Rainbow structures in properly edge-colored graphs and hypergraph systems

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Structures arc-en-ciel dans les graphes proprement arêtes-colorés et les systèmes des hypergraphes

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Titre : Structures arc-en-ciel dans les graphes proprement arêtes-colorés et les systèmes des hypergraphes.

Mots clés : Problèmes extrêmes; système de k-graphes; appariement parfait arc-en-ciel; cycle Hamiltonien arc-en-ciel.

Résumé : La combinatoire extrémal est l'une des branches les plus vigoureuses des mathématiques combinatoires au cours des dernières décennies, et elle a été largement utilisée en informatique, en conception de réseaux et en conception de codage. Elle se concentre sur la détermination de la taille maximale ou minimale possible de certaines structures combinatoires, sous certaines conditions ou contraintes. En particulier, la théorie des graphes extrémaux est une branche importante de la combinatoire extrémale, qui traite principalement de la manière dont les propriétés générales d'un graphe contrôlent la structure locale du graphe.

Un système de k-graphes $\boldsymbol{H} = \{H_i\}_{i\in[m]}$ est une collection de k-graphes sur le même ensemble de sommets V. Pour un système de k-graphes $\boldsymbol{H} = \{H_i\}_{i\in[m]}$ sur V, un graphe H sur V est arc-en-ciel dans $\boldsymbol{H}$ s'il existe une injection $\varphi : E(H) \to [m]$ telle que $e \in E(H_{\varphi(e)})$ pour chaque $e \in E(H)$.

Cette thèse produit une étude en trois parties.

(1) Nous étudions l'existence de cycles Hamiltoniens arc-en-ciel dans les systèmes de k-graphes. Le théorème de Dirac a de nombreuses variantes. Tout d'abord, il a été généralisé dans les systèmes de graphes. Deuxièmement, il a été généralisé dans les hypergraphes. Dans le même esprit, nous voulons trouver un cycle Hamiltonien arc-en-ciel dans un système d'hypergraphes. Étant donné $k \geq 3, \gamma > 0$, un n-vertex k-système de k-graphes $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ avec $\delta_{k-1}(H_i) \geq (1/2+\gamma)n$ pour chaque $i \in [n]$, alors il existe un cycle Hamiltonien $\boldsymbol{H}$-arc-en-ciel. De plus, des chercheurs se sont consacrés à caractériser la condition de degré $(k-2)$ pour l'existence d'un cycle Hamiltonien. Lang et Sanhueza-Matamala, Polcyn, Reiher, Rödl et Schulke ont prouvé indépendamment que pour tout $\gamma > 0$, chaque n-vertex k-graph avec $\delta_{k-2}(H) \geq (5/9+\gamma)\binom{n}{2}$ contient un cycle Hamiltonien. Cependant, la version arc-en-ciel de la conclusion ci-dessus est beaucoup plus difficile. Gupta, Hamann, Muyesser, Parczyk et Sgueglia ont mentionné le problème : étant donné un système de 3-graphes $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ avec une condition de degré minimum de chaque H_i, est-ce que $\boldsymbol{H}$ admet un cycle Hamiltonien arc-en-ciel? Nous résolvons le problème ci-dessus et tirons la conclusion générale pour tout $k \geq 3$.

(2) Nous étudions l'existence d'une appariement parfaite arc-en-ciel dans les systèmes de k-graphes. Soit $c_{k,d}$ le seuil minimum de d-degré pour des appariements fractionnaires parfaites dans les graphiques k, à savoir, pour chaque $\varepsilon > 0$ et suffisamment grand $n \in \mathbb{N}$, chaque n-sommet k-graphe H avec $\delta_d(H) \geq (c_{k,d}+\varepsilon)\binom{n-d}{k-d}$ contient une appariement fractionnaire parfaite. On sait que tout k-graphe de n sommets H avec $\delta_d(H) \geq (\max\{c_{k,d}, 1/2\}+o(1))\binom{n-d}{k-d}$ a un appariement parfait, et cette condition est asymptotiquement optimale. Nous démontrons que dans un k-graphe, les conditions minimales de d-degré pour un appariement parfait garantissent également asymptotiquement la présence d'un appariement parfait rainbow dans le système du k-graphes pour $d \in [k-1]$. Plus généralement, un cadre général pour résoudre l'existence de facteurs transversaux dans les systèmes d'hypergraphes peut également être donné.

(3) Nous étudions l'existence de longs cycles arc-en-ciel dans des graphes proprement arêtes-colorés. En 1989, Andersen a conjecturé que chaque K_n proprement arêtes-colorés admet un chemin arc-en-ciel qui omet un seul sommet. Nous avons prouvé que chaque $K_{n,n}$ proprement arêtes-colorés contient un cycle arc-en-ciel d'au moins $n-28n^{3/4}$ pour n suffisamment grand. La limite ci-dessus est asymptotiquement optimale car chaque classe de couleurs pourrait être un couplage parfait de $K_{n,n}$ et seuls n couleurs apparaissent dans $E(K_{n,n})$.

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Title : Rainbow structures in properly edge-colored graphs and hypergraph systems

Keywords : Extremal problem; k-graph system; rainbow perfect matching; rainbow Hamilton cycle.

Abstract : Extremal Combinatorics is one of the most vigorous branch of Combinatorial Mathematics in recent decades and it has been widely used in Computer Science, Network Design and Coding Design. It focuses on determining the maximum or minimum possible size of certain combinatorial structures, subject to certain conditions or constraints. In particular, Extremal Graph Theory is a significant branch of Extremal Combinatorics, which primarily explores how the overall properties of a graph influence its local structures.

A k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[m]}$ is a collection of not necessarily distinct k-graphs on the same vertex set V. For a k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[m]}$ on V, a graph H on V is rainbow in $\boldsymbol{H}$ if there exists an injection $\varphi : E(H) \to [m]$ such that $e \in E(H_{\varphi(e)})$ for each $e \in E(H)$.

This thesis presents a three-part study.

(1) We study the existence of rainbow Hamilton cycle in k-graph systems. Dirac's theorem has many variants. Firstly, it was generalized in graph systems. Secondly, it was generalized in hypergraphs. Along the same idea, we want to find a rainbow Hamilton cycle in a hypergraph system. Given $k \geq 3, \gamma > 0$, sufficiently large n and an n-vertex k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$, if $\delta_{k-1}(H_i) \geq (1/2 + \gamma)n$ for each $i \in [n]$, then there exists an $\boldsymbol{H}$-rainbow Hamilton cycle. Further, scholars devoted to characterizing the $(k-2)$-degree condition for the existence of a Hamilton cycle. Lang and Sanhueza-Matamala, Polcyn, Reiher, Rödl and Schülke independently proved that for any $\gamma > 0$, every n-vertex k-graph with $\delta_{k-2}(H) \geq (5/9 + \gamma)\binom{n}{2}$ contains a Hamilton cycle. However, the rainbow version of the above conclusion is much more difficult. Gupta, Hamann, Müyesser, Parczyk, and Sgueglia mentioned the following problem : Given a 3-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ with minimum vertex degree condition of each H_i, does $\boldsymbol{H}$ admit a rainbow Hamilton cycle? We settle the above problem, and draw the general conclusion for any $k \geq 3$.

(2) We study the existence of rainbow perfect matching in k-graph systems. Let $c_{k,d}$ be the minimum d-degree threshold for perfect fractional matchings in k-graphs, namely, for every $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$, every n-vertex k-graph H with $\delta_d(H) \geq (c_{k,d} + \varepsilon)\binom{n-d}{k-d}$ contains a perfect fractional matching. It is known that every n-vertex k-graph H with $\delta_d(H) \geq (\max\{c_{k,d}, 1/2\} + o(1))\binom{n-d}{k-d}$ has a perfect matching, and this condition is asymptotically best possible. We proved that a minimum d-degree condition asymptotically forcing a perfect matching in a k-graph also forces rainbow perfect matchings in k-graph systems for $d \in [k-1]$. More generally, a general framework for solving the existence of rainbow factors in hypergraph systems can also be given.

(3) We study the existence of long rainbow cycle in properly edge-colored graphs. In 1989, Andersen conjectured that all proper edge-colorings of K_n admit a rainbow path which omits only one vertex. We proved that every properly edge-colored $K_{n,n}$ contains a rainbow cycle of length at least $n - 28n^{3/4}$ for sufficiently large n. The bound above is asymptotically optimal as each color class could be a perfect matching of $K_{n,n}$ and only n colors occur in $E(K_{n,n})$.

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1 - Introduction

The paper written by Euler on the Seven Bridges of Königsberg and published in 1736 is regarded as the first paper in the history of graph theory [16]. As it continued to develop, graph theory has many branches, including graph coloring, extremal graph theory, algebraic graph theory, topological graph theory, probabilistic graph theory and so on. Since graphs can be used to model many types of relations and processes in physical, biological, social and information systems, graph theory has wide applications in real-world systems. Meanwhile, it also has applications in other areas of mathematics, such as group theory and number theory.

In this thesis, we mainly study the rainbow spanning structures in properly edge-colored graphs and hypergraph systems. For the convenience of the description of our research topic's background, we first give some terminology and notation in the first section of this chapter. Then we will describe background in detail, including motivations, known results and so on.

1.1 . Terminologies and tools

Graph

A *graph* G is an ordered pair $(V(G), E(G))$ with a nonempty set $V(G)$ of *vertices* and a set $E(G)$ of *edges*, where $E(G)$ is made up of some unordered pairs of (not necessarily distinct) vertices. A graph is *finite* if both its vertex set and edge set are finite. A graph G is *simple* if $E(G)$ is a collection of some distinct 2-subsets of $V(G)$. Note that unless otherwise stated, all graphs considered in this thesis are finite and simple. The *order* of a graph G refers to the cardinality of $V(G)$, while the *size* refers to the cardinality of $E(G)$. If there is a path between any two vertices of G, then G is called a *connected* graph. A *complete graph* K_r is a simple graph on r vertices in which every pair of distinct vertices is connected by a unique edge.

Adjacent and incident

Let G be a graph and $u, v \in V(G)$. We say that u, v are *adjacent* if the 2-set $\{u, v\} \in E(G)$. Let $e \in E(G)$ with $e = \{u, v\}$. Then we say that u, v are two *ends* of e and u, v are *incident* with e, respectively.

Degree

Let G be a graph and $u, v \in V(G)$. If u, v are adjacent, then u is called a *neighbor* of v and vice versa. For any vertex $v \in V(G)$, we use $N_G(v)$ to denote the set of all neighbors of v and call $N_G(v)$ the *neighborhood* of v. The cardinality of $N_G(v)$ is the *degree* of v, denoted by $d_G(v)$, i.e. $d_G(v) = |N_G(v)|$. Denote by $\delta(G)$ and $\Delta(G)$ the *minimum degree* and *maximum degree* of G, respectively. Denote the *average degree* $\sum_{v\in V(G)} \frac{d_G(v)}{n}$ of G by $d(G)$.

Subgraph

Let G, H be two graphs. We say H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if H is a subgraph of G and H contains all the edges $\{u, v\} \in E(G)$ with $u, v \in V(H)$, then H is an *induced* subgraph of G. If H is a subgraph of G and $V(H) = V(G)$, then H is a *spanning* subgraph of G. We say G is F*-free* if G does not contain F as a subgraph. Let H_1, H_2 be two subgraphs of G. If $V(H_1) \cap V(H_2) = \emptyset$, then we say H_1 and H_2 are *vertex-disjoint*. If $E(H_1) \cap E(H_2) = \emptyset$, then we say H_1 and H_2 are *edge-disjoint*. Let $V' \subseteq V(G)$ and $E' \subseteq E(G)$. We use $G - V'$ to denote the subgraph induced by $V(G) \setminus V'$ and use $G \setminus E'$ to denote the subgraph of G containing the same vertices as G but with all the elements of E' removed. In particular, if $V' = \{v\}$, then we write $G - v$ for simplicity. If $E' = \{e\}$, then $G \setminus \{e\}$ will be replaced with $G \setminus e$.

Walk, Path and Cycle

Let G be an n-vertex graph. A *walk* in G is defined as a sequence of alternating vertices and edges such as $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$, where each $e_i = \{v_{i-1}, v_i\}$. The *length* of this walk is k. A walk is considered to be *closed* if the starting vertex is the same as the ending vertex, that is $v_0 = v_k$. A walk is considered *open* otherwise. A *path* is defined as an open walk with no repeated vertices. A *cycle* is defined as a closed walk where no other vertices are repeated apart from the starting/ending vertex. We usually use P_k and C_k to denote a path of length of $k-1$ and a cycle of length k, respectively. A path (cycle) is called a *Hamilton path* (*Hamilton cycle*) if it visits each vertex of G exactly once. A cycle of length of 3 is called a *triangle*.

Multipartite graph

Let k be a positive integer. A k*-partite graph* is a graph whose vertex set can be partitioned into k different independent sets, which are called k *parts* of the graph. When $k = 2$, these are the *bipartite graphs*. A k-partite graph is *balanced* if the k parts have same cardinality. A *complete* k-partite graph is a k-partite graph in which there is an edge between every pair of vertices from different independent sets.

Power of a graph

Let G be a graph and k be an integer. The k-th power of G, denoted by G^k, is defined as the graph on the same vertex set whose edges join distinct vertices at distance at most k in G.

Factor

Let G be an n-vertex graph and H be an h-vertex graph. An H*-tiling* is a collection of vertex-disjoint copies of H in G. An H*-factor* is an H-tiling which covers all vertices of G. Note that $n \in h\mathbb{N}$ is a necessary condition for G containing an H-factor. When H is C_k, we call it k*-factor* of G for convenience. In particular, a 1-factor of G is a perfect matching. A 2-factor of G is a collection of vertex-disjoint cycles covering all vertices of G. A connected 2-factor is a Hamilton cycle.

Vertex-coloring

Let k be a positive integer. A *k-coloring* of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest value of k possible to obtain a k-coloring.

Edge-coloring

An *edge-coloring* c of a graph G is an assignment of colors to the edges of G. A *k-edge-coloring* of G is an edge-coloring using k colors overall, while a *local k-edge-coloring* of G is an edge-coloring using at most k colors at each vertex of G. An edge-coloring of G is *proper* if no two adjacent edges receive the same color. An *edge-colored* graph is a graph with an edge-coloring (not necessarily proper). Given an edge-colored graph G, we say G is *monochromatic* if all edges of G have the same color, and G is a *rainbow* graph if all the edges receive pairwise different colors.

For every vertex $v \in V(G)$, the *color degree* of v, denoted by $d^c_G(v)$, is the number of distinct colors appearing on the incident edges of v. The *minimum color degree* of G, denoted by $\delta^c(G)$, is the minimum $d^c_G(v)$ over all vertices $v \in V(G)$. We say that color i is presented at vertex v if some edge incident with v has color i. The color neighborhood $CN(v)$ is the set of different colors that are presented at v.

Digraph

A *digraph* or *directed graph* D is an ordered pair $(V(D), A(D))$ consisting of a nonempty set $V(D)$ of *vertices* and a set $A(D)$ of *arcs*, where $A(D)$ is made up of some ordered pairs of (not necessarily distinct) vertices.

Outdegree and indegree

Let D be a digraph and $u, v \in V(D)$. If $(u, v) \in A(D)$, then we say that v is an *outneighbor* of u and u is an *inneighbor* of v. For any vertex $v \in V(D)$, let $N^+_D(v)$ and $N^-_D(v)$ be its *outneighborhood* and *inneighborhood*, i.e. the set of outneighbors and the set of inneighbors of v, respectively. Let $d^+_D(v) = |N^+_D(v)|$ and $d^-_D(v) = |N^-_D(v)|$ and call $d^+_D(v)$ and $d^-_D(v)$ *outdegree* and *indegree* of v, respectively. Denote by $\delta^+(D)$ and $\delta^-(D)$ the *minimum outdegree* and *minimum indegree*, respectively. Let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and call $\delta(D)$ the *minimum semi-degree* of D. Denote by $\Delta^+(D)$ and $\Delta^-(D)$ the *maximum outdegree* and *maximum indegree*, respectively.

Oriented graph

Let G be an n-vertex graph. If we give every edge of G a direction, then we obtain a digraph and we call this digraph an *oriented graph* of G.

Hypergraphs

A *k-uniform hypergraph* (*k-graph*, hereafter) $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$ which is a family of k-element subsets of $V(H)$, i.e. $E(H) \subseteq \binom{V(H)}{k}$. For any $S \subseteq V(H)$, the *degree* of S in H, denoted by $\deg_H(S)$, is the number of edges containing S. For any integer $\ell \geq 0$, define the *minimum ℓ-degree* $\delta_\ell(H)$ to be $\min\{\deg_H(S) : S \in \binom{V(H)}{\ell}\}$.

Subgraph

Let G, H be two k-graphs. We say H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* $H[V']$ of a k-graph H is a k-graph with vertex set V' and edge set E' where each edge is precisely the edge of H consisting of k vertices in V'. We usually denote $H[V']$ by H. If H is a subgraph of G and $V(H) = V(G)$, then H is a *spanning* subgraph of G. Let $V' \subset V(G)$ and $E' \subset E(G)$. We use $G - V'$ to denote the subgraph induced by $V(G) \setminus V'$ and use $G \setminus E'$ to denote the subgraph of G containing the same vertices as G but with all the elements of E' removed. Let H_1, H_2 be two subgraphs of H. If $V(H_1) \cap V(H_2) = \emptyset$, then we say H_1 and H_2 are *vertex-disjoint*. If $E(H_1) \cap E(H_2) = \emptyset$, then we say H_1 and H_2 are *edge-disjoint*.

k-partite k-graph

A k-graph H is *k-partite* if $V(H)$ can be partitioned into k parts $V_1, \ldots, V_k$ such that every edge consists of exactly one vertex from each part.

Tight path and tight cycle

A *tight path* P is a k-graph whose vertices can be ordered in such a way $v_1v_2\cdots v_t$ that each edge consists of k consecutive vertices and two consecutive edges intersect in exactly $k-1$ vertices. We say that P *connects* $(v_1, \ldots, v_{k-1})$ and $(v_t, \ldots, v_{t-k+2})$. $(v_1, \ldots, v_{k-1})$ and $(v_t, \ldots, v_{t-k+2})$ are called the *ends* of P.

A k-graph is called an *ℓ-cycle* if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In k-graphs, a $(k-1)$-cycle is often called a *tight cycle*, a 1-cycle is often called a *loose cycle*. We say that a k-graph contains a *Hamilton ℓ-cycle* if it contains an ℓ-cycle as a spanning subgraph. Without special instruction, the tight cycle is referred to as cycle for short.

Matching and fractional matching

Given a k-graph H, a *matching* in H is a collection of vertex-disjoint edges of H. A *perfect matching* in H is a matching that covers all vertices of H. a *fractional matching* is a function $f : E(H) \to [0, 1]$, subject to the requirement that $\sum_{e:v\in e} f(e) \leq 1$, for every $v \in V(H)$. Furthermore, if equality holds for every $v \in V(H)$, then we call the fractional matching *perfect*. Denote the maximum size of a fractional matching of H by $\nu^*(H) = \max_f \Sigma_{e\in E(H)} f(e)$.

k-graph systems

A *k-graph system* $\boldsymbol{H} = \{H_i\}_{i\in[m]}$ is a family of not necessarily distinct k-graphs on the same n-vertex set V where $k \geq 2$. Note that each H_i can be seen as the collection of edges with color i, and in this sense $\boldsymbol{H}$ can be regarded as an edge-colored multi-k-graph. Moreover, a k-graph H on V is *rainbow* in

H if there is an injection $\varphi : E(H) \to [m]$ such that $e \in E(H_{\varphi(e)})$ for each $e \in E(H)$. Note that H can be also called ***H***-rainbow.

For terminology and notation not mentioned here, we will give at the beginning of the respective chapters or refer readers to [41].

The following well-known concentration results, i.e. Chernoff bounds, can be found in [10, 75]. Denote a binomial random variable with parameters n and p by $Bi(n, p)$.

Lemma 1.1 (Chernoff Inequality for small deviation [10, 75]) *If $X = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are mutually independent random variables, each X_i has Bernoulli distribution with expectation p_i and $\alpha \le 3/2$, then*

$$\mathbb{P}[|X - \mathbb{E}[X]| \ge \alpha \mathbb{E}[X]] \le 2e^{-\frac{\alpha^2}{3}\mathbb{E}[X]}.$$

In particular, when $X \sim Bi(n, p)$ and $\lambda < \frac{3}{2}np$, then

$$\mathbb{P}[|X - np| \ge \lambda] \le e^{-\Omega(\lambda^2/(np))}.$$

Lemma 1.2 (Chernoff Inequality for large deviation [10, 75]) *If $X = \sum_{i=1}^{n} X_i$ where $X_1, \ldots, X_n$ are mutually independent random variables, each random variable X_i has Bernoulli distribution with expectation p_i and $x \ge 7\mathbb{E}[X]$, then*

$$\mathbb{P}[X \ge x] \le e^{-x}.$$

We also need the Janson's inequality to provide an exponential upper bound for the lower tail of a sum of dependent zero-one random variables.

