H∞ analysis and control of time-delay systems by methods in frequency domain

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ÉCOLE DOCTORALE
SCIENCES ET TECHNOLOGIES DE
L'INFORMATION DES TÉLÉCOMMUNICATIONS ET DES SYSTÈMES

Année : 2011 N° B.U.:

Thèse de Doctorat de l'Université Paris-Sud XI

Spécialité : PHYSIQUE

Présentée et soutenue publiquement par:

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le 28 juin 2011

à l'Université Paris-Sud XI

$\mathcal{H}_\infty$ ANALYSIS AND CONTROL OF TIME-DELAY SYSTEMS BY METHODS IN FREQUENCY DOMAIN

Jury

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Résumé

Dans cette thèse, nous considérons l'analyse et la commande $\mathcal{H}_\infty$ de systèmes continus à retards commensurables par des méthodes fréquentielles. Nous étudions tout d'abord le comportement asymptotique des chaînes de pôles et donnons des conditions de stabilité pour des systèmes de type neutre possédant des chaînes de pôles asymptotiques à l'axe imaginaire. La même analyse est effectuée dans le cas de systèmes fractionnaires. Nous proposons ensuite une méthode numérique qui fournit l'ensemble des fenêtres de stabilité ainsi que le lieu des racines instables pour des systèmes classiques et fractionnaires. Cette première partie de la thèse, dédiée à l'analyse, se termine par une étude des courbes de stabilité d'une classe de systèmes à retards distribués. Dans la deuxième partie de la thèse, qui s'intéresse à la synthèse, nous commençons par déterminer des contrôleurs PID pour des systèmes fractionnaires à retards, à l'aide du théorème du petit gain. Enfin, utilisant la substitution de Rekasius, nous construisons un système de comparaison linéaire invariant dans le temps qui nous fournit des informations sur la stabilité et la norme $\mathcal{H}_\infty$ de systèmes à retards classiques. Cette approche nous permet de mettre au point pour ces systèmes des contrôleurs à retour d'état ou de sortie, ainsi que des filtres linéaires.

Mots clés: Systèmes à retard, Stabilité $\mathcal{H}_\infty$, Systèmes fractionnaires, contrôleurs PID, Systèmes de comparaison

Abstract

This thesis addresses the $\mathcal{H}_\infty$ analysis and control of continuous commensurate time-delay systems by methods in frequency domain. First, the asymptotic behavior of the chains of poles are studied, and the conditions of stability for neutral systems with poles approaching the imaginary axis are given. The same analysis is done for fractional systems. In the sequel, a numerical method for locating all the stability windows as well as the unstable root-locus for classical and fractional system is given. We conclude the analysis part by providing the stability crossing curves of a class of distributed delay system. Starting the synthesis part, we design PID controllers for unstable fractional systems using a small-gain theorem approach. Finally, using the Rekasius substitution, we construct a linear time invariant comparison system that allows us to get information about stability and $\mathcal{H}_\infty$-norm for classical time-delay systems. Using this approach it is possible to design state and output feedback controllers, as well as linear filters for this class of systems.

Keywords: Time-Delay Systems, $\mathcal{H}_\infty$ Stability, Fractional Systems, PID Controllers, Comparison Systems.

“Je suis de ceux qui pensent que la Science a une grande beauté. Un savant dans son laboratoire n’est pas seulement un technicien, c’est aussi un enfant placé en face de phénomènes naturels qui l’impressionnent comme un conte de fées.”

Marie Curie

Acknowledgements

The idea of being unfair with someone is frightening, but nevertheless I'll try to express in the next few words my gratitude to all those who somehow helped me throughout all those years that took me to this path. Initially, I would like to thank to my advisor Catherine Bonnet. Her ubiquitous kindness and concern about every aspect of my stay in France, both in the professional and in the personal parts of my life, are remarkable. I cannot stress enough how much I owe her my deep gratitude.

Also, I would like to thank all members of the jury, who accepted to review my work and travel many hours only to be present in my thesis' defense. To Prof. Jean-Pierre Richard and Prof. Pierre Rouchon, I need to particularly thank for their reports and all the remarks that largely improved the presented text. To Prof. José Cláudio Geromel, Prof. Hitay Özbay and Prof. Jonathan Partington, people I had the great pleasure to spend some time and to work with during these last years, I want to thank them for their invaluable teachings and patience. Their great achievements and indefectible characters are the inspiration and the foundations of my scientific goals and personality. To Prof. Hugues Mounier and Prof. Jean-Jacques Loiseau, I hope our paths will cross again, and I thank them for all the valuable advice they gave to make this a much better work.

Likewise, I could not forget to thank in a special way all my family, who always supported me unconditionally in every possible way during my studies. For this reason, I would like to express my kind wishes to everyone, especially to my parents Beatriz and Hamilton, and my brother Rodrigo Fioravanti.

There were many other researchers that provided me guidance and motivation. To the director of L2S, Silviu Niculescu, I would like to thank for all the support and interesting discussions. Also, to all the members of the project INRIA-DISCO, Frédéric Mazenc, Alban Quadrat and David Avanessoff, I am glad to have had all those discussions during lunch-time. To Suat Gumussoy and Irinel Morãrescu, I am grateful for all the suggestions and projects we still have to develop in the near future.

To my colleagues Rubens Korogui, Alim Gonçalves and Grace Deaecto, I would like to thank all the valuable discussions and suggestions given in the last many years, even when the only possible way was throughout the virtual world. To Islam Boussaada and César Barrios, I appreciated all our moments of discussions and discoveries, even when the subject was not exactly related to our research.

As it is said, *"there is no distance too far between friends, for friendship gives wings to the heart"*. Therefore, I would like to thank all my friends that I had to move away from their everyday lives into this journey. To name just a few, I would like to thank Glauco Pereira and Rafael Vivaldi for keeping on with the same old spirit as we had during our times back into the engineering school. To Liza Cipolato, I appreciated every opportunity we had to discover together new things in the old world. I cannot also forget Mariana Ruske, with whom I shared many certitudes and doubts and Carolina Masiero, who I know is always supporting me in the best way possible even when she is not aware of it.

Whenever you are really far from anyone you know, only someone in the same situation can understand you well. Therefore, I am very happy to have met and shared many happy moments with Bhaskar Biswas, Chun Liu, Daria Kalinkina, Marcos Aurélio, Skander Banaouas, Yasser Toor and Yacine Mezali. Even with all the cultural differences (which sometimes were quite big for both sides), the opportunity to spend time with such nice people coming all from distinct places is a lifetime experience. I am also very grateful that I got to know Claire Mouton, who, aside classical music and general culture, also helped me a lot with the catchy *"langue française"*. I probably gave her more trouble than she would predict, including the correction of the French part of this manuscript. Any mistake still present was surely included by me after her corrections.

I would like to thank my good friend Pedro Kvieska. Over more than 10 years of adventures and misadventures, our friendship remains steadfast even beyond 9 time-zones. From the choice of a good wine to the secrets of Azeroth, going through all scientific, humanistic and humoristic subjects, I have discovered and still learn plenty with him. His ideas and advice play a major role in my life.

To my dear Viviane Gardin, I want to thank her for everything. Her faithfulness, patience and kindness are remarkable. I am very grateful for her ability to redefine some words I have long forgotten.

I would like to thank also the *Institute National de Recherche en Informatique et Automatique - INRIA*, the *Laboratoire de Signaux et Systèmes - L2S*, the *École supérieure d'Électricite - SUPELEC* and the *Université Paris-Sud XI* for providing such fruitful and pleasant environment for practicing science. Moreover, to all their employees, I express my gratitude. Particulary, I would like to thank the previous secretary of the project INRIA-DISCO, Valérie Berthou, for all the help she provided me.

And, with no further distinction, to all those who, even unnamed, have always shown their support for my endeavors, I want to sincerely say: *"Thank you"*.

Index

I Preliminaries **1**

1 Introduction **3**

1.1 Objectives . . . 5
1.2 Preliminaries . . . 5
1.3 Outline of the thesis . . . 9

2 Basic Results **11**

2.1 Classical Time-Delay Systems . . . 13
2.1.1 Class of Systems under consideration . . . 13
2.1.2 $\mathcal{H}_\infty$-stability . . . 13
2.2 Fractional Time-Delay Systems . . . 14
2.2.1 A Brief Historical Perspective on Fractional Calculus . . . 14
2.2.2 Modern Definition . . . 17
2.2.3 Laplace Transform . . . 17
2.2.4 Linear Fractional-Order Systems . . . 19
2.2.5 Stability of Linear Fractional-Order Systems . . . 20
2.2.6 Frequential Properties of Fractional Systems . . . 21
2.2.7 Stability of Linear Fractional-Order System with Delay . . . 23
2.2.8 Time-Varying Linear Systems . . . 24

II Analysis of Time Delay Systems **27**

3 Chains of Poles **29**

3.1 Introduction . . . 31
3.2 Classical Systems . . . 31
3.2.1 Classification of Chains of Poles . . . 32
3.2.2 Asymptotic Location of Neutral Poles of Time-Delay System . . . 36
3.2.3 $\mathcal{H}_\infty$-stability and stabilizability . . . 42
3.2.3.1 When all the roots of $\tilde{c}_d(z)$ are of multiplicity one . . . 45
3.2.3.2 When some roots of $\tilde{c}_d(z)$ are of multiplicity greater than one . . . 47
3.2.4 State-Space Representation . . . 50
3.2.5 The particular case of two delays with illustrative examples . . . 53
3.3 Fractional Systems . . . 60

3.3.1 Asymptotic Location of Neutral Poles of Time-Delay System 60
3.3.2 $\mathcal{H}_\infty$-stability 63
3.3.3 Examples 63
3.4 Final Remarks 66

4 Stability Windows and *Root-Locus* 69
4.1 Introduction 71
4.2 Classical Systems 72
4.2.1 Problem formulation 72
4.2.2 Stability for $\tau = 0$ 73
4.2.3 Location of Chains of Poles 73
4.2.4 Crossing Locations 74
4.2.5 Direction of Crossings 77
4.2.6 Location of Unstable Poles 78
4.2.7 Complete Algorithm 78
4.2.8 Examples 79
4.3 Fractional Systems 82
4.3.1 Problem formulation 82
4.3.2 Stability of fractional-order systems with delay 82
4.3.3 Stability for $\tau = 0$ 84
4.3.4 Location of Chains of Poles 84
4.3.5 Crossing Position 84
4.3.6 Direction of Crossing 86
4.3.7 Location of Unstable Poles 87
4.3.8 Examples 87
4.4 Final Remarks 89

5 Stability Crossing Curves of Shifted Gamma-Distributed Delay Systems 91
5.1 Introduction 93
5.2 Problem Definition 94
5.3 Stability crossing curves characterization 95
5.3.1 Identification of the crossing set 96
5.3.2 Identification of crossing points 97
5.3.3 Classification of the crossing curves 97
5.4 Smoothness of the crossing curves and crossing direction 99
5.5 Examples 101
5.6 Concluding remarks 102

III Control of Time-Delay Systems 107

6 PID Controller Design 109
6.1 Introduction 111
6.2 Problem Definition 111

6.3 PID controller design . . . 113
6.3.1 PD controller design . . . 113
6.3.2 Adding integral action to the PD controller . . . 115
6.4 Example . . . 119
6.5 Final Remarks . . . 122

7 Rational Comparison Systems 125
7.1 Introduction . . . 127
7.2 Rational Comparison System . . . 128
7.2.1 Stability Analysis . . . 129
7.2.2 $\mathcal{H}_\infty$-Norm Calculation . . . 134
7.3 Linear Filter Design . . . 138
7.3.1 Example . . . 144
7.4 State Feedback Control Design . . . 147
7.4.1 Examples . . . 149
7.5 Output Feedback Design . . . 151
7.5.1 Example . . . 156
7.6 Neutral Systems . . . 158
7.7 Final Remarks . . . 159

IV Conclusions and Bibliography 161

8 Conclusions and Perspectives 163
8.1 Conclusions . . . 165
8.2 Perspectives . . . 166

Bibliography 167

V Appendices 179

A Résumé en Français 181

B Resumo em Português 207

List of Figures

2.1 The ω-stability region for fractional systems . . . 21
2.2 Example 2.1: Poles in the ω-domain . . . 22
2.3 Bode diagram for $F(s) = 1/(s^{0.5} + 2)$ . . . 23

3.1 Retarded, Neutral and Advanced Chains of Poles . . . 34
3.2 Distribution Diagram for Example 3.4 . . . 35
3.3 Frequency response for $G(s)$ in Example 3.7. . . . 46
3.4 Neutral Chains of Poles for $G_1(s)$ . . . 56
3.5 Neutral Chains of Poles for $G_2(s)$ . . . 57
3.6 Frequency response for $G_3(s)$ and $\widetilde{G_3}(s)$ . . . 58
3.7 Frequency response for $G_5(s)$, $G_6(s)$ and $G_5(s) * G_6(s)$. . . . 59
3.8 Neutral Chains of Poles for $G_1(s)$ and $\mu = 0.9$ . . . 64
3.9 Neutral Chains of Poles for $G_1(s)$ and $\mu = 0.99$ . . . 65
3.10 Neutral Chains of Poles for $G_2(s)$ and $\mu = 0.6$ . . . 66
3.11 Neutral Chains of Poles for $G_2(s)$ and $\mu = 0.7$ . . . 67
3.12 Bode Diagram for $G_3(s)$ with $t(s) = s^{0.5} + 1$ and $t(s) = 1$ . . . 68

4.1 Real part of the poles of $\widetilde{C}(s, \theta)$ . . . 80
4.2 Root loci for Example 4.1 until $\tau = 0.5$ . . . 81
4.3 Real part of the location of poles of $\widetilde{C}(s, \theta)$ . . . 82
4.4 Root loci for Example 4.3 until $\tau = 20$ . . . 83
4.5 Root-loci for $C_1(s)$ until $\tau = 3.9$ . . . 88

5.1 Example 5.1: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ and $|\omega|^{m/n}$ against ω. . . . 102
5.2 Example 5.1: $\mathcal{T}_1^p$ for $p = 0, \ldots, 9$ (type 3/2). . . . 103
5.3 Example 5.1: $\mathcal{T}_2^p$ for $p = 0, 1, 2, 3$ (type 2/2). . . . 103
5.4 Example 5.2: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ and $|\omega|^{m/n}$ against ω. . . . 104
5.5 Example 5.2: $\mathcal{T}_1^p$ for $p = 0, \ldots, 6$ (type 3/1a). . . . 104
5.6 Example 5.2: $\mathcal{T}_2^p$ for $p = 0, \ldots, 5$ (type 1/a2). . . . 105

6.1 Standard Feedback System . . . 112
6.2 $R_\alpha(\gamma)$ versus γ. . . . 117
6.3 $\phi(\alpha) = \tau^{-\alpha}\psi_o^{-1}$ versus α. . . . 121

7.1 Mapping of $\Psi(\tau, j\omega)$ and $\Phi(\lambda, j\omega)$ . . . 132

7.2 $\mathcal{H}_\infty$-norm versus time delay 137
7.3 $\mathcal{H}_\infty$-norm of the estimation error versus time delay 146
7.4 Maximum singular value in dB versus frequency 147
7.5 Time simulation 148
7.6 Closed loop $\mathcal{H}_\infty$-norm versus time delay 150
7.7 $\mathcal{H}_\infty$ performance versus time delay for $\gamma = 1$. 157

List of Examples

2.1 Poles of a Fractional System 21
2.2 Bode diagram for $F(s) = 1/(s^{0.5} + 2)$ 23
2.3 Time-varying differential equation 26
3.1 Retarded Chain of Poles of $G(s) = 1/(s - e^{-s})$ 32
3.2 Neutral Chain of Poles of $G(s) = 1/(1 + e^{-s})$ 33
3.3 Advanced Chain of Poles of $G(s) = 1/(se^{-s} - 1)$ 33
3.4 Distribution Diagram 33
3.5 Classification of Chains of Poles 35
3.6 Coprime factorization and Bézout factors 44
3.7 $\mathcal{H}_\infty$-norm for systems with $\nu_n = 0$ 45
3.8 Asymptotic analysis of $G(s) = 1/(s + e^{-s} + (s - 1)e^{-2s})$ 54
3.9 Asymptotic analysis of $G(s) = 1/(s^2 + s + 4e^{-s} + (s^2 + 1.1s + 7)e^{-2s})$ 55
3.10 Asymptotic analysis of $G(s) = (2s + 3)/(s^2 + s + (s^2 + 2)e^{-s} - (0.5s + 1)e^{-2s})$ . . 56
3.11 Asymptotic analysis of systems in cascade 57
4.1 Stability Windows and Root-locus for classical systems 79
4.2 Characterization of delay-independent stability 80
4.3 Root-locus with multiple roots 81
4.4 Root Locus for $C(s) = 1/((\sqrt{s})^3 - 1.5(\sqrt{s})^2 + 4\sqrt{s} + 8 - 1.5(\sqrt{s})^2 e^{-\tau s})$ 87
4.5 Stability Windows for $C(s) = 1/(s^{5/6} + (s^{1/2} + s^{1/3})e^{-0.5s} + e^{-s})$ 88
5.1 Stability Crossing Curves for a type 3/2 and 2/2 system 101
5.2 Stability Crossing Curves for a type 3/1a and 1a/2 system 101
7.1 Stability Analysis through Comparison System (1/3) 131
7.2 Stability Analysis through Comparison System (2/3) 132
7.3 Stability Analysis through Comparison System (3/3) 132
7.4 $\mathcal{H}_\infty$-norm Calculation through Comparison System (1/2) 136
7.5 $\mathcal{H}_\infty$-norm Calculation through Comparison System (2/2) 137

List of Symbols

$\mathbb{N}$	-	Set of natural numbers.
$\mathbb{N}_N$	-	Set of the N first natural numbers $\{1, \ldots, N\}$.
$\mathbb{Z}$	-	Set of integer numbers.
$\mathbb{R}$	-	Set of real numbers.
$\mathbb{Z}_+$, $\mathbb{R}_+$	-	Set of non-negative integer (real) numbers, respectively.
$\text{sign}(x)$	-	The sign of a real number. It is defined as 0 if $x = 0$ or $x/\lvert x\rvert$ otherwise.
$\jmath$	-	$\sqrt{-1}$.
$\mathbb{C}$	-	Set of complex numbers.
$\mathbb{C}_+$, $\mathbb{C}_-$	-	The open right half-plane (left half-plane), respectively.
$\overline{\mathbb{C}}_+$, $\overline{\mathbb{C}}_-$	-	The close right half-plane (left half-plane), respectively.
$\jmath\mathbb{R}$	-	The imaginary axis in the complex plane.
$\Re(z)$	-	Real part of the complex number z.
$\Im(z)$	-	Imaginary part of the complex number z.
$\overline{z}$	-	The conjugate of the complex number z.
$\angle(z)$	-	The argument of the complex number z.
$\mathbb{R}^{n\times m}$	-	The set of $n \times m$ real matrices.
X'	-	The transpose of the matrix X.
X^{-1}	-	The inverse of the matrix X.
$X > 0, (X \geq 0)$	-	The symmetric matrix X is positive (semi-)definite.
I	-	The identity matrix of any dimension.
$\sigma(X)$	-	Maximum singular value of the matrix X.
$r_s(X)$	-	The spectral radius of the matrix X.
$\det(X)$	-	Determinant of the matrix X.
$L^2(0,\infty)$ or L^2	-	The space of measurable functions such that $\int_0^\infty \lvert f(t)\rvert^2 dt < \infty$.
$L^\infty(0,\infty)$ or L^∞	-	The space of measurable functions such that $\operatorname{ess\,sup}_{t\in\mathbb{R}_+} \lvert f(t)\rvert < \infty$.
$\mathcal{H}_\infty(\mathbb{C}_+)$ or $\mathcal{H}_\infty$	-	The space of analytic and bounded functions in $\mathbb{C}_+$.
$\lVert G\rVert_{\mathcal{H}_\infty}$	-	$\sup_{\{\Re(s)>0\}} \lvert G(s)\rvert$.
$\mathcal{L}$	-	The Laplace transform operator.
$f(x) \in \mathcal{O}(g(x))$	-	$\lvert f(x)\rvert < K\lvert g(x)\rvert$ for some constant K.
$f(x) \in o(g(x))$	-	$f(x)/g(x) \to 0$ as $\lvert x\rvert \to \infty$.

Part I

Preliminaries

Chapter 1

Introduction

Contents

1.1 **Objectives** 5
1.2 **Preliminaries** 5
1.3 **Outline of the thesis** 9

1.1 Objectives

This thesis has as its main objective the study of methods for analysis and control of time-delay systems. The link between all the themes is the use of *frequential* methods as our basic tools. Even if, in Chapter 7, one may say that the model we will use as well as some techniques such as Riccati equations are commonly referred to time-domain methods, it will be clear how they naturally appear after an ingenious transformation on the Laplace transform of the delay. Moreover, this technique differs completely from what is commonly viewed as the standard *time-domain* methods for time-delay systems, namely the *Lyapunov-Krasovskii* functionals.

Our focus on frequential methods has two main reasons. Although we are restricted to work almost exclusively with linear and time-invariant systems, we can aim to achieve results with a low degree of conservatism in synthesis, and even necessary and sufficient conditions in the analysis, which is, up to now, virtually impossible via the Lyapunov-Krasovskii functionals (Gu, Kharitonov & Chen 2003). Secondly, we can deal in the same framework with the class of fractional systems, which is gaining importance in the last years, as it can be seen by the increasing number of publications dealing with this subject.

During the elaboration of this work, we tried to develop methods in a broad sense, meaning that not only we contributed both to the analysis and synthesis of filters and controllers for such systems, but also that our contributions cover both a theoretical and an applied computational points of view.

1.2 Preliminaries

Most dynamical systems present delays in their inner structure (Malek-Zavarei & Jamshidt 1987), due to phenomena as, for example, transport, propagation or communication, but most of the time they are ignored for the sake of simplicity. On the other hand, those delays can be the cause of bad performance or even instability, and therefore, in order to analyze properly and design controllers for those systems, it is mandatory to take their effects into account. Another important source of delay is the feedback control itself, with this delay induced by the sensors, actuators and, in more modern digital controllers, the time of calculation.

Linear time-delay systems form a class of infinite dimensional systems, and the literature of applications modeled by such systems is vast and dates from long time ago, with an example in our daily life was given in (Faure & Robin 1984). They naturally appear in the modeling of various processes found in physics, mechanics, circuits, economics, biology, and many others. In the time domain, the simplest case with only one delay is described by the following differential equation:

$$\mathcal{G}_r : \begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + E w(t) \\ z(t) &= C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is called the state, $w(t) \in \mathbb{R}^m$ is the external input, $z(t) \in \mathbb{R}^p$ is the output, $\tau > 0$ is the numerical value of the delay and $A_0,\ A_1,\ E,\ C_0,\ C_1$ and D_z are matrices with appropriated dimensions. For the existence and uniqueness of the solution, the initial condition must be given for all $t \in [-\tau, 0]$. Considering that $x(t) = 0$ for all $t \leq 0$, the Laplace transform of $\mathcal{G}_r$ from the input w to the output z is given by

$$G_r(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau})^{-1} E + D_z. \tag{1.2}$$

The subscript "r" in $\mathcal{G}_r$ denotes that this system has a delay in *retarded* form. For these systems, the differential equation (1.1) does not depend explicitly on $\dot{x}(t-\tau)$. This is reflected in the characteristic quasi-polynomial of (1.2), that is, $\det(sI - A_0 - A_1 e^{-s\tau})$, by it not having terms of the form $s^n e^{-ks\tau}$, $k \in [1, n]$. Systems of this type present some nice properties, both in the time-domain as well as in frequency-domain. In the first scenario, for each subsequent time-window $\{\ [0,\tau),\ [\tau, 2\tau),\ \ldots,\ [k\tau, (k+1)\tau)\ \}$, the solution is *smoothed*. In the frequential-domain, as we will see later, whenever we pass from $\tau = 0$ to $\tau = \varepsilon$, for a small ε, the number of poles of $G_r(s)$ goes from finite to infinitely many. However, as we take positive ε approaching 0, we will see that all those infinite new poles will satisfy $\Re(s) \to -\infty$. This implies that for this class of system, there is a sense of continuity in the number of unstable poles with respect to the delay whenever we pass from $\tau = 0$ to a positive value.

Complementing the retarded systems are the so called *neutral* ones. A simple case with one delay can be described by the differential equation:

$$\mathcal{G}_n : \begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + F\dot{x}(t-\tau) + E w(t) \\ z(t) &= C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{1.3}$$

and under the same conditions given for the retarded system, its Laplace transform is given by

$$G_n(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau} - F s e^{-s\tau})^{-1} E + D_z. \tag{1.4}$$

Here, the subscript "n" in $\mathcal{G}_n$ denotes that this system has a delay in *neutral* form. This comes from the dependence of (1.3) with $\dot{x}(t-\tau)$. This can also be seen directly in the characteristic equation of (1.4), through the fact that it has at least one term of the form $s^n e^{-ks\tau}$, $k \in [1, n]$. Differently from the retarded systems, neutral systems do not present the *smoothing* property, and they might not even preserve stability for a small delay if the delay-free system is stable.

We must say that there exists a third class of time-delay system, the one called of *advanced* type. It is characterized by only having delayed terms in the highest derivative of the state-space coordinate or, equivalently in the frequency domain, by having the highest power of s multiplying an exponential term in the characteristic equation. Since those systems have limited practical application and strong difficulties for stabilization for any positive value of the delay, they have been less studied in the last decades. Recently, (Insperger, Stepan & Turi 2010) presented some applications and control schemes for such class of systems.

Historically, although the zeros of quasi-polynomials were already studied by (Pontryagin 1955), the first systematic and complete study of delay systems by frequential methods was produced by (Bellman & Cooke 1963). Many results that we are going to present in the following chapters, especially in Chapter 3, have their bases in early results presented in this book, such as the asymptotic root location of differential-difference equations, both for retarded and neutral systems. From that point on, the study of time-delay systems underwent through a high growth. Among others, we would like to reference the books by (Hale & Lunel 1993), (Niculescu 2001) and (Gu et al. 2003), and the review papers by (Kharitonov 1999) and (Richard 2003) as milestones specific to this area produce in the last decades.

The study of time-delay systems can be developed in a variety of frameworks, each one presenting some advantages and disadvantages. In the next paragraphs, we will provide a classification of such frameworks and some important references for each one of them. Surely, this is not the only way to classify such techniques (see (Niculescu 1997) for a different approach), not even the list of references is exhaustive.

The first category congregates all the time-domain techniques. It can be further divided into two sub-categories, namely finite or infinite dimension. As an example of the first one, we can consider the system $\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bu(t)$, where $x(t) \in \mathbb{R}^n$ is the so-called state vector, being of finite dimension. Most strategies are based in generalized Lyapunov methods, involving some functionals instead of classical positive definite functions. The most investigated generalizations are due to (Krasovskii 1963) and (Razumikin 1956). Among many others, more information on this can be obtained in (Datko 1978), (Niculescu 2001), (Fridman & Shaked 2002), (Richard 2003) and (de Oliveira & Geromel 2004).

The second sub-category of time-domain techniques considers realizations of the above system in infinite dimension given by $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t)$, where $x(t)$ is a function and $\mathcal{A}$, $\mathcal{B}$ are bounded or unbounded operators over these functions. This approach requires the introduction of new concepts of controllability, observability, stabilizability and detectability expressed in terms of operators. See (Curtain & Pritchard 1978), (Hale, Magalhães & Oliva 1984), (Yamamoto

1989), (Yamamoto 1991), (Bensoussan, Prato, Delfour & Mitter 1993) and (Rabah, Sklyar & Rezounenko 2005) for more information on this subject.

The next framework in which time-delay systems can be studied involves frequency-domain techniques. We can also sub-divide this category in two, the first one dealing explicitly with the non-rational transfer functions, as in (Walton & Marshall 1987), (Georgiou & Smith 1990), (Bonnet & Partington 1999) and (Mirkin & Zhong 2003), whereas the second one deals with either a rational comparison system (see (Rekasius 1980), (Zhang, Knospe & Tsiotras 2003) and (Olgac & Sipahi 2002)) or a rational approximate to the infinite-dimensional system (we can mention the Padé techniques (Lam 1990) and (Mäkilä & Partington 1999), the Fourrier-Laguerre series (Partington 1991) and the rational Hankel approximation (Glader, Hognas, Mäkilä & Toivonen 1991)), where techniques used for finite-dimensional systems can be applied.

The last framework gathers all the techniques involving algebraic methods for the study of time-delay systems. The idea is to exploit the structural properties of a given system (e.g. stability, stabilizability, invariants, controllability, observability, flatness, reductions, decomposition, decoupling, equivalences) by means of module theory, homological algebra, algebraic analysis and symbolic computation in order to answer the classical questions of control theory. See, for example, (Desoer & Vidyasagar 1975), (Kamen, Khargonekar & Tannenbaum 1986), (Habets 1994), (Fliess & Mounier 1994), (Mounier 1995), (Glüsing-Luerßen 1997), (Brethé & Loiseau 1997), (Loiseau & Mounier 1998), (Brethé & Loiseau 1998), (Petit & Rouchon 2001), (Petit & Rouchon 2002), (Mounier & Fliess 2004), (Chyzak, Quadrat & Robertz 2005), (Quadrat 2006*b*), (Quadrat 2006*a*) and (Cluzeau & Quadrat 2008) for some theoretical and practical aspects of this framework.

At the same time, fractional systems are also becoming a major subject in many applied areas. As it was discussed in (Monje, Chen & Vinagre 2010), in many fields, such as electrochemistry, biological systems, mass transport and diffusion, frequency domains experiments are performed in order to obtain a model of the dynamic behavior of the system. And it is quite usual that the results are far from what is normally expected from common results, but in fact, they can be decomposed in special elements having a frequency domain model of the form $k/(\jmath\omega)^n$, $n \in \mathbb{R}$. These operators in the Laplace domain can be connected to corresponding operators in the time-domain, opening a whole new class of systems that needs to be understood and controlled.

Throughout the whole text, we use indistinctly the expressions *fractional-order*, *non-integer order* or simply *fractional* systems to designate any transfer-function given by the ration of two pseudo-polynomials, which represents a polynomial in the variable s^α, with $\alpha \in (0, 1)$. For the traditional case, meaning when $\alpha = 1$, we will denote this class as *classical*, *integer* or *non-fractional* systems.

The aim of this work is precisely to provide a number of essential tools to better understand time-delay systems in the frequency domain. Whenever possible, we will present both the integer case and the fractional case, one just after the other, and point out the main similarities and differences, explaining the difficulties for each case. Each individual chapter deals exclusively with a single problem, and therefore, used together with the basic results provided in Chapter 2, should be self-contained.

1.3 Outline of the thesis

This manuscript is divided in five parts. In this first one, we introduce the problem we will be dealing with and provide some basic information that will be used throughout the text. It is divided into two chapters, this first one providing the main objectives and motivation behind the work where in Chapter 2 we present some previous results which form the basis for the development of the following chapters.

The second part deals exclusively with the analysis problem, and it is split into three chapters. Its purpose is to answer the simple question: *"Is the system stable?"*. As it will be seen later, the key to the answer of this question lies in the position of the zeros of a quasi-polynomial, that is a polynomial not only in s but also in $e^{-\tau s}$, with s being the Laplace transform variable and $\tau > 0$ the numerical value of the delay. This irrational transfer function presents an infinite number of poles, being thus generally impossible to calculate all of them.

Nevertheless, the asymptotic position of the poles of a time-delay system, i.e., the zeros of a quasi-polynomial when $|s| \to \infty$, presents an amazing geometrical structure. These poles can be grouped in what we call *"chains of poles"*. The study of those structures is the central point of Chapter 3. This is accomplished initially for classical systems and later expanded for the fractional case. The analysis is taken beyond the location of poles, since for systems presenting them, we can find some situations where $\mathcal{H}_\infty$-stability cannot be guaranteed exclusively by looking at the position of the poles.

The fourth chapter brings some computational methods able to cope with the localization in the complex plane of the unstable poles of the system. Moreover, we are able to provide all the *stability windows* not only for classical systems, where a variety of different methods are already known, but also for the class of fractional systems. As an important side-product of our method, we are able to provide the unstable root-locus for those classes of time-delay systems as a function of the delay.

Although Chapter 5 changes the type of delay we will be dealing with, from *discrete* into *distributed* delay, the main quest remains unaltered. For the important class of γ-distributed delay, we are able to extend some results regarding the stability curves, i.e., the positions in the parameter space where stability is lost or recovered, for the class of fractional systems. This chapter ends the second part.

The third part consists of two chapters dealing with the problem of synthesis of filters and controllers. The first one, Chapter 6, consists of designing classical PID controllers for unstable fractional systems. Using the *small-gain theorem*, we are able to provide some stabilizing PD controllers, and afterwards implement the integral gain maintaining stability. Moreover, some insights on how to tune the parameters are given in order to increase its stability margin.

As our last results, Chapter 7 deals in a single framework with state-feedback, filtering and dynamic output feedback of classical time-delay systems. Based on the Rekasius transformation, we are able to describe a comparison system from where some properties of the time-delay system can be derived. From this comparison system, we can use classical routines on LTI systems in order to provide controllers and filters for the time-delay system.

Finally, Part 4 brings the conclusion and referenced bibliography, and in Part 5 we bring some appendices.

Chapter 2

Basic Results

Contents

2.1 Classical Time-Delay Systems 13

2.2 Fractional Time-Delay Systems 14

2.1 Classical Time-Delay Systems

2.1.1 Class of Systems under consideration

We start by looking at time-delay systems with transfer functions of the form

$$G(s) = \frac{t(s) + \sum_{k=1}^{N'} t_k(s) e^{-ks\tau}}{p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}} = \frac{n(s)}{d(s)} \tag{2.1}$$

where $\tau > 0$ is the time delay, and t, p, q_k for all $k \in \mathbb{N}_N$, and t_k for all $k \in \mathbb{N}_{N'}$, are real polynomials.

Consider a time-delay system $\mathcal{T}$ whose state space realization is given by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + A_2 \dot{x}(t-\tau) + Bw(t) \tag{2.2}$$

$$y(t) = C_{z0} x(t) + C_{z1} x(t-\tau) + Dw(t) \tag{2.3}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^1$ is the external input and $y(t) \in \mathbb{R}^1$ is the output. Assuming that the system evolves from the rest, which means, $x(t) = 0$ for all $t \in [-\tau, 0]$, $\mathcal{T}$ has its transfer function given by

$$\mathcal{T}(s) = (C_{z0} + C_{z1} e^{-\tau s})(sI - A_0 - A_1 e^{-\tau s} - A_2 s e^{-\tau s})^{-1} B + D, \tag{2.4}$$

from where it can be seen that it is in the form given by (2.1). It is important to state that this is not the only way to produce the transfer function given in (2.1).

As we mentioned before, differently from the study of LTI systems, which may be seen as the study of quotients of polynomials, we face here with the ratio of quasi-polynomials, meaning polynomials in the Laplace variable s and $e^{-\tau s}$.

2.1.2 $\mathcal{H}_\infty$-stability

The concept of stability which we will consider throughout this manuscript is the input-output stability, more precisely, when both signals are in $L^2(0, \infty)$. A system has a finite $L^2(0, \infty)$ gain if and only if its transfer function has a finite $\mathcal{H}_\infty$-norm, where $\mathcal{H}_\infty(\mathbb{C}_+)$ is the space of functions which are analytic and bounded in the open right half-plane.

It is well-known that for retarded time-delay systems, exponential stability is equivalent to $\mathcal{H}_\infty$-stability, both having the rule *"no poles in the closed right half-plane"* as a necessary and sufficient condition. For neutral systems, this condition is in general no longer sufficient, both in time-domain (see (Rabah, Sklyar & Rezounenko 2008)) and in the frequency-domain, as it will be presented in Chapter 3 (see also (Partington & Bonnet 2004) for the single delay case).

2.2 Fractional Time-Delay Systems

It is unquestionable that the concepts of traditional integral and derivatives are essential tools in the work of mathematicians, engineers and many other professionals in the areas of science and technology (Loverro 2004). Even though they consist of highly complex concepts, their strong geometric interpretation and physical meaning made them largely known in those aforementioned communities.

The same cannot be said about the field of fractional calculus. The lack of easy geometric interpretation (recently (Podlubny 2002) and (Podlubny, Despotovic, Skovranek & McNaughton 2007) shed some light over this subject) and the complicated physical meaning made this subject much less accessible outside the research area. Some parallelism can be done with the exponential function. From primary school we learn that exponential is a short notation for repeated multiplication of some value, as for $x^3 = x \times x \times x$. But this reasoning cannot be used for the description of $x^{3.5}$, since it is hard to describe the physical meaning of multiplying some value by itself 3.5 times. Nevertheless, the importance of non-integer exponentiation is evident, and can easily be calculated by series expansion, for example.

That reasoning motivates the study of fractional calculus, and specifically, fractional-order systems. It is not driven by a specific purpose in mind, but rather by all the new possibilities it can uncover. But before any development on the properties and particularities of such systems, let's discuss briefly the historical elements of fractional calculus. This presentation is based on the works of (Miller & Ross 1993), (Podlubny 1999*a*), (Cafagna 2007) and (Caponetto, Dongola, Fortuna & Petráš 2010).

2.2.1 A Brief Historical Perspective on Fractional Calculus

Most of mathematics historians would agree that the first mention of Fractional Calculus was made by Gottfried Leibniz in a letter to Guillaume de l'Hôpital, around 1695, when he asked the following question (Oldham & Spanier 1974):

> *Can the meaning of derivatives with integer order be generalized to derivatives with non-integer order?*

That question might have instigated l'Hôpital, as he replied with another question:

What if the order will be 1/2?

What may have just been an ingenious thought over the notation of the n^{th}-derivative (d^n/dx^n), gave birth to fractional calculus in Leibniz's response:

An apparent paradox, from which one day useful consequences will be drawn.

Many other contemporary mathematicians expressed some interest in this new field. Johann Bernoulli also exchanged some letters with Leibniz mentioning derivatives of "general order". But it was not until Euler, in 1772, that his first attempts of interpolating the integer orders of a derivative presented some progress. In the beginning of the 19$^{\text{th}}$ century, the study of fractional calculus started to be more systematic, with Laplace and Lacroix. The latter just developed a mathematical exercise of generalization of the derivative for the power function. Considering $y = x^m$, with m and n not necessarily integer, he stated

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}, \tag{2.5}$$

where $\Gamma(z)$ is the generalization of the factorial function for non-integer values

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.6}$$

Fourier, in 1822, also provided a definition of fractional derivation based on his integral representation of a function, as

$$\frac{d^n}{dx^n} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\alpha) d\alpha \int_{-\infty}^{\infty} p^n \cos(p(x-\alpha) + \frac{1}{2} n\pi) dp. \tag{2.7}$$

Up to that moment, no applications had been found to fractional calculus. It was Niels Abel, in 1823, who applied it for the first time, in order to solve the integral equation of the tautochrone problem. Almost a decade later, Fourier's integral formula and Abel's solution called the attention of Joseph Liouville, who made the first systematic study on fractional calculus. While Lacroix worked with the generalization of the power function, Liouville's starting point was the result for the derivatives of exponential equations

$$\frac{d^n}{dx^n} e^{ax} = a^n e^{ax}. \tag{2.8}$$

So, he assumed that any function $f(x)$ that may be expanded in a series of the form

$$f(x) = \sum_{i=0}^{\infty} c_i e^{a_i x}, \qquad \Re(a_i) > 0 \tag{2.9}$$

has a fractional derivative given by

$$\frac{d^\alpha}{dx^\alpha} f(x) = \sum_{i=0}^{\infty} c_i a_i^\alpha e^{a_i x}, \tag{2.10}$$

where α is any number, rational or irrational (or even complex).

The main disadvantage is that it is only applicable for those functions of the form (2.9) and to the orders where (2.10) converge. Due to this restriction, Liouville formulated a second definition, from where he was able to give a fractional derivative of x^{-a}, whenever x and a are positive. But those definitions where too restrictive, as neither was suitable to be applied to a wide class of functions.

The next steps on fractional calculus were given by Bernhard Riemann during his student days, but those results were not published until after his death. Through a generalization of Taylor series, he derived

$$\frac{d^\alpha}{dx^\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt + \Psi(x), \tag{2.11}$$

where the complementary function $\Psi(x)$ appeared due to the ambiguity in the lower limit of integration a. In fact, this function is an attempt to provide a measure of the deviation from the law of exponents. For example

$$\frac{d^\alpha}{dx^\alpha} \frac{d^\beta}{dx^\beta} f(x) = \frac{d^{\alpha+\beta}}{dx^{\alpha+\beta}} f(x) \tag{2.12}$$

is valid only when the lower limits of integration are equal. Riemann was concerned with the measure of deviation for the case where they are not the same.

Finally, it was the work of H. Laurent in 1884 that put the theory of fractional calculus in a level of development suitable for the starting point in modern mathematics. It considered the n^{th} derivative of Cauchy's integral formula

$$\frac{d^n}{dx^n} f(x) = \frac{n}{2\pi \jmath} \int_C \frac{f(\tau)}{(\tau - x)^{n+1}} d\tau \tag{2.13}$$

and generalized it to the case where n is not a natural number. The question that arises is that whenever n is not an integer, the integrand in equation (2.13) contains not a pole, but a branch point. Therefore, an appropriate contour requires a branch cut. Laurent considered as the contour an open circuit on a Riemann surface, and that formalized what is currently known as Riemann-Liouville fractional integral.

2.2.2 Modern Definition

In modern days, we consider the following fractional integro-differential operator

$$
{}_aD_t^{\mu} = \begin{cases} \frac{d^{\mu}}{dt^{\mu}} & : \mu > 0, \\ 1 & : \mu = 0, \\ \int_a^t (d\tau)^{-\mu} & : \mu < 0. \end{cases} \tag{2.14}
$$

where three definitions are mostly used. The first one is the Grünwald-Letnikov

$$
{}_aD_t^{\mu} f(t) = \lim_{h \to 0} h^{-\mu} \sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\mu}{j} f(t - jh), \tag{2.15}
$$

with [.] meaning the integer part. The second common definition is called as the Riemann-Liouville

$$
{}_aD_t^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\mu-n+1}} d\tau, \tag{2.16}
$$

where $(n - 1 < \mu < n)$. Lastly, the Caputo definition is given by

$$
{}_aD_t^{\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\mu-n+1}} d\tau, \tag{2.17}
$$

where the same restriction on n as in the Riemann-Liouville is used. The main advantage of this last definition comes for fractional order differential equations, as the initial conditions are given in the same form as for the integer-order ones, and therefore they only involve classical derivatives, as it will be seen in the sequel.

2.2.3 Laplace Transform

Before dealing with the Laplace transforms of the fractional derivatives, let us recall one result that will be useful in the sequel.

Remark 2.1 *The Laplace transform of the following equation*

$$
f(t) = t^{a-1} e^{-nt} \tag{2.18}
$$

for $a \in \mathbb{R}^+$ *is given by*

$$
\hat{f}(s) = \frac{\Gamma(a)}{(s+n)^a}, \tag{2.19}
$$

and its domain of convergence is given by $\Re(s) > -n$.

We will calculate the Laplace transform of both the Riemann-Liouville and the Caputo derivatives starting in 0, that means, $\mathcal{L}({}_0D_t^\mu f(t))$ for $\mu > 0$. For the Riemann-Liouville case, we have

$$\mathcal{L}({}_0D_t^\mu f(t)) = \frac{1}{\Gamma(n-\mu)}\mathcal{L}\left(\frac{d}{dt^n}g(t)\right) \tag{2.20}$$

$$= \frac{s^n G(s) - \sum_{m=0}^{n-1} s^m g^{(n-m-1)}(0)}{\Gamma(n-\mu)} \tag{2.21}$$

where

$$g(t) = \int_0^t \frac{f(\tau)}{(t-\tau)^{\mu-n+1}}d\tau. \tag{2.22}$$

One can easily see that $g(t)$ is the convolution of $f(t)$ with $t^{n-\mu-1}$, therefore $\mathcal{L}(g(t)) = F(s)\mathcal{L}(t^{n-\mu-1})$ and from remark 2.1

$$G(s) = \frac{F(s)\Gamma(n-\mu)}{s^{n-\mu}}. \tag{2.23}$$

As $g^{(n-\mu)}(t) = f(t)\Gamma(n-\mu)$, we arrive at the final result

$$\mathcal{L}({}_0D_t^\mu f(t)) = s^\mu F(s) - \sum_{m=0}^{n-1} s^m f^{(\mu-m-1)}(0). \tag{2.24}$$

For the Caputo definition, we have

$$\mathcal{L}({}_0D_t^\mu f(t)) = \frac{1}{\Gamma(n-\mu)}\mathcal{L}\left(\int_0^t \frac{g(t)}{(t-\tau)^{\mu-n+1}}\right) \tag{2.25}$$

$$= \frac{G(s)}{s^{n-\mu}} \tag{2.26}$$

where $g(t) = f^{(n)}(t)$. Therefore

$$\mathcal{L}({}_0D_t^\mu f(t)) = \frac{s^n F(s) - \sum_{m=0}^{n-1} s^m f^{(n-m-1)}}{s^{n-\mu}} = s^\mu F(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0). \tag{2.27}$$

This last result illustrates the fact that the Caputo fractional derivative is the most natural in order to treat initial conditions in dynamical systems, since it does not depend of fractional derivatives for initial conditions, and only requires exactly the same values as in the non-fractional case.

2.2.4 Linear Fractional-Order Systems

Let us consider a general fractional-order system described by its transfer function (Caponetto et al. 2010):

$$G(s) = \frac{b_m s^{\beta_m} + \ldots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \ldots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}. \tag{2.28}$$

We will assume that system (2.28) can be expressed in a commensurate form

$$G(s) = \frac{b_m s^{m/v} + \ldots + b_1 s^{1/v} + b_0}{a_n s^{n/v} + \ldots + a_1 s^{1/v} + a_0} = K_0 \frac{Q(s^{1/v})}{P(s^{1/v})}, \tag{2.29}$$

where $v \in \mathbb{N}$. The domain of $G(s)$ is a Riemann surface with v Riemann sheets.

Before any further development, we need to consider the generalized Mittag-Leffer function, defined as

$$E_{\mu,\nu}(z) = \sum_{i=1}^{\infty} \frac{z^i}{\Gamma(\mu i + \nu)}, \qquad (\mu > 0, \nu > 0). \tag{2.30}$$

This function will play a role for the fractional-order systems symmetric to what the exponential function is for natural systems. Some particular values of this function are:

$$E_{0,1}(z) = \frac{1}{1-z} \tag{2.31}$$

$$E_{1,1}(z) = \exp(z) \tag{2.32}$$

$$E_{2,1}(z) = \cosh(\sqrt{z}) \tag{2.33}$$

$$E_{1/2,1}(z) = e^{z^2}(1 + \operatorname{erf}(z)) \tag{2.34}$$

The importance of this function comes from the Laplace Transform of $t^{\nu-1}E_{\mu,\nu}(at^{\mu})$. Indeed, for $\Re(s) > |a|^{1/\mu}$

$$\mathcal{L}(t^{\nu-1}E_{\mu,\nu}(at^{\mu})) = \int_0^{\infty} e^{-st} t^{\nu-1} E_{\mu,\nu}(at^{\mu}) dt = \frac{s^{\mu-\nu}}{s^{\mu} - a}. \tag{2.35}$$

Defining $E^{(k)}_{\mu,\nu}(z)$ as the k^{th} derivative of $E_{\mu,\nu}(z)$, $k = (0, 1, 2, \ldots)$, we achieve

$$E^{(k)}_{\mu,\nu}(z) = \sum_{i=1}^{\infty} \frac{(i+k)! z^i}{i! \Gamma(\mu i + \mu k + \nu)}, \tag{2.36}$$

which leads to the following Laplace transform

$$\mathcal{L}(t^{\mu k+\nu-1}E^{(k)}_{\mu,\nu}(at^{\mu})) = \int_0^{\infty} e^{-st} t^{\mu k+\nu-1} E^{(k)}_{\mu,\nu}(at^{\mu}) dt = \frac{k! s^{\mu-\nu}}{(s^{\mu} - a)^{k+1}}, \tag{2.37}$$

again, valid for $\Re(s) > |a|^{1/\mu}$.

As an example, consider the case where $n > m$, so $G(s)$ will represent a proper rational function in the variable $s^{1/v}$. Also, consider that the roots of the polynomial P are simple, given by λ_i for $i = (1, \ldots, n)$. In this case, we can expand (2.29) as:

$$G(s) = K_0 \sum_{i=1}^{n} \frac{c_i}{s^{1/v} - \lambda_i}. \tag{2.38}$$

The time-domain solution of system (2.38) is given by

$$y(t) = \mathcal{L}^{-1}\left\{K_0 \sum_{i=1}^{n} \frac{c_i}{s^{1/v} - \lambda_i}\right\} = K_0 \sum_{i=1}^{n} c_i t^{(1-v)/v} E_{1/v,1/v}(\lambda_i t^{1/v}). \tag{2.39}$$

2.2.5 Stability of Linear Fractional-Order Systems

In the analysis on any dynamical system, the first main question concerns its stability. For systems of the type (2.29), let us consider $\omega = s^{1/v}$.

Definition 2.1 *For the pseudo-polynomial*

$$P(s) = \sum_{i=0}^{n} p_i s^{i/v}, \tag{2.40}$$

with $p_n \neq 0$ *and* $v \in \mathbb{N}$, *we call* n *as the* fractional-order *of* $P(s)$.

As we have stated before, the domain of (2.29) is a Riemann surface with v Riemann sheets, where the origin is a branch point of order $v - 1$ and the branch cut is at $\mathbb{R}^-$. The number of poles of this system can be found with help of the following proposition.

Proposition 2.1 *Let* $G(s)$ *be a fractional order system, as defined in (2.29), with denominator a pseudo-polynomial of fractional-order* n. *Then* $G(s)$ *has exactly* n *roots on the Riemann surface.*

With this definition in hand, we can state the conditions for stability of $G(s)$. This was presented in (Matignon 1998).

Theorem 2.2 *A commensurate order system described by a rational transfer function*

$$G(s) = \frac{Q(\omega)}{P(\omega)} \tag{2.41}$$

where $\omega = s^q$, $q \in (0, 2)$, *and* $P(\omega)$ *and* $Q(\omega)$ *are polynomials with respect to the variable* ω, *is stable if and only if*

$$|\arg(\omega_i)| > q\frac{\pi}{2} \tag{2.42}$$

is valid for all $\omega_i \in \mathbb{C}$ *such that* ω_i *is the* i^{th} *solution of* $P(\omega) = 0$.

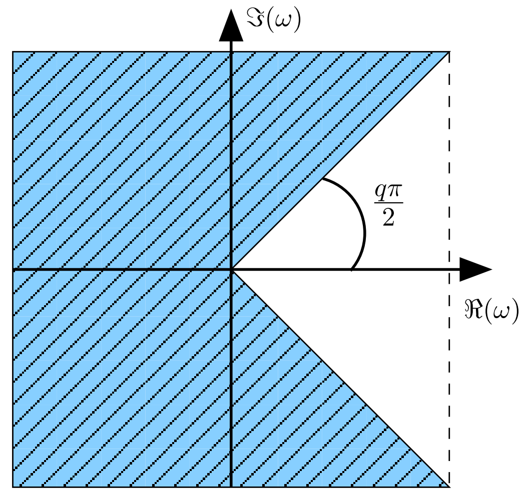

Figure 2.1: The ω-stability region for fractional systems

This shows that the instability region in the s-domain is the closed right half-plane of the first (physical) Riemann sheet. On the other hand, in the ω-domain, one can see that the instability region goes to the real non-negative axis when $q \to 0$ and to $\mathbb{C}/\mathbb{R}_*^-$, that means, the whole complex plane except the negative real axis, when $q \to 2$.

To illustrate the results presented in this section, let us consider the following example given in (Caponetto et al. 2010).

Example 2.1 Consider the linear fractional-order system described by the following transfer function

$$G(s) = \frac{Q(s^{1/10})}{P(s^{1/10})} = \frac{1}{0.8s^{22/10} + 0.5s^{9/10} + 1}. \tag{2.43}$$

Therefore, the domain of $G(s)$ is a Riemann surface with 10 sheets, and $G(s)$ has a fractional-order of 22. Considering $\omega = s^{1/10}$, the poles of $P(\omega)$ are given in Figure 2.2.

The region inside the angular sector inside the solid lines ($|\angle(\omega_i)| < \pi/10$) defines in the ω-domain the equivalent of the first Riemann sheet in the s-domain. Therefore, the region delimited by the dashed line ($|\angle(\omega_i)| \leq \pi/20$) is the unstable region for this system. One can see that none of the 22 poles resides inside this region, and so this system is stable. ♦

2.2.6 Frequential Properties of Fractional Systems

In this subsection we will derive some important properties for fractional systems. In fact, we will focus on Bode Diagrams, that is one of the most used tool for LTI systems. Considering

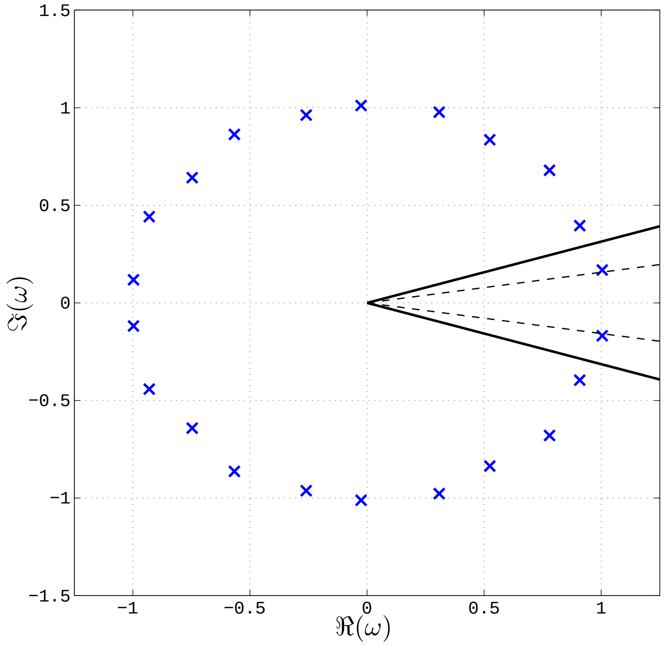

Figure 2.2: Example 2.1: Poles in the ω-domain

$F(s) = k/(s^\mu + a)$, and assuming $s = \jmath\omega$, we obtain:

$$F(\jmath\omega) = \frac{k}{w^\mu \cos(\pi\mu/2) + a + \jmath\omega^\mu \sin(\pi\mu/2)}. \tag{2.44}$$

Therefore, its magnitude in decibels is given by

$$|F(\jmath\omega)|_{dB} = 20\log_{10}(|k|) - 10\log_{10}(w^{2\mu} + 2a\omega^\mu \cos(\pi\mu/2) + a^2). \tag{2.45}$$

It is clear that when $\omega \to \infty$, equation (2.45) becomes

$$|F(\jmath\infty)|_{dB} \approx -20\mu \log_{10}(w), \tag{2.46}$$

which is a line having slope -20μ(dB/dec) on a semi-logarithmic plane.

Regarding the phase plot, we have

$$\angle F(\jmath\omega) = -\arctan\left(\frac{\omega^\mu \sin(\pi\mu/2)}{\omega^\mu \cos(\pi\mu/2) + a}\right). \tag{2.47}$$

Clearly, when $\omega \to \infty$, the angle of $F(\jmath\infty)$ approaches $-\pi\mu/2$. So, compared to the normal case, in the fractional case, a factor of μ multiplies both the slope of the asymptotic magnitude as the phase retard in the bode-plot of the system. One example is considered next:

Example 2.2 Consider a system with transfer function $F(s) = 1/(s^{0.5} + 2)$. It has no poles in the physical Riemann layer, and this is a sufficient condition to guarantee that it is stable. Its bode diagram is presented in Figure 2.3.

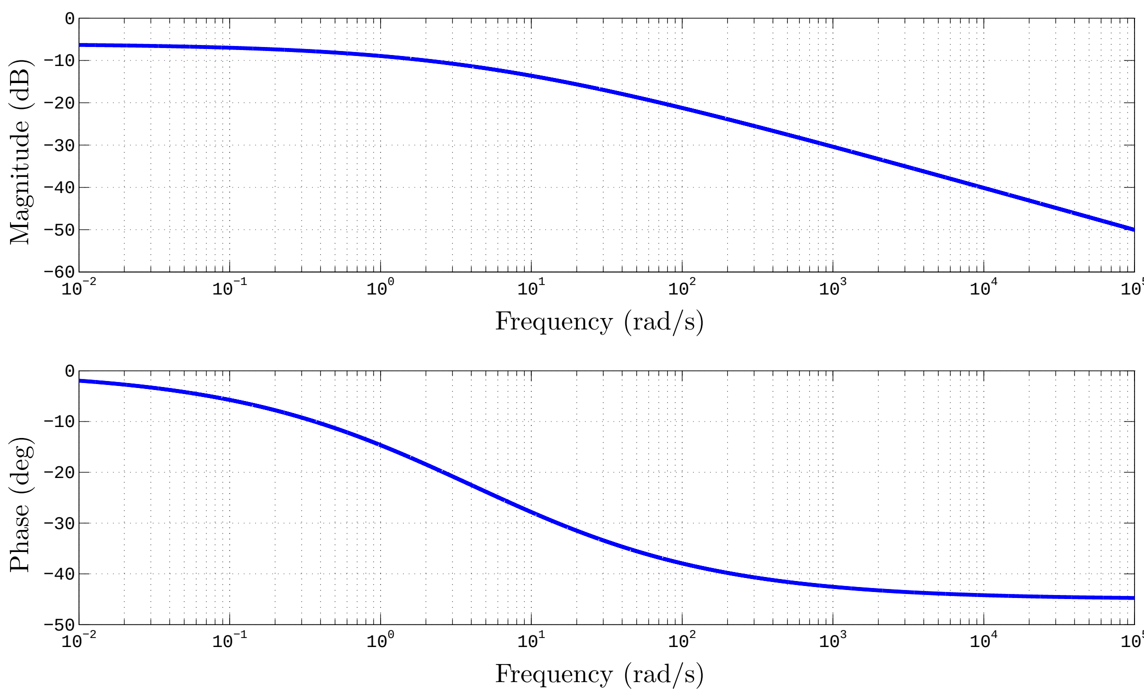

Figure 2.3: Bode diagram for $F(s) = 1/(s^{0.5} + 2)$

One can easily confirm the asymptotic behaviors presented before, that is, the magnitude has a final slope of -10(dB/dec) and the angle approaches $-\pi/4$. ♦

This analysis puts in evidence one strong point of fractional systems. In fact, whenever we are dealing with identification problems, a fractional pole might be better adapted to model a system, since it gives a broader choice of argument and phase change.

2.2.7 Stability of Linear Fractional-Order System with Delay

For this part, we will consider a system that in the frequency-domain has the following characteristic equation:

$$C(s, \tau) = p(s^{\mu}) + \sum_{k=1}^{N} q_k(s^{\mu}) e^{-ks\tau}, \tag{2.48}$$

where the parameter τ is non-negative, $p(s^\mu)$ and $q_k(s^\mu)$ for $k \in \mathbb{N}_N$ are polynomials in s^μ with $\mu \in (0,1)$ and $\deg p \geq \deg q_k$. If $\deg p = \deg q_k$ for at least one $k \in \mathbb{N}_N$, then equation (2.48) defines a *neutral* time-delay system, otherwise it will consist of *retarded* type.

The BIBO-stability (i.e., the system presents a finite L^∞-gain) of fractional systems with delays has been considered in (Bonnet & Partington 2002) where it is shown that BIBO stability conditions already known for delay systems can be extended to the case of fractional delay systems. Their results are summarized in the next two Lemmas:

Lemma 2.3 *Let G be a strictly proper system with characteristic equation given by (2.48) satisfying $\deg p > \deg q_k$ for $k \in 1, \ldots, N$, being thus of retarded type. Then G is BIBO-stable if and only if G has no poles in $\{\Re(s) \geq 0\}$ (in particular, no poles of fractional order at $s = 0$).*

Lemma 2.4 *Let G be a strictly proper system with characteristic equation given by (2.48) satisfying $\deg p \geq \deg q_k$, $k \in 1, \ldots, N$, with the equality holding for at least one polynomial q_k, being thus of neutral type. If there exists $a < 0$ such that G has no poles in $(\mathbb{C}\backslash\mathbb{R}) \cap \{\Re(s) > a\} \cup \{0\}$ then G is BIBO-stable.*

As it is known that BIBO-stability implies $\mathcal{H}_\infty$-stability, see (Mäkilä & Partington 1993, Partington & Mäkilä 1994), similar results can be derived immediately for $\mathcal{H}_\infty$-stability. Finally, $\mathcal{H}_\infty$-stability conditions for systems with poles asymptotically approaching the imaginary axis will be presented in Chapter 3.

2.2.8 Time-Varying Linear Systems

It is not without surprise that we can see the usefulness of fractional systems with delay in the study of a class of time-varying linear systems (Geromel & Palhares 2004). In fact, the whole class of linear differential equations with polynomial coefficients in the independent variable $t > 0$ can be successfully solved with this kind of tools. Since

$$\frac{d}{ds}\hat{f}(s) = -\int_0^\infty t f(t) e^{-st} dt \tag{2.49}$$

we can obtain the following relation:

$$\mathcal{L}(tf(t)) = -\frac{d}{ds}\hat{f}(s). \tag{2.50}$$

As it was done in (Geromel & Palhares 2004), let us consider the following homogeneous equation

$$\sum_{i=0}^{n} a_i(t) y^{(i)}(t) = 0, \tag{2.51}$$

where $a_i(t) = \alpha_i t + \beta_i$ and $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}$ for all $i \in (0, \ldots, n)$, and also $\alpha_n \neq 0$. Applying the Laplace transform in (2.51), and assuming zero initial conditions, we obtain

$$\sum_{i=0}^{n} \beta_i s^i \hat{y}(s) - \sum_{i=0}^{n} \alpha_i \frac{d}{ds}\left(s^i \hat{y}(s)\right) = 0, \tag{2.52}$$

which is equivalent to

$$Q(s)\hat{y}(s) - P(s)\frac{d}{ds}\hat{y}(s) = 0, \tag{2.53}$$

where $P(s)$ and $Q(s)$ are polynomials in s given by

$$P(s) = \sum_{i=0}^{n} \alpha_i s^i, \qquad Q(s) = \beta_0 + \sum_{i=1}^{n} \beta_i s^i - i\alpha_i s^{i-1}. \tag{2.54}$$

Assuming that the n roots $(p_1, \ldots, p_n)$ of $P(s) = 0$ are single, we can decompose $Q(s)/P(s)$ in partial fractions

$$\frac{Q(s)}{P(s)} = d_0 + \sum_{i=1}^{n} \frac{d_i}{s - p_i} \tag{2.55}$$

leading (2.53) to

$$\frac{1}{\hat{y}(s)}\frac{d}{ds}\hat{y}(s) = d_0 + \sum_{i=1}^{n} \frac{d_i}{s - p_i} \tag{2.56}$$

which can be integrated, resulting in

$$\ln(\hat{y}(s)) = d_0 s + \sum_{i=1}^{n} d_i \ln(s - p_i) \tag{2.57}$$

except for a constant value. Finally

$$\hat{y}(s) = e^{d_0 s} \prod_{i=1}^{n} (s - p_i)^{d_i}. \tag{2.58}$$

In order to use (2.58) to actually solve the differential equation, we need to find its inverse transformation. For example, if $d_1, \ldots, d_n$ are integers and $d_1 + \ldots + d_n < 0$, the running product is a rational function, and if $d_0 \leq 0$, then the result is trivial. Another important case is if (2.58) can be decomposed in products of transfer functions of the form given in (2.19). In this case, the solution of the impulse response is the convolution of each individual response. In other cases, even if the exact response cannot be found, some properties of the solution might still be useful, and that will be driven force of the following analysis chapters.

Example 2.3 Consider the differential equation (2.51) with

$$a_0(t) = t, \quad a_1(t) = 1, \quad a_2(t) = t. \tag{2.59}$$

We can determine the polynomials $P(s)$ and $Q(s)$ and their partial decomposition

$$\frac{Q(s)}{P(s)} = \frac{-s}{s^2+1} = \frac{-1/2}{(s+\jmath)} + \frac{-1/2}{(s-\jmath)}, \tag{2.60}$$

and therefore, (2.58) provides

$$\hat{y}(s) = (s+\jmath)^{-1/2}(s-\jmath)^{-1/2} = \frac{!}{\sqrt{s^2+1}} \tag{2.61}$$

where we can identify $y(t) = J_0(t), \ \forall t > 0$, where $J_0(t)$ is a Bessel function. We note that this implies that $J_0(t)$ satisfies the equation $t\ddot{y}(t) + \dot{y}(t) + ty(t) = 0$, known as Bessel equation, with initial conditions $y(0) = 1$ and $\dot{y}(0) = 0$. Even if all the initial conditions are not zero, they do not change the Laplace transform in this case. ♦

Part II

Analysis of Time Delay Systems

Chapter 3

Chains of Poles

Contents

3.1 Introduction 31
3.2 Classical Systems 31
3.3 Fractional Systems 60
3.4 Final Remarks 66

3.1 Introduction

As our first step into the frequential analysis of time-delay systems, we deal with the problem of the asymptotic behavior of the chains of poles. As we have stated before, in the frequency analysis, the main difference when we join a small delay into an LTI system concerns the number of poles. Before the delay, only a finite number of them are present, but as we add a delay (even a small one), an infinite number of extra poles appears. Interesting enough, one may find a strong geometrical structure in the position of those new poles, and this is what we define as a *"chain of poles"*. As it will be seen later in this chapter, these chains of poles play a central role in the stability of any time-delay system. This statement is specially relevant for those of neutral type, where there are situations where $\mathcal{H}_\infty$-stability may not be achieved even in the case where all poles are in the left half-plane.

Most of the techniques applied in this chapter are valid for both classical and fractional systems. But as we will see in the sequel, the final results are far from being the same, and some interesting differences can be spotted. For simplicity of the presentation, we will initially deal with the classical problem completely, and later restart with the fractional case but more in a comparative way.

3.2 Classical Systems

In this section, we will look at time-delay systems with transfer functions of the form

$$G(s) = \frac{t(s) + \sum_{k=1}^{N'} t_k(s) e^{-ks\tau}}{p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}} = \frac{n(s)}{d(s)}, \tag{3.1}$$

where $\tau > 0$ is the delay, and t, p, q_k for all $k \in \mathbb{N}_N$, and t_k for all $k \in \mathbb{N}_{N'}$, are real polynomials. Later on this section we will impose some restrictions about the degrees of the polynomials, but for now let us consider them arbitrary.

Our purpose in this part starts with the general classification of the chains of poles into 3 types: *retarded*, *neutral* or *advanced*. The investigation about the types of chains of poles that a particular time-delay system has is perhaps the most valuable information in the beginning of the comprehension of its behavior. The analysis presented next was introduced in (Bellman & Cooke 1963) and largely used afterwards (for example, in (Partington 2004)). The results presented in the next subsection follow the steps of the latter.

3.2.1 Classification of Chains of Poles

In order to deal with the complete classification of the chains of poles, we need to provide some initial results, which were presented in (Partington 2004).

Lemma 3.1 *Let $a \in \mathbb{C} \setminus \{0\}$. Then the equation $se^s = a$ has an infinite number of solutions, which for a large value of $|s|$ have the form $s = x + \jmath y$ with*

$$x = -\ln(2n\pi) + \ln(|a|) + o(1) \tag{3.2}$$

$$y = \pm 2n\pi \mp \pi/2 + \arg(a) + o(1) \tag{3.3}$$

for n large enough.

Proof: *The real and imaginary parts of the equation $se^s = a$, for $s = x + \jmath y$, are equivalent to:*

$$x + \ln(|x + \jmath y|) = \ln(|a|) \tag{3.4}$$

$$y + \arg(x + \jmath y) = \arg(a) \pm 2n\pi. \tag{3.5}$$

Notice that for any $\delta > 0$ given, all solutions to the given equation with n sufficiently large lie in the double sector

$$S_\delta = \{s \in \mathbb{C} : \pi/2 - \delta < |\arg(s)| < \pi/2 + \delta\}, \tag{3.6}$$

which is centred on the imaginary axis.

