Stabilisation of Highly Nonlinear Hybrid Systems by Feedback Control Based on Discrete-Time State Observations

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***Abstract*— Given an unstable hybrid stochastic differential equation (SDE), can we design a feedback control, based on the discrete-time observations of the state at times $0, \tau, 2\tau, \cdots$, so that the controlled hybrid SDE becomes asymptotically stable? It has been proved that this is possible if the drift and diffusion coefficients of the given hybrid SDE satisfy the linear growth condition. However, many hybrid SDEs in the real world do not satisfy this condition (namely, they are highly nonlinear) and there is no answer to the question yet if the given SDE is highly nonlinear. The aim of this paper is to tackle the stabilization problem for a class of highly nonlinear hybrid SDEs. Under some reasonable conditions on the drift and diffusion coefficients, we show how to design the feedback control function and give an explicit bound on τ (the time duration between two consecutive state observations), whence the new theory established in this paper is implementable.**

***Index Terms*—Highly nonlinear; Itô formula; Markov chain; Asymptotic stability; Lyapunov functional.**

I. Introduction

Many systems in the real word may experience abrupt changes in their structures and parameters due to sudden changes of system factors, for example, a failure of a power station in a network, a change of interest rate in an economic system, an environmental change in an ecological system. Hybrid stochastic differential equations (SDEs; also known as SDEs with Markovian switching) have been widely used to model these systems (see, e.g., [2], [10], [20], [21], [22]).

Hybrid SDEs are in general described by

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t). \quad (1)$$

Here the state $x(t)$ takes values in R^n and the mode $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \cdots, N\}$, $B(t)$ is a Brownian motion and f and g are referred to as the drift and diffusion coefficient, respectively. One of the important issues in the study of hybrid SDEs is the analysis of stability (see, e.g., [7], [22], [20], [26], [27], [28], [29], [31]).

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In the case when a given hybrid SDE is unstable, can we design a feedback control $u(x([t/\tau]\tau), r(t), t)$, based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$, so that the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt + g(x(t), r(t), t)dB(t) \quad (2)$$

becomes stable? Here $\tau > 0$ is a constant which stands for the duration between two consecutive state observations, and $[t/\tau]$ is the integer part of t/τ. This is significantly different from the stabilisation by a continuous-time (regular) feedback control $u(x(t), r(t), t)$, because the regular feedback control requires the continuous observations of the state $x(t)$ for all $t \geq 0$, while the feedback control $u(x([t/\tau]\tau), r(t), t)$ needs only the discrete observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$. The latter is clearly more realistic and costs less in practice. Moreover, a larger of τ means a less frequent observations to be made. It is therefore more desirable in practice to choose larger τ whenever possible. Our aims here are therefore not only to design the control function u but also give an explicit bound, say τ^* on τ in the sense whenever $\tau \leq \tau^*$ the controlled system is stable.

The answer to the stabilization question above is yes when both drift and diffusion coefficients of the given hybrid SDE satisfy the linear growth condition (see, e.g., [16], [17], [18], [25], [30]). However, many hybrid SDEs in the real world do not satisfy this linear growth condition (namely, they are highly nonlinear), for example, the SDEs discussed in Examples 6.1 and 6.2 later ((see, e.g., [2], [10], [4] for more on highly nonlinear hybrid SDEs). Unfortunately, there is so far no answer to the question if the given SDE is highly nonlinear. It is therefore necessary and important to establish a new theory which shows how to design the feedback controls based on the discrete-time state observations in order to stabilise highly nonlinear hybrid SDEs.

The key challenge of this paper lies in the difficulties arisen from the highly nonlinear drift and diffusion coefficients. All papers so far in this direction (see, e.g., [16], [17], [18], [25], [30]) impose the critical linear growth condition on the coefficients. Many known techniques dependent on this linear growth condition does not work in this paper. We need to develop new techniques to overcome the difficulties arisen from the high nonlinearity. We should also mention that there are already papers on the stability of highly nonlinear SDEs (see, e.g., [10], [11], [12], [21]) but the stability criteria in these papers/books are not applicable to the design of feedback controls based on the discrete-time state observations for highly nonlinear SDEs. Comparing with the existing papers, we highlight a number of main contributions of this paper:

(i) This is the first paper that studies the design of a feedback control based on the discrete-time state observations in order to stabilize a given unstable highly nonlinear hybrid SDE.

(ii) In order to make the new theory established in this paper implementable, we propose three conditions on the control function. In particular, one key condition is in terms of M-matrices and hence it can be verified easily. We also explain how to design the control function step by step to meet these conditions.

(iii) Under some mild conditions which guarantee the boundedness of the unique solution of the given SDE, we show that the discrete-time feedback control can preserve the boundedness as long as the control function satisfies the Lipschitz condition. This does not only form the foundation of the paper but also makes the design of the control function become much easier.

(iv) A number of new techniques are developed to overcome the difficulties arisen from the high nonlinearity and discrete-time control. For example, the technique used in the proof of the boundedness of the solution to the controlled system is significantly different from that when the continuous-time feedback control is used.

The paper is organised as follows. We will give the preliminaries on the highly nonlinear hybrid SDEs and impose some standing hypotheses which guarantee the boundedness of the unique solution of the given SDE in Section 2. We will show the discrete-time feedback control can preserve the boundedness as long as the control function satisfies the Lipschitz condition in Section 3. We will in Section 4 propose three conditions and explain, one by one, there are many available controls functions which can meet these conditions, and then show such a discrete-time feedback control can stabilize the given SDE asymptotically. In Section 5 we will further discuss the exponential stabilization. Our theory is illustrated by two examples in Section 6 wile the paper is conclude in Section 7.

II. Controlled System and Standing Hypotheses

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in R^n$, $|x|$ denotes its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm. If A is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. By $A \le 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. Let $R_+ = [0, \infty)$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions φ from $[-h, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-h\le u\le 0} |\varphi(u)|$. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a subset of Ω, denote by I_A its indicator function; that is, $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while

$$\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost all sample paths of $r(t)$ are piecewise constant except for a finite number of simple jumps in any finite subinterval of R_+. We stress that almost all sample paths of $r(t)$ are right continuous.

Suppose that the underlying system is described by a nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \quad (3)$$

on $t \ge 0$ with the initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times S \times R_+ \to R^{n\times m}$$

are Borel measurable functions. As mentioned in the last section, we consider the situation in this paper where either f or g does not satisfy the linear growth condition (namely not bounded by a linear function). The following assumption describes this situation.

Assumption 2.1: Assume that for any real number $b > 0$, there exists a positive constant K_b such that

$$|f(x, i, t) - f(\bar{x}, i, t)| \vee |g(x, i, t) - g(\bar{x}, i, t)| \le K_b(|x - \bar{x}|) \quad (4)$$

for all $x, \bar{x} \in R^n$ with $|x| \vee |\bar{x}| \le b$ and all $(i, t) \in S \times R_+$. Assume also that there exist three constants $K > 0$, $q_1 > 1$ and $q_2 \ge 1$ such that

$$|f(x, i, t)| \le K(|x|+|x|^{q_1}) \quad \text{and} \quad |g(x, i, t)| \le K(|x|+|x|^{q_2}) \quad (5)$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Condition (5) forces that $f(0, i, t) \equiv 0$ and $g(0, i, t) \equiv 0$, which are required for the stability purpose of this paper. Of course, if $q_1 = q_2 = 1$ then condition (5) is the familiar linear growth condition. However, let us stress once again that we are here interested in the hybrid SDEs without the linear growth condition and we will always assume that $q_1 > 1$ in this paper. We will refer to condition (5) as the polynomial growth condition. For the hybrid SDE (71), we see easily that $q_1 = 3$ and $q_2 = 1.5$. This assumption is of course not sufficient to guarantee the existence of the unique global solution of the hybrid SDE (3). We therefore impose another Khasminskii-type condition.

Assumption 2.2: Assume that there exist positive constants p, q, α, β such that

$$q \ge (2q_1)\vee(2q_2+q_1-1) \quad \text{and} \quad p \ge (q_1+1)\vee(2q_2-q_1+1) \quad (6)$$

(where q_1 and q_2 have been specified in Assumption 2.1) while

$$x^T f(x,i,t) + \frac{q-1}{2}|g(x,i,t)|^2 \le -\alpha|x|^p + \beta|x|^2 \tag{7}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

In many hybrid SDEs, p and q are different. In fact, q could be arbitrarily large sometimes. For example, consider the hybrid SDE (71) and let q be arbitrarily large. Then

$$\begin{aligned} &x^T f(x,i,t) + \frac{q-1}{2}|g(x,i,t)|^2 \\ &= \begin{cases} x^2 - 3x^4 + 0.5(q-1)|x|^3 & \text{if } i=1, \\ x^2 - 2x^4 + 0.125(q-1)|x|^3 & \text{if } i=2. \end{cases} \end{aligned} \tag{8}$$

But

$$(q-1)|x|^3 \le |x|^4 + 0.25(q-1)^2 x^2.$$

Hence

$$\begin{aligned} &x^T f(x,i,t) + \frac{q-1}{2}|g(x,i,t)|^2 \\ &\le -1.875x^4 + (1 + 0.125(q-1)^2)x^2. \end{aligned} \tag{9}$$

That is, the hybrid SDE (71) satisfies Assumption 2.2 with any large q and $p = 4$, $\alpha = 1.875$, $\beta = 1 + 0.125(q-1)^2$ (recalling $q_1 = 3$ and $q_2 = 1.5$).