Lemma 1.3 (Theorem 8.7.2 in [10]) *Let Γ be a finite set and $p_i \in [0, 1]$ be a real for $i \in \Gamma$. Let Γ_p be a random subset of Γ such that the elements are chosen independently with $\mathbb{P}[i \in \Gamma_p] = p_i$ for $i \in \Gamma$. Let M be a family of subsets of Γ. For every $A_i \in M$, let $I_{A_i} = 1$ if $A_i \subseteq \Gamma_p$ and 0 otherwise. Let B_i be the event that $A_i \subseteq \Gamma_p$. For $A_i, A_j \in M$, we write $i \sim j$ if B_i and B_j are not pairwise independent, in other words, $A_i \cap A_j \neq \emptyset$. Define $X = \Sigma_{A_i \in M} I_{A_i}$, $\lambda = \mathbb{E}[X]$, $\Delta = \sum_{i \sim j} \mathbb{P}[B_i \wedge B_j]$, then*

$$\mathbb{P}[X \le (1 - \gamma)\lambda] < e^{-\gamma^2 \lambda / [2 + (\Delta/\lambda)]}.$$

Lemma 1.4 (Corollary 2.2, [62]) *Let $\binom{[N]}{r}$ be the set of r-subsets of $\{1, \ldots, N\}$ and let $h : \binom{[N]}{r} \to \mathbb{R}$ be given. Suppose that there exists $\alpha \ge 0$ such that*

$$|h(A) - h(A')| \le \alpha$$

for any $A, A' \in \binom{[N]}{r}$ with $|A \cap A'| = r - 1$. Let $C \subseteq [N]$ be a set of size r chosen uniformly at random. Then

$$\mathbb{E}[e^{h(C)}] = \exp(\mathbb{E}[h(C)] + a), \tag{1.1}$$

where a is a real constant such that $0 \le a \le \frac{\alpha^2}{8} \min\{r, N-r\}$. Furthermore, for any real $t > 0$,

$$\mathbb{P}(|h(C) - \mathbb{E}[h(C)]| \ge t) \le 2\exp\left(-\frac{2t^2}{\min\{r, N-r\}\alpha^2}\right). \tag{1.2}$$

Lemma 1.5 (McDiarmid's inequality [119]) *Suppose $X_1, \ldots, X_m$ are independent Bernoulli random variables and $b_i \in [0, B]$ for $i \in [m]$. Suppose that X is a real-valued random variable determined by $X_1, \ldots, X_m$ such that altering the value of X_i changes X by at most b_i for $i \in [m]$. For all $\lambda > 0$, we have*

$$\mathbb{P}(|X - \mathbb{E}[X]| > \lambda) \le 2\exp\left(\frac{-2\lambda^2}{B\Sigma_{i=1}^m b_i}\right).$$

1.2 . Rainbow structures in properly edge-colored graphs

How global parameters of a graph, such as its edge density or chromatic number, can influence its local substructures? How many edges, for instance, do we have to give a graph on n vertices to ensure that the graph will contain a K_r as a subgraph for some given r, no matter how these edges are arranged? Will some sufficiently high average degree or chromatic number ensure that some structure occurs? Questions of this type are among the most natural ones in Graph Theory, and there is a host of deep and interesting results. Collectively, these are known as Extremal Graph Theory. Extremal Graph Theory lies at the intersection of Extremal Combinatorics and graph theory. In recent years several classical results in Extremal Graph Theory have been improved in a uniform way and their proofs have been simplified and streamlined.

The basic statement of Extremal Graph Theory is Mantel's theorem [117], proved in 1907, which states that any graph on n vertices with no triangle contains at most $n^2/4$ edges. This is clearly best possible, as one may partition the set of n vertices into two sets of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and form the complete bipartite graph between them. This graph has $\lfloor n^2/4 \rfloor$ edges and does not contain a triangle as a subgraph. The natural generalisation of this theorem to cliques of size r is the following, proved by Turán [147], which states that every n-vertex graph does not contain K_{r+1} as a subgraph has at most $(1 - \frac{1}{r})\frac{n^2}{2}$ edges. In 1946, Erdős and Stone [51] generalized Turán's theorem and bounded the number of edges in an H-free graph for a non-complete graph H.

1.2.1 . Extremal problems in properly edge-colored graphs

There has been much research on extremal problems in edge-colored graphs. An example is the canonical Ramsey theorem, proved by Erdős and Rado [48], a special case of which shows that any properly edge-colored K_n admits a rainbow K_m, provided n is large relative to m. Ramsey's theorem states that there exists a positive integer $R(r,s)$ for which every blue-red edge-colored complete graph on $R(r,s)$ vertices contains a blue K_r or a red K_s.

Moreover, the Turán-type problem has a generalization in edge-colored graphs, called rainbow Turán problem. The systematic study of rainbow Turán numbers was initiated in [85] by Keevash, Mubayi, Sudakov and Verstraëte. It can be seen as a generalization of Turán-type problem in edge-colored graphs. For a fixed graph H, the rainbow Turán problem refers to determine the maximum number of edges in a properly edge-colored graph on n vertices which does not contain a rainbow H. This maximum is denoted by $ex^*(n,H)$ and we refer to it as the rainbow Turán number of H. Recall that given a graph H, the maximum number of edges in a graph on n vertices that contains no copy of H is known as the Turán number of H, and is denoted by $ex(n,H)$. Clearly, $ex^*(n,H) \geq ex(n,H)$. They determined $ex^*(n,H)$ asymptotically for any non-bipartite graph H, by showing that $ex^*(n,H) = (1+o(1))ex(n,H)$. For bipartite F with a maximum degree of s in one of the parts, they proved $ex^*(n,F) = O(n^{1/s})$. This matches the upper bound for the (usual) Turán numbers of such graphs. To quote [85], there are two questions that are the most important among the several ones raised therein. The first one is to determine $ex^*(n,\mathcal{C})$, where $\mathcal{C}$ is the class of all cycles. It is shown that $ex^*(n,\mathcal{C}) = \Omega(n\log n)$ in [85] and Das, Lee, and Sudakov [39] obtained an upper bound $O(ne^{(\log n)^{1/2+o(1)}})$. There have been some recent improvements upon the upper bound [76, 91, 144] and the current best one is $O(n\log^2 n)$ appeared in [91]. The second question in [85] concerns with $ex^*(n,C_{2k})$, where C_{2k} is the even cycle of length $2k$. In [85], a general lower bound $ex^*(n,C_{2k}) = \Omega(n^{1+1/k})$ is obtained, whereas the matching upper bounds were only verified for $k=2,3$. This upper bound was subsequently improved by Das, Lee, and Sudakov [39] to $O(n^{1+(1+o_k(1))\log k/k})$ and by Janzer [76] to $O(n^{1+1/k})$. While Janzer's bound matches the lower bound given in [85], the implicit constant is exponential in k. Recently, Kim, Lee, Liu and Tran [91] improved it to a polynomial one.

Some classic problems can be transferred into extremal problems in edge-colored graphs. For example, finding directed cycles can be formulated as a special case of finding properly colored cycles. To see this, consider the following construction which was first introduced by Li [107] and also studied in [42]. Let D be an oriented graph of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Define an edge-coloring τ of G by coloring the edge v_iv_j with j for all arcs (v_i,v_j) in D. The resulting edge-colored graph, denoted by (D,τ), is called the *signature* of D (see Figure 1.1 for an example). Then the following two properties hold : (i) For every vertex $v \in V(G)$, $d^c_{(D,\tau)}(v) = d^+_D(v)$ if $d^-_D(v) = 0$, otherwise $d^c_{(D,\tau)}(v) = d^+_D(v)+1$; (ii) A cycle in G is a directed cycle in D if and only if it is a properly colored cycle in (D,τ). Recall that the well-known Caccetta-Häggkvist Conjecture [24] says that for all positive integers n,r with $n \geq r$, every digraph D of order n with $\delta^+(D) \geq \lceil n/r \rceil$ contains a directed cycle of length at most r. Hence, the study of Caccetta-Häggkvist Conjecture in some sense can be transferred into the study of properly colored cycles in edge-colored graphs with minimum color degree constraints.

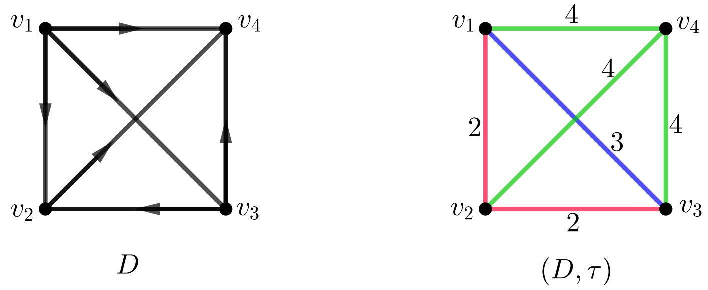


Figure 1.1 – An illustration of the signature (D, τ) of an oriented graph D.

There is another interesting branch, called anti-Ramsey theory [58]. The anti-Ramsey problem is stated as follows : given a positive integer n and a graph H, the anti-Ramsey number $ar(K_n, H)$ is defined to be the minimum number of colors k such that for any edge-coloring of K_n with exactly k colors, there exists a rainbow copy of H. The study of anti-Ramsey theory began with a paper by Erdős, Simonovits, and Sós [50] in 1975 (note that related ideas were studied even earlier in [49]). The anti-Ramsey number $ar(K_n, H)$ is closely related to the Turán number $ex(n, H)$, which is the maximum number of edges in a graph on n vertices with no subgraph isomorphic to H. The main result in [50] states that $ar(K_n, H) = \frac{n^2}{2}\left(1 - \frac{1}{\chi - 1}\right)(1 + o(1))$, where $\chi = \min\{\chi(H \setminus e) : e \in E(H)\}$. Instead of forcing rainbow copies of a given graph H, one can consider forcing properly edge-colored copies of H by using many colors, and study the threshold on the number of colors needed. This is thoroughly studied by Manoussakis, Spyratos, Tuza and Voigt in [116].

Besides a number of applications in graph theory and algorithms, some concepts and results in edge-colored graphs have also appeared in communication network [149], social science [31], biology [44, 45, 123] and so on. For example, edge-colored graphs can be used to model homogeneous faults in networks [149], study the order of chromosomes [44, 45] and DNA physical mapping [123].

1.2.2 . Rainbow cycles in properly edge-colored complete graphs

In 1989, Andersen [13] conjectured that all proper edge-colorings of K_n admit a rainbow path which omits only one vertex.

Conjecture 1.1 (Andersen [13]) *All proper edge-colorings of K_n admit a rainbow path of length $n - 2$.*

It is best possible by a construction of Maamoun and Meyniel [115].

There are many variations of Andersen's Conjecture. The following conjecture was proposed by Hahn [66]. Every edge-colored K_n with at most $n/2$ edges of each color contains a rainbow Hamilton path. In light of the aforementioned construction of Maamoun and Meyniel [115], Hahn and Thomassen [67] suggested the following slightly weaker form of Hahn's Conjecture in 1986 : every edge-colored K_n with less than $n/2$ edges of each color contains a rainbow Hamilton path. However, even this weakening of Hahn's Conjecture is false. Pokrovskiy and Sudakov [127] proved the existence of such edge-colored K_n in which the longest rainbow path has length at most $n - \ln n/42$.

Another direction is to find long rainbow paths or cycles in properly edge-colored complete graphs. In recent years, this problem has been extensively studied and a series of progresses have been made. We can greedily obtain that a rainbow path of length $n/2 - 1$ in every properly edge-colored K_n.

Akbari, Etesami, Mahini and Mahmoody [6] proved that every properly edge-colored K_n has a rainbow cycle of length at least $n/2 - 1$. Gyárfás and Mhalla [64] proved that if the set of edges with every used color forms a perfect matching in K_n, then there exists a rainbow path of length $(2n + 1)/3$. Gyárfás, Ruszinkó, Sárközy and Schelp [65] showed that every properly edge-colored K_n contains a rainbow cycle of length $(4/7 - o(1))n$. Gebauer and Mousset [59] and Chen and Li [26], independently showed that every properly edge-colored K_n contains a rainbow cycle of length $(3/4 - o(1))n$. Alon, Pokrovskiy and Sudakov [9] proved that every properly edge-colored of K_n contains a rainbow cycle with length $n - O(n^{3/4})$, and the error bound has since been improved to $O(\sqrt{n} \cdot \log n)$ by Balogh and Molla [14].

Further support for Conjecture 5.1 and its variants was provided by Montgomery, Pokrovskiy, and Sudakov [121] as well as Kim, Kühn, Kupavskii, and Osthus [90], who considered the decompositions of rainbow spanning structures in properly edge-colored K_n.

1.3 . Dirac-type problems

Problems that relate the minimum degree (in general, minimum ℓ-degree in k-graphs where $\ell \in [k-1]$) to the structure of the (hyper)graphs are often referred to as *Dirac-type* problems. we concentrate on three such problems : Hamilton cycles, perfect matchings and tilings.

Hamilton cycles

A classical theorem of Dirac [43] asserts that for any $n \geq 3$, every n-vertex graph with minimum degree at least $n/2$ contains a Hamilton cycle. Note that the lower bound $n/2$ is best possible, as can be seen by the following example : a complete bipartite graph with parts of sizes k and $k - 1$. The graph has $2k - 1$ vertices and minimum degree $k - 1$, but there is no Hamilton cycle in this graph. The problem of determining the best possible minimum $(k - 1)$-

degree condition forcing Hamilton cycles in k-graphs, was initially researched by Katona and Kierstead [79]. They proved that every n-vertex k-graph H with $\delta_{k-1}(H) > (1-\frac{1}{2k})n+4-k-\frac{5}{2k}$ admits a Hamilton cycle. They also conjectured that the bound on the minimum $(k-1)$-degree can be reduced to roughly $n/2$, which was confirmed asymptotically by Rödl, Ruciński and Szemerédi in [135, 137]. The same authors gave the exact version for $k=3$ in [139].

Theorem 1.1 ([137, 139]) *Let $k \geq 3, \gamma > 0$ and H be an n-vertex k-graph, where n is sufficiently large. If $\delta_{k-1}(H) \geq (1/2+\gamma)n$, then H contains a Hamilton cycle. Furthermore, when $k=3$ it is enough to have $\delta_2(H) \geq \lfloor n/2 \rfloor$.*

More generally, We define the threshold $h_d^\ell(k,n)$ as the smallest integer m such that every k-graph H on n vertices with $\delta_d(H) \geq m$ contains a Hamilton ℓ-cycle. As before, we may omit the subscript when $d=k-1$. Let $h_d^\ell(k) = \limsup_{n\to\infty} h_d^\ell(k,n)/\binom{n-d}{k-d}$. About this parameter, there are many results as follows [70, 82, 97].

Theorem 1.2 ([70, 82, 97]) *For any $k \geq \ell \geq 1$, we have*

$$h^\ell(k) = \begin{cases} \frac{1}{2}, & (k-\ell) \mid k, \\ \frac{1}{\lceil \frac{k}{k-\ell}(k-\ell) \rceil}, & (k-\ell) \nmid k. \end{cases}$$

Theorem 1.3 ([15, 23]) *For integer $k \geq 3$ and any $1 \leq \ell < k-2$, we have*

$$h_{k-2}^\ell(k) = 1-(1-\frac{1}{2(k-\ell)})^2.$$

More generally, Kühn and Osthus [100] and Zhao [152] noted that it is much more difficult to determine the minimum d-degree condition for tight Hamilton cycle for $d \in [k-2]$. Based on the results of Cooley and Mycroft [34], Glebov, Person and Weps [60], Rödl and Ruciński [132] and Rödl, Ruciński, Schacht and Szemerédi [134], Reiher, Rödl, Ruciński, Schacht and Szemerédi [130] gave that $h_{k-2}^{k-1}(k) = 5/9$ when $k=3$. Polcyn, Reiher, Rödl, Ruciński, Schacht, and Schülke [128] gave that $h_{k-2}^{k-1}(k) = 5/9$ when $k=4$. The best bound for general k was given by Lang and Sanhueza-Matamala [105], Polcyn, Reiher, Rödl and Schülke [129] independently. They proved the following theorem.

Theorem 1.4 ([105, 129]) *For any integer $k \geq 3$, $h_{k-2}^{k-1}(k) = 5/9$.*

Perfect matchings

Many open problems in combinatorics can be formulated as a problem of finding perfect matchings in hypergraphs, e.g., Ryser conjectured that every Latin square of odd order has a rainbow, and the existence of combinatorial designs (recently solved by Keevash [80, 81]). A well-known result of Tutte [148] characterized all the graphs with perfect matchings and there are efficient algorithms (e.g., Edmond's algorithm [46]) that determine if a graph has a perfect matching. However, deciding if a 3-partite 3-graph contains a perfect matching is among the first 21 NP-complete problems given by Karp [78]. Therefore it is natural to look for sufficient conditions that guarantee a perfect matching.

Bollobás, Daykin and Erdős [17] first related the minimum (vertex) degree to the existence of a large (but far from perfect) matching in k-graphs. Daykin and Häggkvist [40] extended this result by showing that every k-graph with $\delta_1(H) \geq (1-1/k)\binom{n-1}{k-1}$ contains a perfect matching. Given integers $d < k \leq n$ such that k divides n, define the minimum d-degree threshold $m_d(k,n)$ as the smallest integer m such that every k-graph H on n vertices with $\delta_d(H) \geq m$ contains a prefect matching. A simple greedy argument shows that $m_1(2,n) = n/2$ for all $n \in 2\mathbb{N}$. Given $k \geq 3$, a result of Rödl, Ruciński and Szemerédi [137] on Hamilton cycles implies that $m_{k-1}(k,n) \leq n/2 + o(n)$. Kühn and Osthus [99] sharpened this bound to $m_{k-1}(k,n) \leq n/2 + 3k^2\sqrt{n\log n}$ by reducing the problem to the one for k-partite k-graphs. Rödl, Ruciński and Szemerédi [136] improved it further to $m_{k-1}(k,n) \leq n/2 + O(n\log n)$ by using the absorbing method. Rödl, Ruciński and Szemerédi [133] found a simple proof of $m_{k-1}(k,n) \leq n/2+k/4$. Finally Rödl, Ruciński and Szemerédi [138] determined $m_{k-1}(k,n)$ exactly for all $k \geq 3$ and sufficiently large n (again by the absorbing method). In order to state this and later results, we need the following extremal configurations that are usually referred to as *divisibility barrier*.

Consruction Define $\mathcal{H}_{\text{ext}}(n,k)$ to be the family of all k-graphs $H = (V,E)$, in which there is a partition of V into two parts A, B and $i \in \{0,1\}$ such that $|A| \neq i|V|/k \mod 2$ and $|e \cap A| = i \mod 2$ for all edges $e \in E$.

It is easy to see that no hypergraph $H \in \mathcal{H}_{\text{ext}}(n,k)$ contains a perfect matching. Indeed, suppose H contains a perfect matching M, then $|A| = \sum_{e\in M} |e\cap A| = i|V|/k \mod 2$, contradicting the definition of H. Define $\delta(n,k,d)$ to be the maximum of the minimum d-degrees among all the hypergraphs in $\mathcal{H}_{\text{ext}}(n,k)$ and note that $m_d(k,n) > \delta(n,k,d)$. It is easy to see that

$$\delta(n,k,k-1) = \begin{cases} \frac{n}{2} - k + 2, & \text{if } k/2 \text{ is even and } n/k \text{ is odd,} \\ \frac{n}{2} - k + \frac{3}{2}, & \text{if } k \text{ is odd and } (n-1)/2 \text{ is odd,} \\ \frac{n}{2} - k + \frac{1}{2}, & \text{if } k \text{ is odd and } (n-1)/2 \text{ is even,} \\ \frac{n}{2} - k + 1, & \text{otherwise.} \end{cases}$$

In general, $\delta(n,k,d) = (1/2+o(1))\binom{n-d}{k-d}$ for any fixed $k > d$ but the general formula of $\delta(n,k,d)$ is unknown-this is related to the open problem of finding the minima of binary Krawtchouk polynomials. Nevertheless, Treglown and Zhao [145] determined $m_d(k,n)$ in terms of $\delta(n,k,d)$ for all $d \geq k/2$.

Theorem 1.5 ([145]) *For $k \geq 3$ and $d \geq k/2$, $m_d(k,n) = \delta(n,k,d) + 1$ for all sufficiently large n.*

Another class of extremal constructions are known as *space barrier*.

Consruction Given $s, k, n \in N$ such that $s \leq \lceil n/k \rceil$ (k may not divide n), let $H_s^0(n, k)$ be the k-graph on n vertices whose vertex set is partitioned into two parts A and B such that $|A| = s - 1$, and whose edge set consists of all those edges with at least one vertex in A. When k divides n, let $H^0(n, k) := H_{n/k}^0(n, k)$.

Hàn, Person and Schacht [69] proved that $m_1(3, n) = (5/9 + o(1))n \approx \delta_1(H^0(n, 3))$ for sufficiently large n. Khan [88] and independently Kühn, Osthus and Treglown [101] obtained that $m_1(3, n) = \delta_1(H^0(n, 3)) + 1$ for sufficiently large n. Khan [89] also proved that $m_1(4, n) = \delta_1(H^0(n, 4)) + 1$ for sufficiently large n. Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [8] determined $m_d(k, n)$ asymptotically for all $d \geq k - 4$, including the new cases when $(k, d) = (5, 1), (5, 2), (6, 2)$ and $(7, 3)$. Very recently Treglown and Zhao [146] determined $m_2(5, n)$ and $m_3(7, n)$ exactly for sufficiently large n. All these results point to the following conjecture.

Conjecture 1.2 ([146]) *Let $k, d \in \mathbb{N}$ such that $d \leq k - 1$. Then for sufficiently large $n \in k\mathbb{N}$,*

$$m_d(k, n) = \max\{\delta(n, k, d), \binom{n-d}{k-d} - \binom{(1 - 1/k)n - d + 1}{k - d} + 1\}.$$

When $k \geq 3$ and $1 \leq d < k/2$, Hàn, Person and Schacht [69] gave a general bound : $m_d(k, n) \leq ((k - d)/k + o(1))\binom{n-d}{k-d}$. This was improved by Markström and Ruciński [118] to $m_d(k, n) \leq ((k-d)/k - 1/k^{k-d}) + o(1))\binom{n-d}{k-d}$ and by Kühn, Osthus and Townsend [104] to

$$m_d(k, n) \leq (\frac{k-d}{k} - \frac{k-d-1}{k^{k-d}} + o(1))\binom{n-d}{k-d}.$$

Tilings

Tiling problems have been studied extensively for graphs. Finding sufficient conditions for the existence of an F-factor is one of the central areas of research in Extremal Graph Theory. The celebrated Hajnal–Szemerédi theorem reads as follows.

Theorem 1.6 (Hajnal–Szemerédi [68], Corrádi–Hajnal [35] for $t = 3$) *Every n-vertex graph G with $n \in t\mathbb{N}$ and $\delta(G) \geq (1 - \frac{1}{t})n$ has a K_t-factor. Moreover, the minimum degree condition is sharp.*

The minimum degree threshold forcing an F-factor for arbitrary F was obtained by Kühn and Osthus [102, 103], improving earlier results of Alon and Yuster [11] and Komlós, Sárközy and Szemerédi [95].