Now consider the mapping $u = s + \ln(s)$, constructed in such a way that $e^u = se^s$. Considering R sufficiently large, this mapping takes the elements of $S_{\delta,R} = S_\delta \bigcap \{s \in \mathbb{C} : |s| > R\}$ bijectively to a region in such a way that the points $r\cos(\theta) + \jmath r \sin(\theta)$ are mapped to points $r\cos(\theta) + \ln(r) + \jmath(r\sin(\theta) + \theta)$ with asymptotically the same arguments as s and u tend to infinity.

That shows that for n large, there are solutions u to $e^u = a$ of the form $u = \ln(a) + \jmath 2n\pi$, or, equivalently, solutions s of $se^s = a$ of the form given in (3.2)-(3.3) lying in $S_{\delta,R}$. □

Obviously, asymptotic expressions for the solution of $s^m e^{\lambda s} = a$, for $m \in \mathbb{N}$ and $\lambda \neq 0$, can be directly calculated, since they are obtained by setting $z = \lambda s/m$ and solving $ze^z = \beta\lambda/m$ for each β such that $\beta^m = a$.

With this result in hand, (Partington 2004) provided some examples illustrating the different types of chains of poles.

Example 3.1 Let us consider $G(s) = 1/(s - e^{-s})$. Applying the results of Lemma 3.1, we obtain that the poles have the asymptotic form $s_n = -\ln(2n\pi) \pm \jmath(2n\pi - \pi/2)$, with $n \in \mathbb{Z}$. Therefore, only a finite number of poles can be presented in any extended right half-plane $\Re(s) > c$. Such a chain is called of *retarded* type. Figure 3.1 brings in blue circles its first poles. We can spot that there is a positive real pole around $s = 0.5671$ that does not approach the chain of poles. This kind of pole is what we denominate *pole of small module*, and we will regard them in the next chapter. ♦

Example 3.2 Now let us consider $G(s) = 1/(1 + e^{-s})$. The poles of $G(s)$ lie on the imaginary axis, at the points $s_n = \jmath(2n + 1)\pi$, with $n \in \mathbb{Z}$. This type of chain, where its poles lie in a strip centered on the imaginary axis, is denominated of *neutral* type. Figure 3.1 shows some of those poles in red crosses. As we will see later on this chapter, special care must be taken when dealing with systems with this type of chain, because whenever the poles of the chain are asymptotic to the imaginary axis, the rule "*no poles in the closed right half-plane*" is only a necessary condition to guarantee $\mathcal{H}_\infty$-stability.

♦

Example 3.3 Finally, let us consider $G(s) = 1/(se^{-s} - 1)$. Using Lemma 3.1, we see that the poles have the asymptotic form $s_n = \ln(2n\pi) \mp \jmath(2n\pi - \pi/2)$, with $n \in \mathbb{Z}$. We can note that only a finite number of poles can be presented in any extended left half-plane $\Re(s) < c$. Such a system is said to be of *advanced* type. We show in Figure 3.1 some of the first poles of this system in green plus signs. ♦

For any more complex time-delay system described by equations (3.1), its infinite many poles can be arranged in chains of retarded, neutral and advanced type. In order to obtain these chains, let us define some notations (Partington 2004).

Definition 3.1 *Let $d(s) = p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}$, as defined in (3.1), where p is a polynomial with degree d_0 and leading coefficient c_0, and q_k are polynomials with degree d_k and leading coefficients c_k. The distribution diagram of $d(s)$ is the concave polygonal curve joining the points $P_0 = (0, d_0)$ and $P_N = (N, d_N)$, with vertices at some of the points $P_k = (k, d_k)$ in such a way that no point P_k lies above it.*

Let us consider an example to illustrate the previous definition.

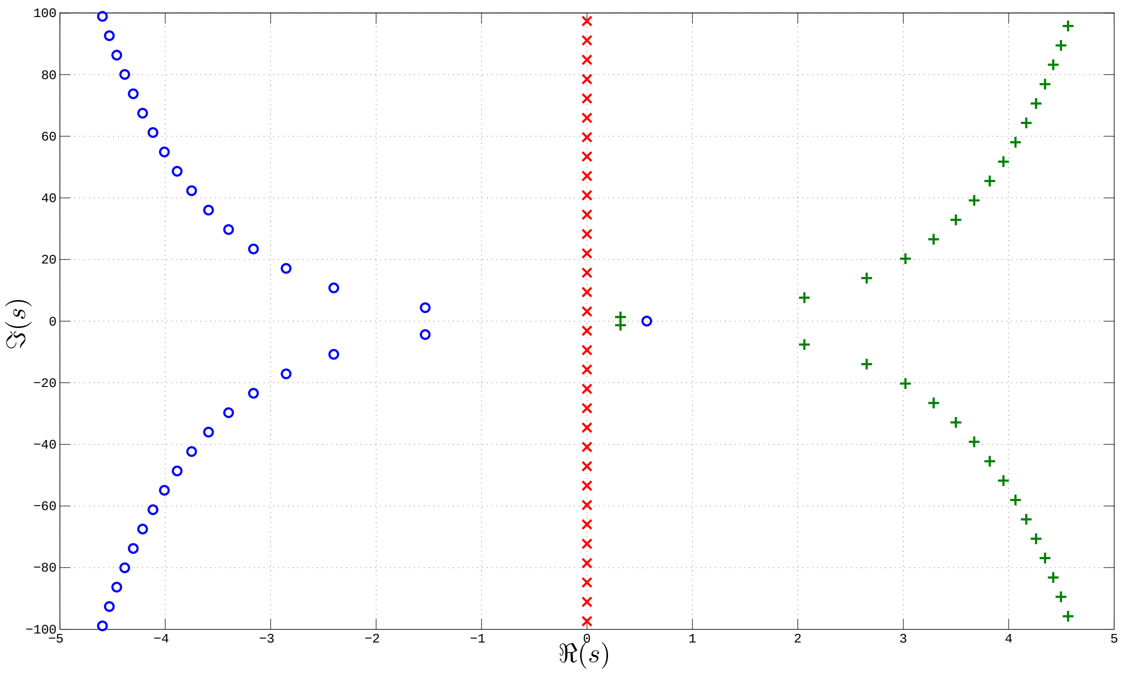

Figure 3.1: Retarded, Neutral and Advanced Chains of Poles

Example 3.4 Let us consider a system with transfer function given by (3.1), with the time delay $\tau = 1$ and

$$n(s) = s - 4s^3e^{-s} + 3s^5e^{-2s} + 6s^4e^{-3s} - 5s^5e^{-4s} + (2s^5 - s^3)e^{-6s} + 2s^3e^{-7s} \tag{3.7}$$

Figure 3.2 brings the distribution diagram for that system. As we can see, the line segments connects some of the points P_k, but not all of them, in order to be a concave curve and have no points above it.

The colors in the line segments anticipate the result given in the next theorem, which will state that each one of those segments will be related to a particular chain of poles, and the sign of their gradients defines their type between retarded, neutral or advanced. ♦

Theorem 3.2 *Let $d(s)$ and its distribution diagram be as given in Definition 3.1. The zeros of $d(s)$ for large values of $|s|$ are asymptotic to the zeros of the functions*

$$f_i(s) = c_{k_i(1)}s^{d_{k_i(1)}}e^{-k_i(1)s\tau} + \ldots + c_{k_i(m)}s^{d_{k_i(m)}}e^{-k_i(m)s\tau}, \tag{3.8}$$

where $P_{k_i(1)}, \ldots, P_{k_i(m)}$ are in the i^{th} edge of the distribution diagram.

Proof: *The proof is based on the fact that a zero of $d(s)$ for a large $|s|$ can only happen if*

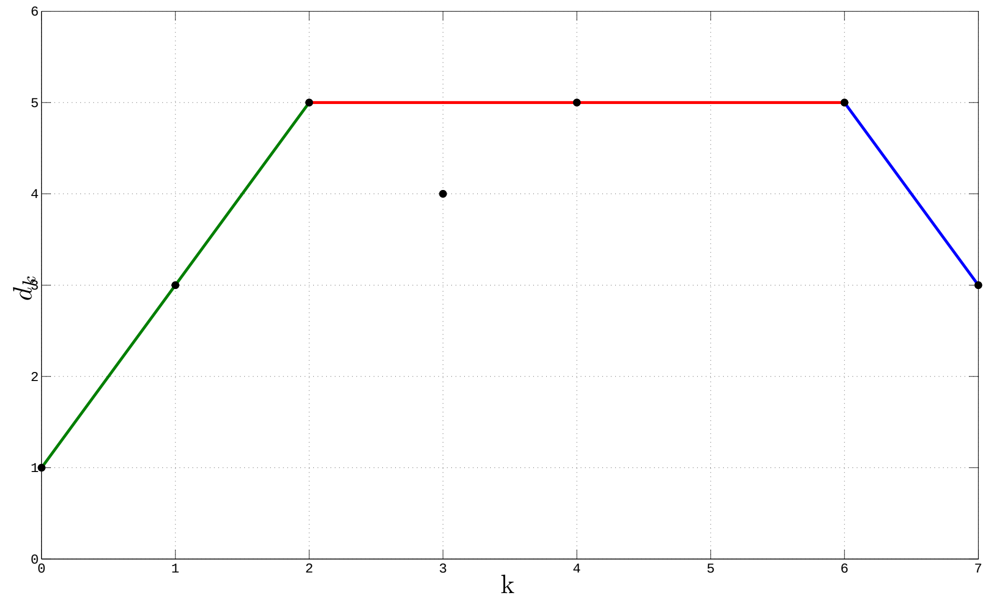

Figure 3.2: Distribution Diagram for Example 3.4

two or more terms are of the same order of magnitude. Since it is rather long, well-known and not the main point of the chapter, we will omit it, leaving the aforementioned books (Bellman & Cooke 1963) and (Partington 2004) as references for this proof. □

Example 3.5 Now we are in position to work out the complete classification of the chains of poles for the system given in Example 3.4. Applying the results of Theorem 3.2, we can say that to find approximations to the poles of large modulus, we only need to solve each of those equations independently:

$$s - 4s^3e^{-s} + 3s^5e^{-2s} = 0; \tag{3.9}$$

$$3s^5e^{-2s} - 5s^5e^{-4s} + 2s^5e^{-6s} = 0; \tag{3.10}$$

$$2s^5e^{-6s} + 2s^3e^{-7s} = 0. \tag{3.11}$$

These 3 equations can be easily reduced to:

$$s^2e^{-s} = 1 \text{ or } \frac{1}{3}; \tag{3.12}$$

$$e^{-2s} = 1 \text{ or } \frac{3}{2}; \tag{3.13}$$

$$s^2e^{s} = -1. \tag{3.14}$$

These last equations can be solved using Lemma 3.1. From there, we can see that the first equation gives four advanced chains of poles corresponding to $se^{-s/2} = \pm 1, \ \pm 1/\sqrt{3}$, the second one gives two neutral chains and finally the third one gives two retarded chains, corresponding to $se^{s/2} = \pm \jmath$. ♦

From the example, we can see that every edge of the distribution diagram where the gradient is positive represents some advanced chains. Whenever the gradient is zero, this implies that some chains are neutral, and negative gradients indicate retarded chains. Therefore, in order to avoid systems with advanced chains, which are unstable for all $\tau > 0$, the necessary and sufficient condition is that $\deg p(s) \geq \deg q_k(s)$ for all $k \in \mathbb{N}_n$. Moreover, if there exists at least one k in $\mathbb{N}_N$ such that $\deg p = \deg q_k$, then the systems will have neutral chains of poles.

This analysis concludes our classification of chains of poles for classical time-delay systems. In the next subsection we will deal exclusively with the neutral chains of poles. Up to now, we are just able to find the asymptotic axis where the neutral chains are asymptotic to, but when dealing with neutral chains, the information of how those poles approach the axis might be crucial. Moreover, as we will see, for some particular cases, the conditions for $\mathcal{H}_\infty$-stability is more involved than just the location of the poles.

3.2.2 Asymptotic Location of Neutral Poles of Time-Delay System

In this part, we continue to look at time-delay systems with transfer functions of the form given by equation (3.1). But in this subsection we will admit some restrictions in the degree of the polynomials, namely $\deg p \geq \deg t$, $\deg p \geq \deg t_k$ for all $k \in \mathbb{N}'_N$, $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$ and $\deg p = \deg q_k$ at least for one $k \in \mathbb{N}_N$. The first two restrictions on the degree guarantee that the system is proper, and therefore $\mathcal{H}_\infty$-stability can be achieved. The other two, as we explained previously, ensure at the same time that no advanced chains of poles are presented and also that at least one neutral chain of poles exists.

The work we will presented in this subsection is in some sense the extension of the analysis developed in (Partington & Bonnet 2004). Their purpose was similar, but the technique developed was able to deal only with numerators without delays and denominators with a single delay term. This means that it was restricted to the case where $N' = 0$ and $N = 1$ above. It is obvious that a different approach is necessary when dealing with the complete case.

The main motivation of this study is the fact already mentioned in the previous chapter that the stability of a system of type (3.1) is linked to the location of its poles. A neutral delay systems will possess a finite number of isolated poles (that we will call "*poles of small modulus*"), which are the ones that cannot be approached by the chains, as well as an infinite number of poles defining its chains of poles. Those poles of small modulus can be found by numerical techniques, and will be the subject of the next chapter.

In order to ease the notation, we will write $z = e^{-s\tau}$. Since $\deg p \geq \deg q_k$ for all $k \in \mathbb{N}_N$, we can suppose that for each k

$$\frac{q_k(s)}{p(s)} = \alpha_k + \frac{\beta_k}{s} + \frac{\gamma_k}{s^2} + \mathcal{O}(s^{-3}) \qquad \text{as} \quad |s| \to \infty, \tag{3.15}$$

where, recalling the previous definition that $\deg p = d_0$ and $\deg q_k = d_k$, we can write the polynomials as $p(s) = \sum_{m=0}^{d_0} \rho_m s^m$ and $q_k(s) = \sum_{m=0}^{d_0} \sigma_{k,m} s^m$, where, by definition, $\sigma_{k,m} = 0$ for all $m > d_k$. In this case, equation (3.15) is satisfied with

$$\alpha_k = \frac{\sigma_{k,d_0}}{\rho_{d_0}}, \tag{3.16}$$

$$\beta_k = \frac{\sigma_{k,d_0-1} - \alpha_k \rho_{d_0-1}}{\rho_{d_0}}, \tag{3.17}$$

$$\gamma_k = \frac{\sigma_{k,d_0-2} - \alpha_k \rho_{d_0-2} - \beta_k \rho_{d_0-1}}{\rho_{d_0}}. \tag{3.18}$$

The coefficient of the highest degree term of $p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}$ can then be written as a multiple of the following polynomial in z:

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \tag{3.19}$$

From now on, in order to avoid the possibility of an infinite number of zero cancelations between the numerator and denominator of G, we will also consider the following:

Hypothesis 3.3 *The numerator of $G(s)$ defined in equation (3.1) satisfies either:*

a) $\deg t(s) > \deg t_k(s)$*; or*

b) $\deg t_k(s) = \deg t(s)$ *for at least one k and the polynomial $\tilde{c}_n$ defined as in (3.19) relatively to the quasi-polynomial $n(s)$ has no root of modulus less than or equal to one, and no common root of modulus strictly greater than one with $\tilde{c}_d$.*

The idea behind this assumption is to guarantee that there is not an infinite number of zeros/poles cancelation that would change the results that we will present in the sequel. It is evident, though, that Hypothesis 3.3 is just a sufficient condition for this purpose.

Our initial concern is to find the position of the vertical lines to which the roots of the neutral chains are asymptotic. This can be derived from (Pontryagin 1955), as it was done in (Bellman & Cooke 1963), and is summarized in the next proposition.

Proposition 3.4 *Let $G(s)$ be a neutral system defined as in (3.1), satisfying Hypothesis 3.3. There exist neutral chains of poles which asymptotically approach the vertical lines*

$$\Re(s) = -\frac{\ln(|r|)}{\tau} \tag{3.20}$$

for each root $z = r$ of the polynomial $\tilde{c}_d(z)$.

Proof: A standard argument involving Rouché's Theorem (Bellman & Cooke 1963), as it was presented in (Partington & Bonnet 2004) and (Rabah et al. 2005), shows that the poles of the neutral chains of $G(s)$ are asymptotic to the solutions of $e^{-s\tau} = r$, which leads to

$$s_n\tau \approx \lambda_n = -\ln(r) + 2\jmath n\pi, \qquad n \in \mathbb{Z}. \tag{3.21}$$

from where the result is immediate. □

The case where all the roots of (3.19) have multiplicity one is much easier to analyze and can be completely treated. In this particular case, let $M \leq N$ denotes the greatest integer such that $\alpha_M \neq 0$. As it can be seen from the analysis of the previous sub-section, in this case, there will be M chains of neutral poles, and since those are asymptotic to vertical lines, it is now necessary to discover on which side the actual poles lie. As it will be seen later, this is especially important for poles such that $|r| = 1$, because this will provide the information on which side of the imaginary axis the poles are, and this plays a crucial role in questions about stability. This analysis is the objective of the next theorem.

Theorem 3.5 *Let $G(s)$ be a neutral delay system defined by (3.1), satisfying Hypothesis 3.3, and suppose that all the roots of (3.19) have multiplicity one. For each r such that $z = r$ is a root of (3.19) and for large enough $n \in \mathbb{Z}$, the solutions asymptotic to (3.20) are given by*

$$s_n\tau = \lambda_n + \mu_n + \mathcal{O}(n^{-2}), \tag{3.22}$$

with λ_n given by (3.21) and

$$\mu_n = -\jmath \frac{\tau \sum_{k=1}^{N} \beta_k r^k}{2\pi n \sum_{k=1}^{N} k\alpha_k r^k} . \tag{3.23}$$

Proof: We have $p(s_n) + \sum_{k=1}^{N} q_k(s_n)e^{-ks_n\tau} = 0$. Dividing both sides by $p(s_n)$ leads to

$$1 + \sum_{k=1}^{N} \frac{q_k(s_n)}{p(s_n)} e^{-ks_n\tau} = 0 . \tag{3.24}$$

Writing $s_n\tau = \lambda_n + \mu_n + \mathcal{O}(n^{-2})$ when s_n is the pole near λ_n, noting that $e^{-\lambda_n} = r$, and using (3.15) together with the fact that $e^{-\mu_n k} = 1 - \mu_n k + \mathcal{O}(n^{-2})$ leads to

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k}{s_n} \right) r^k \left(1 - \mu_n k\right) + \mathcal{O}(n^{-2}) = 0 , \tag{3.25}$$

and hence, assuming n large enough in (3.21)

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k \tau}{\jmath 2\pi n} \right) r^k \left(1 - \mu_n k\right) + \mathcal{O}(n^{-2}) = 0 . \tag{3.26}$$

Considering the approximation up to $\mathcal{O}(n^{-2})$, remembering that r is a root of (3.19), we obtain

$$\frac{\tau}{\jmath 2\pi n} \sum_{k=1}^{N} \beta_k r^k - \mu_n \sum_{k=1}^{N} k\alpha_k r^k + \mathcal{O}(n^{-2}) = 0 , \tag{3.27}$$

which completes the proof under the assumption that r is a root of multiplicity one of (3.19). □

Some conclusions can be drawn by considering μ_n. First of all, associated with each root r of (3.19), let us define K_r as

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k} , \tag{3.28}$$

where, again, K_r is well defined as it is assumed that r has multiplicity one as a root of (3.19).

Since our interest is mainly on which side of the vertical line the poles are, we need to pay attention to $\Im(K_r)$. As α_k and β_k are real numbers for all $k \in \mathbb{N}_N$, if $\Im(r) = 0$ then $\Im(K_r) = 0$. This implies that $\Im(r) \neq 0$ is a necessary (but not sufficient) condition for $\Im(K_r) \neq 0$. But in this case, $\bar{r}$ is also a root of (3.19), and as $\Im(K_r) = -\Im(K_{\bar{r}})$, if $\Im(K_r) \neq 0$, we certainly have chains of poles on both sides of the asymptotic line.

In case $\Im(K_r) = 0$, as, for example, it is always the case when $r \in \mathbb{R}$ or $\beta_i = 0$ for all $i \in \mathbb{N}_N$, then the leading term of μ_n is purely imaginary, and we need to look at the next term. This is the focus of the following theorem.

Theorem 3.6 *Let $G(s)$ be a neutral delay system defined by (3.1), satisfying Hypothesis 3.3 and suppose that all the roots of (3.19) have multiplicity one. For each r such that $z = r$ is a root of (3.19) and large enough $n \in \mathbb{Z}$, the solutions asymptotic to (3.20) are given by*

$$s_n\tau = \lambda_n + \mu_n + \nu_n + \mathcal{O}(n^{-3}), \tag{3.29}$$

with λ_n given by (3.21), μ_n by (3.23) and

$$\nu_n = \frac{\tau^2 \sum_{k=1}^{N} \left(-k^2\alpha_k K_r^2/2 + k\beta_k K_r - \gamma_k - \beta_k \ln(r)/\tau\right) r^k}{4\pi^2 n^2 \sum_{k=1}^{N} k\alpha_k r^k}. \tag{3.30}$$

Proof: We start the same way as in Theorem (3.5), by supposing that $p(s_n)+\sum_{k=1}^{N} q_k(s_n)e^{-ks_n\tau} = 0$ and dividing both sides by $p(s_n)$, leading to (3.24).

Using the same considerations, but now up to $\mathcal{O}(n^{-3})$, i.e., in particular writing $s_n\tau = \lambda_n + \mu_n + \nu_n + \mathcal{O}(n^{-3})$, we obtain

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\beta_k}{s_n} + \frac{\gamma_k}{s_n^2}\right) r^k \left(1 - \mu_n k + \frac{\mu_n^2 k^2}{2} - \nu_n k\right) + \mathcal{O}(n^{-3}) = 0\,, \tag{3.31}$$

and hence, assuming n large enough

$$1 + \sum_{k=1}^{N} \left(\alpha_k + \frac{\tau\beta_k}{\jmath 2\pi n} - \frac{\tau\beta_k \ln(r) + \tau^2\gamma_k}{4\pi^2 n^2}\right) r^k \left(1 - \mu_n k + \frac{\mu_n^2 k^2}{2} - \nu_n k\right) + \mathcal{O}(n^{-3}) = 0\,. \tag{3.32}$$

Considering the approximation up to $\mathcal{O}(n^{-3})$, remembering that r is a root of (3.19), and μ_n is given by (3.23), we obtain

$$\sum_{k=1}^{N} \left(\alpha_k \left(-\frac{\tau^2 K_r^2 k^2}{8\pi^2 n^2} - \nu_n k\right) + \frac{\tau^2 K_r \beta_k k}{4\pi^2 n^2} - \frac{\tau\beta_k \ln(r) + \tau^2\gamma_k}{4\pi^2 n^2}\right) r^k + \mathcal{O}(n^{-3}) = 0\,, \tag{3.33}$$

which completes the proof by using the standard considerations and finally solving for ν_n. □

Again, as before, we will be interested in the sign of the real part of ν_n. If it is negative, then the poles of high modulus will be located on the left of the asymptotic line. Similarly, if it is positive, the poles will be on the right. If it is 0, then it might be necessary to expand the analysis to even higher degrees of n, or it might be the case that the chain of poles is located on the vertical line.

If we look with attention, there are some interesting aspects about equation (3.30). Specifically, the term $\beta_k \ln(r)/\tau$ can indeed change the sign of $\Re(\nu_n)$ as τ increases. This means that there can be a critical value of the delay $\tau^\star$ such that for $\tau < \tau^\star$ the chain approaching asymptotically from one side and for $\tau > \tau^\star$ from the other side. This effect, though, cannot happen when the asymptotic axis is the imaginary axis. This is the purpose of the next proposition.

Proposition 3.7 *Let $G(s)$ be a neutral delay system defined by (3.1), satisfying Hypothesis 3.3 and suppose that all the related roots of (3.19) have multiplicity one. Then, a variation in the delay τ might change the position of the asymptotic axis, but does not change the side (left or right the imaginary axis) where it is lying. Also, if the imaginary axis is the asymptotic axis, it remains one when τ varies to τ'. Finally, suppose also that ν_n, as defined in Theorem 3.6, has a nonzero real part. Then the side where the poles of the chain are lying around the imaginary axis remains the same.*

Proof: The first part of the proposition is trivial. Suppose we have a change in the delay, that might be even big, and that $\tau > 0$ is changed to $\tau' > 0$. Certainly, G is still a delay system with commensurate delays, the delay of reference being now τ'.

This allows us to carry the same analysis and we see that the asymptotic axis are now located at $\Re(s) = -\ln(|r|)/\tau'$. Clearly, $\Re(s)$ cannot change sign when τ varies. Moreover, if the imaginary axis is an asymptotic axis when the delay is equal to τ, it means that $|r| = 1$ and therefore it remains the same when τ moves to τ'.

So we consider the case where $|r| = 1$. If $\Im(K_r) \neq 0$, then necessarily there are chains on both sides of the imaginary axis, for all values of τ. So, for the case of interest, where $\Im(K_r) = 0$, we recall that

$$\nu_n = \tau^2 \frac{\sum_{k=1}^{N} \left(-k^2\alpha_k K_r^2/2 + k\beta_k K_r - \gamma_k\right) r^k}{4\pi^2 n^2 K_r \sum_{k=1}^{N} \beta_k r^k} - \tau \frac{\sum_{k=1}^{N} \beta_k \ln(r) r^k}{4\pi^2 n^2 K_r \sum_{k=1}^{N} \beta_k r^k} = \tau^2 T_1 - \tau T_2, \qquad (3.34)$$

where T_1 and T_2 do not depend on τ. We can easily simplify T_2 as

$$T_2 = \frac{\ln(r)}{4\pi^2 n^2 K_r}. \tag{3.35}$$

Since $|r| = 1$ and $4\pi^2 n^2 K_r$ is real, T_2 is purely imaginary, and therefore

$$\nu_n = \tau^2 \Re\{T_1\} + \jmath \left(\tau^2 \Im\{T_1\} - \tau \frac{\arg(r)}{4\pi^2 n^2 K_r} \right), \tag{3.36}$$

from where we see that the sign of the real part of ν_n is not affected by τ. □

In summary, in the case of perfect commensurate delays, as it is the case when $G(s)$ is the transfer function of linear time delay system described in the time domain, then a change in the delay does not affect the stability analysis. However, if this modeling is not correct and a system initially modeled by, for instance, $G(s) = t(s)/(p(s) + q_1(s)e^{-s\tau_1} + q_2(s)e^{-s\tau_2})$ with $\tau_2 = n\tau_1$, with $n \in \mathbb{N}$, is changed into a system $\tilde{G}(s) = t(s)/(p(s) + q_1(s)e^{-s\tau_1'} + q_2(s)e^{-s\tau_2'})$ where $\tau_2'/\tau_1' \neq n$, then the present analysis does not apply in its present form. On the other hand, if τ_1' and τ_2' are still rationally dependent, then it is possible to find a τ such that the system is in the form given by (3.1), but in this case we have no guarantee that the asymptotic analysis will be close to the original one.

Considering now robustness relatively to changes in the coefficients of the polynomials, we can see that even if there might be some robustness bounds they are not easy to determine if we allow all coefficients to vary at the same time. In the general case, we may conclude that a small change in all coefficients may move the asymptotic position of a chain and the location of its poles from left to right.

With this stability results in hand, in the next subsection, we will consider the $\mathcal{H}_\infty$-stability of $G(s)$.

3.2.3 $\mathcal{H}_\infty$-stability and stabilizability

We are now interested in answering the question of stability of $G(s)$. Let us recall that the notion on which we will concentrate is $\mathcal{H}_\infty$-stability, that is, the system has a finite $L_2(0,\infty)$ gain:

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u \in L^2, u \neq 0} \frac{\|Gu\|_{L^2}}{\|u\|_{L^2}} < \infty \tag{3.37}$$

and that $\mathcal{H}_\infty(\mathbb{C}_+)$ is the space of functions which are analytic and bounded in the open right half-plane $\mathbb{C}_+$.

We will refer to poles in the closed right half-plane $\overline{\mathbb{C}_+}$ as *unstable poles*, and those in the open left half-plane $\mathbb{C}_-$ as *stable poles*.

The case where the polynomial (3.19) possesses only roots of modulus strictly greater than one is easy to handle as there exists $a > 0$ such that the system has a finite number of poles in $\{\Re(s) > -a\}$. Also, the case where polynomial (3.19) possesses at least one root of modulus strictly less than one is obvious.

It is well-known that coprime factorization over $\mathcal{H}_\infty$ allows us to obtain a parametrization of the set of all $\mathcal{H}_\infty$ stabilizing controllers (Bonnet & Partington 1999). For our current case, coprime and Bézout factors can be determined following the same method that was used for retarded systems (Bonnet & Partington 1999).

Proposition 3.8 *Let $G(s)$ be a transfer function of type (3.1) and satisfying Hypothesis 3.3. Then:*

1) If the polynomial (3.19) possesses at least one root of modulus strictly less than one, G cannot be $\mathcal{H}_\infty$-stable.

2) If the polynomial (3.19) possesses only roots of modulus strictly greater than one, then the following statements hold:

(a) G is $\mathcal{H}_\infty$-stable if and only if G has no unstable pole.

(b) G is $\mathcal{H}_\infty$-stabilizable. Moreover, suppose that $n(s)$ and $d(s)$ have no common unstable zeroes, then a coprime factorization of G can be given by

$$N(s) = \frac{n(s)}{(s+1)^\delta}, \qquad D(s) = \frac{d(s)}{(s+1)^\delta}. \tag{3.38}$$

Proof: For the first part, the result follows directly from Proposition 3.4.

In order to show the results for item 2a, notice that if all the roots of $\tilde{c}_d$ are of modulus strictly greater than one, then from proposition 3.4 we know that all chains of roots are asymptotic to axes which are in the left half-plane. We can then determine $\epsilon > 0$ such that G has a finite number of poles in $\{\Re(s) > -\epsilon\}$. In this situation, the absence of poles of G in the closed right half plane is a necessary and sufficient condition for G to be analytic and bounded in the closed right half-plane.

Finally, to prove 2b, suppose that n and d do not possess common unstable zeroes and let $N(s)$ and $D(s)$ be given as in equation (3.38). It is easy to see that $\inf_{\Re(s)>0} (|N| + |D|) > 0$, which means that (N, D) is a coprime factorization of G over $\mathcal{H}_\infty$, and this is equivalent to G being $\mathcal{H}_\infty$-stabilizable (Smith 1989).

In the case where n and d possess common unstable zeroes (they are in finite numbers), we first divide by common factors and then apply the same formula as above to obtain a coprime factorization of (N, D). □

We show now how to calculate the Bézout factors for a simple example.

Example 3.6 Let $G(s)$ be a transfer function of type (3.1) with $t(s) = 1$, $p(s) = 2s^3 - 6s^2 + 4s$, $q_1(s) = s^3 - 2s^2 - s + 2$ and $q_2(s) = s^2 - 3s + 2$ and $h = 1$. This transfer function has an infinite number of poles asymptotic to $\Re(s) = -\ln(2)$ and two unstable poles at $s = 1$ and $s = 2$. A coprime factorization can be given by

$$N(s) = \frac{1}{(s+1)^3}, \tag{3.39}$$

$$D(s) = \frac{p(s) + q_1(s)e^{-s} + q_2(s)e^{-2s}}{(s+1)^3}. \tag{3.40}$$

The Bézout factors X and Y which satisfy the equality $NX + DY = 1$ are given by

$$X(s) = \frac{\mu(s)}{s+1}, \tag{3.41}$$

$$Y(s) = \frac{(s+1)^3 - \mu(s)/(s+1)}{p(s) + q_1(s)e^{-s} + q_2(s)e^{-2s}}, \tag{3.42}$$

where $\mu(s)$ is polynomial of degree one which satisfies

$$(s+1)^4 - \mu(s) = 0 \tag{3.43}$$

at $s = 1$ and $s = 2$, that is

$$X(s) = \frac{65s - 49}{s+1} \tag{3.44}$$

$$Y(s) = \frac{s^4 + 4s^3 + 6s^2 - 61s + 50}{(s+1)(p(s) + q_1(s)e^{-s} + q_2(s)e^{-2s})}. \tag{3.45}$$

♦

On the other hand, the case where (3.19) has at least one root of modulus one is somehow delicate, as even if all poles of $G(s)$ are located in $\mathbb{C}_-$, for $\mathcal{H}_\infty$-stability, it remains to see if G is bounded on the imaginary axis (such an example is given in (Partington & Bonnet 2004)). The results of the single delay case (Partington & Bonnet 2004) easily extend to the multi-delay case.

3.2.3.1 When all the roots of $\tilde{c}_d(z)$ are of multiplicity one

Proposition 3.9 *Let $G(s)$ be a transfer function given as in Theorem 3.6 and suppose that (3.19) has at least one root of modulus one and multiplicity one, the other roots being of modulus strictly greater than one.*

1. *Suppose that $\Re(\nu_n) < 0$ and that G has no unstable pole. Then G is $\mathcal{H}_\infty$-stable if and only if $\deg p \geq \deg t + 2$.*

2. *If $\nu_n = 0$ then the condition $\deg p \geq \deg t + 2$ is still necessary for $\mathcal{H}_\infty$-stability.*

Proof: Let $s = s_n + \eta \in \jmath\mathbb{R}$, we have

$$\left|p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}\right| \approx |\eta| \left|p'(s_n) + \sum_{k=1}^{N} q_k'(s_n)e^{-ks_n\tau} - \sum_{k=1}^{N} q(s_n)khe^{-ks_n\tau}\right| \tag{3.46}$$

$$\approx |\eta||p(s_n)| \left|\frac{p'(s_n)}{p(s_n)} + \sum_{k=1}^{N} \frac{q_k'(s_n)}{p(s_n)}e^{-ks_n\tau} - \sum_{k=1}^{N} \frac{q(s_n)}{p(s_n)}khe^{-ks_n\tau}\right| \tag{3.47}$$

Using (3.15) together with the fact that $e^{-\lambda_n} = r$ we get

$$\left|p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}\right| \approx |\eta||p(s_n)|\tau \left|\sum_{k=1}^{N} \alpha_k r^k k\right|. \tag{3.48}$$

Recall that $\left|\sum_{k=1}^{N} \alpha_k r^k k\right|$ is nonzero by assumption. Therefore, if $\nu_n \neq 0$, η is at least of order n^{-2}, so that evaluating the behavior of the numerator and denominator of G, a necessary and sufficient condition of $\mathcal{H}_\infty$-stability is that $\deg p \geq \deg t + 2$. If $\nu_n = 0$ then we do not know on which side of the imaginary axis the pole lie. However, we already know from the approximation above that we need the condition $\deg p \geq \deg t + 2$ in order to get a bounded transfer function on the imaginary axis. □

With respect to our last claim, let us consider the following example:

Example 3.7 Consider the following system with a single delay

$$G(s) = \frac{s+1}{s^3 + 2s^2 + 3s + (s^3 + 2s^2 + 3s + 1)e^{-s}}. \tag{3.49}$$

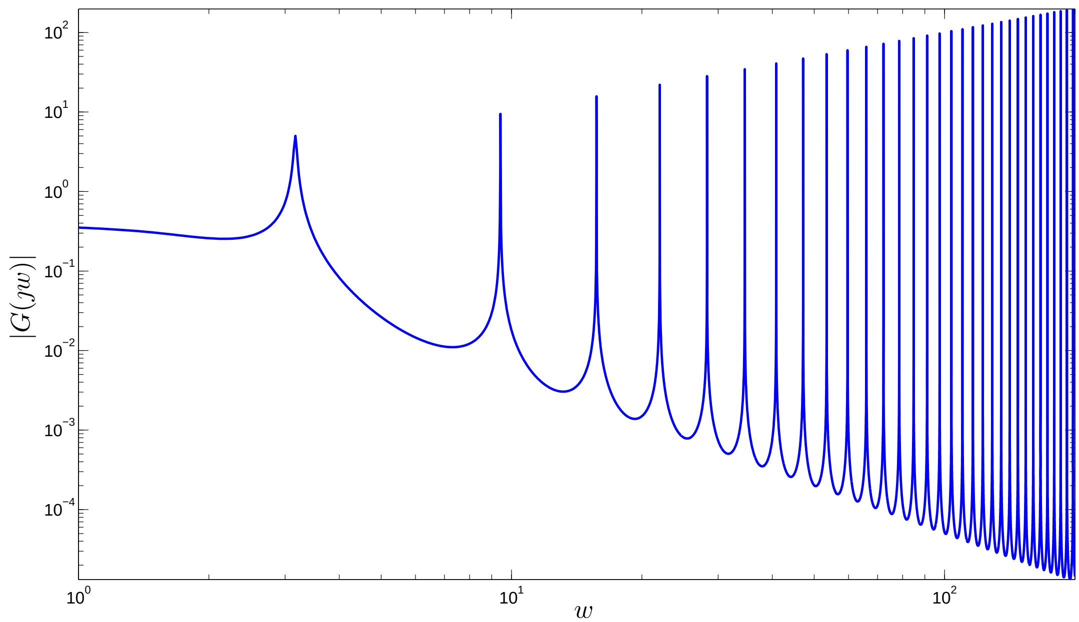

Figure 3.3: Frequency response for $G(s)$ in Example 3.7.

The asymptotic series of the ratio of the polynomials in the denominator of $G(s)$ is given by

$$\frac{q_1(s)}{p(s)} = 1 + s^{-3} + \mathcal{O}(s^{-4}) \qquad \text{as} \quad |s| \to \infty \tag{3.50}$$

There is one chain of neutral poles asymptotic to the imaginary axis. As $\beta_1 = \gamma_1 = 0$, it is easy to see that $\mu_n = \nu_n = 0$. Using the QPmR (Vyhlídal & Zítek 2003) algorithm, it is possible to show that all the poles of the system are in the left half-plane. But even though the difference of the degree between the denominator and the numerator is 2, Figure 3.3 shows that the system does not have a finite $\mathcal{H}_\infty$-norm, illustrating our claim in Proposition 3.9 that this condition is not sufficient when $\nu_n = 0$. ♦

With those results on the characterization of $\mathcal{H}_\infty$-stability in hand we are now able to discuss the stablizability of some neutral systems by rational controllers.

Proposition 3.10 *Let G be a system with transfer function given by (3.1) which satisfies Hypothesis 3.3 and is such that:*

1) the polynomial $\tilde{c}_d$ defined as in (3.19) relatively to the quasi-polynomial $d(s)$ (denominator of $G(s)$) possesses at least one root of modulus less or equal to one, but no root of modulus one with multiplicity strictly greater than one;

2) $\deg p = \deg t + 1$.

Then G is not stabilizable by a rational controller.

Proof: We know from (Partington & Bonnet 2004) that if K is $\mathcal{H}_\infty$-stabilizing G, it is necessarily proper. Therefore, the result when the polynomial $\tilde{c}_d$ possesses at least on root of modulus strictly less than one comes directly from Proposition 3.8, item 1. So from now on, we consider the case when $\tilde{c}_d$ possesses roots of modulus equal or greater than one.

Let us consider a rational controller $K(s) = u(s)/v(s)$ and let v^0 be the coefficient of the highest degree term of v. Looking at the term $G(1+GK)^{-1}$ of the closed-loop $[G, K]$, we see that its numerator nv gives a polynomial $\tilde{c}_{n'} = \tilde{c}_n v^0$ and its denominator $nu + dv$ a polynomial $\tilde{c}_{d'} = \tilde{c}_d v^0$.

Consider the denominator of $G(1+GK)^{-1}$ and let λ_n be defined as in (3.21). Now, let s_n be a pole of G near λ_n and let $s = s_n + \nu \in \jmath\mathbb{R}$. We obtain the following estimates :

$$\begin{aligned} |n(s)u(s) + d(s)v(s)| &\approx |\nu p(s_n)||v(s)|, \\ |n(s)v(s)| &\approx |t(s)v(s)|. \end{aligned}$$

Under the first hypothesis of the proposition and Theorem 3.6, we have that ν is at least of order n^{-2} so that we need $\deg p \geq \deg t + 2$ in order to obtain $\mathcal{H}_\infty$-stability. But this is contrary to the second hypothesis of the proposition, and therefore, such a K does not exist. □

3.2.3.2 When some roots of $\tilde{c}_d(z)$ are of multiplicity greater than one

If $\tilde{c}_d(z)$ has a root of multiplicity greater than one, this means that several chains may be asymptotic to the same vertical axis with poles asymptotic to the same points on this axis.

Although we might be able to adapt the technique developed here to deal with these cases, a general result would present many cases and be very cumbersome to present. Therefore, we will consider here a simple case giving rise to such a situation, the case where $G(s)$ can be factorized as

$$G(s) = \frac{t(s)}{\prod_{k=1}^{N} (p_k(s) + q_k(s)e^{-s\tau})}, \tag{3.51}$$

with p_k and q_k being polynomials of same degree δ, satisfying $\lim_{|s|\to\infty} p_k(s)/q_k(s) = \alpha_k = \pm 1$. Such a system may model the interesting realistic situation of a cascade of N neutral systems.

The system G clearly possesses N chains of poles around the imaginary axis. For sufficiently large integers n, let $\lambda_n = 2ni\pi$ if $\alpha_k = -1$ and let $\lambda_n = (2n+1)i\pi$ if $\alpha_k = 1$ and let

$$s_n^k = \frac{\lambda_n}{\tau} - \frac{\beta_k}{\alpha_k \lambda_n} + \frac{\tau}{\lambda_n^2}\left(\frac{\beta_k^2}{2} - \frac{\gamma_k}{\alpha_k}\right) + \mathcal{O}(n^{-3}). \tag{3.52}$$

be the sequence of roots of $p_k(s) + q_k(s)e^{-ks\tau}$.

Let $0 \leq K_1 \leq N$ be an integer such that $\alpha_k = 1$ for $k = 1, \dots, K_1$, and $\alpha_k = -1$ for $k = K_1 + 1, \dots, N$, with the convention that *if* $K_1 = 0$ *then* $\alpha_k = -1$ *for* $k = 1, \dots, N$. Then the next proposition holds.

Proposition 3.11 *Let $G(s)$ be defined as in (3.51) with p_k and q_k satisfying (3.15) and suppose that $2\gamma_k\alpha_k^{-1} < \beta_k^2$, so that all chains of poles of G are in the left half-plane. Suppose moreover that G does not have any unstable root of small modulus.*

1) If for all $k, k' \in \{1, \dots, K_1\}$, $\beta_k = \beta_{k'}$ and $\gamma_k \neq \gamma_{k'}$, and for all $l, l' \in \{K_1+1, \dots, N\}$ $\beta_l = \beta_{l'}$ and $\gamma_l \neq \gamma_{l'}$, then:

$$G \text{ is } \mathcal{H}_\infty\text{-stable} \iff \deg p_1 \geq \frac{\deg t + 2\max(K_1, N-K_1)}{N}. \tag{3.53}$$

2) Suppose that the β_k $(1 \leq k \leq N)$ are not all identical and let $1 \leq N_1 \leq K_1$ (respectively $1 \leq N_2 \leq N - K_1$) be the maximum number of β_k $(1 \leq k \leq K_1)$ (respectively $K_1 + 1 \leq k \leq N$) which are identical. Moreover, suppose that the N_1 (respectively N_2) chains which have identical β_k all possess different γ_k. In this case:

$$G \text{ is } \mathcal{H}_\infty\text{-stable} \iff \deg p_1 \geq \max\left(\frac{\deg t + N_1 + K_1}{N}, \frac{\deg t + N_2 + N - K_1}{N}\right). \tag{3.54}$$

3) In the particular case where $G(s) = \dfrac{t(s)}{(p_1(s) + q_1(s)e^{-hs})^N}$, we have

$$G \text{ is } \mathcal{H}_\infty\text{-stable} \iff \deg p_1 \geq \frac{\deg t}{N} + 2. \tag{3.55}$$

Proof:

1) Let us evaluate the norm of G on the imaginary axis. Let $1 \leq k^* \leq K_1$ and let us evaluate the norm of G near $s_n^{k^*}$, that is, on $s = s_n^{k^*} + \nu_{k^*} \in \jmath\mathbb{R}$, with $s_n^{k^*}$ being a root of $p_{k^*}(s) + q_{k^*}(s)e^{-s\tau}$. As for all $1 \leq k \leq K_1$, $2\gamma_k\alpha_k^{-1} < \beta_k^2$, we have that ν_{k^*} is of order n^{-2}.

For every $1 \leq k^* \leq K_1$, $k \neq k^*$, we can write

$$s = s_n^{k^*} + \nu_{k^*} = s_n^{k^*} - s_n^k + s_n^k + \nu_{k^*}, \tag{3.56}$$

where s_n^k is a root of $p_k(s) + q_k(s)e^{-hs}$.

As by hypothesis, for all $k, k' \in \{1, \ldots, K_1\}$, $\beta_k = \beta_{k'}$ and $\gamma_k \neq \gamma_{k'}$, the s_n^k $(1 \leq k \leq K_1)$ are at distance of order n^{-2} from each other so that

$$s = s_n^k + \nu_k \tag{3.57}$$

with ν_k of order n^{-2}.

With these results, we obtain:

$$\begin{aligned}
\left|\prod_{k=1}^{K_1}(p_k(s) + q_k(s)e^{-s\tau})\right| &= \left|(p_{k^*}(s) + q_{k^*}(s)e^{-s\tau}) \prod_{k\neq k^*=1}^{N}(p_k(s) + q_k(s)e^{-s\tau})\right| \\
&= \Big|(p_{k^*}(s_n^{k^*} + s - s_n^{k^*}) + q_{k^*}(s_n^{k^*} + s - s_n^{k^*})e^{-\tau(s_n^{k^*}+s-s_n^{k^*})}) \\
&\qquad \prod_{k\neq k^*=1}^{N}(p_k(s_n^k + s - s_n^k) + q_k(s_n^k + s - s_n^k)e^{-\tau(s_n^k+s-s_n^k)})\Big| \\
&\approx \Big|\nu_{k^*}(p'_{k^*}(s_n^{k^*}) + q'_{k^*}(s_n^{k^*})e^{-\tau s_n^{k^*}} - \tau q_{k^*}(s_n^{k^*})e^{-\tau s_n^{k^*}}) \\
&\qquad \prod_{k\neq k^*=1}^{K_1}\nu_k(p'(s_n^k) + q'(s_n^k)e^{-\tau s_n^k} - \tau q(s_n^k)e^{-\tau s_n^k})\Big| \\
&\approx \left|\prod_{k=1}^{N}\nu_k(p'_k(s_n^k) - q'_k(s_n^k)\frac{p_k(s_n^k)}{q_k(s_n^k)} + \tau q_k(s_n^k)\frac{p_k(s_n^k)}{q_k(s_n^k)}\right| \\
&\approx \left|\prod_{k=1}^{K_1}\nu_k\tau p(s_n^k)\right|
\end{aligned}$$

and

$$\left|\prod_{k=1}^{N}(p_k(s) + q_k(s)e^{-\tau s})\right| \approx \prod_{k=1}^{K_1}\tau|\nu_k||p(s_n^k)| \prod_{k=K_1+1}^{N}\left|(p_k(s) + q_k(s)e^{-\tau s})\right| \tag{3.58}$$

with K_1 terms ν_k of order n^{-2}.

In this case, considering the behavior of the numerator and denominator of G, boundedness of G on the axis is equivalent to the condition

$$N \deg p_1 \geq \deg t + 2K_1. \tag{3.59}$$

Now, if $K_1 + 1 \le k^* \le N$, evaluating G near $s_n^{k^*}$ and involving the same arguments as above, we obtain

$$\prod_{k=1}^{N}(p_k(s) + q_k(s)e^{-s\tau}) \approx \prod_{k=1}^{K_1}(p_k(s) + q_k(s)e^{-s\tau}) \prod_{k=K+1}^{N} \nu_k \tau p(s_k), \tag{3.60}$$

and the condition becomes

$$N \deg p_1 \ge \deg t + 2(N - K_1). \tag{3.61}$$

Finally, we obtain the condition

$$G \text{ is } \mathcal{H}_\infty\text{-stable} \iff \deg p_1 \ge \frac{\deg t + 2\max(K_1, N - K_1)}{N}. \tag{3.62}$$

2) In the case where some β_k are identical, let $1 \le N_1 \le K_1$ be the maximum number of β_k $(1 \le k \le K_1)$ which are identical. Then equation (3.56) can be rewritten as:

$$s = s_n^k + \nu_k, \tag{3.63}$$

with ν_k of order n^{-2} for integers k such that $\beta_k = \beta_{k^*}$ and of order n^{-1} for integers k such that $\beta_k \ne \beta_{k^*}$. This means that some roots are at distance of order n^{-1} from each other.

In equation (3.58), we then have N_1 terms of order n^{-2} and $K_1 - N_1$ of order n^{-1} so that boundedness of G occurs if and only if

$$N \deg p_1 \ge 2N_1 + (K_1 - N_1) + \deg t. \tag{3.64}$$

Considering the case of identical β_k for $K_1 + 1 \le k^* \le N$, we finally obtain the result.

3) This is a direct consequence from the single-delay case presented in (Partington & Bonnet 2004). □

We remark that this analysis is also valid in the case where $N = 2$ and $K_1 = 1$, that is the case of two simple roots. We also notice, thanks to Proposition 3.7, that the $\mathcal{H}_\infty$-stability analysis done in this section is not affected by a change in the delay.

3.2.4 State-Space Representation

Consider the delay-differential system

$$E\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + A_2\dot{x}(t-\tau) + Bu(t) \tag{3.65}$$

$$y(t) = Cx(t) \tag{3.66}$$

with zero initial conditions, where $A_0 = (a^0_{ij})$, $A_1 = (a^1_{ij})$, $A_2 = (a^2_{ij})$ and $E = (e_{ij})$ are $N \times N$ matrices. The scalars u and y represent, respectively, the input and output signals of the system. Its transfer function is given by:

$$G(s) = C(Es - A_0 - (A_2 s + A_1)e^{-s\tau})^{-1}B, \tag{3.67}$$

The transfer function of such a system, involving the determinant of $(Es - A_0 - (A_2 s + A_1)e^{-s\tau})$, is of the type (3.1) where t, p, t_k and q_k, for all $k \in \mathbb{N}_N$, are real polynomials of degree lower than N.

The first question which arises from this formulation concerns the type of system we are dealing with. More important, we want to guarantee that we are not dealing with an advanced system, in which case stability cannot be attained for any positive value of the delay. If we suppose that E is invertible, then without loss of generality we can take it to be the identity matrix, which is a common assumption. In this case, the system will be of neutral type if $A_2 \neq 0$ and of retarded type otherwise.

The case where E is not invertible must be taken with care. In fact, depending on the numerical values of each of the matrices, this description can lead to retarded, neutral or advanced systems. For example, if E is not invertible but A_2 is, this descriptor system gives rise to an advanced time-delay system.

When dealing with $\mathcal{H}_\infty$-stability of neutral type systems which have poles clustering the imaginary axis with the tools provided in the previous sessions, one approach is to explicitly calculate the determinant of (3.67). This can be automatized by symbolic packages, but a close look in the problem sheds light on the fact that for the position of the chain of poles and their asymptotic behavior, it is not necessary to obtain all the terms of the determinant of (3.67), but only those of higher degree in s. This can be done in a completely numerical way by exploiting some properties of the determinant. But before that, we need to introduce a generalized version of it.

Definition 3.2 *Let M be a set of m square matrices, with each M_i being of dimension $N \times N$, and v be a vector such that $v \in Z_+^{1\times m}$ and $\sum_{i=1}^m v_i = N$. We define the permutation determinant*

$$\det_p(\{M_1, M_2, \dots, M_m\}, [v_1, v_2, \dots, v_m]) \tag{3.68}$$

as the sum of the determinants of all matrices constructed with v_1 columns of M_1, v_2 columns of M_2, etc., where each one of those columns is positioned at the same place as in the original matrix.

For example, let us take

$$M_1 = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 3 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 3 & 1 & 1 \\ -3 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, \tag{3.69}$$

and $v = [2, 1]$. In this case, defining $M = \{M_1, M_2\}$ we have

$$\det_p(M, v) = \det \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} + \det \begin{bmatrix} 3 & 2 & 0 \\ -3 & 3 & -1 \\ 2 & 1 & 1 \end{bmatrix} = 14. \tag{3.70}$$

This function helps us to calculate some specific terms of the determinant of a sum of matrices. This is the objective of the next proposition.

Proposition 3.12 *Let z be a scalar variable and M_1 and M_2 two $N \times N$ matrices. Then $p(z) = \det(M_0 + M_1 z)$ is a polynomial in z such that $\deg p \leq N$, and the element in z^k of $p(z)$ is given by $\det_p(\{M_0, M_1\}, [N-k, k])$.*

Proof: This results directly from the multilinearity of the determinant. □

Naturally, the result of Proposition 3.12 can be expanded to the case where more than one scalar variable is present, even if we have multiplications of such variables. In these cases, though, in order to find a specific value of some element, it is necessary to calculate the sum of all the *permutation* determinants which will give rise to that specific element. For example, if $p(s, z) = \det(M_0 + M_1 s + M_2 z + M_3 sz)$, with $M = \{M_0, M_1, M_2, M_3\}$, where all indicated matrices are of dimension 3, then the coefficient of $s^2 z^2$ can be calculated by $\det_p(M, [1, 0, 0, 2]) + \det_p(M, [0, 1, 1, 1])$.

The results shown up to now let us calculate directly the values of the asymptotic behavior (3.15) without the explicit calculation of the transfer function. Considering E equals the identity matrix in order to avoid advanced time-delay systems, we have, for $M = \{I, -A_0, -A_1, -A_2\}$:

$$\rho_{d_0-\ell} = \det_p(\{I, -A_0\}, [N-\ell, \ell]) \tag{3.71}$$

$$\sigma_{k,d_0} = \det_p(\{I, -A_2\}, [N-k, k]) \tag{3.72}$$

$$\sigma_{k,d_0-1} = \det_p(M, [N-k-1, 1, 0, k]) + \det_p(M, [N-k, 0, 1, k-1]) \tag{3.73}$$

$$\sigma_{k,d_0-2} = \det_p(M, [N-k-2, 2, 0, k]) + \det_p(M, [N-k-1, 1, 1, k-1]) + + \det_p(M, [N-k, 0, 2, k-2]). \tag{3.74}$$

Notice that in this formulation, by definition, we have $\det_p\{M, v\}$ with $v \in \mathbb{Z}_+^{1\times 4}$. Therefore, every time one element of v is negative, the function is not defined and should not be taken into account.

All that has been presented enables us to show that, when E equals the identity matrix, the non-zero eigenvalues of A_2 are the inverse of the roots of (3.19) calculated from the determinant of (3.67). Considering $e^{-s\tau} = z$, we have

$$\det(sI - A_2 s z) = s^N \left(1 + \sum_{k=1}^{N} \det_p(\{I, -A_2\}, [N-k, k]) z^k\right) \tag{3.75}$$

$$= s^N \left(1 + \sum_{k=1}^{N} \alpha_k z^k\right). \tag{3.76}$$

But we can factor out the element s^N from the determinant, since

$$\det(sI - A_2 s z) = s^N \det(I - Az). \tag{3.77}$$

Therefore, the solutions of $1 + \sum_{k=1}^{N} \alpha_k z^k = 0$ are equivalent to the solutions of $\det(I - A_2 z) = 0$, from which the solution over the variable $z = e^{-s\tau}$ can be directly calculated as

$$z_i = \frac{1}{\mathrm{eig}_i(A_2)}, \tag{3.78}$$

showing that in this particular case, the eigenvalues of A_2 can be directly used to determine the asymptotic position of the chains of poles of the neutral system.

Thus, all the eigenvalues of A_2 inside the unit circle is a necessary and sufficient condition for the non-existence of unstable chains of poles, and any eigenvalue on the unit circle will provide a chain of poles asymptotic to the imaginary axis.

3.2.5 The particular case of two delays with illustrative examples

In this subsection we will illustrate the previous results in the simpler case of only two delays in the denominator and no delays in the numerator. Therefore, let

$$G(s) = \frac{t(s)}{p(s) + q_1(s)e^{-s\tau} + q_2(s)e^{-2s\tau}} \tag{3.79}$$

and consider that

$$\frac{q_k(s)}{p(s)} = \alpha_k + \frac{\beta_k}{s} + \frac{\gamma_k}{s^2} + \mathcal{O}(s^{-3}) \qquad \text{as} \quad |s| \to \infty, \tag{3.80}$$

for constants α_k, β_k and γ_k in $\mathbb{R}$, $k \in \{1, 2\}$ given by equations (3.16)-(3.17).

For the zeroes of $p(s) + q_1(s)z + q_2(s)z^2$ of large modulus, we have

$$s(1 + \alpha_1 z + \alpha_2 z^2) + (\beta_1 z + \beta_2 z^2) + \mathcal{O}(s^{-1}) = 0, \tag{3.81}$$

that is,

$$s = -\frac{\beta_1 z + \beta_2 z^2}{1 + \alpha_1 z + \alpha_2 z^2} + \mathcal{O}(s^{-1}), \tag{3.82}$$

and there are two cases to consider.

1. If $\alpha_2 \neq 0$ then there are only chains of neutral type.
2. If $\alpha_2 = 0$ (and hence $\alpha_1 \neq 0$), there is a chain of neutral type and also a retarded chain of zeroes: for example if $\beta_2 \neq 0$, this is asymptotic to solutions of $s = -\beta_2 z/\alpha_1$, and involves s and e^{-s} being simultaneously large in modulus.

For poles asymptotic to the imaginary axis, we require of course $|\alpha_1| = 1$ in the second case. In the first case we note that a real quadratic equation $\alpha_2 z^2 + \alpha_1 z + 1 = 0$ with $\alpha_2 \neq 0$ will have a root of modulus one if and only if either

1. $|\alpha_2| = 1$ and $\alpha_1^2 < 4\alpha_2$ (two complex conjugate roots of modulus 1); or
2. $\alpha_2 + \alpha_1 + 1 = 0$, so that $z = 1$ is a root; or
3. $\alpha_2 - \alpha_1 + 1 = 0$, so that $z = -1$ is a root.

Further conditions can be given for roots to have modulus bigger than 1 or smaller than 1 by using the Schur–Cohn test (Jacobs 1974). In particular, for stability it is necessary that $\alpha_2 z^2 + \alpha_1 z + 1$ have all its roots $z = e^{-sh}$ of modulus greater or equal than one, so that $x^2 + \alpha_1 x + \alpha_2$ should have all its roots x with $|x| \leq 1$, for which the condition are the ones given for the equality together with $|\alpha_2| < 1$ and $|\alpha_1| < |1 + \alpha_2|$ for the strict case.

In order to fully illustrate the results of this chapter up to this point, let us develop 4 numerical examples.

Example 3.8 As the first example, let us consider the following transfer function

$$G_1(s) = \frac{1}{s + e^{-s} + (s-1)e^{-2s}}. \tag{3.83}$$

One can easily see that it has chains of neutral type, but we seek more accurate information on the location of such poles.

Approximating the system $G_1(s)$ for large modulus s, as in equation (3.15), provides

$$\alpha_1 = \gamma_1 = \gamma_2 = 0, \tag{3.84}$$

$$\alpha_2 = \beta_1 = -\beta_2 = 1, \tag{3.85}$$

which leads the roots of (3.19) to being the solutions of

$$1 + z^2 = 0, \tag{3.86}$$

providing $r = \pm\jmath$ and henceforth $\lambda_n = \jmath(n + \frac{1}{2})\pi$. This implies that the chains of poles are asymptotic to the imaginary axis.

We continue by considering $s_n = \lambda_n + \mu_n$, as in (3.22), which gives a formula for the asymptotic behavior of the pole chains. For this example, we can calculate

$$\mu_n = \frac{\jmath \pm 1}{4\pi n}, \tag{3.87}$$

again, associated with the roots $r = \pm\jmath$ respectively. This leads to the conclusion that the chain of poles are in both sides of the imaginary axis, proving therefore that $G_1(s)$ is unstable.

Using the QPmR (Vyhlídal & Zítek 2003) algorithm, we can see in Figure 3.4 the first poles of the two neutral chains of this systems, which illustrates the analysis.

♦

Example 3.9 The second example consists of the following system

$$G_2(s) = \frac{1}{s^2 + s + 4e^{-s} + (s^2 + 1.1s + 7)e^{-2s}}. \tag{3.88}$$

Considering the polynomial (3.19) for this system, we notice that its roots are $r = \pm\jmath$. As in the last case, this implies that the chains of poles are asymptotic axis.

Although this would at first suggest that, as in the previous example, each chain of poles would contain points on both sides of the imaginary axis, a more careful look shows that, even with $\beta_2 \neq 0$, we have $\Re(\mu_n) = 0$ for all roots.

Using the results of Theorem 3.6, it is possible to see that for both roots

$$\nu_n = \frac{-6.89 + \jmath(\pm 1 - \pi/20)}{8\pi n^2}, \tag{3.89}$$

meaning that $\text{sign}(\Re(\nu_n)) = -1$ for all roots of (3.19) and all $n \in \mathbb{Z}$, and so, as Figure 3.5

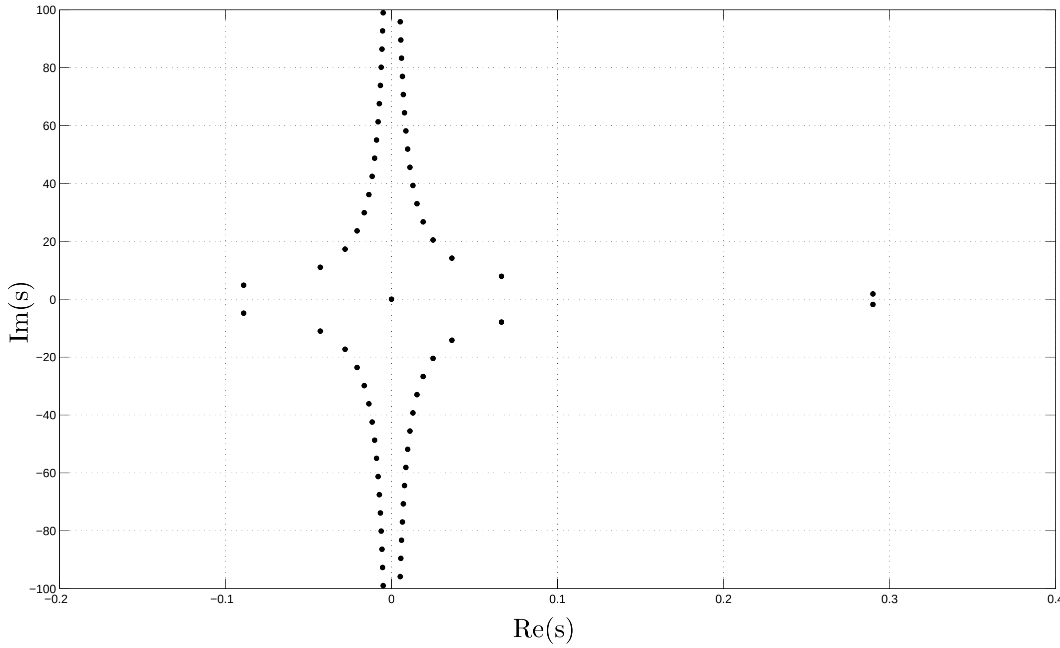

Figure 3.4: Neutral Chains of Poles for $G_1(s)$

illustrates, both chains of poles are located in the left half-plane. Notice, however, that there is a pair of conjugate poles of small modulus at around $s = 0.666 \pm \jmath 1.0325$, implying that this system is also unstable.

♦

Example 3.10 For the third example, consider the following system:

$$G_3(s) = \frac{2s+3}{s^2 + s + (s^2+2)e^{-s} - (0.5s+1)e^{-2s}}. \tag{3.90}$$

Note that $\deg q_2 = 1$, which is smaller than $\deg p = 2$. This indicates that this system, aside from a neutral chain (which comes from the fact that $\deg q_1 = 2$), has one chain of retarded poles.

Following the steps given before, we can easily calculate that $\lambda_n = \jmath\pi(2n+1)$, $\Re(\mu_n) = 0$ and $\Re(\nu_n) = -0.0918/n^2$, which indicates that there is one neutral chain of stable poles asymptotic to the imaginary axis. One may check that there are no right half-plane poles with small modulus either. In spite of these, Proposition 3.9 guarantees that the system $G_3(s)$ is not $\mathcal{H}_\infty$-stable, since the relative degree between $p(s)$ and $t(s)$ is one. But if we now consider the following system:

$$\widetilde{G_3}(s) = \frac{G_3(s)}{s+1} \tag{3.91}$$

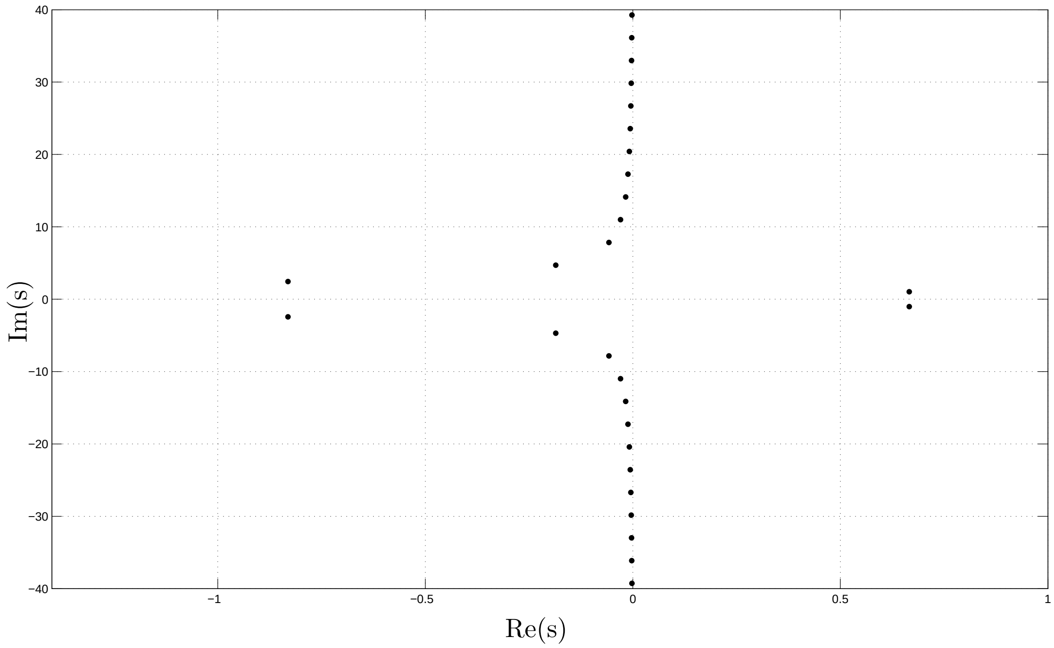

Figure 3.5: Neutral Chains of Poles for $G_2(s)$

we have that $\widetilde{G_3}(s)$ is now $\mathcal{H}_\infty$-stable, and its $\mathcal{H}_\infty$-norm is 4.5604.

Figure 3.6 shows the frequency response of both $G_3(s)$ and $\widetilde{G_3}(s)$, where we can see that the former, although it has all poles in the left half-plane, does not have bounded $\mathcal{H}_\infty$-norm, while the latter does. ♦

Example 3.11 As the last example, let us consider two systems with one delay, given by

$$G_4(s) = \frac{s}{s^2 + 2 + (s^2 + s + 1)e^{-\tau s}}, \tag{3.92}$$

$$G_5(s) = \frac{s + 1}{s^2 + s - 1 + (-s^2 + 3s + 2)e^{-\tau s}}. \tag{3.93}$$

Both $G_4(s)$ and $G_5(s)$ have a chain of neutral poles asymptotic to the imaginary axis, but all the poles of these chains are in the left half-plane. Applying some numerical methods, such as the one of Walton and Marshall (Walton & Marshall 1987), we can see that both systems are stable for $\tau = 0$ and increasing τ the first crossing of a pair of poles from the left to the right half-plane of $G_4(s)$ happens at $\tau = 4.7124$, whereas applying the same method for $G_5(s)$ shows that there are no positive real value of τ such that poles cross the imaginary axis.