It is well known (see, e.g., [21, Theorem 5.3 on page 159]) that under Assumptions 2.1 and 2.2, the hybrid SDE (3) with any initial value $x(0) = x_0 \in R^n$ has a unique global solution such that $\sup_{0\le t<\infty} \mathbb{E}|x(t)|^q < \infty$. Although the qth moment of solution is bounded, the SDE (3) may not be stable. In the case when the given SDE (3) is unstable, we are required to design a feedback control $u(x([t/\tau]\tau), r(t), t)$, based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$, in the drift part so that the controlled system

$$\begin{aligned} dx(t) =& [f(x(t), r(t), t) + u(x(\delta_t), r(t), t)]dt \\ &+ g(x(t), r(t), t)dB(t), \quad t \ge 0, \end{aligned} \tag{10}$$

becomes stable, where $\delta_t = [t/\tau]\tau$ and the control function $u : R^n \times S \times R_+ \to R^n$ is a Borel measurable. In this paper, we will design the control function to satisfy the following assumption.

Assumption 2.3: Assume that there exists a positive number κ such that

$$|u(x,i,t) - u(y,i,t)| \le \kappa|x-y| \tag{11}$$

for all $x, y \in R^n$, $i \in S$ and $t \ge 0$. Moreover, for the stability purpose, assume that $u(0,i,t) \equiv 0$.

This assumption implies

$$|u(x,i,t)| \le \kappa|x|, \quad \forall (x,i,t) \in R^n \times S \times R_+. \tag{12}$$

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III. Boundedness

As pointed out, the qth moment of the solution of the given SDE (3) is bounded. The following theorem, which forms the foundation of this paper, shows that the controlled SDE (10) preserves this nice property.

Theorem 3.1: Under Assumptions 2.1, 2.2 and 2.3, the controlled system (10) with any initial value $x(0) = x_0 \in R^n$ has a unique global solution $x(t)$ on $t \ge 0$ and the solution has the property that

$$\sup_{0\le t<\infty} \mathbb{E}|x(t)|^q < \infty. \tag{13}$$

Proof. We observe that the controlled system (10) is in fact a hybrid stochastic differential delay equation (SDDE) with a bounded variable delay. In fact, if we define the bounded variable delay $\zeta : R_+ \to [0,\tau]$ by

$$\zeta(t) = t - k\tau \quad \text{for } k\tau \le t < (k+1)\tau, \ k = 0,1,2,\cdots,$$

then the controlled system (10) can be written as

$$\begin{aligned} dx(t) =& \big(f(x(t), r(t), t) + u(x(t-\zeta(t)), r(t), t)\big)dt \\ &+ g(x(t), r(t), t)dB(t) \end{aligned} \tag{14}$$

on $t \ge 0$ with the initial value $x(0) = x_0 \in R^n$. Let $\bar{U}(x) = |x|^q$. By the Itô formula,

$$\begin{aligned} d\bar{U}(x(t)) =& \bar{L}\bar{U}(x(t), x(t-\zeta(t)), r(t), t)dt \\ &+ q|x|^{q-2}x^T(t)g(x(t), r(t), t)dB(t), \end{aligned}$$

where the function $\bar{L}\bar{U} : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\begin{aligned} &\bar{L}\bar{U}(x,y,i,t) = \\ &q|x|^{q-2}x^T[f(x,i,t) + u(y,i,t)] + \frac{q}{2}|x|^{q-2}|g(x,i,t)|^2 \\ &+ \frac{q(q-2)}{2}|x|^{q-4}|x^T g(x,i,t)|^2 \\ &\le q|x|^{q-2}\Big[x^T[f(x,i,t) + u(y,i,t)] + \frac{q-1}{2}|g(x,i,t)|^2\Big]. \end{aligned}$$

By Assumptions 2.2 and 2.3,

$$\bar{L}\bar{U}(x,y,i,t) \le -q\alpha|x|^{q+p-2} + q\beta|x|^q + q\kappa|x|^{q-1}|y|.$$

Let us now choose a constant $\varepsilon \in (0,1)$ sufficiently small for

$$e^{-\tau} + \varepsilon\tau < 1. \tag{15}$$

By the well-known Young inequality,

$$\begin{aligned} q\kappa|x|^{q-1}|y| &= \Big(\frac{(q\kappa)^{q/(q-1)}}{(q\varepsilon)^{1/(q-1)}}|x|^q\Big)^{(q-1)/q}\big(q\varepsilon|y|^q\big)^{1/q} \\ &\le \frac{(q-1)(q\kappa)^{q/(q-1)}}{q(q\varepsilon)^{1/(q-1)}}|x|^q + \varepsilon|y|^q. \end{aligned}$$

Hence

$$\begin{aligned} &\bar{L}\bar{U}(x,y,i,t) \\ &\le -q\alpha|x|^{q+p-2} + \Big(q\beta + \frac{(q-1)(q\kappa)^{q/(q-1)}}{q(q\varepsilon)^{1/(q-1)}}\Big)|x|^q + \varepsilon|y|^q \\ &\le C - \bar{U}(x) + \varepsilon\bar{U}(y), \end{aligned} \tag{16}$$

where

$$C := \sup_{u\ge 0}\Big[-q\alpha u^{q+p-2} + \Big(1 + q\beta + \frac{(q-1)(q\kappa)^{q/(q-1)}}{q(q\varepsilon)^{1/(q-1)}}\Big)u^q\Big].$$

By [21, Theorem 7.13 on page 280], we can hence conclude that the SDDE (14), namely the controlled system (10) with any initial value $x(0) = x_0 \in R^n$ has a unique global solution $x(t)$ on $t \ge 0$ and the solution has the property that $\mathbb{E}|x(t)|^q < \infty$ for all $t \ge 0$.

In the remaining proof, we will show the stronger result (13). Set $t_k = k\tau$ for $k = 0, 1, 2 \cdots$. By the Itô formula, we can show that for $t \in [t_k, t_{k+1}]$,

$$e^t \mathbb{E}\bar{U}(x(t)) = e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + \mathbb{E}\int_{t_k}^t e^s[\bar{U}(x(s)) + \bar{L}\bar{U}(x(s), x(s-\zeta(s)), r(s), s)]ds.$$

Using (16), we see

$$\begin{aligned} & e^t \mathbb{E}\bar{U}(x(t)) \\ & \le e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + \mathbb{E}\int_{t_k}^t e^s[C + \varepsilon \bar{U}(x(s-\zeta(s)))]ds \\ & = e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + \mathbb{E}\int_{t_k}^t e^s[C + \varepsilon \bar{U}(x(t_k))]ds \\ & = e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + (e^t - e^{t_k})[C + \varepsilon \mathbb{E}\bar{U}(x(t_k))]. \end{aligned} \tag{17}$$

In particular,

$$\begin{aligned} & e^{t_{k+1}}\mathbb{E}\bar{U}(x(t_{k+1})) \\ & \le e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + (e^{t_{k+1}} - e^{t_k})[C + \varepsilon \mathbb{E}\bar{U}(x(t_k))]. \end{aligned}$$

This implies

$$\begin{aligned} \mathbb{E}\bar{U}(x(t_{k+1})) & \le e^{-\tau}\mathbb{E}\bar{U}(x(t_k)) + (1-e^{-\tau})[C + \varepsilon \mathbb{E}\bar{U}(x(t_k))] \\ & \le C\tau + (e^{-\tau} + \varepsilon\tau)E\bar{U}(x(t_k)). \end{aligned} \tag{18}$$

Consequently

$$\begin{aligned} & \mathbb{E}\bar{U}(x(t_{k+1})) \\ & \le C\tau + (e^{-\tau} + \varepsilon\tau)[C\tau + (e^{-\tau} + \varepsilon\tau)\mathbb{E}\bar{U}(x(t_{k-1}))] \\ & \le C\tau[1 + (e^{-\tau} + \varepsilon\tau) + \cdots + (e^{-\tau} + \varepsilon\tau)^k] \\ & + (e^{-\tau} + \varepsilon\tau)^{k+1}\bar{U}(x(0)) \\ & \le \frac{C\tau}{1 - (e^{-\tau} + \varepsilon\tau)} + |x(0)|^q. \end{aligned} \tag{19}$$

Furthermore, it follows from (17) that

$$\begin{aligned} & \sup_{t_k \le t \le t_{k+1}} \left[e^t \mathbb{E}\bar{U}(x(t))\right] \\ & \le e^{t_k}\mathbb{E}\bar{U}(x(t_k)) + (e^{t_{k+1}} - e^{t_k})[C + \varepsilon \mathbb{E}\bar{U}(x(t_k))]. \end{aligned}$$

This, together with (19), yields

$$\begin{aligned} & \sup_{t_k \le t \le t_{k+1}} \mathbb{E}\bar{U}(x(t)) \\ & \le \mathbb{E}\bar{U}(x(t_k)) + (e^\tau - 1)[C + \varepsilon \mathbb{E}\bar{U}(x(t_k))] \\ & \le C(e^\tau - 1) + [1 + \varepsilon(e^\tau - 1)]\mathbb{E}\bar{U}(x(t_k))] \\ & \le C(e^\tau - 1) + [1 + \varepsilon(e^\tau - 1)]\Big(\frac{C\tau}{1 - (e^{-\tau} + \varepsilon\tau)} + |x(0)|^q\Big). \end{aligned} \tag{20}$$

As this holds for any $k \ge 0$, the required assertion (13) must hold. The proof is complete. □

This theorem implies a number of nice properties of the solution. For example, for any $t \ge 0$, $x(t)$ is bounded in $L^{\bar{q}}$ for any $\bar{q} \in (0, q]$ while both $f(x(t), r(t), t)$ and $g(x(t), r(t), t)$ are in L^2. These properties will play their fundamental roles when we discuss the stabilisation of the SDDE (10) in the next section.