It is not surprising that tiling problems become harder in hypergraphs. Other than the matching problems mentioned above, only a few tiling thresholds are known. Given a k-graph F of order f and an integer n divisible by f, we define the F-tiling threshold $\delta_d(n, F)$ as the smallest integer t such that every n-vertex k-graph H with $\delta_d(H) \geq t$ contains an F-factor. We simply write $\delta(n, F)$ for $\delta_{k-1}(n, F)$. Let K_t^k be the complete k-graph on t vertices. The first step towards a hypergraph is determining $\delta(n, K_4^3)$. Czygrinow and Nagle [38] showed that $\delta(n, K_4^3) \geq 3n/5 + o(n)$. Keevash and Sudakov observed that $\delta(n, K_4^3) \geq 5n/8 + o(n)$. Pikhurko [124] proved $3n/4 - 2 \leq \delta(n, K_4^3) \leq 0.861n$. Lo and Markström [108] showed that $\delta(n, K_4^3) \geq 3n/4 + o(n)$ by the absorbing method. Independently and simultaneously Keevash and Mycroft [86] determined $\delta(n, K_4^3)$ exactly.

Theorem 1.7 ([86]) *For all sufficiently large $n \in 4\mathbb{N}$,*

$$\delta(n, K_4^3) = \begin{cases} \frac{3n}{4} - 2, & \text{if } n \in 8\mathbb{N}, \\ \frac{3n}{4} - 1, & \text{otherwise.} \end{cases}$$

When $t = k + 1$, Lo and Markström [108] showed that $\delta(n, K_{k+1}^k) \leq (1 - 1/2k)n$ for $k \geq 3$. It is plausible that one can prove $\delta(n, K_{k+1}^k) \leq \frac{k}{k+1}n + o(n)$ by applying the approach of [86]. Unfortunately we do not know a matching lower bound (it was shown in [108] that $\delta(n, K_{k+1}^k) \geq 2n/3$ for even k). For arbitrary t, it was shown in [108] that

$$(1 - \frac{193\log(t-1)}{(t-1)^2})n \leq \delta(n, K_t^3) \leq (1 - \frac{2}{t^2 - 3t + 4} + o(1))n,$$

and $\delta(n, K_t^k) \leq (1 - \binom{t-1}{k-1}^{-1} + o(1))n$ for $k \geq 6$ and $t \geq (3 + \sqrt{5})k/2$.

Given positive integers $m_1 \leq \cdots \leq m_k$, let $K_{m_1,\ldots,m_k}^k$ denote the complete k-partite k-graph with parts of sizes $m_1, \ldots, m_k$. In particular, let $K_k^k(m) = K_{m,\ldots,m}^k$. It is clear that $\delta_d(n, K_k^k(m)) \geq m_d(k, n)$, but it is possible to have $\delta_d(n, K_{m_1,\ldots,m_k}^k) < m_d(k, n)$ for certain $m_1, \ldots, m_k$. Other than the matching problems, perhaps the earliest result on hypergraph tiling was on $K_{1,1,2}^3$-tiling (note that $K_{1,1,2}^3$ is the unique 3-graph with four vertices and 2 triples). As a corollary of their main result on loose Hamilton cycles, Kühn and Osthus [98] proved that $\delta(n, K_{1,1,2}^3) = n/4 + o(n)$. Recently Czygrinow, DeBiasio and Nagle [37] determined this threshold exactly for sufficiently large n.

Let us consider hypergraph tiling under vertex degree conditions. Very little is known in addition to [114] used the Local Lemma to derive a general upper bound for $\delta_1(n, F)$ for arbitrary k-graph F as follows.

Theorem 1.8 ([114]) *Let F be a t-vertex m-edge k-graph in which each edge intersects at most d other edges. Then $\delta_1(n, F) \leq (1 - \frac{1}{e(d+1+\frac{m}{t}k^2)})\binom{n-1}{k-1}$, where $e = 2.718$.*

Given a k-graph F, let $\tau_d(n, F)$ denote the minimum integer t such that every k-graph H of order n with $\delta_d(H) \geq t$ has the property that every vertex of H is covered in some copy of F. When F is a graph, it is not hard to see that

$\tau_1(n, F) = (1 - 1/(\chi(F) - 1) + o(1))n$ (see the concluding remarks of [71]). Given a k-graph F, trivially $\mathrm{ex}_d(n, F) < \tau_d(n, F) \le \delta_d(n, F)$, where $\mathrm{ex}_d(n, F)$ is the d-degree Turán number of F, defined as the smallest integer t such that every k-graph H of order n with $\delta_d(H) \ge t + 1$ contains a copy of F.

1.4 . Rainbow structures in (hyper)graph systems

Graph Systems

The most famous transversals are the ones of Latin squares considered by Euler. In 1782, Euler [52] considered a Latin square of order n, which is an $n \times n$ array filled with symbols $1, \ldots, n$, where every symbol appears exactly once in each row and column. A transversal of a Latin square of order n is a collection of cells such that every two cells share no row, column or symbol.

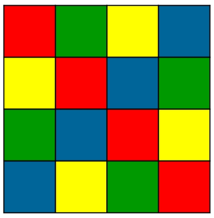


Figure 1.2 – Latin square.

Considering the rows and columns of the Latin square as a bipartite graph $K_{n,n}$, where each symbol in the Latin square represents a color and each cell represents an edge in the graph, the Latin square naturally corresponds to a properly edge-coloring of $K_{n,n}$. Viewing this edge-colored graph as a set of graphs $\{G_i\}_{i\in[n]}$, where each G_i is a graph formed by edges with color i, then a rainbow matching corresponds to a transversal.

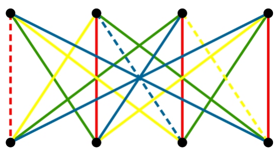


Figure 1.3 – A transversal of latin sqaure.

The following conjecture has become known as the Ryser-Brualdi-Stein conjecture [141, 22, 142] and is the most significant problem on transversals in Latin squares.

Conjecture 1.3 (Ryser-Brualdi-Stein Conjecture [141, 22, 142]) *Every Latin square of order n has a transversal with $n - 1$ cells, and a transversal with n cells if n is odd.*

Towards the above conjecture, Koksma [93] proved the existence of a transversal of size $2n/3$ before Brouwer, De Vries and Wieringa [21] and Woolbright [150] independently showed that every Latin square of order n has a transversal with at least $n - \sqrt{n}$ cells. Hatami and Shor [73] showed that a transversal with $n - 11\log^2 n$ cells exists in any Latin square of order n. This bound stood until the breakthrough work of Keevash, Pokrovskiy, Sudakov and Yepremyan [87] in 2022, which showed that every Latin square of order n has a transversal with $n - O(\log n / \log\log n)$ cells. Recently, Montgomery [131] resolved the above conjecture.

More generally, Aharoni and Berger [1] made the following generalization of the above conjecture.

Conjecture 1.4 (Aharoni-Berger Conjecture [1]) *Let G be a properly edge- colored bipartite multigraph with n colors having at least $n+1$ edges of each color. Then G has a rainbow matching of size n.*

This conjecture attracted a lot of attention since it was made. Aharoni, Charbit and Howard [2] proved that matchings of size $\lfloor 7n/4 \rfloor$ are sufficient to guarantee a rainbow matching of size n. Kotlar and Ziv [96] improved this to $\lfloor 5n/3 \rfloor$. Clemens and Ehrenmüller [32] showed that $3n/2 + o(n)$ is sufficient. The best currently known bound is by Aharoni, Kotlar and Ziv [5] who showed that having $3n/2+1$ edges of each color in an n-edge-colored bipartite multigraph guarantees a rainbow matching of size n. Pokrovskiy [126] approximate version of Conjecture 1.4.

This motivates the study about the existence of rainbow structures in a collection of graphs. Indeed, various interesting results have been proved.

Aharoni, DeVos, Maza, Montejano and Šámal [3] proved that there exists a rainbow triangle in $\{G_1, G_2, G_3\}$ if $e(G_i) > \frac{26-2\sqrt{7}}{81}n^2$ for each $i \in [3]$, which is a Turán type problem over graph systems. Surprisingly, this bound is best possible as $\frac{26-2\sqrt{7}}{81}$ is larger than $1/4$ which we obtained from Mantel's theorem. It is an interesting open problem to generalize this further by determining the tight conditions on $e(G_i)$ for the existence of a $\{G_1, \ldots, G_{\binom{r}{2}}\}$-rainbow isomorphic to K_r with $r > 3$. In the same paper, they proposed the following conjecture.

Conjecture 1.5 ([3]) *For $|V| = n \geq 3$ and graph system $\boldsymbol{G} = \{G_i\}_{i \in [n]}$ on V, if $\delta(G_i) \geq n/2$ for each $i \in [n]$, then there exists a $\boldsymbol{G}$-rainbow Hamilton cycle.*

This was recently verified asymptotically by Cheng, Wang and Zhao [30], and completely by Joos and Kim [77]. In [20], Bradshaw, Halasz and Stacho strengthened the Joos-Kim result by showing that given an n-vertex graph system $\boldsymbol{G} = \{G_i\}_{i \in [n]}$ with $\delta(G_i) \geq n/2$ for $i \in [n]$, then $\boldsymbol{G}$ has exponentially many rainbow Hamilton cycles. Similarly, Bradshaw [19] gave a degree condition for rainbow Hamilton cycle in bipartite graph systems, which generalized the result of Moon and Moser [122]. Moreover, Gupta, Hamann, Müyesser, Parczyk and Sgueglia [63] recently proved that any collection of an n-vertex graph system with at least rn graphs, each with minimum degree at least $(r/(r+1)+o(1))n$, contains a rainbow r-th power of a Hamilton cycle. This can be viewed as a rainbow version of the Pósa-Seymour conjecture, which was proved by Komlós, Sárközy, and Szemerédi [94]. Cheng and Staden [29] developed a version of rainbow blow-up lemma (which can be used when the number of colors is ε-fraction more than the number of edges in H) and obtained a result similar to [63] when the number of colors is ε-fraction more than the number of edges in the power of Hamilton cycle.

Generally, for each graph F, let δ_F be the smallest real number $\delta \geq 0$ such that, for each $\varepsilon > 0$ there exists some n_0 such that, for every $n \geq n_0$ with $|F|$ dividing n, if an n-vertex graph G has minimum degree at least $(\delta+\varepsilon)n$, then G contains an F-factor. Cheng, Han, Wang and Wang [27] proved that the minimum degree bound δ_{K_r} is asymptotically sufficient for the existence of rainbow K_r-factor in graph systems. Montgomery, Müyesser and Pehova [120] generalized the above conclusion for some F satisfying $\delta_F \geq 1/2$ or F has a bridge.

All those graphs above, powers of Hamilton cycles, F-factors and trees, have somewhat bounded maximum degree and have low connectivity. This low connectivity can be captured by the following notion of bandwidth. A graph H has a bandwidth at most b if there exists an ordering $x_1, \ldots, x_n$ of $V(H)$ such that all edges $x_ix_j \in E(H)$ satisfies $|i-j| \leq b$. Indeed, the celebrated bandwidth theorem proved by Böttcher, Schacht and Taraz [18] determines the asymptotically sharp minimum degree condition on G to find such a graph H with bounded maximum degree and low bandwidth as a spanning subgraph. More precisely, the bandwidth theorem states that if an n-vertex k-chromatic graph H has bounded maximum degree and sublinear bandwidth, then every n-vertex graph G with $\delta(G) \geq (1-1/k+o(1))n$ contains a copy of H. Recently, Chakraborti, Im, Kim and Liu [25] made important progress in this direction by proving a 'rainbow bandwidth theorem'.

Theorem 1.9 ([25]) *For every $\varepsilon > 0$ and positive integers Δ, k, there exist $\alpha > 0$ and $h_0 > 0$ satisfying the following for every $h \geq h_0$. Let H be an n-vertex graph with h edges and bandwidth at most αn such that $\Delta(H) \leq \Delta$ and $\chi(H) \leq k$. If $\boldsymbol{G} = \{G_i\}_{i\in[h]}$ is a family of h graphs on the same vertex set of size n such that $\delta(G_i) \geq (1-1/k+\varepsilon)n$ for all $i \in [h]$, then there exists a $\boldsymbol{G}$-rainbow H.*

Hypergraph systems

It is also natural to investigate what can be guaranteed with a lower bound on the minimum degree in hypergraphs. It turns out that even in this more restrictive setting, there can be a discrepancy between the uncolored and the rainbow versions of the problem.

Definition 1.1 (Uncolored minimum degree threshold) *Let $\mathcal{F}$ be an infinite family of k-graphs. By $\delta_{\mathcal{F},d}$ we denote, if it exists, the smallest real number δ such that for all $\alpha > 0$ and for all but finitely many $F \in \mathcal{F}$ the following holds. Let $n = |V(F)|$ and H be any n-vertex k-graph with $\delta_d(H) \geq (\delta + \alpha)\binom{n-d}{k-d}$. Then H contains a copy of F.*

For example, if $\mathcal{F}$ is the family of graphs consisting of a cycle on n vertices for each $n \in \mathbb{N}$, then we have $\delta_{\mathcal{F},1} = 1/2$. Indeed, this follows from Dirac's theorem which states that any graph with minimum degree at least $n/2$ has a Hamilton cycle.

Definition 1.2 (Rainbow minimum degree threshold) *Let $\mathcal{F}$ be an infinite family of k-graphs. By $\delta^{\mathrm{rb}}_{\mathcal{F},d}$ we denote, if it exists, the smallest real number δ such that for all $\alpha > 0$ and for all but finitely many $F \in \mathcal{F}$ the following holds. Let $n = |V(F)|$ and **H** $= \{H_i\}_{i\in[|E(F)|]}$ be any n-vertex k-graph and $\delta_d(H_i) \geq (\delta+\alpha)\binom{n-d}{k-d}$ for each $i \in [|E(F)|]$. Then there exists an **H**-rainbow F.*

Note that $\delta^{\mathrm{rb}}_{\mathcal{F},d} \geq \delta_{\mathcal{F},d}$. Indeed, if H contains no copy of F, the system ***H*** consisting of $|E(F)|$ copies of H does not contain a rainbow copy of F either. However, Montgomery, Müyesser, and Pehova [120] made the following observation which shows that $\delta^{\mathrm{rb}}_{\mathcal{F},d}$ can be much larger than $\delta_{\mathcal{F},d}$. Set $\mathcal{F} = \{k \times (K_{2,3} \cup C_4) : k \in \mathbb{N}\}$ where $k \times G$ denotes the graph obtained by taking k vertex-disjoint copies of G. It follows from a result of Kühn and Osthus [103] that $\delta_{\mathcal{F},1} = 4/9$. Consider the graph system ***G*** $= \{G_1, \dots, G_m\}$ on V obtained in the following way. Partition V into two almost equal vertex subsets, say A and B, and suppose that $G_1 = G_2 = \cdots = G_{m-1}$ are all disjoint unions of a clique on A and a clique on B. Suppose that G_m is a complete bipartite graph between A and B. Observe that each G_i in this resulting graph system has minimum degree $\lfloor |V|/2 \rfloor$. Further observe that if ***G*** contains a rainbow copy of some $F \in \mathcal{F}$, the edge of $K_{2,3}$ or C_4 that gets copied to an edge of G_m would be a bridge (an edge whose removal disconnects the graph) of F. However, neither $K_{2,3}$ nor C_4 contains a bridge. Hence, $\delta^{\mathrm{rb}}_{\mathcal{F},d} \geq 1/2$.

On the other hand, there are many natural instances where $\delta^{\mathrm{rb}}_{\mathcal{F},d} = \delta_{\mathcal{F},d}$. When this equality holds, we say that the corresponding family $\mathcal{F}$ is *d-color-blind*. For example, Joos and Kim [77] showed that the family $\mathcal{F}$ of Hamilton cycles is 1-color-blind. There are many more families of color-blind hypergraphs. In particular, matchings [27, 113, 110, 109], Hamilton ℓ-cycles [28, 143], factors [27, 120] and spanning trees [120] have been extensively studied.

Recently, Gupta, Hamann, Müyesser, Parczyk and Sgueglia [63] gave a unified approach to this problem and proved the following result.

Theorem 1.10 ([63]) *The following families of hypergraphs are all d-color-blind.*

- **(A)** *The family of the r-th powers of Hamilton cycles for fixed $r \geq 2$ (and $d = 1$).*

(B) *The family of k-uniform Hamilton ℓ-cycles for the following ranges of k, ℓ, and d.*

(B1) $1 < \ell < k/2$ *and* $d = k - 2$*;*

(B2) $1 \leq \ell < k/2$ *or* $\ell = k - 1$ *and* $d = k - 1$*;*

(B3) $\ell = k/2$ *and* $k/2 < d \leq k - 1$ *with* k *even.*

Other recent results include works on matchings. The largest size of a matching in a hypergraph H is denoted by $\nu(H)$. A classical problem in Extremal Graph Theory is to determine $\max e(H)$ with $\nu(H)$ fixed. Erdős [47] made the following conjecture : For positive integers k, n, t with $n \geq kt$, every k-graph H on n vertices with $\nu(H) < t$ satisfies $e(H) \leq \max\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\}$. This bound is tight for the complete k-graph on $kt - 1$ vertices and for the k-graph on n vertices in which every edge intersects a fixed set of $t - 1$ vertices. There have been many results about this conjecture, but we mainly focus on the rainbow version. Aharoni and Howard [4] made the following conjecture, also see Huang, Loh, and Sudakov [74].

Conjecture 1.6 ([4, 74]) *Let* $\boldsymbol{H} = \{H_i\}_{i\in[t]}$ *be an n-vertex k-graph system. If* $e(H_i) > \max\{\binom{n}{k} - \binom{n-t+1}{k}, \binom{kt-1}{k}\}$ *for each* $i \in [t]$*, then there exists an* $\boldsymbol{H}$*-rainbow matching.*

Huang, Loh, and Sudakov [74] proved that Conjecture 1.6 holds for $n > 3k^2t$. Recently, Frankl and Kupavskii [56] proved that Conjecture 1.6 holds when $n \geq 12kt\log(e^2t)$, providing an almost linear bound. Lu, Wang and Yu [110] improved it to $n > 2kt$ and t is sufficiently large. More recently, Keevash, Lifshitz, Long and Minzer [83] independently proved a more general version with $n = \Omega(kt)$ using sharp threshold techniques developed in [84].

There are also Dirac-type conditions in hypergraph systems for rainbow matchings. For 3-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n/3]}$, Lu, Yu, and Yuan [113] proved the following result.

Theorem 1.11 ([113]) *For sufficiently large n with $n \equiv 0 (\mathrm{mod}\ 3)$ and a 3-graph system* $\boldsymbol{H} = \{H_i\}_{i\in[n/3]}$*, if* $\delta_1(H_i) > \binom{n-1}{2} - \binom{2n/3}{2}$ *for* $i \in [n/3]$*, then there exists an* $\boldsymbol{H}$*-rainbow perfect matching.*

This implies the result of Kühn, Osthus, and Treglown [101] and Khan [88] on perfect matchings in 3-graphs. In [109], Lu, Wang and Yu proved the following result in 4-graph systems,

Theorem 1.12 ([109]) *Let n be a sufficiently large integer with $n \equiv 0(\mathrm{mod} 4)$. Let* $\boldsymbol{H} = \{H_i\}_{i\in[n/4]}$ *be an n-vertex k-graph system such that for each $i \in [n/4]$, if* $\delta_1(H_i) > \binom{n-1}{3} - \binom{3n/4}{3}$*, then there exists an* $\boldsymbol{H}$*-rainbow perfect matching.*

This gives Khan's result [89] on perfect matchings in 4-graphs as a special case.

Besides, Lu, Wang and Yu [111] also give the co-degree threshold for rainbow perfect matchings in k-graph systems.

Theorem 1.13 ([111]) *Given integers k, d such that $k \geq 3$ and $k/2 \leq d \leq k-1$ and $n \in k\mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that the following holds. Suppose that **H** $= \{H_i\}_{i\in[n/k]}$ is an n-vertex k-graph system with $n \geq n_0$ satisfying $\delta_{k-1}(H_i) > \delta(n, k, k-1)$ for each $i \in [n/k]$. Then there exists an **H**-rainbow perfect matching.*

Recently, You [151] determined the minimum d-degree condition that guarantees the existence of a rainbow perfect matching in k-graph systems for $d \in [k/2, k-1]$.

1.5 . Contribution and outline of the thesis

In this section, we summarize main works and the organization of this dissertation.

(1) Let $c_{k,d}$ be the minimum d-degree threshold for perfect fractional matchings in k-graphs, namely, for every $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$, every n-vertex k-graph H with $\delta_d(H) \geq (c_{k,d}+\varepsilon)\binom{n-d}{k-d}$ contains a perfect fractional matching. It is known that [8] every n-vertex k-graph H with $\delta_d(H) \geq (\max\{c_{k,d}, 1/2\} + o(1))\binom{n-d}{k-d}$ has a perfect matching, and this condition is asymptotically best possible. In Chapter 2, we proved that a minimum d-degree condition forcing a perfect matching in a k-graph also forces rainbow perfect matchings in k-graph systems for $d \in [k-1]$. The degree assumptions in the result is asymptotically best possible (although the minimum d-degree condition forcing a perfect matching in a k-graph is in general unknown). We also give a general framework to prove the existence of rainbow factors in hypergraph systems. This is a joint work with Y. Cheng, J. Han and G. Wang.

(2) A classical theorem of Dirac [43] asserts that for any $n \geq 3$, every n-vertex graph with minimum degree at least $n/2$ contains a Hamilton cycle. The problem of determining the best possible minimum $(k-1)$-degree condition forcing Hamilton cycles in k-graphs, was initially researched by Katona and Kierstead [79]. They proved that every n-vertex k-graph H with $\delta_{k-1}(H) > (1 - \frac{1}{2k})n + 4 - k - \frac{5}{2k}$ admits a Hamilton cycle. They also conjectured that the bound on the minimum $(k-1)$-degree can be reduced to roughly $n/2$, which was confirmed asymptotically by Rödl, Ruciński and Szemerédi in [135, 137]. The same authors gave the exact version for $k = 3$ in [139]. In Chapter 3, we show that given $k \geq 3, \gamma > 0$, sufficiently large n and an n-vertex k-graph system **H** $= \{H_i\}_{i\in[n]}$, if $\delta_{k-1}(H_i) \geq (1/2+\gamma)n$ for each $i \in [n]$, then there exists an **H**-rainbow Hamilton cycle, which is an extension of [137]. This is a joint work with Y. Cheng, J. Han, G. Wang and D. Yang.