Although for each system there exists a range of the delay such that neither has poles in the right half-plane, the relative degree between the numerator and the denominator is 1, and so

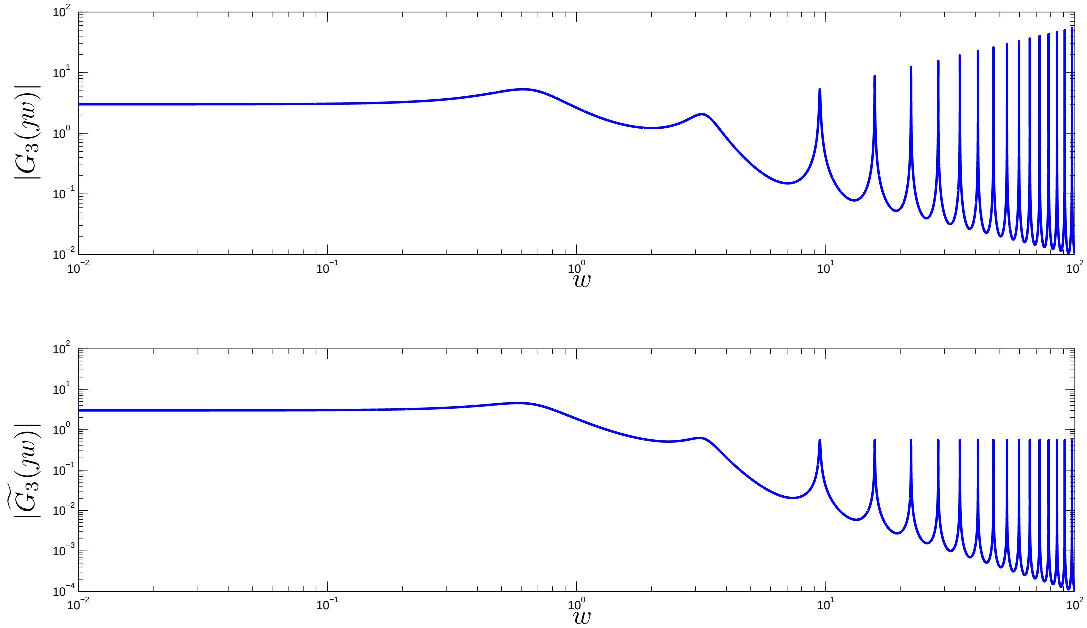

Figure 3.6: Frequency response for $G_3(s)$ and $\widetilde{G_3}(s)$

using equation (3.55) we see that both systems fail to be $\mathcal{H}_\infty$-stable for all values of the delay. The Bode plots of both systems for $\tau = 1$ are shown in the first two subplots of figure (3.7).

If we define $G(s) = G_5(s)G_6(s)$, the position of the poles of $G(s)$ are directly obtained from those of $G_5(s)$ and $G_6(s)$, so both neutral chains are located in the left half-plane and the first stability window of the system ends at $\tau = 4.7124$. But now, as we have the product of two systems with different α_k, $\mathcal{H}_\infty$ stability should be tested by equation (3.53), and in fact, this system is $\mathcal{H}_\infty$ stable. So we can conclude that, even though $G(s)$ is the product of two $\mathcal{H}_\infty$ unstable systems for any delay, it is itself $\mathcal{H}_\infty$ stable for all $\tau < 4.7124$. The Bode plot of $G(s)$ for $\tau = 1$ is shown in the last part of figure (3.7), where we can see that its $\mathcal{H}_\infty$-norm has the value 0.5420.

Graphically, on the logarithmic scale, the frequency response of $G(s)$ is the sum of the frequency responses of $G_5(s)$ and $G_6(s)$. Since $\alpha_1 = 1$ and $\alpha_2 = -1$, the poles of the chains of $G_5(s)$ and $G_6(s)$ occur exactly out of phase, which means that the valleys of the Bode plot of $G_5(s)$ asymptotically match the peaks of $G_6(s)$ and vice-versa, in such a way that it gives rise to two plateaus of local maxima.

On the other hand, we can use (3.54) to show that cascading N unstable systems of the type

$$G_k(s) = \frac{t_k(s)}{p_k(s) + q_k(s)e^{-s\tau}}, \tag{3.94}$$

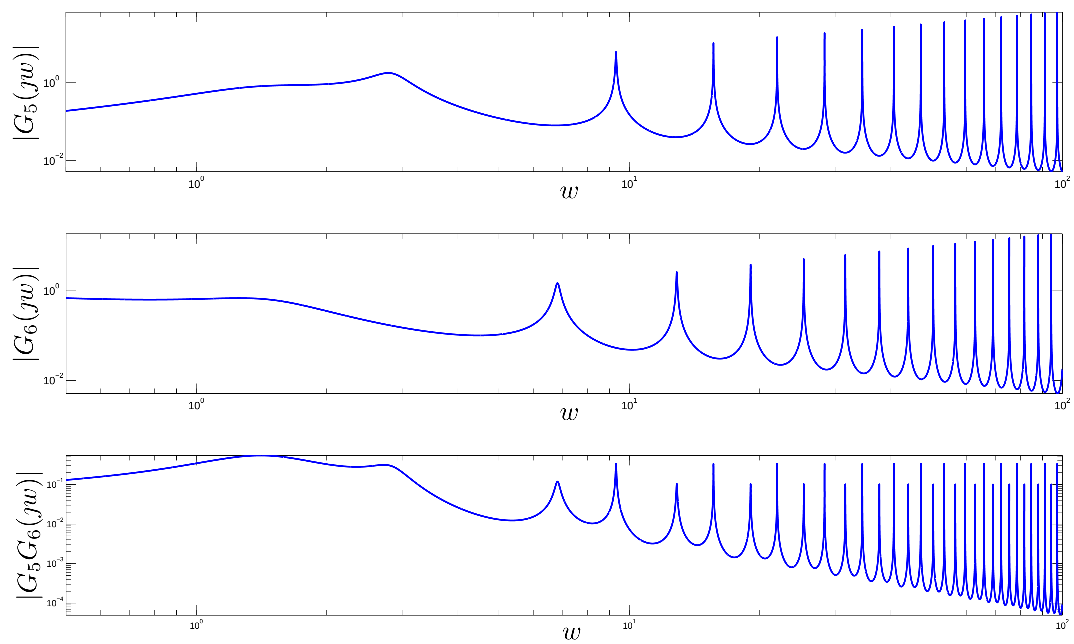

Figure 3.7: Frequency response for $G_5(s)$, $G_6(s)$ and $G_5(s) * G_6(s)$.

where $\deg p_k = \deg q_k = d$ and $\alpha_k = 1$ for all $k = 1, \ldots, N$ will result in a unstable system. In fact, assuming that the $G_k(s)$ have no unstable poles (otherwise the result is trivial), we have already shown that the system is $\mathcal{H}_\infty$-stable only if the relative degree between p_k and t_k is greater or equal than 2. Considering the best case, where all the $G_k(s)$ have relative degree 1, the numerator of the resulting system $G(s)$ will have degree $N(d-1)$. In order to achieve stability, we should have

$$\deg p \geq \frac{\deg t + N_1}{N} + 1 \tag{3.95}$$

where N_1 is the number of equal maximum β_k. This result clearly cannot be achieved for any $N_1 = 1, \ldots, N$, and the same reasoning shows that this result is also valid if all $\alpha_k = -1$. ♦

3.3 Fractional Systems

Now we focus the attention on the analysis of chains of poles of fractional systems. We recall that we consider fractional time-delay systems with transfer functions of the form

$$G(s) = \frac{t(s^\mu) + \sum_{k=1}^{N'} t_k(s^\mu) e^{-ks\tau}}{p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}}, \tag{3.96}$$

where $\tau > 0$, $0 < \mu < 1$, and t, p, t_k for all $k \in \mathbb{N}_{N'}$ and q_k for all $k \in \mathbb{N}_N$, are real polynomials. Note that we define an analytic branch of s^μ on the cut plane $\mathbb{C} \backslash \mathbb{R}_-$ by setting $(re^{\jmath\theta})^\mu = r^\mu e^{\jmath\mu\theta}$ and choosing θ with $-\pi < \theta < \pi$.

Most of the techniques utilized here are derived from the ones presented in the section before. Therefore, except if some novelty is presented, we will omit the proofs and focus on the qualitative differences of the results.

3.3.1 Asymptotic Location of Neutral Poles of Time-Delay System

As we did for the classical systems, from now on we will assume that the fractional system is of neutral type, which means that the polynomials p and q_k further satisfy $\deg p \geq \deg q_k$ with $\deg p = \deg q_k$ for at least one $k \in \mathbb{N}_N$. In order for (3.96) to be a proper neutral type delay system we assume also that $\deg p \geq \deg t$ and $\deg p \geq \deg t_k$ for all $k \in \mathbb{N}_{N'}$. Here, the degree is interpreted as the degree of the polynomial in s^μ, and therefore it is an integer.

The asymptotic behavior of the system differs a little from the result presented before. It can be seen that for each k

$$\frac{q_k(s^\mu)}{p(s^\mu)} = \alpha_k + \frac{\beta_k}{s^\mu} + \mathcal{O}(s^{-2\mu}) \qquad \text{as} \quad |s| \to \infty, \tag{3.97}$$

where, α_k and β_k are calculated exactly in the same as it was done in (3.16)-(3.17). Those differences in the asymptotic behavior will be the cause of the variation in the behavior of the system.

We continue to use the notation $z = e^{-s\tau}$. The coefficient of highest degree of $p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}$, can then be written as a multiple of the same polynomial in z

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \tag{3.98}$$

Therefore, the position of the vertical lines for which the roots of the neutral chains are asymptotic to are the same as those given in Proposition 3.4. Again, we will focus on the case where all the roots of (3.98) have multiplicity one. In other words, for now on we will consider that Hypotheses 3.3 holds.

The objective of the next theorem is to be the counterpart of Theorem 3.5 for the case of fractional systems. We will see, though, that its result is somewhat stronger.

Theorem 3.13 *Let $G(s)$ be a neutral delay system defined by (3.96) and suppose that all the roots of (3.98) have multiplicity one. For each r such that $z = r$ is a solution of (3.98) and for large enough $n \in \mathbb{Z}$, the solutions asymptotic to (3.20) are given by*

$$s_n h = \lambda_n + \delta_n + \mathcal{O}(n^{-2\mu}) \tag{3.99}$$

with λ_n given by (3.21) and

$$\delta_n = \frac{\tau^\mu \sum_{k=1}^{N} \beta_k r^k}{(2\jmath\pi n)^\mu \sum_{k=1}^{N} k\alpha_k r^k} \tag{3.100}$$

Some conclusions can be obtained by considering δ_n. First of all, associated with each root r of (3.98), let us define K_r as it was done before:

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k}\ , \tag{3.101}$$

where, again, K_r is well defined as it is assumed that r has multiplicity one as a root of (3.98).

Our interest is mainly on which side of the vertical line the poles are, in other words, to find out the sign of $\Re(\delta_n)$ for n sufficiently large. Just like the case $\mu = 1$, it is sufficient to look only at K_r to obtain this information. First, let us recall that for the non-fractional case, only the existence of poles on both sides of the line could be assured up to this point, because either $\Re(\delta_n) = 0$, in which case an analysis with further terms was needed, or $\Re(\delta_n) = \pm c$, with the different signs coming from the calculation of δ_n for the complex conjugated root $\overline{r}$ of (3.19).

On the other hand, for our current case, as $0 < \mu < 1$, complex conjugated roots of (3.98) will not always provide complex conjugated δ_n, that means, in general, δ_n associated to a particular r is not equal to the complex conjugate of the one associated with $\overline{r}$. This means that can prove that all poles for some systems are all in one particular half-plane delimited by the vertical line given by (3.20) just with the approximation up to this order. This analysis will be the subject of the next two corollaries.

Corollary 3.14 *Let $0 < \mu < 1$, δ_n be given by (3.100) and its associated K_r by (3.101). Then,* *sign*$(\Re(\delta_n)) < 0$ *for all* $n \in \mathbb{Z}$ *if and only if*

$$\Re(K_r) < -\tan\left(\frac{\mu\pi}{2}\right)|\Im(K_r)| \tag{3.102}$$

Proof: Besides K_r, the only term of interest is $J = (\jmath n)^{-\mu}$, as $\text{sign}(\Re(\delta_n)) = \text{sign}(\Re(JK_r))$. Since n can be both positive or negative, this term is given by

$$J = |n|^{-\mu}\left(\cos(\mu\pi/2) \pm \jmath \sin(\mu\pi/2)\right). \tag{3.103}$$

Multiplying J by K_r and getting its real part leads to

$$\Re(JK_r) = \frac{1}{|n|^{\mu}}\left(\cos\left(\frac{\mu\pi}{2}\right)\Re(K_r) \mp \sin\left(\frac{\mu\pi}{2}\right)\Im(K_r)\right) \tag{3.104}$$

from where (3.102) follows from the fact that $0 < \mu < 1$. □

Some aspects can be seen from this corollary. First, the numerical value of the delay does not appear explicitly in equation (3.102). This means that for all $\tau > 0$ the chains of poles present the same behavior in the sense that they do not change sides with respect to the vertical line in question as a function of an increasing delay. Second, as (3.102) involves only the absolute value of $\Im(K_r)$, the results obtained would be equivalent if we had dealt with the complex conjugate of K_r. But indeed, it is direct to see that complex conjugated roots of the polynomial $c(z)$ in equation (3.98) will define complex conjugated K_r.

Therefore, as stated before, differently from the case $\mu = 1$, in the present context it might be possible to state if all the poles are in the left of the vertical line (3.20) up to this level of approximation. Indeed, as α_k and β_k given in equations (3.16) and (3.17) are independent of μ, the next corollary can be stated.

Corollary 3.15 *Let* $0 < \mu < 1$*,* δ_n *be given by (3.100) and its associated* K_r *by (3.101). Then, if* $\Re(K_r) < 0$*, all poles of the respective chain asymptotic to the vertical line (3.20) will be on the left of this line if*

$$\mu < \frac{2}{\pi}\arctan\left(-\frac{\Re(K_r)}{|\Im(K_r)|}\right). \tag{3.105}$$

Proof: This follows directly from Corollary 3.14. □

Although some cases might still require further analysis, as for example if all β_k are equal to zero, the procedure resembles the one given in the last chapter and therefore will be omitted. With these results in hand, in the next section we will consider the $\mathcal{H}_\infty$-stability for such class of systems.

3.3.2 $\mathcal{H}_\infty$-stability

We are now interested in answering the question of stability of $G(s)$. We continue to aim for the $\mathcal{H}_\infty$-stability, that is, the system has a finite $L_2(0,\infty)$ input/output gain.

The overall analysis resembles the one for the non-fractional system. That is, the case where equation (3.98) possesses only roots of modulus strictly greater than one is easy to handle as there exists $a > 0$ such that the system has a finite number of poles in $\{\Re(s) > -a\}$. Also, the case where equation (3.98) possesses at least one root of modulus strictly less than one is obvious, since there will be a chain of poles asymptotic to a vertical line in the right half-plane, and consequently an infinite number of unstable poles. For the transition case, the problem is dealt by the following Proposition, based on Proposition 3.9.

Proposition 3.16 *Let $G(s)$ be a transfer function given as (3.96) and suppose that (3.98) has at least one simple root of modulus one, the other roots being of modulus strictly greater than one.*

1. *Suppose that $\Re(\delta_n) < 0$ for all $n \in \mathbb{Z}$ and that G has no unstable pole (which could exist only in a finite number), then G is $\mathcal{H}_\infty$-stable if and only if $\deg p \geq \deg t + 1$.*

2. *If $\Re(\delta_n) = 0$, then the condition $\deg p \geq \deg t + 1$ is necessary for $\mathcal{H}_\infty$-stability.*

3.3.3 Examples

As a first example, let us consider the following fractional delay system

$$G_1(s) = (s^\mu + e^{-s} + (-s^\mu + 2)e^{-2s})^{-1}. \tag{3.106}$$

It is in form of (3.96) with $\tau = 1$. Evaluating (3.97) for this system leads to $\alpha_1 = 0$, $\alpha_2 = -\beta_1 = -1$ and $\beta_2 = 2$. The two roots of the polynomial $\tilde{c}_d(z)$ given in (3.98) are $r = \pm 1$, and so, (3.106) has two chains of neutral poles asymptotic to the imaginary axis. The associated values of K_r are $K_r = -1.5$ and $K_r = -0.5$ for $r = 1$ and $r = -1$ respectively.

Since $\Re(K_r) < 0$ and $\Im(K_r) = 0$ for both roots of $\tilde{c}_d(z)$, this system has both its neutral chains of poles on the left of the imaginary axis for all $0 < \mu < 1$. This means that for all these values of μ there are only finitely many poles on the right half-plane. Interesting is to note that, applying the results of the previous section for the case $\mu = 1$, we find out that for this particular value, both chains are located in the right half-plane, and therefore there exist infinitely many unstable poles.

Figures (3.8) and (3.9) show these phenomena. The first one brings the location of the chains of poles for $\mu = 0.9$ and the second for $\mu = 0.99$. Both graphics were obtained by the use of the QPmR algorithm (Vyhlídal & Zítek 2003). One can notice that by increasing $\mu < 1$, even though the infinitely number of poles of the chain are still located on the left half-plane, an increasing number of unstable poles will be present. This somehow agrees with equation (3.105), in the sense that a lower μ is advantageous when dealing with the stability of the neutral chains.

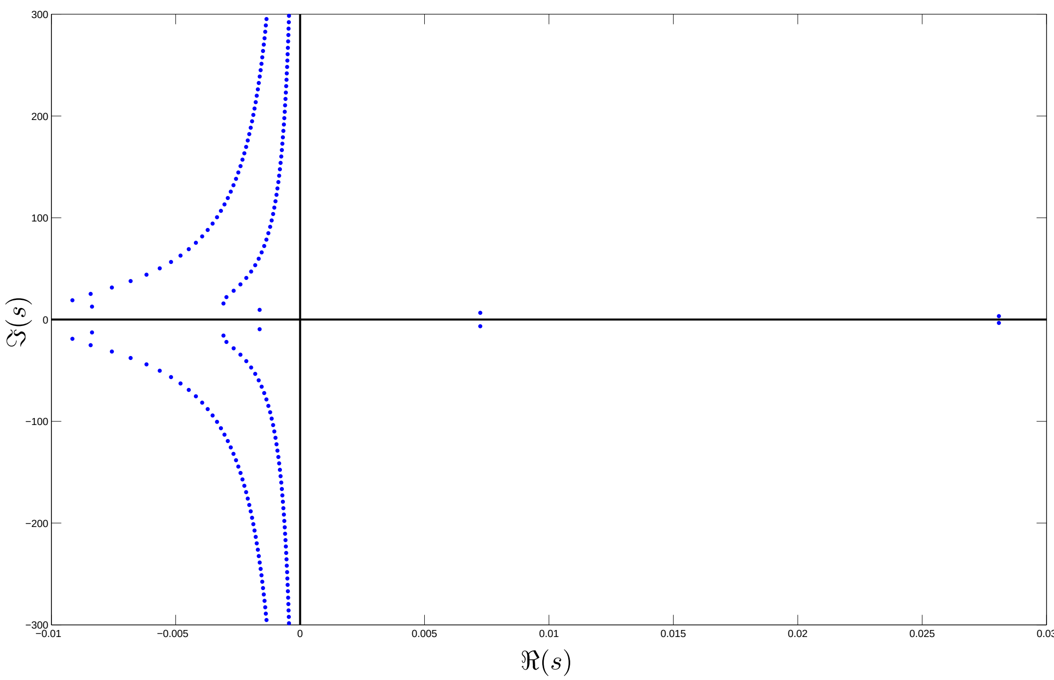

Figure 3.8: Neutral Chains of Poles for $G_1(s)$ and $\mu = 0.9$

The second example consists of the following system

$$G_2 = (s^\mu - s^\mu e^{-s} + (s^\mu - 2)e^{-2s})^{-1}. \tag{3.107}$$

It presents $\tau = 1$, and its asymptotic behavior is described by $\alpha_1 = -\alpha_2 = -1$, $\beta_1 = 0$ and $\beta_2 = -2$. The two roots of the polynomial $\tilde{c}_d(z)$ are $r = 1/2 \pm \jmath\sqrt{3}/2$, and therefore it also has two chains of neutral poles asymptotic to the imaginary axis. The associated values of K_r are $K_r = -1 \pm \jmath 0.5774$.

Since $\Re(K_r) < 0$ and $\Im(K_r) \neq 0$, we know that there exists a $\mu^\star$ such that for $\mu < \mu^\star$ the two chains of poles will be on the left of the imaginary axis. Applying equation (3.105), we

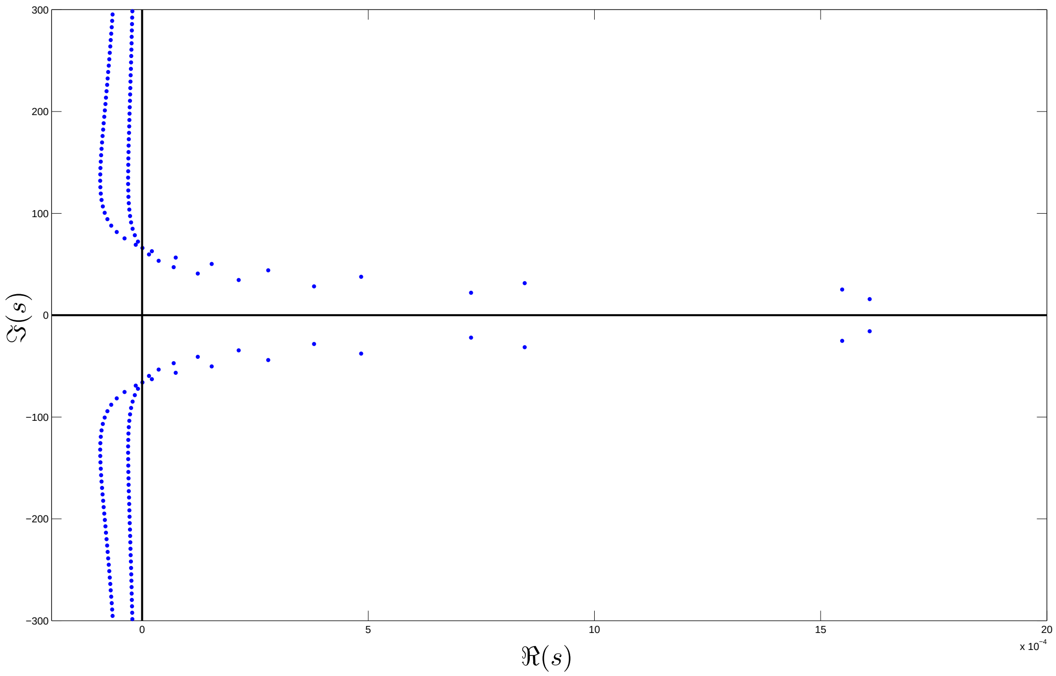

Figure 3.9: Neutral Chains of Poles for $G_1(s)$ and $\mu = 0.99$

discover that this will happen if $\mu < 2/3$. Figure (3.10) brings the location of the chains of poles for $\mu = 0.6$ and Figure (3.11) the one for $\mu = 0.7$. Again, in the first image, we can see the occurrence of some poles of the stable chains already moving towards the right half-plane, creating the shape for the unstable chain that is present for $\mu = 0.7$. The analysis exactly in the transition point, that is, for $\mu = 2/3$, needs a higher order approximation in (3.97).

For the last example, let

$$G_3(s) = \frac{t(s)}{s^{0.5} + 10 - (0.8s^{0.5} + 2)e^{-s} + (s^{0.5} - 5)e^{-2s}} \tag{3.108}$$

This system has two chains of stable poles asymptotic to the imaginary axis and no unstable poles of small modulus. Figure (3.12) brings the Bode plot of the system for two cases of $t(s)$. In the upper part, $t(s) = s^{0.5} + 1$ and in the lower part $t(s) = 1$. One can see that for the first case, even with all poles on the left half-plane, the maximum of the magnitude of the bode plot is unbounded, and therefore the system is not $\mathcal{H}_\infty$-stable. On the other hand, for the second case, we have a bounded maximum value of the magnitude, and with that the $\mathcal{H}_\infty$-stability of the system is achieved.

This figure illustrates the fact that the rule *"No poles in the closed right half-plane"* is a necessary condition for the $\mathcal{H}_\infty$-stability of a system with chains of poles asymptotic to the

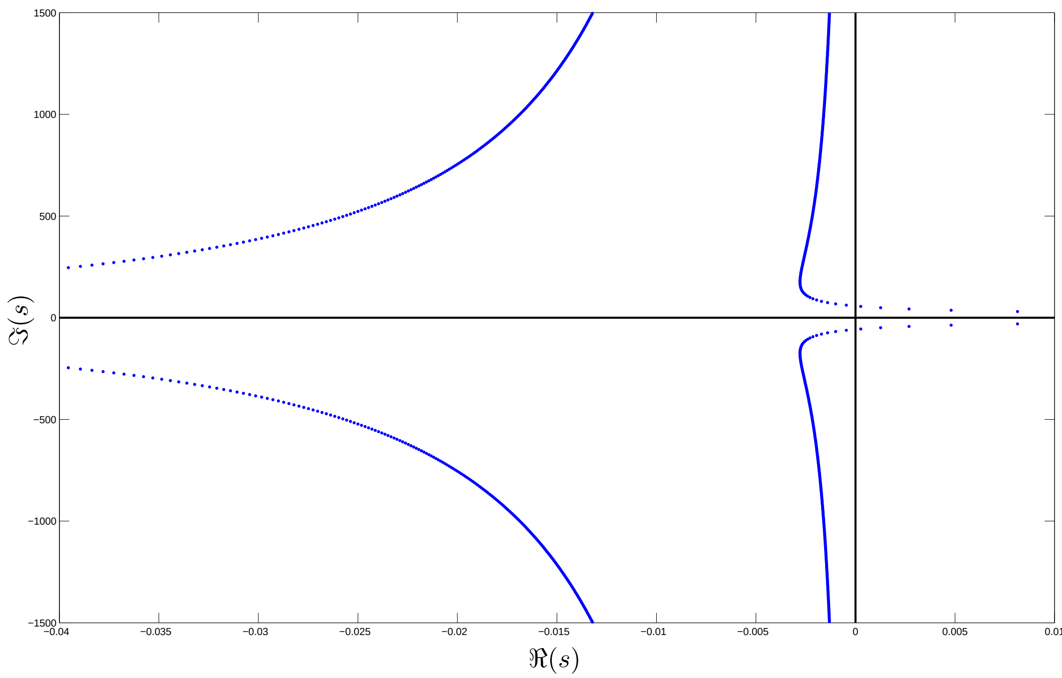

Figure 3.10: Neutral Chains of Poles for $G_2(s)$ and $\mu = 0.6$

imaginary axis. As shown before, for this case, another necessary condition is that the relative degree between the numerator and the denominator interpreted as the degree in s^{μ}, must be at least one. When these two necessary conditions are satisfied then the system is $\mathcal{H}_\infty$- stable.

3.4 Final Remarks

In this chapter we have proposed some procedures to find the asymptotic behavior of the neutral chains of poles of linear time-delay systems. This information is necessary in order to guarantee stability of systems when the chain approaches the imaginary axis. We also derived conditions for systems with all poles in the LHP but a chain asymptotic to the imaginary axis to be $\mathcal{H}_\infty$-stable.

For the classical systems, it is still an open question the link of these frequential results with the time-domain results developed for several years by many researchers as mentioned before. In particular, the important question of strong stability introduced in the time-domain demands us to further investigate the case of changes in the delay which result in systems with delays which are no more commensurate. Also, the $\mathcal{H}_\infty$-stabilizability of neutral systems needs to be further investigated as well as its links with similar time-domain questions.

On the other hand, for the class of fractional systems, we believe that this is the only approach

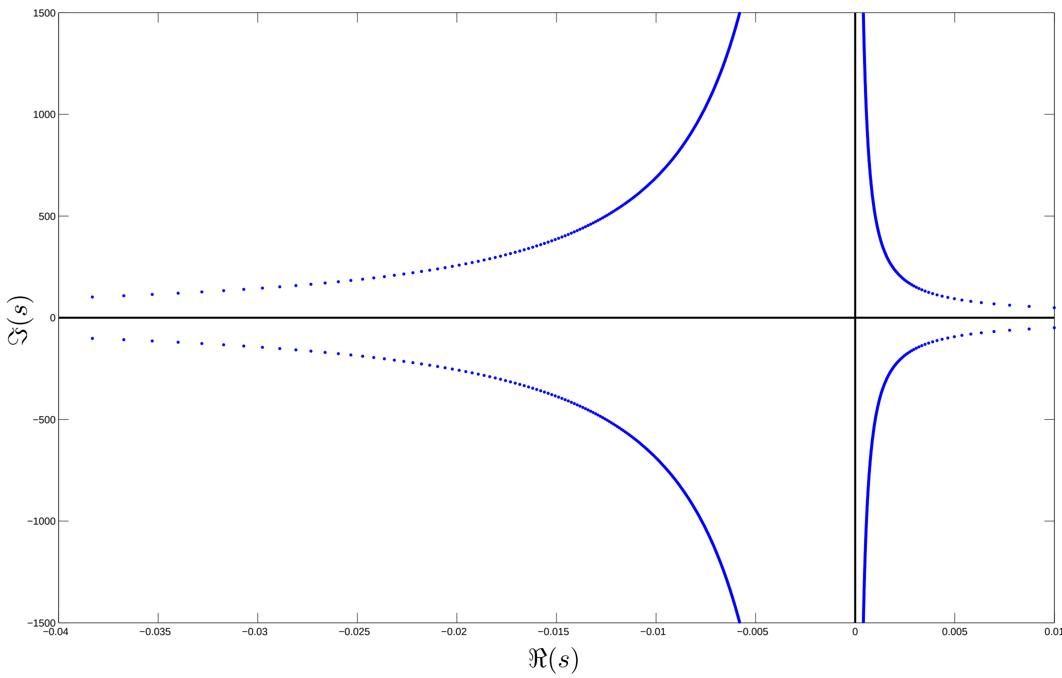

Figure 3.11: Neutral Chains of Poles for $G_2(s)$ and $\mu = 0.7$

that exists up to now in order to deal with the proposed kind of analysis.

The following publications were produced presenting the results of this chapter (Bonnet, Fioravanti & Partington 2010), (Fioravanti, Bonnet & Özbay 2010), (Bonnet, Fioravanti & Partington 2009*b*) and (Bonnet, Fioravanti & Partington 2009*a*):

- C. Bonnet, A. R. Fioravanti and J. R. Partington. 'Stability of Neutral Systems with Multiple Delays and Poles Asymptotic to the Imaginary Axis'. *SIAM Journal on Control and Optimization*, Vol. 49, No. 2, pp. 498-516, March 2011.

- A. R. Fioravanti, C. Bonnet and H. Özbay. 'Stability of fractional neutral systems with multiple delays and poles asymptotic to the imaginary axis'. *49th IEEE Conference on Decision and Control*, Atlanta - USA, December 15-17, 2010.

- C. Bonnet, A. R. Fioravanti and J. R. Partington. 'Stability of Neutral Systems with multiple delays and poles asymptotic to the imaginary axis'. *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pp. 269 - 273, Shanghai - China, December 16-18, 2009.

- C. Bonnet, A. R. Fioravanti and J. R. Partington. 'On the Stability of Neutral Linear Systems with Multiple Commensurate Delays'. *IFAC Workshop on Control of Distributed

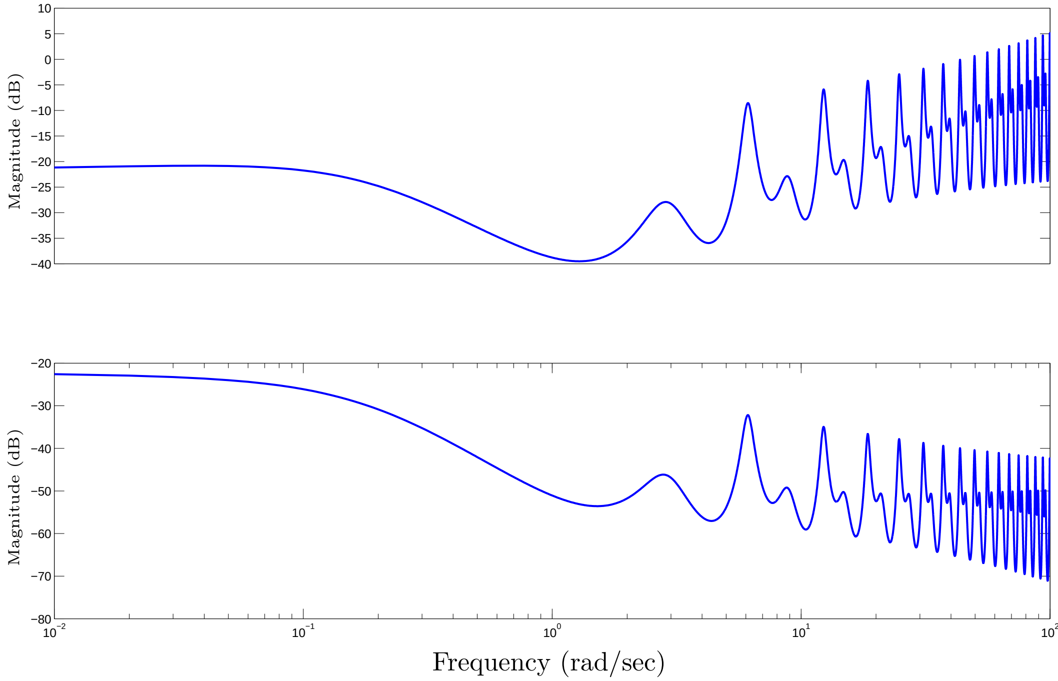

Figure 3.12: Bode Diagram for $G_3(s)$ with $t(s) = s^{0.5} + 1$ and $t(s) = 1$

Parameter Systems, pp. 195-196, Toulouse - France, July 20-24, 2009.

Chapter 4

Stability Windows and *Root-Locus*

Contents

4.1 Introduction 71

4.2 Classical Systems 72

4.3 Fractional Systems 82

4.4 Final Remarks 89

4.1 Introduction

In the previous chapter, we have seen how to locate and analyze the chains of poles of a time-delay system. We have also seen that the stability is connected to the location of the poles, but up to now we have studied only their asymptotic behavior as $|s|$ goes to infinity. It is evident that there are still poles with *small modulus*, that do not fit into any chain, and those poles may also have an important impact in the behavior of the system.

Furthermore, it may be the case that the exact value of the delay is not known. In this situation, it may be interesting to see how stability will vary if we increase the value of the delay. It is known that an interesting phenomenon, namely *stability windows*, might happen. It means that stability might be lost and recovered many times as a function of an increasing delay. There has been a large effort to deal with this problem, as for instance, (Walton & Marshall 1987), (Olgac & Sipahi 2004), among others.

The most natural way to find the position of those poles of small modulus is to solve its correspondent characteristic equation. But since this is a transcendental one, it is quite hard to solve it directly. For this reason, most of the existing procedures study stability of such systems by finding the crossings of poles through the imaginary axis. This fact comes from two important properties of time-delay systems. The first one is the *root continuity argument*, which means that for any positive value of the delay, the position of the poles varies continuously with respect to the delay. This means that any root crossing from the left to the right half-plane will need to pass through the imaginary axis. The second property is the invariance of the tendency of roots crossing (Olgac & Sipahi 2002). This implies that a *manageable* number of root clusters can provide sufficient information to characterize the whole stability of the system.

In order to find the position where the roots cross the imaginary axis, and with that be able to construct these root clusters, many algorithms have been proposed. In (Sipahi & Olgac 2006), an interesting comparison between some of them was provided. All of those share some similarities, meaning that they are based on a particular feature of the roots (for instance, unitary or purely imaginary roots) of some polynomial with the degree higher than the one of the original system. This may increase the numerical difficulties on the calculation of the roots, and searching for a particular property on them might lead to misleading results.

When dealing with the class of neutral systems, one needs to take extra care of what is called *small-delay effect* (Hale & Lunel 2001). As one can derive from the results given in the last chapter, this fact splits the systems between two classes. Whenever all the neutral chains of poles asymptotic to a line located in the left half-plane, one has the same number of unstable roots when the delay passes from 0 to 0^+, and for that it has a chance of being stable for a non-zero value of the delay. On the other hand, whenever at least one neutral chain of poles is located in the right half-plane, infinitely many unstable roots appear when the delay varies from 0 to 0^+. In this last case, even if the delay free system is stable, these systems cannot recover stability for any finite positive delay.

Finally, all the algorithms mentioned above can successfully provide the number of unstable poles at any given value of the delay, but not the location of them. Regarding this problem, some methods have been proposed, such as a gridding procedure in (Vyhlídal & Zítek 2003) or discretization methods in (Engelborghs, Luzyanina & Roose 2002), among others.

On the other hand, for the fractional case, the number of available tools is not so numerous. Although the method of (Walton & Marshall 1987) can be successfully expanded to cope with multiple delays and also fractional systems, each extra commensurate term of the delay after the first one needs to be reduced, and this process potentially doubles at each step the degree of the polynomial we need to solve. So, even if we were originally dealing with low degree system with multiple delays, this method will require the zeros of a high order polynomial, which can be a challenging and perhaps unreliable numerical problem.

Other techniques have been proposed, as (Hwang & Cheng 2006), where a numerical procedure based on Cauchy's integral theorem was given to test the stability of such systems, and (Hwang & Cheng 2005), where a technique based on the Lambert W function was used for the same purpose. But the complete characterization of all stability windows is hard when using those methods.

So, in the present chapter, we intend to develop a general technique to deal initially with the problem of finding the stability windows, both for the integer as well as for the fractional case. Later on, we will see how those results can be successfully applied to provide the position of all unstable poles for a given value of the delay τ, provided that all the neutral chains of poles are located in the left half-plane.

The same style used in Chapter 3 will be used here. First, we will completely treat the more standard problem of integer order. In the sequel, we will deal with the fractional case, with the focus on pointing out the major differences needed to be taken into account.

4.2 Classical Systems

4.2.1 Problem formulation

Let us consider time-delay systems with characteristic equation of the form:

$$C(s,\tau) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}, \tag{4.1}$$

where $\tau > 0$, and p and q_k, for all $k \in \mathbb{N}_N$, are polynomials with real coefficients which satisfy $\deg p \geq \deg q_k$. According to the classification of roots given in Chapter 3, if $\deg p = \deg q_k$ for at least one $k \in \mathbb{N}_N$, then equation (4.1) defines a *neutral* time-delay system, otherwise it will consist of the *retarded* type.

Related to the system (4.1), from now on we assume that:

Hypothesis 4.1 *The polynomials $p(s)$ and $q_k(s)$ for all $k \in \mathbb{N}_N$ do not have common zeros.*

It is obvious that if Hypothesis 4.1 is violated then $p(s)$, $q_k(s)$ have a common factor $c(s) \neq$ constant. The roots of $c(s)$ are invariant with the delay, that means, they are roots of (4.1) for all values of τ. Simplifying by $c(s)$ we get a system described by (4.1) satisfying Hypothesis 4.1.

The other hypothesis we will assume from now on is:

Hypothesis 4.2 *The polynomials $p(s)$ and $q_k(s)$ satisfy*

$$p(0) + \sum_{k=1}^{N} q_k(0) \neq 0. \tag{4.2}$$

This means that $s = 0$ is not a pole for system (4.1) for all finite values of τ. At the same time, this hypothesis also guarantees that poles at $s = 0$ can only happen for $\tau \to \infty$.

4.2.2 Stability for $\tau = 0$

In the same way as many other procedures, we need to start by the study of the delay free system. Considering $\tau = 0$ in (4.1), we get a polynomial with real coefficients, whose roots can be easily found. The number of unstable poles of $C(s, 0)$ as well as their location will be crucial in the following developments.

4.2.3 Location of Chains of Poles

The second part of the algorithm is mainly the analysis provided in Chapter 3. This said, if the system we are dealing with is of the *retarded* type, all the infinite new poles that appear when we pass from $\tau = 0$ to $\tau = 0^+$ will be in the extreme left half-plane, that means, $\Re(s) \to -\infty$, and so this step can be ignored.

The case of *neutral* systems needs to be dealt with care. In fact, whenever a chain of neutral poles is asymptotic to some vertical line lying in the right half-plane, an infinite number of unstable poles appear for any infinitesimal delay. As we have already seen that no positive value of the delay can change the side where any particular chain is lying, it means that the crossings through the imaginary axis will happen only at a finite rate, and therefore, these systems will be always unstable for any non-zero value of the delay.

The case where all chains of poles are in the left half-plane but there are poles asymptotic to the imaginary axis can also be dealt with. In fact, as we have already seen, position of poles brings only a necessary condition for $\mathcal{H}_\infty$-stability, but sufficient conditions can be given in the expense of further analysis. As this kind of problem is not the main interest of this chapter and would only provide more particular cases, from now on we consider the following hypothesis:

Hypothesis 4.3 *There are no chains of poles in an extended right half-plane.*

4.2.4 Crossing Locations

In order to find the location on the imaginary axis where the crossings occur, we will rely on a transformation of variables which decouples the polynomials and the exponential part. Since for $s = \jmath\omega$ the exponential terms in (4.1) will have modulus equal to one, the idea is to replace $e^{-\jmath\omega\tau k}$ with $e^{-\jmath\theta k}$ and find the roots of the resulting complex polynomial as a function of $\theta \in [0, \pi]$. In other words, we find all the roots of the complex polynomial in s

$$\widetilde{C}(s, \theta) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-\jmath k\theta}, \tag{4.3}$$

varying θ in the closed interval $[0, \pi]$.

The following two lemmas will provide some basic results needed for the complete understanding of the effects of this transformation of variables:

Lemma 4.4 *For any τ and $s = \jmath\omega$ such that $\widetilde{C}(s, \theta) = 0$, then $\widetilde{C}(\overline{s}, 2\pi - \theta) = 0$.*

Proof: It is a matter of simple substitution to show that $\widetilde{C}(\jmath\omega, \theta) = \overline{\widetilde{C}(-\jmath\omega, 2\pi - \theta)}$. □

Lemma 4.5 *There exist $s = \jmath\omega$ and $\tau > 0$ such that $C(s, \tau) = 0$ if and only if there exists $\theta \in [0, 2\pi)$ such that $\widetilde{C}(s, \theta) = 0$.*

Proof: For the sufficiency, recall that by the Hypothesis 4.2 the case $\omega = 0$ can be neglected. So letting

$$\tau(\omega, \theta, \ell) = \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega}, \tag{4.4}$$

and choosing $\ell = \{0, 1, \ldots\}$ if $\omega > 0$ or $\ell = \{-1, -2, \ldots\}$ if $\omega < 0$, will provide $\tau > 0$ such that $C(s, \tau)$ given in (4.1) satisfies $C(s, \tau) = 0$.

For the necessity, choose θ as $\angle e^{\jmath\omega\tau}$, taken from $[0, 2\pi)$, and notice that for all $k \in \mathbb{N}$

$$e^{-\jmath k\omega\tau} = e^{-\jmath k\theta}. \tag{4.5}$$

□

It is important to notice that for a fixed $\theta \in [0, \pi]$, $\widetilde{C}$ is a polynomial with complex coefficients, but no delays. This implies that, due to the variable transformation in the exponential term, a solution $s^\star$ of $\widetilde{C}$ does not imply that $\overline{s^\star}$ is another solution.

Since by the Hypothesis 4.2 we can neglect the roots at the origin, all roots of (4.1) on the imaginary axis occur in complex conjugate pairs, and for that we can finally state our main result.

Theorem 4.6 *Let Assumptions 4.1-4.3 hold. Let Ω be the set of all ordered pairs (ω, θ), with $\omega \in \mathbb{R}$ and $\theta \in [0, \pi]$ such that $\widetilde{C}(\jmath\omega, \theta) = 0$. Let*

$$\tau(\omega, \theta, \ell) = \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega} \tag{4.6}$$

for all $(\omega, \theta) \in \Omega$. Choose $\ell = \{0, 1, \ldots\}$ if $\omega > 0$, and $\ell = \{-1, -2, \ldots\}$ if $\omega < 0$. Let Δ be defined as the set of all the ordered pairs $(\pm\jmath\omega, \tau(\omega, \theta, \ell))$. Then Δ is the complete set of roots of (4.1) on the imaginary axis for all $\tau > 0$.

Proof: First we need to show that any element of Δ is a root for (4.1). The result for the term "$+\omega$" comes directly from the sufficiency of Lemma 4.5, whereas the "$-\omega$" comes from the aforementioned fact that the poles in the imaginary axis of (4.1) occur in complex conjugate pairs. Finally, it lacks to show that any root of (4.1) is an element of Δ. But from Lemma 4.4, we see that for the complex conjugated solutions of $\widetilde{C}(\jmath\omega, \theta) = 0$, at least one of the θ will be in the $[0, \pi]$ interval. The rest follows from the necessity of Lemma 4.5. □

With these results in hand, it is easy to check if a system is stable independent of delay.

Corollary 4.7 *If the system given by (4.1) is stable for $\tau = 0$ and $\Omega = \emptyset$, then the system is stable for all positive values of τ.*

Proof: If comes directly from the fact that there are no roots crossing the imaginary axis. □

The set Ω together with the root tendency of each ordered pair is what we call the *root cluster*. The root tendency will be better explained in the sequel, but as stated in (Olgac & Sipahi 2004), it is constant with respect to any sequential crossings in (4.6). Another important aspect studied in (Olgac & Sipahi 2002) is the number of elements of Ω. In that paper an upper bound is provided, but numerical experiments show much lower results, which makes this procedure very attractive.

To be able to use the results of Theorem 4.6, we need to be able to find all $\omega \in \mathbb{R}$ and $\theta \in [0, \pi]$ such that $\widetilde{C}(\jmath\omega, \theta) = 0$. To this matter, we propose two distinct approaches.

The most direct one consists in sampling θ in its interval $[0, \pi]$, and for each fixed value $\theta^\star$ calculate the roots of the resulting complex polynomial $\widetilde{C}(s, \theta^\star) = 0$. Although we need to solve a large number of polynomial equations to solve this problem, two main advantages appear when we compare with the procedures described in (Sipahi & Olgac 2006). First, all polynomials we need to solve will be of the same degree of $p(s)$, whereas even in their best procedure, (Sipahi & Olgac 2006) still needs to deal with polynomials of order at least twice larger than that. This brings a real advantage when reliability of the numerical results are concerned.

Second, the root continuity argument still holds for $\widetilde{C}(s, \theta)$ as a function of θ. This means that plotting the curve of the real part of the roots of $\widetilde{C}(s, \theta)$ as a function of θ brings a useful graphical information of where the actual crossings through the imaginary axis are, allowing a better use of Newton's method in order to improve numerical accuracy without increasing the possibility of getting false results.

The second method we propose tries to keep the advantages of the previous one without increasing the computational burden. If (s, θ) is a simple root of (4.3), then a small perturbation on $\theta^\star = \theta + \epsilon$ will provide a solution of $\widetilde{C}(s^\star, \theta^\star) = 0$ with the form

$$s^\star = s + \sum_{k=1}^{\infty} \lambda_k \epsilon^k \tag{4.7}$$

where

$$\lambda_1 = \jmath \frac{\sum_{k=1}^{N} k q_k(s) e^{-\jmath\theta k}}{p'(s) + \sum_{k=1}^{N} q_k'(s) e^{-\jmath\theta k}} = T(s, \theta). \tag{4.8}$$

Here, $p'(s)$ and $q_k'(s)$ denote the derivative of the polynomials $p(s)$ and $q_k(s)$ in s respectively.

So, a method to provide the same curves as the ones calculated with the gridding method is to numerically integrate $T(s, \theta)$ for θ varying from 0 to π and the starting positions of s being the solutions of $\widetilde{C}(s, 0) = 0$, which are the same as the solutions of $C(s, 0) = 0$ already calculated for the test of stability in $\tau = 0$. A strategy to integrate this function is to predict the following step of the numerical integration using $T(s, \theta)$, and by getting close enough to the real solution, being able to correct it with some iterations of Newton's method. The size of the step can be tuned during execution by regarding the distance between the predicted part and the real solution of the corrected one.

The cases where $T(s,\theta) = 0$ or where we have multiple poles can still be dealt with, see (Chen, Fu, Niculescu & Guan 2010*a*) and (Chen, Fu, Niculescu & Guan 2010*b*). But this will need a higher order analysis or a different approach for the definition of (4.7). On the other hand, in both cases, when these events happen, we can always stop the numerical integration, deal locally with the gridding method provided before, and restart the integration method once we achieve simple roots with $T(s,\theta) \neq 0$.

4.2.5 Direction of Crossings

The objective now is to find for each crossing of roots through the imaginary axis, if it is a stabilizing or a destabilizing one. Notice that the use of the expressions *destabilizing* and *stabilizing crossings* only means that a pair of poles is crossing the imaginary axis in the defined direction, and not that the system is becoming unstable or stable, respectively. For that, it is necessary to know the number of unstable poles before these crossings. As it was shown in (Olgac & Sipahi 2002), the direction of crossing is constant with respect to sequential crossings (ℓ in (4.6)), and therefore it is denoted as *root tendency*.

We can deal with this problem in the same way we dealt before. Assume that $(s,\, \tau)$ is a simple root of $C(s,\tau) = 0$. For a small variation of $\tau^\star = \tau + \epsilon$ then a solution of $C(s^\star, \tau^\star) = 0$ can be found with the form

$$s^\star = s + \sum_{k=1}^{\infty} \mu_k \epsilon^k \tag{4.9}$$

where

$$\mu_1 = s \frac{\sum_{k=1}^{N} k q_k(s) e^{-\tau s k}}{p'(s) + \sum_{k=1}^{N} (q_k'(s) - \tau k q_k(s)) e^{-\tau s k}} = V(s,\tau). \tag{4.10}$$

The root tendency is given by $\text{sign}(\Re(V(j\omega,\tau)))$, where $(j\omega,\tau) \in \Delta$. If it is positive, then it is a destabilizing crossing, whereas if it is negative, this means it is a stabilizing crossing. In case the result is 0, a higher order analysis is needed, since this might be the case where the root just touches the imaginary axis and returns to its original half-plane.

To confirm the root tendency, we will show that

$$\text{sign}\left(\Re\left(V\left(j\omega, \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega}\right)\right)\right) \tag{4.11}$$

is independent of ℓ. Notice that for those given values, the exponential term

$$e^{-\tau s k} = e^{-j\theta k} e^{-j 2\pi \ell k} = e^{-j\theta k} \tag{4.12}$$

is independent of ℓ. So, the only part ℓ still appears is in the denominator of (4.10). But since $\text{sign}(\Re(z)) = \text{sign}(\Re(1/z))$ for $z \in \mathbb{C}\backslash\{0\}$, we can calculate the root tendency over the inverse of V, and we readily see that ℓ just enters in the imaginary part of it.

From this point, it is easy to determine for each value of the delay if the system has unstable roots or not. Start counting from the number of unstable poles of the delay free system. Sort Δ by the value of the delay of the crossing, and for each value of it such that the root tendency is positive, add two for the counting of unstable poles (the roots will always appear in pairs), or subtract two if the root tendency is negative. Repeat the procedure until the maximum value of the delay. Finally, identify the values of τ where the number of unstable poles is zero. Those are the stable regions.

4.2.6 Location of Unstable Poles

Suppose now that the problem is not finding the values of τ such that the system is stable, but in fact finding the actual position of the unstable poles for a given value of delay $\tau^\star$. This can be achieved by an adaptation of the same techniques used before.

From the definition of Ω, we can calculate a subset $\Delta_{\tau^\star}$ of Δ containing only the elements of Δ with the delay smaller than $\tau^\star$ and with positive root tendency.

Integrating $V(s,\tau)$ with respect to τ for each element of $(s^\circ, \tau^\circ) \in \Delta_{\tau^\star}$ for $\tau \in [\tau^\circ, \tau^\star]$ and s° as starting point, and the unstable poles of the delay free system for $\tau \in [0, \tau^\star]$ will generate the curves of the root loci. We can even back integrate from the elements of $\Delta_{\tau^\star}$ for τ decreasing from τ° to $\tau^\bullet$ in order to see their dynamics before the crossing, although back integrating until $\tau^\bullet = 0$ can be a bad idea since there is a chance that this might lead to solutions asymptotic to $\Re(s) \to -\infty$. A good trade-off seems to be choosing $\tau^\bullet = \tau^\circ/2$.

This procedure will provide the position of all unstable poles of (4.1), as well as some information about the stable ones. If one needs more information about the stable ones, one can always perform the same integration procedure over the stable poles for $\tau = 0$, the stabilizing crossings, and even also back integrating from a number of future crossings. But no guarantee can be given that those will be the ones closer to the imaginary axis. For this purpose, the methods provided by (Engelborghs et al. 2002) and (Vyhlídal & Zítek 2003) are more suitable.

4.2.7 Complete Algorithm

To recapitulate, the main points of the proposed procedure are summarized here:

- Calculate the position of the poles for the delay free system.

- Calculate the roots of (3.19) in order to get the asymptotic position of the neutral chains of poles.

- From all the roots of the delay free system, integrate (4.8) for $\theta \in [0, \pi]$ and find the locations where the solution has zero real part.

- With the result of the previous item, calculate the root cluster, the root tendency and the location of all crossings happening before the desired value of delay. This solves the question about the stability of the system.

- Integrate equation (4.10) from the points just calculated if they have positive tendency as well as the unstable poles for the delay free system until the target delay. This results in the location of the unstable poles.

4.2.8 Examples

Example 4.1 Our first example is from (Olgac & Sipahi 2004). Let the neutral system be described as in (3.19) with the state matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ -137.52 & -116.41 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3.8 & -2.2 \\ 142.45 & 117.68 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0.3 \\ -1.208 & -0.2253 \end{bmatrix},$$

and $E = I$ the identity matrix.

The roots of the delay free system are located in $s = 0.9976 \pm \jmath 3.0036$. The roots of (3.19) are located in $z = (3.6550, 1.9956)$, so both the neutral chains are located in left half-plane.

Integrating (4.8) for this system gives the result shown in Figure 4.1. From that we can see three crossings through $\Re(s) = 0$.

Calculating the root cluster and root tendency gives rise to Table 4.1. The first column gives the first τ such that that particular crossing happens, while the second column shows $\delta_\tau = 2\pi/|\omega|$ which is the distance between the sequential crossings. The third column shows the information of where the crossing is happening whereas the fourth one shows the root tendency.

From this table, we see that, the systems remains unstable for $0 \leq \tau < 0.0244$, and then becomes stable for $0.0244 < \tau < 0.1895$. And finally, it remains unstable for all $\tau > 0.1895$. These results are very similar to those obtained in (Olgac & Sipahi 2004) except for a small

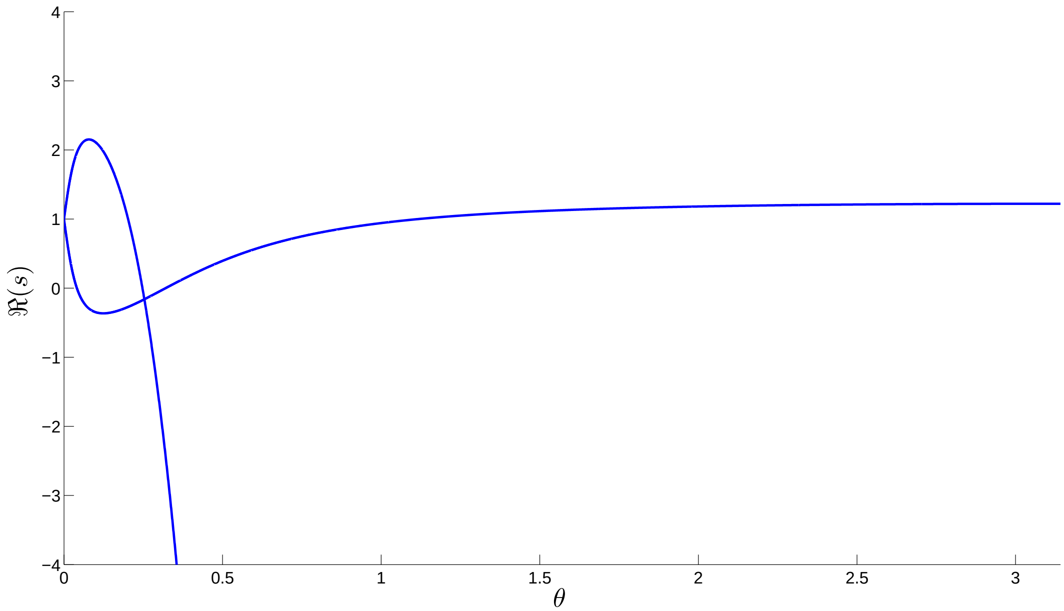

Figure 4.1: Real part of the poles of $\widetilde{C}(s, \theta)$

τ_0	δ_τ	ω	RT
0.0244	3.7577	1.6721	-1
0.1895	0.1973	31.8483	$+1$
0.2797	5.4839	1.1458	$+1$

Table 4.1: Root Cluster for Example

variation on the upper bound due to numerical precision. Using the algorithm of [(Vyhlídal & Zítek 2003)], our results have been verified.

We continue by supposing that we are interested in the case $\tau = 0.5$. From Table 4.1, we already know that the system is unstable, and moreover, that there will be six unstable poles. Plotting the root-locus of the system as in (4.10) provides Figure (4.2). In this picture, the numerical value of the delay is represented in colors, blue being $\tau = 0$ and red $\tau = 0.5$.

From the image we can see the initial unstable poles crossing the axis before that the first pair of stable ones arrives at $\omega = \pm 31.85$. In sequence, those two stabilized poles return to the right half-plane and finally new destabilizing crossings happen.

♦

Example 4.2 The second example is from (Marshall, Górecki, Walton & Korytowski 1992). Let $C(s, \tau) = s^2 + 4s + 4 - (1/4)e^{-s\tau}$. The poles for the delay free system are all in the left half-plane, and since this is a *retarded* time-delay system, we do not have issues with the chain of poles.

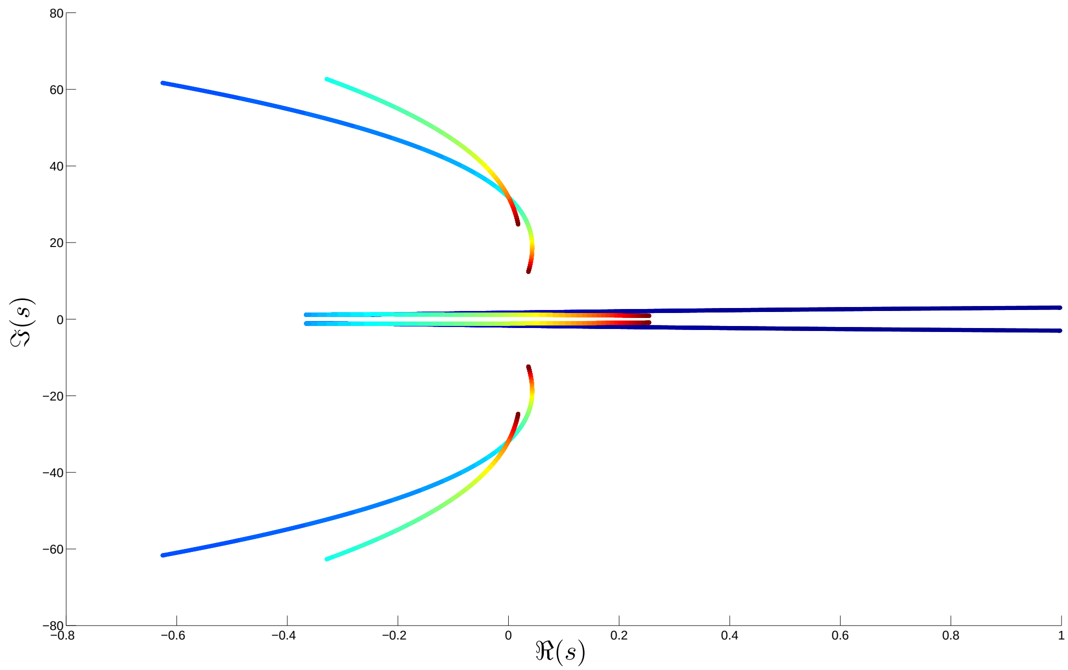

Figure 4.2: Root loci for Example 4.1 until $\tau = 0.5$

Integrating (4.8) for this system gives the result shown in Figure 4.3, which shows that there are no crossings over $\Re(s) = 0$. Therefore, this system is stable independent of delay.

♦

Example 4.3 As our last example, let $E = I$, $A_2 = 0$ and

$$A_1 = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix} \; A_2 = \begin{bmatrix} -0.0471 & -0.0504 & -0.0607 \\ -0.0942 & -0.1008 & -0.1214 \\ 0.0471 & 0.0504 & 0.0607 \end{bmatrix}.$$

One can see that this system is stable for $\tau = 0$ and remains stable until $\tau < 7.7117$, when the only destabilizing crossing happens at $\omega = \pm \jmath 0.0611$. The second crossing over the same point will happen only for $\tau = 110.62$.

Plotting the root locus for $\tau = 20$ shows an interesting situation. In Figure 4.4 we can see that around $\tau = 14.2$ two complex conjugated poles arrive at the same place and split into different directions in the real axis. As we stated before, this brings difficulties in the integration method. We solved this problem by detecting that (4.10) was becoming large and by searching for solutions over the real axis for a slightly larger value of τ. The errors of the final roots are of

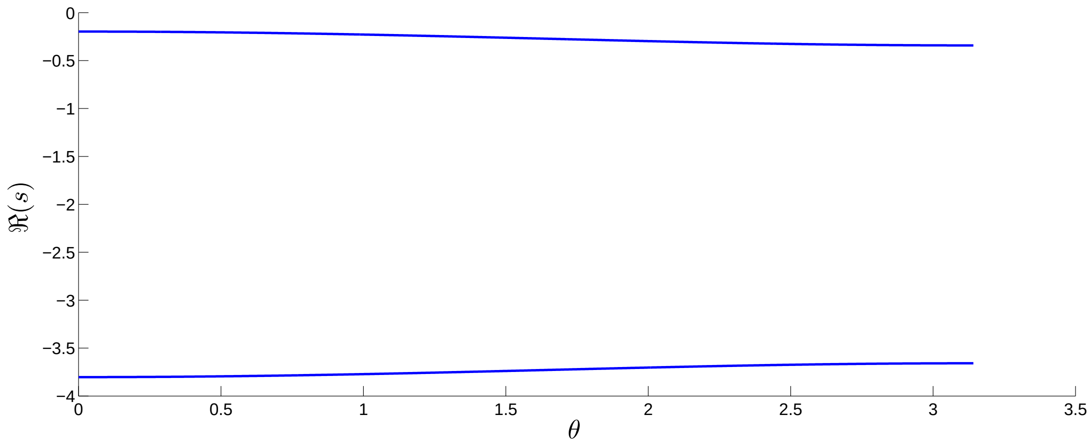

Figure 4.3: Real part of the location of poles of $\widetilde{C}(s, \theta)$

order 10^{-9}, which demonstrates that this circumvention may be effective in real problems. ♦

4.3 Fractional Systems

4.3.1 Problem formulation

Consider a system that in the frequency-domain has the following characteristic equation

$$C(s, \tau) = p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}, \tag{4.13}$$

where the parameter τ is non-negative, $p(s^\mu)$ and $q_k(s^\mu)$ for $k \in \mathbb{N}_N$ are polynomials in s^μ with $\mu \in (0, 1)$ and $\deg p \geq \deg q_k$. Here, the degree is interpreted as the degree in s^μ, and therefore it is an integer. If $\deg p = \deg q_k$ for at least one $k \in \mathbb{N}_N$, then equation (4.13) defines a *neutral* time-delay system, otherwise it will consist of the *retarded* type. Furthermore, we assume that both Hypothesis 4.1 and 4.2 are still valid.

4.3.2 Stability of fractional-order systems with delay

Before any development, let us recall some basic results given in Chapter 2. For fractional-order systems, a practical test for stability can be achieved if we use the variable substitution $\varsigma = s^\mu$. Applying this substitution, the characteristic equation (4.13) becomes

$$C_\varsigma(\varsigma, \tau) = p(\varsigma) + \sum_{k=1}^{N} q_k(\varsigma) e^{-k\tau\varsigma^{1/\mu}}. \tag{4.14}$$

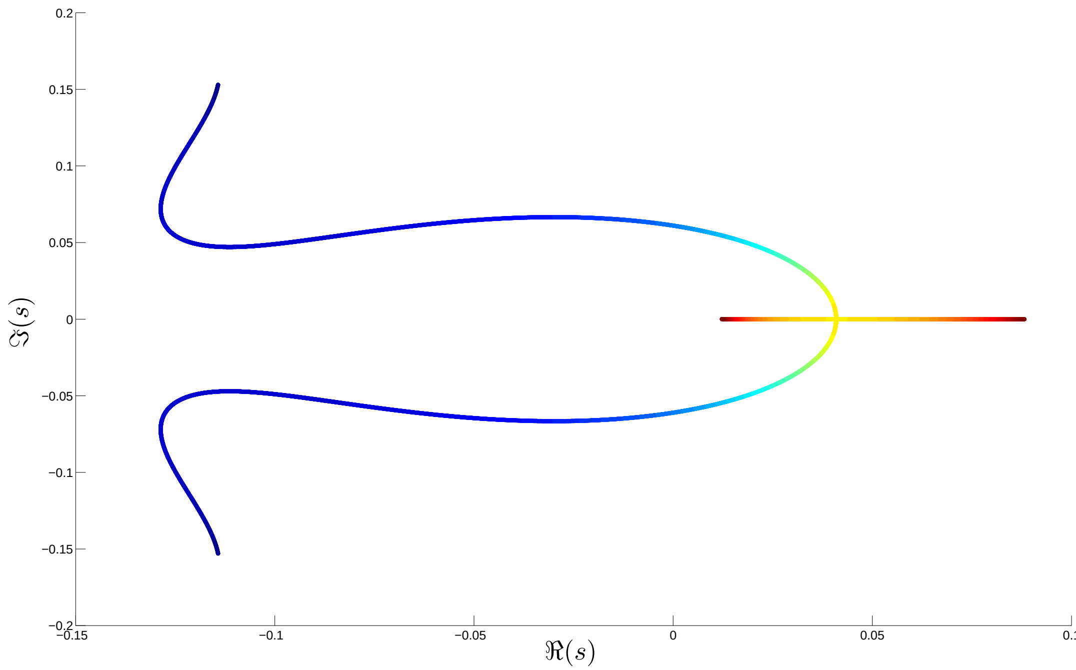

Figure 4.4: Root loci for Example 4.3 until $\tau = 20$

This will transform the domain of the system from a multi-sheeted Riemann surface into the complex plane, where its poles can be easily calculated. In this new variable, the instability region of the original system is not given by the right half-plane, but in fact by the region described as:

$$|\angle\varsigma| \leq \mu\frac{\pi}{2}, \tag{4.15}$$

with $\varsigma \in \mathbb{C}$, as illustrated by Figure 2.1.

The importance here is to notice that under this transformation, the imaginary axis in the s-domain is mapped into the lines

$$\angle\varsigma = \pm\mu\frac{\pi}{2}, \tag{4.16}$$

in the ς-domain, and therefore a solution of the type $\varsigma^\star = |\varsigma^\star|\angle \pm \mu\pi/2$, meaning $C_\varsigma(\varsigma^\star, \tau) = 0$ implies that the original system has a purely imaginary solution of the type

$$s^\star = \pm j|\varsigma^\star|^{1/\mu}. \tag{4.17}$$

Also, we can notice that every solution ς^* such that $|\angle\varsigma^*| < \mu\pi$ is mapped into the physical Riemann sheet in the s-domain through the inverse transform $s = \varsigma^{1/\mu}$.

Compared to the work presented in the previous section, the one proposed here is a little more involved. One major difference is the fact that roots can cross from one Riemann sheet to the other, which makes the procedure of following the roots in the s-domain much more complicated. To circumvent this complexity, we first apply the transformation to work in the ς-domain, and after completing the root-locus we map just part of the result which will fall back into the physical Riemann sheet of the s-domain. But as it will be clear in the sequel, this transformation changes the form of the delay term, and this must be taken into account.

Nevertheless, as we did in Chapter 3, any result which follows directly from its counterpart in the classical systems will be given without a complete proof, in order to focus only on the major differences presented.

4.3.3 Stability for $\tau = 0$

Considering $\tau = 0$ in (4.14), we get a polynomial with real coefficients. Again, the number of unstable poles of $C_\varsigma(\varsigma, 0)$ and their location will be used afterwards.

4.3.4 Location of Chains of Poles

This part does not differ from the equivalent on in the classical system. Indeed, if the system we are dealing with is of the *retarded* type, there will be no new unstable poles when we pass from $\tau = 0$ to $\tau = 0^+$. On the *neutral* case, the same considerations about the position of the asymptotic axis must be taken. Therefore, we will also consider that Hypothesis 4.3 is valid in the present context.