IV. Asymptotic Stabilisation

We have just shown that the controlled system (10) preserves the boundedness of the given SDE (3) as long as the control function satisfies Assumption 2.3. However, such a control may not be able to stabilise the given SDE. We need more carefully design the control function in order for the controlled system (10) to be stable. In this section, we will step by step explain how to design the control function to meet a number of conditions under our standing Assumptions 2.1-2.3, and then show such designed control function will indeed guarantee the asymptotic stability of the controlled system (10). Let us begin to state our first condition.

Condition 4.1: Design the control function $u : R^n \times S \times R_+ \to R^n$ so that we can find constants $\alpha_i > 0$, $\bar{\alpha}_i > 0$ and $\beta_i, \bar{\beta}_i \in R$ $(i \in S)$ for both

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 \le -\alpha_i|x|^p + \beta_i|x|^2 \tag{21}$$

and

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 \le -\bar{\alpha}_i|x|^p + \bar{\beta}_i|x|^2 \tag{22}$$

to hold for all $(x,i,t) \in R^n \times S \times R_+$ and for both

$$\begin{aligned} \mathcal{A}_1 &:= -2\text{diag}(\beta_1, \cdots, \beta_N) - \Gamma, \\ \mathcal{A}_2 &:= -(q_1+1)\text{diag}(\bar{\beta}_1, \cdots, \bar{\beta}_N) - \Gamma \end{aligned} \tag{23}$$

to be nonsingular M-matrices.

Regarding the theory on M-matrices we refer the reader to [21, Section 2.6]. Let us explain that there are lots of such control functions available under Assumption 2.2. For example, in the case when the state $x(t)$ of the given SDE (3) is observable in any mode $i \in S$ (otherwise it is more complicated and we will explain later), we could, for example, design the control function $u(x,i,t) = Ax$, where A is a symmetric $n\times n$ real-valued matrix such that $\lambda_{\max}(A) \le -2\beta$. Then

$$x^T u(x,i,t) \le -2\beta|x|^2, \quad \forall (x,i,t) \in R^n \times S \times R_+.$$

By Assumption 2.2, we further have

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2 \le -\alpha|x|^p - \beta|x|^2$$

as well as

$$x^T[f(x,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2 \le -\alpha|x|^p - \beta|x|^2$$

while

$$\mathcal{A}_1 = 2\text{diag}(\beta, \cdots, \beta) - \Gamma \ \text{ and } \ \mathcal{A}_2 = (q_1+1)\text{diag}(\beta, \cdots, \beta) - \Gamma$$

which are nonsingular M-matrices (see, e.g., [21, Theorem 2.10]). That is, the control function $u(x,i,t) = Ax$ meets Condition 4.1. Of course, in application, we need to make full use of the special forms of both coefficients f and g to design the control function u more wisely in order to meet our further conditions more easily.

To state our second condition, we set

$$(\theta_1, \cdots, \theta_N)^T := \mathcal{A}_1^{-1}(1, \cdots, 1)^T,$$
$$(\bar{\theta}_1, \cdots, \bar{\theta}_N)^T := \mathcal{A}_2^{-1}(1, \cdots, 1)^T. \quad (24)$$

As $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonsingular M-matrices, all θ_i and $\bar{\theta}_i$ are positive. Define a function $U : R^n \times S \to R_+$ by

$$U(x, i) = \theta_i |x|^2 + \bar{\theta}_i |x|^{q_1+1}, \quad (x, i) \in R^n \times S \quad (25)$$

while define a function $LU : R^n \times S \times R_+ \to R$ by

$$\begin{aligned} &LU(x, i, t) \\ &= 2\theta_i \Big[x^T [f(x, i, t) + u(x, i, t)] + \frac{1}{2}|g(x, i, t)|^2 \Big] \\ &+ (q_1 + 1)\bar{\theta}_i |x|^{q_1-1} \Big[x^T [f(x, i, t) + u(x, i, t)] + \frac{q_1}{2}|g(x, i, t)|^2 \Big] \\ &+ \sum_{j=1}^N \gamma_{ij}(\theta_j |x|^2 + \bar{\theta}_j |x|^{q_1+1}). \end{aligned} \quad (26)$$

Please note that LU is a single function (not L acting on U). By (21)-(24), we observe

$$\begin{aligned} &LU(x, i, t) \\ &\le 2\theta_i(-\alpha_i |x|^p + \beta_i |x|^2) + \sum_{j=1}^N \gamma_{ij}\theta_j |x|^2 \\ &+ (q_1 + 1)\bar{\theta}_i |x|^{q_1-1}(-\bar{\alpha}_i |x|^p + \bar{\beta}_i |x|^2) + \sum_{j=1}^N \gamma_{ij}\bar{\theta}_j |x|^{q_1+1} \\ &\le -2\theta_i\alpha_i |x|^p - |x|^2 - (q_1 + 1)\bar{\theta}_i\bar{\alpha}_i |x|^{p+q_1-1} - |x|^{q_1+1}. \end{aligned} \quad (27)$$

This observation makes the following condition possible.

Condition 4.2: Find four positive constants γ_j, $j = 1, 2, 3, 4$, such that

$$\begin{aligned} &LU(x, i, t) + \gamma_1 \big(2\theta_i |x| + (q_1 + 1)\bar{\theta}_i |x|^{q_1} \big)^2 \\ &+ \gamma_2 |f(x, i, t)|^2 + \gamma_3 |g(x, i, t)|^2 \le -\gamma_4 |x|^2 \end{aligned} \quad (28)$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Let us explain why it is always possible to meet this condition. In fact, by Assumption 2.1 and (27), we have

$$\begin{aligned} &LU(x, i, t) + \gamma_1 \big(2\theta_i |x| + (q_1 + 1)\bar{\theta}_i |x|^{q_1} \big)^2 \\ &+ \gamma_2 |f(x, i, t)|^2 + \gamma_3 |g(x, i, t)|^2 \\ &\le -|x|^{q_1+1} - |x|^2 - (q_1 + 1)\bar{\theta}_i\bar{\alpha}_i |x|^{p+q_1-1} \\ &+ 8\gamma_1\theta_i^2 |x|^2 + 2\gamma_1(q_1 + 1)^2\bar{\theta}_i^2 |x|^{2q_1} \\ &+ 2\gamma_2 K^2(|x|^2 + |x|^{2q_1}) + 2\gamma_3 K^2(|x|^2 + |x|^{2q_2}). \end{aligned} \quad (29)$$

Recalling (6), we have $p + q_1 - 1 \ge 2(q_1 \vee q_2)$ and hence

$$|x|^{2q_1} \vee |x|^{2q_2} \le |x|^2 + |x|^{p+q_1-1}.$$

It then follows from (29) that

$$\begin{aligned} &LU(x, i, t) + \gamma_1 \big(2\theta_i |x| + (q_1 + 1)\bar{\theta}_i |x|^{q_1} \big)^2 \\ &+ \gamma_2 |f(x, i, t)|^2 + \gamma_3 |g(x, i, t)|^2 \\ &\le -|x|^{q_1+1} - \big[(q_1 + 1)\bar{\theta}_i\bar{\alpha}_i \\ &- 2\gamma_1(q_1 + 1)^2\bar{\theta}_i^2 - 2K^2(\gamma_2 + \gamma_3) \big] |x|^{p+q_1-1} \\ &- \big[1 - 8\gamma_1\theta_i^2 - 2\gamma_1(q_1 + 1)^2\bar{\theta}_i^2 - 4K^2(\gamma_2 + \gamma_3) \big] |x|^2. \end{aligned} \quad (30)$$

If we choose positive constants γ_1-γ_3 sufficiently small for

$$(q_1 + 1)\min_{i\in S}\bar{\theta}_i\bar{\alpha}_i \ge 2\gamma_1(q_1 + 1)^2 \max_{i\in S}\bar{\theta}_i^2 + 2K^2(\gamma_2 + \gamma_3)$$

and

$$0.5 \ge 8\gamma_1 \max_{i\in S}\theta_i^2 + 2\gamma_1(q_1 + 1)^2 \max_{i\in S}\bar{\theta}_i^2 + 4K^2(\gamma_2 + \gamma_3)$$

then

$$\begin{aligned} &LU(x, i, t) + \gamma_1 \big(2\theta_i |x| + (q_1 + 1)\bar{\theta}_i |x|^{q_1} \big)^2 \\ &+ \gamma_2 |f(x, i, t)|^2 + \gamma_3 |g(x, i, t)|^2 \\ &\le -0.5|x|^2 - |x|^{q_1+1}, \end{aligned} \quad (31)$$

namely we can have $\gamma_4 = 0.5$. (Please note that (31) is stronger than (28) but it will illustrate Condition 4.6) later.) Of course, in application, we need to make full use of the special forms of both coefficients f and g to choose γ_1 - γ_4 more wisely in order to have a larger bound on τ, which is the duration between the two consecutive state observations, as stated in our third condition.

Condition 4.3: Make sure the duration between the two consecutive state observations satisfies

$$\tau < \frac{\sqrt{\gamma_4\gamma_1}}{2\kappa^2} \quad \text{and} \quad \tau \le \frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{2}\kappa} \wedge \frac{\gamma_1\gamma_3}{\kappa^2} \wedge \frac{1}{4\kappa}. \quad (32)$$

In the introduction, we have explained that a larger of τ means a less frequent observations to be made so is more desirable in practice. However, a large τ could also mean information received via discrete-time state observations is not enough for the feedback control to stabilize the given unstable system. There is hence a balance on τ. Condition 4.3 means that the feedback control can certainly stabilize the given system as long as the discrete-time observations are frequently enough.