(3) Gupta, Hamann, Müyesser, Parczyk, and Sgueglia [63] mentioned the following problem as "there is a well-known (uncolored) Dirac-type result whose rainbow version is missing" and "it would be an interesting challenge to obtain this result" : Given a 3-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ with minimum vertex degree condition of each H_i, does $\boldsymbol{H}$ admit a rainbow Hamilton cycle? In Chapter 4, we develop a sequentially Hamilton framework, which is of independent interest, settling the above problem, and draw the general conclusion for any $k \geq 3$. We show that given $\gamma > 0$, $k \geq 3$, sufficiently large n and an n-vertex k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$, if $\delta_{k-2}(H_i) \geq (5/9+\gamma)\binom{n}{2}$ for $i \in [n]$, then there exists an $\boldsymbol{H}$-rainbow Hamilton cycle. This result implies the conclusion in a single graph, which was proved by Lang and Sanhueza-Matamala [105], Polcyn, Reiher, Rödl and Schülke [129] independently. This is a joint work with Y. Tang, G. Wang and G. Yan.

(4) In 1989, Andersen [13] conjectured that all proper edge-colorings of K_n admit a rainbow path which omits only one vertex. It is best possible by a construction of Maamoun and Meyniel [115]. In Chapter 5, we proved that every properly edge-colored $K_{n,n}$ contains a rainbow cycle of length at least $n - 28n^{3/4}$ for sufficiently large n. The bound above is asymptotically optimal as each color class could be a perfect matching of $K_{n,n}$ and only n colors occur in $E(K_{n,n})$. This is a joint work with H. Li and G. Wang.

2 - Rainbow perfect matchings in hypergraph systems with minimum d-degree

It is well-known that perfect matchings are closely related to its fractional counterpart. Given a k-graph H, a *fractional matching* is a function $f : E(H) \to [0,1]$, subject to the requirement that $\sum_{e:v\in e} f(e) \leq 1$, for every $v \in V(H)$. Furthermore, if equality holds for every $v \in V(H)$, then we call the fractional matching *perfect*. Denote the maximum size of a fractional matching of H by $\nu^*(H) = \max_f \Sigma_{e\in E(H)} f(e)$.

Let $c_{k,d}$ be the minimum d-degree threshold for perfect fractional matchings in k-graphs, namely, for every $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$, every n-vertex k-graph H with $\delta_d(H) \geq (c_{k,d} + \varepsilon)\binom{n-d}{k-d}$ contains a perfect fractional matching. It is known that [8] every n-vertex k-graph H with $\delta_d(H) \geq (\max\{c_{k,d}, 1/2\}+o(1))\binom{n-d}{k-d}$ has a perfect matching, and this condition is asymptotically best possible. However, determining the parameter $c_{k,d}$ is a major open problem in this field and we refer to [56] for related results and discussions.

Theorem 2.1 *For every $\varepsilon > 0$ and integer $d \in [k-1]$, there exists $n_0 \in \mathbb{N}$, such that the following holds for all integers $n \geq n_0$ and $n \in k\mathbb{N}$. Every n-vertex k-graph system $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ with $\delta_d(G_i) \geq (\max\{c_{k,d}, 1/2\} + \varepsilon)\binom{n-d}{k-d}$ for each i contains a rainbow perfect matching.*

2.1 . Notation and preliminaries

Given a k-graph system $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ on vertex V and a subset $V' \subseteq V$. Let $\boldsymbol{G}[V'] = \{G_i[V']\}_{i\in[n/k]}$ be the *induced* k-graph system on V'. If $|V'| \in k\mathbb{N}$ and there exists a rainbow perfect F-tiling inside $\boldsymbol{G}[V']$ whose color set is $C \subseteq [n/k]$, then we say that V' *spans* a rainbow F-tiling in $\boldsymbol{G}$ with color set C. Next, we give some definition needed in this chapter.

Definition 2.1 (Rainbow F-absorber) *Let $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ be a k-graph system on V. For every k-set B in V and every color C in $[n/k]$, $A = A_1 \cup A_2$ is called a rainbow edge-absorber for (B, C) if*

- $V(A) = B\dot{\cup}L$,
- *A_1 is a rainbow perfect matching L with color set C_1 and A_2 is a rainbow perfect matching on $B \cup L$ with color set $C_1 \cup C$.*

Definition 2.2 *We call a hypergraph H a $(1, b)$-graph, if $V(H)$ can be partitioned into $A \cup B$ and $E(H)$ is a family of $(1 + b)$-sets each of which contains exactly one vertex in A and b vertices in B.*

For a $(1,b)$-graph H with partition $A\dot{\cup}B$, a $(1,d)$-subset D of $V(H)$ is a $(d+1)$-tuple where $|D\cap A| = 1$ and $|D\cap B| = d$. A $(1,b)$-graph H with partition classes A, B is *balanced* if $b|A| = |B|$. We say that a set $S\subseteq V(H)$ is *balanced* if $b|S\cap A| = |S\cap B|$.

Given an n-vertex k-graph system $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ on V, we construct auxiliary $(1,k)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ as follows.

Definition 2.3 *Let $H_{\boldsymbol{G}}$ be an auxiliary $(1,k)$-graph of $\boldsymbol{G}$ with vertex set $V' = [n/k]\cup V$ and edge set $\{\{i\}\cup e : i\in[n/k], e\in G_i\}$.*

For a hypergraph H, the 2-degree of a pair of vertices is the number of edges containing this pair and $\Delta_2(H)$ denotes the maximum 2-degree in H. For reals a, b and c, we write $a = (1\pm b)c$ for $(1-b)c\le a\le(1+b)c$. We need the following result which was attributed to Pippenger [125](see Theorem 4.7.1 in [10]), following Frankl and Rödl. A *cover* in a hypergraph H is a set of edges such that each vertex of H is in at least one edge of the set.

Lemma 2.1 ([125]) *For every integer $k\ge 2$, $r\ge 1$ and $a>0$, there exist $\gamma = \gamma(k,r,a)>0$ and $d_0 = d_0(d,r,a)$ such that the following holds for every $n\in\mathbb{N}$ and $D\ge d_0$. Every k-graph $H = (V,E)$ on V of n vertices in which all vertices have positive degrees and which satisfies the following conditions :*

- *For all vertices $x\in V$ but at most γn of them, $d_H(x) = (1\pm\gamma)D$.*
- *For all $x\in V$, $d_H(x) < rD$.*
- *$\Delta_2(H) < \gamma D$.*

contains a cover of at most $(1+a)(n/k)$ edges.

2.2 . Rainbow absorption method

Given an n-vertex k-graph system $\boldsymbol{G} = \{G_i\}_{i\in[n]}$ on V with $\delta_d(G_i)\ge(1/2+\varepsilon)\binom{n-d}{k-d}$ for $i\in[n/k]$, $d\in[k-1]$, we first construct a $(1,k)$-graph $H_{\boldsymbol{G}}$ with vertex set $[n/k]\cup V$ and edge set $\{\{i\}\cup e : e\in H_i, i\in[n/k]\}$. Next, we construct a specific rainbow edge-absorber. For any k-set $T = \{v_1,\dots,v_k\}$ in V and every color $c_1\in[n/k]$, we give a rainbow absorber $A = A_1\cup A_2$ for (T,c_1) as follows.

- $A_1 = \{M_2,\dots,M_k\}$ is a set of $k-1$ disjoint edges in $H_{\boldsymbol{G}}$ where $c_i\in M_i (i\in[2,k])$.
- There is a vertex $u_i (i\in[2,k])$ from each $V(M_i)$ such that $\{u_2,\dots,u_k,v_1, c_1\}\in E(H_{\boldsymbol{G}})$ and $(V(M_i)\setminus\{u_i\})\cup\{v_i\}\in E(H_{\boldsymbol{G}})$ for $i\in[2,k]$. Let A_2 be $\{\{u_2,\dots,u_k,v_1,c_1\}, (V(M_2)\setminus\{u_2\})\cup\{v_2\},\dots,(V(M_k)\setminus\{u_k\})\cup\{v_k\}\}$.

For any k-set T in V and every color $c_1\in[n/k]$, we denote the family of such rainbow edge-absorbers for (T,c_1) by $\mathcal{A}(T,c_1)$.

Claim 2.1 $|\mathcal{A}(T,c_1)|\ge\varepsilon^{2k-2}n^{k-1}\binom{n-1}{k-1}^k/2$.

Proof. Fix $c_1 \in [n/k]$ and $T = \{v_1, \ldots, v_k\} \subseteq V$. Choose $(c_2, \ldots, c_k)$ arbitrarily from $[n/k]$ and there are at least $(\frac{n}{k} - 1) \cdots (\frac{n}{k} - (k-1)) \geq \varepsilon^{k-1} n^{k-1}$ choices. Fix such $(c_2, \ldots, c_k)$. Next, we construct $M_2, \ldots, M_k$ and note that there are at most $(k-1)\binom{n-1}{k-2} \leq \varepsilon \binom{n-1}{k-1}$ edges which contain c_1, v_1 and v_j for some $j \in [2, k]$. Due to the minimum degree assumption, there are at least $\frac{1}{2}\binom{n-1}{k-1}$ edges containing v_1 and c_1 but none of $v_2, \ldots, v_k$. We fix such one edge $\{c_1, v_1, u_2, \ldots, u_k\}$ and set $U_1 = \{u_2, \ldots, u_k\}$. For each $i \in [2, k]$ and each pair $\{u_i, v_i\}$, suppose we succeed in choosing a set U_i such that U_i is disjoint with $W_{i-1} = \cup_{j \in [i-1]} U_j \cup T$ and both $U_i \cup \{u_i, c_i\}$ and $U_i \cup \{v_i, c_i\}$ are edges in $H_{\boldsymbol{G}}$, then for a fixed $i \in [2, k]$, we call such a choice U_i good.

Note that in each step $i \in [2, k]$, there are $k + (i-1)(k-1) \leq k^2$ vertices in W_{i-1}, thus the number of edges with color c_i intersecting u_i and at least one other vertex in W_{i-1} is at most $k^2\binom{n-1}{k-2}$. So the minimum degree assumption implies that for each $i \in [2, k]$, there are at least $2\varepsilon\binom{n-1}{k-1} - 2k^2\binom{n-1}{k-2} \geq \varepsilon\binom{n-1}{k-1}$ good choices for U_i and in total we obtain $\varepsilon^{2k-2} n^{k-1} \binom{n-1}{k-1}^k / 2$ rainbow absorbers for (T, c_1). □

For any edge $e \in E(H_{\boldsymbol{G}})$, If $A \subseteq V(H_{\boldsymbol{G}})$ and $|A|$ is divisible by $k+1$, then $A \in \binom{(k+1)n}{|A|}$ is an *absorber* for e if $e \subseteq A$, there is a perfect matching in $H_{\boldsymbol{G}}[A]$ and there is a perfect matching in $H_{\boldsymbol{G}}[A \setminus e]$. Let $\mathcal{L}(e)$ denote the set of absorbers for e in $H_{\boldsymbol{G}}$.

Lemma 2.2 (Rainbow Absorption Lemma) *Let A^0 be a rainbow edge-absorber as above. For every $\varepsilon > 0$, there exist γ, γ_1 and n_0 such that the following holds for all integers $n \geq n_0$. Suppose that $\boldsymbol{G} = \{G_i\}_{i \in [n/k]}$ is an n-vertex k-graph system on V and $\delta_d(G_i) \geq (1/2 + \varepsilon)\binom{n-d}{k-d}$ and $H_{\boldsymbol{G}}$ is the auxiliary $(1, k)$-graph of $\boldsymbol{G}$, then there exists a matching M in $H_{\boldsymbol{G}}$ with size at most $2\gamma(k-1)n$ such that for every balanced set $U \subseteq ([n/k] \cup V) \setminus V(M)$ of size at most $\gamma_1 n$, $V(M) \cup U$ spans a matching in $H_{\boldsymbol{G}}$.*

Proof. Let $1/n \ll \gamma_1 \ll \alpha \ll \gamma \ll \varepsilon' \ll \varepsilon$. Note that a matching of size k in $H_{\boldsymbol{G}}$ corresponds to a rainbow edge-absorber in $\boldsymbol{G}$. Choose a family $\mathcal{F}$ of matchings of size $k-1$ from $H_{\boldsymbol{G}}$ by including each matching of size $k-1$ independently at random with probability

$$p = \gamma / n^{(k-1)(k+1)-1}.$$

Note that $|\mathcal{F}|, |\mathcal{L}(e) \cap \mathcal{F}|$ are binomial random variables with expectations

$$\mathbb{E}[|\mathcal{F}|] \leq \gamma n \text{ and}$$

$$\mathbb{E}[|\mathcal{L}(e) \cap \mathcal{F}|] \geq \gamma \varepsilon' n \text{ for any } e \in E(H_{\boldsymbol{G}}).$$

The latter inequality holds since for any edge e of $H_{\boldsymbol{G}}$, $|\mathcal{L}(e)| \geq \varepsilon' n^{(k-1)(k+1)}$ by the minimum degree assumption and Claim 2.1. By Lemma 1.1, with probability $1 - o(1)$, the family $\mathcal{F}$ satisfies the following properties.

1. $|\mathcal{F}| \le 2\mathbb{E}[|\mathcal{F}|] \le 2\gamma n$,
2. $|\mathcal{L}(e) \cap \mathcal{F}| \ge \frac{1}{2}\mathbb{E}[|\mathcal{L}(e) \cap \mathcal{F}|] \ge \frac{1}{2}\gamma\varepsilon' n$ for any $e \in E(H_{\boldsymbol{G}})$.

Moreover, we can also bound the expected number of pairs of intersecting members of $\mathcal{F}$ by

$$n^{(k-1)(k+1)}(k-1)^2(k+1)^2 n^{(k-1)(k+1)-1}p^2 \le \frac{1}{8}\gamma\varepsilon' n.$$

Thus, by Markov's inequality, we derive that with probability at least $1/2$, $\mathcal{F}$ contains at most $\frac{1}{4}\gamma\varepsilon' n$ pairs of intersecting members of $\mathcal{F}$. Remove one member from each of the intersecting pairs in $\mathcal{F}$. Thus, the resulting family, say $\mathcal{F}'$, consists of pairwise disjoint matchings of size $m-1$ that satisfies

1. $|\mathcal{F}'| \le 2\gamma n$,
2. $|\mathcal{L}(e) \cap \mathcal{F}| \ge \frac{1}{2}\gamma\varepsilon' n - \frac{1}{4}\gamma\varepsilon' n \ge \alpha n$ for any $e \in E(H_{\boldsymbol{G}})$.

Therefore, the union of members in $\mathcal{F}'$ is a matching in $H_{\boldsymbol{G}}$ of size at most $2\gamma(k-1)n$ and can (greedily) absorb a balanced set U of size at most $\gamma_1 n$ since $\gamma_1 \ll \alpha$. □

2.3 . Rainbow matching cover

The goal of this section is to prove the following lemma, an important component of the proof of Theorem 2.1.

Lemma 2.3 (Rainbow Almost Cover Lemma) *For every $\varepsilon, \phi > 0$ and integer $d \in [k-1]$, the following holds for sufficiently large $n \in b\mathbb{N}$. Suppose that $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ is an n-vertex k-graph system on V such that $\delta_d(G_i) \ge (c_{k,d}+\varepsilon)\binom{n-d}{k-d}$ for $i \in [n/k]$, then $\boldsymbol{G}$ contains a rainbow matching covering all but at most ϕn vertices.*

For a k-graph H, a *fractional cover* is a function $\omega : V(H) \to [0,1]$, subject to the requirement $\sum_{v:v\in e}\omega(v) \ge 1$ for every $e \in E(H)$. Denote the minimum fractional cover size by $\tau^*(H) = \min_\omega \Sigma_{v\in V(H)}\omega(v)$. The conclusion $\nu^*(H) = \tau^*(H)$ for any hypergraph follows from the LP-duality. For n-vertex k-graphs we trivially have $\nu^*(H) = \tau^*(H) \le \frac{n}{k}$.

Given an n-vertex k-graph system $\boldsymbol{G} = \{G_i\}_{i\in[n/k]}$ on V. Let $H_{\boldsymbol{G}}$ be the auxiliary $(1,k)$-graph of $\boldsymbol{G}$. Let $\delta_{1,k-1}(H_{\boldsymbol{G}}) := \min\{\deg_{H_{\boldsymbol{G}}}(S) : S$ is a $(1,k-1)$-subset of $V(H_{\boldsymbol{G}})\}$ where $\deg_{H_{\boldsymbol{G}}}(S)$ denotes the number of edges in $H_{\boldsymbol{G}}$ containing S.

The proof of the following claim is by now a standard argument on fractional matchings and covers.

Claim 2.2 *If each G_i contains a perfect fractional matching for $i \in [\frac{n}{k}]$, then the auxiliary $(1,k)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ contains a perfect fractional matching.*

Proof. By the duality theorem, we transform the maximum fractional matching problem into the minimum fractional cover problem. Since $\tau^*(H_{\boldsymbol{G}}) = \nu^*(H_{\boldsymbol{G}}) \le \frac{n}{k}$, it suffices to show that $\tau^*(H_{\boldsymbol{G}}) \ge \frac{n}{k}$ to obtain $\nu^*(H_{\boldsymbol{G}}) = \frac{n}{k}$. Let ω be the minimum fractional cover of $H_{\boldsymbol{G}}$ and take $i_1 \in [n/k]$ such that $\omega(i_1) := \min_{i\in[n/k]} \omega(i)$. We may assume that $\omega(i_1) = 1 - x < 1$, since otherwise $\omega([n/k]) \ge \frac{n}{k}$ and we are done. By definition we get $\omega(e) \ge 1 - \omega(i_1) = x$ for every $e \in G_{i_1}$. We define a new weight function ω' on V by setting $\omega'(v) = \frac{\omega(v)}{x}$ for every vertex $v \in V$. Thus, ω' is a fractional cover of G_{i_1} because for each $e \in G_{i_1}$, $\omega'(e) = \frac{\omega(e)}{x} \ge 1$. Recall that G_{i_1} has a perfect fractional matching, and thus $\omega'(V) \ge \tau^*(G_{i_1}) \ge \frac{n}{k}$ which implies that $\omega(V) \ge \frac{xn}{k}$. Therefore,

$$\omega([\tfrac{n}{k}] \cup V) \ge (1-x)\tfrac{n}{k} + \tfrac{xn}{k} = \tfrac{n}{k}.$$

Hence, $\tau^*(H_{\boldsymbol{G}}) = \frac{n}{k}$, i.e. $H_{\boldsymbol{G}}$ contains a perfect fractional matching. □

Given an n-vertex k-graph system $\boldsymbol{G}$, we shall construct an auxiliary $(1,b)$-graph $H_{\boldsymbol{G}}$ of $\boldsymbol{G}$ and a sequence of random subgraphs of $H_{\boldsymbol{G}}$. Then, we use the properties of them to get a "near regular" spanning subgraph for the sake of applying Lemma 2.1.

The proof is based on a two-round randomization which is already used in [8, 110, 27]. Since we work with balanced $(1,k)$-graphs, we need to make sure that each random graph is balanced. In order to achieve this we modify the randomization process by fixing an arbitrarily small and balanced set $S \subseteq V(H_{\boldsymbol{G}})$. This is done in Fact 2.1.

Let $H_{\boldsymbol{G}}$ be the auxiliary $(1,k)$-graph of $\boldsymbol{G}$ with partition classes A, B and $k|A| = |B|$ where A is the color set and $B = V$. Let $S \subseteq V(H_{\boldsymbol{G}})$ be a set of vertices such that $|S \cap A| = n^{0.99}/b$ and $|S \cap B| = n^{0.99}$. The desired subgraph H'' is obtained by two rounds of randomization. As a preparation to the first round, we choose every vertex randomly and uniformly with probability $p = n^{-0.9}$ to get a random subset R of $V(H_{\boldsymbol{G}})$. Take $n^{1.1}$ independent copies of R and denote them by R_{i+}, $i \in [n^{1.1}]$, i.e. each R_{i+} is chosen in the same way as R independently. Define $R_{i-} = R_{i+} \setminus S$ for $i \in [n^{1.1}]$.

Fact 2.1 *Let $n, H_{\boldsymbol{G}}, A, B, S$ and R_{i-}, R_{i+} be given as above. Then, with probability $1 - o(1)$, there exist subgraphs $R_i, i \in [n^{1.1}]$, such that $R_{i-} \subseteq R_i \subseteq R_{i+}$ and R_i is balanced.*

Proof. Recall that $|A| = n/k, |B| = n, |S \cap A| = n^{0.99}/b$ and $|S \cap B| = n^{0.99}$, thus

$$\mathbb{E}[|R_{i+} \cap A|] = n^{0.1}/k,$$

$$\mathbb{E}[|R_{i+} \cap A \cap S|] = n^{0.09}/k,$$

$$\mathbb{E}[|R_{i+} \cap B|] = n^{0.1},$$

$$\mathbb{E}[|R_{i+} \cap B \cap S|] = n^{0.09}.$$

By Lemma 1.1, we have

$$\mathbb{P}[||R_{i+} \cap A| - n^{0.1}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.06})},$$

$$\mathbb{P}[||R_{i+} \cap A \cap S| - n^{0.09}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.07})},$$

$$\mathbb{P}[||R_{i+} \cap B| - n^{0.1}| \geq n^{0.08}] \leq e^{-\Omega(n^{0.06})},$$

$$\mathbb{P}[||R_{i+} \cap B \cap S| - n^{0.09}/b| \geq n^{0.08}] \leq e^{-\Omega(n^{0.07})}.$$

Thus, with probability $1 - o(1)$, for all $i \in [n^{1.1}]$,

$$|R_{i+} \cap A| \in [n^{0.1}/k - n^{0.08}, n^{0.1}/b + n^{0.08}],$$

$$|R_{i+} \cap A \cap S| = (1 + o(1))n^{0.09}/k,$$

$$|R_{i+} \cap B| \in [n^{0.1} - n^{0.08}, n^{0.1} + n^{0.08}],$$

$$|R_{i+} \cap B \cap S| = (1 + o(1))n^{0.09}.$$

Therefore, $|b|R_{i+} \cap A| - |R_{i+} \cap B|| \leq (k+1)n^{0.08} < \min\{|R_{i+} \cap A \cap S|, |R_{i+} \cap B \cap S|\}$. Hence, with probability $1 - o(1)$, R_i can be balanced for $i \in [n^{1.1}]$. $\square$

The following two lemmas together construct the desired sparse regular k-graph we need.