4.3.5 Crossing Position

Again, we will rely on a transformation of variables which decouples the pseudo-polynomials and the exponential part. Since for $s = \jmath\omega$ the exponential terms in (4.13) will have modulus equal to one, we replace $e^{-\jmath\omega\tau k}$ with $e^{-\jmath\theta k}$ and find the roots of the resulting complex pseudo-polynomial as a function of $\theta \in [0, \pi]$. That means, we find all the roots of the complex pseudo-polynomial in s

$$\widetilde{C}(s, \theta) = p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-jk\theta}, \tag{4.18}$$

varying θ in the interval $[0, \pi]$.

The following results are direct adaptation from the classical counterpart, therefore we will only state them. But how they will be applied differs from what we presented in the previous section, and this will be discussed in the sequel.

Lemma 4.8 *For any τ and $s = \jmath\omega$ such that $\widetilde{C}(s,\theta) = 0$, then $\widetilde{C}(\bar{s}, 2\pi - \theta) = 0$.*

Lemma 4.9 *There exist $s = \jmath\omega$ and $\tau > 0$ such that $C(s,\tau) = 0$ if and only if there exists $\theta \in [0,\pi]$ such that $\widetilde{C}(s,\theta) = 0$.*

Theorem 4.10 *Let Hypothesis 4.1-4.3 hold. Let Ω be the set of all ordered pairs (ω,θ), with $\omega \in \mathbb{R}$ and $\theta \in [0,\pi]$ such that $\widetilde{C}(\jmath\omega,\theta) = 0$. Let*

$$\tau(\omega,\theta,\ell) = \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega} \tag{4.19}$$

for all $(\omega,\theta) \in \Omega$. Choose $\ell = \{0,1,\ldots\}$ if $\omega > 0$, and $\ell = \{-1,-2,\ldots\}$ if $\omega < 0$. Let Δ be defined as the set of all the ordered pairs $(\pm\jmath\omega, \tau(\omega,\theta,\ell))$. Then Δ is the complete set of roots of (4.13) on the imaginary axis for $\tau > 0$.

Corollary 4.11 *If the system given by (4.13) is stable for $\tau = 0$ and $\Omega = \emptyset$, then the system is stable for all positive values of τ.*

We still denote the set Ω together with the root tendency of each ordered pair by the name of *root cluster*. As in the non-fractional case, the root tendency is constant with respect to any sequential crossings in (4.19).

To be able to use the results of Theorem 4.10, we need to be able to find all $\omega \in \mathbb{R}$ and $\theta \in [0,\pi]$ such that $\widetilde{C}(\jmath\omega,\theta) = 0$. As the direct calculation in the s-domain is not standard, we need to treat the problem completely into the ς-domain. To this matter, we propose two distinct approaches.

The most direct one consists in sampling θ in its interval $[0,\pi]$, and for each fixed value $\theta^\star$, make the variable substitution $\varsigma = s^\mu$ and calculate the roots of the resulting complex polynomial in ς

$$\widetilde{C}_\varsigma(\varsigma,\theta^\star) = p(\varsigma) + \sum_{k=1}^{N} q_k(\varsigma) e^{-jk\theta^\star} \tag{4.20}$$

As we have guaranteed that no poles in the origin will happen, this means that plotting the absolute value of the argument of those roots as a function of θ brings a useful graphic information. Also, as by assumption there are no chains of poles asymptotic to the imaginary axis, these plots will be continuous. Therefore, searching the positions where these curves cross the line $\theta = \mu\pi/2$ provide the information necessary to calculate where the actual crossings through the imaginary axis are in the s-domain.

The second method is the adaptation of the one presented for classical systems. If $(\varsigma,\ \theta)$ is a simple root of (4.20), then a small perturbation on $\theta^\star = \theta + \epsilon$ will provide a solution of

$\widetilde{C}(\varsigma^\star, \theta^\star) = 0$ with the form

$$\varsigma^\star = \varsigma + \sum_{k=1}^{\infty} \lambda_k \epsilon^k \tag{4.21}$$

where

$$\lambda_1 = \jmath \frac{\sum_{k=1}^{N} k q_k(\varsigma) e^{-\jmath\theta k}}{p'(\varsigma) + \sum_{k=1}^{N} q_k'(\varsigma) e^{-\jmath\theta k}} = T(\varsigma, \theta). \tag{4.22}$$

Here, $p'(\varsigma)$ and $q_k'(\varsigma)$ denote the derivative of the polynomials $p(\varsigma)$ and $q_k(\varsigma)$ in ς respectively.

So, a method to provide the same curves as the one calculated with the gridding method is to numerically integrate $T(\varsigma, \theta)$ for θ varying from 0 to π and the starting positions of ς being the roots of $\widetilde{C}_\varsigma(\varsigma, 0) = 0$, which are the same as the roots of $C_\varsigma(\varsigma, 0) = 0$ already calculated for the test of stability in $\tau = 0$. Whenever multiple roots occur, the solution described for the classical case is still applicable.

4.3.6 Direction of Crossing

The objective now is to find for each crossing of roots through the imaginary axis, if it is a stabilizing or a destabilizing one. As in the classical case, this is constant with respect to sequential crossings (ℓ in (4.19)), and therefore it is denoted as *root tendency*.

It is more convenient to calculate this point in the s-domain, but we can still deal with this problem in the same way we dealt before. Assume that $(s,\ \tau)$ is a simple root of $C(s, \tau) = 0$. For a small variation of $\tau^\star = \tau + \epsilon$ then a solution of $C(s^\star, \tau^\star) = 0$ can be found with the form

$$s^\star = s + \sum_{k=1}^{\infty} \mu_k \epsilon^k \tag{4.23}$$

where

$$\begin{aligned} \mu_1 &= s \frac{\sum_{k=1}^{N} k q_k(s) e^{-\tau s k}}{\mu s^{\mu-1} p'(s) + \sum_{k=1}^{N} (\mu s^{\mu-1} q_k'(s) - \tau k q_k(s)) e^{-\tau s k}} \\ &= V(s, \tau). \end{aligned} \tag{4.24}$$

The root tendency is given by $\text{sign}(\Re(V(j\omega, \tau)))$, where $(j\omega, \tau) \in \Delta$. A positive result means that it is a destabilizing crossing, whereas a negative ones spots a stabilizing crossing. In case the result is 0, a higher order analysis is needed, since this might be the case where the root just touches the imaginary axis and returns to its original half-plane.

From this point, we can easily apply the same counting algorithm applied before in order to determine for each value of the delay if the system has unstable roots or not.

4.3.7 Location of Unstable Poles

Although up to now the results from the classical systems and the fractional systems are somehow equivalent, the problem of finding the position of the unstable poles in the current case is more involved. We start from the same point, that is, from the definition of Ω, we can calculate a subset $\Delta_{\tau^\star}$ of Δ containing only the elements of Δ with the delay smaller than $\tau^\star$ and with positive root tendency. But to avoid any issues around the Riemann surfaces, it is more adequate to deal with this problem in the ς-domain. So assuming that $(\varsigma,\,\tau)$ is a simple root of $C_\varsigma(\varsigma,\tau)=0$, then for a small variation of $\tau^\star=\tau+\epsilon$, a solution of $C_\varsigma(\varsigma^\star,\tau^\star)=0$ can be found with the form $\varsigma^\star=\varsigma+\sum_{k=1}^{\infty}\nu_k\epsilon^k$, where

$$\begin{aligned}\nu_1 &= \frac{\varsigma^{1/\mu}\sum_{k=1}^{N}kq_k(\varsigma)e^{-\tau\varsigma^{1/\mu}k}}{p'(\varsigma)+\sum_{k=1}^{N}(q_k'(\varsigma)-\tau\mu^{-1}\varsigma^{(1-\mu)/\mu}kq_k(\varsigma))e^{-\tau\varsigma^{1/\mu}k}} \\ &= V_\varsigma(\varsigma,\tau). \end{aligned} \tag{4.25}$$

Integrating $V_\varsigma(\varsigma,\tau)$ with respect to τ for each element of $((s^\circ)^\mu,\tau^\circ)\in\Delta_{\tau^\star}$ for $\tau\in[\tau^\circ,\tau^\star]$ and $(s^\circ)^\mu$ as starting point, and the unstable poles (ς_k) of the delay free system for $\tau\in[0,\tau^\star]$ will generate the curves of the root locus in the ς-domain. Applying the inverse transformation $s=\varsigma^{1/\mu}$ for all the points in the ς *root-locus* with argument between $(-\mu\pi,\mu\pi)$ generates the root-locus in the physical layer in the s-domain. We can still back integrate from the elements of $\Delta_{\tau^\star}$ for τ varying from τ° to $\tau^\bullet$ in order to see their dynamics before the crossing, but we still need to consider only the part that will fall into the physical Riemann sheet.

4.3.8 Examples

Example 4.4 Our first example comes from (Ozturk & Uraz 1985) and (Hwang & Cheng 2006). Let us consider the system defined by

$$C_1(s)=\frac{1}{(\sqrt{s})^3-1.5(\sqrt{s})^2+4\sqrt{s}+8-1.5(\sqrt{s})^2e^{-\tau s}}. \tag{4.26}$$

Utilizing a heavy computation scheme based on the Cauchy's integral, (Hwang & Cheng 2006) showed that this system is unstable for $\tau=0.99$ but stable for $\tau=1$.

Applying the first part of the algorithm, we can see that there is a destabilizing crossing of poles at $\tau=0.7854k$ occurring at $s=\pm\jmath 8.0$ and a stabilizing crossing at $\tau=0.0499+0.9485k$ for $s=\pm\jmath 6.6246$, for all $k\in\{0,1,\ldots\}$. Therefore, we have the following 5 stability windows; $0.0499<\tau<0.7854$, $0.9983<\tau<1.5708$, $1.9486<\tau<2.3562$, $2.8953<\tau<3.1416$ and $3.8437<\tau<3.9270$, which agrees with the result given by both papers and in some sense explains it.

We further continue by considering $\tau = 3.9$. As this value is inside the last stability window, we know beforehand that the system is stable. But we search a better understand of the *root-loci* of the system as a function of the delay. Figure 4.5 brings this picture, where the colors represent the chosen τ, with deep-blue for $\tau = 0$ and strong red for $\tau = 3.9$.

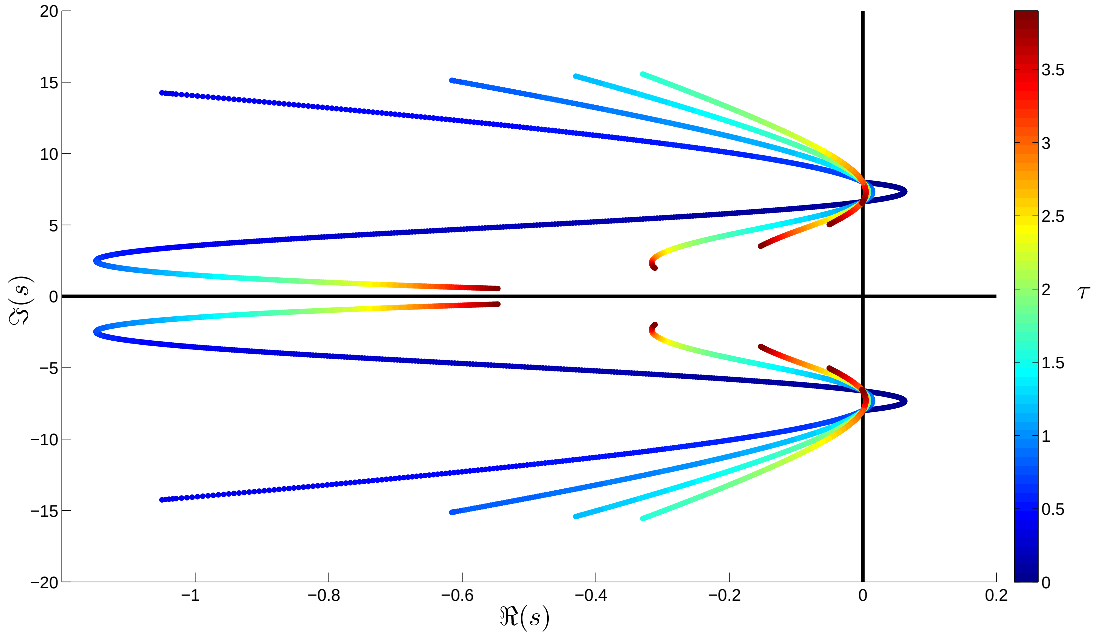

Figure 4.5: Root-loci for $C_1(s)$ until $\tau = 3.9$

One can see that although the depth inside the right half-plane for each destabilizing cross becomes less expressive when τ increases, the stabilizing crossing happens closer to the following destabilizing one, and after the pair of poles cross at $\tau = 3.9270$ to the right half-plane, the next pair of unstable poles arrive at $\tau = 4.7124$ before the previous ones exit this half-plane at $\tau = 4.7922$, and therefore the system cannot recover its stability. ♦

Example 4.5 The second example also comes from (Hwang & Cheng 2006). Considering the fractional-order system with two delays

$$C_2(s) = \frac{1}{s^{5/6} + (s^{1/2} + s^{1/3})e^{-0.5s} + e^{-s}} \tag{4.27}$$

It is stated that this system is stable. In order to apply our procedure, we transformed $C_2(s)$

into the following equivalent system

$$\widetilde{C}_2(s,\tau) = \frac{1}{s^{5/6} + (s^{3/6} + s^{2/6})e^{-\tau s} + e^{-2\tau s}} \tag{4.28}$$

and we have to study the stability for $\tau = 0.5$.

This system has no unstable pole for $\tau = 0$ (in fact, it has no pole in the physical Riemann sheet). Applying the methodology described before, we achieve that crossings through the imaginary axis happens for $\tau = 2.3562 + 6.2832k$ and $\tau = 2.6180 + 6.2832k$, and both of them are destabilizing crosses. That means that the only stability window for this system is $0 \leq \tau < 2.3562$. As $\tau = 0.5$ is included in this window, we can ensure that the original system $C_2(s)$ is stable. ♦

4.4 Final Remarks

In this chapter, a new method for calculating stability windows and location of the unstable poles was proposed for a large class of time-delay systems. As the main advantages, we just deal with polynomials of the same order as that of the original system, and we have a graphical representation of the number of elements in the root cluster. All the results up to now seem promising and consistent with the existing literature.

To the best of the author's knowledge, this is the first method able to deal with the proposed problem both from classical and fractional systems in the same framework. This helps us provide more insights about similarities and differences from those classes of systems.

The following publications were produced presenting the results of this chapter (Fioravanti, Bonnet, Özbay & Niculescu 2010), (Fioravanti 2010), (Fioravanti, Bonnet, Özbay & Niculescu 2011):

- A. R. Fioravanti, C. Bonnet, H. Özbay and S.-I. Niculescu, 'A Numerical Method to Find Stability Windows and Unstable Poles for Linear Neutral Time-Delay Systems'. *9th IFAC Workshop on Time Delay Systems*, Prague - Czech Republic, June 7 - 9, 2010.

- A. R. Fioravanti. 'Une méthode de continuation numérique pour des systèmes complexes paramétrés'. *3rd Digiteo Annual Forum*, École Polytechnique, Palaiseau - France, October 12, 2010.

- A. R. Fioravanti, C. Bonnet, H. Özbay and S.-I. Niculescu. 'Stability windows and unstable root-loci for linear fractional time-delay systems'. *The 18th IFAC World Congress*, Milan - Italy, August 28 - September 02, 2011. Accepted.

Chapter 5

Stability Crossing Curves of Shifted Gamma-Distributed Delay Systems

Contents

5.1 Introduction 93

5.2 Problem Definition 94

5.3 Stability crossing curves characterization 95

5.4 Smoothness of the crossing curves and crossing direction 99

5.5 Examples 101

5.6 Concluding remarks 102

5.1 Introduction

Up to this point, we focused our analysis on systems with discrete delays. But it is important to state that during the last decades, many authors such as (May 1974), (MacDonald 1978) and (Cushing 1981) worked with the idea that in some applications, like in biology, the use of distributed delays would lead to more realistic models. These pioneers started a huge development in the theory of those kinds of delay systems, specially in the quest of the study of diverse delay kernels, typically represented by general probability measures over time. This relies on the fact that, in the biological domain for example, even if the delay for some particular individual has a precise value, it may have a statistical distribution over the values in the whole population.

Many applications, both inside as outside the biological domain, can be modeled in this framework (see (Adimy, Crause & Abdllaoui 2008) and (Özbay, Benjelloun, Bonnet & Clairambault 2010) for some examples). The thesis (Morărescu 2006) also presents an excellent review of applications modeled with distributed delays.

We will focus our analysis on linear systems with shifted fractional gamma-distributed delay kernel. They were first introduced in (Cushing 1981), from where we can obtain the linearized model

$$\dot{x}(t) = -\alpha x(t) + \beta \int_0^t g(t-\theta)x(\theta)d\theta, \tag{5.1}$$

where α was related to some death rate and β to a maternity function. The integration kernel of the distribution delay is the gamma-distribution

$$g(\xi) = \frac{a^{n+1}}{\Gamma(n+1)}\xi^n e^{-a\xi}, \tag{5.2}$$

where Γ is the Gamma-function defined in Chapter 2 and n and a are positive real parameters. Such kernels have been used in different applicative areas since they offer an excellent fit to the data in many situations from biology to networks control. As some examples, the kernel may represent the distribution of maturation delay of hematopoïetic stem cells as in (Hearn, Haurie & Mackey 1998), (Haurie, Dale, Rudnicki & Mackey 2000) or (Bernard, Bélair & Mackey 2001), or the total network delay where the gamma distribution is used to model queueing delay as discussed in (Li & Mason 2007).

For zero initial conditions, applying the Laplace transform of (5.1) yields a characteristic equation of the form

$$D(s) = (s+\alpha)\left(1 + s\frac{\overline{\tau}}{n+1}\right)^{n+1} - \beta = 0, \tag{5.3}$$

where $\overline{\tau} = (n+1)/a$ is the *mean delay*.

Later, (Nisbet & Gurney 1986) introduced what is called *gamma-distribution with a gap*. It is expressed by the following delay kernel

$$\hat{g}(\xi) = \begin{cases} 0, & \xi < \tau \\ \dfrac{a^{n+1}}{\Gamma(n+1)}(\xi-\tau)e^{-a(\xi-\tau)}, & \xi \geq \tau. \end{cases} \tag{5.4}$$

This imposes a minimum delay τ and a mean delay $\hat{\tau} = \tau + (n+1)/a$. The characteristic equation becomes a quasi-polynomial of the form

$$\hat{D}(s) = (s+\alpha)\left(1 + s\frac{\hat{\tau}}{n+1}\right)^{n+1} - \beta e^{-s\tau} = 0, \tag{5.5}$$

which will be of fractional type whenever n is not an integer.

The objective of this chapter is to analyze stability in the parameter space defined by the delay terms, that is, the average delay $\hat{\tau}$, and the gap τ. The approach follows the lines of (Gu, Niculescu & Chen 2005) and (Morărescu, Niculescu & Gu 2007), which considered exclusively classical systems, that is, $n \in \mathbb{Z}_+$. We will characterize the *stability crossing curves*, that means, the set of parameters such that there is at least one pair of characteristic roots on the imaginary axis. Such stability crossing curves divide the parameter space $\mathbb{R}^2_+$ into different regions, such that within each such region, the number of strictly unstable characteristic roots is constant.

5.2 Problem Definition

Inspired by the motivation of the last section, consider the problem of stability analysis of a general class of delay differential equations that can be described in frequency-domain by the following characteristic function:

$$H(s, \alpha, \tau) = Q(s) + P(s)\frac{1}{(s+\alpha)^{m/n}}e^{-s\tau} \tag{5.6}$$

where P and Q denote polynomials and α, τ are strictly positive real parameters affecting the behavior of the system. Moreover, we take $s \in \mathbb{C} \setminus (-\infty, -\alpha]$ and $-\pi < \arg(s) < \pi$.

In order to restrict the analysis to fractional delay systems of retarded type, we need the following hypothesis:

Hypothesis 5.1 *The polynomials $P(s)$ and $Q(s)$ satisfy*

$$\deg P - \frac{m}{n} < \deg Q. \tag{5.7}$$

From now on we will also consider that:

Hypothesis 5.2 *The polynomials $P(s)$ and $Q(s)$ do not have common zeros.*

It is obvious that if the Hypothesis 5.2 is violated then P, Q have a common factor $c(s) \neq$ constant. Simplifying by $c(s)$ we get a system described by (5.6) satisfying the related Hypothesis.

As it can be deduced from (Bonnet & Partington 2002), $\mathcal{H}_\infty$-stability of such systems is equivalent to the condition *"no poles in the closed right half-plane"*. Therefore, as mentioned in the introduction, our aim is to present how the location of the roots of the characteristic function of the type (5.6) depends on the *parameter-space* defined by the pair (α, τ).

From (El'sgolts' & Norkin 1973) and (Datko 1978), we can establish the continuity dependence of the roots of the characteristic function with respect to the parameters α and τ. This property can be expanded for the fractional case, as seen in (Bonnet & Partington 2007). Therefore, the stability analysis reduces to the following problems:

- First, detect crossings with respect to the imaginary axis $\jmath\mathbb{R}$ since such crossing are related to changes of the stability property. In other words, we need to compute the *frequency crossing set* denoted by Ω, which consists of all frequencies corresponding to the existence of at least one critical characteristic root. As we shall see later, the frequency crossing set is reduced to a finite collection of intervals. This set will be derived by using geometric arguments.

- Second, describe the behavior of critical roots under changes of parameters in (α, τ) parameter space. More precisely, we will detect switches and reversals, corresponding to the situation when critical characteristic roots cross $\jmath\mathbb{R}$ towards instability or stability, respectively. Excepting some explicit computation of the crossing direction, we will also briefly discuss the smoothness properties as well as some appropriate classification of the stability crossing boundaries.

- Finally, another useful related concept is represented by the *characteristic crossing curves* consisting of all pairs (α, τ) for which there exists at least one value $\omega \in \Omega$ such that $H(\jmath\omega, \alpha, \tau) = 0$.

5.3 Stability crossing curves characterization

First of all, let us define precisely the terms which will be used in the sequel:

Definition 5.1 *Let $\mathcal{T}$ denote the set of all $(\alpha, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that (5.6) has at least one zero on $\jmath\mathbb{R}$. Any $(\alpha, \tau) \in \mathcal{T}$ is known as a* crossing point*. The set $\mathcal{T}$, which is the collection of all crossing points, is called the* stability crossing curves*.*

Notice that as $\omega \in \mathbb{R}$ and $(\alpha, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have $H(\jmath\omega, \alpha, \tau) = \overline{H(-\jmath\omega, \alpha, \tau)}$. Therefore, in order to avoid redundancy, we only need to consider positive ω. With this remark, we can define the following:

Definition 5.2 *Let Ω be defined as the collection of all $\omega > 0$ such that there exists a parameter pair (α, τ) such that $H(\jmath\omega, \alpha, \tau) = 0$. The set Ω is called the* crossing set*.*

It is important to notice that, by the last definition, as the parameters α and τ vary, the characteristic roots may cross the imaginary axis at $\jmath\omega$ if and only if $\omega \in \Omega$.

Now let us characterize the stability crossing curves in the (α, τ) parameter space.

5.3.1 Identification of the crossing set

Our first result regards the crossing set Ω.

Proposition 5.3 *The frequency crossing set Ω consists of a finite union of bounded intervals.*

Proof: Let $\omega \in \Omega$ be a crossing frequency. Considering $s = \jmath\omega$ and applying the modulus to (5.6) one gets

$$|Q(\jmath\omega)| = \left| \frac{P(\jmath\omega)}{(\jmath\omega + \alpha)^{m/n}} \right|. \tag{5.8}$$

If $Q(\jmath\omega) = 0$ one can easily derive that either $P(\jmath\omega) = 0$ or $\alpha \to +\infty$ and $m/n > 0$. The first case is excluded by Hypothesis 5.2. In a similar way $P(\jmath\omega) = 0$ implies that either $Q(\jmath\omega) = 0$ (which is also excluded by the same hypothesis) or $\alpha \to +\infty$ and $m/n < 0$. Therefore, equation (5.8) rewrites as

$$\left| \frac{P(\jmath\omega)}{Q(\jmath\omega)(\jmath\omega + \alpha)^{m/n}} \right| = 1. \tag{5.9}$$

Since $\deg(P) < \deg(Q) + m/n$ the left hand side of (5.9) tends to 0 when $\omega \mapsto \infty$. This means that there exists $\omega_M > 0$ such that $\Omega \subset (0, \omega_M]$. Furthermore (5.9) has solutions $(\omega, \alpha(\omega))$ if and only if

$$\begin{cases} |\omega|^{m/n} \leq \left| \dfrac{P(\jmath\omega)}{Q(\jmath\omega)} \right|, & m/n > 0, \\ |\omega|^{-m/n} \leq \left| \dfrac{Q(\jmath\omega)}{P(\jmath\omega)} \right|, & m/n < 0. \end{cases} \tag{5.10}$$

In this case

$$\alpha^2 = \left|\frac{P(\jmath\omega)}{Q(\jmath\omega)}\right|^{2n/m} - \omega^2. \tag{5.11}$$

It now becomes clear that Ω consists of a finite union of bounded intervals. Supposing that ω_0 is a limit point of one interval, the respective bound will be closed whenever (5.11) holds and opened when either $P(\jmath\omega_0)$ or $Q(\jmath\omega_0)$ is equal to zero. □

5.3.2 Identification of crossing points

Proposition 5.4 *Given $\omega \in \Omega$, the corresponding crossing points are given by:*

$$\alpha = \sqrt{\left|\frac{P(j\omega)}{Q(j\omega)}\right|^{2n/m} - \omega^2}, \tag{5.12}$$

$$\tau = \frac{1}{\omega}\Big((2p+1)\pi + \angle P(j\omega) - \angle Q(j\omega) - \frac{m}{n}\arctan\left(\omega\alpha^{-1}\right)\Big), \tag{5.13}$$

where $p \in \mathbb{Z}$.

Proof: The definition of α follows directly from (5.11). The definition of τ is obtained from (5.6) since $\angle Q(j\omega)$ must be equal to an integer multiple of 2π with $\angle\left(\frac{P(j\omega)}{(j\omega+\alpha)^{m/n}}e^{-j\omega\tau}\right)$. □

In the sequel we consider $\Omega = \bigcup_{k=1}^{N}\Omega_k$ and we do not restrict $\angle Q(j\omega)$ and $\angle P(j\omega)$ to a 2π range. Rather we allow them to vary continuously within each interval Ω_k. Thus, for each fixed $p \in \mathbb{Z}$, (5.12) and (5.13) define a continuous curve. Let us denote by $\mathcal{T}_k^p$ the curve corresponding to Ω_k with $p \in \mathbb{Z}$. We note that we have an infinite number of curves corresponding to each interval.

5.3.3 Classification of the crossing curves

Let the left and right end points of interval Ω_k be denoted as ω_k^ℓ and ω_k^r, respectively. Due to Hypothesis 5.1 and 5.2, it is not difficult to see that each end point ω_k^ℓ or ω_k^r must belong to one, and only one, of the following three types:

Type 1a. It satisfies the equation $Q(j\omega) = 0$ and $m/n > 0$.

Type 1b. It satisfies the equation $P(j\omega) = 0$ and $m/n < 0$.

Type 2. It satisfies the equation

$$|\omega|^{m/n} = \left|\frac{P(j\omega)}{Q(j\omega)}\right|. \tag{5.14}$$

Type 3. It equals 0.

Let us denote a generic end-point by ω_0, which may be either a left end or a right end of an interval Ω_k. Then the corresponding points in $\mathcal{T}_k^p$ may be described as follows:

If ω_0 is of type **1a**, then as $\omega \to \omega_0$, $\alpha \to +\infty$ and

$$\tau \to \frac{1}{\omega_0}\Big((2p+1)\pi + \angle P(j\omega_0) - \lim_{\omega\to\omega_0} \angle Q(j\omega)\Big). \tag{5.15}$$

Obviously,

$$\lim_{\omega\to\omega_0} \angle Q(j\omega) = \angle \left[\frac{d}{d\omega} Q(j\omega)\right]_{\omega\to\omega_0} \tag{5.16}$$

if ω_0 is the left end point ω_k^ℓ of Ω_k, and

$$\lim_{\omega\to\omega_0} \angle Q(j\omega) = \angle \left[\frac{d}{d\omega} Q(j\omega)\right]_{\omega\to\omega_0} + \pi \tag{5.17}$$

if ω_0 is the right end point ω_k^r of Ω_k. In other words, $\mathcal{T}_k^p$ approaches a horizontal line.

If ω_0 is of type **1b**, then as $\omega \to \omega_0$, $\alpha \to +\infty$ and

$$\tau \to \frac{1}{\omega_0}\Big((2p+1)\pi - \angle Q(j\omega_0) + \lim_{\omega\to\omega_0} \angle P(j\omega)\Big). \tag{5.18}$$

Again $\mathcal{T}_k^p$ approaches a horizontal line.

If ω_0 is of type **2**, then $\alpha = 0$. In other words, $\mathcal{T}_k^m$ intersects the τ-axis at $\omega = \omega_0$.

Obviously, only ω_1^ℓ may be of type **3**. In this case, $\alpha \to |P(0)/Q(0)|^{n/m}$ and $\tau \to \infty$ as $\omega \to 0$. Therefore (α, τ) approaches a vertical line. It is noteworthy that in some particular cases for $p = 0$ one gets $\pi + \angle P(j\omega) - \angle Q(j\omega) = 0$ implying:

$$\tau \underset{\omega\to 0}{\longrightarrow} \left[\frac{d}{d\omega}\angle P(j\omega)\right]_{\omega\to 0} - \left[\frac{d}{d\omega}\angle Q(j\omega)\right]_{\omega\to 0} \frac{m}{n}\left|\frac{P(0)}{Q(0)}\right|^{-n/m},$$

that is

$$\tau \underset{\omega\to 0}{\longrightarrow} \frac{P'(0)}{P(0)} - \frac{Q'(0)}{Q(0)} - \frac{m}{n}\left|\frac{P(0)}{Q(0)}\right|^{-n/m}, \tag{5.19}$$

resulting in (α, τ) approaching a limit point.

Moreover, we can see from (5.6) that whenever there exists $\alpha \in \mathbb{R}_*^+$ such that $\alpha^{m/n} = -P(0)/Q(0)$, then $\omega = 0$ will be a solution of the characteristic equation independently of the value of τ so that $\mathcal{T}_k^p$ includes a vertical line.

We say an interval Ω_k is of type ℓ/r if its left end is of type ℓ and its right end is of type r. We may accordingly divide these intervals into 12 types, 6 for when $m/n > 0$ and 6 when $m/n < 0$. The classification for the first 6 goes as follows:

- **Type 1a/1a**. In this case, both ends of $\mathcal{T}_k^p$ approach horizontal lines.
- **Type 1a/2**. $\mathcal{T}_k^p$ starts at ∞ along a horizontal line at ends at the τ-axis
- **Type 2/1a**. This is the reversal of type 1a/2. $\mathcal{T}_k^p$ starts at a point on the τ-axis and the other end approaches ∞ along a horizontal line.
- **Type 2/2**. Both ending points of $\mathcal{T}_k^p$ are on the τ-axis.
- **Type 3/1a**. In this case, $\mathcal{T}_k^p$ starts at ∞ along a vertical line, where the other end approaches a horizontal line.
- **Type 3/2**. Finally, $\mathcal{T}_k^p$ starts at ∞ along a vertical line, where the other end is at the τ-axis.

5.4 Smoothness of the crossing curves and crossing direction

In the sequel, let us consider that the frequency crossing set Ω is given and the stability crossing curves are described by the smooth mappings $\omega \mapsto \alpha(\omega)$, $\omega \mapsto \tau(\omega)$. Denote also by $\mathcal{T}_h$ an arbitrary crossing curve and consider the following decompositions into real/imaginary parts:

$$\begin{aligned} R_0 + \jmath I_0 &= \jmath \left.\frac{\partial H(s,\alpha,\tau)}{\partial s}\right|_{s=\jmath\omega}, \\ R_1 + \jmath I_1 &= \left.\frac{\partial H(s,\alpha,\tau)}{\partial \alpha}\right|_{s=\jmath\omega}, \\ R_2 + \jmath I_2 &= \left.\frac{\partial H(s,\alpha,\tau)}{\partial \tau}\right|_{s=\jmath\omega}. \end{aligned}$$

By the implicit function theorem, we have that the tangent of $\mathcal{T}_h$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}\alpha}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix}_{s=\jmath\omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - R_2 I_0 \\ R_1 I_0 - R_0 I_1 \end{pmatrix}, \tag{5.20}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{5.21}$$

It follows that $\mathcal{T}_h$ is smooth everywhere except possibly at the points where either (5.21) is not satisfied, or when

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{5.22}$$

If (5.22) is satisfied then straightforward computations show us that $R_0 = I_0 = 0$. In other words $s = \jmath\omega$ is a multiple solution of (5.6).

We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*. To establish the direction of crossing we need to consider α and τ as functions of $s = \sigma + \jmath\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_h$ along the positive direction is $\left(\frac{\partial\alpha}{\partial\omega}, \frac{\partial\tau}{\partial\omega}\right)$, the normal to $\mathcal{T}_h$ pointing to the left hand side of positive direction is $\left(-\frac{\partial\tau}{\partial\omega}, \frac{\partial\alpha}{\partial\omega}\right)$. Corresponding to a pair of complex conjugate solutions crossing the imaginary axis along the horizontal direction into the right half-plane, the pair (α, τ) moves along the direction $\left(\frac{\partial\alpha}{\partial\sigma}, \frac{\partial\tau}{\partial\sigma}\right)$. So, as (α, τ) crosses the stability crossing curves from the right hand side to the left hand side, a pair of complex conjugate solutions crosses the imaginary axis to the right half-plane, if

$$\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial\tau}{\partial\sigma} - \frac{\partial\tau}{\partial\omega}\frac{\partial\alpha}{\partial\sigma}\right)_{s=\jmath\omega} > 0, \tag{5.23}$$

i.e. the region on the left of $\mathcal{T}_h$ gains two solutions on the right half plane. If the inequality (5.23) is reversed then the region on the left of $\mathcal{T}_h$ loses two right half plane solutions. Similar to (5.21) we can express

$$\left(\begin{array}{c}\frac{\partial \alpha}{\partial \sigma}\\ \frac{\partial \tau}{\partial \sigma}\end{array}\right)_{s=\jmath\omega} = \left(\begin{array}{cc} R_1 & R_2 \\ I_1 & I_2 \end{array}\right)^{-1}\left(\begin{array}{c} I_0 \\ -R_0 \end{array}\right) = \frac{1}{R_1I_2 - R_2I_1}\left(\begin{array}{c} R_0R_2 + I_0I_2 \\ -R_0R_1 - I_0I_1 \end{array}\right). \tag{5.24}$$

Proposition 5.5 *Assume $\omega \in \Omega$, and $s = \jmath\omega$ is a simple root of (5.6) and $H(j\omega', \alpha, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (α, τ) is not an intersection point of two curves or different sections of a single curve). Then as (α, τ) moves from the region on the right to the region on the left of the corresponding crossing curve, a pair of solutions of (5.6) crosses the imaginary axis to the right (through $s = \pm\jmath\omega$) if $R_1I_2 - R_2I_1 > 0$. The crossing is to the left if the inequality is reversed.*

5.5 Examples

In order to illustrate the proposed results, we shall consider two academic examples:

Example 5.1 (types 3/2 and 2/2). Let $m = 1$, $n = 2$, $P(s) = 3s^2+2s+3$ and $Q(s) = s^2+s-1$. Figure 5.1 plots $|P(\jmath\omega)|/|Q(\jmath\omega)|$ and $|\omega|^{m/n}$ against ω. From the plot, it can be seen that the crossing set Ω contains two intervals, $\Omega_1 = [0,\ 0.8895]$ of type 3/2 and $\Omega_2 = [1.34,\ 8.4489]$ of type 2/2. Figure 5.2 shows the stability crossing curves for Ω_1. With $\omega \to 0$, we can notice that for $p = 0$, the curve is in the special case where (α, τ) goes first to a limit point and then join a vertical axis. But for all other cases, the graphic is asymptotic to a vertical line. Finally, when $\omega \to \omega_1^r = 0.8895$, all curves end on the τ-axis. Figure 5.3 shows the stability crossing curves for Ω_2, which is of type 2/2. As it can be easily remarked, it consists of a series of curves with both ends on the τ-axis. ♦

Example 5.2 (types 3/1a and 1a/2). For the second example, let us consider $m = 1$, $n = 2$, $P(s) = 2s + 5$ and $Q(s) = s^2 + 2$. Figure 5.4 plots $|P(\jmath\omega)|/|Q(\jmath\omega)|$ and $|\omega|^{m/n}$ against ω. From the plot, it can be seen that the crossing set Ω contains two intervals, $\Omega_1 = [0, \sqrt{2}]$ of type 3/1a and $\Omega_2 = [\sqrt{2},\ 2.5441]$ of type 1a/2. Figure 5.5 shows the stability crossing curves for Ω_1. We precise that as $\omega \to 0$ the graphic asymptotically approaches the vertical line $\alpha = 6.25$. When $\omega \to \omega_1^r = \sqrt{2}$, all the curves are asymptotic to a horizontal line. To conclude the academic examples, Figure 5.6 shows the stability crossing curves for Ω_2, which is of type 1/a2, and consists

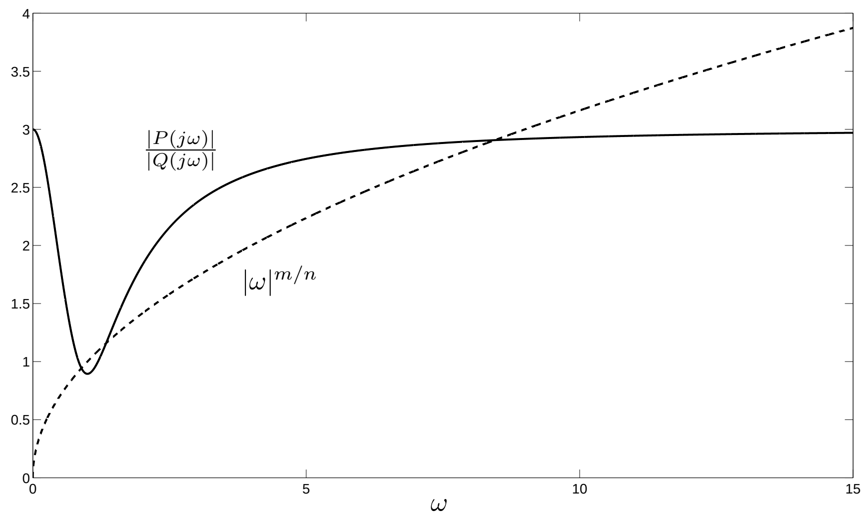

Figure 5.1: Example 5.1: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ and $|\omega|^{m/n}$ against ω.

of a series of curves which starts on a horizontal line and ends on the τ-axis. It can be shown that this system is stable for $(\alpha, \tau) = (0, 0)$, but there is no value of α for which the system does not cross the stability curves when τ increases. ♦

5.6 Concluding remarks

In this chapter, we addressed the asymptotic stability of a class of linear systems with fractional gamma-distributed delays. More precisely, we gave the complete characterizations of the stability crossing curves and corresponding crossing frequency set for that class of systems. As we stated in the beginning of the chapter, there are some strong practical applications dealing with distributed delays. Our contribution was to introduce the concepts of crossing curves, largely studied for classical systems, to the class of fractional ones.

The following publication was produced presenting the results of this chapter (Morărescu, Fioravanti, Niculescu & Bonnet 2009):

- I. C. Morarescu, A. R. Fioravanti, S.-I. Niculescu and C. Bonnet. 'Stability Crossing Curves of Linear Systems with Shifted Fractional Gamma-Distributed Delays'. *8th IFAC Workshop on Time Delay Systems*, Sinaia - Romania, September 1-3, 2009.

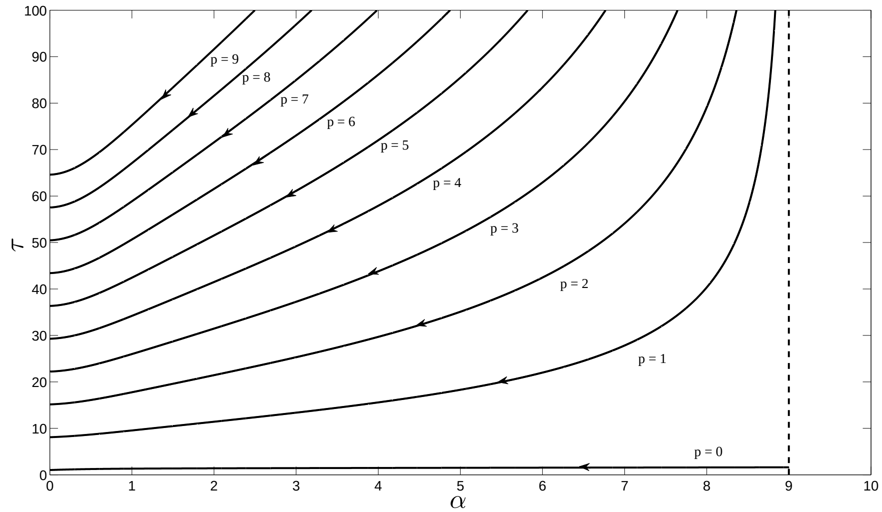

Figure 5.2: Example 5.1: $\mathcal{T}_1^p$ for $p = 0, \ldots, 9$ (type 3/2).

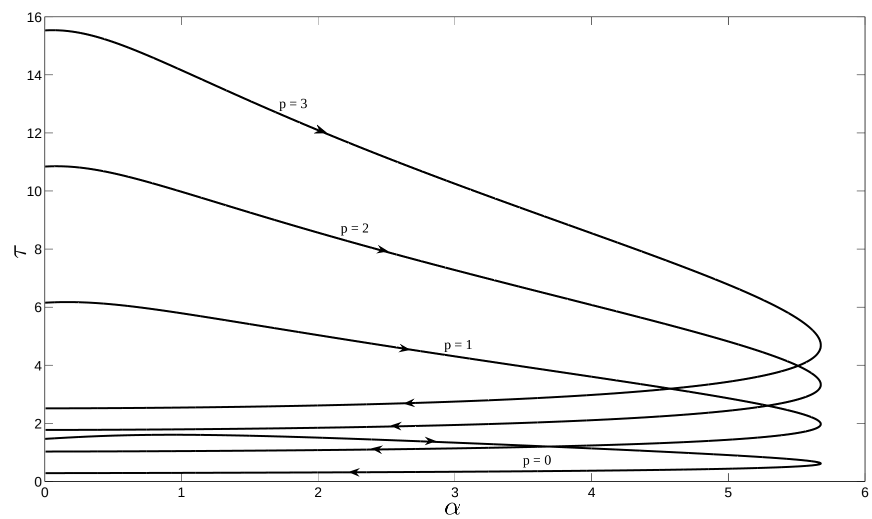

Figure 5.3: Example 5.1: $\mathcal{T}_2^p$ for $p = 0, 1, 2, 3$ (type 2/2).

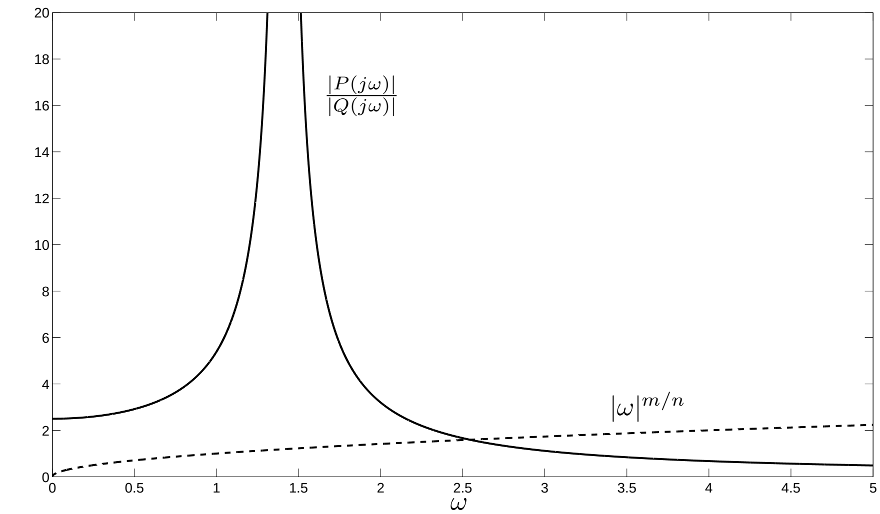

Figure 5.4: Example 5.2: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ and $|\omega|^{m/n}$ against ω.

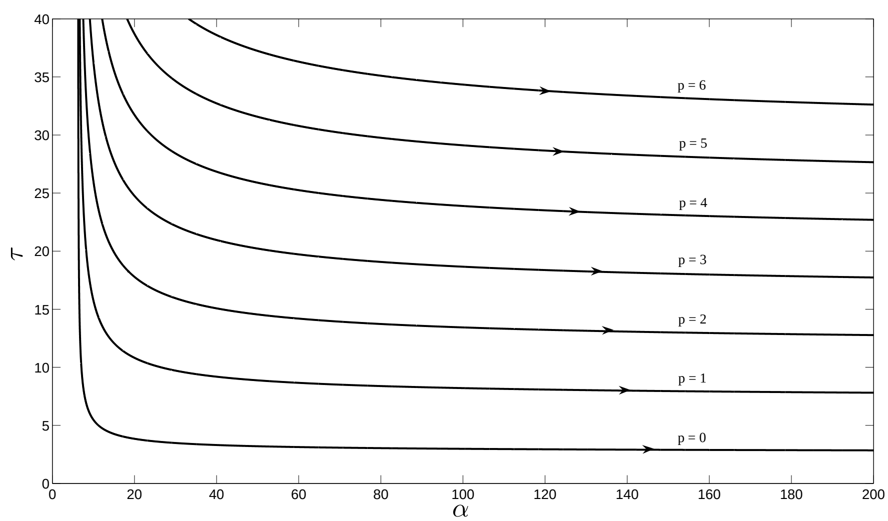

Figure 5.5: Example 5.2: $\mathcal{T}_1^p$ for $p = 0, \ldots, 6$ (type 3/1a).

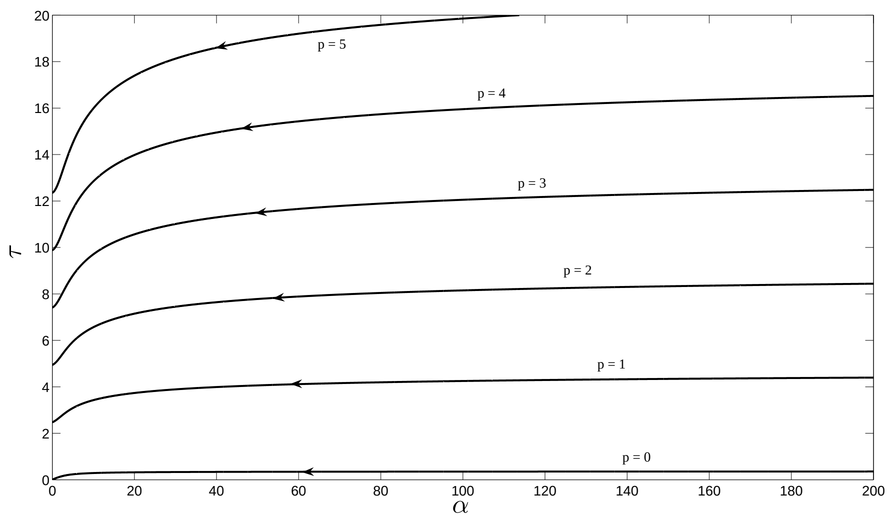

Figure 5.6: Example 5.2: $\mathcal{T}_2^p$ for $p = 0, \ldots, 5$ (type 1/a2).

Part III

Control of Time-Delay Systems

Chapter 6

PID Controller Design

Contents

6.1 Introduction **111**

6.2 Problem Definition **111**

6.3 PID controller design **113**

6.4 Example **119**

6.5 Final Remarks **122**

6.1 Introduction

It goes without questioning that PID controllers are the most successful ones in industry. They can be easily implemented, tuned and serve well for a large amount of applications. Many techniques have been proposed to provide PID controllers which are able to cope with time delays, e.g. (Gündeş, Özbay & Özgüler 2007), (Lin, Wang & Lee 2004), (Özbay & Gündeş 2007), (Silva, Datta & Bhattacharyya 2005) among many others.

When dealing with fractional systems, two possible subjects might be taken into account. The first concerns fractional controllers, which are commonly denoted as $PI^{\alpha}D^{\beta}$ (Podlubny 1999*b*), meaning that the transfer function of the controller is given by

$$C(s) = K_p + \frac{K_i}{s^{\alpha}} + K_d s^{\beta}, \tag{6.1}$$

with α and β in $(0, 1)$.

This family of controllers has 5 parameters that can be tuned, namely K_p, K_i, K_d, α and β. This clearly opens more space to the designer for stability, performance or robustness, but at the expense of a harder implementation issue. Indeed, even if some analog fractional controllers already exist (Bohannan 2008), they are still far from being easily accessible, and lack some of the properties that made conventional PID controllers widely spread.

On the other hand, as we have already illustrated, fractional systems can appear naturally in many situations, both in engineering and science. Therefore, designing classic PID controllers able to cope with delays for such systems might have a large practical impact, especially if the rules for an initial tuning of the parameters are easily comprehensible and do not require any deep knowledge of specific optimization tools or techniques. This is our main motivation for the chapter.

6.2 Problem Definition

Consider the standard single input single output feedback system shown in Figure 6.1, where C is the controller to be designed for the plant P.

We assume that the plant is linear and time invariant. Its dynamical behavior is represented by the transfer function

$$P(s) = e^{-s\tau} \frac{G(s^{\mu})}{s^{\mu} - p} \tag{6.2}$$

where $\tau > 0$ is the total input-output time delay, $\mu \in (0,\ 1)$ is the fractional order, $p^{1/\mu} \geq 0$ is the location of the unstable pole of the plant, and $G(\sigma)$ is a rational stable transfer function

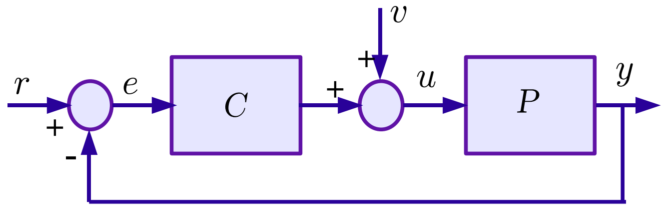

Figure 6.1: Standard Feedback System

in the variable $\sigma = s^\mu$ with $G(p) \neq 0$ and $G(0) \neq 0$. It is clear that we need $G(0) \neq 0$ for stabilizability of (6.2) by a controller which includes an integrator. We assume that μ is a rational number, i.e., we are restricting ourselves to the class of *fractional systems of commensurate order*, (Matignon 1998).

Given all the parameters of the plant, (6.2), our goal is to design a classical Proportional + Integral + Differential (PID) controller in the form

$$C(s) = K_p + \frac{K_i}{s} + K_d \frac{s}{\tau_d s + 1} \tag{6.3}$$

where $K_p,\ K_i,\ K_d$ are free parameters and τ_d is an arbitrarily small positive number making the controller proper.

The feedback system formed by the controller C and the plant P is stable if $(1+PC)^{-1}$, $C(1+PC)^{-1}$, $P(1+PC)^{-1}$ are stable transfer functions. It has been proved (Bonnet & Partington 2002) that $\mathcal{H}_\infty$-stability of systems of type (6.2) is equivalent to their BIBO-stability, and the necessary and sufficient condition being that the system has no poles in the right half-plane, including no pole of fractional order at $s = 0$. This information can be easily tested with the numerical algorithm we provided in Chapter 4.

Let us briefly recall the stability of such systems given in Chapter 2. We start by taking $\sigma = s^\mu$ and assuming that $T(\sigma)$ is a rational function with poles $\sigma_1, \ldots, \sigma_n$. We continue by enumerating the poles so that $\sigma_1, \ldots, \sigma_{2n_c}$ are complex conjugate, with $\sigma_{n_c+k} = \bar{\sigma}_k$ and $\sigma_k = |\sigma_k| e^{j\theta_k}$ where $\theta \in (0,\ \pi)$ for $k = 1, \ldots, n_c$, and $\sigma_{2n_c+1}, \ldots, w_n$ are real. Then, the system $T(s^\mu)$ is stable if and only if

$$\begin{aligned} &\mu \frac{\pi}{2} < \theta_k && \text{for} \quad k = 1, \ldots, n_c, \text{ and} \\ &\sigma_k < 0 && \text{for} \quad k = 2n_c + 1, \ldots, n. \end{aligned} \tag{6.4}$$

We say that C is a stabilizing controller for the plant P if the feedback system formed by this pair is stable.

6.3 PID controller design

In this section we will design classical PID controllers in the form (6.3) for the plant (6.2). It is inspired by the work given in (Gündeş et al. 2007), in the way that the design will be done in two steps: first PD controllers will be investigated, then the integral action will be added.

6.3.1 PD controller design

Typical PD controller can be written in the form

$$C_{pd}(s) = K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{6.5}$$

We can express the non-delayed part of the plant as the ratio of two stable factors:

$$P(s) = e^{-s\tau} Y(s)^{-1} X(s) \quad \text{with} \quad Y(s) := \frac{s^\mu - p}{s^\mu + x} \qquad X(s) := \frac{G(s^\mu)}{s^\mu + x} \tag{6.6}$$

where $x > 0$ is a free parameter. While it is an arbitrary positive number at this stage, x plays an important role in the controller design.

With the notation introduced in (6.6) the feedback system stability is equivalent to stability of U^{-1}, where

$$U(s) := Y(s) + e^{-s\tau} X(s) C_{pd}(s). \tag{6.7}$$

Inserting C_{pd}, X and Y into (6.7) we have

$$U(s) = 1 - \frac{p + x}{s^\mu + x} + e^{-s\tau} \frac{G(s^\mu)}{s^\mu + x} K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{6.8}$$

By choosing

$$K_p = (p + x) G(0)^{-1} \tag{6.9}$$

we obtain

$$\begin{aligned} U(s) &= 1 - \frac{p + x}{s^\mu + x} \left(1 - e^{-s\tau} G(s^\mu) G(0)^{-1} \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right)\right) \\ &= 1 - \frac{(p + x)s^\mu}{s^\mu + x} \left(\frac{1 - e^{-s\tau} G(s^\mu) G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau} G(s^\mu) G(0)^{-1} \frac{s^{1-\alpha}}{\tau_d s + 1}\right). \end{aligned} \tag{6.10}$$

Since $\left\|\dfrac{s^\mu}{s^\mu + x}\right\|_\infty = 1$ for all $x > 0$, by the small gain theorem, U^{-1} is stable if

$$\left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\mu}}{\tau_d s + 1}\right\|_\infty < \frac{1}{p + x}. \tag{6.11}$$

The following results are immediate consequences of the above discussion.

Lemma 6.1 *For the plant* (6.2)*, there exists a stabilizing proportional controller,* $C(s) = K_p$*, if*

$$p < \left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu}\right\|_\infty^{-1} =: \psi_o. \tag{6.12}$$

When (6.12) *holds, all proportional controllers in the form* (6.9) *are stabilizing, where* x *satisfies* $0 < x < (\psi_o - p)$.

Lemma 6.2 *Suppose there exist* $\widetilde{K}_d \in \mathbb{R}$ *and* $\tau_d > 0$*, such that*

$$p < \left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\mu}}{\tau_d s + 1}\right\|_\infty^{-1} =: \psi_d. \tag{6.13}$$

Then, the controller $C_{pd}(s)$ *given in 6.5 is a stabilizing controller for the plant* (6.2) *with* $K_p = (p + x)G(0)^{-1}$ *for all* x *satisfying* $0 < x < (\psi_d - p)$.

From the PD controller design method proposed in Lemma 6.2, we see that the allowable values of the proportional gain are in the range

$$K_p^{\min} := pG(0)^{-1} < K_p < \psi_d G(0)^{-1} =: K_p^{\max}. \tag{6.14}$$

Therefore, we would like to maximize ψ_d in order to maximize the allowable range for K_p. This problem is equivalent to finding the optimal $\widetilde{K}_d \in \mathbb{R}$ so that

$$\psi_d^{-1} = \left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\mu}}{\tau_d s + 1}\right\|_\infty \tag{6.15}$$

is minimized for a given fixed $\tau_d > 0$. A similar problem has been studied in (Özbay & Gündeş 2007) for the case $\mu = 1$, i.e., for rational systems with time delays. In general, minimization of ψ_d^{-1} is a two-dimensional search: for each fixed $\widetilde{K}_d \in \mathbb{R}$ compute the infinity norm by a frequency sweep. In (Özbay & Gündeş 2007) it is shown that, for a large class of rational systems with time delays, this computation can be reduced to a one dimensional search. Unfortunately, it seems that this is not the case for the class of plants studied here.

Once ψ_d is maximized, we would like to choose K_p so that the gain margin is maximized, i.e.,

$$\min\left\{\frac{K_p}{K_p^{\min}}, \frac{K_p^{\max}}{K_p}\right\} \tag{6.16}$$

is maximized, (Özbay 2000). Clearly the optimal choice is $K_p^{\text{opt}} = \sqrt{K_p^{\min} K_p^{\max}}$, i.e.

$$K_p^{\text{opt}} = \sqrt{p\psi_d}\, G(0)^{-1}. \tag{6.17}$$

6.3.2 Adding integral action to the PD controller

Assume that the condition (6.13) of Lemma 6.2 is satisfied and hence a stabilizing PD controller C_{pd} can be found for the plant (6.2). We now try to find

$$C_i(s) = \frac{K_i}{s} \tag{6.18}$$

so that $C_{pid}(s) = C_{pd}(s) + C_i(s)$ is a stabilizing controller for the plant. This is a two step design process and it works as follows, see e.g. (Gündeş et al. 2007) or (Vidyasagar 1985). Define

$$H(s) := P(s)(1 + P(s)C_{pd}(s))^{-1} \tag{6.19}$$

and note that $H(0) = G(0)/x$ which is non-zero by the assumption that $G(0) \neq 0$ and by design $x > 0$. If C_i defined by (6.18) is a stabilizing controller for the "new plant" H, (6.19), then C_{pid} is a stabilizing controller for the original plant P. Now let

$$K_i := \gamma H(0)^{-1}, \quad \text{with} \quad \gamma > 0 \tag{6.20}$$

then

$$\begin{aligned} (1 + C_i(s)H(s))^{-1} &= \left(\frac{s + \gamma H(s)H(0)^{-1} + \gamma - \gamma}{s}\right)^{-1} \\ &= \frac{s}{s+\gamma}\left(1 + \frac{\gamma s^{\alpha}}{s+\gamma}\left(\frac{H(s)H(0)^{-1} - 1}{s^{\alpha}}\right)\right)^{-1}. \end{aligned} \tag{6.21}$$

Let us define

$$R_{\alpha}(\gamma) := \left\| \frac{\gamma s^{\alpha}}{s+\gamma} \right\|_{\infty}. \tag{6.22}$$

Then by the small gain theorem $C_i(s) = \gamma H(0)^{-1}/s$ is a stabilizing controller for $H(s)$ if

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1}. \tag{6.23}$$

Note that for $\alpha = 1$ we have $R_1(\gamma) = \gamma$, and for the rational case the function $(H(s)H(0)^{-1}-1)/s$ is stable. However, when $H(s)$ is a fractional transfer function, $(H(s)H(0)^{-1} - 1)/s$ might be unstable due to problems of boundedness at zero. Therefore, writing

$$(1 + C_i(s)H(s))^{-1} = \frac{s}{s+\gamma}\left(1 + \frac{\gamma s}{s+\gamma}\left(\frac{H(s)H(0)^{-1} - 1}{s}\right)\right)^{-1}, \tag{6.24}$$

rather than (6.21), and then applying the small gain theorem, as was done in (Gündeş et al. 2007), does not work in the fractional systems case. So, we have to compute $R_\alpha(\gamma)$ as a function of γ for the specific α value appearing in the plant transfer function.

Lemma 6.3 *The value of $R_\alpha(\gamma)$ as defined in 6.22 is given by*

$$R_\alpha(\gamma) = \alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}\gamma^\alpha. \tag{6.25}$$

Proof: Let us consider

$$T(\omega) = \frac{(\jmath\omega)^\alpha}{\jmath\omega + \gamma}. \tag{6.26}$$

We can calculate the square of its absolute value as

$$|T(\omega)|^2 = \frac{\omega^{2\alpha}}{\gamma^2 + \omega^2} \tag{6.27}$$

If we derive equation (6.27) with respect to ω and find the zeros of that equation, we can calculate $\omega^{\text{opt}} = \gamma\sqrt{\alpha/(1-\alpha)}$, which from the test of the second derivative is proven to be a local maximum.

Therefore, $R_\alpha(\gamma)$ is given by

$$R_\alpha(\gamma) = \gamma \left| \frac{(\jmath\omega^{\text{opt}})^\alpha}{\jmath\omega^{\text{opt}} + \gamma} \right| \tag{6.28}$$

which can be easily simplified into the form given in equation (6.25). □

The graphs of $R_\alpha(\gamma)$ versus γ for different values of α are shown in Figure 6.2. Another observation we can make from (6.21) is that if $\|H(s)H(0)^{-1}-1\|_\infty < 1$ then all $C_i(s) = \gamma H(0)^{-1}/s$ stabilize H, for any $\gamma > 0$.

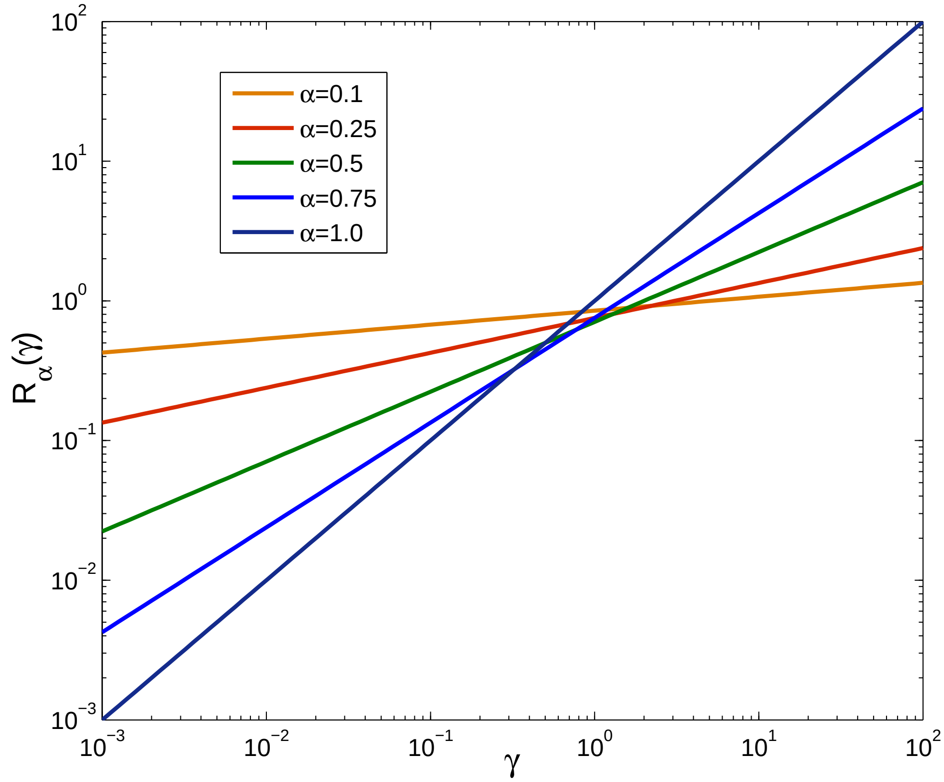

Figure 6.2: $R_\alpha(\gamma)$ versus γ.

The above discussion is summarized with the following results.

Lemma 6.4 *Assume that condition* (6.12) *of Lemma 6.1 is satisfied and the proportional controller* $K_p = (p+x)G(0)^{-1}$ *is designed to stabilize the plant* $P(s) = e^{-s\tau}G(s^\mu)(s^\mu - p)^{-1}$. *Then the PI controller*

$$C_{pi}(s) = K_p + \frac{\gamma H(0)^{-1}}{s} = \left((p+x) + \frac{\gamma x}{s}\right) G(0)^{-1} \tag{6.29}$$

is a stabilizing controller for the plant P *for all* γ *satisfying*

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1}. \tag{6.30}$$

where $H(s) = P(s)(1 + K_p P(s))^{-1}$.

Lemma 6.5 *Assume that condition* (6.13) *is satisfied for some* $\widetilde{K}_d \in \mathbb{R}$ *and* $\tau_d > 0$. *Let* C_{pd} *be a stabilizing controller for the plant,* $P(s) = e^{-s\tau}G(s^\mu)(s^\mu - p)^{-1}$, *as designed in Lemma 6.2.*

Then the PID controller

$$\begin{aligned} C_{pid}(s) &= C_{pd}(s) + \frac{\gamma H(0)^{-1}}{s} \\ &= \left((p+x)\left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right) + \frac{\gamma x}{s} \right) G(0)^{-1} \end{aligned} \tag{6.31}$$

is a stabilizing controller for P for all γ satisfying

$$0 < R_\alpha(\gamma) < \left\| \frac{H(s)H(0)^{-1} - 1}{s^\alpha} \right\|_\infty^{-1} \tag{6.32}$$

where $H(s) = P(s)(1 + C_{pd}(s)P(s))^{-1}$. □

The above PI and PID controller design methods lead to an interesting question: what are the optimal choices of $x > 0$ such that the ranges of allowable γ, i.e. the right hand sides of (6.30) and (6.32), are the largest possible? For example, in the PI design, for each fixed x in the range $0 < x < (\psi_o - p)$ one can compute the upper bound in (6.30) numerically. Therefore, the largest allowable γ range and the corresponding optimal x can be found from a one dimensional numerical search. Clearly, it is not possible to find an analytical solution for this problem.

On the other hand, we can find a suboptimal analytical solution as follows. Recall that $H(0) = G(0)/x$ and

$$H(s) = \frac{e^{-s\tau}G(s^\mu)}{s^\mu - p + x - x + (p+x)G(0)^{-1}e^{-s\tau}G(s^\mu)}. \tag{6.33}$$

Then we have

$$\begin{aligned} \frac{H(s)H(0)^{-1} - 1}{s^\mu} &= \frac{\dfrac{e^{-s\tau}G(s^\mu)G(0)^{-1}x}{s^\mu + x}\left(1 - \left(\dfrac{(p+x)s^\mu}{s^\mu + x}\right)\dfrac{1 - e^{-hs}G(s^\mu)G(0)^{-1}}{s^\mu}\right)^{-1} - 1}{s^\mu} \\ &= \frac{\dfrac{p}{(s^\mu + x)}\left(\dfrac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu}\right) - \dfrac{1}{s^\mu + x}}{1 - \dfrac{(p+x)s^\mu}{s^\mu + x}\left(\dfrac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu}\right)}. \end{aligned}$$

Recall that

$$\psi_o = \left\| \frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} \right\|_\infty^{-1}. \tag{6.34}$$

So, from the above

$$\left\| \frac{H(s)H(0)^{-1} - 1}{s^{\mu}} \right\|_{\infty} \leq \frac{p\psi_o^{-1} + 1}{x} \left(1 - (p+x)\psi_o^{-1}\right)^{-1}. \tag{6.35}$$

Thus we have the following lower bound for the upper bound in (6.30),

$$\widetilde{\gamma} := x\frac{\psi_o - (p+x)}{\psi_o + p} \leq \left\| \frac{H(s)H(0)^{-1} - 1}{s^{\mu}} \right\|_{\infty}^{-1}. \tag{6.36}$$

Now we can maximize $\widetilde{\gamma}$ by an appropriate choice of x. It is a simple exercise to show that the optimal choice of x maximizing $\widetilde{\gamma}$ is

$$x_{\text{opt}} = \frac{\psi_o - p}{2} \tag{6.37}$$

and the corresponding maximal $\widetilde{\gamma}$ is

$$\widetilde{\gamma}_{\text{opt}} = \frac{x_{\text{opt}}^2}{\psi_o + p}. \tag{6.38}$$

This means that, by equation (6.25), γ should be in the range

$$0 < \gamma < \frac{c_\alpha x_{\text{opt}}^{2/\alpha}}{(\psi_o + p)^{1/\alpha}} =: \gamma_{\max} \quad \text{where} \quad c_\alpha := \left(\sqrt{\alpha}\ (1-\alpha)^{(1-\alpha)/2\alpha} \right)^{-1}. \tag{6.39}$$

For example $c_{0.5} = 2$. We propose to choose

$$\gamma_{\text{opt}} := \frac{\gamma_{\max}}{2}. \tag{6.40}$$

as the (sub)optimal γ value to be used in the PI controller. Inserting (6.37) into the PI controller expression (6.29), we obtain

$$C_{pi}(s) = \left(1 + \frac{\gamma_{\text{opt}}}{s}\right) x_{\text{opt}} G(0)^{-1} \tag{6.41}$$

as the suboptimal PI controller, where x_{opt} is given by (6.37) and γ_{opt} is determined from (6.39) and (6.40).