We can now state our first stabilisation result in this paper.

Theorem 4.4: Under Assumptions 2.1, 2.2 and 2.3, we can design the control function u to satisfy Condition 4.1 and then choose four positive constants γ_j, $j = 1, 2, 3, 4$, to meet condition 4.2. If we further make sure τ to be sufficiently small for Condition 4.3 to hold, then the solution of the controlled system (10) has the property that for any initial value $x(0) = x_0 \in R^n$,

$$\int_0^\infty \mathbb{E}|x(t)|^2 dt < \infty. \quad (33)$$

That is, the controlled system (10) is H_∞-stable in L^2.

Proof. To make the proof more understandable, we divide it into three steps.

Step 1. We will use the method of Lyapunov functionals to prove the theorem. For this purpose, we define two segments $\hat{x}_t := \{x(t + s) : -2\tau \le s \le 0\}$ and $\hat{r}_t := \{r(t + s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = x_0$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. The Lyapunov functional used in this proof will be of the form

$$\begin{aligned} &V(\hat{x}_t, \hat{r}_t, t) = U(x(t), r(t)) \\ &+ c\int_{-\tau}^0 \int_{t+s}^t \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\ &+ |g(x(v), r(v), v)|^2 \Big] dv ds \end{aligned} \quad (34)$$

for $t \geq 0$, where U has been defined by (25) and c is a positive constant to be determined later while we set

$$f(x,i,v) = f(x,i,0),\ \ g(x,i,v) = g(x,,i,0),$$

$$u(x,i,v) = u(x,i,0)$$

for $(x,i,v) \in R^n \times S \times [-2\tau, 0)$. We claim that $V(\hat{x}_t, \hat{r}_t, t)$ is an Itô process on $t \geq 0$. In fact, by the generalised Itô formula (see, e.g., [21]), we have

$$dU(x(t), r(t)) = \mathcal{L}U(x(t), x(\delta_t), r(t), t)dt + dM(t) \quad (35)$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here but it can be found in [21, Theorem 1.45 on page 48]) and $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\begin{aligned}
&\mathcal{L}U(x,y,i,t) \\
&= 2\theta_i\Big[x^T[f(x,i,t) + u(y,i,t)] + \frac{1}{2}|g(x,i,t)|^2\Big] \\
&+ (q_1+1)\bar{\theta}_i|x|^{q_1-1}\Big[x^T[f(x,i,t) + u(y,i,t)] + \frac{1}{2}|g(x,i,t)|^2\Big] \\
&+ \frac{(q_1+1)(q_1-1)}{2}\bar{\theta}_i|x|^{q_1-3}|x^T g(x,i,t)|^2 \\
&+ \sum_{j=1}^N \gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{q_1+1}).
\end{aligned}$$

On the other hand, the fundamental theory of calculus shows

$$\begin{aligned}
&d\Big(c\int_{-\tau}^0\int_{t+s}^t \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&\quad + |g(x(v), r(v), v)|^2\Big]dvds\Big) \\
&= \Big(c\tau\Big[\tau|f(x(t), r(t), t) + u(x(\delta_v), r(v), v)|^2 \\
&\quad + |g(x(t), r(t), t)|^2\Big] \\
&\quad - c\int_{t-\tau}^t \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&\quad + |g(x(v), r(v), v)|^2\Big]dv\Big)dt. \qquad (36)
\end{aligned}$$

Summing (35) and (36) yields

$$\begin{aligned}
&dV(\hat{x}_t, \hat{r}_t, t) = \mathcal{L}U(x(t), x(\delta_t), r(t), t)dt + dM(t) \\
&+ \Big(c\tau\Big[\tau|f(x(t), r(t), t) + u(x(\delta_v), r(v), v)|^2 \\
&\quad + |g(x(t), r(t), t)|^2\Big] \\
&- c\int_{t-\tau}^t \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&\quad + |g(x(v), r(v), v)|^2\Big]dv\Big)dt. \qquad (37)
\end{aligned}$$

That is, $V(\hat{x}_t, \hat{r}_t, t)$ is an Itô process as claimed. Furthermore, it is easy to see that

$$\begin{aligned}
&\mathcal{L}U(x,y,i,t) \leq \\
&LU(x,i,t) + [2\theta_i + (q_i+1)\bar{\theta}_i|x|^{q_1-1}]x^T[u(y,i,t) - u(x,i,t)],
\end{aligned}$$

where the function LU has been defined by (26). It then follows from (37) that

$$dV(\hat{x}_t, \hat{r}_t, t) \leq \mathbb{L}V(\hat{x}_t, \hat{r}_t, t)dt + dM(t), \qquad (38)$$

where

$$\begin{aligned}
&\mathbb{L}V(\hat{x}_t, \hat{r}_t, t) = LU(x(t), r(t), t) \\
&+ [2\theta_{r(t)} + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1-1}]x^T(t)[u(x(\delta_t), r(t), t) \\
&- u(x(t), r(t), t)] \\
&+ c\tau\Big[\tau|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^2 \\
&+ |g(x(t), r(t), t)|^2\Big] \\
&- c\int_{t-\tau}^t \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&+ |g(x(v), r(v), v)|^2\Big]dv. \qquad (39)
\end{aligned}$$

Moreover, by Theorem 3.1 and Assumptions 2.1 and 2.3, it is straightforward to see that

$$\sup_{0 \leq t < \infty} \mathbb{E}|\mathbb{L}V(\hat{x}_t, \hat{r}_t, t)| < \infty. \qquad (40)$$

Step 2. Let us now estimate $\mathbb{L}V(\hat{x}_t, \hat{r}_t, t)$. Let $c = \kappa^2/\gamma_1$. (Please recall that c is a free parameter in the definition of the Lyapunov functional.) By Assumption 2.3, we have

$$\begin{aligned}
&[2\theta_{r(t)} + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1-1}]x^T(t)[u(x(\delta_t), r(t), t) \\
&- u(x(t), r(t), t)] \\
&\leq \gamma_1\big[2\theta_{r(t)}|x(t)| + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1}\big]^2 \\
&+ \frac{\kappa^2}{4\gamma_1}|x(t) - x(\delta_t)|^2. \qquad (41)
\end{aligned}$$

By Condition (4.3), we also have

$$2c\tau^2 \leq \gamma_2 \quad \text{and} \quad c\tau \leq \gamma_3. \qquad (42)$$

It then follows from (39) along with Condition 4.2 and inequality (12) that

$$\begin{aligned}
&\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \leq LU(x(s), r(s), s) \\
&+ \gamma_1\big[2\theta_{r(s)}|x(s)| + (q_1+1)\bar{\theta}_{r(s)}|x(s)|^{q_1}\big]^2 \\
&+ \gamma_2|f(x(s), r(s), s)|^2 + \gamma_3|g(x(s), r(s), s)|^2 \\
&+ \frac{2\kappa^4\tau^2}{\gamma_1}|x(\delta_s)|^2 + \frac{\kappa^2}{4\gamma_1}|x(s) - x(\delta_s)|^2 \\
&- \frac{\kappa^2}{\gamma_1}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&+ |g(x(v), r(v), v)|^2\Big]dv \\
&\leq -\gamma_4|x(s)|^2 + \frac{2\tau^2\kappa^4}{\gamma_1}|x(\delta_s)|^2 + \frac{\kappa^2}{4\gamma_1}|x(s) - x(\delta_s)|^2 \\
&- \frac{\kappa^2}{\gamma_1}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 \\
&+ |g(x(v), r(v), v)|^2\Big]dv.
\end{aligned}$$

But, noting $\kappa\tau \leq 1/4$ from Condition 4.3, we have

$$\frac{2\tau^2\kappa^4}{\gamma_1}|x(\delta_s)|^2 \leq \frac{4\tau^2\kappa^4}{\gamma_1}|x(s)|^2 + \frac{\kappa^2}{4\gamma_1}|x(s) - x(\delta_s)|^2.$$

Consequently,

$$\mathbb{L}V(\hat{x}_s,\hat{r}_s,s) \le -\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)|x(s)|^2 + \frac{\kappa^2}{2\gamma_1}|x(s)-x(\delta_s)|^2 - \frac{\kappa^2}{\gamma_1}\int_{s-\tau}^{s}\Big[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2 + |g(x(v),r(v),v)|^2\Big]dv. \quad (43)$$

Step 3. Fix the initial value x_0 arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $|x_0| < k_0$. For each integer $k \ge k_0$, define the stopping time

$$\zeta_k = \inf\{t \ge 0 : |x(t)| \ge k\},$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). By Theorem 3.1, we see that ζ_k is increasing to infinity with probability 1 as $k \to \infty$. By the generalised Itô formula (see, e.g., [21, Lemma 1.9 on page 49]), we obtain from (38) that

$$\mathbb{E}V(\hat{x}_{t\wedge\zeta_k},\hat{r}_{t\wedge\zeta_k},t\wedge\zeta_k) = V(\hat{x}_0,\hat{r}_0,0) + \mathbb{E}\int_0^{t\wedge\zeta_k}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)ds \quad (44)$$

for any $t \ge 0$ and $k \ge k_0$. Recalling (40), we can let $k \to \infty$ and then apply the dominated convergence theorem as well as the Fubini theorem to get

$$\mathbb{E}V(\hat{x}_t,\hat{r}_t,t) = V(\hat{x}_0,\hat{r}_0,0) + \int_0^t \mathbb{E}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s)ds \quad (45)$$

for any $t \ge 0$. By (43), we have

$$\mathbb{E}\mathbb{L}V(\hat{x}_s,\hat{r}_s,s) \le -\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)\mathbb{E}|x(s)|^2 + \frac{\kappa^2}{2\gamma_1}\mathbb{E}|x(s)-x(\delta_s)|^2 - \frac{\kappa^2}{\gamma_1}\mathbb{E}\int_{s-\tau}^{s}\Big[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2 + |g(x(v),r(v),v)|^2\Big]dv. \quad (46)$$