Lemma 2.4 *Given an n-vertex k-graph system $\boldsymbol{G} = \{G_i\}_{i \in [n/k]}$ on V, let $H_{\boldsymbol{G}}$ be the auxiliary $(1,k)$-graph of $\boldsymbol{G}$. For each $X \subseteq V(H_{\boldsymbol{G}})$, let $Y_X^+ := |\{i : X \subseteq R_{i+}\}|$ and $Y_X := |\{i : X \subseteq R_i\}|$. Then with probability at least $1 - o(1)$, we have*

1. $|R_i| = (1/b + 1 + o(1))n^{0.1}$ *for all* $i \in [n^{1.1}]$.
2. $Y_{\{v\}} = (1 + o(1))n^{0.2}$ *for* $v \in V(H_{\boldsymbol{G}}) \setminus S$ *and* $Y_{\{v\}} \leq (1 + o(1))n^{0.2}$ *for* $v \in S$.
3. $Y_{\{u,v\}} \leq 2$ *for all* $\{u, v\} \subseteq V(H_{\boldsymbol{G}})$.
4. $Y_e \leq 1$ *for all* $e \in E(H_{\boldsymbol{G}})$.
5. *Suppose that $V(R_i) = C_i \cup V_i$, we have $\delta_{1,d}(H_{\boldsymbol{G}}[V(R_i)]) \geq (c_{k,d} + \varepsilon/4)$ $\binom{|R_{i+} \cap B| - d}{k-d} - |R_{i+} \cap B \cap S|\binom{|R_{i+} \cap B| - d - 1}{k-d-1} \geq (c_{k,d} + \varepsilon/8)\binom{|R_i \cap B| - d}{k-d}$.*

Proof. Note that $\mathbb{E}[|R_{i+}|] = (1/k+1)n^{0.1}, \mathbb{E}[|R_{i-}|] = ((1/k+1)n - (1/k+1)n^{0.99})n^{-0.9} = (1/k+1)n^{0.1} - (1/k+1)n^{0.09}$. By Lemma 1.1, we have

$$\mathbb{P}[|\ |R_{i+}| - n^{0.1}(1/k+1)\ | \geq n^{0.095}] \leq e^{-\Omega(n^{0.09})},$$

$$\mathbb{P}[|\ |R_{i-}| - ((1/k+1)n^{0.1} - (1/k+1)n^{0.09})\ | \geq n^{0.095}] \leq e^{-\Omega(n^{0.09})}.$$

Hence, with probability at least $1 - O(n^{1.1})e^{-\Omega(n^{0.09})}$, for the given sequence R_i in Fact 2.1, $i \in [n^{1.1}]$, satisfying $R_{i-} \subseteq R_i \subseteq R_{i+}$, we have $|R_i| = (1/k + 1 + o(1))n^{0.1}$.

For each $X \subseteq V(H_{\boldsymbol{G}})$, let $Y_X^+ := |\{i : X \subseteq R_{i+}\}|$ and $Y_X := |\{i : X \subseteq R_i\}|$. Note that the random variables Y_X^+ have binomial distributions $Bi(n^{1.1}, n^{-0.9|X|})$ with expectations $n^{1.1-0.9|X|}$ and $Y_X \leq Y_X^+$. In particular, for each $v \in V(H_{\boldsymbol{G}})$, $\mathbb{E}[Y_{\{v\}}^+] = n^{0.2}$, by Lemma 1.1, we have

$$\mathbb{P}[|\ |Y_{\{v\}}^+| - n^{0.2}\ | \geq n^{0.19}] \leq e^{-\Omega(n^{0.18})}.$$

Hence, with probability at least $1-O(n)e^{-\Omega(n^{0.18})}$, we have $Y_{\{v\}} = (1+o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $Y_{\{v\}} \leq (1+o(1))n^{0.2}$ for $v \in S$.

Let $Z_{p,q} = |X \in \binom{V(H_{\boldsymbol{G}})}{p} : Y_X^+ \geq q|$. Then,

$$\mathbb{E}[Z_{p,q}] \leq \binom{\frac{n}{k}+n}{p}\binom{n^{1.1}}{q}(n^{-0.9pq}) \leq Cn^{p+1.1q-0.9pq}.$$

Hence, by Markov's inequality we have

$$\mathbb{P}[Z_{2,3} = 0] = 1 - \mathbb{P}[Z_{2,3} \geq 1] \geq 1 - \mathbb{E}[Z_{2,3}] = 1 - o(1),$$

$$\mathbb{P}[Z_{1+k,2} = 0] = 1 - \mathbb{P}[Z_{1+k,2} \geq 1] \geq 1 - \mathbb{E}[Z_{1+k,2}] = 1 - o(1),$$

i.e. with probability at least $1 - o(1)$, every pair $\{u, v\} \subseteq V(H_{\boldsymbol{G}})$ is contained in at most two sets R_{i+}, and every edge is contained in at most one set R_{i+}. Thus, the conclusions also hold for R_i.

Fix a $(1, d)$-subset $D \subseteq V(H_{\boldsymbol{G}})$ and let $N_D(H_{\boldsymbol{G}})$ be the neighborhood of D in $H_{\boldsymbol{G}}$. Recall that R is obtained by choosing every vertex randomly and uniformly with probability $p = n^{-0.9}$, let DEG_D be the number of edges $\{f | f \subseteq R$ and $f \in N_D(H_{\boldsymbol{G}})\}$. Therefore $DEG_D = \sum_{f \in N_D(H_{\boldsymbol{G}})} X_f$, where $X_f = 1$ if f is in R and 0 otherwise. We have

$$\mathbb{E}[DEG_D] = d_{H_{\boldsymbol{G}}}(D) \times (n^{-0.9})^{k-d} \geq (c_{k,d} + \varepsilon)\binom{n-d}{k-d}n^{-0.9(k-d)}$$

$$\geq (c_{k,d} + \varepsilon/3)\binom{|R \cap B| - d}{k-d} = \Omega(n^{0.1(k-d)}).$$

For two distinct intersecting edges $f_i, f_j \in N_D(H_{\boldsymbol{G}})$ with $|f_i \cap f_j| = \ell$ for $\ell \in [k-d-1]$, the probability that both of them are in R is

$$\mathbb{P}[X_{f_i} = X_{f_j} = 1] = p^{2(k-d)-\ell},$$

for any fixed ℓ, we have

$$\Delta = \sum_{f_i \cap f_j \neq \emptyset} \mathbb{P}[X_{f_i} = X_{f_j} = 1] \leq \sum_{\ell=1}^{k-d-1} p^{2(k-d)-\ell}\binom{n-d}{k-d}\binom{k-d}{\ell}\binom{n-k}{k-d-\ell}$$

$$\leq \sum_{\ell=1}^{k-d-1} p^{2(k-d)-\ell}O(n^{2(k-d)-\ell}) = O(n^{0.1(2(k-d)-1)}).$$

Applying Lemma 1.3 with $\Gamma = B$, $\Gamma_p = R \cap B$ and $M = N_{H_{\boldsymbol{G}}}(D)$(a family of $(k-d)$-sets) , we have

$$\mathbb{P}[DEG_D \leq (1-\varepsilon/12)\mathbb{E}[DEG_D]] \leq e^{-\Omega((\mathbb{E}[DEG_D])^2/\Delta)} = e^{-\Omega(n^{0.1})}.$$

Therefore by the union bound, with probability $1-o(1)$, for all $(1,d)$-subsets $D \subseteq V(H_{\boldsymbol{G}})$, we have

$$DEG_D > (1-\varepsilon/12)\mathbb{E}[DEG_D] \geq (c_{k,d}+\varepsilon/4)\binom{|R \cap B| - d}{k-d}.$$

Summarizing, with probability $1-o(1)$, for the sequence $R_i, i \in [n^{1.1}]$, satisfying $R_{i-} \subseteq R_i \subseteq R_{i+}$, all of the following hold.

1. $|R_i| = (1/b + 1 + o(1))n^{0.1}$ for all $i \in [n^{1.1}]$.
2. $Y_{\{v\}} = (1+o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $Y_{\{v\}} \leq (1+o(1))n^{0.2}$ for $v \in S$.
3. $Y_{\{u,v\}} \leq 2$ for all $\{u,v\} \subseteq V(H_{\boldsymbol{G}})$.
4. $Y_e \leq 1$ for all $e \in E(H_{\boldsymbol{G}})$.
5. $DEG_D^{(i)} \geq (c_{k,d}+\varepsilon/4)\binom{|R_{i+} \cap B|-d}{k-d}$ for all $(1,d)$-subsets of $D \subseteq V(H_{\boldsymbol{G}})$ and $i \in [n^{1.1}]$.

Thus, by property 5 above, we conclude that suppose $V(R_i^+) = C_i^+ \cup V_i^+$ and $V(R_i) = C_i \cup V_i$, the following holds.

$$\delta_{1,d}(H_{\boldsymbol{G}}[V(R_i^+)]) \geq (c_{k,d}+\varepsilon/4)\binom{|R_{i+} \cap B| - d}{k-d}.$$

After the modification, we still have

$$\begin{aligned}\delta_{1,d}(H_{\boldsymbol{G}}[V(R_i)]) &\geq (c_{k,d}+\varepsilon/4)\binom{|R_{i+} \cap B| - d}{k-d} - |R_{i+} \cap B \cap S|\binom{|R_{i+} \cap B| - d - 1}{k-d-1} \\ &\geq (c_{k,d}+\varepsilon/8)\binom{|R_i \cap B| - d}{k-d}.\end{aligned}$$

□

Lemma 2.5 *Let $n, H_{\boldsymbol{G}}, S, R_i$, $i \in [n^{1.1}]$ be given as in Lemma 2.4 such that each $H_{\boldsymbol{G}}[V(R_i)]$ is a balanced $(1,k)$-graph and has a perfect fractional matching ω_i. Then there exists a spanning subgraph H'' of $H^* = \cup_i H_{\boldsymbol{G}}[V(R_i)]$ such that*

- $d_{H''}(v) \leq (1+o(1))n^{0.2}$ *for $v \in S$,*
- $d_{H''}(v) = (1+o(1))n^{0.2}$ *for all $v \in V(H_{\boldsymbol{G}}) \setminus S$,*
- $\Delta_2(H'') \leq n^{0.1}$.

The proofs follow the lines as in [8, 110, 27].

Proof. By the condition that each $H_{\boldsymbol{G}}[V(R_i)]$ has a perfect fractional matching ω_i, we select a generalized binomial subgraph H'' of H^* by independently choosing each edge e with probability $\omega_{i_e}(e)$ where i_e is the index i such that $e \in H_{\boldsymbol{G}}[V(R_i)]$. Recall that property 4 guarantees the uniqueness of i_e.

For $v \in V(H'')$, let $I_v = \{i : v \in R_i\}$, $E_v = \{e \in H^* : v \in e\}$ and $E_v^i = E_v \cap H_{\boldsymbol{G}}[V(R_i)]$, then $E_v^i, i \in I_v$ forms a partition of E_v and $|I_v| = Y_{\{v\}}$. Hence, for $v \in V(H'')$,

$$d_{H''}(v) = \sum_{e \in E_v} 1 = \sum_{i \in I_v} \sum_{e \in E_v^i} X_e,$$

where X_e is the Bernoulli random variable with $X_e = 1$ if $e \in E(H'')$ and $X_e = 0$ otherwise. Thus its expectation is $\omega_{i_e}(e)$. Therefore

$$\mathbb{E}[d_{H''}(v)] = \sum_{i \in I_v} \sum_{e \in E_v^i} \omega_{i_e}(e) = \sum_{i \in I_v} 1 = Y_{\{v\}}.$$

Hence, $\mathbb{E}[d_{H''}(v)] = (1 + o(1))n^{0.2}$ for $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $\mathbb{E}[d_{H''}(v)] \leq (1 + o(1))n^{0.2}$ for $v \in S$. Now by Chernoff's inequality, for $v \in V(H_{\boldsymbol{G}}) \setminus S$,

$$\mathbb{P}[|d_{H''}(v) - n^{0.2}| \geq n^{0.15}] \leq e^{-\Omega(n^{0.1})},$$

and for $v \in S$,

$$\mathbb{P}[d_{H''}(v) - n^{0.2} \geq n^{0.15}] \leq e^{-\Omega(n^{0.1})}.$$

Taking a union bound over all vertices, we conclude that with probability $1 - o(1)$, $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for $v \in S$.

Next, note that for distinct $u, v \in V(H_{\boldsymbol{G}})$,

$$d_{H''}(\{u, v\}) = \sum_{e \in E_u \cap E_v} 1 = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} X_e,$$

and

$$\mathbb{E}[d_{H''}(\{u, v\})] = \sum_{i \in I_u \cap I_v} \sum_{e \in E_u^i \cap E_v^i} \omega_i(e) \leq |I_u \cap I_v| \leq 2.$$

Thus, by Lemma 1.2,

$$\mathbb{P}[d_{H''}(\{u, v\}) \geq n^{0.1}] \leq e^{-n^{0.1}},$$

then by a union bound we have $\Delta_2(H'') \leq n^{0.1}$ with probability $1 - o(1)$. □

Proof. [Proof of Lemma 2.3] By the definition of $c_{k,d}$, Lemma 2.4 (5) and Claim 2.2, there exists a perfect fractional matching ω_i in every subgraph $H_{\boldsymbol{G}}[V(R_i)]$, $i \in [n^{1.1}]$. By Lemma 2.5, there is a spanning subgraph H'' of $H^* = \cup_i H_{\boldsymbol{G}}[V(R_i)]$

such that $d_{H''}(v) \leq (1+o(1))n^{0.2}$ for each $v \in S$, $d_{H''}(v) = (1+o(1))n^{0.2}$ for all $v \in V(H_{\boldsymbol{G}}) \setminus S$ and $\Delta_2(H'') \leq n^{0.1}$. Hence, by Lemma 2.1 (by setting $D = n^{0.2}$), H'' contains a cover of at most $\frac{n+n/k}{1+k}(1+a)$ edges which implies that H'' contains a matching of size at least $\frac{n+n/k}{1+k}(1-a(1+k-1))$, where a is a constant satisfying $0 < a < \phi/(1+k-1)$. Hence $H_{\boldsymbol{G}}$ contains a matching covering all but at most $\phi(n+n/k)$ vertices. $\square$

Proof. [The proof of Theorem 2.1] Suppose that $\frac{1}{n} \ll \phi \ll \gamma_1 \ll \gamma \ll \varepsilon' \ll \varepsilon$ where $\varepsilon', \gamma, \gamma_1$ are defined in Lemma 2.2 and ϕ, ε in Lemma 2.3. Let $H_{\boldsymbol{G}}$ be the auxiliary $(1,k)$-graph of $\boldsymbol{G}$. By Lemma 2.2, we get a matching M in $H_{\boldsymbol{G}}$ of size at most $2\gamma(k-1)n$ such that for every balanced set $U \subseteq [n/k] \cup V \setminus V(M)$ of size at most $\gamma_1 n$, $V(M) \cup U$ spans a matching in $H_{\boldsymbol{G}}$. Let $\boldsymbol{G}' = \{G_i'\}_{i\in[n/k]}$ be the induced k-graph system of $\boldsymbol{G}$ on V' where $V' := V \setminus V(M)$. Denote the subsystem of $\boldsymbol{G}'$ by $\boldsymbol{G}'_I = \{G_i'\}_{i\in I}$, where $I = [n/k] \setminus V(M)$. We still have $\delta_d(G_i') \geq (\max\{c_{k,d}, 1/2\} + \frac{\varepsilon}{2})\binom{n-d}{k-d}$ for $i \in I$, since $2\gamma(k-1)n\binom{n-d-1}{k-d-1} \leq \frac{\varepsilon}{2}\binom{n-d}{k-d}$. Then, we construct the new auxiliary $(1,k)$-graph $H_{\boldsymbol{G}'_I}$ of $\boldsymbol{G}'_I$.

By Lemma 2.3, $H_{\boldsymbol{G}'_I}$ contains a matching M_1 covering all but at most $\phi|V'| \leq \phi(n+n/k)$ vertices. Suppose $W_1 = [n/k] \cup V \setminus (V(M) \cup V(M_1))$, hence $|W_1| \leq \phi(n+n/k) \leq \gamma_1 n$ and W_1 is balanced. By Lemma 2.2, $V(M) \cup W_1$ spans a matching M_2 in $H_{\boldsymbol{G}}$ and therefore $M_1 \cup M_2$ is a perfect matching in $H_{\boldsymbol{G}}$, which yields a rainbow perfect matching in $\boldsymbol{G}$. $\square$

2.4 . Concluding remarks

Let F be a k-graph with b vertices and f edges. We first define an absorber without colors. Given a set B of b vertices, a k-graph $A^0 = A_1^0 \cup A_2^0$ is called an *F-absorber for B* if

- $V(A^0) = B \dot\cup L$[1],
- A_1^0 is an F-factor on L and A_2^0 is an F-factor on $B \cup L$.

Note that $|V(A^0)|$ is a constant. Naturally, we give the definition of rainbow F-absorber as follows.

Definition 2.4 (Rainbow F-absorber) *Let $\boldsymbol{G} = \{G_i\}_{i\in[nf/b]}$ be a k-graph system on V and F be a k-graph with b vertices and f edges. For every b-set B in V and every f-set C in $[nf/b]$, $A = A_1 \cup A_2$ is called a rainbow F-absorber for (B, C) if*

- *$V(A) = B \dot\cup L$,*
- *A_1 is a rainbow F-factor on L with color set C_1 and A_2 is a rainbow F-factor on $B \cup L$ with color set $C_1 \cup C$.*

Now we introduce one of the main parameters $c_{d,F}^{\mathrm{abs}}$. Roughly speaking, it is the minimum degree threshold such that all b-sets are contained in many rainbow F-absorbers.

1. As usual, $A \dot\cup B$ denotes the disjoint union of A and B.

Definition 2.5 (**$c_{d,F}^{\mathrm{abs}}$: Rainbow absorption threshold**) *Fix an F-absorber $A^0 = A_1^0 \cup A_2^0$ and let m be the number of vertex disjoint copies of F in A_2^0. Let $c_{d,F,A^0} \in (0,1)$ be the infimum of reals $c > 0$ such that for every $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$ where $d \in [k-1]$. Let **G** $= \{G_i\}_{i\in[nf/b]}$ be an n-vertex k-graph system on V. If $\delta_d(G_i) \geq (c+\varepsilon)\binom{n-d}{k-d}$ for $i \in [nf/b]$, then for every b-set B in V and every f-set C in $[nf/b]$ with the form $[(i-1)f, if]$ for some $i \in [n/b]$, there are at least $\varepsilon' n^{(m-1)(b+1)}$ rainbow F-absorbers A with color set $C(A_1^0) \cup C$ whose underlying graph is isomorphic to A^0 such that $C(A_1^0) = [(i_1-1)f+1, i_1 f] \cup [(i_2-1)f+1, i_2 f] \cup \cdots \cup [(i_{m-1}-1)f+1, i_{m-1}f]$ where $i_j \in [n/b]$ for each $j \in [m-1]$ and $i_{j_1} \neq i_{j_2}$ for distinct $j_1, j_2 \in [m-1]$. Let $c_{d,F}^{\mathrm{abs}} := \inf c_{d,F,A^0}$ where the infimum is over all F-absorbers A^0.*

We next define a threshold parameter for the rainbow almost F-factor in a similar fashion. We use the following auxiliary b-graph H_F. Given a k-graph F with b vertices and f edges, and an n-vertex k-graph system ***H*** $= \{H_i\}_{i\in[f]}$ on V, let H_F be the b-graph with vertex set $V(H_F) = V$ and edge set $E(H_F) = \{V(F') : F' \text{ is a rainbow copy of } F \text{ with color set } [f]\}$.

Definition 2.6 (**$c_{d,F}^{\mathrm{cov}}$: Rainbow almost F-factor threshold**) *Let $c_{d,F}^{\mathrm{cov}} \in (0,1)$ be the infimum of reals $c > 0$ such that for every $\varepsilon > 0$, the following holds for sufficiently large $n \in \mathbb{N}$. Let **H** $= \{H_i\}_{i\in[f]}$ be an n-vertex k-graph system. If $\delta_d(H_i) \geq (c+\varepsilon)\binom{n-d}{k-d}$ for every $i \in [f]$, then the b-graph H_F has a perfect fractional matching.*

Now we are ready to state our general result on rainbow F-factors. The proof process is similar with Theorem 2.1 and we will omit the details.

Theorem 2.2 *Let F be a k-graph with b vertices and f edges. For any $\varepsilon > 0$ and integer $d \in [k-1]$, the following holds for sufficiently large $n \in b\mathbb{N}$. Let **G** $= \{G_i\}_{i\in[nf/b]}$ be an n-vertex k-graph system on V. If $\delta_d(G_i) \geq (\max\{c_{d,F}^{\mathrm{abs}}, c_{d,F}^{\mathrm{cov}}\} + \varepsilon)\binom{n-d}{k-d}$ for $i \in [nf/b]$, then there is a **G**-rainbow F-factor.*

3 - Rainbow Hamilton cycles in hypergraph systems with minimum $(k-1)$-degree

The main goal of this chapter is to extend Theorem 1.1 to the rainbow setting. For every $k \geq 3, \gamma > 0$, we say that an n-vertex k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ is a (k, n, γ)-*graph system* if $\delta_{k-1}(H_i) \geq (1/2+\gamma)n$ for each $i \in [n]$.