6.4 Example

In this section we consider the plant

$$P(s) = \frac{e^{-s\tau}}{s^{\mu} - p}, \qquad \text{with} \qquad \tau > 0, \quad p \geq 0 \tag{6.42}$$

and design PID controllers using the method developed in the previous Section.

For P and PI controller design we need to compute the quantity

$$\psi_o = \left\| \frac{1 - e^{-s\tau} G(s^\mu) G(0)^{-1}}{s^\mu} \right\|_\infty^{-1}.$$

Note that when $\mu = 1$, we have $\psi_o = \tau^{-1}$. In the case $0 < \mu < 1$ we compute ψ_o from

$$\begin{aligned} \psi_o^{-1} &= \sup_{\omega \in \mathbb{R}} \frac{|1 - e^{-\jmath\tau\omega}|}{|(\jmath\omega)^\mu|} = \sup_{\omega \in \mathbb{R}} \frac{\left((1-\cos(\tau\omega))^2 + \sin^2(\tau\omega)\right)^{1/2}}{\omega^\mu} \\ &= \tau^\mu \sqrt{2} \sup_{\tilde{\omega} \in \mathbb{R}} \frac{\sqrt{1 - \cos(\tilde{\omega})}}{\tilde{\omega}^\mu}. \end{aligned}$$

Therefore,

$$\tau^{-\mu} \psi_o^{-1} = \sqrt{2} \sup_{\tilde{\omega} \in \mathbb{R}} \frac{\sqrt{1 - \cos(\tilde{\omega})}}{\tilde{\omega}^\mu} =: \phi(\mu). \tag{6.43}$$

Figure 6.3 shows how $\phi(\mu)$ varies with μ. As expected, for $\mu = 1$ we have $\phi = 1$. But it is interesting to observe that behavior of ϕ is not monotonic, and there is a minimum value near $\mu = 0.9$.

According to Lemma 6.1 there is a stabilizing controller for the plant (6.42) if $p < \psi_o$, i.e., if

$$p\tau^\mu < \frac{1}{\phi(\mu)}$$

where $\phi(\mu)$ is as shown in Figure 6.3. In particular, for $\mu = 0.5$ we have $\phi = 1.2$. Therefore, we can find a stabilizing proportional controller using Lemma 1 if

$$\tau < \frac{1}{1.2^2 p^2} = \frac{0.6944}{p^2}.$$

Recall that the sufficient conditions of Section 6.3 are obtained using the small gain arguments, so there is some conservatism. We can also use the results of (Marshall et al. 1992) and find that there exists a stabilizing proportional controller for all $h < h_{\max}$ as follows:

The stability for $\tau = 0$ is guaranteed with $K_p > p$. When τ increases, the position of the infinite number of new poles poses no restriction, since for a delay system of retarded type they appear in the left-half plane. The exact value of the delay for which some poles cross the imaginary axis are related to the non-negative real roots ω_R of the quasi-polynomial

$$W(\omega) = \omega - p\sqrt{2\omega} + p^2 - K_p^2$$

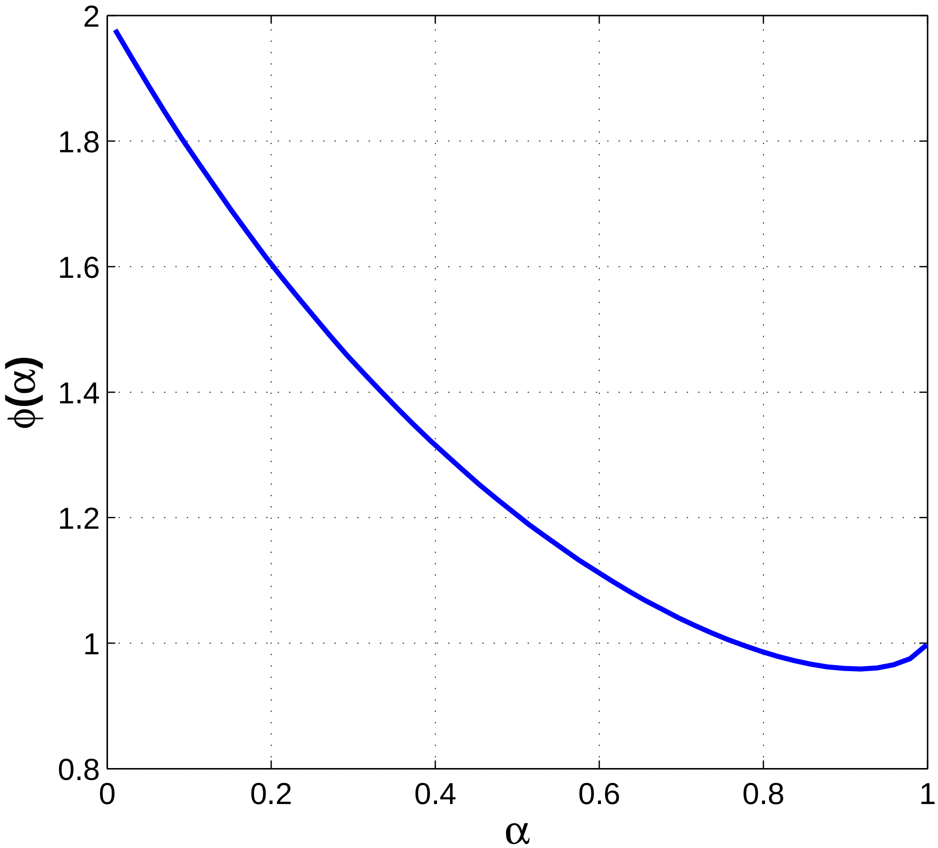

Figure 6.3: $\phi(\alpha) = \tau^{-\alpha}\psi_o^{-1}$ versus α.

which leads to

$$\omega_R = K_p^2 + p\sqrt{2K_p^2 - p^2}$$

The maximum delay τ is given by

$$\tau = \frac{1}{\omega_R} \arcsin\left(\frac{\sqrt{2\omega_R}}{2K_p}\right) \tag{6.44}$$

and, maximizing (6.44) with respect to $K_p > p$ results in $K_p \to p$, and hence $\tau \to \tau_{\max}$ with

$$\tau_{\max} = \frac{\pi}{4p^2}$$

The value of $\tau_{\max}$ is exact, in the sense that if $\tau \geq \tau_{\max}$ then there does not exist a stabilizing proportional controller. Thus the level of conservatism in our approach is less than 12% in this case.

The suboptimal PI controller (6.41) for

$$P(s) = \frac{e^{-s\tau}}{\sqrt{s} - p}$$

can be computed from

$$\psi_o = \frac{1}{1.2\sqrt{\tau}}, \quad x_{\text{opt}} = \frac{\psi_o - p}{2} \quad \gamma_{\text{opt}} = \frac{1}{4}\left(\frac{\psi_o - p}{\psi_o + p}\right)^2 \left(\frac{\psi_o - p}{2}\right)^2.$$

In particular, when $p = 0$, we have

$$C_{pi}(s) = \frac{1}{2.4\sqrt{\tau}}\left(1 + \frac{1/16}{1.2^2 s\tau}\right).$$

For the optimal PD controller proposed in Section 6.3, we need to find the optimal $\widetilde{K}_d \in \mathbb{R}$, say $\widetilde{K}_d^{\text{opt}}$, so that ψ_d^{-1}, (6.15), is minimized for a small fixed value of $\tau_d > 0$.

Considering $h = 1$, we calculated the optimal PD control which results in the parameters $\tau_d = 4.2$ and $\widetilde{K}_d^{\text{opt}} = -1.7346$, and hence $\psi_d^{-1} = 0.9873$. Then the optimal PD controller is given by

$$G(0)^{-1}\sqrt{p\psi_d}(1 + \widetilde{K}_d^{\text{opt}}\frac{s}{\tau_d s + 1})$$

where stability is assured for all systems with $p < \psi_d = 1.0165$. Notice that with just the proportional controller, we could only guarantee stability for systems with $p < 0.8333$, which indicates an increase of about 22%.

6.5 Final Remarks

In this chapter we developed a method to design classical PID controllers for a class of fractional order plants with time delays. The main idea behind this approach was to use the small gain type of arguments, as it was introduced in (Gündeş et al. 2007). The fractional order plant is factored into a stable part and an unstable part, where the unstable part is in the form $(s^\mu - p)^{-1}$ with $p > 0$. There is no restriction on the stable part $G(s^\mu)$ except that $G(0) \neq 0$ and $G(p) \neq 0$. It may be possible to extend this method to fractional order plants with a higher degree unstable part, but in that situation there are some technical difficulties even for the rational plants case, see (Gündeş et al. 2007) and its references.

The (sub)optimal PD and PI controller design method proposed here also works for rational plants with time delays and single pole in $\mathbb{R}_+$, see (Özbay & Gündeş 2007). However, in the fractional systems case there is a major difference for the minimization of ψ_d^{-1},(6.15): when $\alpha \neq 1$ we cannot let $\tau_d = 0$, because, otherwise $s^{1-\alpha}$ term multiplying $\widetilde{K}_d$ will make the norm equal to infinity unless $\widetilde{K}_d = 0$. Therefore, the selection of a small positive τ_d plays an important role in this case, and hence, searching for the optimal $\widetilde{K}_d$ and τ_d pair is more difficult compared to the problems studied before.

The following publications were produced presenting the results of this chapter (Özbay, Bonnet & Fioravanti 2011), (Özbay, Bonnet & Fioravanti 2009):

- H. Özbay, C. Bonnet and A. R. Fioravanti. 'PID Controller Design for Fractional-Order Systems with Delays'. Under revision.

- H. Özbay, C. Bonnet and A. R. Fioravanti. 'PID Controller Design for Unstable Fractional-Order Systems with Time Delays: A Small Gain Based Approach'. *Systems Theory: Modeling, Analysis and Control*, pp. 351-358, Fes - Morocco, May 25-28, 2009.

Chapter 7

Rational Comparison Systems

Contents

7.1 Introduction 127

7.2 Rational Comparison System 128

7.3 Linear Filter Design 138

7.4 State Feedback Control Design 147

7.5 Output Feedback Design 151

7.6 Neutral Systems 158

7.7 Final Remarks 159

7.1 Introduction

After dealing with the PID controller for fractional systems, we will focus our attention on some standard problems in the $\mathcal{H}_\infty$ framework. Filtering, state and dynamic output feedback problems are some of the most important aspect of it, and since the development of efficient tools to handle Riccati equations and Linear Matrix Inequalities, they have become major aspects in the study of any class of systems. Unfortunately, although some aspects of dealing with fractional systems with this kind of tools are already presented in the literature, we are still far from completely adapting all the methodology available for normal systems. Therefore, as some of the results we need are still not known for the class of fractional systems, in this chapter we will focus completely on classical systems.

Differently from a time-domain view of the problem that would likely treat these problems in a Lyapunov-Krazovskii framework, we still want to deal with them in the frequential domain, in an attempt to reduce the conservatism that always comes with the use of bounds in order to linearize the Lyapunov-Krazovskii functionals. On the other hand, it is unquestionable that when dealing with optimization aspects, the time-domain approach is much more appealing. Therefore, in this chapter, we will treat the problem in a hybrid framework, where even if all the calculations are made with tools adapted from the time-domain, we exploit properties directly from the frequential-domain in order to deal with the delay. With this approach we try to both be less conservative than pure time-domain approaches but still keeping its simplicity and fast optimization algorithms.

This chapter, to some extent, follows the same stream proposed in (Zhang et al. 2003). There, adopting a comparison system approach, the well known Padé approximation is used to determine linear time invariant systems of increasing but finite order, allowing the direct determination of stability margin and bounds for the $\mathcal{H}_\infty$-norm performance of the time-delay system. It is shown that the quality of the results is better whenever the order of the Padé approximation increases.

We will propose a method that can simultaneously improve the results of a first-order Padé approximation but without increasing its order. To this behalf, a linear time invariant comparison system of order twice the number of state variables of the time-delay system, built from the Rekasius substitution, is introduced, and the relationship between the stability of the comparison system and the time-delay system is established. This is accomplished by the Nyquist criterion applied to some specific characteristic equations, related to the comparison and to the time-delay systems respectively, see also (Youcef-Toumi & Bobbett 1991) and (Brockett & Byrnes 1981). Moreover, the $\mathcal{H}_\infty$-norm of the comparison system provides a precise and useful lower bound to the $\mathcal{H}_\infty$-norm of the time-delay system. This property is used for delay-dependent linear filter and control design and it is shown how lower and upper bounds on the $\mathcal{H}_\infty$-norm of the estimation error or controlled output can be imposed.

Exclusively, rational transfer functions of LTI systems will be denoted as

$$C(sI-A)^{-1}B+D=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{7.1}$$

where the matrices A, B, C, D are real and of compatible dimensions.

7.2 Rational Comparison System

In this section we define an LTI system that serves as a comparison system for both stability analysis and $\mathcal{H}_\infty$-norm calculation for a general time-delay one of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + E_0 w(t) \tag{7.2}$$

$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau) \tag{7.3}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $w(t) \in \mathbb{R}^r$ is the exogenous input and $z(t) \in \mathbb{R}^q$ is the output. It is assumed that the system evolves from the rest, which means that $x(t) = 0$ for all $t \in [-\tau, 0]$, and the delay $\tau \geq 0$ is constant with respect to time.

The basic idea stems from the fact, already denominated by several authors as Rekasius substitution, that for $s = \jmath\omega$ with $\omega \in \mathbb{R}$, the equality

$$e^{-s\tau} = \frac{1-\lambda^{-1}s}{1+\lambda^{-1}s} \tag{7.4}$$

is not a mere approximation, but it holds for some $\lambda \in \mathbb{R}$ such that $\omega/\lambda = \tan(\omega\tau/2)$. It is important to notice that for a given pair (λ, ω) there exist many $\tau \geq 0$ satisfying the relationship $\tau = (2/\omega)\mathrm{atan}(\omega/\lambda)$. Based on this discussion we are able to introduce what we call a rational comparison system for (7.2)-(7.3) as follows:

$$\begin{aligned} H(\lambda, s) &= \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ A_0 + A_1 & A_0 - A_1 - \lambda I & E_0 \\ \hline C_{z0} + C_{z1} & C_{z0} - C_{z1} & 0 \end{array}\right]. \end{aligned} \tag{7.5}$$

Denoting by $T(\tau, s)$ the non-rational transfer function of the time-delay system from the input w

to the output z, this LTI system has been determined in such a way that the equality $H(\lambda, \jmath\omega) = T(\tau, \jmath\omega)$ holds whenever the constants $\lambda \in \mathbb{R}$, $\tau \geq 0$ and the frequency $\omega \in \mathbb{R}$ are related by the nonlinear relationship $\omega/\lambda = \tan(\omega\tau/2)$ that emerges from (7.4). Indeed, simple matrix manipulations provide

$$\begin{aligned} H(\lambda, s) &= C_z(sI - A_\lambda)^{-1}E \\ &= ((C_{z0} + C_{z1})\lambda + (C_{z0} - C_{z1})s)\Big(s^2 I - (A_0 - A_1 - \lambda I)s - (A_0 + A_1)\lambda\Big)^{-1} E_0 \\ &= (C_{z0}(\lambda + s) + C_{z1}(\lambda - s))\Big((sI - A_0)(\lambda + s) - A_1(\lambda - s)\Big)^{-1} E_0 \\ &= T(\tau, s) \end{aligned} \tag{7.6}$$

where the last equality follows from factorizing the term $\lambda + s$ in the third one and making use of (7.4). At this point, we are interested in verifying if it is possible to design filters and controllers based on $H(\lambda, s)$ that assure some desired properties to the time-delay system under consideration defined by the non-rational transfer function $T(\tau, s)$. To this end, we need to make clear the relationship between the comparison system and the time-delay system as far as stability and $\mathcal{H}_\infty$-norm calculation are concerned.

7.2.1 Stability Analysis

In this section, we focus our attention on the determination, in this framework, of the maximum delay $\tau^* > 0$ such that the system (7.2)-(7.3) remains asymptotically stable for all $\tau \in [0, \tau^*)$. As we have seen throughout this manuscript, the determination of τ^* depends on being able to calculate the poles of the transfer function $T(\tau, s)$, which are all roots of the characteristic equation

$$\Delta_T(\tau, s) = \det\Big(sI - A_0 - A_1 e^{-s\tau}\Big). \tag{7.7}$$

Whenever $\tau > 0$, equation (7.7) is transcendental and admits, generally, infinitely many roots, being thus hard to solve. In order to solve the problem in question, we could directly apply the technique discussed in Chapter 4, however, in this context, we want to be able to use a similar technique both for the delay system and for the comparison system. Therefore, writing it in the alternative form

$$\Psi(\tau, s) = \frac{\Delta_T(\tau, s)}{\Delta_T(0, s)} \tag{7.8}$$

where

$$\Psi(\tau, s) = \det\Big(I + (sI - A_0 - A_1)^{-1} A_1 (1 - e^{-s\tau})\Big) \tag{7.9}$$

the Nyquist criterion (Brockett & Byrnes 1981), based on the variation of the argument of $\Psi(\tau, s)$ can be applied, even though this transfer function is not rational, see (Youcef-Toumi & Bobbett 1991). A similar reasoning can be adopted to the comparison system whose characteristic equation is $\Delta_H(\lambda, s) = 0$ where

$$\Delta_H(\lambda, s) = \det\Big(sI - A_\lambda\Big) \tag{7.10}$$

is a $2n$-th order polynomial equation with real coefficients which admits $2n$ roots. Some algebraic manipulations enable us to factorize it as

$$\Phi(\lambda, s) = \frac{\Delta_H(\lambda, s)}{(s+\lambda)^n \det(sI - A_0 - A_1)} \tag{7.11}$$

where

$$\Phi(\lambda, s) = \det\left(I + (sI - A_0 - A_1)^{-1} A_1 \frac{2s}{s+\lambda}\right) \tag{7.12}$$

satisfies the equality $\Psi(\tau, \jmath\omega) = \Phi(\lambda, \jmath\omega)$, provided that $e^{-s\tau}$ is replaced by the rational ratio defined by the Rekasius formula (7.4), valid for $s = \jmath\omega$, $\omega \in \mathbb{R}$.

Assuming that the time-delay system is asymptotically stable for $\tau = 0$, it is guaranteed that all the n roots of $\Delta_T(0, s) = \det(sI - A_0 - A_1) = 0$ are located in the open left-hand side of the complex plane. Hence, the multivariate Nyquist criterion states that all roots of the characteristic equation $\Delta_T(\tau, s) = 0$ are also located in the same region provided that the mapping of $\Psi(\tau, \jmath\omega)$ for all $\omega \in \mathbb{R}$ does not encircle the origin. Hence, by increasing $\tau \geq 0$, it is possible to determine the pair (τ^*, ω^*) corresponding to the first occurrence of $\Psi(\tau^*, j\omega^*) = 0$, which defines the so-called stability margin of the time-delay system.

Similarly, the analysis of the comparison system leads to two conclusions. First, for any fixed $\lambda > 0$ such that the origin is not encircled by the mapping of $\Phi(\lambda, \jmath\omega)$ for all $\omega \in \mathbb{R}$, there are no roots of $\Delta_H(\lambda, s) = 0$ located in the right-hand side of the complex plane, that is matrix A_λ is Hurwitz. Second, for any fixed $\lambda < 0$ such that the origin is not encircled, n roots of $\Delta_H(\lambda, s) = 0$ are located in the right-hand side of the complex plane, which means that only n (among $2n$) eigenvalues of matrix A_λ have negative real part. As a consequence, all roots of $\Delta_H(\lambda, s) = 0$ are located in the open left-hand part of the complex plane for all $\lambda > \lambda^o > 0$ whenever $\Phi(\lambda^o, j\omega^o) = 0$ corresponds to the first zero crossing of the mapping $\Phi(\lambda, j\omega)$ obtained by decreasing λ from $+\infty$.

Once the pair (τ^*, ω^*) is obtained, we can readily calculate λ^* from $\omega^*/\lambda^* = \tan(\omega^*\tau^*/2)$, which satisfies $\Phi(\lambda^*, j\omega^*) = 0$ and consequently $\lambda^* \leq \lambda^o$. Whenever $\lambda^* = \lambda^o > 0$ we can ensure that for all $\lambda \in (\lambda^*, \infty)$ and $\tau \in [0, \tau^*)$ both the time-delay and the comparison systems are asymptotically stable. Conversely, from the pair (λ^o, ω^o) we can calculate the minimum $\tau^o > 0$ which satisfies the relation $\omega^o/\lambda^o = \tan(\omega^o\tau^o/2)$ and also guarantees that $\Psi(\tau^o, j\omega^o) = 0$ and consequently $\tau^o \geq \tau^* > 0$. When $\tau^o = \tau^*$ and $\lambda^o > 0$ the equivalence between the stability of the time-delay system and the comparison system is again established.

Equivalent conclusions are already available in several references, as it can be seen, for instance, in (Gu et al. 2003), (Niculescu & Gu 2004), (Olgac & Sipahi 2002) and (Chen, Gu & Nett 1995). However, the novelty here is the formulation based on the state space realization (7.5), the equality (7.6) and the use of the Nyquist criterion, without the necessity to calculate explicitly the transcendental characteristic equation of the time-delay system under consideration. The conclusion is that the comparison system can be used for stability analysis of the time-delay system whenever the first zero crossing point is such that $\lambda^o = \lambda^* > 0$. If this is not the case, A_λ Hurwitz for all $\lambda > 0$ is not sufficient to ensure that the time-delay system is stable for all $\tau \geq 0$.

Due to the fact that

$$\begin{aligned} \Delta_H(0, s) &= \det \begin{bmatrix} sI & 0 \\ -A_0 - A_1 & sI - A_0 + A_1 \end{bmatrix} \\ &= s^n \det(sI - A_0 + A_1) \end{aligned} \tag{7.13}$$

if we consider $A_0 + A_1$ Hurwitz, by continuity, the existence of $\lambda^o > 0$ is assured whenever matrix $A_0 - A_1$ has at least one eigenvalue with positive real part. Or, in other words, the existence of $\lambda^o > 0$ is assured by the fact that matrices $A_0 + A_1$ and $A_0 - A_1$ are not simultaneously Hurwitz. Clearly, this is not a necessary condition for the existence of $\lambda^o > 0$.

The next three examples illustrate the results obtained so far concerning the relationship between stability of the comparison and time-delay systems. The application of the Nyquist criterion as previously discussed is simplified by the fact that both mappings $\Psi(\tau, j\omega)$ and $\Phi(\lambda, j\omega)$ begin $(\omega = 0)$ and end $(\omega \to \infty)$ at the same point $1 + j0$ of the complex plane.

Example 7.1 The first example has been borrowed from (Olgac & Sipahi 2002) where the time-delay system (7.2) with zero input is defined by matrices

$$A_0 = \begin{bmatrix} -1.0 & 13.5 & -1.0 \\ -3.0 & -1.0 & -2.0 \\ -2.0 & -1.0 & -4.0 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2.0 & -1.0 & 5.0 \\ 2.0 & 0.0 & 6.0 \end{bmatrix}$$

for which $A_0 + A_1$ is Hurwitz but this is not the case of $A_0 - A_1$. From the first mapping of Figure 7.1 we have calculated the pairs $(\lambda^o, \omega^o) = (12.0692, 3.0352)$ and $(\tau^*, \omega^*) = (0.1623, 3.0352)$

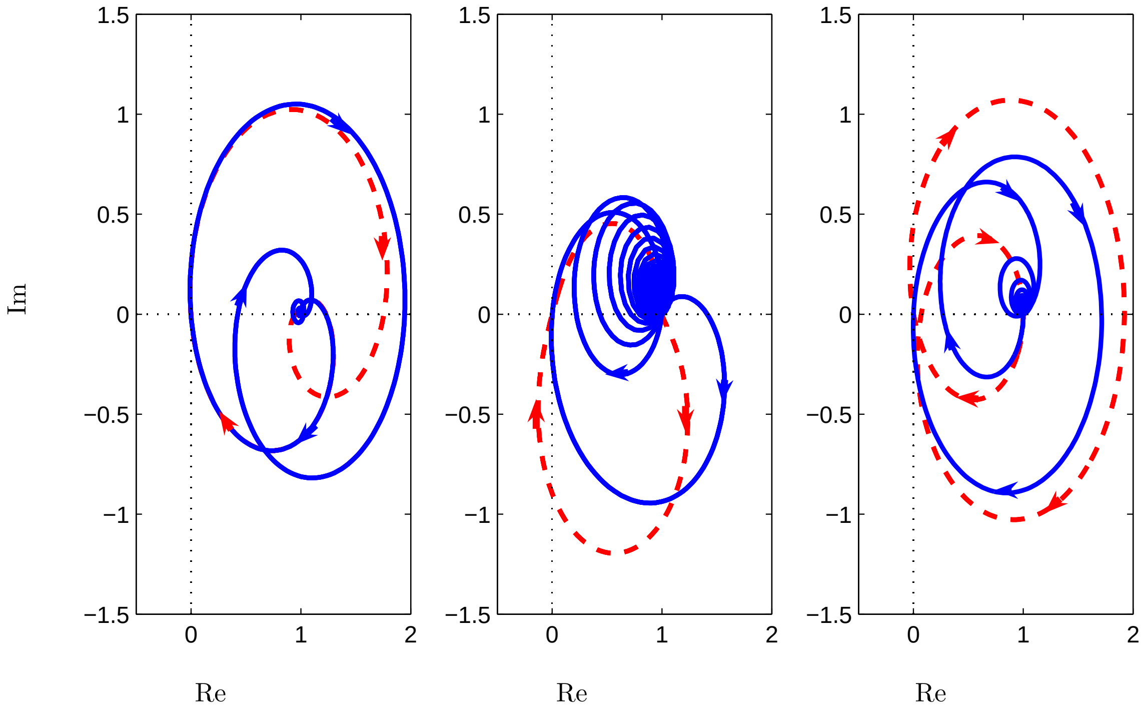

Figure 7.1: Mapping of $\Psi(\tau, j\omega)$ and $\Phi(\lambda, j\omega)$

yielding that the stability of the comparison system implies the stability of the time-delay system and vice-versa. In agreement with (Olgac & Sipahi 2002), both are stable for all $\lambda > \lambda^o$ and $0 \leq \tau < \tau^*$, respectively. ♦

Example 7.2 Consider (7.2) with zero input and matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -5 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

As it can be verified, matrices $A_0 + A_1$ and $A_0 - A_1$ are Hurwitz. From the second mapping in Figure 7.1 we have determined $(\lambda^o, \omega^o) = (-0.1478, 1.6813)$ and $(\tau^*, \omega^*) = (1.9729, 1.6813)$. In this case, since $\lambda^o = \lambda^* < 0$, for all $\lambda > 0$ matrix A_λ is Hurwitz and for $\lambda \in (\lambda^*, 0)$ only three (among six) eigenvalues of A_λ have negative real part. ♦

Example 7.3 Once again we consider the time-delay system (7.2) with zero input defined by matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.05 & -0.38 & -0.05 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.06 & 0.22 & -0.65 \end{bmatrix}$$

where $A_0 + A_1$ but not $A_0 - A_1$ is Hurwitz. From the third mapping shown in Figure 7.1, we have determined the following pairs $(\lambda^o, \omega^o) = (0.6912, 0.0801)$ and $(\tau^*, \omega^*) = (2.0838, 1.0013)$. Notice that in this case $\lambda^o > \lambda^* = 0.5834$ and consequently for $\lambda = \lambda^*$ the comparison system is unstable. ♦

Remark 7.1 *For a given $\tau \geq 0$, the evaluation of $\arg \Psi(\tau, j\omega)$ maintaining continuity (as it is done by the phase$(\cdot)$ routine of Matlab) enables us to decide whether or not the origin is encircled. Hence, starting from $\tau = 0$, the time delay is increased up to τ^* by verifying that no encirclement has occurred.*

These three simple examples confirm our previous claims and imply that, in some cases, the stability of the comparison system is not sufficient to ensure the stability of the time-delay system and vice-versa. Indeed, in Example 7.2 the stability of the comparison system for all $\lambda > 0$ is not sufficient to ensure that the time-delay system is stable for all $\tau \geq 0$. Furthermore, Example 7.3 shows that the stability of the comparison system for all $\lambda > \lambda^o > 0$ does not imply that the time-delay system is stable for all $\tau \in [0, \tau^o)$, where $\tau^o = 2.8807$ is determined from $\omega^o/\lambda^o = \tan(\omega^o\tau^o/2)$, because $\tau^o > \tau^*$.

The complete characterization of all time-delay systems for which the equality $\lambda^o = \lambda^* > 0$ holds is not simple. For this reason, the following lemma gives a subinterval of (λ^o, ∞) such that the stability of the comparison system and the time-delay system is simultaneously preserved.

Lemma 7.1 *Assume that matrix $A_0 + A_1$ is Hurwitz and the time-delay system (7.2)-(7.3) is stable for all $\tau \in [0, \tau^*)$. Then, for all $\lambda \in (2/\tau^*, \infty)$ the comparison system is stable and for any $\omega \in \mathbb{R}$ the value of the time delay τ satisfying $\omega/\lambda = \tan(\omega\tau/2)$ belongs to the interval $[0, \tau^*)$.*

Proof: The Nyquist criterion applied to the comparison system provides the pair (λ^o, ω^o). If $\lambda^o \leq 0$ the first part follows trivially since the comparison system is stable for all $\lambda > 0$. On the other hand, considering that $\lambda^o > 0$ the comparison system is stable for all $\lambda \in (\lambda^o, \infty)$. In this case, the equality $\omega^o/\lambda^o = \tan(\omega^o\tau^o/2)$ provides $\tau^o \geq \tau^*$ because $\Psi(\tau^o, j\omega^o) = \Phi(\lambda^o, j\omega^o) = 0$ and it cannot exist $0 \leq \tau < \tau^*$ satisfying $\Psi(\tau, j\omega) = 0$ for some $\omega \in \mathbb{R}$. Since[1] $\tau^o = (2/\omega^o)\text{atan}(\omega^o/\lambda^o) \leq 2/\lambda^o$ we conclude that $\lambda^o \leq 2/\tau^o \leq 2/\tau^*$, proving thus the first claim.

[1] This property follows from the fact that the continuous function $f(x) = \text{atan}(x)/x$ satisfies the bounds $0 < f(x) \leq 1$ for all $x \in \mathbb{R}$.

Finally, invoking again that $\tau = (2/\omega)\text{atan}(\omega/\lambda) \leq 2/\lambda$ for all $\omega \in \mathbb{R}$ and that $\lambda > 2/\tau^*$ the second part follows. □

This result will be important afterwards. It states that for the class of time-delay systems such that matrix $A_0 + A_1$ is Hurwitz, for $\lambda > 0$ big enough the comparison system is stable and the delay extracted from the Rekasius formula maintain the time-delay system stable whatever the value of the frequency $\omega \in \mathbb{R}$.

7.2.2 $\mathcal{H}_\infty$-Norm Calculation

This section aims to show how to calculate the norm

$$\|T(\tau, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \ \sigma(T(\tau, j\omega)) \tag{7.14}$$

for a given $\tau \in [0, \tau^*)$. In this framework, a line search with respect to $\omega \geq 0$ together with a singular value decomposition of $T(\tau, j\omega)$ is an immediate, although certainly not the best, way to evaluate the supremum appearing in (7.14). However, it must be clear that the situation is much more involved for filter and controller design since the direct manipulation of $T_F(\tau, j\omega)$ is virtually impossible. The purpose of this section is to show that the rational transfer function $H(\lambda, s)$ can be successfully used for $\mathcal{H}_\infty$-norm calculation and does not present any incovenience to be adopted for linear filter synthesis. In this context, the previous assumption on the existence of $\lambda^o > 0$ assuring that A_λ is Hurwitz for all $\lambda \in (\lambda^o, \infty)$ is essential because, otherwise, the equality $\|H(\lambda, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \ \sigma(H(\lambda, j\omega))$ does not hold and the norm $\|H(\lambda, s)\|_\infty$ is unbounded. For this reason we define the positive scalar $\lambda_o = \inf \{\lambda \mid A_\xi \text{ is Hurwitz } \forall \xi \in (\lambda, \infty)\}$ to be used in the next theorem. Under the assumption that matrix $A_0 + A_1$ is Hurwitz, the scalar λ_o exists and is given by $\lambda_o = \lambda^o$ or $\lambda_o = 0^+$ whenever $\lambda^o > 0$ or $\lambda^o < 0$, respectively.

Theorem 7.2 *Assume that $A_0 + A_1$ is Hurwitz. For each $\lambda \in (\lambda_o, \infty)$ define $\alpha \geq 0$ such that*

$$\alpha = \arg \sup_{\omega \in \mathbb{R}} \ \sigma(H(\lambda, j\omega)) \tag{7.15}$$

and determine $\tau(\lambda)$ from $\alpha/\lambda = \tan(\alpha\tau/2)$. If $\tau(\lambda) \in [0, \tau^*)$ then $\|H(\lambda, s)\|_\infty \leq \|T(\tau(\lambda), s)\|_\infty$.

Proof: The proof follows from the fact that for $\lambda \in (\lambda_o, \infty)$ the asymptotic stability of $H(\lambda, s)$ is assured which implies, from (7.15), that $\|H(\lambda, s)\|_\infty = \sigma(H(\lambda, j\alpha))$. Since $\tau(\lambda) \in [0, \tau^*)$, the transfer function $T(\tau(\lambda), s)$ is also asymptotically stable, in which case (7.6) together with (7.14) yields

$$\begin{aligned}\|H(\lambda,s)\|_\infty &= \sigma(H(\lambda,j\alpha))\\ &= \sigma(T(\tau(\lambda),j\alpha))\\ &\leq \|T(\tau(\lambda),s)\|_\infty\end{aligned} \tag{7.16}$$

and the proof is concluded. □

At this point, it is relevant to analyze the previous result for $\lambda \to \infty$. Defining the first order transfer function $\Theta(\lambda,s) = (\lambda - s)/(\lambda + s)$ the approximation

$$\begin{aligned}H(\lambda,s) &= (C_{z0} + C_{z1}\Theta(\lambda,s))\Big(sI - A_0 - A_1\Theta(\lambda,s)\Big)^{-1} E_0\\ &\approx (C_{z0} + C_{z1})\Big(sI - A_0 - A_1\Big)^{-1} E_0\end{aligned} \tag{7.17}$$

is valid for all $|s|$ finite whenever λ goes to infinity because $\lim_{\lambda\to\infty} \Theta(\lambda,s) = 1$. In this case, all poles of $H(\infty,s)$ are in the open left-hand side of the complex plane and we get $\tau = 0$. As a consequence, the result provided by Theorem 7.2 turns out to be exact since the transfer functions $H(\infty,s)$ and $T(0,s)$ are asymptotically stable, equal and, consequently, their norms satisfy $\|H(\infty,s)\|_\infty = \|T(0,s)\|_\infty < \infty$. Moreover, notice that we cannot discard the possibility that for some $\lambda \in (\lambda_o,\infty)$ the value of the time delay calculated through (7.15) be such that $\tau(\lambda) \notin [0,\tau^*)$. In this case, the lower bound provided by Theorem 7.2 remains valid but only in a subset of (λ_o,∞). This aspect is treated in the next corollary of Theorem 7.2 where, as usual, the goal is to impose a finite upper bound to the $\mathcal{H}_\infty$-norm of the time-delay system.

Corollary 7.3 *Assume that A_0+A_1 is Hurwitz. For any given positive parameter $\gamma > \|H(\infty,s)\|_\infty$ there exist $\lambda_\gamma \geq \lambda_o > 0$ and $0 \leq \tau_\gamma \leq \tau^*$ such that the inequality*

$$\|H(\lambda,s)\|_\infty \leq \|T(\tau,s)\|_\infty < \gamma \tag{7.18}$$

holds $\forall\ \lambda \in (\lambda_\gamma,\infty)$ whenever the time-delay function $\tau(\lambda)$ given by Theorem 7.2 is continuous in the same interval.

Proof: Indeed, for $\lambda = \infty$, the transfer function $H(\infty,s)$ is asymptotically stable and the formula (7.15) gives $\tau(\infty) = 0$, independently of α because $\tau(\lambda) = (2/\alpha)\text{atan}(\alpha/\lambda) \leq 2/\lambda$ for all non-negative α. Therefore $\tau(\lambda)$ is continuous at $\lambda = \infty$. Since $H(\infty,s) = T(0,s)$, we obtain $\|H(\infty,s)\|_\infty = \|T(0,s)\|_\infty < \gamma$ which allows the conclusion that, whenever $\tau(\lambda)$ is continuous, $\lambda > 0$ can be reduced from $+\infty$ to some $\lambda_\gamma \geq \lambda_o$, yielding $\tau_\gamma = \tau(\lambda_\gamma) \leq \tau^*$ and preserving (7.18).

□

Theorem 7.2 makes clear that it may not be possible to generate through $\lambda \in (\lambda_o, \infty)$ a lower bound valid for all $\tau \in [0, \tau^*)$. However, from Corollary 7.3 we know that it is possible to determine a sub-interval of $\lambda > 0$ such that the lower and upper bounds $\|H(\lambda, s)\|_\infty \leq \|T(\tau, s)\|_\infty < \gamma$ hold. It is important to keep in mind that the determination of the sub-interval defined by λ_γ must be done with care due to the eventual occurrence of multiple solutions $\alpha(\lambda)$ to problem (7.15), which may cause discontinuities on the associated value of the time delay $\tau(\lambda)$ extracted from the nonlinear relationship provided in Theorem 7.2.

From Corollary 7.3, a possible numerical procedure to calculate these bounds is as follows: for each element of a strictly decreasing sequence $\lambda_k = \{\infty, \cdots, \lambda_o\}$ the time-delay value $\tau_k = \tau(\lambda_k)$ is computed. The index k is increased whenever $-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0$ at $\lambda = \lambda_k$ and $\|T(\tau_k, s)\|_\infty < \gamma$. When this procedure stops we get $\lambda_\gamma = \lambda_{k-1}$ and $\tau_\gamma = \tau_{k-1}$. An important note about this algorithm is that the calculation of $\|T(\tau_k, s)\|_\infty$ is essential to impose the desired upper bound and to compute the limits λ_γ and τ_γ. Moreover, the existence of the derivative $d\tau(\lambda)/d\lambda < 0$ implies the continuity and monotonicity of $\tau(\lambda)$ and avoids its sudden variation with respect to the variation of λ. This allows us to identify any unboundedness tendency of $\|T(\tau(\lambda), s)\|_\infty$ and also the stability margin τ^* since this norm is continuous within the entire interval (λ_γ, ∞). The constraint $-2/\lambda^2 < d\tau(\lambda)/d\lambda < 0$ at $\lambda = \lambda_k$, inspired by inequality $0 \leq \tau(\lambda) < 2/\lambda$, is numerically implemented through the simple test $0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2$ whose accuracy is controlled by taking $|\lambda_k - \lambda_{k-1}| < \epsilon$ sufficiently small. This procedure is applied to the next illustrative examples.

Remark 7.2 *From the above discussion, the first element of the sequence $\{\lambda_k\}$ can be chosen as $2/\epsilon$, where $\epsilon > 0$ is such that the norms of the finite order systems, namely $\|H(2/\epsilon, s)\|_\infty$ and $\|T(0, s)\|_\infty$, are close to each other, which means that their distance is within some precision defined by the designer. Such an $\epsilon > 0$ satisfying this condition always exists.*

Example 7.4 This example has been borrowed from (Fridman & Shaked 2002). The time-delay system (7.2)-(7.3) is given by

$$A_0 = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, \; A_1 = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}$$

$E_0 = [-0.5 \; 1.0]'$, $C_{z0} = [1 \; 0]$ and $C_{z1} = D_0 = 0$ where, as it can be verified, matrix $A_0 + A_1$ but not $A_0 - A_1$ is Hurwitz. Firstly, we have determined $\lambda^o = \lambda^* = 0.1001$ and $\omega^o = \omega^* = 0.4361$

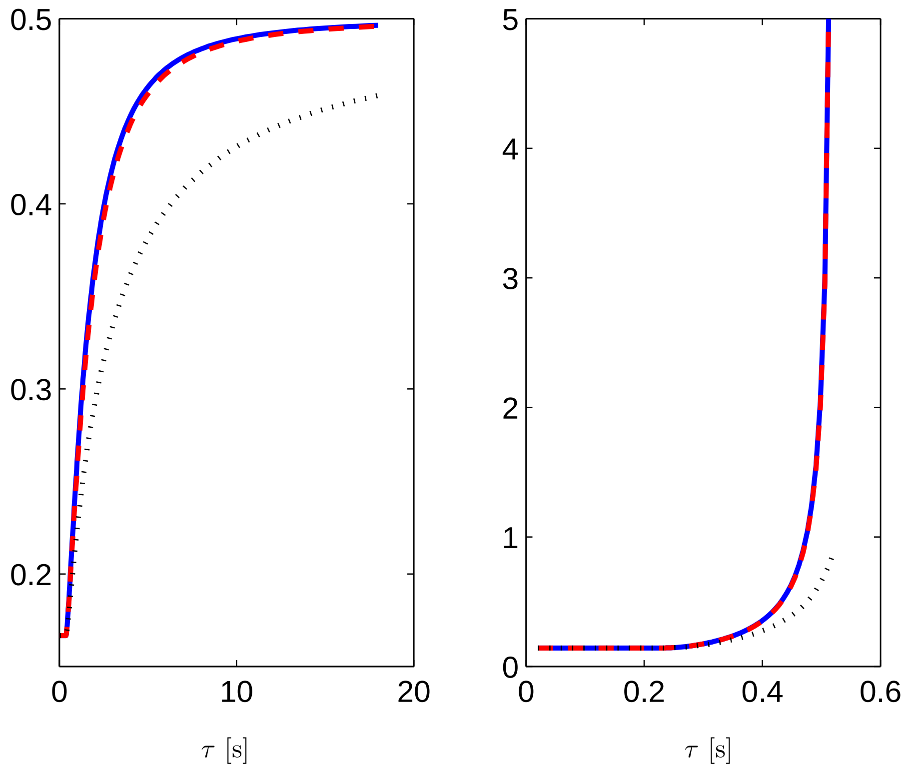

Figure 7.2: $\mathcal{H}_\infty$-norm versus time delay

corresponding to $\tau^* = 6.1690$ which substantially improves the best result $\tau^* = 4.47$ reported in (Fridman & Shaked 2002). However, it is a matter of simple calculation to verify that the transfer function $T(\tau, s)$ reduces to $T(\tau, s) = -0.5/(s + 2 + e^{-\tau s})$. Hence, instead of 4-th order, the transfer function $H(\lambda, s)$ is always of second order. Considering the scalar time-delay system defined by $A_0 = -2.0$, $A_1 = -1.0$, $E_0 = -0.5$, $C_0 = 1.0$ and $C_1 = D_0 = 0$ we have verified that $\lambda^o = \lambda^* = 0^+$ and $\tau^* = \infty$. The left-hand side of Figure 7.2 shows in solid line the value of $\|T(\tau, s)\|_\infty$ against τ, in dashed line the corresponding lower bound provided by Theorem 7.2 and in dotted line the lower bound provided by the first order Padé approximation as proposed in (Zhang et al. 2003). It is important to mention that the Padé approximations of order three and higher practically coincide with the true value of the $\mathcal{H}_\infty$-norm. ♦

Example 7.5 This example has been proposed in (Niculescu 1998). From the state space realization given by matrices

$$A_0 = \begin{bmatrix} -2.0 & 0.0 \\ 0.6 & 1.0 \end{bmatrix}, \ A_1 = \begin{bmatrix} -0.8 & 0.0 \\ -0.6 & -2.4 \end{bmatrix}$$

$E_0 = [0.0\ 0.2]'$ and

$$C_{z0} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ C_{z1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ D_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

it is seen once again that matrix $A_0 + A_1$ is Hurwitz but $A_0 - A_1$ is not. We have determined $\lambda^o = \lambda^* = 3.4001$ and $\omega^o = \omega^* = 2.1818$ associated to $\tau^* = 0.5230$. The right-hand side of Figure 7.2 shows that the function $\|T(\tau, s)\|_\infty$ and the lower bound proposed by Theorem 7.2 coincides for all practical purposes. It also shows in dotted line the first order Padé approximation, (Zhang et al. 2003). The same figure illustrates that when $\lambda \to \lambda^*$ then $\tau \to \tau^*$ and the norm of the time-delay system as well as its lower bound become unbounded. ♦

In these two examples, the lower bounds provided by Theorem 7.2 are very accurate for all $\tau \in [0, \tau^*)$. Comparing with the ones proposed in (Zhang et al. 2003), the same precision is attained only for Padé approximations of higher order, in general, greater than or equal to two. This feature is particularly important for linear filter synthesis.

7.3 Linear Filter Design

Consider a time-delay system whose state space minimal realization is given by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + E_0 w(t) \tag{7.19}$$
$$y(t) = C_{y0} x(t) + C_{y1} x(t-\tau) + D_y w(t) \tag{7.20}$$
$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau) \tag{7.21}$$

where, as before, $x(t) \in \mathbb{R}^n$ denotes the state, $w(t) \in \mathbb{R}^r$ is the exogenous input, $y(t) \in \mathbb{R}^p$ is the measured signal and $z(t) \in \mathbb{R}^q$ is the output to be estimated. We still assume that the system evolves from the rest, and that the delay $\tau \geq 0$ is constant with respect to time.

The aim of this part is to design a full order filter with the following structure

$$\dot{\hat{x}}(t) = \hat{A}_0 \hat{x}(t) + \hat{A}_1 \hat{x}(t-\tau) + \hat{B}_0 y(t) \tag{7.22}$$
$$\hat{z}(t) = \hat{C}_0 \hat{x}(t) + \hat{C}_1 \hat{x}(t-\tau) \tag{7.23}$$

where $\hat{x}(t) \in \mathbb{R}^n$. When connected to (7.19)-(7.21) this filter yields the minimal realization of the estimation error $\varepsilon(t) = z(t) - \hat{z}(t)$, in terms of the state space equations

$$\dot{\xi}(t) = F_0 \xi(t) + F_1 \xi(t-\tau) + G_0 w(t) \tag{7.24}$$
$$\varepsilon(t) = J_0 \xi(t) + J_1 \xi(t-\tau) \tag{7.25}$$

where $\xi(t) = [x(t)' \ \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the estimation error state vector and the indicated matrices stand for

$$F_0 = \begin{bmatrix} A_0 & 0 \\ \hat{B}_0 C_{y0} & \hat{A}_0 \end{bmatrix}, \ F_1 = \begin{bmatrix} A_1 & 0 \\ \hat{B}_0 C_{y1} & \hat{A}_1 \end{bmatrix} \tag{7.26}$$

$G_0' = [E_0' \ \ D_y'\hat{B}_0']$, $J_0 = [C_{z0} \ \ -\hat{C}_0]$ and $J_1 = [C_{z1} \ \ -\hat{C}_1]$. The transfer function from the external input w to the estimation error ε, which is not rational whenever $\tau > 0$, is given by

$$T_F(\tau, s) = \Big(J_0 + J_1 e^{-\tau s}\Big)\Big(sI - F_0 - F_1 e^{-\tau s}\Big)^{-1} G_0 \tag{7.27}$$

where the subindex "F" indicates its dependence on a given filter of the form (7.22)-(7.23). Hence, the goal is to design a filter with state space realization (7.22)-(7.23) such that $\|T_F(\tau, s)\|_\infty < \gamma$ where $\tau \in [0, \tau^*)$ and $\gamma > 0$ is given. Our strategy is to replace the transfer function $T_F(\tau, s)$ by the rational transfer function $H_F(\lambda, s)$ of the comparison system, that is

$$\begin{aligned} H_F(\lambda, s) &= \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ F_0 + F_1 & F_0 - F_1 - \lambda I & G_0 \\ \hline J_0 + J_1 & J_0 - J_1 & 0 \end{array}\right] \end{aligned} \tag{7.28}$$

solve the corresponding $\mathcal{H}_\infty$ filtering design problem and extract the time delay as indicated in Corollary 7.3. This is possible because the state space realization (7.28), with the state space matrices defined in (7.26), admits an important and interesting property that is the key for filter design in the present context. Indeed, even though the matrices of the state space realization of $H_F(\lambda, s)$ depend on an intricate manner of the filter state space realization matrices, applying the similarity transformation

$$S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{7.29}$$

one can rewrite (7.28) in the equivalent form

$$\begin{aligned} H_F(\lambda,s) &= \left[\begin{array}{c|c} SF_\lambda S^{-1} & SG \\ \hline JS^{-1} & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\lambda & 0 & E \\ \hat{L}C_y & \hat{A}_\lambda & \hat{L}D_y \\ \hline C_z & -\hat{C} & 0 \end{array}\right] \end{aligned} \tag{7.30}$$

where the system matrices (A_λ, E, C_z) have been defined in (7.5), $C_y = [C_{y0} + C_{y1} \quad C_{y0} - C_{y1}]$ and the filter matrices are given by

$$\hat{A}_\lambda = \left[\begin{array}{cc} 0 & \lambda I \\ \hat{A}_0 + \hat{A}_1 & \hat{A}_0 - \hat{A}_1 - \lambda I \end{array}\right], \quad \hat{L} = \left[\begin{array}{c} 0 \\ \hat{B}_0 \end{array}\right] \tag{7.31}$$

and $\hat{C} = [\hat{C}_0 + \hat{C}_1 \quad \hat{C}_0 - \hat{C}_1]$, indicating that they are in the comparison form. Hence, the filter (7.22)-(7.23), whenever connected to the time-delay system (7.19)-(7.21), produces an LTI comparison system associated to the estimation error (7.24)-(7.25). Its transfer function can be alternatively determined from the connection of the LTI comparison system of (7.19)-(7.21) and the LTI comparison system of the filter (7.22)-(7.23).

At first glance (7.30) leads to the conclusion that the state space realization of $H_F(\lambda, s)$ has the classical structure of the estimation error. This is true, however, matrices $\hat{A}_\lambda$ and $\hat{L}$ must be constrained to have the particular structures given in (7.31) in order to be a comparison system for the filter (7.22)-(7.23). To circumvent this difficulty, we propose to start by designing an LTI full order filter replacing the matrices variables $(\hat{A}_\lambda, \hat{L}, \hat{C})$ in (7.30) by general matrices variables (A_F, L_F, C_F) and solve $\|H_F(\lambda, s)\|_\infty < \gamma$ for a given $\gamma > 0$, which is a classical problem in $\mathcal{H}_\infty$ Theory. The second step is to determine a non-singular matrix $V \in \mathbb{R}^{2n\times 2n}$ such that $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$, which naturally implies that the estimation error transfer function of the comparison system remains unchanged. For a given $\gamma > 0$ and under the usual assumption $D_yD_y' = I$, imposed just for simplicity, we obtain the so called central filter in the observer form

$$\left[\begin{array}{c|c} A_F & L_F \\ \hline C_F & 0 \end{array}\right] = \left[\begin{array}{c|c} A_\lambda - L_FC_y & L_F \\ \hline C_z & 0 \end{array}\right] \tag{7.32}$$

where the filter gain is given by $L_F = PC_y' + ED_y'$ and $P = P' > 0$ satisfies the Riccati inequality

$$\tilde{A}_\lambda P + P\tilde{A}_\lambda' + \tilde{E}\tilde{E}' - P\Big(C_y'C_y - \gamma^{-2}C_z'C_z\Big)P < 0 \tag{7.33}$$

in which $\tilde{A}_\lambda = A_\lambda - ED_y'C_y$ and $\tilde{E} = E\left(I - D_y'D_y\right)$. It is a well-known fact in $\mathcal{H}_\infty$ Theory that, for $\gamma > 0$ given, the existence of $P > 0$ satisfying (7.33) is a necessary and sufficient condition for the existence of a full order LTI filter satisfying the norm constraint $\|H_F(\lambda, s)\|_\infty < \gamma$. In the affirmative case, the desired filter is given by (7.32) (see, for instance, (Colaneri, Geromel & Locatelli 1997) and (Zhou & Doyle 1998) for more details).

Lemma 7.4 *Assume that* $\dim(y) = p \le n = \dim(x)$, $\lambda > 0$ *and matrix*

$$V = \begin{bmatrix} N \\ NA_F/\lambda \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{7.34}$$

where $N' \in \mathbb{R}^{2n\times n}$ *belongs to the null space of* L_F'*, is non-singular. Under these conditions, the equality* $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$ *holds.*

Proof: Since $L_F \in \mathbb{R}^{2n\times p}$, the existence of a full rank N imposes $p \le n$. The proof follows from the observation that for the equality to hold, the state space realization (7.31) requires the conditions $[I\ 0]VL_F = 0$ and $[I\ 0]VA_FV^{-1} = [0\ \lambda I]$. Clearly, the first condition follows from the definition of N. The second one follows from $[0\ \lambda I]V - [I\ 0]VA_F = NA_F - NA_F = 0$. □

Once matrix V is determined, it yields the filter state space representation $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$ from which the state space matrices of the time-delay filter (7.22) – (7.23) are extracted. Notice, however, that in general we will have $p < n$ and, in this case, the similarity transformation defined by matrix V is not unique because $L_F'N' = 0$ admits many full rank solutions. It is clear that such similarity transformation does not change the value of $\|H_F(\lambda, s)\|_\infty$ but, on the contrary, it may change the value of $\|T_F(\tau, s)\|_\infty$. Indeed, for this last transfer function, the matrix V does not define a similarity transformation. In our numerical experiments, matrix $N' \in \mathbb{R}^{2n\times n}$ has always been chosen as the first n columns of the null space of L_F'.

Theorem 7.5 *Consider* $\gamma > \min_F \|H_F(\infty, s)\|_\infty$ *given. For* $\lambda > 0$ *big enough, the central solution of (7.33) defined by the pair* (L_F, P) *is such that* P *has the particular structure*

$$P = \begin{bmatrix} Z+Q & -Q \\ -Q & Q \end{bmatrix} \tag{7.35}$$

where $Z \in \mathbb{R}^{n\times n}$ *and* $Q \in \mathbb{R}^{n\times n}$ *are positive definite matrices. Furthermore, the relations* $\|H_F(\infty, s)\|_\infty = \|T_F(0, s)\|_\infty < \gamma$ *hold.*

Proof: Partitioning $P \in \mathbb{R}^{2n\times 2n}$ in four $n \times n$ matrix blocks such that

$$P = \begin{bmatrix} X & U \\ U' & \hat{X} \end{bmatrix} > 0 \tag{7.36}$$

it is a matter of simple calculations to verify that the Riccati inequality (7.33) can be rewritten as $\Pi + \lambda\Gamma < 0$, where $\Pi, \Gamma \in \mathbb{R}^{2n\times 2n}$ depend on the four blocks of P but not on λ. In addition, considering the non-singular matrix

$$\Sigma = \begin{bmatrix} I & (1/2)(U - \hat{X})\hat{X}^{-1} \\ 0 & I \end{bmatrix} \tag{7.37}$$

it is verified that the factorization

$$\Gamma = \Sigma \begin{bmatrix} (1/2)(U + \hat{X})\hat{X}^{-1}(U + \hat{X})' & 0 \\ 0 & -2\hat{X} \end{bmatrix} \Sigma' \tag{7.38}$$

holds. The fact that the matrix placed in the first diagonal block is semidefinite positive puts in evidence that two conditions are required in order to the Riccati inequality $\Pi + \lambda\Gamma < 0$ be satisfied for $\lambda > 0$ big enough. The first one is that $\hat{X} = -U = Q > 0$. Consequently, defining $Z = X - Q > 0$ the structure (7.35) follows. The second one is that the first diagonal block of the matrix $\Sigma^{-1}\Pi\Sigma'^{-1} \in \mathbb{R}^{2n\times 2n}$ must be negative definite, that is

$$\begin{aligned}(\tilde{A}_0 + \tilde{A}_1)Z + Z(\tilde{A}_0 + \tilde{A}_1)' + \tilde{E}_0\tilde{E}_0' \; - Z(C_{y0} + C_{y1})'(C_{y0} + C_{y1})Z + \\ + \gamma^{-2}Z(C_{z0} + C_{z1})'(C_{z0} + C_{z1})Z < 0\end{aligned} \tag{7.39}$$

where $\tilde{A}_0 = A_0 - E_0D_y'C_{y0}$, $\tilde{A}_1 = A_1 - E_0D_y'C_{y1}$ and $\tilde{E}_0 = E_0\left(I - D_y'D_y\right)$. Furthermore, calculating the filter gain $L_F = PC_y' + ED_y'$, with P given in (7.35), we obtain $L_{F1} + L_{F2} = Z(C_{y0} + C_{y1})' + E_0D_y'$, implying that such a filter gain imposes $\|T_F(0, s)\|_\infty < \gamma$. Actually, defining $\Xi_\lambda = (sI - A_\lambda)^{-1}L_F$, it can be verified that as λ goes to infinity the approximation

$$\Xi_\lambda \approx \begin{bmatrix} (sI - A_0 - A_1)^{-1}(L_{F1} + L_{F2}) \\ 0 \end{bmatrix} \tag{7.40}$$

is valid for any L_F and $|s|$ finite. When plugged into $C_F(sI - A_F)^{-1}L_F = C_z\left(I + \Xi_\lambda C_y\right)^{-1}\Xi_\lambda$, it shows that the filter (7.32) is nothing else than the central filter associated to the time-delay system with $\tau = 0$. As a consequence, the equality $\|H_F(\infty, s)\|_\infty = \|T_F(0, s)\|_\infty$ holds. □

This theorem states that for $\lambda > 0$ big enough the central solution of the Riccati inequality (7.33) has a particular structure and provides a filter such that the estimation error norms, namely $\|H_F(\infty, s)\|_\infty$ and $\|T_F(0, s)\|_\infty$, coincide. Hence, from Corollary 7.3 there exists an interval $\lambda \in (\lambda_\gamma, \infty)$, such that

$$\|H_F(\lambda, s)\|_\infty \leq \|T_F(\tau, s)\|_\infty < \gamma \tag{7.41}$$

for all $\tau \in [0, \tau_\gamma)$ where λ_γ and τ_γ are determined by the following algorithm similar to the one used for $\mathcal{H}_\infty$-norm calculation.

- Define a strictly decreasing sequence $\lambda_k \in \{\infty, \cdots, \lambda_o\}$.
- For an arbitrary element λ_k, determine the central filter $(A_{F_k}, L_{F_k}, C_{F_k})$ and the time delay $\tau_k = \tau(\lambda_k)$.
- From the similarity transformation matrix V_k, determine the time-delay filter $(\hat{A}_{\lambda_k}, \hat{L}_k, \hat{C}_k)$.
- Increase k whenever $0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2$ and $\|T_{F_k}(\tau_k, s)\|_\infty < \gamma$.

Adopting $|\lambda_k - \lambda_{k-1}| \leq \epsilon$ small enough, the continuity of $\|T_F(\tau(\lambda), s)\|_\infty$ is assured at $\lambda = \lambda_k$ whenever the similarity transformation $V(\lambda)$ is continuous at the same point. As discussed before, this property allows us to detect the stability margin for the estimation error. From (7.34) the continuity of $V(\lambda)$ at $\lambda = \lambda_k$ is a consequence of the continuity of $N(\lambda)$ at the same point, which can be numerically evaluated by the norm test $\|N_k - N_{k-1}\|_\infty < \epsilon$. Finally, notice that the same $\epsilon > 0$ can also be used to get a solution of the inequality (7.33) from the one of a Riccati equation by replacing matrix $\tilde{E}\tilde{E}'$ by $\tilde{E}\tilde{E}' + \epsilon I$.

Remark 7.3 *The discussion of Remark 7.2 concerning the determination of $\epsilon > 0$ is still valid in the present context. In addition, it is not difficult to show that the stabilizing solution of the Riccati equation has the structure*

$$P = \begin{bmatrix} Z & \mathcal{O}(\lambda^{-1}) \\ \mathcal{O}(\lambda^{-1}) & \mathcal{O}(\lambda^{-1}) \end{bmatrix} \tag{7.42}$$

for $\lambda > 0$ large enough, which assures once again the relations $\|H_F(\infty, s)\|_\infty = \|T_F(0, s)\|_\infty < \gamma$. In the context of time-delay systems, the use of filters that are directly calculated from classical $\mathcal{H}_\infty$ design methods is an important contribution.

Let us now analyze the impact of the similarity transformation provided in Lemma 7.4 to the inequality (7.41) when $p < n$ since, in this case, the matrix $V \in \mathbb{R}^{2n\times 2n}$ is not uniquely determined. The lower bound in (7.41) is clearly invariant with the choice of V because diag$\{I, V\}$ defines a similarity transformation for the LTI system (7.30). Consequently, with the central filter (7.32) we obtain

$$\begin{aligned} H_F(\lambda, s) &= \left[\begin{array}{c|c} A_H & E_H \\ \hline C_H & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} A_\lambda & 0 & E \\ L_F C_y & A_F & L_F D_y \\ \hline C_z & -C_F & 0 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_\lambda - L_F C_y & E - L_F D_y \\ \hline C_z & 0 \end{array}\right] \end{aligned} \tag{7.43}$$

On the other hand, with the filter realization provided by the similarity transformation, after cumbersome algebraic manipulations we obtain

$$T_F(\tau, s) = C_H \Lambda(s)\Big(sI - (A_H + \lambda I)\Lambda(s)\Big)^{-1} E_H \tag{7.44}$$

where $\Lambda(s) = \text{diag}\{\Lambda_0(s), V^{-1}\Lambda_0(s)V\}$ and

$$\Lambda_0(s) = \frac{1}{2}\begin{bmatrix} (1+e^{-\tau s})I \\ (1-e^{-\tau s})I \end{bmatrix}\begin{bmatrix} 0 \\ I \end{bmatrix}' \tag{7.45}$$

which makes clear that the transfer function of the time-delay system is not invariant with respect to the similarity transformation matrix V used to put the filter in the canonical form $(\hat{A}_\lambda, \hat{L}, \hat{C})$. In conclusion, for each V provided by Lemma 7.4, the inequality $\|H_F(\lambda, s)\|_\infty \leq \|T_F(\tau, s)\|_\infty$ holds. Its left-hand side is constant and the right-hand side varies depending on each choice of V. Hence, with (7.44) the optimal similarity transformation should be determined from $V(\lambda) = \arg\inf_V \|T_F(\tau(\lambda), s)\|_\infty$, assuring continuity. This problem remains still open. An application of the proposed algorithm to an illustrative time-delay system filter design is presented in the next subsection.

7.3.1 Example

In this subsection, a practical application inspired in the classical filtering problem consisting on recovering the signal after been corrupted by an additive noise in a transmission channel with time delay, see also (Li & Fu 1997), (Geromel & Regis Filho 2005). The matrices of the space state model (7.19)-(7.21) are $A_0 = \text{diag}\{A_{0s}, A_{0n}\}$, $A_1 = \text{diag}\{A_{sn}, A_{sn}\}$ and $E_0 = \text{diag}\{E_{0s}, E_{0n}\}$ where

$$A_{0s} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}, \ A_{0n} = \begin{bmatrix} 0 & 1 \\ -2 & -0.5 \end{bmatrix}$$

$$A_{sn} = \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix}, \ E_{0s} = E_{0n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The output matrices are of the form $C_{y0} = C_{z0} = [C_s \ 0 \ 0]$, $C_{y1} = [0 \ 0 \ C_n]$, $C_{z1} = [0 \ 0 \ 0 \ 0]$ and $D_y = [0 \ D_{yn}]$, where

$$C_s = \begin{bmatrix} 2 & 0 \end{bmatrix}, \ C_n = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D_{yn} = 0.5$$

This is a fourth order time-delay system with two inputs. The first one is the signal corrupted by a delayed noise and the second one is the signal to be estimated[2]. Applying the Nyquist criterion, we have determined $\lambda_o = \lambda^* = 1.000$ and the corresponding maximum delay $\tau^* = 1.570$ [s]. Beyond this point the time-delay system becomes unstable. For this system, $L_F \in \mathbb{R}^{8\times 1}$ and hence we have to choose $N' \in \mathbb{R}^{8\times 4}$ in the null space of L_F' which is defined by seven easy to calculate 8-th dimensional orthogonal vectors. As commented before, in the next experiments, we have adopted N simply as the first four vectors provided by the Matlab null space routine, which always passed on the proposed continuity test. Moreover, following the discussion of Remark 7.2, we have considered $\epsilon = 2 \times 10^{-3}$.

Figure 7.3 shows the estimation error as a function of τ calculated by the proposed algorithm. For $\gamma = 3$ we have obtained $\lambda_\gamma = 1.052$ and $\tau_\gamma \approx 1.395$ [s], and for $\gamma = 4$ we have determined $\lambda_\gamma = 1.026$ and $\tau_\gamma \approx 1.418$ [s]. With the value of $\lambda_o = 1.000$ already calculated, which depends only on the system under consideration and not on the actual value of γ, it is apparent that although τ_γ and λ_γ are limited by the pre-specified $\mathcal{H}_\infty$-norm level of the estimation error, both are close to their theoretical limits. On the other hand, for $\gamma = 2$ the maximum delay is much smaller ($\tau_\gamma \approx 0.412$ [s]) and it is only limited by the pre-specified $\mathcal{H}_\infty$-norm level of the estimation error. It is important to remark that in all the three cases shown in Figure 7.3, the lower bound and the true value of the $\mathcal{H}_\infty$-norm of the estimation error practically coincide.

For the filter designed with $\gamma = 3$ and the delay $\tau = 1.385$ [s], the actual $\mathcal{H}_\infty$-norm of the estimation error $\|T_F(\tau, s)\|_\infty \approx 2.732$ at the frequency $\omega_T \approx 1.234$ [rad/s] is quite close to the guaranteed norm level γ previously fixed. For this filter, Figure 7.4 shows the maximum singular value of its transfer function. The left-hand side part shows the central LTI filter (7.32) while the

[2] In this case, a simple change of variables must be performed in order to get a model satisfying the normalization constraint $D_y D_y' = I$.

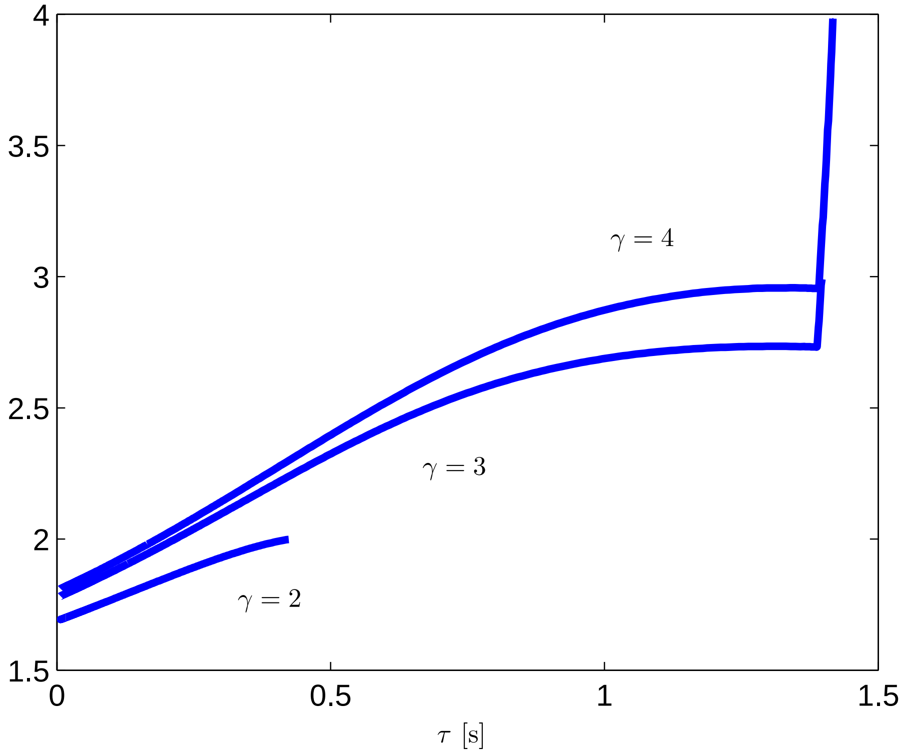

Figure 7.3: $\mathcal{H}_\infty$-norm of the estimation error versus time delay

right-hand side part shows the original time-delay filter (7.22)-(7.23) obtained from the similarity transformation provided by Lemma 7.4. Both are similar but the second filter is the only one able to cope with the time delay.