On the other hand, it follows from the SDDE (10) that

$$\begin{aligned}&\mathbb{E}|x(s)-x(\delta_s)|^2\\ =&\mathbb{E}\Big|\int_{\delta_s}^{s}[f(x(v),r(v),v)+u(x(\delta_v),r(v),v)]dv + \int_{\delta_s}^{s} g(x(v),r(v),v)dB(v)\Big|^2\\ \le&2\mathbb{E}\int_{\delta_s}^{s}\Big(\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2 + |g(x(v),r(v),v)|^2\Big)dv\\ \le&2\mathbb{E}\int_{s-\tau}^{s}\Big(\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2 + |g(x(v),r(v),v)|^2\Big)dv. \quad (47)\end{aligned}$$

Substituting (43) into (45) yields

$$\mathbb{E}V(\hat{x}_t,\hat{r}_t,t) \le V(\hat{x}_0,\hat{r}_0,0) - \Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)\int_0^t \mathbb{E}|x(s)|^2 ds. \quad (48)$$

By Condition (4.3), $\gamma_4 - 4\tau^2\kappa^4/\gamma_1 > 0$. Hence

$$\int_0^t \mathbb{E}|x(s)|^2 ds \le \frac{\gamma_1 V(\hat{x}_0,\hat{r}_0,0)}{\gamma_4\gamma_1 - 4\tau^2\kappa^4}.$$

Letting $t \to \infty$ we obtain that

$$\int_0^\infty \mathbb{E}|x(s)|^2 ds \le \frac{\gamma_1 V(\hat{x}_0,\hat{r}_0,0)}{\gamma_4\gamma_1 - 4\tau^2\kappa^4} \quad (49)$$

as required. The proof is therefore complete. $\Box$

In general, it does not follow from (33) that $\lim_{t\to\infty}\mathbb{E}|x(t)|^2 = 0$. However, in our case, this is possible. In fact, we can show a stronger result that $\lim_{t\to\infty}\mathbb{E}|x(t)|^{\bar{q}} = 0$ for any $\bar{q} \in [2,q)$. We state this as our second theorem in this section.

Theorem 4.5: Under the same conditions of Theorem 4.4, the solution of the controlled hybrid SDDE (10) has the property that for any $\bar{q} \in [2,q)$ and any initial value $x_0 \in R^n$,

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^{\bar{q}} = 0. \quad (50)$$

That is, the controlled system (10) is asymptotically stable in $L^{\bar{q}}$.

Proof. Fix the initial vale $x_0 \in R^n$ arbitrarily. By Theorem 3.1,

$$C_1 := \sup_{0\le t<\infty}\mathbb{E}|x(t)|^q < \infty. \quad (51)$$

For any $0 \le t_1 < t_2 < \infty$, the Itô formula shows

$$\begin{aligned}&\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\\ =&\mathbb{E}\int_{t_1}^{t_2}\Big(2x^T(t)[f(x(t),r(t),t)+u(x(\delta_t),r(t),t)] + |g(x(t),r(t),t)|^2\Big)dt.\end{aligned}$$

By conditions (5) and (12), we see

$$\begin{aligned}&\big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\big|\\ \le&\mathbb{E}\int_{t_1}^{t_2}\Big(2|x(t)|\big[K(|x(t)|+|x(t)|^{q_1}) + \kappa|x(\delta_t)|\big] + K^2\big[|x(t)|+|x(t)|^{q_2}\big]^2\Big)dt\\ \le&\int_{t_1}^{t_2} C_2\big(1+\mathbb{E}|x(t)|^q + \mathbb{E}|x(\delta_t)|^q\big)dt,\end{aligned}$$

where C_2 is a constant independent of t_1 and t_2. This, together with (51), implies

$$\big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\big| \le C_2(1+2C_1)(t_2-t_1).$$

That is, $\mathbb{E}|x(t)|^2$ is uniformly continuous in t on R_+. It then follows from (33) that

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^2 = 0. \quad (52)$$

That is, the assertion (50) holds when $\bar{q} = 2$. Let us now fix any $\bar{q} \in (2,q)$. For a constant $\sigma \in (0,1)$, the Hölder inequality shows

$$\begin{aligned}\mathbb{E}|x(t)|^{\bar{q}} &= \mathbb{E}\big(|x(t)|^{2\sigma}|x(t)|^{\bar{q}-2\sigma}\big)\\ &\le \big(\mathbb{E}|x(t)|^2\big)^{\sigma}\big(\mathbb{E}|x(t)|^{(\bar{q}-2\sigma)/(1-\sigma)}\big)^{1-\sigma}.\end{aligned}$$

In particular, letting $\sigma = (q-\bar{q})/(q-2)$, we get

$$\begin{aligned}\mathbb{E}|x(t)|^{\bar{q}} &\le \big(\mathbb{E}|x(t)|^2\big)^{(q-\bar{q})/(q-2)}\big(\mathbb{E}|x(t)|^q\big)^{(\bar{q}-2)/(q-2)} \\ &\le C_1^{(\bar{q}-2)/(q-2)}\big(\mathbb{E}|x(t)|^2\big)^{(q-\bar{q})/(q-2)}. \qquad (53)\end{aligned}$$

This, along with (52), implies the required assertion (50). □

Theorem 4.4 shows that it is possible to design a control function for the controlled system (10) to become H_∞-stable in L^2. We now show it is also possible to make the controlled system become H_∞-stable in $L^{\bar{q}}$ for some $\bar{q} > 2$. For this purpose, we will replace Condition 4.2 by the following one.

Condition 4.6: Find five positive constants γ_j, $j = 1, \cdots, 5$, such that

$$\begin{aligned}&LU(x,i,t) + \gamma_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ &+ \gamma_2|f(x,i,t)|^2 + \gamma_3|g(x,i,t)|^2 \\ &\le -\gamma_4|x|^2 - \gamma_5|x|^{q_1+1} \qquad (54)\end{aligned}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Recalling the paragraph below Condition 4.2, in particular, inequality (31), we see it is always possible to find such five positive constants provided the control function u meets condition 4.1 under our standing Assumptions 2.1–2.3.

Theorem 4.7: Under Assumptions 2.1, 2.2 and 2.3, we can design the control function u to satisfy Condition 4.1 and then choose five positive constants γ_j, $j = 1, \cdots, 5$, to meet condition 4.6. If we further make sure τ to be sufficiently small for Condition 4.3 to hold, then the solution of the controlled system (10) has the property that for any $\bar{q} \in [2, q_1+1]$ and any initial value $x(0) = x_0 \in R^n$,

$$\int_0^\infty \mathbb{E}|x(t)|^{\bar{q}} dt < \infty. \qquad (55)$$

That is, the controlled system (10) is H_∞-stable in $L^{\bar{q}}$ for any $\bar{q} \in [2, q_1+1]$.

Proof. We use the same notation as in the proof of Theorem 4.4. Bearing in mind of our new Condition 4.6, we can see from the proof there that

$$\begin{aligned}&\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \le -\gamma_5|x(s)|^{q_1+1} \\ &-\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)|x(s)|^2 - \frac{\kappa^2}{2\gamma_1}|x(s)-x(\delta_s)|^2 \\ &-\frac{\kappa^2}{\gamma_1}\int_{s-\tau}^s \Big[\tau|f(x(v),r(v),v) + u(x(\delta_v),r(v),v)|^2 \\ &+ |g(x(v),r(v),v)|^2\Big]dv \qquad (56)\end{aligned}$$

instead of (43). We can then further have

$$\begin{aligned}&\mathbb{E}V(\hat{x}_t, \hat{r}_t, t) \le V(\hat{x}_0, \hat{r}_0, 0) - \gamma_5\int_0^t \mathbb{E}|x(s)|^{q_1+1}ds \\ &-\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)\int_0^t \mathbb{E}|x(s)|^2 ds \qquad (57)\end{aligned}$$

instead of (48). It then follows easily that

$$\int_0^\infty \mathbb{E}(|x(s)|^2 + |x(s)|^{q_1+1})ds < \infty.$$

But for any $\bar{q} \in [2, q_1+1]$, $|x(s)|^{\bar{q}} \le |x(s)|^2 + |x(s)|^{q_1+1}$. We hence obtain the required assertion (55). The proof is complete. □

V. Exponential Stabilisation

In the previous section we have shown that under Assumptions 2.1-2.3, it is possible to design a feedback control based on the discrete-time state observations to make the controlled system (10) become H_∞-stable in $L^{\bar{q}}$ ($\bar{q} \in [2, q_1+1]$ or asymptotic stable in $L^{\bar{q}}$ ($\bar{q} \in [2,q)$). Although both stabilities are important and widely used in applications, they do not reveal the rate at which the solution tends to zero. In this section, we will further show that it is also possible to design a feedback control based on the discrete-time state observations to make the controlled system (10) become exponentially stable either in $L^{\bar{q}}$ ($\bar{q} \in [2,q)$) or almost surely. For this purpose, we need to replace Conditions 4.2 and 4.3 by stronger conditions.