Theorem 3.1 *For every $k \geq 3, \gamma > 0$ and sufficiently large $n \in \mathbb{N}$, every (k, n, γ)-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ admits an $\boldsymbol{H}$-rainbow Hamilton cycle.*

3.1 . Notation and preliminaries

Given a k-graph H and a k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ on the same vertex set with H, we define $\{i : E(H_i) \cap E(H) \neq \emptyset\}$ as the *color set* of H, denoted by $C(H)$. We call $P = x_1 \cdots x_{2k-2}$ an $\boldsymbol{H}$-rainbow path *with color pattern* $(c_1, \ldots, c_{k-1})$ if $\{x_i, \ldots, x_{i+k-1}\} \in E(H_{c_i})$ for $i \in [k-1]$. Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a family of vertex-disjoint paths. If each $P_i, i \in [m]$, is an $\boldsymbol{H}$-rainbow path and $C(P_i) \cap C(P_j) = \emptyset$ for distinct $i, j \in [m]$, then we call this family an $\boldsymbol{H}$-*rainbow family of paths*. Denote $\bigcup_{i\in[m]} V(P_i)$ by $V(\mathcal{P})$. The *size* of $\mathcal{P}$ is the number of paths in the family.

When we write $\alpha \ll \beta$, we mean that α, β are constants in $(0,1)$, and for every β we have chosen, there exists $\alpha_0 = \alpha_0(\beta)$ such that the subsequent arguments hold for all $\alpha \leq \alpha_0$. While multiple constants appear in a hierarchy, they are chosen from right to left.

Besides, we require the following concentration inequalities.

In this section we give an outline of the proof of Theorem 3.1. Our proof is under the framework of the absorption method, systematised by the work of Rödl, Ruciński and Szemerédi [135, 138], which reduces the problem of finding a spanning subgraph to building an absorption structure and an almost spanning structure. Tailored to our problem, the idea is to build a rainbow absorbing cycle and a rainbow path cover. Moreover, the rainbow absorbing cycle will be able to swallow an arbitrary leftover of vertices, a leftover of colors as well as an $\boldsymbol{H}$-rainbow family of paths so that we obtain a rainbow Hamilton cycle. This motivates us to append the color information and the connecting technique into the rainbow absorption method, which is our contribution compared with the proof in [137].

Lemma 3.1 (Absorbing lemma) *Given $k \geq 3, \gamma > 0$, there exists $\kappa > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Let $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ be a*

*(k,n,γ)-graph system on V. Then there exists an **H**-rainbow cycle A with at most $\gamma n/2$ vertices such that for any **H**-rainbow family of paths $\mathcal{P}$ and any vertex set U in $V\backslash V(A)$ with $|\mathcal{P}|,|U|\leq\kappa n$, there exists an **H**-rainbow cycle A' with vertex set $V(A)\cup U\cup V(\mathcal{P})$ and $C(A)\subseteq C(A')$.*

We first define two versions of absorbers as follows .

Definition 3.1 *Given a (k,n,γ)-graph system **H**, a vertex x and a color c, we say that a path P is a rainbow absorber for (x,c) in an n-vertex k-graph system if the following holds :*

- *$x\notin V(P)$;*
- *$P=x_1\cdots x_{2k-2}$ is an **H**-rainbow path with color pattern $(c_1,\ldots,c_{k-1})$;*
- *$x_1\cdots x_{k-1}xx_k\cdots x_{2k-2}$ is an **H**-rainbow path with color pattern $(c,c_1,\ldots,c_{k-1})$.*

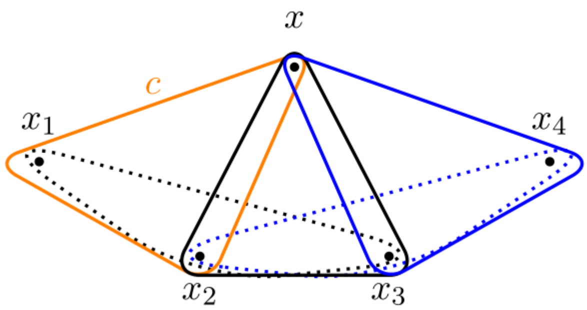


Figure 3.1 – Absorber for (x,c) when $k=3$

Definition 3.2 *Given a (k,n,γ)-graph system **H**, two disjoint $(k-1)$-tuples of vertices $\mathbf{u}=(u_1,\ldots,u_{k-1})$, $\mathbf{v}=(v_1,\ldots,v_{k-1})$ and a $(k-1)$-tuple $(o_1,\ldots,o_{k-1})$ of colors, we say that a path P is a rainbow absorber for $(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})$ in an n-vertex k-graph system if the following holds :*

- *$V(P)\cap\{u_1,\ldots,u_{k-1},v_1,\ldots,v_{k-1}\}=\emptyset$;*
- *$P=x_1\cdots x_{2k-2}$ is an **H**-rainbow path with color pattern $(c_1,\ldots,c_{k-1})$;*
- *$x_1\cdots x_{k-1}u_1\cdots u_{k-1}$ and $v_1\cdots v_{k-1}x_k\cdots x_{2k-2}$ are **H**-rainbow paths with color patterns $(c_1,\ldots,c_{k-1})$, $(o_1,\ldots,o_{k-1})$ respectively.*

The second task is to connect the absorbers to a path. The following lemma helps us to connect any two disjoint paths (by connecting their ends) and its proof is in Section 3.4.

Lemma 3.2 (Connecting lemma) *For every $k\geq 3,\gamma>0$, there exists $c\in\mathbb{N}$ such that the following holds for sufficiently large $n\in\mathbb{N}$. Let **H** $=\{H_i\}_{i\in[c]}$ be a (k,n,γ)-graph system and $\mathbf{u}$, $\mathbf{v}$ be two disjoint $(k-1)$-tuples of vertices. Then, there exists an **H**-rainbow path from $\mathbf{u}$ to $\mathbf{v}$ with at most $c+k-1$ vertices.*

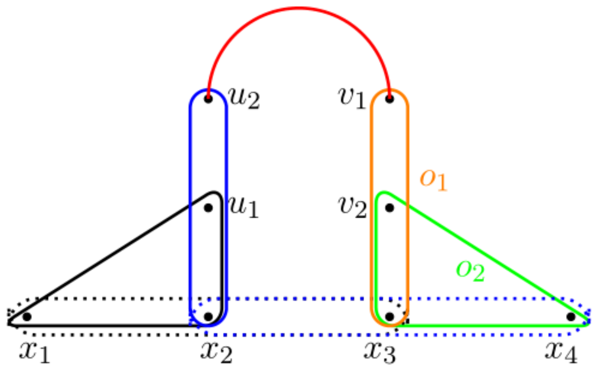


Figure 3.2 – Absorber for $((u_1, u_2), (v_1, v_2); o_1, o_2)$ when $k = 3$

Given a (k, n, γ)-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$, we need to construct an $\boldsymbol{H}$-rainbow family of paths, covering almost all vertices of $V\backslash V(A)$ and almost all colors of $[n]\backslash C(A)$. To achieve this, we use the regularity lemma for hypergraphs and a trick of Ferber and Kwan [54].

Lemma 3.3 (Path cover lemma) *For every $k \geq 3$, γ, $\delta > 0$, there exists $L > 0$ such that the following holds for sufficiently large $n \in \mathbb{N}$. Every (k, n, γ)-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ on V contains an $\boldsymbol{H}$-rainbow family of paths $\mathcal{P}$ of size at most L, covering at least $(1 - \delta)n$ vertices of V.*

Proof. [Proof of Theorem 3.1] For any $k \geq 3$ and $\gamma > 0$, we choose $1/n \ll 1/L, \kappa \ll \gamma, 1/k$, and fix $\boldsymbol{H}$ to be a (k, n, γ)-graph system on V.
Step 1. By Absorbing lemma, we obtain an $\boldsymbol{H}$-rainbow cycle A with at most $\gamma n/2$ vertices such that the following property holds. For any $\boldsymbol{H}$-rainbow family of paths $\mathcal{P}$ and any vertex set U in $V\backslash V(A)$ with $|\mathcal{P}|, |U| \leq \kappa n$, there exists an $\boldsymbol{H}$-rainbow cycle A' with vertex set $V(A) \cup U \cup V(\mathcal{P})$ and $C(A) \subseteq C(A')$.
Step 2. Set $\boldsymbol{H}' = \{H'_i\}_{i\in C'}$ where $C' = [n]\backslash C(A), H'_i = H_i[V\backslash V(A)]$ for $i \in C'$. Let $n' = n - |V(A)|$. Note that $\boldsymbol{H}'$ is a $(k, n', \gamma/2)$-graph system where $n' > (1 - \gamma/2)n$. Applying path cover lemma to $\boldsymbol{H}'$ with $\delta = \kappa$, we obtain an $\boldsymbol{H}'$-rainbow family of paths $\mathcal{P} = \{P_1, \ldots, P_p\}$, where $p \leq L \leq \kappa n$, covering all but at most $\kappa n'$ vertices of $V\backslash V(A)$. Denote the set of uncovered vertices by T. Thus, $|T| \leq \kappa n' \leq \kappa n$.
Step 3. Using property in **Step 1**, we obtain a rainbow cycle with vertex set $V(A) \cup T \cup V(\mathcal{P})$, which is actually an $\boldsymbol{H}$-rainbow Hamilton cycle. □

3.2 . Rainbow absorption method

Given a vertex $x \in V$ and a color $c \in [n]$, let $\mathcal{L}(x; c)$ be the family of rainbow absorbers for (x, c). Similarly, given two disjoint $(k-1)$-tuples **u** and **v** of V and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $[n]$, let $\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})$ be the set of rainbow absorbers for $(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})$. We need the following simple result.

Fact 3.1 *Let $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ be a (k, n, γ)-graph system on V, S be a $(k-1)$-subset of V and $V_0 \subseteq V \backslash S$. For any $i \in [n]$, we have*

$$|N_{H_i}(S) \cap V_0| \geq |V_0| - \frac{1}{2}n + \gamma n + k - 1.$$

In particular, for two $(k-1)$-subsets of vertices S_1 and S_2, we obtain that for any $i, j \in [n]$,

$$|N_{H_i}(S_1) \cap N_{H_j}(S_2)| \geq 2\gamma n + |S_1 \cap S_2|.$$

Proof. We have $|N_{H_i}(S) \cup V_0| \leq n - k + 1$ and thus

$$|N_{H_i}(S) \cap V_0| \geq |V_0| + |N_{H_i}(S)| - (n + k - 1) \geq |V_0| - \frac{1}{2}n + \gamma n + k - 1.$$

For the second statement, we apply the first one with $S = S_1$ and $V_0 = N_{H_i}(S_2) \setminus S_1$ and note that $|V_0| \geq (\frac{1}{2} + \gamma)n - (k - 1 - |S_1 \cap S_2|)$. □

Next we show lower bounds on the number of absorbers in a (k, n, γ)-graph system.

Proposition 3.1 *For any $k \geq 3, \gamma > 0$, there exists $\zeta > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. Suppose $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ is a (k, n, γ)-graph system on V, then $|\mathcal{L}(x; c)| \geq \zeta n^{3k-3}$ for every vertex $x \in V$ and color $c \in [n]$, $|\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})| \geq \zeta n^{3k-3}$ for every two disjoint $(k-1)$-tuples $\mathbf{u}$ and $\mathbf{v}$ of V and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $[n]$.*

Proof. Given k, γ, we choose $1/n \ll \zeta \ll \gamma/k$. Fixing vertex $x \in V$ and color $c \in [n]$, we construct a rainbow absorber $P = x_1 \cdots x_{2k-2}$ with color pattern $(c_1, \ldots, c_{k-1})$ for (x, c). We choose $(c_1, \ldots, c_{k-1})$ arbitrarily, so there are $(n-1) \cdots (n-k+1) \geq 2^{1-k} n^{k-1}$ choices. Furthermore, $x_1, \ldots, x_{k-2}$ can be chosen arbitrarily in $(n-1) \cdots (n-k+2) \geq 2^{2-k} n^{k-2}$ ways. For x_{k-1}, there are at least $(\frac{1}{2} + \gamma)n$ choices such that $\{x_1, \ldots, x_{k-1}, x\} \in E(H_c)$. By Fact 3.1, there are at least $2\gamma n + k - 2$ choices for x_j, $j \in [k, 2k-2]$, such that $\{x_{j-k+1}, \ldots, x_j\}, \{x_{j-k+2}, \ldots, x_j, x\} \in E(H_{c_{j-k+1}})$. For $j \in [k+1, 2k-2]$, x_j should be different from $x_1, \ldots, x_{j-k}$. Thus, the number of choices for each x_j is at least $2\gamma n + k - 2 - (j - k) \geq 2\gamma n$, $j \in [k, 2k-2]$, yielding together at least $2^{1-k} n^{k-1} 2^{2-k} n^{k-2} (\frac{1}{2} + \gamma) n (2\gamma n)^{k-1} \geq \zeta n^{3k-3}$ rainbow absorbers for (x, c).

Given $\mathbf{u} = (u_1, \ldots, u_{k-1})$, $\mathbf{v} = (v_1, \ldots, v_{k-1})$ and $(o_1, \ldots, o_{k-1})$, we construct a rainbow absorber $P = x_1 \cdots x_{2k-2}$ with color pattern $(c_1, \ldots, c_{k-1})$ for $(\mathbf{u}, \mathbf{v}; o_1,$

$\ldots,\, o_{k-1})$. There are $(n-k+1)\cdots(n-2k+2) \geq 2^{1-k}n^{k-1}$ choices for $(c_1,\ldots,c_{k-1})$. There are at least $(\frac{1}{2}+\gamma)n-(k-1) \geq \gamma n$ choices for x_{k-1} such that $\{u_1,\ldots,u_{k-1},x_{k-1}\} \in E(H_{c_{k-1}})$ and x_{k-1} should be different from $v_1,\ldots,v_{k-1}$. For x_i, $i \in [k-2]$, there are at least $(\frac{1}{2}+\gamma)n-(2k-3) \geq \gamma n$ choices such that $\{u_{k-i},\ldots,u_{k-1},x_{k-1},\ldots,x_{i+1},x_i\} \in E(H_{c_i})$, and it should be different from $v_1,\ldots,v_{k-1},u_1,\ldots,u_{k-1-i}$.

By Fact 3.1, there are at least $2\gamma n$ choices for x_k such that $\{x_1,\cdots,x_{k-1},x_k\} \in E(H_{c_1})$, $\{v_1,\cdots,v_{k-1},x_k\} \in E(H_{o_1})$ and it is different from $u_1,\ldots,u_{k-1}$. For x_i, $i \in [k+1,2k-2]$, the number of choices is at least $2\gamma n+k-2-(k-1+2(i-k)) \geq \gamma n$, such that $\{x_{i-(k-1)},\ldots,x_i\} \in E(H_{c_{i-(k-1)}})$, $\{v_{i-(k-1)},\ldots,v_{k-1},x_k,\ldots,x_i\} \in E(H_{o_{i-(k-1)}})$ and it should be different from $u_1,\ldots,u_{k-1},x_1,\ldots,x_{i-k},v_1,\ldots,v_{i-k}$. Thus, there are at least $2^{1-k}n^{k-1}(\gamma n)^{k-1}(\gamma n)^{k-1} \geq \zeta n^{3k-3}$ rainbow absorbers for $(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})$. □

Now we show that we can construct a family of disjoint absorbers, with all different colors.

Lemma 3.4 *For any $k \geq 3$ and $\alpha,\zeta > 0$, there exists $\beta > 0$ such that the following holds for all sufficiently large $n \in \mathbb{N}$. Let **H** $= \{H_i\}_{i\in[n]}$ be an n-vertex k-graph system on V. If $|\mathcal{L}(x;c)| \geq \zeta n^{3k-3}$ for every vertex $x \in V$, $c \in [n]$ and $|\mathcal{L}(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})| \geq \zeta n^{3k-3}$ for all disjoint $(k-1)$-tuples $\mathbf{u}$ and $\mathbf{v}$ of V and $(k-1)$-tuple $(o_1,\ldots,o_{k-1})$ of $[n]$, then there exists an **H**-rainbow family $\mathcal{F}'$ of paths of length $k-1$, satisfying*

$$|\mathcal{F}'| \leq \alpha n, \quad |\mathcal{F}' \cap \mathcal{L}(x;c)| \geq \beta n, \text{ and}$$

$$|\mathcal{F}' \cap \mathcal{L}(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})| \geq \beta n,$$

for every vertex $x \in V$, $c \in [n]$, two disjoint $(k-1)$-tuples $\mathbf{u}$ and $\mathbf{v}$ of V and $(o_1,\ldots,o_{k-1})$ of $[n]$.

Proof. Let $1/n \ll \beta \ll \varepsilon \ll \alpha,\zeta$. Each **H**-rainbow path $x_1x_2\cdots x_{2k-2}$ with color pattern $(c_1,\ldots,c_{k-1})$ can be viewed as a $(3k-3)$-tuple $(x_1,x_2,\ldots,x_{2k-2},c_1,\ldots,c_{k-1})$. Choose a family $\mathcal{F}$ of $(3k-3)$-tuples from $\binom{V}{2k-2}\times\binom{[n]}{k-1}$ by including each possible $(3k-3)$-tuple independently at random with probability

$$p = \varepsilon\frac{(n-(2k-2))!\cdot(n-(k-1))!}{(n-1)!\cdot n!} \geq \varepsilon n^{-(3k-4)}.$$

Note that $|\mathcal{F}|$, $|\mathcal{L}(x,c)\cap\mathcal{F}|$, $|\mathcal{L}(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})\cap\mathcal{F}|$ are binomial random variables with

$$\mathbb{E}[|\mathcal{F}|] = p\frac{n!\cdot n!}{(n-(2k-2))!\cdot(n-(k-1))!} = \varepsilon n,$$

$$\mathbb{E}[|\mathcal{L}(x,c)\cap\mathcal{F}|] = p|\mathcal{L}(x;c)| \geq \varepsilon\zeta n,$$

$$\mathbb{E}[|\mathcal{L}(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})\cap\mathcal{F}|] = p|\mathcal{L}(\mathbf{u},\mathbf{v};o_1,\ldots,o_{k-1})| \geq \varepsilon\zeta n,$$

for every vertex $x \in V$, $c \in [n]$, two disjoint $(k-1)$-tuples **u** and **v** of V and $(o_1, \ldots, o_{k-1})$ of $[n]$. By Lemma 1.1, with probability $1 - o(1)$, the family $\mathcal{F}$ satisfies the following properties

$$|\mathcal{F}| \leq 2\mathbb{E}[|\mathcal{F}|] = 2\varepsilon n \leq \alpha n,$$

$$|\mathcal{L}(x;c) \cap \mathcal{F}| \geq 2^{-1}\mathbb{E}[|\mathcal{L}(x;c) \cap \mathcal{F}|] \geq 2^{-1}\varepsilon\zeta n,$$

$$|\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1}) \cap \mathcal{F}| \geq 2^{-1}\mathbb{E}[|\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})|] \geq 2^{-1}\varepsilon\zeta n,$$

for every vertex $x \in V$, $c \in [n]$, two disjoint $(k-1)$-tuples **u** and **v** of V and $(o_1, \ldots, o_{k-1})$ of $[n]$. We say that two $(3k-3)$-tuples $(x_1, x_2, \ldots, x_{2k-2}, c_1, \ldots, c_{k-1})$ and $(y_1, y_2, \ldots, y_{2k-2}, f_1, \ldots, f_{k-1})$ are *intersecting* if $x_i = y_j$ for some $i, j \in [2k-2]$ or $c_m = f_\ell$ for some $m, \ell \in [k-1]$. We can bound the expected number of pairs of $(3k-3)$-tuples in $\mathcal{F}$ that are intersecting from above by

$$\frac{n! \cdot n!}{(n-(2k-2))! \cdot (n-(k-1))!}(3k-3)^2\frac{(n-1)! \cdot n!}{(n-(2k-2))! \cdot (n-(k-1))!}p^2$$
$$= (3k-3)^2\varepsilon^2 n.$$

Thus, using Markov's inequality, we derive that with probability at least 1/2, $\mathcal{F}$ contains at most $2(3k-3)^2\varepsilon^2 n$ intersecting pairs of $(3k-3)$-tuples. Remove one $(3k-3)$-tuple from every intersecting pair in $\mathcal{F}$ and remove the $(3k-3)$-tuples that can not absorb any (x, c) or $(\mathbf{u}, \mathbf{v}, o_1, \ldots, o_{k-1})$ where $x \in V$, $c \in [n]$, **u** and **v** are $(k-1)$-tuples of V and $(o_1, \ldots, o_{k-1})$ is a $(k-1)$-tuple of $[n]$. Thus the resulting subfamily, say $\mathcal{F}'$, consists of pairwise disjoint $(3k-3)$-tuples, which satisfies

$$|\mathcal{L}(x;c) \cap \mathcal{F}'| \geq 2^{-1}\varepsilon\zeta n - 2(3k-3)^2\varepsilon^2 n \geq \beta n,$$

for any $x \in V$, $c \in [n]$, and a similar statement holds for $|\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1}) \cap \mathcal{F}'|$ for any two disjoint $(k-1)$-tuples **u** and **v** of V and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $[n]$. Since each $(3k-3)$-tuple in $\mathcal{F}'$ induces a rainbow absorber, $\mathcal{F}'$ is an ***H***-rainbow family of paths, where each path is of length $k-1$. □

Now we are ready to prove Lemma 3.1, assuming Lemma 3.2 holds.

Proof. [Proof of Lemma 3.1] Given $1/n \ll \kappa \ll \beta \ll \alpha, \zeta \ll \gamma, 1/k$, let $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ be a (k, n, γ)-graph system on V. By Proposition 3.1, we obtain $|\mathcal{L}(x;c)| \geq \zeta n^{3k-3}$ for every vertex $x \in V$ and $c \in [n]$, and $|\mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})| \geq \zeta n^{3k-3}$ for every two disjoint $(k-1)$-tuples **u** and **v** of V and a $(k-1)$-tuple $(o_1, \ldots, o_{k-1})$ of $[n]$. By Lemma 3.4, there is an ***H***-rainbow family of paths $\mathcal{F}' = \{P_1, \ldots, P_{qn}\}$, where $q \leq \alpha$ and $|V(P_i)| = 2k-2$ for $i \in [qn]$, $|\mathcal{F}' \cap \mathcal{L}(x;c)| \geq \beta n$ for every vertex $x \in V$, $c \in [n]$ and $|\mathcal{F}' \cap \mathcal{L}(\mathbf{u}, \mathbf{v}; o_1, \ldots, o_{k-1})| \geq \beta n$ for every two disjoint $(k-1)$-tuples **u** and **v** of V and $(o_1, \ldots, o_{k-1})$ of $[n]$.