For the final qualification of the designed filter we have calculated the time simulations given in Figure 7.5. In each subplot the signal to be estimated is represented in dashed line while the filter output is represented in solid line. In order to generate some representative cases, as far as the estimation error of the filter is concerned, in both noise input channels we have injected the same perturbation $w_1(t) = w_2(t) = e^{-0.1t}\sin(\omega_n t)$ where $\omega_n \in \{\omega_T, \omega_T/2, 2\omega_T\}$. This corresponds to the injection of a slowly decaying sinusoidal perturbation not only at the frequency where the pick of the maximum singular value of the estimation error is attained, but also one half and twice the value of this frequency, respectively. From the top to the bottom, Figure 7.5 gives the time simulation of each case. As it can be seen, the filter is effective and reduces substantially the estimation error after a relative small period of time.

Finally, it is important to stress that there exists a filter such that $\|H_F(\lambda, s)\|_\infty < \gamma$ if and only if the Riccati inequality (7.33) is feasible and, in the affirmative case, a possible filter is the one given in (7.32). Hence, whenever a filter satisfying the equality $\|H_F(\lambda, s)\|_\infty = \|T_F(\tau, s)\|_\infty$

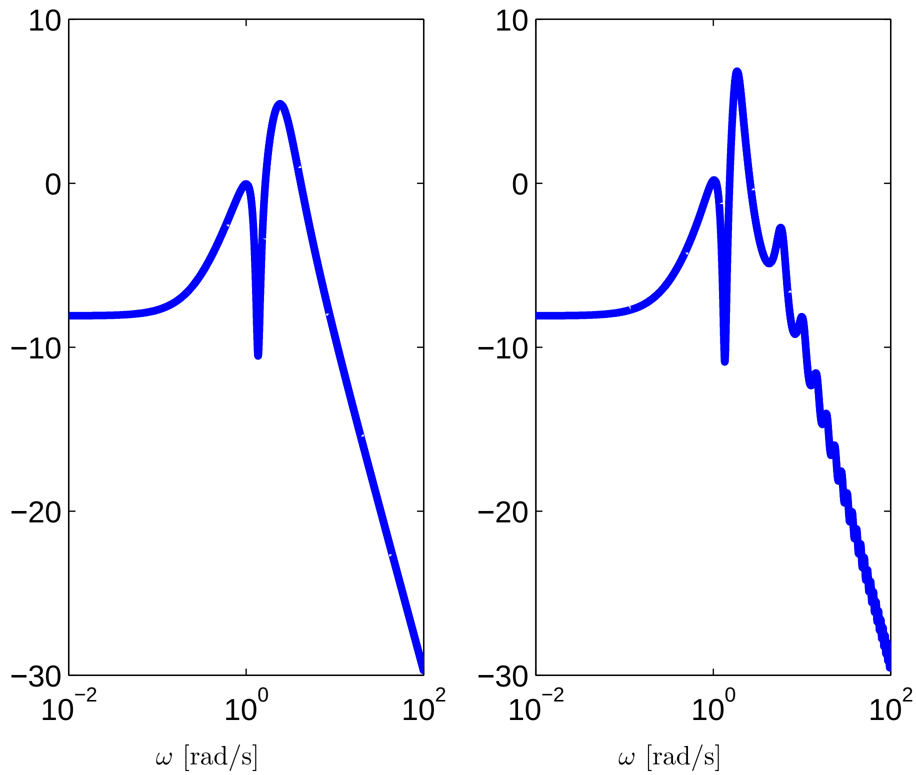

Figure 7.4: Maximum singular value in dB versus frequency

is obtained, this characterizes, in our context, the best filter for a time-delay system as far as the lower bound provided by Corollary 7.3 is concerned.

7.4 State Feedback Control Design

We now move our attention to the time delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + B_0u(t) + E_0w(t) \tag{7.46}$$

$$z(t) = C_0x(t) + C_1x(t-\tau) + G_0u(t) + D_0w(t) \tag{7.47}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $w(t) \in \mathbb{R}^r$ is the exogenous input and $z(t) \in \mathbb{R}^q$ is the controlled output. As before, it is assumed that the system evolves from the rest and the time delay $\tau \geq 0$ is constant with respect to time. The goal is to design a state feedback control $u(t) = K_0x(t) + K_1x(t-\tau)$ which connected to the open loop system (7.46)-(7.47) produces a closed loop system state space realization of the form of (7.2)-(7.3), with transfer function $T_K(\tau, s)$. Furthermore, adopting the same reasoning as before, the associated rational

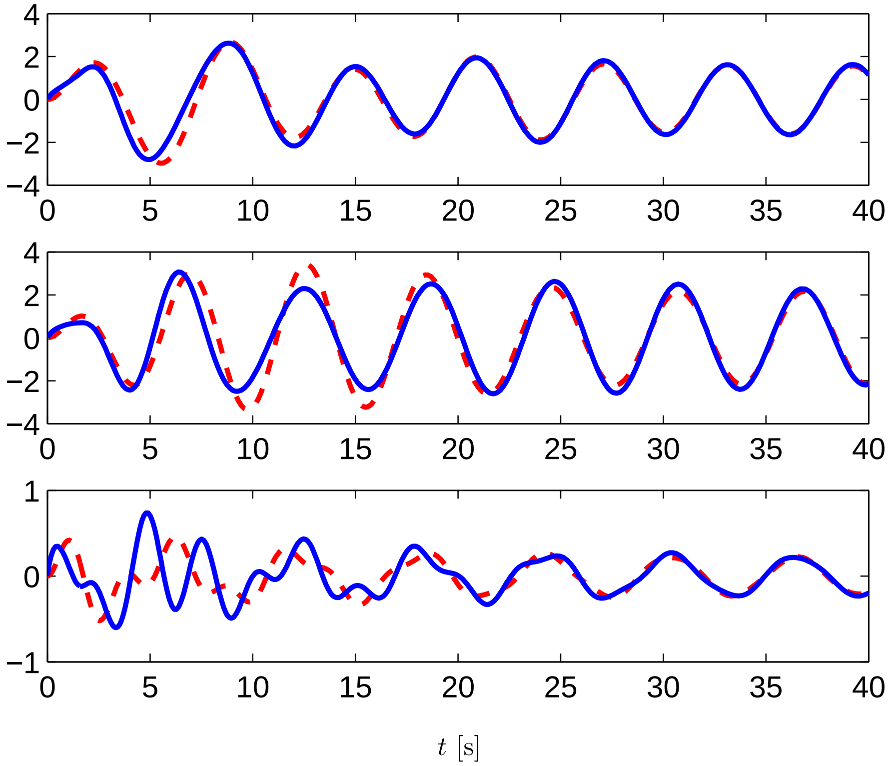

Figure 7.5: Time simulation

transfer function $H_K(\lambda, s)$ is given by

$$H_K(\lambda, s) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C + GK & D \end{array}\right] \tag{7.48}$$

where $B' = [0\ B_0']$, $G = G_0$, $D = D_0$ and

$$K = \left[\begin{array}{cc} K_0 + K_1 & K_0 - K_1 \end{array}\right] \tag{7.49}$$

At this point, it is important to observe that the transfer function $H_K(\lambda, s)$ has a closed loop structure and once the matrix gain $K \in \mathbb{R}^{m\times 2n}$ is determined it immediately provides the time delay system closed loop gains $K_0 \in \mathbb{R}^{m\times n}$ and $K_1 \in \mathbb{R}^{m\times n}$. Hence, we turn now our attention to the $\mathcal{H}_\infty$ control design problem consisting on the determination, if any, of a state feedback gain $K \in \mathbb{R}^{m\times 2n}$ such that $\|T_K(\tau, s)\|_\infty < \gamma$ which from the results of the previous section is replaced by the determination of $K \in \mathbb{R}^{m\times 2n}$ such that

$$\|H_K(\lambda, s)\|_\infty < \gamma \tag{7.50}$$

Doing this, the state feedback gain is readily determined. Indeed, provided that $\gamma^2 I - D'D > 0$, $G'G$ is nonsingular and the open loop system satisfies the orthogonality conditions $G'[C\ D] = 0$, $D'C = 0$ which are assumed for simplicity only, the so called central solution of (7.50) is given by $K = -(G'G)^{-1}B'P$ (Colaneri et al. 1997), where $P > 0$ satisfies the Riccati inequality

$$A'_\lambda P + PA_\lambda + C'C - P\Big(B(G'G)^{-1}B' - E(\gamma^2 I - D'D)^{-1}E'\Big)P < 0 \tag{7.51}$$

Moreover, the observability of (A_λ, C) and the controllability of (A_λ, B) assure that the Riccati equation obtained by replacing the inequality in (7.51) by equality admits a positive definite stabilizing solution as well.

Theorem 7.6 *There exists a symmetric and positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ satisfying the Riccati inequality (7.51) for $\lambda > 0$ big enough if and only if there exists $K \in \mathbb{R}^{m\times 2n}$ such that $\|T_K(0,s)\|_\infty < \gamma$. In the affirmative case, the central solution of (7.50) defined by the pair (K, P) is such that P^{-1} has the particular structure*

$$P^{-1} = \begin{bmatrix} Z+Q & -Q \\ -Q & Q \end{bmatrix} \tag{7.52}$$

where matrices $Z \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ are positive definite and the equality $\|H_K(\infty,s)\|_\infty = \|T_K(0,s)\|_\infty$ holds.

This theorem is of particular importance since, together with Theorem 7.2, it states that there exists an interval such that for $\lambda \in (\lambda_\gamma, \infty)$, then

$$\|H_K(\lambda,s)\|_\infty \le \|T_K(\tau,s)\|_\infty < \gamma \tag{7.53}$$

for all $\tau \in [0, \tau_\gamma)$ where λ_γ and τ_γ are calculated by starting from $\lambda > 0$ big enough. For each value of $\lambda \in (\lambda_\gamma, \infty)$ the state feedback gain $L(\tau) = K(\lambda)$ corresponding to $\tau = (2/\alpha(\lambda))\text{atan}(\alpha(\lambda)/\lambda) \in [0, \tau_\gamma)$ is determined.

7.4.1 Examples

The first example is the one proposed in (Niculescu 1998). The matrices of the state space realization are those already given and $B_0 = [0\ 1]'$, $G_0 = [1\ 0]'$. The left hand side of the Figure 7.6 has been calculated for $\gamma = 0.4$. It shows the lower and the upper bounds (dashed lines) of $\|T_{L(\tau)}(\tau,s)\|_\infty$ (solid line) for each value of the time delay in the interval $\tau \in (0, 10)$. For $\tau = 1.9702$ the state feedback gains are $K_0 = [-0.6164 \quad -3.9562]$ and

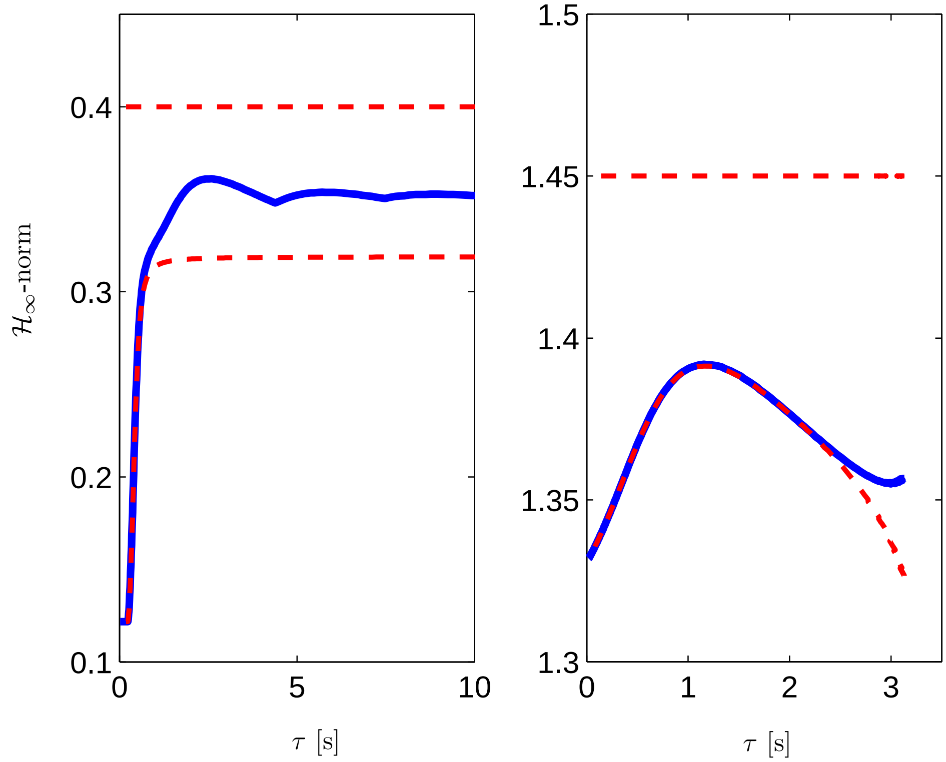

Figure 7.6: Closed loop $\mathcal{H}_\infty$-norm versus time delay

$K_1 = [1.2489 \;\; 3.6279]$ and the closed loop system transfer function norm with the linear state feedback gain is $\|T_K(\tau, s)\|_\infty = 0.3568$. For the same value of the time delay, the lower bound is $\|H_K(\lambda, s)\|_\infty = 0.3177$ determined with $\lambda = 0.65$.

The second example consists on the control of a system composed by the cascade connection of two transfer functions with a delay between them. The state space matrices are given by

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and

$$C_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; G_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The right hand side of Figure 7.6 shows once again the lower and the upper bounds (dashed lines) of $\|T_{L(\tau)}(\tau, s)\|_\infty$ (solid line) for each value of the time delay in the interval $\tau \in (0, 3)$ calculated for $\gamma = 1.45$. As it can be seen the procedure is very effective for the determination of a state feedback gain such that the closed loop time delay system satisfies an *a priori* fixed $\mathcal{H}_\infty$-norm bound. Finally, we want to stress a difference between the two examples solved. In the first one it was possible to determine a state feedback control assuring a pre-specified $\mathcal{H}_\infty$-norm upper bound for all time delays in an interval much greater than the one obtained in the second example. We have verified numerically that the time delay interval clearly depends on the system data but, mainly, on the value of the upper bound $\gamma > 0$ fixed by the designer.

7.5 Output Feedback Design

Finally, let us consider a time-delay system whose state space minimum realization is given by

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + B_0u(t) + E_0w(t) \tag{7.54}$$

$$y(t) = C_{y0}x(t) + C_{y1}x(t-\tau) + D_{yw}w(t) \tag{7.55}$$

$$z(t) = C_{z0}x(t) + C_{z1}x(t-\tau) + D_{zu}u(t) \tag{7.56}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^r$ is the exogenous input, $y(t) \in \mathbb{R}^p$ is the measured signal and $z(t) \in \mathbb{R}^q$ is the controlled output. We continue to assume that $x(t) = 0\ \forall t \in [-\tau, 0]$, and the delay $\tau \geq 0$ is constant with respect to time. The aim now is to design a full order dynamic output feedback controller with the following structure

$$\dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{A}_1\hat{x}(t-\tau) + \hat{B}_0y(t) \tag{7.57}$$

$$u(t) = \hat{C}_0\hat{x}(t) + \hat{C}_1\hat{x}(t-\tau) \tag{7.58}$$

where $\hat{x}(t) \in \mathbb{R}^n$. When connected to (7.54)-(7.56) this controller yields the minimal realization of the regulated output $z(t)$, in terms of the state space equations

$$\dot{\xi}(t) = F_0\xi(t) + F_1\xi(t-\tau) + G_0w(t) \tag{7.59}$$

$$z(t) = J_0\xi(t) + J_1\xi(t-\tau) \tag{7.60}$$

where $\xi(t) = [x(t)'\ \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the state vector and the indicated matrices stand for

$$F_0 = \begin{bmatrix} A_0 & B_0\hat{C}_0 \\ \hat{B}_0C_{y0} & \hat{A}_0 \end{bmatrix},\ F_1 = \begin{bmatrix} A_1 & B_0\hat{C}_1 \\ \hat{B}_0C_{y1} & \hat{A}_1 \end{bmatrix} \tag{7.61}$$

$G_0' = [E_0',\ D_{yw}'\hat{B}_0']$, $J_0 = [C_{z0},\ D_{zu}\hat{C}_0]$ and $J_1 = [C_{z1},\ D_{zu}\hat{C}_1]$.

We extend the previous results to cope with the problem consisting on the determination of a controller with state space realization (7.57)-(7.58) such that $\|T_C(\tau,s)\|_\infty < \gamma$ where $\tau \in [0,\tau^*)$ and $\gamma > 0$ is given. Our strategy again is to replace the transfer function $T_C(\tau,s)$ by the rational transfer function $H_C(\tau,s)$ of the comparison system, that is

$$H_C(\lambda,s) = \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ F_0+F_1 & F_0-F_1-\lambda I & G_0 \\ \hline J_0+J_1 & J_0-J_1 & 0 \end{array}\right] \tag{7.62}$$

solve the corresponding $\mathcal{H}_\infty$ output feedback design problem and extract the time-delay as indicated in Corollary 7.3. This is possible because the state space realization (7.62), with the state space matrices defined in (7.61), admits an important and interesting property that is the key for control design in the present context. Indeed, even though the matrices of the state space realization of $H_C(\lambda,s)$ depend on an intricate manner of the control state space realization matrices, applying the similarity transformation

$$S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{7.63}$$

one can rewrite (7.62) in the equivalent form

$$H_C(\lambda,s) = \left[\begin{array}{c|c} SF_\lambda S^{-1} & SG \\ \hline JS^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_\lambda & B\hat{C} & E \\ \hat{B}C_y & \hat{A}_\lambda & \hat{B}D_{yw} \\ \hline C_z & D_{zu}\hat{C} & 0 \end{array}\right] \tag{7.64}$$

where $B' = [0,\ B_0']$ and the controller matrices are given by

$$\hat{A}_\lambda = \begin{bmatrix} 0 & \lambda I \\ \hat{A}_0+\hat{A}_1 & \hat{A}_0-\hat{A}_1-\lambda I \end{bmatrix},\ \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix} \tag{7.65}$$

and $\hat{C} = [\hat{C}_0+\hat{C}_1 \quad \hat{C}_0-\hat{C}_1]$, indicating that they are in the comparison form. Hence, the controller (7.57)-(7.58) whenever connected to the time-delay system (7.54)-(7.56) produces an LTI comparison system associated to the regulated output (7.59)-(7.60) whose transfer function can be alternatively determined from the connection of the LTI comparison system of the system (7.54)-(7.56) and the LTI comparison system of the controller (7.57)-(7.58).

Again, the state space realization of $H_C(\lambda, s)$ has the classical structure of the regulated output, but matrices $\hat{A}_\lambda$ and $\hat{B}$ must be constrained to have the particular structures given in (7.65) in order to be a comparison system for the controller (7.57)-(7.58). To circumvent this difficult, we will repeat the strategy utilized for the filter design, that is, design an LTI full order output feedback controller replacing the matrices variables $(\hat{A}_\lambda, \hat{B}, \hat{C})$ in (7.64) by general matrices variables (A_C, B_C, C_C) and solve $\|H_C(\lambda, s)\|_\infty < \gamma$ for a given $\gamma > 0$, which is a classical problem in $\mathcal{H}_\infty$ Theory. The second step is to determine a non-singular matrix $V \in \mathbb{R}^{2n\times 2n}$ such that $(\hat{A}_\lambda, \hat{B}, \hat{C}) = (VA_CV^{-1}, VB_C, C_CV^{-1})$, which naturally implies that the regulated output transfer function of the comparison system remains unchanged. Once we have the controller matrices at hand, it is a simple matter of computation to determine whether $\|T_C(\tau(\lambda), s)\|_\infty < \gamma$ holds.

For a given $\gamma > 0$ and under the usual assumptions $D_{zu}'C_z = 0$, $ED_{yw}' = 0$, $D_{yw}D_{yw}' = I$ and $D_{zu}'D_{zu} = I$, imposed just for simplicity, it is a well known fact in $\mathcal{H}_\infty$ theory that the existence of a stabilizing matrix $P = P' > 0$ and $\Pi = \Pi' > 0$ satisfying

$$A_\lambda\Pi + \Pi A_\lambda' + EE' - \Pi\Big(C_y'C_y - \gamma^{-2}C_z'C_z\Big)\Pi = 0 \tag{7.66}$$

$$A_\lambda'P + PA_\lambda + C_z'C_z - P\Big(BB' - \gamma^{-2}EE'\Big)P < 0 \tag{7.67}$$

and spectral radius constraint $r_s(P\Pi) < \gamma^2$ is a necessary and sufficient condition for the existence of a full order LTI controller (depending on λ) such that $\|H_C(\lambda, s)\|_\infty < \gamma$. In the affirmative case, the desired controller has the state space realization defined by

$$\begin{aligned} B_C &= -\Pi C_y' \\ C_C &= B'P(I - \gamma^{-2}\Pi P)^{-1} \\ A_C &= A_\lambda + \gamma^{-2}\Pi C_z'C_z + B_CC_y - BC_C \end{aligned} \tag{7.68}$$

Lemma 7.7 *For $\lambda > 0$ large enough, the stabilizing positive definite solution of (7.66) and any positive definite feasible solution of (7.67) exhibit the structures*

$$\Pi = \begin{bmatrix} Z & \lambda^{-1}Q \\ \lambda^{-1}Q' & \lambda^{-1}W \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} Y + R & -R \\ -R & R \end{bmatrix} \tag{7.69}$$

where $Z > 0$, $W > 0$, Q, $Y > 0$ and $R > 0$ are $n \times n$ matrices.

Proof: First, to evaluate the behavior of Π with respect to λ, let us consider the matrix function

$$\Phi_\lambda = A_\lambda \Pi + \Pi A_\lambda' + EE' - \Pi\Big(C_y' C_y - \gamma^{-2} C_z' C_z\Big)\Pi \tag{7.70}$$

where Π is given in (7.69). Calculating $\lim_{\lambda\to\infty}\Phi_\lambda = \Phi_\infty$ we obtain the following matrix blocks identities:

$$\begin{aligned}
\Phi_\infty^{11} &= Q + Q' - Z(C_{y0}+C_{y1})'(C_{y0}+C_{y1})Z + \gamma^{-2} Z(C_{z0}+C_{z1})'(C_{z0}+C_{z1})Z \\
\Phi_\infty^{12} &= W + Z(A_0+A_1)' - Q \\
\Phi_\infty^{22} &= -2W + E_0E_0'
\end{aligned} \tag{7.71}$$

Assuming $E_0E_0' > 0$, otherwise perturb it slightly, the condition $\Phi_\infty = 0$ is satisfied whenever $Z > 0$ is the stabilizing solution of the Riccati equation

$$\begin{aligned}
&(A_0+A_1)Z + Z(A_0+A_1)' + E_0E_0' - Z\Big((C_{y0}+C_{y1})' \times \\
&\qquad \times (C_{y0}+C_{y1}) - \gamma^{-2}(C_{z0}+C_{z1})'(C_{z0}+C_{z1})\Big)Z = 0
\end{aligned} \tag{7.72}$$

Moreover, this solution is possible since the condition $\Pi > 0$ for $\lambda \to \infty$ is equivalent to $Z > 0$ and $W > 0$. Hence, the first part of Lemma 7.7 follows.

For the second part, considering any $P > 0$ feasible for inequality (7.67), multiplying it by P^{-1} on both sides we can apply the same procedure as in the state-feedback case, which yields the conclusion that (7.67) is satisfied for $\lambda \to \infty$ if and only if P^{-1} has the structure given in (7.69) with $Y > 0$ satisfying

$$(A_0+A_1)'Y^{-1} + Y^{-1}(A_0+A_1) - Y^{-1}(B_0B_0' - \gamma^{-2}E_0E_0')Y^{-1} + (C_{z0}+C_{z1})'(C_{z0}+C_{z1}) < 0 \tag{7.73}$$

and $R > 0$ arbitrary. □

This lemma is used in the proof of the next theorem where, similarly to the state feedback case, a central result valid for $\lambda \to \infty$ (the behavior at infinity) is established. It states that at infinity the close-loop delay system and the associated comparison system have transfer functions with equal $\mathcal{H}_\infty$-norm.

Theorem 7.8 *Consider $\gamma > \min_C \|H_C(\infty, s)\|_\infty$. For $\lambda > 0$ large enough, we have that the relations $\|H_C(\infty, s)\|_\infty = \|T_C(0, s)\|_\infty < \gamma$ hold.*

Proof: For matrices $\Pi > 0$ and $P > 0$ with the structures (7.69) and satisfying $r_s(\Pi P) < \gamma^2$ we determine the gains B_C and C_C from (7.68). The first gain can be partitioned as $B_C = [B_{C0}',\ 0]'$

where $B_{C0} = -Z(C_{y0}+C_{y1})'$, while the second one can be written as $C_C = [C_{C0},\ C_{C0}+B_0'R^{-1}]$ where $C_{C0} = B_0'Y^{-1}(I-\gamma^{-2}ZY^{-1})^{-1}$. For $\lambda \to \infty$ and $|s|$ finite, the central controller transfer function can be written as $C_\infty(s) = C_C(sI-A_C)^{-1}B_C = C_C(I-\Xi(B_CC_y+\gamma^{-2}\Pi C_z'C_z))^{-1}\Xi B_C$ where

$$\Xi \approx \begin{bmatrix} (sI-(A_0+A_1)+B_0C_{C0})^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} I & I \end{bmatrix} \tag{7.74}$$

Defining $A_{C0} = A_0+A_1-B_0C_{C0}+B_{C0}(C_{y0}+C_{y1})'+\gamma^{-2}Z(C_{z0}+C_{z1})'(C_{z0}+C_{z1})$, we conclude that $C_\infty(s) = C_{C0}(sI-A_{C0})^{-1}B_{C0}$, which guarantees $\|T_C(0,s)\|_\infty < \gamma$ since from Lemma 7.7 the spectral radius condition $r_s(\Pi P) < \gamma^2$ reduces to $r_s(ZY^{-1}) < \gamma^2$ for $\lambda > 0$ large enough. Indeed, $C_\infty(s)$ is the central controller associated to the time-delay system when $\tau = 0$. As a consequence $\|H_C(\infty,s)\|_\infty = \|T_C(0,s)\|_\infty < \gamma$. □

An important point concerning the controller design is how to obtain a suitable similarity transformation $V \in \mathbb{R}^{2n\times 2n}$. It is important to note that V must put A_C and B_C in the appropriate form. Moreover it must guarantee that the closed loop system is stable, since V does not define a similarity transformation for the time-delay system and thus must be computed with care.

Lemma 7.9 *Assume that* $\dim(y) = p \le n = \dim(x)$*,* $\lambda > 0$ *and the non-singular matrix*

$$V = \begin{bmatrix} N \\ \lambda^{-1}NA_C \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{7.75}$$

where $N' \in \mathbb{R}^{2n\times n}$ *belongs to the null space of* B_C'*. Under these conditions,* $(\hat{A}_\lambda, \hat{B}, \hat{C}) = (VA_CV^{-1}, VB_C, C_CV^{-1})$ *is true.*

An important fact concerning this lemma is the influence of the similarity transformation V in (7.75) as $\lambda \to \infty$. Partitioning matrix $N = [N_1\ N_2]$, where N_1 is assumed to be non-singular, we have that the central controller obtained from Theorem 7.8, after applying the similarity transformation defined in (7.75), is given by

$$C_\infty(s) = \left[\begin{array}{c|c} N_1A_{C0}N_1^{-1} & N_1B_{C0} \\ \hline C_{C0}N_1^{-1} & 0 \end{array}\right] \tag{7.76}$$

This fact indicates, as expected, that the similarity transformation does not affect the controller transfer function when $\tau = 0$, and consequently $\|H_C(\infty,s)\|_\infty = \|T_C(0,s)\|_\infty$ for any nonsingular $V \in \mathbb{R}^{2n\times 2n}$.

Under these discussions we are able to extend the algorithm proposed to calculate the $\mathcal{H}_\infty$-norm of a time-delay system in order to obtain intervals $\lambda \in (\lambda_\gamma, \infty)$ and $\tau(\lambda) \in [0, \tau_\gamma)$ such that there exists a controller with realization (7.57)-(7.58) for each pair $(\lambda, \tau(\lambda))$ that guarantees $\|H_C(\lambda, s)\|_\infty \leq \|T_C(\tau(\lambda), s)\|_\infty < \gamma$. At each iteration k we must calculate not only the time delay $\tau_k = \tau(\lambda_k)$, but also the central controller (A_{Ck}, B_{Ck}, C_{Ck}) and the similarity transformation V_k. In order to assure the continuity of $\|T_C(\tau(\lambda), s)\|_\infty$ in the considered intervals, what enables us to detect any unboundedness tendency, we must compute matrix N_k in (7.75) with care. We have chosen N_k' as the first n column vectors provided by the Matlab null space routine applied to B_{Ck}'. In order to guarantee the continuity of $\|T_C(\tau(\lambda), s)\|_\infty$ at λ_k we propose to verify the continuity in determining N_k from the null space of B_{Ck}' by evaluating the norm condition $\|N_k - N_{k-1}\| \leq \epsilon$, with ϵ sufficiently small. In fact, the problem of generating a continuous null space basis for matrix B_C depending on the parameter λ is not a simple task, as is discussed in (Byrd & Schnabel 1986), and here we circumvent this drawback by applying a numerical procedure for verify this fact.

7.5.1 Example

In this section we consider a second order example borrowed from (Fridman & Shaked 2002) for illustration and comparison. The time-delay system (7.54)-(7.56) matrices are defined as follows[3]:

$$\left[\begin{array}{c|c|c|c} A_0 & A_1 & E_0 & B_0 \end{array}\right] = \left[\begin{array}{cc|cc|cc|c} 0 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 0 & 1 \end{array}\right] \tag{7.77}$$

$$\left[\begin{array}{c|c|c} C_{y0} & C_{y1} & D_{yw} \end{array}\right] = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0.1 \end{array}\right] \tag{7.78}$$

$$\left[\begin{array}{c|c|c} C_{z0} & C_{z1} & D_{zu} \end{array}\right] = \left[\begin{array}{cc|cc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 \end{array}\right] \tag{7.79}$$

For this system we have applied the proposed algorithm to generate a sequence of stabilizing controllers for each pair $(\lambda_k, \tau(\lambda_k))$ such that $\lambda_k \in (\lambda_\gamma, \infty)$ and $\tau(\lambda_k) \in [0, \tau_\gamma)$. For the upper bound $\gamma = 1$ we have determined $\lambda_\gamma = 1.0100$ and $\tau_\gamma = 1.2477$ [s]. In Figure 7.7 we present in dashed line the lower bound $\|H_C(\lambda, s)\|_\infty$ and in solid line the exact value $\|T_C(\tau(\lambda), s)\|_\infty$ when imposing $\gamma = 1$. It is interesting to note that for $\tau(\lambda) \in [0, 0.5)$ the values of the lower bound and the true value of $\|T_C(\tau(\lambda), s)\|_\infty$ are identical and we verify that, for the remaining interval, the maximum difference between them is about 3.2%.

For comparison purposes, the point mark in $\|T_C(\tau(\lambda), s)\|_\infty$ curve, in Figure 7.7, refers to

[3]In this case, a simple change of variables must be performed in order to get a model satisfying $D_{yw}' D_{yw} = I$ and $D_{zu} D_{zu}' = I$.

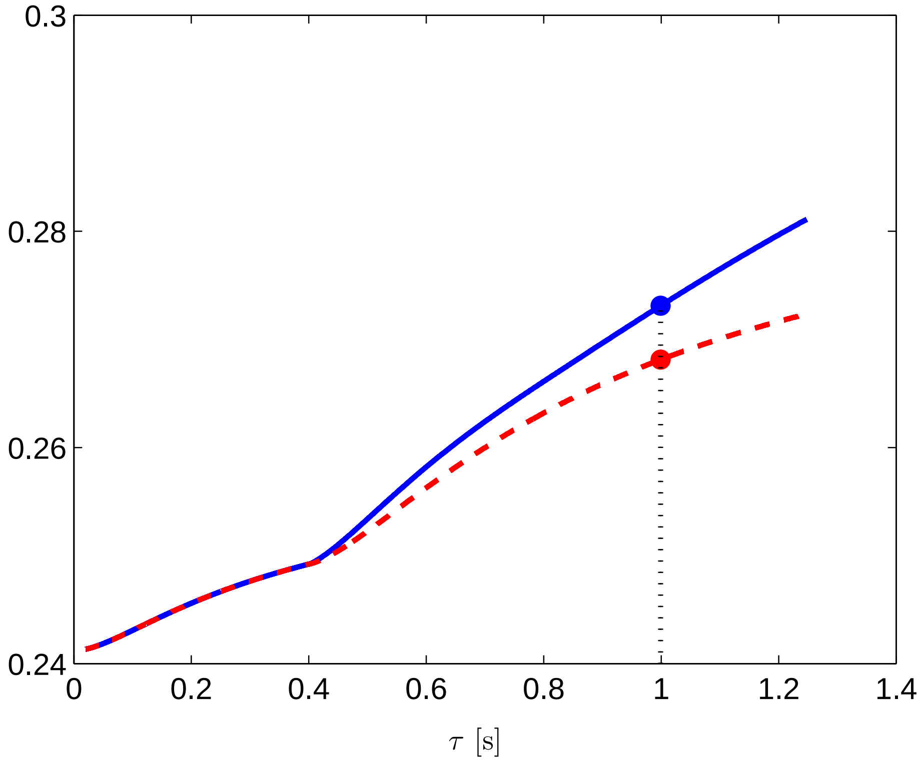

Figure 7.7: $\mathcal{H}_\infty$ performance versus time delay for $\gamma = 1$.

the time delay $\tau = 0.9990$ [s], for $\lambda = 1.40438$, and the corresponding norm $\|T_C(0.9990, s)\|_\infty = 0.2731$, which is 68% smaller than the $\mathcal{H}_\infty$-norm obtained by (Fridman & Shaked 2002). In the same figure we point out the lower bound $\|H_C(1.40438, s)\|_\infty = 0.2681$, which is only 1.81% smaller than the true norm value. Also for $\tau = 0.9990$ [s] we have determined the controller matrices given as follows:

$$\left[\begin{array}{c|c} A_{C0} & A_{C1} \end{array}\right] = \left[\begin{array}{cc|cc} -28.6072 & 1.4110 & 3.6807 & -2.4378 \\ -76.1020 & 3.8891 & 11.2365 & -7.4419 \end{array}\right] \tag{7.80}$$

$$\left[\begin{array}{c|c|c} B_C & C'_{C0} & C'_{C1} \end{array}\right] = \left[\begin{array}{c|c|c} 15.0420 & -10.5733 & 2.2117 \\ 36.8268 & 0.4678 & -0.9181 \end{array}\right] \tag{7.81}$$

As it can be verified, this time-delay controller makes the closed-loop system asymptotically stable with the transfer function $\mathcal{H}_\infty$-norm previously calculated.

7.6 Neutral Systems

The theoretical results reported so far can be generalized to cope with neutral systems of the class defined by the state space realization

$$\dot{x}(t) - F_1\dot{x}(t-\tau) = A_0x(t) + A_1x(t-\tau) + E_0w(t) \tag{7.82}$$

$$z(t) = C_0x(t) + C_1x(t-\tau) + D_0w(t) \tag{7.83}$$

where, as before, it is assumed that it evolves from the rest and the time delay $\tau \geq 0$ is constant with respect to time. In this case, the comparison system is defined as follows

$$\begin{aligned} H(\lambda, s) &= \left[\begin{array}{c|c} A_\lambda & E \\ \hline C & D \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \tilde{A}_0 + \tilde{A}_1 & \tilde{A}_0 - \tilde{A}_1 - \lambda\Pi & \tilde{E}_0 \\ \hline C_0 + C_1 & C_0 - C_1 & D_0 \end{array}\right] \end{aligned} \tag{7.84}$$

where, matrices $\tilde{A}_0, \tilde{A}_1, \tilde{E}_0$ are obtained by simply multiplying A_0, A_1, E_0 to the left by $(I+F_1)^{-1}$ and $\Pi = (I+F_1)^{-1}(I-F_1)$. As before, the equality $H(\lambda, \jmath\omega) = T(\tau, j\omega)$ holds whenever the constants $\lambda > 0$, $\tau > 0$ and the frequency $\omega \in \mathbb{R}$ are related by $\omega/\lambda = \tan(\omega\tau/2)$.

Indeed, it may be verified that the equality

$$\begin{aligned} H(\lambda, s) &= D + C(sI - A_\lambda)^{-1}E \\ &= D_0 + \Big(C_0(\lambda+s) + C_1(\lambda-s)\Big)\Big((sI - A_0)(\lambda+s) - (F_1s + A_1)(\lambda-s)\Big)^{-1}E_0 \\ &= T(\tau, s) \end{aligned} \tag{7.85}$$

holds. Furthermore, assuming that all eigenvalues of the matrix F_1 are inside the unity circle in order to have all chains of poles in the extended left half-plane, the matrix $-\Pi$ is well defined and its eigenvalues are located in the open left side of the complex plane. The consequence is that when $\lambda > 0$ goes to infinity, the poles of $H(\lambda, s)$ given in (7.84) go to the eigenvalues of $\Pi^{-1}(\tilde{A}_0 + \tilde{A}_1)$ and to the eigenvalues of $-\lambda\Pi$. Hence, whenever the neutral system is asymptotically stable for $\tau = 0$, the same is true for $H(\infty, s)$. As a consequence, the previous results can be adapted for neutral systems of the class under consideration.

7.7 Final Remarks

In this chapter we have proposed a procedure for time-delay filter and control design. It is based on what we call comparison system, which is an LTI system with order twice the number of state variables of the time-delay system estimation error.

The most important feature of the comparison system is that it makes possible the filter or controller design by manipulating finite order LTI systems, exclusively. As a consequence, the classical routines for filter design can be applied, opening the possibility to handle time-delay systems with high number of state variables.

The following publications were produced presenting the results of this chapter (Korogui, Fioravanti & Geromel 2011*b*), (Korogui, Fioravanti & Geromel 2011*a*), (Korogui, Fioravanti & Geromel 2010) and (Korogui, Fioravanti & Geromel 2009):

- R. H. Korogui, A. R. Fioravanti and J. C. Geromel. 'On a Rational Transfer Function-Based Approach to $\mathcal{H}_\infty$ Filtering Design for Time-Delay Linear Systems'. *IEEE Transactions on Signal Processing*, Vol. 59, No. 3, pp. 979-988, March 2011.

- R. H. Korogui, A. R. Fioravanti and J. C. Geromel. '$\mathcal{H}_\infty$ Control Design for Time-Delay Linear Systems: a Rational Transfer Function Based Approach'. Under revision.

- R. H. Korogui, A. R. Fioravanti and J. C. Geromel. 'Análise de Estabilidade e Controle $\mathcal{H}_\infty$ de Sistemas Lineares com Atraso'. *XVIII Congresso Brasileiro de Automática (CBA2010)*, Bonito - Brazil, September 12-16, 2010.

- R. H. Korogui, A. R. Fioravanti and J. C. Geromel. 'On a Rational Transfer Function-Based Approach to $\mathcal{H}_\infty$ Filter Design for Time-Delay Linear Systems'. *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pp. 1854 - 1859, Shanghai - China, December 16-18, 2009.

Part IV

Conclusions and Bibliography

Chapter 8

Conclusions and Perspectives

Contents

8.1 Conclusions 165

8.2 Perspectives 166

8.1 Conclusions

We conclude this manuscript by revisiting the key points presented in the previous chapters.

In the beginning of Chapter 2, we introduced the classes of systems that were studied. Some basic results for classical time-delay systems were presented. The same was made for fractional systems, but in a much slower pace, due to the relative novelty of the subject. Aside from the historical development and general properties, we have seen the unexpected connection between the Laplace transform of some particular time-varying systems and fractional time-delayed systems.

Chapter 3 brought our first results. Starting from some asymptotic analysis, we calculated the position of chains of poles for neutral commensurate time-delay systems. From that result, much more information could be derived, such as stability and stabilizability, and whenever possible, the Bézout factors were provided. The same strategy was applied for fractional systems, where the results differed in its nature, showing that, for these cases, fractional systems have a stronger capability of being stable. We finished by giving some expressions enabling the utilization of such analysis for systems described in time-domain equations, avoiding the need to completely calculate its transfer function.

While the focus in Chapter 3 was over the poles with high modulus, in Chapter 4 we came to study the other ones, giving special attention to the ones placed in the right half-plane. We developed a numerical algorithm able to locate the poles crossing the imaginary axis and track them through continuation methods, both for classical and fractional systems. Although some other results dealing with the same problem already exist for the classical case, our technique can be more adequate to deal with cases presenting multiples delays. Also, we believe that our method is the first one specifically proposed for this problem in the case of fractional systems. Together with the results of Chapter 3, the ones presented there completed the characterization of stability of linear (possibly fractional) systems with multiple commensurate delays.

Chapter 5 continued with the development of numerical algorithms, but now for a class of fractional systems with distributed delays. For this class, two parameters are essential, namely the mean-delay and the gap. Therefore, we provided ways to calculate the stability crossing curves with respect to those two parameters, and with this chart, locate all places where the system is stable or unstable.

From this point on, we started to deal with the problem of synthesis of filters and controllers. In the first one, Chapter 6, we designed classical PID controllers for fractional system with one unstable pole. The strategy was done in two steps, first a stabilizing PD controller was found, and afterwards the integral component was added. Some rules to tune the open parameters were given in order to increase the stability margin.

Finally, in Chapter 7, we worked with a comparison system developed from the Rekasius transformation in order to tackle the problems of state-feedback, filtering and dynamic output feedback design. The main point was the use of classical LTI routines to minimize the $\mathcal{H}_\infty$-norm of the estimation error or the closed-loop output, and not routines based directly on (Krasovskii 1963) or (Razumikin 1956). This led to filters and controllers with better performance.

8.2 Perspectives

Our initial goal is to complete the deployment of the Matlab toolbox YALTA, incorporating, among other features for time-delay systems, all the numerical routines presented in this manuscript and implemented by the author. It is under responsibility of the INRIA-DISCO team and shall be launched in the second semester of 2011.

The second point that will be the subject of future research efforts is the generalization of the results presented in Chapters 3 and 4 for the case of non-commensurate delays. Although we believe that the theoretical results of Chapter 3 can indeed be generalized for this case, our plan for the numerical method of Chapter 4 is to provide some bounds for the error on the values of the stability windows when we numerically approximate the non-commensurate system by a commensurate one.

Finally, a subject that deserves to be fully explored is the application of the methods presented here for systems with uncertainties. This robustness analysis and synthesis have different meanings for the different cases we have presented. For example, we might consider that the transfer functions in Chapters 3 and 4 have some coefficients that are not perfectly known, or that the delay in Chapter 7 is not precisely known in order to be used for the feedback loop. In all these cases, new techniques need to be developed both theoretically and numerically.

Bibliography

Adimy, M., Crause, F. & Abdllaoui, A. E. (2008), 'Discrete maturity-structured model of cell differentiation with applications to acute myelogenous leukemia', *Journal of Biological Systems* **16(3)**, 395–424.

Bellman, R. & Cooke, K. (1963), *Differential-difference equations*, Academic Press, New York–London.

Bensoussan, A., Prato, G. D., Delfour, M. C. & Mitter, S. K. (1993), *Representation and Control of Infinite Dimensional Systems*, Birkhäuser, Boston.

Bernard, S., Bélair, J. & Mackey, M. C. (2001), 'Sufficient conditions for stability of linear differential equations with distributed delay', *Discrete and continuous dynamical systems - Series B* **1**, 233–256.

Bohannan, G. W. (2008), 'Analog fractional order controller in temperature and motor control applications', *Journal of Vibration Control* **14(9-10)**, 1487–1498.

Bonnet, C., Fioravanti, A. R. & Partington, J. R. (2009*a*), On the stability of neutral linear systems with multiple commensurate delays, *in* 'Proceedings of the IFAC Workshop on Control of Distributed Parameter Systems', Toulouse - France, pp. 195–196.

Bonnet, C., Fioravanti, A. R. & Partington, J. R. (2009*b*), Stability of neutral systems with multiple delays and poles asymptotic to the imaginary axis, *in* 'Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference', Shanghai, P.R. China, pp. 269–273.

Bonnet, C., Fioravanti, A. R. & Partington, J. R. (2010), 'Stability of neutral systems with commensurate delays and poles asymptotic to the imaginary axis', *SIAM Journal on Control and Optimization* . Submitted.

Bonnet, C. & Partington, J. R. (1999), 'Bézout factors and L^1-optimal controllers for delay systems using a two-parameter compensator scheme', *IEEE Transactions on Automatic Control* **44(8)**, 1512–1521.

Bonnet, C. & Partington, J. R. (2002), 'Analysis of fractional delay systems of retarded and neutral type', *Automatica* **38(7)**, 1133–1138.

Bonnet, C. & Partington, J. R. (2007), 'Stabilization of some fractional delay systems of neutral type', *Automatica* **43**, 2047–2053.

Brethé, D. & Loiseau, J.-J. (1997), 'Stabilization of linear time-delay systems', *JESA-RAIRO-APII* **6**, 1025–1047.

Brethé, D. & Loiseau, J.-J. (1998), 'An effective algorithm for finite spectrum assignment of single-input systems with delays', *Mathematics and Computers in Simulation* **45**, 339–348.

Brockett, R. & Byrnes, C. (1981), 'Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint', *IEEE Trans. on Automatic Control* **26**(1), 271–284.

Byrd, R. H. & Schnabel, R. B. (1986), 'Continuity of the null space basis and constrained optimization', *Matemathical Programming* **35**(1), 32–41.

Cafagna, D. (2007), 'Past and present - fractional calculus: A mathematical tool from the past for present engineers', *IEEE Industrial Electronics Magazine* **1(2)**, 35–40.

Caponetto, R., Dongola, G., Fortuna, L. & Petráš, I. (2010), *Fractional Order Systems: Modeling and Control Applications*, World Scientific, Singapore.

Chen, J., Fu, P., Niculescu, S. I. & Guan, Z. (2010*a*), 'An eigenvalue perturbation approach to stability analysis,part i: Eigenvalue series of matrix operators', *SIAM Journal on Control and Optimization* **48(8)**, 5564–5582.

Chen, J., Fu, P., Niculescu, S. I. & Guan, Z. (2010*b*), 'An eigenvalue perturbation approach to stability analysis,part ii: When will zeros of time-delay systems cross imaginary axis?', *SIAM Journal on Control and Optimization* **48(8)**, 5583–5605.

Chen, J., Gu, G. & Nett, C. N. (1995), 'A new method for computing delay margins for stability of linear delay systems', *Systems & Control Letters* **26**(2), 107–117.

Chyzak, F., Quadrat, A. & Robertz, D. (2005), 'Effective algorithms for parametrizing linear control systems over Ore algebras', *Appl. Algebra Engrg. Comm. Comput.* **16**, 319–376.

Cluzeau, T. & Quadrat, A. (2008), 'Factoring and decomposing a class of linear functional systems', *Linear Algebra Appl.* **428**, 324–381. the OreMorphisms project `http://www-sop.inria.fr/members/Alban.Quadrat/OreMorphisms/index.html`.

Colaneri, P., Geromel, J. C. & Locatelli, A. (1997), *Control Theory and Design: An $RH_2 - RH_\infty$ viewpoint*, Academic Press Inc., London, UK.

Curtain, R. F. & Pritchard, A. J. (1978), *Infinite dimensional linear systems theory*, Springer-Verlag, Berlin.

Cushing, J. M. (1981), *Volterra integrodiferential equations in population dynamics*, Ligouri Editore, pp. 81–148.

Datko, R. (1978), 'A procedure for determination of the exponential stability of certain differential-difference equations', *Quarterly of Applied Mathematics* **36**, 279–292.

de Oliveira, M. C. & Geromel, J. C. (2004), 'Synthesis of non-rational controllers for linear delay systems', *Automatica* **40(2)**, 171–188.

Desoer, C. A. & Vidyasagar, M. (1975), *Feedback systems: Input-output properties*, Academic Press.

El'sgolts', L. & Norkin, S. (1973), *Introduction to the theory and applications of differential equations with deviating arguments*, Academic Press, New York.

Engelborghs, K., Luzyanina, T. & Roose, D. (2002), 'Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL', *ACM Trans. Math. Softw.* **28(1)**, 1–21.

Faure, P. & Robin, M. (1984), *Éléments d'automatique*, Dunod-Bordas, Paris.

Fioravanti, A. R. (2010), Une méthode de continuation numérique pour des systèmes complexes paramétrés, *in* '3rd Digiteo Annual Forum', Palaiseau - France.

Fioravanti, A. R., Bonnet, C. & Özbay, H. (2010), Stability of fractional neutral systems with multiple delays and poles asymptotic to the imaginary axis, *in* 'Proceedings to the 49th IEEE Conference on Decision and Control', Atlanta - USA.

Fioravanti, A. R., Bonnet, C., Özbay, H. & Niculescu, S.-I. (2010), A numerical method to find stability windows and unstable poles for linear neutral time-delay systems, *in* 'Proceedings of the 9th IFAC Workshop on Time Delay Systems', Prague - Czech Republic.

Fioravanti, A. R., Bonnet, C., Özbay, H. & Niculescu, S.-I. (2011), Stability windows and unstable root-loci for linear fractional time-delay systems, *in* 'Proceedings of the 18th IFAC World Congress', Milan - Italy. Submitted.

Fliess, M. & Mounier, H. (1994), 'Quelques propriétés structurelles des systèmes linéaires à retards constants', *Comptes Rendus de l'Académie des Sciences de Paris* **I-319**, 429–442.

Fridman, E. & Shaked, U. (2002), 'A descriptor system approach to H_∞ control of linear time-delay systems', *IEEE Trans. Automatic Control* **47**(2), 253–270.

Georgiou, T. T. & Smith, M. C. (1990), 'Optimal robustness in the gap metric', *IEEE Transactions on Automatic Control* **6**, 673–686.

Geromel, J. C. & Palhares, A. G. B. (2004), *Análise Linear de Sistemas Dinâmicos*, Edgar Blücher Ltda., São Paulo.

Geromel, J. C. & Regis Filho, L. A. V. (2005), '$\mathcal{H}_2$ optimal robust filtering', *European Journal of Control* **12**(1), 30–39.

Glader, C., Hognas, G., Mäkilä, P. & Toivonen, H. T. (1991), 'Approximation of delay systems - a case study', *International Journal of Control* **53**, 369–390.

Glüsing-Luerßen, H. (1997), 'A behavioral approach to delay-differential systems', *SIAM Journal on Control and Optimization* **35(2)**, 480–499.

Gündeş, A. N., Özbay, H. & Özgüler, A. B. (2007), 'PID controller synthesis for a class of unstable MIMO plants with I/O delays', *Automatica* **43(1)**, 135–142.

Gu, K., Kharitonov, V. & Chen, J. (2003), *Stability of time-delay systems*, Birkhauser, Boston.

Gu, K., Niculescu, S.-I. & Chen, J. (2005), 'On stability of crossing curves for general systems with two delays', *Journal of Mathematical Analysis and Applications* **311**, 231–253.

Habets, L. (1994), Algebraic and computational aspects of time-delay systems, PhD thesis, Eindhoven University of Technology.

Hale, J. K. & Lunel, S. M. V. (1993), *Introduction to Functional Differential Equations*, Springer Verlag, New York.

Hale, J. K. & Lunel, S. M. V. (2001), 'Efects of small delays on stability and control', *Operator Theory; Advances and Applications* **122**, 275–301.

Hale, J. K., Magalhães, L. T. & Oliva, W. M. (1984), *An introduction to infinite dimensional dynamical systems–geometric theory*, Springer-Verlag, New York.

Haurie, C., Dale, D. C., Rudnicki, R. & Mackey, M. C. (2000), 'Modeling complex neutrophil dynamics in the grey collie', *Journal of Theoretical Biology* **204**, 505–519.

Hearn, T., Haurie, C. & Mackey, M. C. (1998), 'Cyclical neutropenia and the peripheral control of white blood cell production', *Journal of Theoretical Biology* **192**, 226–232.

Hwang, C. & Cheng, Y. C. (2005), 'A note on the use of the Lambert W function in the stability analysis of time-delay systems', *Automatica* **41(11)**, 1979–1985.

Hwang, C. & Cheng, Y. C. (2006), 'A numerical algorithm for stability testing of fractional delay systems', *Automatica* **42(5)**, 825–831.

Insperger, T., Stepan, G. & Turi, J. (2010), 'Delayed feedback of sampled higher derivatives', *Philosophical Transactions of the Royal Society* **368**, 469–482.

Jacobs, O. L. R. (1974), *Introduction to control theory*, Oxford University Press, London.

Kamen, E. W., Khargonekar, P. P. & Tannenbaum, A. (1986), 'Proper stable bézout factorization and feedback control of linear time-delay systems', *International Journal of Control* **43**, 837–857.

Kharitonov, V. L. (1999), 'Robust stability analysis of time delay systems: a survey', *Annual Reviews in Control Engineering Practice* **23**, 185–196.

Korogui, R. H., Fioravanti, A. R. & Geromel, J. C. (2009), On a rational transfer function-based approach to $\mathcal{H}_\infty$ filter design for time-delay linear systems, *in* 'Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference', Shanghai, China, pp. 1854–1859.

Korogui, R. H., Fioravanti, A. R. & Geromel, J. C. (2010), Análise de estabilidade e controle $\mathcal{H}_\infty$ de sistemas lineares com atraso, *in* 'Proceedings of the XVIII Congresso Brasileiro de Automática (CBA2010)', Bonito - Brazil.

Korogui, R. H., Fioravanti, A. R. & Geromel, J. C. (2011*a*), '$\mathcal{H}_\infty$ control design for time-delay linear systems: a rational transfer function based approach', *IEEE Transactions on Signa* **59(3)**, 979–988. Under Revision.

Korogui, R. H., Fioravanti, A. R. & Geromel, J. C. (2011*b*), 'On a rational transfer function-based approach to $\mathcal{H}_\infty$ filtering design for time-delay linear systems', *IEEE Transactions on Signal Processing* .

Krasovskii, N. N. (1963), *Stability of motion: applications of Lyapunov's second method to differential systems and equations with delays*, Stanford University Press, Stanford CA, USA. (translation by J. Brenner).

Lam, J. (1990), 'Convergence of a class of Padé approximations for delay systems', *International Journal of Control* **52**, 989–1008.

Li, H. & Fu, M. (1997), 'A linear matrix inequality approach to robust $\mathcal{H}_\infty$ filtering', *IEEE Trans. Automatic Control* **45**(9), 2338–2350.

Li, H. & Mason, L. (2007), Synthesis of networks delays for voice packets in service overlay networks, *in* 'Proceedings of the QShine: International Conference on Heterogeneous Networking for Quality, Reliability, Security and Robustness', Vancouver, Canada.

Lin, C., Wang, Q.-G. & Lee, T. H. (2004), 'An improvement on multivariable PID controller design via iterative LMI approach', *Automatica* **40**, 519–525.

Loiseau, J.-J. & Mounier, H. (1998), ''stabilisation de l'équation de la chaleur commandée en flux', *ESAIM: Proceedings* **5**, 131–144.

Loverro, A. (2004), Fractional calculus: History, definitions and applications for the engineer, Technical report, Department of Aerospace and Mechanical Engineering, University of Notre Dame.

MacDonald, N. (1978), *Time Lags in Biological Models*, Vol. 27 of *Lecture Notes in Biomath*, Springer-Verlag, New York.

Mäkilä, P. M. & Partington, J. R. (1993), 'Robust stabilization-BIBO stability, distance notions and robustness optimization', *Automatica* **23**(3), 681–693.

Mäkilä, P. M. & Partington, J. R. (1999), 'Shift operator induced approximations of delay systems', *SIAM Journal on Control and Optimization* **37(6)**, 1897–1912.

Malek-Zavarei, M. & Jamshidt, M. (1987), *Time-Delay Systems: Analysis, Optimization and Applications*, Vol. 9, North-Holland Systems and Control Series, Amsterdam.

Marshall, J. E., Górecki, H., Walton, K. & Korytowski, A. (1992), *Time Delay Systems: stability and performance criteria with applications*, Ellis Horwood, England.

Matignon, D. (1998), 'Stability properties for generalized fractional differential systems', *ESAIM: Proceedings* **5**, 145–158.

May, R. M. (1974), *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ.

Miller, K. S. & Ross, B. (1993), *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA.

Mirkin, L. & Zhong, Q.-C. (2003), '2DOF controller parametrization for systems with a single I/O delay', *IEEE Transactions on Automatic Control* **48**, 1999–2004.

Monje, C. A., Chen, Y. & Vinagre, B. M. (2010), *Fractional-order Systems and Controls: Fundamentals and Applications*, Springer.

Morărescu, C.-I. (2006), Qualitative Analysis of Distributed Delay Systems: Methodology and Algorithms, PhD thesis, Université de Technologie de Compiègne and Universitatea din Bucure sti.

Morărescu, C.-I., Niculescu, S.-I. & Gu, K. (2007), 'Stability crossing curves of shifted gamma-distributed delay systems', *SIAM Journal on Dynamical Systems* **6**, 475–493.

Morărescu, I. C., Fioravanti, A. R., Niculescu, S.-I. & Bonnet, C. (2009), Stability crossing curves of linear systems with shifted fractional gamma-distributed delays, *in* 'Proceedings of the 8th IFAC Workshop on Time Delay Systems', Sinaia - Romania.

Mounier, H. (1995), Propriétés structurelles des systèmes linéaires à retard: aspects théoriques et pratiques, PhD thesis, Université Paris Sud, Orsay.

Mounier, H. & Fliess, M. (2004), 'An algebraic framework for infinite dimensional linear systems', *e-STA: Sciences et Technologies de l'Automatique* **1(1)**.

Niculescu, S.-I. (1997), *Systèmes à Retard: Aspects qualitatifs sur la stabilité et la stabilisation*, Diderot.

Niculescu, S.-I. (1998), '$\mathcal{H}_\infty$ memoryless control with an α-stability constraint for time-delay systems: An LMI approach', *IEEE Trans. Automatic Control* **43**(5), 739–743.

Niculescu, S.-I. (2001), *Delay effects on stability. A robust control approach*, Vol. 269, Springer-Verlag, Heidelberg.

Niculescu, S.-I. & Gu, K. (2004), *Advances in Time-Delay Systems*, Springer.

Nisbet, R. M. & Gurney, W. S. C. (1986), *The formulation of age-structure models*, Mathematical Ecology, Springer-Verlag, pp. 95–115.

Oldham, K. B. & Spanier, J. (1974), *The Fracional Calculus*, Academic Press, New York.

Olgac, N. & Sipahi, R. (2002), 'An exact method for the stability analysis of time delayed LTI systems.', *IEEE Transactions on Automatic Control* **47(5)**, 793–797.

Olgac, N. & Sipahi, R. (2004), 'A practical method for analyzing the stability of neutral type LTI-time delayed systems', *Automatica* **40**, 847–853.

Özbay, H. (2000), *Introduction to Feedback Control Theory*, CRC Press LLC, Boca Raton FL.

Özbay, H., Benjelloun, H., Bonnet, C. & Clairambault, J. (2010), Stability conditions for a system modeling cell dynamics in leukemia, *in* 'Proceedings of IFAC TDS', Prague, Czech Republic.

Özbay, H., Bonnet, C. & Fioravanti, A. R. (2009), PID controller design for unstable fractional-order systems with time delays: A small gain based approach, *in* 'Proceedings of Systems Theory: Modeling, Analysis and Control', Fes - Morocco.

Özbay, H., Bonnet, C. & Fioravanti, A. R. (2011), PID controller design for fractional-order systems with delays. Under Revision.

Özbay, H. & Gündeş, A. N. (2007), 'Resilient PI and PD controller designs for a class of unstable plants with I/O delays', *Applied and Computational Mathematics* **6**, 18–26.

Ozturk, N. & Uraz, A. (1985), 'An analysis stability test for a certain class of distributed parameter systems with delays', *IEEE Transactions on Circuits and Systems* **32(4)**, 393 – 396.

Partington, J. R. (1991), 'Approximation of unstable infinite-dimensional systems using coprime factors', *System & Control Letters* **16(2)**, 89–96.

Partington, J. R. (2004), *Linear Operators and Linear Systems: An Analytical Approach to Control Theory*, Cambridge University Press.

Partington, J. R. & Bonnet, C. (2004), 'H_∞ and BIBO stabilization of delay systems of neutral type', *System & Control Letters* **52(3–4)**, 283–288.

Partington, J. R. & Mäkilä, P. M. (1994), 'Worst-case analysis of identification–BIBO robustness for closed-loop data', *IEEE Transactions on Automatic Control* **39**, 2171–2176.

Petit, N. & Rouchon, P. (2001), 'Flatness of heavy chain systems', *SIAM Journal on Control and Optimization* **40(2)**, 475–495.

Petit, N. & Rouchon, P. (2002), 'Dynamics and solutions to some control problems for water-tank systems', *IEEE Transactions on Automatic Control* **47(4)**, 594–609.

Podlubny, I. (1999*a*), *Fractional Differential Equations*, Vol. 198 of *Mathematics in Science and Engineering*, Academic Press, USA.

Podlubny, I. (1999*b*), 'Fractional-order systems and $pi^\lambda d^\mu$-controllers', *IEEE Transactions on Automatic Control* **44**, 208–214.

Podlubny, I. (2002), 'Geometric and physical interpretation of fractional integration and fractional differentiation', *Fractional Calculus and Applied Analysis* **5(4)**, 367–386.

Podlubny, I., Despotovic, V., Skovranek, T. & McNaughton, B. H. (2007), 'Shadows on the walls: Geometric interpretation of fractional integration', *Journal of Online Mathematics and its Applications* **7**, ID 1664.

Pontryagin, L. S. (1955), 'On the zeros of some elementary transcendental functions', *American Mathematical Society Translations* **2(1)**, 95–110.

Quadrat, A. (2006*a*), 'A lattice approach to analysis and synthesis problems', *Mathematics of Control, Signals, and Systems* **18**(2), 147–186.

Quadrat, A. (2006*b*), 'On a generalization of the Youla-Kučera parametrization. Part II: The lattice approach to MIMO systems', *Mathematics of Control, Signals, and Systems* **18**(3), 199–235.

Rabah, R., Sklyar, G. M. & Rezounenko, A. V. (2005), 'Stability analysis of neutral type systems in hilbert space', *Journal of Differential Equations* **214**, 391–428.

Rabah, R., Sklyar, G. M. & Rezounenko, A. V. (2008), 'On strong regular stabilizability for linear neutral type systems', *Journal of Differential Equations* **245**, 569–593.

Razumikin, B. S. (1956), 'On the stability of systems with a delay', *Prikladnava Matematika i Mekhanika* **20**, 500–512. (in Russian).

Rekasius, Z. V. (1980), A stability test for systems with a delay, *in* 'Proceedings on joint automatic control conferences', San Francisco, CA, USA.

Richard, J. P. (2003), 'Time-delay systems: An overview of some recent advances and open problems', *Automatica* **39**, 1667–1694.

Silva, G. J., Datta, A. & Bhattacharyya, S. P. (2005), *PID Controllers for Time-Delay Systems*, Birkhäuser, Boston.

Sipahi, R. & Olgac, N. (2006), 'Stability robustness of retarded LTI systems with single delay and exhaustive determination of their imaginary spectra', *SIAM Journal on Control and Optimization* **45**, 1680–1696.

Smith, M. C. (1989), 'On stabilization and the existence of coprime factorizations', *IEEE Transactions on Automatic Control* **34(9)**, 1005–1007.

Vidyasagar, M. (1985), *Control System Synthesis: A Factorization Approach*, MIT Press, Cambridge MA.

Vyhlídal, T. & Zítek, P. (2003), Quasipolynomial mapping based rootfinder for analysis of time delay systems, *in* 'Time Delay Systems - A Proceedings volume from the 4th IFAC workshop', Rocquencourt, France, pp. 227–232.

Walton, K. & Marshall, J. E. (1987), 'Direct method for TDS stability analysis', *IEE Proceedings, Part D* **134**, 101–107.

Yamamoto, Y. (1989), 'Reachability of a class of infinite-dimensional linear systems: an external approach with applications to general neutral systems', *SIAM Journal on Control and Optimization* **27**, 217–234.

Yamamoto, Y. (1991), 'Equivalence of internal and external stability for a class of distributed systems', *Mathematics of Control, Signals, and Systems* **4**, 391–409.

Youcef-Toumi, K. & Bobbett, J. (1991), 'Stability of uncertain linear systems with time delay', *Journal of Dynamic Systems, Measurement and Control - Trans. of the ASME* **113**(4), 558–567.

Zhang, J., Knospe, C. R. & Tsiotras, P. (2003), 'New results for the analysis of linear systems with time-invariant delays', *Int. Journal Rob. Nonlin. Control* **13**(12), 1149–1175.

Zhou, K. & Doyle, J. C. (1998), *Essentials of Robust Control*, Prentice Hall, New Jersey.

Part V

Appendices

Appendix A

Résumé en Français

Introduction

Objectifs

L'objectif de cette thèse est l'étude des méthodes d'analyse et de synthèse pour les systèmes dynamiques avec retard. Le lien essentiel entre tous les thèmes est l'utilisation d'un outil de base commun, les méthodes fréquentielles. Deux raisons principales justifient ce choix. Tout d'abord, pour le problème de la synthèse, on peut aspirer à avoir des résultats avec faible conservatisme, et pour l'analyse, nous pouvons même obtenir des résultats qui sont nécessaires et suffisants. Ces faits sont difficiles à obtenir avec l'utilisation de fonctionnelles de Lyapunov-Krasovskii (Gu et al. 2003). Enfin, de nombreuses méthodes peuvent être adaptées pour traiter la classe des systèmes fractionnaires, qui est de plus en plus étudiée dans le domaine de la théorie comme de la pratique.

Présentation

De nombreux systèmes dynamiques ont des retards dans leur structure, en raison de phénomènes tels que le transport, la communication ou la propagation, mais souvent on les ignore pour des questions de simplicité. Mais ces retards peuvent entraîner une dégradation de performance, voire de l'instabilité, et donc il faut les considérer avant de concevoir des contrôleurs. Une autre source de retard provient du contrôle lui-même, avec des retards dûs aux capteurs, aux actionneurs, et aussi au temps de calcul dans les contrôleurs numériques.

Les systèmes avec retard appartiennent à la classe des systèmes de dimension infinie. Dans le domaine temporel, le cas le plus simple avec un seul retard peut être représenté par l'équation différentielle suivante:

$$\mathcal{G}_r : \begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + E w(t) \\ z(t) &= C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{A.1}$$

où $x(t) \in \mathbb{R}^n$ est l'état, $w(t) \in \mathbb{R}^m$ est l'entrée extérieure, $z(t) \in \mathbb{R}^p$ représente la sortie, $\tau > 0$ est la valeur numérique du retard et $A_0,\ A_1,\ E,\ C_0,\ C_1$ et D_z sont des matrices de dimensions appropriées. Pour la définition complète du système, avec existence et unicité de la solution, la condition initiale $x(t) = x_0(t)$ doit être fournie pour tous les $t \in [-\tau, 0)$. Dans le cas où $x_0(t) = 0$ pour tout l'intervalle $t \in [-\tau, 0]$, la transformée de Laplace de $\mathcal{G}_r$ entre l'entrée w et la sortie z est donnée par :

$$G_r(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau})^{-1} E + D_z \tag{A.2}$$

L'indice "r" sur $\mathcal{G}_r$ dénote qu'il s'agit d'un système du type *retardé*, dû au fait que, dans (A.1) l'équation différentielle ne dépend pas de $\dot{x}(t-\tau)$. Ce fait se reflète dans l'équation caractéristique de (A.2), c'est-à-dire, $\det(sI - A_0 - A_1 e^{-s\tau}) = 0$, avec le fait qu'il ne présente pas de termes du type $s^n e^{-ks\tau}$, $k \in [1, n]$. Cette classe de systèmes présente des propriétés intéressantes dans le domaine temporel et fréquentiel, comme le lissage de la réponse à l'avancée du temps et de la continuité dans le nombre de pôles instables dans le passage de la valeur du retard de $\tau = 0$ à une petite valeur positive.