Condition 5.1: Find five positive constants γ_j, $j = 1,2,3,4,5$, such that

$$\begin{aligned}&LU(x,i,t) + \gamma_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ &+ \gamma_2|f(x,i,t)|^2 + \gamma_3|g(x,i,t)|^2 \\ &\le -\gamma_4|x|^2 - \gamma_5|x|^{p+q_1-1} \qquad (58)\end{aligned}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Condition 5.2: Make sure the duration between the two consecutive state observations satisfies

$$\tau < \frac{\sqrt{\gamma_4\gamma_1}}{2\kappa^2} \quad \text{and} \quad \tau \le \frac{\sqrt{\gamma_1\gamma_2}}{\sqrt{2}\kappa} \wedge \frac{\gamma_1\gamma_3}{\kappa^2} \wedge \frac{1}{4\sqrt{2}\kappa}. \qquad (59)$$

We should point out that the last term $1/4\kappa$ in (32) is now replaced by $1/4\sqrt{2}\kappa$ in (59) so the bound on τ here could be smaller than before. We should also point out that it is always possible to meet Condition 5.1 under Assumption 2.1 - 2.3. For example, if we choose positive constants γ_1-γ_3 sufficiently small for

$$0.5(q_1+1)\min_{i\in S}\bar{\theta}_i\bar{\alpha}_i \ge 2\gamma_1(q_1+1)^2\max_{i\in S}\bar{\theta}_i^2 + 2K^2(\gamma_2+\gamma_3)$$

and

$$0.5 \ge 4\gamma_1\max_{i\in S}\theta_i^2 + 2\gamma_1(q_1+1)^2\max_{i\in S}\bar{\theta}_i^2 + 4K^2(\gamma_2+\gamma_3),$$

it then follows from (30) that

$$\begin{aligned}&LU(x,i,t) + \gamma_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ &+ \gamma_2|f(x,i,t)|^2 + \gamma_3|g(x,i,t)|^2 \\ &\le -0.5|x|^2 - \big(0.5(q_1+1)\min_{i\in S}\bar{\theta}_i\bar{\alpha}_i\big)|x|^{p+q_1-1},\end{aligned}$$

namely we can have $\gamma_4 = 0.5$ and $\gamma_5 = 0.5(q_1+1)\min_{i\in S}\bar{\theta}_i\bar{\alpha}_i$. In application, we naturally need to make full use of the special forms of both coefficients f and g to choose γ_1 - γ_5 more wisely.

Theorem 5.3: Under Assumptions 2.1,2.2 and 2.3, we can design the control function u to satisfy Condition 4.1 and then choose five positive constants γ_j, $j = 1,2,3,4,5$, to meet condition 5.1. If we further make sure τ to be sufficiently small for Condition (59) to hold, then the solution of the controlled system (10) has the property that for any $\bar{q} \in [2,q)$ and any initial value $x(0) = x_0 \in R^n$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^{\bar{q}}) < 0. \qquad (60)$$

That is, the controlled system (10) is exponentially stable in $L^{\bar{q}}$.

Proof. We will use the same Lyapunov functional $V(\hat{x}_t, \hat{r}_t, t)$ as defined by (34) with the same $c = \kappa^2/\gamma_1$. Fix any initial value $x_0 \in R^n$. By the method of stopping times as we did in Step 3 of the proof of Theorem 4.4, we can show that

$$e^{\varepsilon t}\mathbb{E}V(\hat{x}_t, \hat{r}_t, t) \le V(\hat{x}_0, \hat{r}_0, 0) + \int_0^t e^{\varepsilon s}\mathbb{E}\Big(\varepsilon V(\hat{x}_s, \hat{r}_s, s) + \mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\Big)ds \tag{61}$$

for all $t \ge 0$, where ε is a sufficiently small positive number to be determined later. Setting

$$a_1 = \min_{i\in S}\theta_i, \quad a_2 = \max_{i\in S}\theta_i, \quad a_3 = \max_{i\in S}\bar{\theta}_i,$$

we then have

$$a_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le V(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\kappa^2}{\gamma_1}\Psi_1(t) + \int_0^t e^{\varepsilon s}\Big(\varepsilon a_2\mathbb{E}|x(s)|^2 + \varepsilon a_3\mathbb{E}|x(s)|^{q_1+1} + \mathbb{E}\mathbb{L}V(\hat{x}_s, \hat{r}_s, s)\Big)ds, \tag{62}$$

where

$$\Psi_1(t) = \mathbb{E}\int_0^t e^{\varepsilon s}\Big(\int_{-\tau}^0\int_{s+u}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dvdu\Big)ds.$$

As we did in Step 2 of the proof of Theorem 4.4, we can show that

$$\begin{aligned}\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \le& -\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)|x(s)|^2 - \gamma_5|x(s)|^{p+q_1-1} + \frac{3\kappa^2}{8\gamma_1}|x(s) - x(\delta_s)|^2\\ &- \frac{\kappa^2}{\gamma_1}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv.\end{aligned} \tag{63}$$

Making use of (47), we get

$$\begin{aligned}\mathbb{E}\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \le& -\Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}\Big)\mathbb{E}|x(s)|^2 - \gamma_5\mathbb{E}|x(s)|^{p+q_1-1}\\ &- \frac{\kappa^2}{4\gamma_1}\mathbb{E}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv.\end{aligned} \tag{64}$$

Moreover, we clearly have

$$\mathbb{E}|x(s)|^{q_1+1} \le \mathbb{E}|x(s)|^2 + \mathbb{E}|x(s)|^{p+q_1-1}. \tag{65}$$

Substituting (64) and (65) into (62) yields

$$\begin{aligned}a_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le& V(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\kappa^2}{\gamma_1}\Psi_1(t) - \frac{\kappa^2}{4\gamma_1}\Psi_2(t)\\ &- \Big(\gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1} - \varepsilon a_2 - \varepsilon a_3\Big)\int_0^t e^{\varepsilon s}\mathbb{E}|x(s)|^2 ds\\ &- (\gamma_5 - \varepsilon a_3)\int_0^t e^{\varepsilon s}\mathbb{E}|x(s)|^{p+q_1-1}ds,\end{aligned} \tag{66}$$

where

$$\Psi_2(t) = \mathbb{E}\int_0^t e^{\varepsilon s}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv.$$

On the other hand, it is easy to see that

$$\begin{aligned}\Psi_1(s) &\le \mathbb{E}\int_0^t e^{\varepsilon s}\Big(\tau\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv\Big)ds\\ &= \tau\Psi_2(t).\end{aligned}$$

We can now choose $\varepsilon > 0$ so small for

$$\varepsilon\tau \le \frac{1}{4}, \quad \varepsilon(a_2 + a_3) \le \gamma_4 - \frac{4\tau^2\kappa^4}{\gamma_1}, \quad \varepsilon a_3 \le \gamma_5.$$

Consequently, we obtain from (66) that

$$\mathbb{E}|x(t)|^2 \le \big(V(\hat{x}_0, \hat{r}_0, 0)/a_1\big)e^{-\varepsilon t}, \quad \forall t \ge 0. \tag{67}$$

Finally, for any $\bar{q} \in [2, q)$, by (53) and (67), we get

$$\mathbb{E}|x(t)|^{\bar{q}} \le C_1^{(\bar{q}-2)/(q-2)}\big(V(\hat{x}_0, \hat{r}_0, 0)/a_1\big)^{(q-\bar{q})/(q-2)}e^{-\varepsilon t(q-\bar{q})/(q-2)}. \tag{68}$$

This implies the required assertion (60). The proof is complete.

In general, it is not possible to imply the almost surely exponential stability from the $\bar{q}$th moment exponential stability. However, in our situation, this is possible as described in the following theorem.

Theorem 5.4: Let all the conditions of Theorem 5.3 hold. Then the solution of the controlled system (10) has the property that any initial value $x(0) = x_0 \in R^n$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) < 0 \quad a.s. \tag{69}$$

That is, the controlled system (10) is almost surely exponentially stable.