Next, we shall connect all the paths in $\mathcal{F}'$ into an ***H***-rainbow cycle. Suppose we have connected $P_1, \ldots, P_j$ into one path P, by using each time at most

$\lceil 8k\gamma^{-2}\rceil - (2k-2)$ vertices from outside $V(\mathcal{F}')$. Let **e** $= (u_1, \ldots, u_{k-1})$ be an end of P and **f** $= (v_1, \ldots, v_{k-1})$ be an end of P_{j+1}. Let H_i' be the induced subgraph of H_i obtained by removing the vertices of $V(\mathcal{F}') \cup V(P)$ except **e** and **f**. The number of vertices removed is at most

$$|V(\mathcal{F}') \cup V(P)| \leq (2k-2)qn + \left(\left\lceil \frac{8k}{\gamma^2} \right\rceil - (2k-2)\right)(qn-1) < \left\lceil \frac{8k}{\gamma^2} \right\rceil qn < \frac{\gamma n}{2},$$

where the last inequality holds since $\alpha \ll \gamma$ and $k \geq 3$.

We get a $(k, n', \gamma/2)$-graph system $\boldsymbol{H}' = \{H_i'\}_{i \in C}$ where $C = [n] \setminus (C(P) \cup C(\mathcal{F}')$ and $n' = |V(H_i')|$. Taking a $(\lceil 8k\gamma^{-2}\rceil - (k-1))$-subset C' of C, we apply Lemma 3.2 to $\{H_i'\}_{i \in C'}$ with **e**$' = (u_{k-1}, \ldots, u_1)$ and **f**$' = (v_{k-1}, \ldots, v_1)$, obtaining an $\boldsymbol{H}$-rainbow path P' connecting **e**$'$ and **f**$'$ such that $|V(P')| \leq \lceil 8k\gamma^{-2}\rceil$. Thus, $P \cup P' \cup P_{j+1}$ forms an $\boldsymbol{H}$-rainbow path.

After connecting all paths in $\mathcal{F}'$ in a cyclic order, we obtain an $\boldsymbol{H}$-rainbow $(k-1)$-cycle A with at most

$$(2k-2)qn + \left(\left\lceil \frac{8k}{\gamma^2} \right\rceil - (2k-2)\right) qn \leq \frac{\gamma n}{2}$$

vertices. Finally, fix any $\boldsymbol{H}$-rainbow family of paths $\mathcal{P}$ and any vertex set U in $V \backslash V(A)$ with $|\mathcal{P}|, |U| \leq \kappa n$. We may assume that $U \cap V(\mathcal{P}) = \emptyset$ as otherwise we just replace U be $U \setminus V(\mathcal{P})$. Since the paths in $\mathcal{P}$ are vertex disjoint, and $V(\mathcal{P})$, U and $V(A)$ are pairwise disjoint, we infer that the number of colors in $\boldsymbol{H}$ not used in $\mathcal{P}$ or A is at least $n - |V(A)| - (|V(\mathcal{P})| - (k-1)|\mathcal{P}|) \geq n - |V(A)| - (n - |V(A)| - |U| - (k-1)|\mathcal{P}|) = |U| + (k-1)|\mathcal{P}|$. Thus, by the property of $\mathcal{F}'$ and the fact that $\kappa \ll \beta$, there is an $\boldsymbol{H}$-rainbow cycle A' with vertex set $V(A) \cup U \cup V(\mathcal{P})$ and $C(A) \subseteq C(A')$. $\square$

3.3 . Rainbow path cover lemma

In this section, we prove our path cover lemma, Lemma 3.3. A k-graph H is k-partite if there is a partition $V(H) = V_1 \cup \cdots \cup V_k$ such that every edge of H intersects each set V_i in precisely one vertex for $i \in [k]$. Given a k-partite k-graph H on $V_1 \cup \cdots \cup V_k$ and subsets $A_i \subset V_i$, $i \in [k]$, we define $e_H(A_1, \ldots, A_k)$ to be the number of edges in H with one vertex in each A_i and the $density$ of H with respect to $(A_1, \ldots, A_k)$ as

$$d_H(A_1, \ldots, A_k) = \tfrac{e_H(A_1, \ldots, A_k)}{|A_1| \cdots |A_k|}.$$

We say that a k-partite k-graph H is ε-*regular* if for all $A_i \subseteq V_i$ with $|A_i| \geq \varepsilon |V_i|$, $i \in [k]$, we have

$$|d_H(A_1, \ldots, A_k) - d_H(V_1, \ldots, V_k)| \leq \varepsilon.$$

We give a straightforward generalization of the graph regularity lemma.

Lemma 3.5 (Weak regularity lemma for hypergraphs [70]) *For any $k \geq 2$, $\varepsilon > 0$ and $t_0 \in \mathbb{N}$, there exists $T_0 \in \mathbb{N}$ such that the following holds. For every k-graph H on sufficiently large $n \in \mathbb{N}$ vertices, there is, for some $t \in \mathbb{N}$ with $t_0 \leq t \leq T_0$, a partition $V(H) = V_0 \cup V_1 \cup \cdots \cup V_t$ such that $|V_0| \leq \varepsilon n$, $|V_1| = |V_2| = \cdots = |V_t|$ and for all but at most εt^k sets $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$, the induced k-partite k-graph $H[V_{i_1}, \ldots, V_{i_k}]$ is ε-regular.*

The partition in Lemma 3.5 is called an $\varepsilon\text{-}regular\ partition$ of H. For an ε-regular partition of H and $d \geq 0$, we refer to the sets V_i, $i \in [t]$ as *clusters* and define the $reduced\ hypergraph$ $K = K(\varepsilon, d)$ with vertex set $[t]$ and $\{i_1, \ldots, i_k\} \in \binom{[t]}{k}$ being an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geq d$. Next we provide a proof sketch of Lemma 3.3 and highlight key ideas. We need the following definition.

Definition 3.3 *A hypergraph H^* is a $(1,k)$-graph ($(1,k)$-partite, in other words), if there is a partition of $V(H^*) = V_1 \cup V_2$ such that every edge contains exactly one vertex of V_1 and k vertices of V_2.*

Given a partition of $V(H^*) = V_1 \cup V_2$, a $(1, k-1)$-*subset* S of $V(H^*)$ contains one vertex in V_1 and $k-1$ vertices in V_2. Let $\delta_{1,k-1}(H^*) := \min\{\deg_{H^*}(S) : S \text{ is a } (1, k-1)\text{-subset of } V(H^*)\}$.

Step 1. Construct an auxiliary $(1,k)$-graph. Given a (k, n, γ)-graph system ***H*** $= \{H_i\}_{i \in [n]}$ on V, we construct the auxiliary hypergraph H^* with vertex set $V(H^*) = [n] \cup V$ and edge set $E(H^*) = \{\{i\} \cup e |\ e \in E(H_i), i \in [n]\}$. By the definition of (k, n, γ)-graph system, we have $\delta_{k-1}(H_i) \geq (1/2 + \gamma)n$ for each $i \in [n]$. Thus, H^* is a $(1,k)$-graph with $\delta_{1,k-1}(H^*) \geq (1/2 + \gamma)n$.

Step 2. Obtain a reduced hypergraph K. With an initial partition $[n] \cup V$ of $V(H^*)$, we apply the Weak Regularity Lemma (Lemma 3.5) to H^*, and obtain a partition $V(H^*) = V_0^* \cup I_1 \cup \cdots \cup I_{t_1} \cup W_1 \cup \cdots \cup W_{t_2}$ where $I_i \subseteq [n], W_j \subseteq V$, $|I_i| = |W_j| = m$ for every $i \in [t_1], j \in [t_2]$, $|V_0^*| \leq 2\varepsilon n$. By moving at most $2\varepsilon n/m$ clusters to V_0^* and renaming as V_0 if necessary, we may assume that $t_1 = t_2$, but $|V_0| \leq 4\varepsilon n$, $I_i \subseteq [n]$ and $W_j \subseteq V$ for every $i, j \in [t]$. Let K be the reduced hypergraph for the partition with vertex set $\mathcal{I} \cup \mathcal{W}$ where $\mathcal{I} = \{I_1, \ldots, I_t\}$ and $\mathcal{W} = \{W_1, \ldots, W_t\}$. Note that K is a $(1,k)$-graph. We will prove that K almost 'inherits' the $(1, k-1)$-degree condition of H^*.

Step 3. Obtain many matchings in K. We equally split $\mathcal{I}$ into k parts $\mathcal{I}_i = \{I_{(i-1)t/k+1}, \ldots, I_{it/k}\}$ for $i \in [k]$. For each $\mathcal{I}_i \cup \mathcal{W}$, we randomly partition it to balanced smaller pieces, namely, into parts of form $\mathcal{I}' \cup \mathcal{W}'$, where $|\mathcal{I}'| = Q/k$ and $|\mathcal{W}'| = Q$. Denote the family of vertex-disjoint $(1,k)$-subgraphs of K induced on all parts from the partition of $\mathcal{I}_i \cup \mathcal{W}$ by $\mathcal{F}_i, i \in [k]$. Note that the size of $\mathcal{F}_i$ is t/Q. We shall see in Section 3.3 that almost all members in $\mathcal{F}_i$ are 'nice' in the sense that they inherit the $(1, k-1)$-degree condition of H^*. For each such member in $\mathcal{F}_i$, $i \in [k]$, we use the following lemma, a combination of Theorem 2.1 and Theorem 1.2 in [8], and obtain a perfect matching. This yields for each $i \in [k]$ a large matching (in K), say M_i, by taking the union of the resulting matchings over all members in $\mathcal{F}_i$.

Lemma 3.6 ([8, 27]) *For every $\gamma > 0, k \in \mathbb{N}$, the following holds for all sufficiently large $n \in k\mathbb{N}$. every $(1,k)$-graph H on $[n/k] \cup V$ with $\delta_{1,k-1}(H) \geq (1/2+\gamma)n$ admits a perfect matching, where $|V| = n$.*

Step 4. Embed the paths. Now back to the original $(1,k)$-graph H^*, each matching edge in $\bigcup_{i\in[k]} M_i$ can be blown up and we obtain an ***H***-rainbow family of paths. This is achieved in Lemma 3.8. However, note that distinct matchings M_i, M_j may intersect on vertices in $\mathcal{W}$. To overcome this, we build the ***H***-rainbow family of paths in H piece by piece by zooming in each matching M_i one by one. The following proposition shows that the reduced hypergraph almost inherits the minimum degree property of the original hypergraph.

Proposition 3.2 *For any $\gamma > 0, k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that the following holds for sufficiently large $t \in \mathbb{N}$. Given a $(1,k)$-graph H^* with $\delta_{1,k-1}(H^*) \geq (1/2+\gamma)n$ and an ε-regular partition $V(H^*) = V_0 \cup I_1 \cup \cdots \cup I_t \cup W_1 \cup \cdots \cup W_t$, let $K := K(\varepsilon, \gamma/6)$ be the reduced hypergraph. The number of $(1,k-1)$-subsets S of $V(K)$ violating $\deg_K(S) \geq (1/2+\gamma/4)t$ is at most $k\sqrt{\varepsilon}t^k$.*

Proof. Let $1/t, \varepsilon \ll \gamma$. Note that the reduced hypergraph $K(\varepsilon, \gamma/6)$ can be written as the intersection of two hypergraphs $D := D(\gamma/6)$ and $R := R(\varepsilon)$ both defined on the vertex set $\{I_1, \ldots, I_t, W_1, \ldots, W_t\}$ where

- D consists of all sets $\{I_{i_0}, W_{i_1}, \ldots, W_{i_k}\}$ such that $d(I_{i_0}, W_{i_1}, \ldots, W_{i_k}) \geq \gamma/6$,
- R consists of all sets $\{I_{i_0}, W_{i_1}, \ldots, W_{i_k}\}$ such that $H^*[I_{i_0}, W_{i_1}, \ldots, W_{i_k}]$ is ε-regular.

For any $(1,k-1)$-set S, assuming $S = \{I_1, W_1, W_2, \ldots, W_{k-1}\}$, we first show that

$$\deg_D(S) \geq \left(\frac{1}{2} + \frac{\gamma}{2}\right) t. \tag{3.1}$$

Note that $n/t \geq m := |W_i| = |I_j|$ for $i, j \in [t]$. We now consider the number z of edges in H^* which intersect each of $I_{i_0}, W_{i_1}, \ldots, W_{i_{k-1}}$ in exactly one vertex. If (3.1) does not hold, then from the condition on $\delta_{1,k-1}(H^*)$, we have

$$\begin{aligned} tm^{k+1}\left(\frac{1}{2} + \frac{2\gamma}{3}\right) \leq m^k \left(\left(\frac{1}{2} + \gamma\right) n - (k-1)m\right) \leq z \qquad (3.2) \\ < \left(\frac{1}{2} + \frac{\gamma}{2}\right) tm^{k+1} + t\frac{\gamma}{6}m^{k+1}, \end{aligned}$$

a contradiction.

Note that there are at most εt^{k+1} edges in $\overline{R}$ (the complement of R). Let $\mathcal{S}$ be the family of all $(1,k-1)$-element subsets S for which $\deg_{\overline{R}}(S) > \sqrt{\varepsilon}t$.

We have $|\mathcal{S}| \le k\sqrt{\varepsilon}t^k$. This, together with (3.1) and $\varepsilon \ll \gamma$, implies that all but at most $k\sqrt{\varepsilon}t^k$ $(1, k-1)$-sets $S \subseteq V(K)$ satisfy $\deg_K(S) \ge \deg_D(S) - \sqrt{\varepsilon}t \ge (\frac{1}{2} + \frac{\gamma}{4})t$. $\square$

Ferber and Kwan [55] showed that if we randomly partition the vertex set of a k-graph H, then the subgraph of H induced on almost all parts inherits the minimum degree of H. Here we need such a result for our $(1, k)$-graphs, whose proof follows almost identical as that in [55]. We include a proof for completeness.

Lemma 3.7 (Partition lemma) *Suppose that $k \ge 3$, $\lambda, \gamma > 0$, there exist $\eta > 0$ and $Q \in k\mathbb{N}$ such that the following holds for $t \in Q\mathbb{N}$. If H^* is a $(1, k)$-graph on $[\frac{t}{k}] \cup V$ with $|V| = t$ where all but at most $\frac{\eta t}{k}\binom{t}{k-1}$ of the $(1, k-1)$-subsets of $V(H^*)$ have degree at least $(1/2+\gamma)(t-k+1)$, then there is a partition $V(H^*) = S_1 \cup \cdots \cup S_{t/Q}$ such that all but at most $\lambda t/Q$ classes S_i satisfy $\delta_{1,k-1}(H^*[S_i]) \ge (1/2 + \gamma/2)(Q - k + 1)$ where each S_i consists of a Q/k-subset of $[t/k]$ and a Q-subset of V.*

Proof. Let $\eta \ll 1/Q \ll \lambda, \gamma$. Partition $[t/k]$ into t/Q sets $I_1, \dots, I_{t/Q}$ randomly such that $|I_i| = Q/k$ for $i \in [t/Q]$. We randomly order V as $v_1, \dots, v_t$ and let $V_i = \{v_{(i-1)Q+1}, \dots, v_{iQ}\}$ for $i \in [t/Q]$. Let $S_i = I_i \cup V_i$ for $i \in [t/Q]$. Note that each V_i is a random subset of V. Let M^* be the collection of $(1, k-1)$-subsets with degree less than $(1/2+\gamma)(t-k+1)$ in H^*. Then $|M^*| \le \frac{\eta t}{k}\binom{t}{k-1}$. We will prove that for $i \in [t/Q]$ and every $(1, k-1)$-subset S of S_i,

$$\mathbb{P}\left[\deg_{H^*[S_i]}(S) < \left(\frac{1}{2} + \frac{\gamma}{2}\right)(Q - k + 1)\right] \le \eta + e^{-\Omega(\gamma^2 Q)}. \tag{3.3}$$

First note that the probability of the event $S \in M^*$, is at most η. Now let A_S be the event that S is not in M^*. The set $V_i \backslash S$ is equivalent to a uniformly random set of size $Q - (k-1)$ in $V \backslash S$. Let A denote the event that a vertex v in $V_i \backslash S$ such that $S \cup \{v\} \in E(H^*)$. Note that

$$\mathbb{P}[A|A_S] \ge \frac{(\frac{1}{2} + \gamma)(t-k+1)\binom{t-k}{Q-k}}{(Q-(k-1))\binom{t-(k-1)}{Q-(k-1)}} = \frac{1}{2} + \gamma,$$

then we have

$$\mathbb{E}\left[\deg_{H^*[S_i]}(S)|A_S\right] \ge (\frac{1}{2} + \gamma)(Q - k + 1).$$

Exchanging any element with an element outside $V_i \backslash S$ affects $\deg_{H^*[S_i]}(S)$ by at most 1. Fixing i, we apply Lemma 1.4 with $S \notin M^*$, the probability that S has degree less than $(1/2 + \gamma/2)(Q - k + 1)$ in $H^*[S_i]$ is at most

$$2\exp\left(-\frac{2\left(\frac{\gamma}{2}(Q-k+1)\right)^2}{(Q-k+1)}\right) = e^{-\Omega(\gamma^2 Q)}.$$

Thus, (3.3) is proved.

We say that S_i is $poor$ if some $(1, k-1)$-set in the induced graph $H^*[S_i]$ has degree less than $(1/2+\gamma/2)(Q-k+1)$. By (3.3), $\mathbb{P}[S_i \text{ is poor}] \leq \frac{Q}{k}\binom{Q}{k-1}(\eta + e^{-\Omega(\gamma^2 Q)})$ for $i \in [t/Q]$. Let X be the number of poor classes in our partition, then $\mathbb{E}[X] \leq \frac{t}{k}\binom{Q}{k-1}(\eta + e^{-\Omega(\gamma^2 Q)})$. By Markov's inequality, we obtain

$$\mathbb{P}\left[X \geq \lambda \frac{t}{Q}\right] \leq \frac{Q}{\lambda k}\binom{Q}{k-1}\left(\eta + e^{-\Omega(\gamma^2 Q)}\right).$$

By the choice of $\eta \ll 1/Q \ll \lambda, \gamma$, it follows that

$$\frac{Q}{k}\binom{Q}{k-1}\left(\eta + e^{-\Omega(\gamma^2 Q)}\right) < \lambda,$$

and thus $\mathbb{P}[X \geq \lambda \frac{t}{Q}] < 1$. Therefore, there is a partition $V(H^*) = S_1 \cup \cdots \cup S_{t/Q}$, where $S_i = I_i \cup V_i$, such that at least $(1-\lambda)t/Q$ classes of them satisfy $\delta_{1,k-1}(H^*[S_i]) \geq (1/2+\gamma/2)(Q-k+1)$. □

Given a $(k+1)$-partite $(k+1)$-graph H on $V_0 \cup V_1 \cup \cdots \cup V_k$, we call that a $(k-1)$-subset S of $V(H)$ is *legal* if $|S \cap V_i| \leq 1$ for $i \in [k]$ and $|S \cap V_0| = 0$. An *expanded path* P of length t in H is a $(k+1)$-graph with vertex set $\{c_1, \ldots, c_t\} \cup \{v_1, \ldots, v_{t+k-1}\}$ where $\{c_1, \ldots, c_t\} \subseteq V_0$, $\{v_1, \ldots, v_{t+k-1}\} \subseteq V_1 \cup \cdots \cup V_k$ and edge set $\{e_1, \ldots, e_t\}$ such that $e_i = \{c_i, v_i, \ldots, v_{i+k-1}\}$. Note that $|V(P) \cap V_j| = \lfloor \frac{t+k-1}{k} \rfloor$ or $\lceil \frac{t+k-1}{k} \rceil$ for $j \in [k]$.

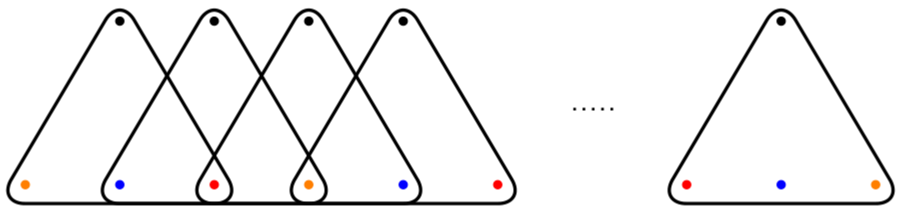


Figure 3.3 – An expanded path in a 4-partite 4-graph (the vertices with the same color are from the same part)

Lemma 3.8 *Given $c, m > 0$ and $k \geq 2$, every $(k+1)$-partite $(k+1)$-graph H on $V_0 \cup V_1 \cup \cdots \cup V_k$ with at most m vertices in each part and with at least cm^{k+1} edges contains an expanded path of at least cm/k vertices.*

Proof. There are at most $k \cdot m^{k-1}$ legal $(k-1)$-subsets of $V(H)$. We proceed the following process iteratively. If there is a legal $(k-1)$-subset S, which is contained in less than cm^2/k edges in the current hypergraph, then we delete all the edges containing S. The process terminates at a nonempty hypergraph

H_0 since less than $km^{k-1}(cm^2/k) = cm^{k+1}$ edges have been deleted in total. In H_0, every legal $(k-1)$-subset has degree either zero or at least cm^2/k.

