M2P2 systems may enable $\tilde{\beta} > 1$, which allows direct escape along a hyperbolic path. Note, however, that the hyperbola does not contain the sun at its focus because the net inverse square force is repulsive. In fact, the sun is located at the center of the two opposing hyperbolas that define the classical conic section. If the initial conditions are representative of a circular of radius r_o , Eq. (10) then provides the orbit equation as

$$r(\theta) = r_o/[1 - \tilde{\beta}(1 - \cos\theta)] \tag{47}$$

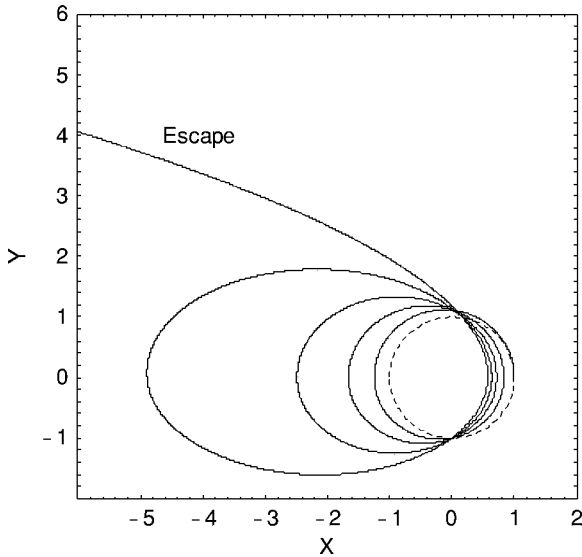


Fig. 8 Escape trajectory with $\tilde{\beta}=0.1$.

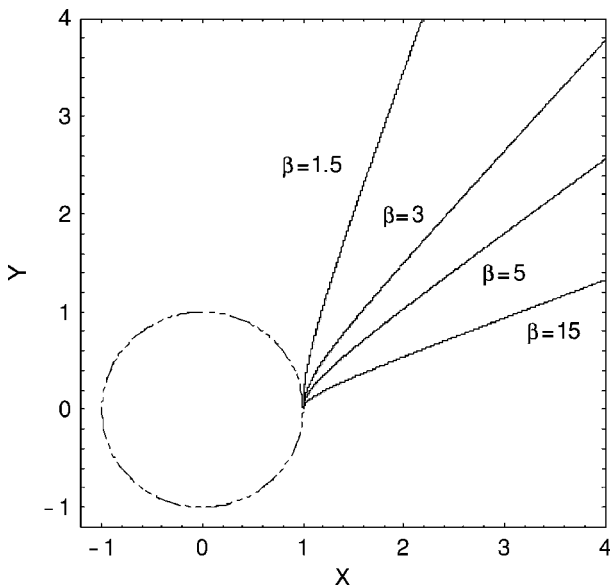


Fig. 9 Family of escape hyperbolas.

so that the orbit has an asymptote at θ_∞ (as $r \rightarrow \infty$) given by $\cos \theta_\infty = 1 - 1/\tilde{\beta}$. It can be seen that Eq. (19) is recovered if $\tilde{\beta} = 1$. In addition, Eq. (11) provides time t as a function of polar angle as

$$t(\theta) = \frac{r_o^{\frac{3}{2}}}{\sqrt{\mu}} \frac{2(\tilde{\beta} - 1)[1 - \tilde{\beta}(1 - \cos \theta)] \tanh^{-1} \left[\sqrt{2\tilde{\beta} - 1} \tan(\theta/2) \right] + \tilde{\beta} \sqrt{2\tilde{\beta} - 1} \sin \theta}{(2\tilde{\beta} - 1)[1 - \tilde{\beta}(1 - \cos \theta)]} \quad (48)$$

Last, equating the energy at the beginning of the orbit arc [$v_o^2/2 - \mu(1 - \beta)/r_o$] to the energy as $r \rightarrow (v_\infty^2/2)$ yields the hyperbolic

excess speed v_∞ as

$$v_\infty = v_o \sqrt{2\tilde{\beta} - 1} \quad (49)$$

where $v_o = \sqrt{(\mu/r_o)}$ is the speed on the initial circular orbit at radius r_o . A range of escape hyperbolas are shown in Fig. 9 with $r_o = 1$. It can be seen that a rectilinear orbit, defined by Eq. (19), will have an asymptote with $\theta_\infty = \pi/2$, whereas $\theta_\infty \rightarrow 0$ as $\beta \rightarrow \infty$.

Conclusions

An extension of the classical two-body problem has been investigated that considers the addition of a modulated, radial, inverse square force. The force is assumed to be the modulated thrust from a solar sail or M2P2 system. It has been shown that the forward integration problem can be solved in closed form, whereas an inverse problem can be constructed that allows orbits to be defined a priori. Both of these approaches have been used to investigate transfer between circular, coplanar orbits and open escape orbits. For escape orbits, a switching strategy has been defined that allows motion along an escape ladder in the $a-e$ plane, allowing energy gain, while orbital angular momentum is conserved.

Acknowledgments

Aspects of this work were supported by funding from the Levrulme Trust and the Lockheed Martin Corporation, to whom the author expresses his thanks.

References

- ¹McInnes, C. R., *Solar Sailing: Technology, Dynamics and Mission Applications*, Springer-Verlag, London, 1999, pp. 56–111.
- ²Winglee, R. M., Slough, J., Ziemba, T., and Goodson, A., “Mini-Magnetospheric Plasma Propulsion: Tapping the Energy of the Solar Wind for Spacecraft Propulsion,” *Journal of Geophysics Research*, Vol. 105, No. A9, 2000, pp. 21067–21077.
- ³Zubrin, R., and Andrews, D., “Magnetic Sails and Interplanetary Travel,” *Journal of Spacecraft and Rockets*, Vol. 28, No. 2, 1991, pp. 197–203.
- ⁴Tsien, H. S., “Take-Off from Satellite Orbit,” *Journal of the American Rocket Society*, Vol. 23, No. 4, 1953, pp. 233–236.
- ⁵Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, AIAA, New York, 1987, pp. 408–415.
- ⁶Modi, V. J., “On the Semi-Passive Attitude Control and Propulsion of Space Vehicles Using Solar Radiation Pressure,” *Acta Astronautica*, Vol. 35, No. 2/3, 1995, pp. 231–246.
- ⁷Prussing, J. E., and Coverstone-Carroll, V., “Constant Radial Thrust Acceleration Redux,” *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 3, 1998, pp. 516–518.
- ⁸Akella, M. R., “On the Existence of Almost Periodic Orbits in Low Radial Thrust Spacecraft Motion,” *Advances in the Astronautical Sciences*, Vol. 106, 2000, pp. 41–52.
- ⁹Broucke, R., “Notes on the Central Force r^N ,” *Astrophysics and Space Science*, Vol. 27, No. 1, 1980, pp. 33–53.
- ¹⁰Sweetser, T. H., and Sauer, C. G., “Advanced Propulsion Options for Missions to the Kuiper Belt,” *Advances in the Astronautical Sciences*, Vol. 109, 2001, pp. 2297–2306.

¹¹Fowles, G. R., *Analytical Mechanics*, Holt-Saunders International, New York, 1977, pp. 164–165.