Proof. Fix the initial vale $x_0 \in R^n$ arbitrarily. Let $t_k = k\tau$ for $k = 0, 1, 2, \cdots$. By the Itô formula and the Burkholder-Davis-Gundy inequality (see, e.g., [21, pp.70–76]), we can show that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big) \le& \mathbb{E}|x(t_k)|^2 + \mathbb{E}\int_{t_k}^{t_{k+1}}\Big(2|x(t)||f(x(t), r(t), t) + u(x(\delta_t), r(t), t)| + |g(x(t), r(t), t)|^2\Big)dt\\ &+6\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}|x(t)|^2|g(x(t), r(t), t)|^2 dt\Big)^{1/2}.\end{aligned}$$

But

$$\begin{aligned}&6\mathbb{E}\Big(\int_{t_k}^{t_{k+1}}|x(t)|^2|g(x(t), r(t), t)|^2 dt\Big)^{1/2}\\ \le&6\mathbb{E}\Big[\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|\Big)\Big(\int_{t_k}^{t_{k+1}}|g(x(t), r(t), t)|^2 dt\Big)^{1/2}\Big]\\ \le&0.5\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big) + 18\mathbb{E}\int_{t_k}^{t_{k+1}}|g(x(t), r(t), t)|^2 dt.\end{aligned}$$

Hence

$$\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le 2\mathbb{E}|x(t_k)|^2 + \mathbb{E}\int_{t_k}^{t_{k+1}}\Big(4|x(t)||f(x(t),r(t),t)+u(x(\delta_t),r(t),t)| + 38|g(x(t),r(t),t)|^2\Big)dt. \qquad (70)$$

Let $\bar{q}=(q_1+1)\vee(2q_2)$. Recalling (6) and $q_1>1$, we see $\bar{q}\in[2,q)$. By Assumption 2.1, it is almost straightforward to show from (70) that

$$\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le 2\mathbb{E}|x(t_k)|^2 + C_3\int_{t_k}^{t_{k+1}}\Big(\mathbb{E}|x(t)|^2+\mathbb{E}|x(\delta_t)|^2+\mathbb{E}|x(t))|^{\bar{q}}\Big)dt,$$

where C_3 and the following C_4 are all positive constants independent of k. Using (67) and (68), we hence have

$$\mathbb{E}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|^2\Big)\le C_4e^{-\bar{\varepsilon}t_k},$$

where $\bar{\varepsilon}=\varepsilon(q-\bar{q})/(q-2)$. Consequently

$$\sum_{k=0}^{\infty}\mathbb{P}\Big(\sup_{t_k\le t\le t_{k+1}}|x(t)|>e^{-0.25\bar{\varepsilon}t_k}\Big)\le\sum_{k=0}^{\infty}C_4e^{-0.5\bar{\varepsilon}t_k}<\infty.$$

The well-known Borel-Cantelli lemma (see, e.g., [21, p.10]) shows that for almost all $\omega\in\Omega$, there is positive integer $k_0=k_0(\omega)$ such that

$$\sup_{t_k\le t\le t_{k+1}}|x(t)|\le e^{-0.25\bar{\varepsilon}t_k},\quad k\ge k_0.$$

Hence, for almost all $\omega\in\Omega$,

$$\frac{1}{t}\log(|x(t)|)\le-\frac{0.25\bar{\varepsilon}\tau k}{\tau(k+1)},\quad t\in[t_k,t_{k+1}],\ k\ge k_0.$$

This implies

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)\le-0.25\bar{\varepsilon}<0\quad a.s.$$

which is the assertion. The proof is complete.

VI. Examples

To illustrate our theoretical results, we will discuss a couple of examples.

Example 6.1: Let us consider a scalar hybrid SDE

$$dx(t)=f(x(t),r(t),t)dt+g(x(t),r(t),t)dB(t), \qquad (71)$$

where the coefficients f and g are defined by

$$f(x,1,t)=x-3x^3,\ f(x,2,t)=x-2x^3,$$
$$g(x,1,t)=|x|^{3/2},\ g(x,2,t)=0.5|x|^{3/2}, \qquad (72)$$

$B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S=\{1,2\}$ with its generator

$$\Gamma=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}. \qquad (73)$$

This is a simple version of hybrid SDE models appeared frequently in finance and population systems (see, e.g., [2], [10], and [4] for more on highly nonlinear hybrid SDEs).

Recalling the discussions after Assumptions 2.1 and 2.2, we know that the SDE (71) satisfies Assumptions 2.1 and 2.2 with any large q and $p=4$, $\alpha=1$, $\beta=1+0.5(q-1)^2$, $q_1=3$ and $q_2=1.5$.

We first consider the case where the system is fully observable and controllable in both mode 1 and 2. That is, we could use a feedback control in both modes to stabilise the given unstable hybrid SDE (71). In our notation, we will use the control function $u:R\times S\times R_+\to R$ define by

$$u(x,1,t)=-3x,\quad u(x,2,t)=-2x. \qquad (74)$$

Obviously, Assumption 2.3 is satisfied with $\kappa=3$. By Theorem 3.1, the controlled system

$$dx(t)=[f(x(t),r(t),t)+u(x(\delta_t),r(t),t)]dt + g(x(t),r(t),t)dB(t) \qquad (75)$$

has a unique global solution on $t\ge0$ for any initial value $x_0\in R$ and the solution has the property that

$$\sup_{0\le t<\infty}\mathbb{E}|x(t)|^q<\infty\quad\forall q>2. \qquad (76)$$

Let us now verify Condition 4.1. It is straightforward to show that, for $(x,i,t)\in R\times S\times R_+$,

$$x[f(x,i,t)+u(x,t,i)]+\frac{1}{2}|g(x,t,i)|^2 \le\begin{cases}-2.75x^4-1.75x^2 & \text{if } i=1,\\ -1.9375x^4-0.9375x^2 & \text{if } i=2,\end{cases}$$

and

$$x[f(x,i,t)+u(x,t,i)]+\frac{q_1}{2}|g(x,t,i)|^2 \le\begin{cases}-2.25x^4-x^2 & \text{if } i=1,\\ -1.8125x^4-0.8125x^2 & \text{if } i=2.\end{cases}$$

Namely, (21) and (22) hold with

$$\alpha_1=2.75,\ \beta_1=-1.75,\ \alpha_2=1.9375,\ \beta_2=-0.9375$$

and

$$\bar{\alpha}_1=2.25,\ \bar{\beta}_1=-1,\ \bar{\alpha}_2=1.8125,\ \bar{\beta}_2=-0.8125$$

respectively. Moreover,

$$\mathcal{A}_1=\begin{pmatrix}4.5 & -1\\ -1 & 2.875\end{pmatrix}\ \text{ and }\ \mathcal{A}_2=\begin{pmatrix}5 & -1\\ -1 & 4.25\end{pmatrix},$$

which are both M-matrices. That is, Condition 4.1 is satisfied. By (24), we then have

$$\theta_1=0.3246073,\ \theta_2=0.4607330,$$
$$\bar{\theta}_1=0.2592593,\ \bar{\theta}_2=0.2962963.$$

The function U defined by (25) becomes

$$U(x,i)=\begin{cases}0.3246073x^2+0.2592593x^4 & \text{if } i=1,\\ 0.4607330x^2+0.2962963x^4 & \text{if } i=2.\end{cases}$$

By (27), we also have

$$LU(x,i,t)\le\begin{cases}-2.78534x^4-x^2-2.333334x^6 & \text{if } i=1,\\ -2.78534x^4-x^2-2.148148x^6 & \text{if } i=2.\end{cases}$$

Choosing $\gamma_1 = 0.5$, $\gamma_2 = 0.1$ and $\gamma_3 = 1$, we can then further show (by elementary calculations) that

$$\begin{aligned} &LU(x,i,t) + \gamma_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ &+ \gamma_2|f(x,i,t)|^2 + \gamma_3|g(x,i,t)|^2 \\ &\quad \le -0.4442002x^2 - 0.895611x^6. \end{aligned} \tag{77}$$

That is, Condition 5.1 is satisfied with additional $\gamma_4 = 0.4442002$ and $\gamma_5 = 0.895611$. Consequently, Condition 5.2 becomes $\tau < 0.02618194$. By Theorems 5.3 and 5.4, we can therefore conclude that the controlled system (75) with the control function (74) is not only exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \ge 2$ but also almost surely provided $\tau < 0.02618194$.

We perform a computer simulation with $\tau = 0.02$ and the initial vale $x(0) = 1$ and $r(0) = 1$. The sample paths of the Markov chain and the solution of the SDDE (75) are plotted in Figure 6.1. The simulation supports our theoretical results clearly.

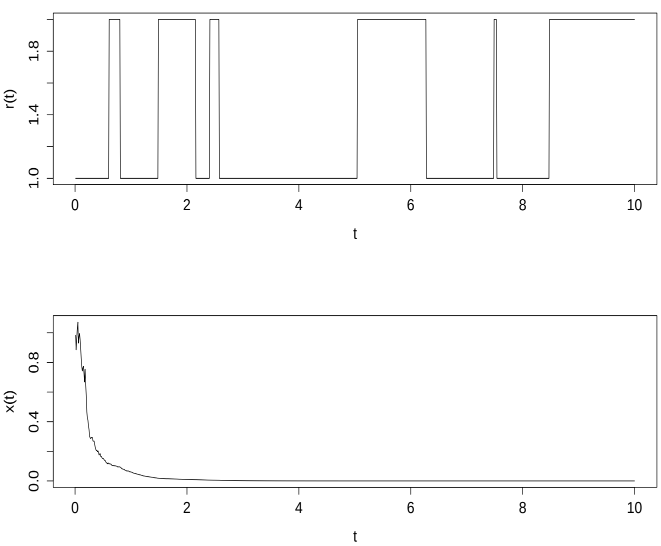

Figure 6.1: The computer simulation of the sample paths of the Markov chain and the SDDE (75) with the control function (74) and $\tau = 0.02$ using the Euler–Maruyama method with step size 10^{-4}.