Dans un autre contexte, nous avons des systèmes nommés *neutres*. En considérant à nouveau le cas avec un seul retard, nous obtenons :

$$\mathcal{G}_n : \begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + F\dot{x}(t-\tau) + E w(t) \\ z(t) &= C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{A.3}$$

et sous les mêmes hypothèses présentées pour le système retardé, sa transformée de Laplace est donnée par

$$G_n(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau} - F s e^{-s\tau})^{-1} E + D_z. \tag{A.4}$$

Nous notons que dans ces systèmes, il y a une relation de (A.4) avec $\dot{x}(t-\tau)$. Cette relation se propage dans l'équation caractéristique via l'existence d'au moins un terme $s^n e^{-ks\tau}$, $k \in [1, n]$. Contrairement aux systèmes retardés, les systèmes neutres ne possèdent pas la propriété de lissage de la réponse, et peuvent même perdre la stabilité pour tout retard positif, même si le système sans retard est stable.

Il y a encore un troisième groupe de systèmes à retard, nommés *avancés*. Sa caractéristique est d'avoir seulement des termes avec du retard dans des termes de dérivées plus élevées dans la représentation d'état, ou, ce qui revient au même, d'avoir l'élément de plus haut degré en s

multiplié par un terme exponentiel dans l'équation caractéristique. Toutefois, ces systèmes ont une application pratique limitée, et leur stabilisation est extrêmement compliquée pour toute valeur du retard.

Historiquement, la première étude complète et systématique des méthodes de systèmes à retard avec des fréquences a été faite par (Bellman & Cooke 1963). Depuis, l'étude des systèmes à retard a fortement augmenté, comme en témoigne le grand nombre de livres et d'articles scientifiques dans les décennies suivantes. Entre autres, on voudrait citer les livres (Hale & Lunel 1993), (Niculescu 2001) et (Gu et al. 2003), et les articles (Kharitonov 1999) et (Richard 2003) en guise de références importantes dans le domaine.

Chaînes de Pôles

Notre première étape dans l'analyse fréquentielle des systèmes à retard concernera le problème du comportement asymptotique des chaînes de pôles. Ces structures ont un très fort lien géométrique dans le plan complexe, et sont d'une importance primordiale pour la question de la stabilité du système. Ceci est particulièrement pertinent pour les systèmes neutres, pour lesquels il existe des situations où la stabilité $\mathcal{H}_\infty$ ne peut être obtenue même si tous les pôles sont placés dans le demi-plan gauche.

Systèmes Classiques

Dans cette partie, on va étudier les systèmes à retard dont la fonction de transfert est de la forme

$$G(s) = \frac{t(s) + \sum_{k=1}^{N'} t_k(s) e^{-ks\tau}}{p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}} = \frac{n(s)}{d(s)}, \tag{A.5}$$

où $\tau > 0$ est le retard, et t, p, q_k pour tous les $k \in \mathbb{N}_N$, et t_k pour $k \in \mathbb{N}_{N'}$, sont des polynômes réels.

Avant de donner le classement complet des chaînes de pôles, nous avons besoin de quelques résultats antérieurs (Partington 2004).

Lemme 1 *Soit $a \in \mathbb{C} \setminus \{0\}$. L'équation $se^s = a$ a un nombre infini de solutions, qui pour les* *grandes valeurs de* $|s|$ *ont la forme* $s = x + \jmath y$ *avec*

$$x = -\ln(2n\pi) + \ln(|a|) + o(1) \quad (A.6)$$

$$y = \pm 2n\pi \mp \pi/2 + \arg(a) + o(1) \quad (A.7)$$

et n *suffisamment grand.*

Les expressions asymptotiques pour la solution de $s^m e^{\lambda s} = a$, pour $m \in \mathbb{N}$ et $\lambda \neq 0$, peuvent être calculées directement en observant qu'elles peuvent être obtenues par $z = \lambda s/m$ et $ze^z = \beta\lambda/m$ pour chaque β tel que $\beta^m = a$.

Pour un système à retard donné par une équation plus complexe, ses pôles en nombre infini peuvent être classés en un certain nombre de chaînes retardées, neutres ou avancées. Pour cela, nous définissons :

Définition 2 *Soit* $d(s) = p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}$*, où* p *est un polynôme de degré* d_0 *dont le coefficient du terme de plus haut degré est donné par* c_0*, et* q_k *sont des polynômes de degré* d_k *avec le coefficient de plus haut degré* c_k*. Le diagramme de distribution de* $n(s)$ *est la courbe polygonale concave joignant les points* $P_0 = (0, d_0)$ *et* $P_N = (N, d_N)$*, avec des sommets en quelques points* $P_k = (k, d_k)$ *de telle sorte qu'aucun point de* P_k *ne reste au-dessus de la courbe.*

Chaque arête du diagramme de distribution dont la pente est positive représente des chaînes avancées. Lorsque le gradient est nul, cela implique qu'il y a des chaînes neutres, et avec la pente négative, on a des chaînes retardées. Ainsi, pour éviter les systèmes avec des chaînes avancées, nous avons la condition nécessaire et suffisante que $\deg p(s) \geq \deg q_k(s)$ pour tout $k \in \mathbb{N}_N$. En plus, s'il existe au moins un k dans $\mathbb{N}_N$ tel que $\deg p = \deg q_k$, le système a des chaînes neutres.

Nous allons maintenant mettre l'accent sur le problème de la recherche de la position asymptotique des pôles de chaînes neutres. Pour faciliter la notation, nous appellerons $z = e^{-s\tau}$. En considérant que $\deg p \geq \deg q_k$ pour tous les $k \in \mathbb{N}_N$, on peut supposer que pour chaque k

$$\frac{q_k(s)}{p(s)} = \alpha_k + \frac{\beta_k}{s} + \frac{\gamma_k}{s^2} + \mathcal{O}(s^{-3}) \qquad \text{si} \quad |s| \to \infty. \quad (A.8)$$

Le coefficient de plus haut degré en s de $p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}$ peut être écrit comme un multiple du polynôme dans la variable z :

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \quad (A.9)$$

Notre premier objectif est de trouver les positions des lignes verticales que les racines des chaînes neutres approchent asymptotiquement.

Proposition 3 *Soit $G(s)$ un système neutre défini comme en (A.5). Dans ce cas, il y a des chaînes de pôles neutres qui approchent asymptotiquement les lignes verticales*

$$\Re(s) = -\frac{\ln(|r|)}{\tau} \tag{A.10}$$

pour chaque racine $z = r$ du polynôme $\tilde{c}_d(z)$.

Le cas où toutes les racines de (A.9) sont de multiplicité un est plus simple à analyser et sera entièrement traité. Dans ce cas particulier, soit $M \leq N$ le plus grand nombre entier tel que $\alpha_M \neq 0$. Dans ce cas, il y a M chaînes de pôles neutres, et étant donné que ceux-ci sont asymptotiques aux lignes verticales, il faut tout d'abord comprendre par quel côté cette approche se passe. Comme nous le verrons ci-dessous, cette information est cruciale pour les pôles qui ont $|r| = 1$, parce que cela fournira le demi-plan dans lequel les pôles sont situés, ce qui affecte directement la question de la stabilité. Cette analyse est l'objectif du théorème suivant :

Théorème 4 *Soit $G(s)$ un système à retard neutre défini par (A.5), et supposons que toutes les racines de (A.9) sont de multiplicité un. Pour chaque r tel que $z = r$ est une racine de (A.9) et pour $n \in \mathbb{Z}$ suffisamment grand, les solutions asymptotiques à (A.10) sont données par :*

$$s_n \tau = \lambda_n + \mu_n + \mathcal{O}(n^{-2}), \tag{A.11}$$

avec λ_n donné par

$$\lambda_n = -\ln(r) + 2\jmath n\pi, \qquad n \in \mathbb{Z} \tag{A.12}$$

et

$$\mu_n = -\jmath \frac{\tau \sum_{k=1}^{N} \beta_k r^k}{2\pi n \sum_{k=1}^{N} k\alpha_k r^k}. \tag{A.13}$$

Nous pouvons tirer certaines conclusions en observant μ_n. Mais d'abord, pour chaque racine r de (A.9), on définit K_r par

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k}, \tag{A.14}$$

où K_r est bien défini parce que r, en tant que racine de (A.9), est de multiplicité un.

L'information principale dont nous avons besoin est de savoir de quel côté de cette ligne verticale se rencontrent les pôles. Pour cela, nous devons examiner $\Im(K_r)$. Comme α_k et β_k sont des nombres réels pour chaque $k \in \mathbb{N}_N$, si $\Im(r) = 0$ alors $\Im(K_r) = 0$. Cela implique que $\Im(r) \neq 0$ est une condition nécessaire (mais pas suffisante) pour $\Im(K_r) \neq 0$. Mais dans ce cas, $\overline{r}$ est aussi une racine de (A.9), et en remarquant que $\Im(K_r) = -\Im(K_{\overline{r}})$, nous obtenons que si $\Im(K_r) \neq 0$, il y aura sûrement des chaînes de pôles asymptotiques des deux côtés de la ligne verticale.

Dans le cas où $\Im(K_r) = 0$, le coefficient de plus grande valeur de μ_n est purement imaginaire, et nous avons besoin de regarder le prochain terme. C'est l'objet du théorème suivant :

Théorème 5 *Soit $G(s)$ un système à retard neutre défini par (A.5), et supposons que toutes les racines de (A.9) sont de multiplicité un. Pour chaque r tel que $z = r$ est une racine de (A.9) et pour $n \in \mathbb{Z}$ suffisamment grand, les solutions asymptotiques à (A.10) sont données par :*

$$s_n\tau = \lambda_n + \mu_n + \nu_n + \mathcal{O}(n^{-3}), \tag{A.15}$$

avec λ_n donné par (A.12), μ_n par (A.13) et

$$\nu_n = \frac{\tau^2 \sum_{k=1}^{N} \left(-k^2\alpha_k K_r^2/2 + k\beta_k K_r - \gamma_k - \beta_k \ln(r)/\tau\right) r^k}{4\pi^2 n^2 \sum_{k=1}^{N} k\alpha_k r^k}. \tag{A.16}$$

Comme auparavant, nous sommes intéressés par le signe de la partie réelle de ν_n. S'il est négatif, les pôles de grand module seront à gauche de la ligne asymptotique. S'il est positif, ils sont à droite. S'il est nul, nous avons besoin d'élargir l'analyse à des degrés plus élevés de n. Dans ce cas, il est possible que les chaînes de pôles soient situées exactement sur l'axe vertical.

Si on regarde attentivement, on verra certains aspects intéressants dans l'équation (A.16). Tout d'abord, $\beta_k \ln(r)/\tau$ peut changer le signe de $\Re(\nu_n)$ lorsque τ augmente. Ceci implique qu'il peut y avoir une valeur critique du retard $\tau^\star$ telle que pour $\tau < \tau^\star$ une chaîne tend asymptotiquement d'un côté, et pour $\tau > \tau^\star$ de l'autre côté. Cet effet, cependant, ne peut se produire lorsque l'axe asymptotique est l'axe imaginaire, comme indiqué dans la proposition suivante :

Proposition 6 *Soit $G(s)$ un système à retard neutre défini par (A.5), et supposons que toutes les racines de (A.9) sont de multiplicité un. Une variation du retard τ peut faire varier la position de l'axe asymptotique, mais ne modifie pas le demi-plan dans lequel il est situé. En outre, si l'axe asymptotique est l'axe imaginaire, il le sera pour toutes les valeurs de τ. Enfin, supposons que ν_n défini dans 5 a une partie réelle non nulle. Dans ce cas, le côté où les pôles sont situés autour de l'axe imaginaire ne varie pas avec τ.*

Notre intérêt à ce point est de répondre à la question de la stabilité de $G(s)$. On rappelle que la notion de stabilité que nous considérons est appelée la stabilité $\mathcal{H}_\infty$, à savoir le système dynamique a un gain $L_2(0,\infty)$ fini

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u \in L^2, u \neq 0} \frac{\|Gu\|_{L^2}}{\|u\|_{L^2}} < \infty. \tag{A.17}$$

Nous reprenons aussi que $\mathcal{H}_\infty(\mathbb{C}_+)$ est l'espace des fonctions analytiques et limitées dans le demi-plan ouvert droit $\mathbb{C}_+$. Nous appellerons les pôles dans le demi-plan droit fermé $\overline{\mathbb{C}_+}$ *pôles instables*, et ceux dans le demi-plan gauche ouvert $\mathbb{C}_-$ *pôles stables*.

Le cas où le polynôme (A.9) n'a que des racines de module strictement supérieur à un est plus simple à prendre en considération, parce qu'il y a un nombre $a > 0$ tel que le système a un nombre fini de pôles dans $\{\Re(s) > -a\}$. Également le cas où le polynôme (A.9) a au moins une racine de module inférieur à un est simple.

Il est un fait connu que la factorisation copremier $\mathcal{H}_\infty$ nous permet de donner la paramétrisation de l'ensemble de tous les contrôleurs stabilisateurs $\mathcal{H}_\infty$ (Bonnet & Partington 1999). Pour notre cas actuel, les facteurs copremiers et les facteurs de Bézout peuvent être déterminés d'une manière similaire à celle présentée pour les systèmes retardés (Bonnet & Partington 1999).

Proposition 7 *Soit $G(s)$ une fonction de transfert du type (A.5). Alors*

1) *Si le polynôme (A.9) a au moins une racine de module strictement inférieur à un, G ne peut pas être stable dans le sens $\mathcal{H}_\infty$.*

2) *Si le polynôme (A.9) n'a que des racines de module supérieur à un :*

   (a) *G est $\mathcal{H}_\infty$ stable si et seulement si G n'a pas de pôle instable.*

   (b) *G est stabilisable dans le sens $\mathcal{H}_\infty$. En plus, en supposant que $n(s)$ et $d(s)$ n'ont pas de zéro instable en commun, une factorisation copremier de G peut être donnée par*

$$N(s) = \frac{n(s)}{(s+1)^\delta}, \qquad D(s) = \frac{d(s)}{(s+1)^\delta}. \tag{A.18}$$

D'autre part, le cas où (A.9) a au moins une racine de module un nécessite plus d'attention, même si tous les pôles du système $G(s)$ sont situés dans $\mathbb{C}_-$. Pour la stabilité $\mathcal{H}_\infty$, il est encore nécessaire de voir si G est limité sur l'axe imaginaire (comme dans l'exemple donné dans (Partington & Bonnet 2004)).

Proposition 8 *Soit $G(s)$ une fonction de transfert donnée comme dans le théorème 5 et supposons que (A.9) a au moins une racine de module un de multiplicité un, avec toutes les autres racines de module supérieur à un.*

1. *Supposons que $\Re(\nu_n) < 0$ et que G n'a pas de pôles instables. Dans ce cas, G est stable dans le sens $\mathcal{H}_\infty$ si et seulement si* $\deg p \geq \deg t + 2$.

2. *Si $\nu_n = 0$,* $\deg p \geq \deg t + 2$ *est toujours nécessaire pour la stabilité $\mathcal{H}_\infty$.*

Avec ces résultats, nous pouvons discuter de la stabilisabilité des systèmes neutres par contrôleurs rationnels.

Proposition 9 *Soit G un système avec la fonction de transfert donnée par (A.5) tel que :*

1) le polynôme $\tilde{c}_d$ défini dans (A.9) sur la quasi-polynomiale $d(s)$ (dénominateur de $G(s)$) a au moins une racine de module inférieur ou égal à un, mais aucune racine de module égal à un n'est de multiplicité supérieure à un ;

2) $\deg p = \deg t + 1$.

Dans ce cas, G n'est pas stabilisable par un contrôleur rationnel.

Systèmes Fractionnaires

Intéressons-nous désormais à l'analyse des chaînes de pôles pour les systèmes fractionnaires de la forme :

$$G(s) = \frac{t(s^\mu) + \sum_{k=1}^{N'} t_k(s^\mu) e^{-ks\tau}}{p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}}, \tag{A.19}$$

où $\tau > 0$, $0 < \mu < 1$, et p, t_k pour tout $k \in \mathbb{N}_{N'}$ et q_k pour tous $k \in \mathbb{N}_N$ sont des polynômes réels. Comme précédemment, à partir de maintenant nous supposerons que ce système est neutre.

Le comportement asymptotique ne diffère pas beaucoup des résultats présentés ci-dessus. Nous avons calculé que pour chaque k

$$\frac{q_k(s^\mu)}{p(s^\mu)} = \alpha_k + \frac{\beta_k}{s^\mu} + \mathcal{O}(s^{-2\mu}) \qquad \text{quand} \quad |s| \to \infty. \tag{A.20}$$

Le coefficient de plus haut degré de $p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu)e^{-ks\tau}$ peut être écrit comme un multiple du même polynôme en z

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \tag{A.21}$$

Par conséquent, les positions des lignes verticales pour lesquelles les racines du système neutre sont asymptotiques sont les mêmes que celles données dans la proposition 3. Prenons à nouveau le cas où toutes les racines de (A.21) sont de multiplicité un.

Théorème 10 *Soit $G(s)$ un système à retard neutre défini par (A.19), et supposons que toutes les racines de (A.21) sont de multiplicité un. Pour chaque r tel que $z = r$ est une racine de (A.21) et pour $n \in \mathbb{Z}$ suffisamment grand, les solutions asymptotiques à (A.10) sont données par :*

$$s_n h = \lambda_n + \delta_n + \mathcal{O}(n^{-2\mu}) \tag{A.22}$$

avec λ_n donné par (A.12) et

$$\delta_n = \frac{\tau^\mu \sum_{k=1}^{N} \beta_k r^k}{(2\jmath\pi n)^\mu \sum_{k=1}^{N} k\alpha_k r^k} \tag{A.23}$$

Certaines conclusions sont obtenues par l'observation de δ_n. Mais d'abord, associé à chaque r de (A.21), nous définissons K_r comme avant :

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k}\ . \tag{A.24}$$

Encore une fois, l'information principale dont nous avons besoin est de savoir de quel côté de cette ligne verticale se rencontrent les pôles, ce qui revient à trouver le signe de $\Re(\delta_n)$ pour n suffisamment grand. Il suffit de regarder K_r pour obtenir cette information. Dans le cas fractionnaire, les racines complexes conjuguées de (A.21) ne donnent pas nécessairement δ_n complexes conjugués, ce qui implique que chaque δ_n associé à un r n'est pas le complexes conjugué du δ_n associé à $\overline{r}$. C'est-à-dire, on peut conclure que le système a tous les pôles dans un seul demi-plan avec une approximation de cet ordre.

Corollaire 11 *Soit $0 < \mu < 1$, δ_n donné par (A.23) et K_r par (A.24). Dans ce cas, sign$(\Re(\delta_n)) < 0$ pour tout $n \in \mathbb{Z}$ si et seulement si*

$$\Re(K_r) < -\tan\left(\frac{\mu\pi}{2}\right)|\Im(K_r)| \tag{A.25}$$

Certains aspects importants peuvent être obtenus de ce corollaire . Tout d'abord, la valeur numérique du retard ne figure pas explicitement dans l'équation (A.25), ce qui implique que pour chaque $\tau > 0$, les chaînes de pôles ont un comportement unique par rapport au retard, dans le sens qu'elles ne changent pas de côté par rapport à l'axe vertical en question. En plus, comme (A.25) n'utilise que la valeur absolue de $\Im(K_r)$, les résultats sont similaires si nous traitons le complexe conjugué de K_r. Mais en fait, les racines complexes conjuguées de $c(z)$ dans l'équation (A.21) donnent K_r complexe conjugué.

Par conséquent, contrairement au cas $\mu = 1$, il est possible que tous les pôles soient à gauche de l'axe vertical (A.10) en utilisant une approximation de tel ordre. De plus, comme α_k et β_k sont indépendants de μ, nous présentons aussi le résultat suivant :

Corollaire 12 *Soient $0 < \mu < 1$ et δ_n donnés par (A.23) et leur K_r par (A.24). Si $\Re(K_r) < 0$, tous les pôles de sa chaîne seront asymptotiques vers la gauche de la ligne verticale (A.10) si*

$$\mu < \frac{2}{\pi} \arctan\left(-\frac{\Re(K_r)}{|\Im(K_r)|}\right). \tag{A.26}$$

Avec ces résultats, nous pouvons traiter le problème de la stabilité $\mathcal{H}_\infty$ de cette classe de systèmes. Le cas où le polynôme (A.21) n'a que des racines de module strictement supérieur à un est plus simple à prendre en considération, parce qu'il y a un nombre $a > 0$ tel que le système a un nombre fini de pôles dans $\{\Re(s) > -a\}$. Également le cas où le polynôme (A.21) a au moins une racine de module inférieur à un est simple, car il existe un nombre infini de pôles instables. Pour le cas de la transition, nous avons :

Proposition 13 *Soit $G(s)$ une fonction de transfert donnée par (A.19) et supposons que (A.21) a au moins une racine de module un et aucune racine de module moins grand que un.*

1. *Si $\Re(\delta_n) < 0$ pour tout $n \in \mathbb{Z}$ et G n'a pas de pôles instables, alors G est stable dans le sens $\mathcal{H}_\infty$ si et seulement si $\deg p \geq \deg t + 1$.*
2. *Si $\Re(\delta_n) = 0$, la condition $\deg p \geq \deg t + 1$ est toujours nécessaire pour la stabilité $\mathcal{H}_\infty$.*

Fenêtres de Stabilité et *Lieu des Racines*

Introduction

Jusqu'à présent, on a étudié seulement le comportement asymptotique des pôles, quand $|s|$ tend vers l'infini. Évidemment, il y a des pôles de *petit module*, qui n'appartiennent à aucune chaîne, et qui sont importants pour le comportement du système. L'étude de ces pôles et de leur comportement quand τ varie est l'objet de la présente partie.

La façon la plus directe de trouver la position de ces pôles serait de résoudre l'équation caractéristique correspondante. Mais comme il s'agit d'une équation transcendante, avec un nombre infini de racines, nous ne pouvons pas les obtenir directement. C'est pour cette raison que la plupart des procédures étudient la stabilité de ces systèmes pour trouver les moments où les pôles traversent l'axe imaginaire. Cela est dû à deux propriétés importantes des systèmes à retard. Tout d'abord, nous avons *l'argument de continuité des racines*, qui implique que pour toute valeur positive du retard, la position des pôles varie de façon continue par rapport au retard, ce qui implique que toute racine qui change de demi-plan a besoin de passer par l'axe imaginaire. La seconde propriété est due à l'invariance de la tendance des intersections de racines (Olgac & Sipahi 2002). Ensemble, ces propriétés impliquent qu'un certain nombre d'ensembles de racines peuvent fournir les informations nécessaires pour la caractérisation complète de la stabilité du système. Plusieurs algorithmes ont été proposés par l'exploitation de ces propriétés. D'autre part, pour le cas fractionné, le nombre d'outils disponibles n'est pas si grand. Même si certains algorithmes peuvent être adaptés à ce cas (Walton & Marshall 1987), ils ont un handicap qui devient plus prononcé dans le cas fractionnaire.

Notre objectif est de développer une technique pour étudier la connexion entre la valeur du retard et la stabilité du système, connu sous le nom de "fenêtres de la stabilité". Avec ces résultats, dans plusieurs cas, nous pouvons localiser la position exacte de tous les pôles instables du système.

Systèmes Classiques

On considère les systèmes à retard avec l'équation caractéristique :

$$C(s,\tau) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}, \tag{A.27}$$

où $\tau > 0$, et p et q_k, pour tout $k \in \mathbb{N}_N$, sont des polynômes à coefficients réels satisfaisant $\deg p \geq \deg q_k$. Nous supposons que $p(s)$ et $q_k(s)$ pour tout $k \in \mathbb{N}_N$ n'ont pas de zéro commun.

Nous commençons par examiner $\tau = 0$ dans (A.27). Dans ce cas, nous avons un polynôme à coefficients réels, dont les racines peuvent être facilement calculées. Ensuite, pour le cas neutre, nous avons besoin de l'analyse développée dans la section précédente. Comme ce n'est pas l'objet de cette section, nous supposons que toutes les chaînes de pôles se situent dans le demi-plan gauche étendu.

Pour trouver la position dans l'axe imaginaire où les croisements ont lieu, nous allons effectuer une transformation de variables qui découple les polynômes et la partie exponentielle. L'idée est de remplacer $e^{-\jmath\omega\tau k}$ par $e^{-\jmath\theta k}$ et de trouver les racines du polynôme complexe obtenu en focntion de $\theta \in [0, \pi]$. En d'autres termes, on va trouver toutes les racines du polynôme en s

$$\widetilde{C}(s, \theta) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-\jmath k\theta}, \tag{A.28}$$

en faisant varier θ dans l'intervalle fermé $[0, \pi]$.

On peut noter que pour $\theta \in [0, \pi]$, $\widetilde{C}$ est un polynôme à coefficients complexes, mais sans retard. Cela implique que, en raison de la transformation de variable, une solution $s^\star$ de $\widetilde{C}$ ne signifie pas que $\overline{s^\star}$ soit une autre solution. Par aileurs, toutes les solutions de (A.27) sur l'axe imaginaire se produisent en paires complexes conjuguées. Cela nous amène à notre résultat principal :

Théorème 14 *Soit Ω l'ensemble des couples (ω, θ), avec $\omega \in \mathbb{R}$ et $\theta \in [0, \pi]$ tel que $\widetilde{C}(\jmath\omega, \theta) = 0$. Soit*

$$\tau(\omega, \theta, \ell) = \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega} \tag{A.29}$$

pour tout $(\omega, \theta) \in \Omega$. On va choisir $\ell = \{0, 1, \ldots\}$ si $\omega > 0$, et $\ell = \{-1, -2, \ldots\}$ si $\omega < 0$. Soit Δ défini comme l'ensemble de tous les couples $(\pm\jmath\omega, \tau(\omega, \theta, \ell))$. Donc Δ est l'ensemble complet des racines de (A.27) sur l'axe imaginaire pour tout $\tau > 0$.

On note que si le système (A.27) est stable pour tout $\tau = 0$ et $\Omega = \emptyset$, alors le système est stable pour toutes les valeurs positives de τ. L'ensemble Ω ainsi que les tendances des racines est ce qu'on appelle un *groupe de racines*. Cette tendance est constante par rapport à tous les croisement de (A.29).

Pour utiliser les résultats du théorème 14, nous devons trouver tous les $\omega \in \mathbb{R}$ et $\theta \in [0, \pi]$ tel que $\widetilde{C}(\jmath\omega, \theta) = 0$. Pour cela, nous proposons deux méthodes. La plus directe consiste à échantillonner θ dans son intervalle $[0, \pi]$, et pour chaque $\theta^\star$ calculer les racines du polynôme résultant $\widetilde{C}(s, \theta^\star) = 0$. On note que le degré du polynôme est celui de $p(s)$, et que l'argument de la continuité est valide pour $\widetilde{C}(s, \theta)$.

La seconde méthode vient du fait que, si $(s,\ \theta)$ est une racine simple de (A.28), une petite perturbation dans $\theta^\star = \theta + \epsilon$ fournira une solution $\widetilde{C}(s^\star, \theta^\star) = 0$ de la forme $s^\star = s + \sum_{k=1}^{\infty} \lambda_k \epsilon^k$ où

$$\lambda_1 = \jmath \frac{\sum_{k=1}^{N} k q_k(s) e^{-\jmath\theta k}}{p'(s) + \sum_{k=1}^{N} q_k'(s) e^{-\jmath\theta k}} = T(s, \theta). \tag{A.30}$$

Ainsi, la stratégie consiste à intégrer numériquement $T(s, \theta)$ pour θ de 0 à π et en prenant comme valeurs initiales des solutions de $\widetilde{C}(s, 0) = 0$, qui sont exactement les solutions de $C(s, 0) = 0$ déjà calculées pour le test de stabilité à $\tau = 0$.

Le prochain objectif est de découvrir le sens de chacun des croisements sur l'axe imaginaire. Comme (Olgac & Sipahi 2002) le montre, cette tendance est constante par rapport à chaque nouveau passage au même endroit (ℓ dans (A.29)). Nous pouvons résoudre ce problème d'une manière similaire à celle faite plus haut. En supposant que $(s,\ \tau)$ est une racine simple de $C(s,\tau) = 0$, pour une petite variation de $\tau^\star = \tau + \epsilon$, la solution de $C(s^\star, \tau^\star) = 0$ a la forme $s^\star = s + \sum_{k=1}^{\infty} \mu_k \epsilon^k$, où

$$\mu_1 = s \frac{\sum_{k=1}^{N} k q_k(s) e^{-\tau s k}}{p'(s) + \sum_{k=1}^{N} (q'_k(s) - \tau k q_k(s)) e^{-\tau s k}} = V(s,\tau). \tag{A.31}$$

La tendance est donnée par $\mathrm{sign}(\Re(V(j\omega,\tau)))$, où $(j\omega,\tau) \in \Delta$. Si elle est positive, nous avons un croisement déstabilisant, sinon, si elle est négative, nous avons un croisement stabilisant. Dans le cas nul, nous devons poursuivre avec une analyse d'ordre supérieur. Sachant cela, déterminer pour chaque valeur du retard si un système a des pôles instables est juste une question de comptage.

Si nous voulons non seulement calculer les valeurs de τ telles que le système soit stable, mais aussi trouver la position réelle des pôles instables, nous pouvons adapter les techniques utilisées jusqu'à maintenant. À partir de la définition de Ω, on peut calculer un sous-ensemble $\Delta_{\tau^\star}$ ne contenant que les éléments de Δ avec les valeurs de retard inférieur à $\tau^\star$ et des tendances positives. On intègre $V(s,\tau)$ pour chaque élément $(s^\circ, \tau^\circ) \in \Delta_{\tau^\star}$ dans $\tau \in [\tau^\circ, \tau^\star]$ et avec s° comme point de départ, et aussi pour les pôles instables du système avec un retard nul dans $\tau \in [0, \tau^\star]$. Dans ce cas, nous avons le lieu des racines. Cela donne les positions de tous les pôles instables de (A.27).

Systèmes Fractionnaires

Tous les résultats présentés pour les systèmes classiques peuvent être adaptés pour les systèmes fractionnaires, en apportant les modifications nécessaires. Une nouvelle transformation doit être appliquée pour que nous traitions le fait que les pôles peuvent changer de surface de Riemann en faisant varier le retard. Toutefois, en général, et avec des techniques similaires, nous pouvons atteindre le même degré d'analyse pour les systèmes fractionnaires.

Courbes de Stabilité pour des Systèmes avec Retard γ-Distribué

Jusqu'ici, nous nous sommes concentrés sur l'analyse des systèmes avec des retards discrets. Mais il est important de savoir que durant les dernières décennies, plusieurs auteurs ont travaillé avec l'idée que dans certaines applications, l'utilisation des retards distribués conduit à des modèles plus réalistes. Ces pionniers ont commencé une évolution majeure dans la théorie des systèmes à retard, en particulier dans la question de l'étude du noyau du retard, généralement représenté par des mesures de probabilité dans le temps.

Cela dit, le but de cette partie du travail est de caractériser les *courbes de stabilité*, c'est-à-dire, l'ensemble des paramètres tels qu'il y a au moins une paire de racines sur l'axe imaginaire. Ces courbes divisent l'espace des paramètres $\mathbb{R}_+^2$ en différentes régions, avec la propriété que dans chacune d'entre elles, le nombre de racines instable est constant.

Définition du problème

Considérons le problème de l'analyse de la stabilité d'une classe d'équations différentielles à retard qui peut être décrite dans le domaine fréquentiel par l'équation caractéristique:

$$H(s, \alpha, \tau) = Q(s) + P(s)\frac{1}{(s+\alpha)^{m/n}}e^{-s\tau} \tag{A.32}$$

où P et Q sont des polynômes avec $\deg P - m/n < \deg Q$, $m, n \in \mathbb{Z}$ et α, τ sont des paramètres réels positifs qui contrôlent le comportement du système. On note que la stabilité $\mathcal{H}_\infty$ de tels systèmes est équivalente à la condition de ne pas avoir de pôle dans le demi-plan droit. Nous pouvons également établir la dépendance continue des racines de la fonction caractéristique par rapport aux paramètres α e τ, même pour le cas fractionnaire. Par conséquent, l'analyse de la stabilité se résume à:

- Initialement, détecter tous les croisements sur l'axe imaginaire. En d'autres termes, calculer *l'ensemble des fréquences de croisement*, noté Ω, correspondqnt à toutes les fréquences positives tel qu'il existe au moins une racine caractéristique critique.

- Puis, décrire le comportement des racines critiques en raison de changements dans l'espace des paramètres (α, τ). Plus précisément, nous détectons les points critiques.

- Enfin, un autre concept lié est représenté par *les courbes de croisements de stabilité*, composé de toutes les paires (α, τ) tel qu'il existe au moins une valeur de $\omega \in \Omega$ avec

$H(\jmath\omega, T, \tau) = 0$.

Caractérisation des Courbes de Stabilité

Avant tout développement, nous allons définir quelques termes qui seront utilisés après :

Définition 15 *Nous notons $\mathcal{T}$ l'ensemble des $(\alpha, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ tel que* (A.32) *a au moins un zéro dans $\jmath\mathbb{R}$. Tout $(\alpha, \tau) \in \mathcal{T}$ sera appelé* point de croisement*. L'ensemble $\mathcal{T}$, qui représente la collection de tous les points de croisement, est appelé* courbes de croisement de stabilité *ou* courbes de stabilité*.*

Définition 16 *Nous notons Ω la collection de tous les $\omega > 0$ tel qu'il existe un élément (α, τ) pour lequel $H(\jmath\omega, \alpha, \tau) = 0$. L'ensemble Ω est appelé* l'ensemble de croisement*.*

Proposition 17 *Pour $\omega \in \Omega$, les points de croisement correspondants sont donnés par :*

$$\alpha = \sqrt{\left|\frac{P(j\omega)}{Q(j\omega)}\right|^{2n/m} - \omega^2}, \tag{A.33}$$

$$\tau = \frac{1}{\omega}\Big((2p+1)\pi + \angle P(j\omega) - \angle Q(j\omega) - \frac{m}{n}\arctan\left(\omega\alpha^{-1}\right)\Big), \tag{A.34}$$

où $p \in \mathbb{Z}$.

Nous considérons que $\Omega = \bigcup_{k=1}^{N} \Omega_k$ et on note $\mathcal{T}_k^p$ la courbe correspondant à Ω_k avec $p \in \mathbb{Z}$. Soient les extrémités de gauche et de droite de l'intervalle Ω_k notés respectivement ω_k^ℓ et ω_k^r. Chaque point ω_k^ℓ ou ω_k^r doit appartenir à un et seulement à un des trois types suivants :

Type 1a. Répond à l'équation $Q(j\omega) = 0$ et $m/n > 0$.

Type 1b. Répond à l'équation $P(j\omega) = 0$ et $m/n < 0$.

Type 2. Répond à l'équation

$$|\omega|^{m/n} = \left|\frac{P(j\omega)}{Q(j\omega)}\right|. \tag{A.35}$$

Type 3. Est égal à 0.

Nous appelons ω_0 un point limite quelconque, qui peut représenter soit le point extrême gauche soit le point extrême droit d'un intervalle Ω_k. On dit qu'un intervalle Ω_k est de type ℓ/r si le point extrême gauche est de type ℓ et celui de droite est de type r. Par conséquent, nous pouvons diviser tous les intervalles en 12 types, 6 pour $m/n > 0$ et 6 pour $m/n < 0$.

Lissage dans les courbes et directions de croisement

Nous considérons que l'ensemble des fréquences de croisement Ω est donné et que les courbes de croisement de stabilité sont décrites par des mappages lisses $\omega \mapsto \alpha(\omega)$, $\omega \mapsto \tau(\omega)$. Nous allons noter $\mathcal{T}_h$ l'une de ces courbes et considérer la décomposition suivante en parties réelles et imaginaires :

$$R_0 + \jmath I_0 = \jmath \frac{\partial H(s,\alpha,\tau)}{\partial s}\bigg|_{s=\jmath\omega},\ R_1 + \jmath I_1 = \frac{\partial H(s,\alpha,\tau)}{\partial \alpha}\bigg|_{s=\jmath\omega},\ R_2 + \jmath I_2 = \frac{\partial H(s,\alpha,\tau)}{\partial \tau}\bigg|_{s=\jmath\omega}.$$

Dans ce cas, le théorème des fonctions implicites dit que la tangente à $\mathcal{T}_h$ peut être décrite par :

$$\begin{aligned} \begin{pmatrix} \dfrac{\mathrm{d}\alpha}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix}_{s=\jmath\omega} &= \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} \\ &= \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - R_2 I_0 \\ R_1 I_0 - R_0 I_1 \end{pmatrix}, \end{aligned} \tag{A.36}$$

dans le cas où $R_1 I_2 - R_2 I_1 \neq 0$.

Nous concluons que $\mathcal{T}_h$ est lisse en tout point sauf aux points où $R_1 I_2 - R_2 I_1 \neq 0$ n'est pas remplie ou lorsque

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{A.37}$$

Pour établir la direction des croisements, nous devons considérer α et τ comme des fonctions de $s = \sigma + \jmath\omega$, et donc la notation en dérivées partielles doit être adoptée. Lorsque (α, τ) croise les courbes de stabilité de droite à gauche, une paire de solutions complexes conjuguées traverse l'axe imaginaire en direction du demi-plan droit, si

$$\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial\tau}{\partial\sigma} - \frac{\partial\tau}{\partial\omega}\frac{\partial\alpha}{\partial\sigma}\right)_{s=\jmath\omega} > 0. \tag{A.38}$$

L'inverse se produit si l'inégalité est inversée. Nous concluons donc avec la proposition suivante :

Proposition 18 *Supposons que $\omega \in \Omega$, et que $s = \jmath\omega$ est une racine simple de (A.32) et $H(j\omega', \alpha, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$. Lorsque (α, τ) se déplace d'une région à droite vers une région à gauche de la courbe de croisement correspondante, une paire de solutions de (A.32) traverse l'axe imaginaire vers le demi-plan droit si $R_1 I_2 - R_2 I_1 > 0$. Le sens de la traversée est inversée si l'inégalité l'est.*

Synthèse de contrôleurs PID

Les contrôleurs PID sont les plus utilisés dans l'industrie car ils sont facilement réalisables et réglables, avec succès pour un grand nombre d'applications. Plusieurs techniques proposent des contrôleurs PID pouvant même traiter des systèmes à retard.

Quand on parle de systèmes fractionnaires dans ce contexte, deux voies peuvent être suivies. La première concerne les contrôleurs fractionnaires, connus sous le nom de $PI^\alpha D^\beta$. La deuxième traite des contrôleurs PID classiques pour les systèmes fractionnaires, et cela est le point que nous allons étudier.

Définition du problème

Nous allons considérer le problème de retour de sortie SISO où C est le contrôleur que l'on va concevoir pour le système P. Nous supposons que le système est linéaire et invariant dans le temps, avec son comportement représenté par la fonction de transfert suivante :

$$P(s) = e^{-s\tau} \frac{G(s^\mu)}{s^\mu - p} \tag{A.39}$$

où $\tau > 0$ est le retard total entre l'entrée et la sortie, $\mu \in (0,\ 1)$ est l'ordre fractionnaire, $p^{1/\mu} \geq 0$ est le lieu du pôle instable du système, et $G(\sigma)$ est une fonction de transfert rationnelle stable dans la variable $\sigma = s^\mu$ avec $G(p) \neq 0$ et $G(0) \neq 0$.

Compte tenu de tous les paramètres du système, notre objectif est de concevoir un régulateur PID dans la forme classique

$$C(s) = K_p + \frac{K_i}{s} + K_d \frac{s}{\tau_d s + 1} \tag{A.40}$$

où $K_p,\ K_i,\ K_d$ sont des paramètres libres et τ_d est une valeur arbitraire, petite et positive utilisée pour la réalisation propre de la loi de commande. Le système en boucle fermée formée par le contrôleur C et le système P est stable si $(1 + PC)^{-1},\ C(1 + PC)^{-1},\ P(1 + PC)^{-1}$ sont des fonctions de transfert stables.

La stratégie adoptée sera en deux étapes : d'abord, les contrôleurs PD seront étudiés et ensuite on ajoutera l'action intégrale.

Synthèse de contrôleurs PD

Un contrôleur de type PD a la forme

$$C_{pd}(s) = K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{A.41}$$

Nous pouvons écrire la partie sans retard du système comme le rapport de deux facteurs stables

$$P(s) = e^{-s\tau} Y(s)^{-1} X(s) \quad \text{avec} \quad Y(s) := \frac{s^\mu - p}{s^\mu + x} \quad X(s) := \frac{G(s^\mu)}{s^\mu + x} \tag{A.42}$$

où $x > 0$ est un paramètre libre. La stabilité du système en boucle fermée est équivalente à la stabilité de U^{-1}, où

$$U(s) := Y(s) + e^{-s\tau} X(s) C_{pd}(s). \tag{A.43}$$

En injectant C_{pd}, X et Y dans (A.43), nous obtenons

$$U(s) = 1 - \frac{p+x}{s^\mu + x} + e^{-s\tau} \frac{G(s^\mu)}{s^\mu + x} K_p \left(1 + \widetilde{K}_d \frac{s}{\tau_d s + 1}\right). \tag{A.44}$$

On choisit $K_p = (p+x)G(0)^{-1}$, et donc

$$U(s) = 1 - \frac{(p+x)s^\mu}{s^\mu + x} \left(\frac{1 - e^{-s\tau} G(s^\mu) G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau} G(s^\mu) G(0)^{-1} \frac{s^{1-\alpha}}{\tau_d s + 1} \right). \tag{A.45}$$

Comme $\left\| \frac{s^\mu}{s^\mu + x} \right\|_\infty = 1$ pour tout $x > 0$, par le théorème du petit gain, U^{-1} est stable si

$$\left\| \frac{1 - e^{-s\tau} G(s^\mu) G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau} G(s^\mu) G(0)^{-1} \frac{s^{1-\mu}}{\tau_d s + 1} \right\|_\infty < \frac{1}{p+x}. \tag{A.46}$$

À partir de là, on trouve directement les conditions pour les contrôleurs P et PD ($\widetilde{K}_d = 0$). Pour ce dernier cas, nous pouvons voir que les valeurs possibles pour le gain proportionnel sont dans la gamme

$$K_p^{\min} := pG(0)^{-1} < K_p < \psi_d G(0)^{-1} =: K_p^{\max}. \tag{A.47}$$

Nous tenons à maximiser ψ_d afin de maximiser la gamme disponible pour K_p. Ce problème est équivalent à trouver le $\widetilde{K}_d \in \mathbb{R}$ optimal pour que

$$\psi_d^{-1} = \left\| \frac{1 - e^{-s\tau} G(s^\mu) G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau} G(s^\mu) G(0)^{-1} \frac{s^{1-\mu}}{\tau_d s + 1} \right\|_\infty \tag{A.48}$$

soit minimisé pour $\tau_d > 0$ fixé, ce qui ne peut être résolu qu'avec une recherche en deux dimensions.

Avec ψ_d maximisée, le choix de K_p pour que la marge de gain soit maximisée, c'est-à-dire,

$$\min \left\{ \frac{K_p}{K_p^{\min}}, \frac{K_p^{\max}}{K_p} \right\} \tag{A.49}$$

maximum, peut être résolu avec $K_p^{\mathrm{opt}} = \sqrt{K_p^{\min} K_p^{\max}}$.

L'ajout de l'action intégrale

Supposons qu'un régulateur PD stabilisateur puisse être trouvé pour le système (A.39). Nous allons essayer de trouver maintenant

$$C_i(s) = K_i s^{-1} \tag{A.50}$$

tel que $C_{pid}(s) = C_{pd}(s) + C_i(s)$ soit un contrôleur stabilisateur. Pour cela, on définit

$$H(s) := P(s)(1 + P(s)C_{pd}(s))^{-1} \tag{A.51}$$

et on observe que $H(0) = G(0)/x$ est non nul. Si C_i défini dans (A.50) est un contrôleur stabilisateur pour le "nouveau système" H (A.51), C_{pid} est un contrôleur stabilisateur pour le système original P. En supposant

$$K_i := \gamma H(0)^{-1}, \quad \text{avec} \quad \gamma > 0 \tag{A.52}$$

nous avons

$$(1 + C_i(s)H(s))^{-1} = \frac{s}{s+\gamma}\left(1 + \frac{\gamma s^\alpha}{s+\gamma}\left(\frac{H(s)H(0)^{-1} - 1}{s^\alpha}\right)\right)^{-1}. \tag{A.53}$$

On définit

$$R_\alpha(\gamma) := \left\|\frac{\gamma s^\alpha}{s+\gamma}\right\|_\infty, \tag{A.54}$$

et on obtient que, par le théorème du petit gain, $C_i(s) = \gamma H(0)^{-1}/s$ est un contrôleur stabilisateur de $H(s)$ si

$$0 < R_\alpha(\gamma) < \left\|\frac{H(s)H(0)^{-1} - 1}{s^\alpha}\right\|_\infty^{-1}. \tag{A.55}$$

Lorsque $H(s)$ est une fonction de transfert fractionnaire, $(H(s)H(0)^{-1} - 1)/s$ peut être instable en raison des questions sur la limite en zéro. Nous avons donc besoin de calculer $R_\alpha(\gamma)$ en fonction de γ pour la valeur de α que le système présente.

Lemme 19 *La valeur de $R_\alpha(\gamma)$ défini dans (A.54) est donnée par :*

$$R_\alpha(\gamma) = \alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}\gamma^\alpha. \tag{A.56}$$

On peut se demander quel est le meilleur choix pour $x > 0$, de telle sorte que l'intervalle des valeurs possibles de γ soit aussi grand que possible. Par exemple, pour le régulateur PI, pour chaque valeur fixe de x dans la gamme $0 < x < (\psi_o - p)$, on peut calculer cet intervalle numériquement. Par conséquent, l'intervalle le plus grand de γ et la valeur optimale correspondante x peuvent être trouvés avec une recherche numérique unidimensionnelle.

Cependant, nous pouvons trouver une solution analytique sous-optimale, en raison de la limite inférieure suivante :

$$\widetilde{\gamma} := x\frac{\psi_o - (p+x)}{\psi_o + p} \leq \left\| \frac{H(s)H(0)^{-1} - 1}{s^{\mu}} \right\|_{\infty}^{-1}. \tag{A.57}$$

Nous devons maximiser $\widetilde{\gamma}$ avec le choix approprié de x. Sa valeur optimale est

$$x_{\text{opt}} = \frac{\psi_o - p}{2} \tag{A.58}$$

pour $\widetilde{\gamma}$ correspondant

$$\widetilde{\gamma}_{\text{opt}} = \frac{x_{\text{opt}}^2}{\psi_o + p}. \tag{A.59}$$

Ce qui implique que γ sera dans l'invervalle

$$0 < \gamma < \frac{c_\alpha x_{\text{opt}}^{2/\alpha}}{(\psi_o + p)^{1/\alpha}} =: \gamma_{\max} \quad \text{où} \quad c_\alpha := \left(\sqrt{\alpha}\ (1-\alpha)^{(1-\alpha)/2\alpha} \right)^{-1}. \tag{A.60}$$

Système Rationnel de Comparaison

Dans cette section, nous définissons un système LTI qui servira de système de comparaison pour l'analyse de stabilité et le calcul de la norme $\mathcal{H}_\infty$ pour un système avec retard, afin d'être utilisé pour la conception de filtres et de contrôleurs, de la forme

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + E_0 w(t) \tag{A.61}$$

$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau) \tag{A.62}$$

où $x(t) \in \mathbb{R}^n$ représente l'état, $w(t) \in \mathbb{R}^r$ est l'entrée externe et $z(t) \in \mathbb{R}^q$ la sortie. Nous supposons que le système évolue à partir du zéro et que le retard $\tau \geq 0$ est une constante.

Il est connu par *substitution de Rekasius* que, pour $s =$ *jmath omega* avec $\omega \in \mathbb{R}$, l'égalité

$$e^{-s\tau} = \frac{1 - \lambda^{-1}s}{1 + \lambda^{-1}s} \tag{A.63}$$

est satisfaite pour $\lambda \in \mathbb{R}$ tel que $\omega/\lambda = \tan(\omega\tau/2)$. Ainsi, nous pouvons introduire un système de comparaison pour (A.61)-(A.62) dans la forme :

$$H(\lambda, s) = \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ A_0 + A_1 & A_0 - A_1 - \lambda I & E_0 \\ \hline C_{z0} + C_{z1} & C_{z0} - C_{z1} & 0 \end{array}\right] \tag{A.64}$$

On appelle $T(\tau, s)$ la fonction de transfert non-rationnelle du système avec retard entre w et z. Ce système LTI a été déterminé de telle sorte que l'égalité $H(\lambda, \jmath\omega) = T(\tau, \jmath\omega)$ soit satisfaite lorsque les constantes $\lambda \in \mathbb{R}$, $\tau \geq 0$ et la fréquence $\omega \in \mathbb{R}$ sont liées par la relation $\omega/\lambda = \tan(\omega\tau/2)$.

Analyse de stabilité

Nous voulons déterminer le plus grand retard $\tau^* > 0$ tel que le système (A.61)-(A.62) soit stable pour tout $\tau \in [0, \tau^*)$. Cela dépend du calcul des racines de l'équation caractéristique de $T(\tau, s)$

$$\Delta_T(\tau, s) = \det\Big(sI - A_0 - A_1 e^{-s\tau}\Big) \tag{A.65}$$

Toutefois, ici, on peut la réécrire d'une autre façon

$$\Psi(\tau, s) = \frac{\Delta_T(\tau, s)}{\Delta_T(0, s)} = \det\Big(I + (sI - A_0 - A_1)^{-1} A_1 (1 - e^{-s\tau})\Big) \tag{A.66}$$

et on peut appliquer le critère de Nyquist dans ce système. La même stratégie peut être adoptée pour l'équation caractéristique du système de comparaison $\Delta_H(\lambda, s) = 0$, où

$$\Delta_H(\lambda, s) = \det\Big(sI - A_\lambda\Big) \tag{A.67}$$

est un polynôme avec $2n$ racines. On peut la factoriser ainsi

$$\Phi(\lambda, s) = \frac{\Delta_H(\lambda, s)}{(s + \lambda)^n \det(sI - A_0 - A_1)} = \det\left(I + (sI - A_0 - A_1)^{-1} A_1 \frac{2s}{s + \lambda}\right) \tag{A.68}$$

qui satisfait l'égalité $\Psi(\tau, \jmath\omega) = \Phi(\lambda, \jmath\omega)$, dans le cas où $e^{-s\tau}$ est remplacé par une relation définie par la substitution Rekasius (A.63), valable pour $s = \jmath\omega$, $\omega \in \mathbb{R}$.

En supposant que le système est asymptotiquement stable pour $\tau = 0$, il est possible de déterminer le couple (τ^*, ω^*) qui correspond à la première occurrence de $\Psi(\tau^*, j\omega^*) = 0$, qui définit la marge de stabilité du système avec retard. Aussi, toutes les racines de $\Delta_H(\lambda, s) = 0$ sont situés dans le demi-plan gauche ouvert pour tous les $\lambda > \lambda^o > 0$ si $\Phi(\lambda^o, j\omega^o) = 0$ correspond à la première traversée du mapping $\Phi(\lambda, j\omega)$, qui est obtenue en réduisant λ de $+\infty$.

En obtenant le couple (τ^*, ω^*), on peut calculer λ^* par $\omega^*/\lambda^* = \tan(\omega^*\tau^*/2)$, ce qui satisfait $\Phi(\lambda^*, j\omega^*) = 0$ et donc $\lambda^* \leq \lambda^o$. Lorsque $\lambda^* = \lambda^o > 0$, nous pouvons garantir que pour chaque $\lambda \in (\lambda^*, \infty)$ et $\tau \in [0, \tau^*)$, le système avec retard et le système de comparaison sont stables. La caractérisation complète de tous les systèmes avec retard qui satisfont l'égalité $\lambda^o = \lambda^* > 0$ n'est pas simple. Pour cette raison, le lemme suivant fournit un sous-intervalle de (λ^o, ∞) tel que la stabilité de chacun des deux systèmes soit simultanément préservée.

Lemme 20 *Supposons que la matrice $A_0 + A_1$ est Hurwitz et le système avec retard (A.61)-(A.62) est stable pour tout $\tau \in [0, \tau^*)$. Dans ce cas, pour chaque $\lambda \in (2/\tau^*, \infty)$ le système de comparaison est stable, et pour tout $\omega \in \mathbb{R}$ la valeur du retard τ qui satisfait $\omega/\lambda = \tan(\omega\tau/2)$ appartient à l'intervalle $[0, \tau^*)$.*

Calcul de la norme de $\mathcal{H}_\infty$

Nous voulons maintenant calculer la norme

$$\|T(\tau, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(T(\tau, j\omega)) \tag{A.69}$$

pour un $\tau \in [0, \tau^*)$ donné. L'existence de $\lambda^o > 0$ qui assure que A_λ est Hurwitz pour chaque $\lambda \in (\lambda^o, \infty)$ est essentiel parce que, autrement, $\|H(\lambda, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda, j\omega))$ n'est pas valide et la norme $\|H(\lambda, s)\|_\infty$ est illimitée. Pour cela, nous définissons le scalaire positif $\lambda_o = \inf \{\lambda \mid A_\xi \text{ est Hurwitz } \forall \xi \in (\lambda, \infty)\}$. En supposant que la matrice $A_0 + A_1$ est Hurwitz, le scalaire λ_o existe et est donné par $\lambda_o = \lambda^o$ ou $\lambda_o = 0^+$, lorsque $\lambda^o > 0$ ou $\lambda^o < 0$, respectivement.

Théorème 21 *Supposons que $A_0 + A_1$ est Hurwitz. Pour chaque $\lambda \in (\lambda_o, \infty)$ définissons $\alpha \geq 0$ tel que*

$$\alpha = \arg \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda, j\omega)) \tag{A.70}$$

et déterminons $\tau(\lambda)$ à partir de $\alpha/\lambda = \tan(\alpha\tau/2)$. Si $\tau(\lambda) \in [0, \tau^*)$ alors $\|H(\lambda, s)\|_\infty \leq \|T(\tau(\lambda), s)\|_\infty$.

Corollaire 22 *Supposons que $A_0 + A_1$ est Hurwitz. Pour tout paramètre positif $\gamma > \|H(\infty, s)\|_\infty$ donné, il existe $\lambda_\gamma \geq \lambda_o > 0$ et $0 \leq \tau_\gamma \leq \tau^*$ tels que l'inégalité*

$$\|H(\lambda, s)\|_\infty \leq \|T(\tau, s)\|_\infty < \gamma \tag{A.71}$$

est valide $\forall\ \lambda \in (\lambda_\gamma, \infty)$ dans le cas où la fonction $\tau(\lambda)$ donnée par le théorème 21 est continue dans l'intervalle.

Concept de Filtres Linéaires

Considérons un système avec retard dont la réalisation dans l'espace d'état est donnée par

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + E_0w(t) \tag{A.72}$$

$$y(t) = C_{y0}x(t) + C_{y1}x(t-\tau) + D_yw(t) \tag{A.73}$$

$$z(t) = C_{z0}x(t) + C_{z1}x(t-\tau) \tag{A.74}$$

où $x(t) \in \mathbb{R}^n$ est appelé l'état, $w(t) \in \mathbb{R}^r$ l'entrée externe, $y(t) \in \mathbb{R}^p$ et le signal mesuré $y(t) \in \mathbb{R}^p$ la sortie à estimer. Nous supposons que le système évolue à partir de zéro et que le retard $\tau \geq 0$ est constant.

Notre objectif est de concevoir un filtre d'ordre complet

$$\dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{A}_1\hat{x}(t-\tau) + \hat{B}_0y(t) \tag{A.75}$$

$$\hat{z}(t) = \hat{C}_0\hat{x}(t) + \hat{C}_1\hat{x}(t-\tau) \tag{A.76}$$

où $\hat{x}(t) \in \mathbb{R}^n$. Lorsqu'il est connecté à (A.72)-(A.74), le filtre définit la réalisation minimale de l'erreur d'estimation $\varepsilon(t) = z(t) - \hat{z}(t)$, en termes de

$$\dot{\xi}(t) = F_0\xi(t) + F_1\xi(t-\tau) + G_0w(t) \tag{A.77}$$

$$\varepsilon(t) = J_0\xi(t) + J_1\xi(t-\tau) \tag{A.78}$$

où $\xi(t) = [x(t)' \ \hat{x}(t)']' \in \mathbb{R}^{2n}$ est le vecteur d'état de l'erreur d'estimation, et les matrices sont données par

$$F_0 = \begin{bmatrix} A_0 & 0 \\ \hat{B}_0C_{y0} & \hat{A}_0 \end{bmatrix}, \ F_1 = \begin{bmatrix} A_1 & 0 \\ \hat{B}_0C_{y1} & \hat{A}_1 \end{bmatrix} \tag{A.79}$$

$G_0' = [E_0' \ \ D_y'\hat{B}_0']$, $J_0 = [C_{z0} \ \ -\hat{C}_0]$ e $J_1 = [C_{z1} \ \ -\hat{C}_1]$. Sa fonction de transfert entre w et ε, est donnée par

$$T_F(\tau,s) = \Big(J_0 + J_1e^{-\tau s}\Big)\Big(sI - F_0 - F_1e^{-\tau s}\Big)^{-1}G_0. \tag{A.80}$$

Notre objectif est de créer un filtre avec la réalisation dans l'espace d'état donné par (A.75)-(A.76) tel que $\|T_F(\tau,s)\|_\infty < \gamma$ où $\tau \in [0,\tau^*)$ et $\gamma > 0$. La stratégie est de remplacer la fonction de transfert $T_F(\tau, s)$ par l'équivalent $H_F(\lambda, s)$ du système de comparison

$$H_F(\lambda, s) = \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ F_0 + F_1 & F_0 - F_1 - \lambda I & G_0 \\ \hline J_0 + J_1 & J_0 - J_1 & 0 \end{array}\right] \tag{A.81}$$

résoudre le problème du filtrage $\mathcal{H}_\infty$ et extraire le retard comme indiqué dans le corollaire 22. Cela est possible parce que même si les matrices de la réalisation de l'espace d'état de $H_F(\lambda, s)$ dépendent d'une manière compliquée des matrices du filtre, il y a une matrice de transformation de similarité S tel que (A.81) est équivalent à

$$H_F(\lambda, s) = \left[\begin{array}{c|c} SF_\lambda S^{-1} & SG \\ \hline JS^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_\lambda & 0 & E \\ \hat{L}C_y & \hat{A}_\lambda & \hat{L}D_y \\ \hline C_z & -\hat{C} & 0 \end{array}\right] \tag{A.82}$$

où (A_λ, E, C_z) ont été définis dans(A.64), $C_y = [C_{y0} + C_{y1} \;\; C_{y0} - C_{y1}]$,

$$\hat{A}_\lambda = \begin{bmatrix} 0 & \lambda I \\ \hat{A}_0 + \hat{A}_1 & \hat{A}_0 - \hat{A}_1 - \lambda I \end{bmatrix}, \; \hat{L} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix} \tag{A.83}$$

et $\hat{C} = [\hat{C}_0 + \hat{C}_1 \;\; \hat{C}_0 - \hat{C}_1]$.

À première vue (A.82) amène à la conclusion que la réalisation dans l'espace d'état de $H_F(\lambda, s)$ a la même structure que l'erreur d'estimation classique. Bien que correctes, les matrices $\hat{A}_\lambda$ et $\hat{L}$ doivent avoir une structure particulière, donnée par (A.83). Pour faire face à cela, nous commençons par concevoir un filtre LTI en remplaçant les variables $(\hat{A}_\lambda, \hat{L}, \hat{C})$ dans (A.82) par des variables générales (A_F, L_F, C_F) et résoudre $\|H_F(\lambda, s)\|_\infty < \gamma$ pour un $\gamma > 0$ donné. La deuxième étape consiste à déterminer une matrice inversible $V \in \mathbb{R}^{2n\times 2n}$ tel que $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$, ce qui garantit que la fonction de transfert de l'erreur d'estimation pour le système de comparaison reste inchangée. Pour un $\gamma > 0$ donné, et en supposant que $D_yD_y' = I$, on obtient le filtre central en forme d'observateur

$$\left[\begin{array}{c|c} A_F & L_F \\ \hline C_F & 0 \end{array}\right] = \left[\begin{array}{c|c} A_\lambda - L_FC_y & L_F \\ \hline C_z & 0 \end{array}\right] \tag{A.84}$$

où le gain du filtre est donné par $L_F = PC_y' + ED_y'$ et $P = P' > 0$ vérifie l'inégalité de Riccati

$$\tilde{A}_\lambda P + P\tilde{A}_\lambda' + \tilde{E}\tilde{E}' - P\Big(C_y'C_y - \gamma^{-2}C_z'C_z\Big)P < 0 \tag{A.85}$$

où $\tilde{A}_\lambda = A_\lambda - ED'_y C_y$ et $\tilde{E} = E\left(I - D'_y D_y\right)$.

Lemme 23 *Supposons que* $\dim(y) = p \leq n = \dim(x)$, $\lambda > 0$ *et la matrice*

$$V = \begin{bmatrix} N \\ NA_F/\lambda \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \tag{A.86}$$

où $N' \in \mathbb{R}^{2n\times n}$ *est inversible et appartient à l'espace nul de* L'_F. *Dans ces conditions, l'égalité* $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$ *est vérifiée.*

Une fois la matrice V déterminée, on obtient la représentation d'état du filtre $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$, d'où les matrices du filtre (A.75) – (A.76) peuvent être extraites.

Théorème 24 *Considérons que* $\gamma > \min_F \|H_F(\infty, s)\|_\infty$ *est donné. Pour* $\lambda > 0$ *suffisamment grand, la solution centrale de (A.85) définie par la paire* (L_F, P) *est telle que* P *a la structure particulière*

$$P = \begin{bmatrix} Z+Q & -Q \\ -Q & Q \end{bmatrix} \tag{A.87}$$

où $Z \in \mathbb{R}^{n\times n}$ *et* $Q \in \mathbb{R}^{n\times n}$ *sont des matrices définies positives. En outre, les relations* $\|H_F(\infty, s)\|_\infty = \|T_F(0, s)\|_\infty < \gamma$ *sont vérifiées.*

Ce théorème montre que pour $\lambda > 0$ suffisamment grand, la solution centrale de l'inégalité Riccati (A.85) a une structure particulière et génère un filtre tel que la norme de l'erreur d'estimation de $\|H_F(\infty, s)\|_\infty$ et $\|T_F(0, s)\|_\infty$ coïncident. Par conséquent, par le corollaire 22, il existe un intervalle $\lambda \in (\lambda_\gamma, \infty)$, tel que

$$\|H_F(\lambda, s)\|_\infty \leq \|T_F(\tau, s)\|_\infty < \gamma \tag{A.88}$$

pour chaque $\tau \in [0, \tau_\gamma)$ où λ_γ et τ_γ sont déterminés par l'algorithme suivant :

- Définir une suite strictement décroissante $\lambda_k \in \{\infty, \cdots, \lambda_o\}$.
- Pour chaque élément λ_k, déterminer le filtre central $(A_{F_k}, L_{F_k}, C_{F_k})$ et le retard $\tau_k = \tau(\lambda_k)$.
- À partir de la matrice de transformation de similarité V_k, déterminer le filtre $(\hat{A}_{\lambda_k}, \hat{L}_k, \hat{C}_k)$.
- Augmenter k lorsque $0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2$ et $\|T_{F_k}(\tau_k, s)\|_\infty < \gamma$.

Concept de Contrôleurs par Retour d'État

Intéressons-nous désormais au système avec retard

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + B_0u(t) + E_0w(t) \tag{A.89}$$

$$z(t) = C_0x(t) + C_1x(t-\tau) + G_0u(t) + D_0w(t) \tag{A.90}$$

où $x(t) \in \mathbb{R}^n$ est l'état, $u(t) \in \mathbb{R}^m$ le contrôle, $w(t) \in \mathbb{R}^r$ l'entrée externe et $z(t) \in \mathbb{R}^q$ la sortie contrôlée. Les hypothèses sur le système sont les mêmes. L'objectif est de concevoir un contrôleur par retour d'état via $u(t) = K_0x(t) + K_1x(t-\tau)$, qui connecté au système en boucle ouverte (A.61)-(A.62) produit un système en boucle fermée avec la réalisation d'état de la forme (A.61)-(A.62), et la fonction de transfert $T_K(\tau, s)$. En adoptant la même stratégie que ci-dessus, la fonction rationnelle associée $H_K(\lambda, s)$ est donnée par

$$H_K(\lambda, s) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C + GK & D \end{array}\right] \tag{A.91}$$

où $B' = [0 \ B_0']$, $G = G_0$, $D = D_0$ et $K = \left[\begin{array}{cc} K_0 + K_1 & K_0 - K_1 \end{array}\right]$.

On observe que la fonction de transfert $H_K(\lambda, s)$ a une structure en boucle fermée, et une fois que le gain $K \in \mathbb{R}^{m\times 2n}$ est déterminé, il fournit des gains du système avec retard $K_0 \in \mathbb{R}^{m\times n}$ et $K_1 \in \mathbb{R}^{m\times n}$. Par conséquent, le problème se résume à déterminer un gain de retour d'état $K \in \mathbb{R}^{m\times 2n}$ tel que $\|T_K(\tau, s)\|_\infty < \gamma$, qui est remplacé par la détermination de $K \in \mathbb{R}^{m\times 2n}$ tel que $\|H_K(\lambda, s)\|_\infty < \gamma$. En considérant que $\gamma^2 I - D'D > 0$, $G'G$ ne sont pas singuliers et que le système en boucle ouverte remplit les conditions d'orthogonalité $G'[C \ D] = 0$, $D'C = 0$, la solution centrale est donnée par $K = -(G'G)^{-1}B'P$ où $P > 0$ vérifie l'inégalité de Riccati

$$A_\lambda' P + PA_\lambda + C'C - P\Big(B(G'G)^{-1}B' - E(\gamma^2 I - D'D)^{-1}E'\Big)P < 0 \tag{A.92}$$

Concept de Contrôleurs par Retour de Sortie

Les techniques sont similaires à celles développées pour la synthèse de filtres linéaires et les contrôleurs par retour d'état peuvent être adaptés pour faire face au problème de la conception des contrôleurs par retour de sortie.

Appendix B

Resumo em Português

Introdução

Objetivos

O objetivo desta tese é o estudo de métodos de análise e síntese para sistemas dinâmicos com atraso. O elemento de ligação entre todos os temas é o uso de métodos frequenciais como nossa ferramenta básica. Esta escolha se deve a dois motivos principais. Primeiramente, para o problema de síntese, podemos almejar resultados com baixo grau de conservatismo, quanto que para o caso de análise, podemos obter resultados que são necessários e suficientes. Tais fatos são difíceis de se obter com o uso de funcionais de Lyapunov-Krasovskii (Gu et al. 2003). Por fim, muitos dos métodos podem ser adaptados para lidar com a classe de sistemas fracionários, que ultimamente vêm sendo cada vez mais estudada tanto em teoria quanto na prática.

Apresentação

Vários sistemas dinâmicos apresentam atrasos na sua estrutura, devido a fenômenos como, por exemplo, transporte, propagação ou comunicação, mas várias vezes os ignoramos por questão de simplicidade. Mas estes atrasos podem ocasionar uma degradação de performance ou mesmo instabilidade, e desta forma, para se analisar e projetar controladores de forma correta para tais sistemas, o atraso precisa ser considerado. Outra fonte de atrasos é proveniente do controle propriamente dito, com tais atrasos sendo criados pelos sensores, atuadores, e também no tempo de cálculo em controladores digitais.