Example 6.2: We continue with the hybrid SDE (71) but consider the case where the system is observable only in mode 1 but not in mode 2 so we could only use a feedback control in mode 1 (namely the system is not controllable or observable in mode 2 so have to set the control function to be 0 in mode 2). As the system is not controllable in mode 2, we will need to assume that the system will switch to mode 1 from 2 sufficiently faster than that from mode 1 to 2. Accordingly, instead of (73), we now assume the Markov chain $r(t)$ has the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}. \tag{78}$$

Moreover, we now design the control function

$$u(x,1,t) = -4x, \quad u(x,2,t) = 0. \tag{79}$$

Obviously, Assumption 2.3 is satisfied with $\kappa = 4$. The global solution of the controlled system (75) still has property (76). It is straightforward to show that, for $(x,i,t) \in R \times S \times R_+$,

$$\begin{aligned} &x[f(x,i,t) + u(x,t,i)] + \frac{1}{2}|g(x,t,i)|^2 \\ &\le \begin{cases} -2.75x^4 - 2.75x^2 & \text{if } i = 1, \\ -1.9375x^4 + 1.0625x^2 & \text{if } i = 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &x[f(x,i,t) + u(x,t,i)] + \frac{q_1}{2}|g(x,t,i)|^2 \\ &\le \begin{cases} -2.25x^4 - 2.25x^2 & \text{if } i = 1, \\ -1.8125x^4 + 1.1875x^2 & \text{if } i = 2. \end{cases} \end{aligned}$$

Namely, (21) and (22) hold with

$$\alpha_1 = 2.75,\ \beta_1 = -2.75,\ \alpha_2 = 1.9375,\ \beta_2 = 1.0625$$

and

$$\bar{\alpha}_1 = 2.25,\ \bar{\beta}_1 = -2.25,\ \bar{\alpha}_2 = 1.8125,\ \bar{\beta}_2 = 1.1875$$

respectively. Moreover,

$$\mathcal{A}_1 = \begin{pmatrix} 6.5 & -1 \\ -6 & 3.875 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_2 = \begin{pmatrix} 10 & -1 \\ -6 & 1.25 \end{pmatrix},$$

which are both M-matrices. That is, Condition 4.1 is satisfied. By (24), we then have

$$\theta_1 = 0.2540717,\ \theta_2 = 0.6514658,$$

$$\bar{\theta}_1 = 0.3461538,\ \bar{\theta}_2 = 2.4615385.$$

The function U defined by (25) becomes

$$U(x,i) = \begin{cases} 0.2540717x^2 + 0.3461538x^4 & \text{if } i = 1, \\ 0.6514658x^2 + 2.4615385x^4 & \text{if } i = 2. \end{cases}$$

By (27), we also have

$$LU(x,i,t) \le \begin{cases} -2.397394x^4 - x^2 - 3.115384x^6 & \text{if } i = 1, \\ -3.52443x^4 - x^2 - 17.84615x^6 & \text{if } i = 2. \end{cases}$$

Choosing $\gamma_1 = 0.1$, $\gamma_2 = 0.25$ and $\gamma_3 = 1$, we can then further show

$$\begin{aligned} &LU(x,i,t) + \gamma_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ &+ \gamma_2|f(x,i,t)|^2 + \gamma_3|g(x,i,t)|^2 \\ &\quad \le -0.4741703x^2 - 0.6736681x^6. \end{aligned} \tag{80}$$

That is, Condition 5.1 is satisfied with additional $\gamma_4 = 0.4741703$ and $\gamma_5 = 0.6736681$. Consequently, Condition 5.2 becomes $\tau < 0.00625$. By Theorems 5.3 and 5.4, we can therefore conclude that the controlled system (75) with the control function (79) is not only exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \ge 2$ but also almost surely provided $\tau < 0.00625$.

We perform a computer simulation with $\tau = 0.005$ and the initial vale $x(0) = 1$ and $r(0) = 1$. The sample paths of the Markov chain and the solution of the SDDE (75) are plotted in Figure 6.2. The simulation supports our theoretical results once again.

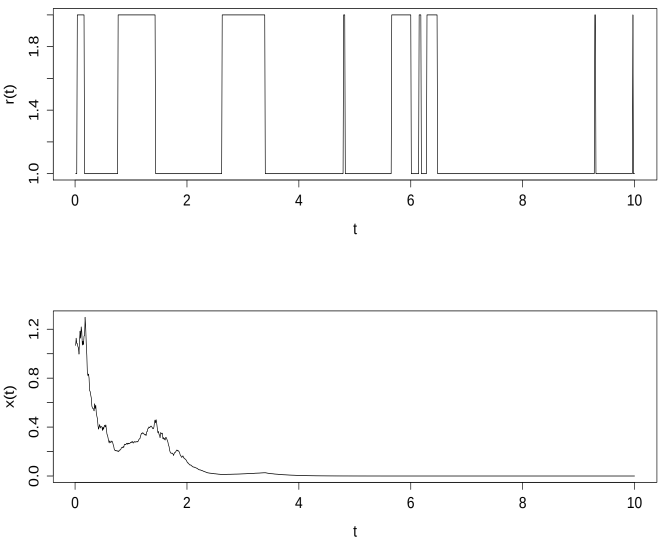

Figure 6.2: The computer simulation of the sample paths of the Markov chain and the SDDE (10) with the control function (78) and $\tau = 0.005$ using the Euler–Maruyama method with step size 10^{-4}.

VII. Conclusion

In this paper we have discussed the stabilisation of highly nonlinear hybrid SDEs by the feedback controls based on the discrete-time observations of the states. We pointed out that the existing results on the stabilisation of nonlinear hybrid SDEs require the coefficients of the underlying SDEs satisfy the linear growth condition. On the other hand, many hybrid SDE models in the real world do not fulfil this linear growth condition (namely, they are highly nonlinear). There is hence a need to develop a new theory on the stabilisation for the highly nonlinear SDE models. In this paper we consider a class of hybrid SDEs which are not stable but their solutions are bounded in qth moment. We then show that the controlled SDEs preserve the moment boundedness as long as the control functions satisfy the Lipschitz condition. We then show how to design the control functions more wisely so that the controlled SDEs become stable. The stability discussed in this paper include the H_∞-stable in $L^{\bar{q}}$, asymptotic stability in $\bar{q}$th moment, qth moment exponential stability and almost surely exponential stability. The key technique used is this paper is the method of Lyapunov functionals. A couple of examples and computer simulations have been used to illustrate our theory.

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References

[1] Ahlborn, A. and Parlitz, U., Stabilizing unstable steady states using multiple delay feedback control, *Physical Review Letters* **93**, 264101 (2004).

[2] Bahar A. and Mao X., Stochastic delay population dynamics, *Journal of International Applied Math.* **11(4)** (2004), 377–400.

[3] Cao, J., Li, H.X. and Ho, D.W.C., Synchronization criteria of Lur's systems with time-delay feedback control, *Chaos, Solitons and Fractals* **23** (2005) 1285–1298.

[4] Fei, W., Hu, L., Mao, X. and Shen, M., Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* **82** (2017), 65–70.

[5] Feng, J., Lam, J. and Shu, Z., Stabilization of Markovian systems via probability rate synthesis and output feedback, *IEEE Trans. Auto. Control* **55** (2010), 772–777.

[6] Hu, L., Mao, X. and Shen, Y., Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Systems & Control Letters* **62** (2013), 178–187.

[7] Ji, Y. and Chizeck, H.J., Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control* **35** (1990), 777–788.

[8] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, 1986.

[9] Ladde, G.S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, 1980.

[10] Lewis A.L., *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2000.

[11] Mao, X., *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[12] Mao, X., *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[13] Mao, X., *Stochastic Differential Equations and Their Applications*, 2nd Edition, Chichester: Horwood Pub., 2007.

[14] Mao, X., Stability of stochastic differential equations with Markovian switching, *Sto. Proc. Their Appl.* **79** (1999), 45–67.

[15] Mao, X., Exponential stability of stochastic delay interval systems with Markovian switching, *IEEE Trans. Auto. Control* **47(10)** (2002), 1604–1612.

[16] Mao, X., Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica* **49(12)** (2013), 3677–3681.

[17] Mao, X., Almost sure exponential stabilization by discrete-time stochastic feedback control, *IEEE Transactions on Automatic Control* **61(6)** (2016), 1619–1624.

[18] Mao, X., Liu, W., Hu, L., Luo,Q. and Lu, J., Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations, *Systems and Control Letters* **73** (2014), 88–95.

[19] Mao, X., Lam, J. and Huang, L., Stabilisation of hybrid stochastic differential equations by delay feedback control, *Systems & Control Letters* **57** (2008), 927–935.

[20] Mao, X., Matasov, A. and Piunovskiy, A.B., Stochastic differential delay equations with Markovian switching, *Bernoulli* **6(1)** (2000), 73–90.

[21] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[22] Mariton, M., *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.

[23] Mohammed, S.-E.A., *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1984.

[24] Pyragas, K., Control of chaos via extended delay feedback, *Physics Letters A* **206** (1995), 323–330.

[25] Qiu, Q., Liu, W., Hu, L., Mao, X. and You, S., Stabilisation of stochastic differential equations with Markovian switching by feedback control based on discrete-time state observation with a time delay, *Statistics and Probability Letters* **115** (2016), 16–26.

[26] Shaikhet, L., Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes* **2(18)** (1996), 180–184.

[27] Shi, P., Mahmoud, M.S., Yi, J. and Ismail, A., Worst case control of uncertain jumping systems with multi-state and input delay information, *Information Sciences* **176** (2006), 186–200.

[28] Sun, M., Lam, J., Xu, S. and Zou, Y., Robust exponential stabilization for Markovian jump systems with mode-dependent input delay, *Automatica* **43** (2007), 1799–1807.

[29] Wei, G., Wang, Z., Shu, H. and Fang, J., Robust H_∞ control of stochastic time-delay jumping systems with nonlinear disturbances, *Optim. Control Appl. Meth.* **27** (2006), 255–271.

[30] You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q., Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Control and Optimization* **53(2)** (2015), 905–925.

[31] Yue, D. and Han, Q., Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Trans. Automat. Control* **50** (2005), 217–222.

[32] Zhang, L, Boukas, E. K. and Lam, J., Analysis and synthesis of Markov jump linear systems with time-varying delays and partially known transition probabilities, *IEEE Trans. Autom. Control* **53** (2008), 2458–2464.

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