Sistemas com atraso formam uma classe de sistemas de dimensão infinita. No domínio do tempo, o caso mais simples com apenas um atraso pode ser representado pela seguinte equação diferencial:

$$\mathcal{G}_r : \begin{cases} \dot{x}(t) & = & A_0 x(t) + A_1 x(t-\tau) + Ew(t) \\ z(t) & = & C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{B.1}$$

onde $x(t) \in \mathbb{R}^n$ é denominado de estado, $w(t) \in \mathbb{R}^m$ é a entrada externa, $z(t) \in \mathbb{R}^p$ representa a saída, $\tau > 0$ é o valor numérico do atraso e $A_0,\ A_1,\ E,\ C_0,\ C_1$ e D_z são matrizes de dimensões apropriadas. Para a definição completa do sistema, com existência e unicidade da solução, a condição inicial $x(t) = x_0(t)$ precisa ser fornecida para todo $t \in [-\tau, 0]$. Considerando condições iniciais nulas, a transformada de Laplace de $\mathcal{G}_r$ da entrada w para a saída z é dada por

$$G_r(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau})^{-1} E + D_z. \tag{B.2}$$

O elemento "r" subscrito em $\mathcal{G}_r$ denota que este é um sistema com *retardo*, resultado do fato de que em (B.1) a equação diferencial não depende de $\dot{x}(t-\tau)$. Este fato é refletido na equação característica de (B.2), ou seja, $\det(sI - A_0 - A_1 e^{-s\tau}) = 0$, com o fato de que ela não apresenta termos do tipo $s^n e^{-ks\tau}$, $k \in [1, n]$. Esta classe de sistema apresenta propriedades interessantes tanto no domínio do tempo como da frequência, como a suavização da resposta com o avanço do tempo e a continuidade no número de pólos instáveis na passagem de $\tau = 0$ para um pequeno valor positivo.

Em outro contexto temos os sistemas denominados *neutros*. Considerando novamente o caso com um atraso, temos:

$$\mathcal{G}_n : \begin{cases} \dot{x}(t) & = & A_0 x(t) + A_1 x(t-\tau) + F\dot{x}(t-\tau) + Ew(t) \\ z(t) & = & C_0 x(t) + C_1 x(t-\tau) + D_z w(t) \end{cases} \tag{B.3}$$

e sob as mesmas hipóteses dadas para o sistema com retardo, sua trasformada de Laplace é dada por

$$G_n(s) = (C_0 + C_1 e^{-s\tau})(sI - A_0 - A_1 e^{-s\tau} - Fse^{-s\tau})^{-1} E + D_z. \tag{B.4}$$

Podemos observar que tais sistemas possuem uma dependência de (B.3) com $\dot{x}(t-\tau)$. Isto se propaga na equação característica através do fato de que existe ao menos um termo da forma $s^n e^{-ks\tau}$, $k \in [1, n]$. Diferente dos sistemas com retardo, os sistemas neutros não possuem a propriedade de suavizar a resposta e podem mesmo perder a estabilidade para qualquer atraso positivo mesmo sendo o sistema com atraso nulo estável.

Existe ainda um terceiro grupo de sistema com atraso, denominado *avançado*. Ele é caracterizado por ter apenas termos com atraso nos termos de maior derivada na representação em espaço de estados, ou, de forma equivalente, por ter o elemento de maior grau em s multiplicado por um termo exponencial na equação característica. No entanto, estes sistemas tem uma aplicação prática limitada, e sua estabilização é extremamente complicada para qualquer valor do atraso.

Historicamente, o primeiro estudo completo e sistemático de sistemas com atraso por métodos frequenciais foi feito por (Bellman & Cooke 1963). A partir deste ponto, o estudo de sistemas com atraso cresceu fortemente, conforme mostra o grande número de livros, artigos de jornal nas décadas seguintes. Entre muitos outros, gostaríamos de referenciar os livros (Hale & Lunel 1993), (Niculescu 2001) e (Gu et al. 2003), além dos artigos de revisão (Kharitonov 1999) e (Richard 2003) como marcas importantes na área.

Cadeias de Pólos

Nosso primeiro passo na análise frequencial dos sistemas com atraso será dado em direção ao problema do comportamento assintótico das cadeias de pólos. Estas estruturas possuem uma forte ligação geométrica no plano complexo, e são de extrema importância no quesito de estabilidade do sistema. Esta afirmação é especialmente relevante para os sistemas neutros, para os quais existem situações onde a estabilidade $\mathcal{H}_\infty$ pode não ser obtida mesmo no caso de todos os pólos estarem no semiplano esquerdo.

Sistemas Clássicos

Nesta parte, olharemos para sistemas com atraso cuja função de transferência é da forma

$$G(s) = \frac{t(s) + \sum_{k=1}^{N'} t_k(s) e^{-ks\tau}}{p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}} = \frac{n(s)}{d(s)}, \tag{B.5}$$

onde $\tau > 0$ é o atraso, e t, p, q_k para todo $k \in \mathbb{N}_N$, e t_k para $k \in \mathbb{N}_{N'}$, são polinômios reais.

Antes de fornecer a classificação completa das cadeias de pólos, necessitamos de alguns resultados prévios (Partington 2004).

Lema 1 *Seja $a \in \mathbb{C} \setminus \{0\}$. A equação $se^s = a$ possui um número infinito de soluções, que para grandes valores de $|s|$ possuem a forma $s = x + \jmath y$ com*

$$x = -\ln(2n\pi) + \ln(|a|) + o(1) \tag{B.6}$$

$$y = \pm 2n\pi \mp \pi/2 + \arg(a) + o(1) \tag{B.7}$$

e n grande o suficiente.

As expressões assintóticas para a solução de $s^m e^{\lambda s} = a$, para $m \in \mathbb{N}$ e $\lambda \neq 0$, podem ser diretamente calculadas ao observarmos que elas podem ser obtidas se fizermos $z = \lambda s/m$ e resolvermos $ze^z = \beta\lambda/m$ para cada β tal que $\beta^m = a$.

Para um sistema com atraso dado por uma equação mais complexa, seus infinitos pólos podem ser classificados em um número de cadeias de retardo, neutras ou avançadas. Para isso, definimos:

Definição 2 *Seja $d(s) = p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}$, onde p é um polinômio com grau d_0 cujo coeficiente do termo de maior grau é dado por c_0, e q_k são polinômios com grau d_k e coeficientes de maior grau c_k. O diagrama de distribuição de $n(s)$ é a curva poligonal concava unindo os pontos $P_0 = (0, d_0)$ e $P_N = (N, d_N)$, com vértices em alguns dos pontos $P_k = (k, d_k)$ de forma que nenhum ponto de P_k fique acima dele.*

Cada aresta do diagrama de distribuição cujo gradiente é positivo representa cadeias avançadas. Sempre que o gradiente é zero, implica-se que algumas cadeias são neutras, e com gradiente negativos temos cadeias com retardo. Desta forma, para se evitar sistemas com cadeias avançadas, temos a condição necessária e suficiente de que $\deg p(s) \geq \deg q_k(s)$ para todo $k \in \mathbb{N}_N$. Ademais, se existir ao menos um k em $\mathbb{N}_N$ tal que $\deg p = \deg q_k$, o sistema possuirá cadeias neutras.

Agora vamos nos concentrar no problema de encontrar a posição assintótica das cadeias de pólos neutras. Para facilitar a notação, vamos denominar $z = e^{-s\tau}$. Considerando $\deg p \geq \deg q_k$ para todo $k \in \mathbb{N}_N$, podemos supor que para todo k

$$\frac{q_k(s)}{p(s)} = \alpha_k + \frac{\beta_k}{s} + \frac{\gamma_k}{s^2} + \mathcal{O}(s^{-3}) \qquad \text{quando} \quad |s| \to \infty. \tag{B.8}$$

O coeficiente de maior grau em s de $p(s) + \sum_{k=1}^{N} q_k(s)e^{-ks\tau}$ pode ser escrito como um múltiplo do seguinte polinômio em z:

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \tag{B.9}$$

Nosso primeiro objetivo é encontrar as posições das linhas verticais para as quais as raízes das cadeias neutras se aproximam assintoticamente.

Proposição 3 *Seja $G(s)$ un sistema neutro definido como em (B.5). Neste caso, existem cadeias neutras de pólos aproximando assintoticamente as linhas verticais*

$$\Re(s) = -\frac{\ln(|r|)}{\tau} \tag{B.10}$$

para cada raiz $z = r$ do polinômio $\tilde{c}_d(z)$.

O caso onde todas as raízes de (B.9) possuem multiplicidade um é mais simples de ser analizado, e será completamente tratado. Neste caso particular, seja $M \leq N$ o maior número inteiro tal que $\alpha_M \neq 0$. Neste caso, existem M cadeias de pólos neutros, e dado que estas são assintóticas à linhas verticais, é preciso descobrir por que lado esta aproximação ocorre. Como veremos em seguida, esta informação é crucial para os pólos que possuem $|r| = 1$, pois isto fornecerá em qual semiplano os pólos se localizam, o que interfere diretamente na questão da estabilidade. Esta análise é o objetivo do teorema a seguir:

Teorema 4 *Seja $G(s)$ um sistema com atraso neutro definido por (B.5), e suponha que todas as raízes de (B.9) possuem multiplicidade um. Para cada r tal que $z = r$ é uma raiz de (B.9) e para $n \in \mathbb{Z}$ grande o suficiente, as soluções assintóticas a (B.10) são dadas por:*

$$s_n \tau = \lambda_n + \mu_n + \mathcal{O}(n^{-2}), \tag{B.11}$$

com λ_n dado por

$$\lambda_n = -\ln(r) + 2\jmath n\pi, \qquad n \in \mathbb{Z} \tag{B.12}$$

e

$$\mu_n = -\jmath \frac{\tau \sum_{k=1}^{N} \beta_k r^k}{2\pi n \sum_{k=1}^{N} k\alpha_k r^k}. \tag{B.13}$$

Podemos chegar a algumas conclusões observando μ_n. Porém, primeiramente, associado a cada raiz r de (B.9), definimos K_r como

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k}, \tag{B.14}$$

onde K_r é bem definido dado que r, como raiz de (B.9), possui multiplicidade um.

A informação principal que necessitamos é descobrir em qual lado delimitado por tal linha vertical que os pólos se encontram. Para isso, necessitamos olhar para $\Im(K_r)$. Como α_k e β_k são números reais para todo $k \in \mathbb{N}_N$, se $\Im(r) = 0$ então $\Im(K_r) = 0$. Isto implica que $\Im(r) \neq 0$ é uma condição necessária (porém não suficiente) para $\Im(K_r) \neq 0$. Mas neste caso, $\overline{r}$ também é uma raiz de (B.9), e observando que $\Im(K_r) = -\Im(K_{\overline{r}})$, temos que se $\Im(K_r) \neq 0$, certamente existirão cadeias de pólos assintóticas pelos dois lados da linha vertical.

No caso onde $\Im(K_r) = 0$, o coeficiente de maior valor de μ_n é puramente imaginário, e necessitamos olhar para o próximo termo. Este é o foco do próximo teorema:

Teorema 5 *Seja $G(s)$ um sistema com atraso neutro definido por (B.5), e suponha que todas as raízes de (B.9) possuem multiplicidade um. Para cada r tal que $z = r$ é uma raiz de (B.9) e para $n \in \mathbb{Z}$ grande o suficiente, as soluções assintóticas a (B.10) são dadas por:*

$$s_n\tau = \lambda_n + \mu_n + \nu_n + \mathcal{O}(n^{-3}), \tag{B.15}$$

com λ_n dado por (B.12), μ_n por (B.13) e

$$\nu_n = \frac{\tau^2 \sum_{k=1}^{N} \left(-k^2\alpha_k K_r^2/2 + k\beta_k K_r - \gamma_k - \beta_k \ln(r)/\tau\right) r^k}{4\pi^2 n^2 \sum_{k=1}^{N} k\alpha_k r^k}. \tag{B.16}$$

Como antes, estamos interessados no sinal da parte real de ν_n. Se negativo, os pólos de grande módulo estarão à esquerda da linha assintótica. Se positivo, eles estarão à direita, e no caso de valor nulo, precisamos expandir a análise para para maiores graus de n. Existe neste caso a possibilidades de que as cadeias de pólos estejam localizadas exatamente sobre o eixo vertical.

Se observarmos com atenção, veremos alguns aspectos interessantes sobre a equação (B.16). Primeiramente, $\beta_k \ln(r)/\tau$ pode mudar o sinal de $\Re(\nu_n)$ quando τ aumenta. Isto implica que pode existir um valor crítico do atraso $\tau^\star$ tal que para $\tau < \tau^\star$ a cadeia se aproxima assintoticamente por um lado e para $\tau > \tau^\star$ pelo outro lado. Este efeito, no entanto, não pode ocorrer quando o eixo assintótico é o eixo imaginário, como veremos na próxima proposição:

Proposição 6 *Seja $G(s)$ um sistema com atraso neutro definido por (B.5), e suponha que todas as raízes de (B.9) possuem multiplicidade um. Uma variação no atraso τ pode variar a posição do eixo assintótico, mas não muda o semiplano em que ele se localiza. Ademais, se o eixo assintótico é o eixo imaginário, assim ele se mantém para todo valor de τ. Finalmente, suponha que ν_n, como definido no Teorema 5, possua parte real não nula. Neste caso, o lado onde os pólos se localizam ao redor do eixo imaginário não varia com τ.*

Nosso interesse a partir deste ponto é responder a questão sobre a estabilidade de $G(s)$. Lembramos que a noção de estabilidade de consideramos é denominada de estabilidade $\mathcal{H}_\infty$, ou seja, o sistema dinâmico apresenta um ganho $L_2(0,\infty)$ finito

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u \in L^2, u \neq 0} \frac{\|Gu\|_{L^2}}{\|u\|_{L^2}} < \infty. \tag{B.17}$$

Retomamos também que $\mathcal{H}_\infty(\mathbb{C}_+)$ denota o espaço de funções analíticas e limitadas no semiplano aberto direito $\mathbb{C}_+$. Nós iremos denominar os pólos no semiplano direito fechado $\overline{\mathbb{C}_+}$ como *pólos instáveis*, e aqueles no semiplano aberto esquerdo $\mathbb{C}_-$ como *pólos estáveis*.

O caso onde o polinômio (B.9) possui apenas raízes de módulo estritamente maior do que um é mais simples de ser considerado, pois existe um número $a > 0$ tal que o sistema tem um número finito de pólos em $\{\Re(s) > -a\}$. Também o caso onde o polinômio (B.9) possui pelo menos uma raiz de módulo menor que um é direto.

É um fato conhecido que a fatorização coprima sobre $\mathcal{H}_\infty$ nos permite a parametrização do conjunto de todos os controladores $\mathcal{H}_\infty$ estabilizantes (Bonnet & Partington 1999). Para o nosso caso atual, os fatores coprimos e os fatores de Bézout podem ser determinados de forma similar àquela apresentada para sistemas com retardo em (Bonnet & Partington 1999).

Proposição 7 *Seja $G(s)$ uma função de transferência do tipo (B.5). Então*

1) *Se o polinômio (B.9) possui ao menos uma raiz de módulo estritamente menor que um, G não pode ser estável no sentido $\mathcal{H}_\infty$.*

2) *Se o polinômio (B.9) possui apenas raízes de módulo estritamente maiores que um, então:*

   (a) *G é estável no sentido $\mathcal{H}_\infty$ se e somente se G não possui pólos instáveis.*

   (b) *G é estabilizável no sentido $\mathcal{H}_\infty$. Além do mais, supondo que $n(s)$ e $d(s)$ não possuem zeros instáveis em comum, uma fatorização coprima de G pode ser dada por*

$$N(s) = \frac{n(s)}{(s+1)^\delta}, \qquad D(s) = \frac{d(s)}{(s+1)^\delta}. \tag{B.18}$$

Por outro lado, no caso em que (B.9) possui ao menos uma raiz de módulo um exige mais cuidado, mesmo se todos os pólos do sistema $G(s)$ se localizem em $\mathbb{C}_-$. Para a estabilidade $\mathcal{H}_\infty$, ainda é necessário ver se G é limitada no eixo imaginário (como no exemplo dado em (Partington & Bonnet 2004)).

Proposição 8 *Seja $G(s)$ uma função de transferência dada como no Teorema 5 e suponha que (B.9) possua ao menos uma raiz de módulo um e multiplicidade um, com todas as outras raízes tendo módulo maior do que um.*

1. *Suponha que* $\Re(\nu_n) < 0$ *e que* G *não possua pólos instáveis. Neste caso,* G *é estável no sentido* $\mathcal{H}_\infty$ *se e somente se* $\deg p \geq \deg t + 2$.

2. *Se* $\nu_n = 0$*, a condição* $\deg p \geq \deg t + 2$ *ainda é necessária para a estabilidade* $\mathcal{H}_\infty$.

Com tais resultados em mão, podemos discorrer sobre a estabilizabilidade de alguns sistemas neutros através de controladores racionais.

Proposição 9 *Seja* G *um sistema com função de transferência dada por (B.5) tal que:*

1) *o polinômio* $\tilde{c}_d$ *definido em (B.9) relativo ao quasi-polinômio* $d(s)$ *(denominador de* $G(s)$*) possui ao menos uma raiz de módulo menor ou igual a um, mas nenhuma raiz de módulo um possui mutiplicidade maior do que um;*

2) $\deg p = \deg t + 1$.

Neste caso, G *não é estabilizável por um controlador racional.*

Sistemas Fracionários

Mudamos agora nosso foco para a análise das cadeias de pólos para sistemas fracionários com forma:

$$G(s) = \frac{t(s^\mu) + \sum_{k=1}^{N'} t_k(s^\mu) e^{-ks\tau}}{p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}}, \tag{B.19}$$

onde $\tau > 0$, $0 < \mu < 1$, e t, p, t_k para todo $k \in \mathbb{N}_{N'}$ e q_k para todo $k \in \mathbb{N}_N$ são polinômios reais. Como feito anteriormente, a partir de agora vamos assumir que este sistema é do tipo neutro.

O comportamento assintótico não difere muito do resultado apresentado anteriormente. Podemos calcular que para cada k

$$\frac{q_k(s^\mu)}{p(s^\mu)} = \alpha_k + \frac{\beta_k}{s^\mu} + \mathcal{O}(s^{-2\mu}) \qquad \text{quando} \quad |s| \to \infty. \tag{B.20}$$

O coeficiente de maior grau de $p(s^\mu) + \sum_{k=1}^{N} q_k(s^\mu) e^{-ks\tau}$ pode ser escrito como sendo um múltiplo do mesmo polinômio em z

$$\tilde{c}_d(z) = 1 + \sum_{i=1}^{N} \alpha_i z^i. \tag{B.21}$$

Portanto, as posições das linhas verticais para as quais são assintóticas as raízes do sistema neutro são as mesmas dadas na Proposição 3. Vamos considerar novamente o caso onde todas as raízes de (B.21) têm multiplicidade um.

Teorema 10 *Seja $G(s)$ um sistema com atraso neutro definido por (B.19) e suponha que todas as raízes de (B.21) possuem multiplicidade um. Para cada r tal que $z = r$ é uma raiz de (B.21) e para $n \in \mathbb{Z}$ grande o suficiente, as soluções assintóticas a (B.10) são dadas por:*

$$s_n h = \lambda_n + \delta_n + \mathcal{O}(n^{-2\mu}) \tag{B.22}$$

com λ_n dado por (B.12) e

$$\delta_n = \frac{\tau^\mu \sum_{k=1}^{N} \beta_k r^k}{(2\jmath\pi n)^\mu \sum_{k=1}^{N} k\alpha_k r^k} \tag{B.23}$$

Algumas conclusões são obtidas observando δ_n. Mas antes, associado a cada r de (B.21), vamos definir K_r como feito anteriormente:

$$K_r = \frac{\sum_{k=1}^{N} \beta_k r^k}{\sum_{k=1}^{N} k\alpha_k r^k}\,. \tag{B.24}$$

Nosso interesse principal reside em saber de qual lado ao redor do eixo vertical se localizam os pólos, ou seja, descobrir o sinal de $\Re(\delta_n)$ para n suficientemente grande. Novamente, precisamos apenas olhar para K_r para obter essa informação. No caso fracionário, no entanto, como $0 < \mu < 1$, raízes complexas conjugadas de (B.21) não irão necessariamente gerar δ_n complexos conjugados, o que implica que cada δ_n associado com um r particular não é igual ao complexo conjugado daquele associado com $\bar{r}$. Ou seja, pode ser possível concluir que um sistema possui todos os pólos em apenas um dos semiplanos já com uma aproximação desta ordem.

Corolário 11 *Seja $0 < \mu < 1$, δ_n dado por (B.23) e seu K_r associado por (B.24). Neste caso, $sign(\Re(\delta_n)) < 0$ para todo $n \in \mathbb{Z}$ se e somente se*

$$\Re(K_r) < -\tan\left(\frac{\mu\pi}{2}\right)|\Im(K_r)| \tag{B.25}$$

Alguns aspectos importantes podem ser obtidos deste corolário. Primeiro, o valor numérico do atraso não aparece explicitamente na equação (B.25), o que implica que para todo $\tau > 0$ as cadeias de pólos apresentam um comportamento único, no sentido de que elas não trocam de lado com respeito ao eixo vertical em questão, em função de um outro valor do atraso. Além do mais, como (B.25) envolve apenas o valor absoluto de $\Im(K_r)$, os resultados obtidos seriam equivalentes se tratássemos com o complexo conjugado de K_r. Mas de fato observamos que raízes complexas conjugadas de $c(z)$ na equação (B.21) definem K_r complexos conjugados.

Portanto, diferentemente do caso $\mu = 1$, existe a chance de que possamos afirmar que todos os pólos estejam à esquerda do eixo vertical (B.10) utilizando aproximação até tal ordem. E dado que α_k e β_k são independentes de μ, podemos ainda apresentar o seguinte resultado:

Corolário 12 *Sejam $0 < \mu < 1$ e δ_n dados por (B.23) e seu K_r associado por (B.24). Se $\Re(K_r) < 0$, todos os pólos da respectiva cadeia serão assintóticos pela esquerda da linha vertical (B.10) se*

$$\mu < \frac{2}{\pi} \arctan \left(-\frac{\Re(K_r)}{|\Im(K_r)|} \right). \tag{B.26}$$

Com tais resultados, podemos tratar do problema da estabilidade $\mathcal{H}_\infty$ desta classe de sistemas. Em âmbitos gerais, ela se assemelha àquela do sistema convencional. Ou seja, no caso onde (B.21) apresenta apenas raízes de módulo maior do que um é fácil de tratar, pois existe um $a > 0$ tal que o sistema possua um número finito de pólos em $\{\Re(s) > -a\}$. O caso onde a equação (B.21) possui pelo menos uma raiz com módulo menor do que um também não apresenta dificuldades, dado que existem um número infinito de pólos instáveis. Para o caso de transição, temos que:

Proposição 13 *Seja $G(s)$ uma função de transferência dada por (B.19) e suponha que (B.21) possui ao menos uma raiz de módulo um, sendo as outras de módulo maior que um.*

1. *Se $\Re(\delta_n) < 0$ para todo $n \in \mathbb{Z}$ e G não possua pólos instáveis, então G é estável no sentido $\mathcal{H}_\infty$ se e somente se $\deg p \geq \deg t + 1$.*

2. *Se $\Re(\delta_n) = 0$, a condição $\deg p \geq \deg t + 1$ ainda é necessária para a estabilidade $\mathcal{H}_\infty$.*

Janelas de Estabilidade e *Lugar das Raízes*

Introdução

Até o presente momento, estudamos apenas o comportamento assintótico dos pólos com $|s|$ indo ao infinito. É evidente que existem pólos de *pequeno módulo*, que não pertencem a nenhuma cadeia, e que têm impactos importantes no comportamento do sistema. O estudo destes pólos e como eles se comportam quando τ varia é o objeto desta parte.

A forma mais direta de se encontrar a posição de tais pólos seria resolver a equação característica correspondente. Mas como se trata de uma equação transcendental, com infinitas raízes, não podemos obtê-las diretamente. Por este motivo, a maioria dos procedimentos estuda a estabilidade destes sistemas encontrando os momentos onde os pólos cruzam o eixo imaginário.

Isto deve-se a duas propriedades importantes dos sistemas com atraso. Primeiramente, temos o *argumento da continuidade das raízes*, que implica que para qualquer valor positivo do atraso, a posição dos pólos varia continuamente com respeito ao atraso, implicando que qualquer raiz mudando de semiplano necessita passar pelo eixo imaginário. A segunda propriedade deve-se à invariância na tendência dos cruzamentos das raízes (Olgac & Sipahi 2002). Juntas, essas propriedades implicam que um número *tratável* de aglomerações de raízes pode fornecer a informação necessária para a caracterização completa da estabilidade do sistema. Vários algoritmos foram propostos explorando tais propriedades, e (Sipahi & Olgac 2006) apresenta uma comparação entre alguns destes. Por outro lado, para o caso fracionário, o número de ferramentas disponíveis não é tão numeroso. Mesmo que alguns algoritimos possam ser adaptados para tal caso (Walton & Marshall 1987), eles possuem deficiências que se tornam mais acentuadas no caso fracionário.

Nosso objetivo é desenvolver uma técnica para tratar da ligação entre o valor do atraso e a estabilidade do sistema, denominada janelas de estabilidade. Com tais resultados, em diversos casos, podemos localizar a posição exata de todos os pólos instáveis do sistema.

Sistemas Clássicos

Vamos considerar sistemas com atraso com a seguinte equação característica:

$$C(s,\tau) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-ks\tau}, \tag{B.27}$$

onde $\tau > 0$, e p e q_k, para todo $k \in \mathbb{N}_N$, são polinômios com coeficientes reais que satisfazem $\deg p \geq \deg q_k$. Assumimos que $p(s)$ e $q_k(s)$ para todo $k \in \mathbb{N}_N$ não possuem zeros comuns.

Começamos por considerar $\tau = 0$ em (B.27). Nesta caso, temos um polinômio com coeficientes reais, cujas raízes podem ser facilmente calculadas. Em seguida, precisamos da análise desenvolvida anteriormente se tratarmos de sistemas neutros. Como não é o foco desta seção, vamos considerar que não há cadeias de pólos no semiplano direito estendido.

Para encontrarmos a posição no eixo imaginário onde ocorrem os cruzamentos, vamos efetuar uma transformação de variáveis que desacopla os polinômios e a parte exponencial. A idéia será de substituir $e^{-\jmath\omega\tau k}$ por $e^{-\jmath\theta k}$ e encontrar as raízes do polinômio complexo resultante como uma função de $\theta \in [0,\pi]$. Em outras palavras, achar todas as raízes do polinômio em s

$$\widetilde{C}(s,\theta) = p(s) + \sum_{k=1}^{N} q_k(s) e^{-\jmath k\theta}, \tag{B.28}$$

variando θ no intervalo fechado $[0,\pi]$.

Podemos notar que para $\theta \in [0, \pi]$, $\widetilde{C}$ é um polinômio com coeficientes complexos, mas sem atrasos. Isto implica que, devido à transformação de variável no termo exponencial, uma solução $s^\star$ de $\widetilde{C}$ não implica que $\overline{s^\star}$ seja outra solução. No entanto, todas as soluções de (B.27) no eixo imaginário ocorrem em pares complexos conjugados. Isto nos leva ao nosso resultado principal:

Teorema 14 *Seja Ω o conjunto de todos os pares ordenados (ω, θ), com $\omega \in \mathbb{R}$ e $\theta \in [0, \pi]$ tal que $\widetilde{C}(\jmath\omega, \theta) = 0$. Seja*

$$\tau(\omega, \theta, \ell) = \frac{\theta}{\omega} + \frac{2\pi\ell}{\omega} \tag{B.29}$$

para todo $(\omega, \theta) \in \Omega$. Escolha $\ell = \{0, 1, \ldots\}$ se $\omega > 0$, e $\ell = \{-1, -2, \ldots\}$ se $\omega < 0$. Seja Δ definido como o conjunto de todos os pares ordenados $(\pm\jmath\omega, \tau(\omega, \theta, \ell))$. Então Δ é o conjunto completo de raízes de (B.27) no eixo imaginário para todo $\tau > 0$.

Observe que se o sistema (B.27) é estável para $\tau = 0$ e $\Omega = \emptyset$, então o sistema é estável para todos os valores positivos de τ. O conjunto Ω junto com a tendências das raízes para cada par ordenado é o que denominamos de uma *aglomeração de raízes.* Esta tendência é constante com respeito a todos os cruzamentos seguintes em (B.29).

Para usarmos os resultados do Teorema 14, precisamos encontrar todos os $\omega \in \mathbb{R}$ e $\theta \in [0, \pi]$ tais que $\widetilde{C}(\jmath\omega, \theta) = 0$. Para isso, propomos dois métodos. O mais direto consiste em amostrar θ no seu intervalo $[0, \pi]$, e para cada valor fixo $\theta^\star$ calcular as raízes do polinômio resultante $\widetilde{C}(s, \theta^\star) = 0$. Observe que o grau do polinômio é o mesmo de $p(s)$, e que o argumento de continuidade é válido para $\widetilde{C}(s, \theta)$ em relação a θ.

O segundo método vem do fato de que se $(s,\ \theta)$ é uma raiz simples de (B.28), então uma pequena perturbação em $\theta^\star = \theta + \epsilon$ irá fornecer uma solução de $\widetilde{C}(s^\star, \theta^\star) = 0$ na forma $s^\star = s + \sum_{k=1}^{\infty} \lambda_k \epsilon^k$ onde

$$\lambda_1 = \jmath \frac{\sum_{k=1}^{N} k q_k(s) e^{-\jmath\theta k}}{p'(s) + \sum_{k=1}^{N} q'_k(s) e^{-\jmath\theta k}} = T(s, \theta). \tag{B.30}$$

Desta forma, a estratégia é integrar numericamente $T(s, \theta)$ para θ variando de 0 até π e tendo como valores iniciais as soluções de $\widetilde{C}(s, 0) = 0$, que são exatamente as mesmas soluções de $C(s, 0) = 0$ já calculadas para o teste de estabilidade em $\tau = 0$.

O próximo objetivo é descobrir a direção de cada um dos cruzamentos pelo eixo imaginário. Como mostrado em (Olgac & Sipahi 2002), esta tendência é constante com relação a cada novo cruzamento pelo mesmo local (ℓ em (B.29)). Podemos tratar deste problema de uma maneira similar ao feito anteriormente. Assumindo que $(s,\ \tau)$ é uma raiz simples de $C(s, \tau) = 0$, para uma pequena variação de $\tau^\star = \tau + \epsilon$, a solução de $C(s^\star, \tau^\star) = 0$ possui a forma $s^\star = s + \sum_{k=1}^{\infty} \mu_k \epsilon^k$

onde

$$\mu_1 = s\frac{\sum_{k=1}^{N} kq_k(s)e^{-\tau sk}}{p'(s) + \sum_{k=1}^{N}(q'_k(s) - \tau kq_k(s))e^{-\tau sk}} = V(s,\tau). \tag{B.31}$$

A tendência é dada por $\text{sign}(\Re(V(j\omega,\tau)))$, onde $(j\omega,\tau) \in \Delta$. Se positiva, então temos um cruzamento destabilizante, caso contrário, se negativa, significa um cruzamento estabilizante. No caso de obtermos 0, precisamos seguir com uma análise de maior ordem. A partir deste ponto, determinar para cada valor do atraso se um sistema tem pólos instáveis é apenas uma questão de contagem.

Se quisermos não apenas calcular os valores de τ tais que o sistema é estável, mas sim encontrar a real posição dos pólos instáveis, podemos adaptar as técnicas utilizadas até agora. A partir da definição de Ω, podemos calcular um subconjunto $\Delta_{\tau^\star}$ de Δ contendo apenas os elementos de Δ com valores de atraso menores que $\tau^\star$ e tendência positiva. Integrando $V(s,\tau)$ para cada elemento $(s^\circ,\tau^\circ) \in \Delta_{\tau^\star}$ em $\tau \in [\tau^\circ,\tau^\star]$ e s° como ponto inicial, além dos pólos instáveis do sistema com atraso nulo em $\tau \in [0,\tau^\star]$, teremos as curvas do lugar das raízes. Este procedimento fornece a posição de todos os pólos instáveis de (B.27).

Sistemas Fracionários

Todos os resultados apresentados para sistemas clássicos podem ser adaptados para os sistemas fracionários, com os devidos cuidados tomados. Uma nova transformação precisa ser efetuada para podermos lidar com o fato de que pólos podem trocar de superfície de Riemann ao variarmos o atraso. Porém, de uma maneira geral, e com técnicas similares, podemos obter o mesmo grau de análise para os sistemas fracionários.

Curvas de Estabilidade para Sistemas com Atrasos com Distribuição γ

Até este momento nós focamos a análise em sistemas com atrasos discretos. Mas é importante salientar que durante as últimas décadas, vários autores trabalharam com a idéia de que em algumas aplicações, o uso de atrasos distribuídos levaria a modelos mais realísticos. Estes pioneiros começaram um grande desenvolvimento na teoria destes sistemas com atraso, especialmente no questão do estudo de vários núcleos do atraso, normalmente representado por medidas de probabilidade sobre o tempo.

Posto isto, o objetivo desta parte do trabalho é caracterizar as *curvas de cruzamento de estabilidade*, ou seja, o conjunto de parâmetros tais que existe ao menos um par de raízes características no eixo imaginário. Tais curvas dividem o espaço dos parâmetros $\mathbb{R}^2_+$ em diferentes regiões, com a propriedade de que em cada uma destas, o número de raízes instáveis é constante.

Definição do Problema

Considere o problema da análise de estabilidade de uma classe de equações diferenciais com atraso que podem ser descritas no domínio da frequência pela equação característica:

$$H(s,\alpha,\tau) = Q(s) + P(s)\frac{1}{(s+\alpha)^{m/n}}e^{-s\tau} \tag{B.32}$$

onde P e Q denotam polinômios com $\deg P - m/n < \deg Q$, $m, n \in \mathbb{Z}$ e α, τ são parâmetros reais positivos controlando o comportamento do sistema. Note que a estabilidade $\mathcal{H}_\infty$ de tais sistemas é equivalente à condição de não ter pólos no semiplano direito. Nós podemos também estabelecer a dependência contínua das raízes da função característica com respeito aos parâmetros α e τ, inclusive para o caso fracionário. Portanto, a análise de estabilidade se resume a:

- Inicialmente, detectar todos os cruzamentos com respeito ao eixo imaginário. Em outras palavras, calcular o *conjunto de frequências de cruzamento* denominado por Ω, que consiste de todas as frequências positivas tais que existe ao menos uma raiz característica crítica.
- Em seguida, descrever o comportamento das raízes críticas devido a mudanças no espaço de parâmetros (α, τ). Mais precisamente, vamos detectar os pontos críticos
- Finalmente, outro conceito relacionado é representado pelas *curvas de cruzamentos característicos* consistindo de todos os pares (α, τ) tais que existe ao menos um valor $\omega \in \Omega$ com $H(\jmath\omega, T, \tau) = 0$.

Caracterização das curvas de cruzamento de estabilidade

Antes de qualquer desenvolvimento, vamos definir alguns termos que serão utilizados:

Definição 15 *Denotamos por* $\mathcal{T}$ *o conjunto de todos os* $(\alpha, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ *tal que* (B.32) *possui ao menos um zero em* $\jmath\mathbb{R}$*. Qualquer* $(\alpha, \tau) \in \mathcal{T}$ *será denominado como um* ponto de cruzamento*. O conjunto* $\mathcal{T}$*, que representa a coleção de todos os pontos de cruzamento, é chamado de* curvas de cruzamento de estabilidade*.*

Definição 16 *Denotamos por* Ω *a coleção de todos os* $\omega > 0$ *tal que existe um elemento* (α, τ) *para o qual* $H(\jmath\omega, \alpha, \tau) = 0$*. O conjunto* Ω *é chamado de* conjunto de cruzamento*.*

Proposição 17 *Dados $\omega \in \Omega$, os pontos de cruzamento correspondentes são dados por:*

$$\alpha = \sqrt{\left|\frac{P(j\omega)}{Q(j\omega)}\right|^{2n/m} - \omega^2}, \tag{B.33}$$

$$\tau = \frac{1}{\omega}\Big((2p+1)\pi + \angle P(j\omega) - \angle Q(j\omega) - \frac{m}{n}\arctan\left(\omega\alpha^{-1}\right)\Big), \tag{B.34}$$

onde $p \in \mathbb{Z}$.

Consideramos que $\Omega = \bigcup_{k=1}^{N} \Omega_k$ e denotamos por $\mathcal{T}_k^p$ a curva correspondente a Ω_k com $p \in \mathbb{Z}$. Sejam os pontos extremos a esquerda e a direita do intervalo Ω_k denotados por ω_k^ℓ e ω_k^r, respectivamente. Cada ponto ω_k^ℓ ou ω_k^r precisa pertencer a um e apenas um dos três tipos seguintes.

Tipo 1a. Satisfaz a equação $Q(j\omega) = 0$ e $m/n > 0$.

Tipo 1b. Satisfaz a equação $P(j\omega) = 0$ e $m/n < 0$.

Tipo 2. Satisfaz a equação

$$|\omega|^{m/n} = \left|\frac{P(j\omega)}{Q(j\omega)}\right|. \tag{B.35}$$

Tipo 3. É igual a 0.

Iremos denominar qualquer ponto limite por ω_0, que pode representar tanto o ponto extremo a esquerda quanto a direita de um intervalo Ω_k. Dizemos que um intervalo Ω_k é do tipo ℓ/r se o ponto extremo a esquerda é do tipo ℓ e a direita do tipo r. Portanto, podemos dividir todos os intervalos em 12 tipos, 6 para $m/n > 0$ e 6 para $m/n < 0$.

Suavidade nas curvas e direções de cruzamento

Iremos considerar que o conjunto de frequências de cruzamento Ω seja dado e que as curvas de cruzamento de estabilidade sejam descritas por mapeamentos suaves $\omega \mapsto \alpha(\omega)$, $\omega \mapsto \tau(\omega)$. Chamaremos por $\mathcal{T}_h$ um destas curvas arbitrárias e consideramos a seguinte decomposição em partes real e imaginária:

$$R_0 + \jmath I_0 = \jmath \left.\frac{\partial H(s,\alpha,\tau)}{\partial s}\right|_{s=\jmath\omega},\ R_1 + \jmath I_1 = \left.\frac{\partial H(s,\alpha,\tau)}{\partial \alpha}\right|_{s=\jmath\omega},\ R_2 + \jmath I_2 = \left.\frac{\partial H(s,\alpha,\tau)}{\partial \tau}\right|_{s=\jmath\omega}.$$

Neste caso, o teorema da função implícita indica que a tangente de $\mathcal{T}_h$ pode ser escrita por

$$\begin{pmatrix} \dfrac{\mathrm{d}\alpha}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix}_{s=\jmath\omega} = \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - R_2 I_0 \\ R_1 I_0 - R_0 I_1 \end{pmatrix}, \tag{B.36}$$

no caso em que $R_1 I_2 - R_2 I_1 \neq 0$.

Concluímos que $\mathcal{T}_h$ é suave em todos os pontos exceto nos possíveis pontos onde ou $R_1 I_2 - R_2 I_1 \neq 0$ não é satisfeito ou quando

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{B.37}$$

Para estabelecer a direção dos cruzamentos, precisamos considerar α e τ como funções de $s = \sigma + \jmath\omega$, e portanto a notação em derivadas parciais precisa ser adotada. De forma que quando (α, τ) cruza as curvas de estabilidade da direita para a esquerda, um par de soluções complexas conjugadas cruza o eixo imaginário para o semiplano direito se

$$\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial\tau}{\partial\sigma} - \frac{\partial\tau}{\partial\omega}\frac{\partial\alpha}{\partial\sigma} \right)_{s=\jmath\omega} > 0. \tag{B.38}$$

O inverso ocorre se a desigualdade é invertida. Desta forma concluímos com a seguinte proposição:

Proposição 18 *Assuma que $\omega \in \Omega$, e também que $s = \jmath\omega$ é uma raiz simples de (B.32) e $H(j\omega', \alpha, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$. Então, quando (α, τ) mover de uma região à direita para uma região à esquerda da curva de cruzamento correspondente, um par de soluções de (B.32) cruza o eixo imaginário para a direita se $R_1 I_2 - R_2 I_1 > 0$. O cruzamento é invertido se a desigualdade também for.*

Síntese de Controladores PID

Controladores do tipo PID são os mais bem sucedidos na indústria, dado que são facilmente implementáveis, ajustáveis e são bem sucedidos uma grande quantidade de aplicações. Várias técnicas propõem inclusive controladores PID capazes de lidar com atrasos.

Quando falamos de sistemas fracionários neste contexto, dois caminhos podem ser seguidos. O primeiro trata de controladores fracionários, conhecidos por $PI^{\alpha}D^{\beta}$. O segundo trata de controladores PID clássicos para sistemas fracionários, e será este o ponto que iremos investigar.

Definição do Problema

Considere o problema de realimentação de saída SISO onde C é o controlador a ser projetado para a planta P. Assumimos que a planta é linear e invariante com o tempo, com seu comportamento representado pela seguinte função de transferência:

$$P(s) = e^{-s\tau}\frac{G(s^{\mu})}{s^{\mu} - p} \tag{B.39}$$

onde $\tau > 0$ é o atraso total de entrada-saída, $\mu \in (0,\ 1)$ é a ordem fracionária, $p^{1/\mu} \geq 0$ é a localização do pólo instável da planta, e $G(\sigma)$ uma função de transferência racional estável na variável $\sigma = s^{\mu}$ com $G(p) \neq 0$ e $G(0) \neq 0$.

Dados todos os parâmetros da planta, nosso objetivo é projetar um controlador clássico PID na forma

$$C(s) = K_p + \frac{K_i}{s} + K_d\frac{s}{\tau_d s + 1} \tag{B.40}$$

onde $K_p,\ K_i,\ K_d$ são parâmetros livres e τ_d é um valor arbitrário, pequeno e positivo utilizado para a realização própria do controlador. O sistema realimentado formado pelo controlador C e a planta P é estável se $(1+PC)^{-1}$, $C(1+PC)^{-1}$, $P(1+PC)^{-1}$ são funções de transferência estáveis.

A estratégia adotada será em dois passos: primeiro, controladores PD serão investigados para em seguida se adicionar a ação integral.

Síntese de controladores PD

Um controlador típico PD tem a forma

$$C_{pd}(s) = K_p\left(1 + \widetilde{K}_d\frac{s}{\tau_d s + 1}\right). \tag{B.41}$$

Podemos exprimir a parte sem atraso da planta como a razão de dois fatores estáveis

$$P(s) = e^{-s\tau}Y(s)^{-1}X(s) \quad \text{com} \quad Y(s) := \frac{s^{\mu} - p}{s^{\mu} + x} \quad X(s) := \frac{G(s^{\mu})}{s^{\mu} + x} \tag{B.42}$$

onde $x > 0$ é um parâmetro livre. A estabilidade do sistema realimentado é equivalente a estabilidade de U^{-1}, onde

$$U(s) := Y(s) + e^{-s\tau}X(s)C_{pd}(s). \tag{B.43}$$

Inserindo C_{pd}, X e Y em (B.43), obtemos

$$U(s) = 1 - \frac{p+x}{s^\mu + x} + e^{-s\tau}\frac{G(s^\mu)}{s^\mu + x}K_p\left(1 + \widetilde{K}_d\frac{s}{\tau_d s + 1}\right). \tag{B.44}$$

Escolhendo $K_p = (p+x)G(0)^{-1}$, temos

$$U(s) = 1 - \frac{(p+x)s^\mu}{s^\mu + x}\left(\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\alpha}}{\tau_d s + 1}\right). \tag{B.45}$$

Dado que $\left\|\frac{s^\mu}{s^\mu + x}\right\|_\infty = 1$ para todo $x > 0$, pelo teorema do pequeno ganho, U^{-1} é estável se

$$\left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\mu}}{\tau_d s + 1}\right\|_\infty < \frac{1}{p+x}. \tag{B.46}$$

Deste resultado, derivam-se diretamente as condições para os controladores P e PD ($\widetilde{K}_d = 0$). Para este último caso, podemos ver que os valores possíveis para o ganho proporcional estão na faixa

$$K_p^{\min} := pG(0)^{-1} < K_p < \psi_d G(0)^{-1} =: K_p^{\max}. \tag{B.47}$$

Gostaríamos de maximizar ψ_d para maximizar a faixa disponível para K_p. Este problema equivale a encontrar o $\widetilde{K}_d \in \mathbb{R}$ ótimo para que

$$\psi_d^{-1} = \left\|\frac{1 - e^{-s\tau}G(s^\mu)G(0)^{-1}}{s^\mu} - \widetilde{K}_d e^{-s\tau}G(s^\mu)G(0)^{-1}\frac{s^{1-\mu}}{\tau_d s + 1}\right\|_\infty \tag{B.48}$$

seja minimizado para $\tau_d > 0$ fixo, o que só pode ser resolvido com uma busca bi-dimensional.

Com ψ_d maximizado, a escolha de K_p para que a margem de ganho seja maximizada, i.e.,

$$\min\left\{\frac{K_p}{K_p^{\min}}, \frac{K_p^{\max}}{K_p}\right\} \tag{B.49}$$

máximo, pode ser resolvida com $K_p^{\text{opt}} = \sqrt{K_p^{\min}K_p^{\max}}$.

Adicionando a ação integral

Assuma que um controlador estabilizante PD possa ser encontrado para a planta (B.39). Tentaremos achar agora

$$C_i(s) = K_i s^{-1} \tag{B.50}$$

tal que $C_{pid}(s) = C_{pd}(s) + C_i(s)$ é um controlador estabilizante geral. Para tal, definindo

$$H(s) := P(s)(1 + P(s)C_{pd}(s))^{-1} \tag{B.51}$$

observamos que $H(0) = G(0)/x$ é não-nulo. Se C_i definido em (B.50) for um controlador estabilizante para a "nova planta" H, (B.51), C_{pid} será um controlador estabilizante apara a planta original$P$. Assumindo

$$K_i := \gamma H(0)^{-1}, \quad \text{com} \quad \gamma > 0 \tag{B.52}$$

temos

$$(1 + C_i(s)H(s))^{-1} = \frac{s}{s+\gamma}\left(1 + \frac{\gamma s^\alpha}{s+\gamma}\left(\frac{H(s)H(0)^{-1} - 1}{s^\alpha}\right)\right)^{-1}. \tag{B.53}$$

Definindo

$$R_\alpha(\gamma) := \left\|\frac{\gamma s^\alpha}{s+\gamma}\right\|_\infty, \tag{B.54}$$

temos, pelo teorema do pequeno ganho, que $C_i(s) = \gamma H(0)^{-1}/s$ é um controle estabilizante de $H(s)$ se

$$0 < R_\alpha(\gamma) < \left\|\frac{H(s)H(0)^{-1} - 1}{s^\alpha}\right\|_\infty^{-1}. \tag{B.55}$$

Quando $H(s)$ é uma função de tranferência fracionária, $(H(s)H(0)^{-1} - 1)/s$ pode ser instável devido a questões de seu limite em zero. Portanto, precisamos calcular $R_\alpha(\gamma)$ como uma função de γ para o valor de α que a planta apresenta.

Lema 19 *O valor de $R_\alpha(\gamma)$ como definido em B.54 é dado por*

$$R_\alpha(\gamma) = \alpha^{\alpha/2}(1-\alpha)^{(1-\alpha)/2}\gamma^\alpha. \tag{B.56}$$

Podemos nos perguntar qual é a melhor escolha para $x > 0$ para que o intervalo de valores possíveis para γ seja o maior possível. Por exemplo, para o controlador PI, para cada valor fixo de x no intervalo $0 < x < (\psi_o - p)$ podemos computar este intervalo numéricamente. Portanto, o maior intervalo em γ e o valor ótimo correspondente x podem ser encontrados com uma procura numérica unidimensional.

Podemos no entanto encontrar uma solução sub-ótima analítica, devido ao seguinte limitante inferior:

$$\tilde{\gamma} := x\frac{\psi_o - (p+x)}{\psi_o + p} \leq \left\|\frac{H(s)H(0)^{-1} - 1}{s^\mu}\right\|_\infty^{-1}. \tag{B.57}$$

Temos que maximizar $\tilde{\gamma}$ com a escolha apropriada de x. Seu valor ótimo é

$$x_{\text{opt}} = \frac{\psi_o - p}{2} \tag{B.58}$$

para o correspondente $\tilde{\gamma}$

$$\tilde{\gamma}_{\text{opt}} = \frac{x_{\text{opt}}^2}{\psi_o + p}. \tag{B.59}$$

O que determina que γ estará no intervalo

$$0 < \gamma < \frac{c_\alpha x_{\text{opt}}^{2/\alpha}}{(\psi_o + p)^{1/\alpha}} =: \gamma_{\max} \quad \text{onde} \quad c_\alpha := \left(\sqrt{\alpha} \; (1-\alpha)^{(1-\alpha)/2\alpha} \right)^{-1}. \tag{B.60}$$

Sistema Racional de Comparação

Nesta parte, vamos definir um sistema LTI que irá servir como um sistema de comparação para a análise de estabilidade e cálculo da norma $\mathcal{H}_\infty$ para um sistema com atraso, com o intuito de em frente ser utilizado para o projeto de filtros e controladores, da forma

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + E_0 w(t) \tag{B.61}$$

$$z(t) = C_{z0} x(t) + C_{z1} x(t-\tau) \tag{B.62}$$

onde $x(t) \in \mathbb{R}^n$ representa o estado, $w(t) \in \mathbb{R}^r$ a entrada externa e $z(t) \in \mathbb{R}^q$ a saída. Assumimos que o sistema evolui do zero e que o atraso $\tau \geq 0$ é constante.

É conhecido por *substituição de Rekasius* que, para $s = \jmath\omega$ com $\omega \in \mathbb{R}$, a igualdade

$$e^{-s\tau} = \frac{1 - \lambda^{-1}s}{1 + \lambda^{-1}s} \tag{B.63}$$

é satisfeita para $\lambda \in \mathbb{R}$ tal que $\omega/\lambda = \tan(\omega\tau/2)$. Desta forma, podemos introduzir um sistema de comparação para (B.61)-(B.62) da forma:

$$H(\lambda, s) = \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ A_0 + A_1 & A_0 - A_1 - \lambda I & E_0 \\ \hline C_{z0} + C_{z1} & C_{z0} - C_{z1} & 0 \end{array}\right] \tag{B.64}$$

Chamando de $T(\tau, s)$ a função de transferência não racional do sistema com atraso de w para z. este sistema LTI foi determinado de forma que a igualdade $H(\lambda, \jmath\omega) = T(\tau, \jmath\omega)$ é satisfeita sempre que as constantes $\lambda \in \mathbb{R}$, $\tau \geq 0$ e a frequência $\omega \in \mathbb{R}$ estão relacionadas pela relação

$\omega/\lambda = \tan(\omega\tau/2)$.

Análise de Estabilidade

Queremos determinar o maior atraso $\tau^* > 0$ tal que o sistema (B.61)-(B.62) permanece estável para todo $\tau \in [0, \tau^*)$. Isto depende do cálculo das raízes da equação característica de $T(\tau, s)$

$$\Delta_T(\tau, s) = \det\Big(sI - A_0 - A_1 e^{-s\tau}\Big) \tag{B.65}$$

Porém, reescrevendo em uma forma alternativa

$$\Psi(\tau, s) = \frac{\Delta_T(\tau, s)}{\Delta_T(0, s)} = \det\Big(I + (sI - A_0 - A_1)^{-1} A_1 (1 - e^{-s\tau})\Big) \tag{B.66}$$

podemos aplicar o critério de Nyquist neste sistema. A mesma estratégia pode ser adotada para a equação característica do sistema de comparação $\Delta_H(\lambda, s) = 0$, onde

$$\Delta_H(\lambda, s) = \det\Big(sI - A_\lambda\Big) \tag{B.67}$$

é um polinômio que admite $2n$ raízes. Podemos fatorizá-la como

$$\Phi(\lambda, s) = \frac{\Delta_H(\lambda, s)}{(s+\lambda)^n \det(sI - A_0 - A_1)} = \det\left(I + (sI - A_0 - A_1)^{-1} A_1 \frac{2s}{s+\lambda}\right) \tag{B.68}$$

que satisfaz a igualdade $\Psi(\tau, \jmath\omega) = \Phi(\lambda, \jmath\omega)$, desde que $e^{-s\tau}$ seja substituída pela relação definida pela susbstituição de Rekasius (B.63), válida para $s = \jmath\omega$, $\omega \in \mathbb{R}$.

Assumindo que o sistema com atraso é assintoticamente estável para $\tau = 0$, é possível determinar o par (τ^*, ω^*) correspondente à primeira ocorrência de $\Psi(\tau^*, j\omega^*) = 0$, definindo a chamada margem de estabilidade do sistema com atraso. De forma similar, todas as raízes de $\Delta_H(\lambda, s) = 0$ estão localizadas no semiplano aberto esquerdo para todo $\lambda > \lambda^o > 0$ sempre que $\Phi(\lambda^o, j\omega^o) = 0$ corresponder ao primeiro cruzamento do mapeamento $\Phi(\lambda, j\omega)$ obtido ao se diminuir λ de $+\infty$.

Obtido o par (τ^*, ω^*), podemos calcular λ^* através de $\omega^*/\lambda^* = \tan(\omega^*\tau^*/2)$, que sastisfaz $\Phi(\lambda^*, j\omega^*) = 0$ e portanto $\lambda^* \leq \lambda^o$. Sempre que $\lambda^* = \lambda^o > 0$ podemos garantir que para todo $\lambda \in (\lambda^*, \infty)$ e $\tau \in [0, \tau^*)$ tanto o sistema com atraso como o sistema de comparação são assintoticamente estáveis. A caracterização completa de todos os sistemas com atraso para os quais vale a igualdade $\lambda^o = \lambda^* > 0$ não é simples. Por esta razão, o lema seguinte fornece um sub-intervalo de (λ^o, ∞) tal que a estabilidade de ambos os sistemas são simultaneamente preservadas.

Lema 20 *Assuma que a matriz* $A_0 + A_1$ *é Hurwitz e que o sistema com atraso (B.61)-(B.62) é estável para todo* $\tau \in [0, \tau^*)$. *Neste caso, para todo* $\lambda \in (2/\tau^*, \infty)$ *o sistema de comparação é estável, e para qualquer* $\omega \in \mathbb{R}$ *o valor do atraso* τ *que satisfaz* $\omega/\lambda = \tan(\omega\tau/2)$ *pertence ao intervalo* $[0, \tau^*)$.

Cálculo da Norma $\mathcal{H}_\infty$

Pretendemos agora calcular a norma

$$\|T(\tau, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(T(\tau, j\omega)) \tag{B.69}$$

para um $\tau \in [0, \tau^*)$ dado. A existência de $\lambda^o > 0$ assegurando que A_λ é Hurwitz para todo $\lambda \in (\lambda^o, \infty)$ é essencial dado que, caso contrário, a igualdade $\|H(\lambda, s)\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda, j\omega))$ não é válida e a norma $\|H(\lambda, s)\|_\infty$ é ilimitada. Para tanto, definimos o escalar positivo $\lambda_o = \inf \{\lambda \mid A_\xi \text{ is Hurwitz } \forall \xi \in (\lambda, \infty)\}$. Assumindo que a matriz $A_0 + A_1$ é Hurwitz, o escalar λ_o existe e é dado por $\lambda_o = \lambda^o$ ou $\lambda_o = 0^+$ sempre que $\lambda^o > 0$ ou $\lambda^o < 0$, respectivamente.

Teorema 21 *Assuma que* $A_0 + A_1$ *é Hurwitz. Para cada* $\lambda \in (\lambda_o, \infty)$ *defina* $\alpha \geq 0$ *tal que*

$$\alpha = \arg \sup_{\omega \in \mathbb{R}} \sigma(H(\lambda, j\omega)) \tag{B.70}$$

e determine $\tau(\lambda)$ *a partir de* $\alpha/\lambda = \tan(\alpha\tau/2)$. *Se* $\tau(\lambda) \in [0, \tau^*)$ *então* $\|H(\lambda, s)\|_\infty \leq \|T(\tau(\lambda), s)\|_\infty$.

Corolário 22 *Assuma que* $A_0 + A_1$ *é Hurwitz. Para qualquer parâmetro positivo* $\gamma > \|H(\infty, s)\|_\infty$ *dado, existem* $\lambda_\gamma \geq \lambda_o > 0$ *e* $0 \leq \tau_\gamma \leq \tau^*$ *tais que a desigualdade*

$$\|H(\lambda, s)\|_\infty \leq \|T(\tau, s)\|_\infty < \gamma \tag{B.71}$$

é válida $\forall\ \lambda \in (\lambda_\gamma, \infty)$, *sempre que a função* $\tau(\lambda)$ *dada pelo Teorema 21 for contínua no intervalo.*

Projeto de Filtros Lineares

Considere um sistema com atraso cuja realização mínima do espaço de estado é dada por

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + E_0w(t) \quad \text{(B.72)}$$

$$y(t) = C_{y0}x(t) + C_{y1}x(t-\tau) + D_yw(t) \quad \text{(B.73)}$$

$$z(t) = C_{z0}x(t) + C_{z1}x(t-\tau) \quad \text{(B.74)}$$

onde $x(t) \in \mathbb{R}^n$ denomina o estado, $w(t) \in \mathbb{R}^r$ a entrada externa, $y(t) \in \mathbb{R}^p$ o sinal medido e $z(t) \in \mathbb{R}^q$ a saída a ser estimada. Assumimos que o sistema evolui do repouso e que o atraso $\tau \geq 0$ é constante.

Nosso objetivo é projetar um filtro de ordem completa

$$\dot{\hat{x}}(t) = \hat{A}_0\hat{x}(t) + \hat{A}_1\hat{x}(t-\tau) + \hat{B}_0y(t) \quad \text{(B.75)}$$

$$\hat{z}(t) = \hat{C}_0\hat{x}(t) + \hat{C}_1\hat{x}(t-\tau) \quad \text{(B.76)}$$

onde $\hat{x}(t) \in \mathbb{R}^n$. Quando conectado a (B.72)-(B.74) o filtro define a realização mínima do erro de estimação $\varepsilon(t) = z(t) - \hat{z}(t)$, em termos de

$$\dot{\xi}(t) = F_0\xi(t) + F_1\xi(t-\tau) + G_0w(t) \quad \text{(B.77)}$$

$$\varepsilon(t) = J_0\xi(t) + J_1\xi(t-\tau) \quad \text{(B.78)}$$

onde $\xi(t) = [x(t)' \ \hat{x}(t)']' \in \mathbb{R}^{2n}$ é o vetor de estado do erro de estimação, e as matrizes indicadas são dadas por

$$F_0 = \begin{bmatrix} A_0 & 0 \\ \hat{B}_0C_{y0} & \hat{A}_0 \end{bmatrix}, \ F_1 = \begin{bmatrix} A_1 & 0 \\ \hat{B}_0C_{y1} & \hat{A}_1 \end{bmatrix} \quad \text{(B.79)}$$

$G_0' = [E_0' \ \ D_y'\hat{B}_0']$, $J_0 = [C_{z0} \ \ -\hat{C}_0]$ e $J_1 = [C_{z1} \ \ -\hat{C}_1]$. Sua função de tranferência de w para ε, é dada por

$$T_F(\tau, s) = \left(J_0 + J_1e^{-\tau s}\right)\left(sI - F_0 - F_1e^{-\tau s}\right)^{-1}G_0. \quad \text{(B.80)}$$

Nosso objetivo é projetar um filtro com realização em estado de espaço dada por (B.75)-(B.76) tal que $\|T_F(\tau, s)\|_\infty < \gamma$ onde $\tau \in [0, \tau^*)$ e $\gamma > 0$. A estratégia será substituir a função de transferência $T_F(\tau, s)$ pela equivalente do sistema de comparação $H_F(\lambda, s)$, ou seja

$$H_F(\lambda, s) = \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ F_0 + F_1 & F_0 - F_1 - \lambda I & G_0 \\ \hline J_0 + J_1 & J_0 - J_1 & 0 \end{array}\right] \tag{B.81}$$

resolver o problema de filtragem $\mathcal{H}_\infty$ e extrair o atraso como indicado no Corolário 22. Isto é possível pois apesar das matrizes da realização do estado de espaço de $H_F(\lambda, s)$ dependerem de uma maneira complicada das matrizes do filtro, existe uma matriz de transformação de similaridade S tal que (B.81) é equivalente a

$$H_F(\lambda, s) = \left[\begin{array}{c|c} SF_\lambda S^{-1} & SG \\ \hline JS^{-1} & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_\lambda & 0 & E \\ \hat{L}C_y & \hat{A}_\lambda & \hat{L}D_y \\ \hline C_z & -\hat{C} & 0 \end{array}\right] \tag{B.82}$$

onde (A_λ, E, C_z) foram definidas em (B.64), $C_y = [C_{y0} + C_{y1} \;\; C_{y0} - C_{y1}]$,

$$\hat{A}_\lambda = \begin{bmatrix} 0 & \lambda I \\ \hat{A}_0 + \hat{A}_1 & \hat{A}_0 - \hat{A}_1 - \lambda I \end{bmatrix}, \; \hat{L} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix} \tag{B.83}$$

e $\hat{C} = [\hat{C}_0 + \hat{C}_1 \;\; \hat{C}_0 - \hat{C}_1]$.

À primeira vista (B.82) leva a conclusão de que a realização em espaço de estado de $H_F(\lambda, s)$ possui a mesma estrutura do erro de estimação clássico. Apesar de correto, as matrizes $\hat{A}_\lambda$ e $\hat{L}$ precisam ter uma estrutura particular dada em (B.83). Para lidar com esse fato, começamos por projetar um filtro LTI substituindo as variáveis matriciais $(\hat{A}_\lambda, \hat{L}, \hat{C})$ em (B.82) por variáveis matriciais genéricas (A_F, L_F, C_F) e resolver $\|H_F(\lambda, s)\|_\infty < \gamma$ para um dado $\gamma > 0$. O segundo passo é determinar uma matriz não singular $V \in \mathbb{R}^{2n \times 2n}$ tal que $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$, o que garante que a função de transferência do erro de estimação para o sistema de comparação se mantém inalterada. Para um dado $\gamma > 0$ e assumindo $D_yD_y' = I$, obtemos o filtro central na forma de observador

$$\left[\begin{array}{c|c} A_F & L_F \\ \hline C_F & 0 \end{array}\right] = \left[\begin{array}{c|c} A_\lambda - L_FC_y & L_F \\ \hline C_z & 0 \end{array}\right] \tag{B.84}$$

onde o ganho do filtro é dado por $L_F = PC_y' + ED_y'$ e $P = P' > 0$ satisfaz a desigualdade de Riccati

$$\tilde{A}_\lambda P + P\tilde{A}_\lambda' + \tilde{E}\tilde{E}' - P\Big(C_y'C_y - \gamma^{-2}C_z'C_z\Big)P < 0 \tag{B.85}$$

onde $\tilde{A}_\lambda = A_\lambda - ED_y'C_y$ e $\tilde{E} = E\left(I - D_y'D_y\right)$.

Lema 23 *Assuma que* $\dim(y) = p \leq n = \dim(x)$*,* $\lambda > 0$ *e a matriz*

$$V = \begin{bmatrix} N \\ NA_F/\lambda \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{B.86}$$

onde $N' \in \mathbb{R}^{2n \times n}$ *é não singular e pertence ao espaço nulo de* L_F'*. Nestas condições, a igualdade* $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$ *é verificada.*

Uma vez determinada a matriz V, obtemos a representação em estado de espaço do filtro $(\hat{A}_\lambda, \hat{L}, \hat{C}) = (VA_FV^{-1}, VL_F, C_FV^{-1})$ de onde as matrizes do filtro (B.75) – (B.76) podem ser extraídas.

Teorema 24 *Considere* $\gamma > \min_F \|H_F(\infty, s)\|_\infty$ *dado. Para* $\lambda > 0$ *grande o suficiente, a solução central de (B.85) definido pelo par* (L_F, P) *é tal que* P *possui a estrutura particular*

$$P = \begin{bmatrix} Z + Q & -Q \\ -Q & Q \end{bmatrix} \tag{B.87}$$

onde $Z \in \mathbb{R}^{n \times n}$ *e* $Q \in \mathbb{R}^{n \times n}$ *são matrizes definidas positivas. Além do mais, as relações* $\|H_F(\infty, s)\|_\infty = \|T_F(0, s)\|_\infty < \gamma$ *são verificadas.*

Este teorema mostra que, para $\lambda > 0$ grande o suficiente, a solução central da desigualdade de Riccati (B.85) possui uma estrutura particular e gera um filtro tal que as normas do erro de estimação, $\|H_F(\infty, s)\|_\infty$ e $\|T_F(0, s)\|_\infty$, coincidem. Portanto, pelo Corolário 22, existe um intervalo $\lambda \in (\lambda_\gamma, \infty)$, tal que

$$\|H_F(\lambda, s)\|_\infty \leq \|T_F(\tau, s)\|_\infty < \gamma \tag{B.88}$$

para todo $\tau \in [0, \tau_\gamma)$ onde λ_γ e τ_γ são determinados pelo seguinte algoritmo:

- Defina uma sequência decrescente estrita $\lambda_k \in \{\infty, \cdots, \lambda_o\}$.
- Para cada elemento λ_k, determine o filtro central $(A_{F_k}, L_{F_k}, C_{F_k})$ e o atraso $\tau_k = \tau(\lambda_k)$.
- A partir da matriz de transformação de similaridade V_k, determine o filtro $(\hat{A}_{\lambda_k}, \hat{L}_k, \hat{C}_k)$.
- Aumente k sempre que $0 < \tau_k - \tau_{k-1} < 2(\lambda_{k-1} - \lambda_k)/\lambda_k^2$ e $\|T_{F_k}(\tau_k, s)\|_\infty < \gamma$.

Projeto de Controladores via Realimentação de Estado

Mudamos nossa atenção para o sistema com atraso

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau) + B_0u(t) + E_0w(t) \quad \text{(B.89)}$$

$$z(t) = C_0x(t) + C_1x(t-\tau) + G_0u(t) + D_0w(t) \quad \text{(B.90)}$$

onde $x(t) \in \mathbb{R}^n$ denomina o estado, $u(t) \in \mathbb{R}^m$ o controle, $w(t) \in \mathbb{R}^r$ a entrada externa e $z(t) \in \mathbb{R}^q$ a saída controlada. As hipóteses gerais sobre o sistema são mantidas. O objetivo é projetar um controlador via realimentação de estado $u(t) = K_0x(t)+K_1x(t-\tau)$ que conectado ao sistema em malha aberta (B.61)-(B.62) produz um sistema em malha fechada com realização em espaço de estado da forma (B.61)-(B.62), e função de transferência $T_K(\tau, s)$. Adotando a mesma estratégia anterior, a função racional associada $H_K(\lambda, s)$ é dada por

$$H_K(\lambda, s) = \left[\begin{array}{c|c} A_\lambda + BK & E \\ \hline C + GK & D \end{array}\right] \quad \text{(B.91)}$$

onde $B' = [0 \ B_0']$, $G = G_0$, $D = D_0$ e $K = \left[\begin{array}{cc} K_0 + K_1 & K_0 - K_1 \end{array}\right]$.

Observamos que a função de transferência $H_K(\lambda, s)$ possui uma estrutura de malha fechada, e uma vez o ganho $K \in \mathbb{R}^{m\times 2n}$ determinado, ela fornece os ganhos do sistema com atraso $K_0 \in \mathbb{R}^{m\times n}$ e $K_1 \in \mathbb{R}^{m\times n}$. Portanto, o problema se resume à determinação de um ganho de realimentação de estado $K \in \mathbb{R}^{m\times 2n}$ tal que $\|T_K(\tau, s)\|_\infty < \gamma$, que é substituído pela determinação de $K \in \mathbb{R}^{m\times 2n}$ tal que $\|H_K(\lambda, s)\|_\infty < \gamma$. Considerando que $\gamma^2 I - D'D > 0$, $G'G$ não são singulares e que o sistema em malha aberta satisfaz as condições de ortogonalidade $G'[C \ D] = 0$, $D'C = 0$, sua solução central é dada por $K = -(G'G)^{-1}B'P$, onde $P > 0$ satisfaz a desigualdade de Riccati

$$A_\lambda' P + PA_\lambda + C'C - P\Big(B(G'G)^{-1}B' - E(\gamma^2 I - D'D)^{-1}E'\Big)P < 0 \quad \text{(B.92)}$$

Projeto de Controladores via Realimentação de Saída

Técnicas similares às desenvolvidas para a síntese de filtros lineares e controladores via realimentação de estado podem ser adaptadas para lidar com o problema do projeto de controladores de saída.