Let P be a longest expanded path in H_0 with vertex set $\{c_1, \dots, c_t\} \cup \{v_1, \dots, v_{t+k-1}\}$ for some integer t. We have $|V(P) \cap V_0| = t$ and $|V(P) \cap V_i| \le t$ since each edge contains exactly one vertex of V_i for each $i \in [k]$. Consider $S_t = \{v_{t+1}, \dots, v_{t+k-1}\}$, which is a legal $(k-1)$-subset of $V(H)$. Furthermore, $\deg_{H_0}(S_t) \ge cm^2/k$ since S_t has positive degree. All the edges containing S_t must intersect $(V(P) \cap V_0) \cup (V(P) \cap V_j)$ by the maximality of P, where the index $j > 0$ such that $S_t \cap V_j = \emptyset$. Thus, we have

$$\frac{cm^2}{k} \le |V(P) \cap V_0| \cdot |V_j| + |V(P) \cap V_j| \cdot |V_0| \le 2tm, \tag{3.4}$$

which implies $t \ge cm/(2k)$. Note that $|V(P)| = t + t + k - 1$ and thus $|V(P)| \ge cm/k$. $\square$

The next result enables us to find a family of long vertex-disjoint expanded paths which covers almost all vertices in V_0 in an ε-regular $(k+1)$-partite $(k+1)$-graph.

Lemma 3.9 *For any $\alpha > 0, k \in \mathbb{N}$, there exists $\varepsilon > 0$ such that the following holds for sufficiently large $m \in \mathbb{N}$. Suppose H is an ε-regular $(k+1)$-partite $(k+1)$-graph with density at least α and $V(H) = V_0 \cup V_1 \cup \dots \cup V_k$ where $|V_0| = m$, $m/k \le |V_i| \le m$ for $i \in [k]$. Then we obtain that H contains a family $\mathcal{P}$ of vertex-disjoint expanded paths such that for each $P \in \mathcal{P}$, $|V(P)| \ge \varepsilon(\alpha - \varepsilon)m/k$ and $\sum_{P \in \mathcal{P}} |V(P) \cap V_0| \ge (1 - 2k\varepsilon)m$.*

Proof. Let $1/m \ll \varepsilon \ll \alpha, 1/k$. We call an expanded path P *good* if $|V(P)| \ge \varepsilon(\alpha - \varepsilon)m/k$. Let $\mathcal{P} = \{P_1, \dots, P_p\}$ be a largest family of good, vertex-disjoint expanded paths and $|V(P_i) \cap V_0| = t_i$ for $i \in [p]$. Note that $|V(P_i) \cap V_j| = \lfloor \frac{t_i+k-1}{k} \rfloor$ or $\lceil \frac{t_i+k-1}{k} \rceil$ for $i \in [p]$ and $j \in [k]$. Suppose to the contrary that $\mathcal{P}$ covers less than $(1-2k\varepsilon)m$ vertices of V_0 and thus $\sum_{i \in [p]} t_i < (1-2k\varepsilon)m$. Let $W = V(H) - \bigcup_{P \in \mathcal{P}} V(P)$ be the set of vertices uncovered by $\mathcal{P}$. Then we have $|W \cap V_0| \ge 2k\varepsilon m$. Hence, by the observation that $|V(P_i) \cap V_j| \le \lceil \frac{t_i+k-1}{k} \rceil \le \frac{t_i}{k} + 2$ for each $i \in [p]$, $j \in [k]$ and the fact that $p = |\mathcal{P}| \le (k+1)m/(\varepsilon(\alpha - \varepsilon)m/k) = k(k+1)(\varepsilon(\alpha-\varepsilon))^{-1}$, we have that

$$|W \cap V_i| = |V_i| - |V_i \cap V(\mathcal{P})| \ge \frac{m}{k} - \sum_{i \in [p]} \left(\frac{t_i}{k} + 2 \right) \ge \frac{m}{k} - \frac{(1-2k\varepsilon)m}{k} - 2p \ge \varepsilon m + 1.$$

Let $W_i \subseteq W \cap V_i, i \in \{0, 1, \dots, k\}$ be such that

$$|W_0| = |W_1| = \dots = |W_k| = \varepsilon m \ge \varepsilon |V_i|.$$

Finally, let $\hat{H}$ be the subhypergraph of H induced on the vertex set $W_0 \cup W_1 \cup \dots \cup W_k$. Since H is ε-regular, we have

$$d_H(W_0, W_1, \dots, W_k) \ge d_H(V_0, V_1, \dots, V_k) - \varepsilon \ge \alpha - \varepsilon,$$

or equivalently,

$$|E(\hat{H})| \geq (\alpha - \varepsilon)(\varepsilon m)^{k+1},$$

and then Lemma 3.8 implies that there is an expanded path in $\hat{H}$ on at least $\varepsilon(\alpha - \varepsilon)m/k$ vertices, contrary to the maximality of $\mathcal{P}$. $\square$

Proof. [Proof of Lemma 3.3] We choose the following parameters

$$1/n \ll 1/T_0 \ll \varepsilon, 1/t_0 \ll 1/Q \ll \lambda \ll \delta, \gamma, 1/k.$$

Given a (k, n, γ)-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ on V, we construct a $(1, k)$-graph H^* with vertex set $[n] \cup V$ and edge set $\{\{i\} \cup e : e \in E(H_i), i \in [n]\}$. With an initial partition $[n] \cup V$ of $V(H^*)$, we apply Lemma 3.5 to H^*, and obtain a partition $V(H^*) = V_0^* \cup I_1 \cup \cdots \cup I_{t_1} \cup W_1 \cup \cdots \cup W_{t_2}$ where $t_0 \leq t_1, t_2 \leq T_0$, $|I_i| = |W_j| = m$ for $i \in [t_1]$ and $j \in [t_2]$, $|V_0^*| \leq 2\varepsilon n$. By moving at most $2\varepsilon n/m$ clusters to V_0^* and renaming if necessary, we obtain a partition $V(H^*) = V_0 \cup I_1 \cup \cdots \cup I_t \cup W_1 \cup \cdots \cup W_t$, where $t = \min\{t_1, t_2\}$, $|V_0| \leq 4\varepsilon n$, $I_i \subseteq [n]$ and $W_j \subseteq V$ for every $i, j \in [t]$. Let $L = \left\lceil \frac{3kT_0}{\varepsilon(\gamma/6-\varepsilon)} \right\rceil$.

Let $K := K(\varepsilon, \gamma/6)$ be the $(1, k)$-partite reduced hypergraph on $\mathcal{I} \cup \mathcal{W}$ where $\mathcal{I} = \{I_1, \ldots, I_t\}$ and $\mathcal{W} = \{W_1, \ldots, W_t\}$. We get a family of $(1, k)$-graphs $\mathcal{F} = \{F_1, \ldots, F_k\}$ where $F_i = K[\{I_{(i-1)t/k+1}, \ldots, I_{it/k}\} \cup \mathcal{W}]$ for $i \in [k]$.

For each $i \in [k]$, applying Proposition 3.2 and Lemma 3.7 to F_i with $\eta = k\sqrt{\varepsilon}$, we obtain a partition $V(F_i) = S_{i,1} \cup \cdots \cup S_{i,t/Q}$ such that each $S_{i,j}$ consists of Q/k vertices in $\mathcal{I}$ and Q vertices in $\mathcal{W}$, and all but at most $\lambda t/Q$ classes $S_{i,j}$ satisfy $\delta_{1,k-1}(F_i[S_{i,j}]) \geq (1/2 + \gamma/2)(Q - k + 1)$ where $j \in [t/Q]$. We call such classes $S_{i,j}$ *nice*. Denote by $\mathcal{S}_i$ the set of indices j such that $S_{i,j}$ is nice. Then $|\mathcal{S}_i| \geq (1 - \lambda)t/Q$. Applying Lemma 3.6 to each $F_i[S_{i,\ell}]$ for $i \in [k]$, $\ell \in \mathcal{S}_i$, we obtain a perfect matching $M_{i,\ell}$. Let $M_i = \bigcup_{\ell\in\mathcal{S}_i} M_{i,\ell}$ and $M = \bigcup_{i\in[k]} M_i$. Note that each M_i is a matching in F_i. For each $W_j \in \mathcal{W}$, let p_j be the number of edges in M that contain W_j, $j \in [t]$. Then $\sum_{j\in[t]} p_j \geq k\cdot(1-\lambda)\frac{t}{Q}\cdot Q = (1-\lambda)kt$. Next, we proceed the following process.

Path Embedding Process :

Given H^*, $\mathcal{I} = \{I_1, \ldots, I_t\}$, $\mathcal{W} = \{W_1, \ldots, W_t\}$, $M_1, \ldots, M_k$, we initialize $W_j^* := W_j$ for $j \in [t]$ and $i := 1$.

Step 1. For each $e \in M_i$, let H_e be the subgraph of H^* induced on the corresponding clusters constituting the edge e, we denote by $I_e, W^*_{j_1(e)}, \ldots, W^*_{j_k(e)}$ where $I_e \in \mathcal{I}$.

Step 2. Applying Lemma 3.9 to each H_e, $e \in M_i$ with $\alpha = \gamma/6$, we obtain a family $\mathcal{P}_e$ of vertex-disjoint expanded paths that covers all but at most $2k\varepsilon m$ vertices in I_e and for each $P \in \mathcal{P}_e$, $|V(P)| \geq \varepsilon(\gamma/6 - \varepsilon)m/k.$

Step 3. Let $\mathcal{P}_i = \bigcup_{j\leq i} \bigcup_{e\in M_j} \mathcal{P}_e$ and update W_j^* by deleting the vertices used in $\mathcal{P}_i$ for $j \in [t]$.

Step 4. Update $i := i + 1$ and if $i \leq k$, go to **Step 1**; otherwise terminate the process.

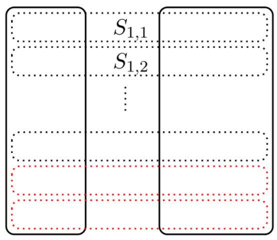
After the process, we obtain $\mathcal{P} := \mathcal{P}_k$. It follows from the definition of p_j that the size of uncovered vertices of each W_j is

$$|W_j^*| = m - \sum_{W_j \in e, e \in M} |\mathcal{P}_e \cap W_j| \leq m - p_j \lfloor \frac{(1-2k\varepsilon)m + k - 1}{k} \rfloor \leq m - p_j \frac{(1-2k\varepsilon)m}{k}.$$

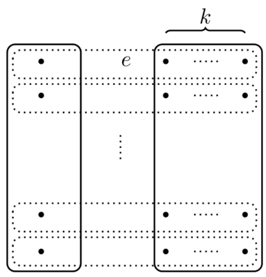
Recall that $\sum_{j\in[t]} p_j \geq (1-\lambda)kt$. We obtain that $\mathcal{P}$ covers all but

$$|V_0| + \sum_{j\in[t]} |W_j^*| \leq 4\varepsilon n + \sum_{j\in[t]} \left(m - p_j \frac{(1-2k\varepsilon)m}{k} \right) \leq (4(k+1)\varepsilon + \lambda)\, n \leq \delta n$$

vertices of V. Moreover, since $|V(P)| \geq \frac{\varepsilon(\frac{\gamma}{6}-\varepsilon)}{k} \lfloor \frac{n}{t} \rfloor$ for each path $P \in \mathcal{P}$ and $t \leq T_0$, we have $|\mathcal{P}| < 2n/(\frac{\varepsilon(\frac{\gamma}{6}-\varepsilon)}{k} \lfloor \frac{n}{t} \rfloor) < L$. Finally, observe that $\mathcal{P}$ gives rise to an ***H***-rainbow family of paths which completes the proof. $\square$



F_1(the black dotted lines represent nice classes)



$F_1[S_{1,1}]$(with a perfect matching)

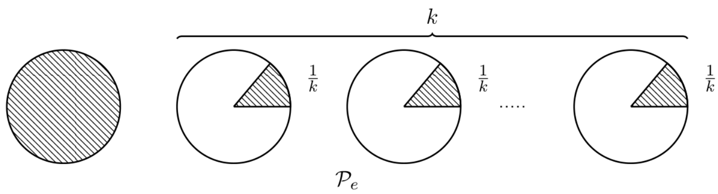


Figure 3.4 – The proof sketch of Lemma 3.3

3.4 . The connecting lemma

In this section, we prove Lemma 3.2. The idea of the proof is to grow tree-like structures (called cascades) from both designated ends e_1 and e_2 until they meet, forming the ***H***-rainbow path as desired. Our proof follows almost identical as that in [106, 137]. Before we describe the cascades, it is convenient to introduce the following notation. For two sequences of vertices

$$\omega_1 = (v_1, \ldots, v_r, w_1, \ldots, w_s) \text{ and } \omega_2 = (w_1, \ldots, w_s, u_1, \ldots, u_t)$$

where $r, t \geq 1, s \geq 0$ and all vertices are distinct, we define their $concatenation$ as

$$\omega_1\omega_2 = (v_1, \dots, v_r, w_1, \dots, w_s, u_1, \dots, u_t).$$

This operation can be iterated. For instance, if $\omega_1 = (w_1, \dots, w_{k-2})$, $\omega_2 = (w_2, \dots, w_{k-1})$ and $\omega_3 = (w_3, \dots, w_k)$ where all w_i are distinct, then $\omega_1\omega_2\omega_3 = (w_1, \dots, w_k)$. We could write $\omega_1\omega_2 w_k$ instead of $\omega_1\omega_2\omega_3$. Let $e_0 = (v_1, \dots, v_{k-1})$ be a given $(k-1)$-tuple of vertices. We will define the *rainbow* e_0-$cascade$ as an auxiliary sequence of bipartite graphs $G_j, j = 1, 2, \dots$, with bipartitions (A_{j-1}, A_j), whose vertices are $(k-2)$-tuples of the vertices of H and the edges correspond to some $(k-1)$-tuples of the vertices of H. Each node $f \in A_j$ belongs to two graphs G_j and G_{j+1}. Its neighbors in G_j belongs to A_{j-1}, while its neighbors in G_{j+1} belongs to A_{j+1}. For a node $f = (v_1, \dots, v_{k-2})$ of the rainbow cascade, the vertex v_1 is called the $prefix$, while v_{k-2} is called the $suffix$ of f.

We define the rainbow cascade recursively as follows. Let $e_0 = (v_1, \dots, v_{k-1})$, $f_0 = (v_2, \dots, v_{k-1})$ and $A_0 = \{f_0\}$. For every vertex $v \notin e_0$, we include the node $g = (v_3, \dots, v_{k-1}, v)$ in the set A_1 if and only if $v_1 f_0 g = e_0 v \in H_{c_1}$ for $c_1 \in [c]$. The graph G_1 is the star with center f_0 and the arms leading to all the nodes $g \in A_1$.

Further, let A_2 be the set of all $(k-2)$-tuples h such that for some node $g \in A_1$ we have $f_0 gh \in H_{c_2}$ where $c_2 \neq c_1$ and $c_2 \in [c]$. Note that each $h \in A_2$ is obtained from a node $g \in A_1$ by dropping the prefix of g and adding a new suffix u, we denote such node by g_u. The graph G_2 consists of all edges $\{g, h\}$ where $g \in A_1$, $h \in A_2$ and $f_0 gh \in H_{c_2}$, it is equal to say G_2 consists of all edges $\{g, g_u\}$ where $f_0 gu \in H_{c_2}$.

For $j = 3, \dots, k-2$, we similarly define

$$\begin{aligned} A_j = \{h : &\exists\, f \in A_{j-2}, g \in A_{j-1} \text{ such that } \{f, g\} \in G_{j-1}, fgh \in H_{c_j} \text{ where} \\ &c_j \neq c_\ell \text{ for} \ell \in [j-1]\} \end{aligned}$$

and G_j as the bipartite graph with bipartition (A_{j-1}, A_j) and the edge set

$$\begin{aligned} \{\{g, h\} : &\exists\, f \in A_{j-2} \text{ such that } \{f, g\} \in G_{j-1} \text{ and } fgh \in H_{c_j}, \text{ where } c_j \neq c_\ell \\ &\text{for } \ell \in [j-1]\}. \end{aligned}$$

In other words, A_j and G_j correspond to the sets of $(k-2)$-tuples and $(k-1)$-tuples of the vertices of V which can be reached from e_0 in j steps by an ***H***-rainbow path.

First refinement. Having defined A_j and G_j for $j \leq k$, beginning with $j = k-1$ we change the recursive mechanism by getting rid of the nodes in A_j with too small degree in G_j. We define auxiliary

$$\begin{aligned} A'_{k-1} = \{h : &\exists\, f \in A_{k-3}, g \in A_{k-2} \text{ such that } \{f, g\} \in G_{k-2}, fgh \in H_{c_{k-1}} \text{ where} \\ &c_{k-1} \neq c_\ell \text{ for } \ell \in [k-2]\} \end{aligned}$$

and G'_{k-1} as the bipartite graph with bipartition (A_{k-2}, A'_{k-1}) and the edge set

$$\{\{g,h\} : \exists\, f \in A_{k-3} \text{ such that } \{f,g\} \in G_{k-2} \text{ and } fgh \in H_{c_{k-1}} \text{ where } c_{k-1} \neq c_\ell \\ \text{for } \ell \in [k-2]\}.$$

Then let A_{k-1} be the subset of A'_{k-1} consisting of all nodes h with $\deg_{G'_{k-1}}(h) \geq \sqrt{n}$ and set $G_{k-1} = G'_{k-1}[A_{k-2} \cup A_{k-1}]$.

Second refinement. For $j \geq k$, to form an edge $\{g,h\}$ of G_j we will now require not one but many nodes $f \in A_{j-2}$ to fulfil the above definition.

Set $m = \lceil n^{1/4} \rceil$. Having defined G_{j-1}, let $A'_j = \{h : \exists\, f_1, \dots, f_m \in A_{j-2}, g \in A_{j-1} \text{ such that for all } i \in [m], \{f_i, g\} \in G_{j-1} \textit{ and } f_i gh \in H_{c_j} \text{ where } c_j \neq c_\ell \text{ for } \ell \in [j-1]\}$ and let G'_j be the bipartite graph with bipartition (A_{j-1}, A'_j) and the edge set $\{\{g,h\} : \exists f_1, \dots, f_m \in A_{j-2} \text{ such that for all } i \in [m], \{f_i, g\} \in G_{j-1} \textit{ and } f_i gh \in H_{c_j} \text{ where } c_j \neq c_\ell \text{ for } \ell \in [j-1]\}$.

Finally, let A_j be the subset of A'_j consisting of all nodes h with $\deg_{G'_j}(h) \geq \sqrt{n}$ and let $G_j = G'_j[A_{j-1} \cup A_j]$. The sequence $(G_j), j = 1, 2, \dots$, will be called the rainbow e_0-cascade.

Claim 3.1 ([137]) *For every $j \geq k-1$ and every edge $\{g,h\}$ of G_j where $g = (w_1, \dots, w_{k-2}) \in A_{j-1}, h = (w_2, \dots, w_{k-1}) \in A_j$ and $(g \cup h) \cap e_0 = \emptyset$ and for every set of vertices $W \subset V \setminus (g \cup h \cup e_0)$ such that $j + |W| \leq n^{1/4}$, there is an **H**-rainbow path P of length j which connects $(w_{k-1}, \dots, w_1)$ with $e_0 = (v_1, \dots, v_{k-1})$ and $V(P) \cap W = \emptyset$.*

Degrees. Recall that $G'_j = G_j$ for $j \leq k-2$. For a node $g \in A_j$, we set

$$d^+(g) = \deg_{G'_{j+1}}(g) \text{ and } d^-(g) = \deg_{G_j}(g)$$

for the *forward* and *backward* degree of g in the cascade. Note that in the definition of $d^+(g)$ we consider the forward degree before some small degree vertices of A'_{j+1} are removed. The reason is that we have no control over the effects of the removal on individual forward degrees. On the other hand, for all $f \in A_j$, $\deg_{G_j}(f) = \deg_{G'_j}(f)$, so the backward degree is unaffected unless the node is removed. It is trivial that $d^-(g), d^+(g) \leq n - k + 2$. Observe that $G_1 \cup \dots \cup G_{k-2}$ is a tree, thus, $d^-(g) = 1$ for all $g \in A_j, j = 1, \dots, k-2$. Recall that for $j \geq k-1$ the graph G_j is obtained from G'_j by removing nodes g with $\deg_{G'_j}(g) < \sqrt{n}$. Hence our construction guarantees that for all $g \in A_j, j \geq k-1$, we have $d^-(g) \geq \sqrt{n}$.

For all $j \leq k-2$ and all $g \in A_j$,

$$d^+(g) \geq \left(\frac{1}{2} + \gamma\right) n, \tag{3.5}$$

since there are at least $(\frac{1}{2} + \gamma)n$ vertices u such that $fgu \in H_{c_{j+1}}$ where f is the neighbor of g in A_{j-1}. Each such vertex u corresponds to a neighbor g_u of g in A_{j+1}.

For $j \geq k$, the second refinement affects and no lower bound on $d^+(g)$ is obvious. However, the lower bound $d^-(g) \geq \sqrt{n}$ introduced by the first refinement maintains.

Growth. By inequality (3.5), for each $j \in [k-2]$, we have

$$|G_j| = |A_j| \geq \left(\frac{1}{2} + \gamma\right)^j n^j, \tag{3.6}$$

$$|G_{k-1}| \geq \left(\frac{1}{2} + \gamma\right)^{k-1} n^{k-1}. \tag{3.7}$$

Call a node $f \in A_j$ *small* if $d^-(f) < \frac{1}{2}n$ and denote by S_j the subset of A_j consisting of the small nodes. Assume for simplicity that $1/\varepsilon^2$ is an integer.

Claim 3.2 ([137]) *There exists an index j_0, $k-1 \leq j_0 \leq k-1+(k-1)/\gamma^2$ such that for all $j \in [j_0, j_0+k-2]$ we have $|S_j| \leq 2\gamma n^{k-2}$.*

Claim 3.3 ([137]) *Let*

$$k\gamma^{2^{2-k}} < 2^{-k}$$

and let j_0 be as in Claim 3.2. Then $|A_{j_0+k-2} \backslash S_{j_0+k-2}| \geq (n-k+2-\gamma^{2^{2-k}}n)^{k-2}$.

Proof. [The proof of Lemma 3.2] Let γ_0 satisfy the condition in Claim 3.3, i.e. $\gamma_0 := \gamma^{2^{2-k}}$ and $k\gamma_0 < 2^{-k}$. Given two disjoint $(k-1)$-tuples of vertices e_1 and e_2, we build the rainbow e_1-cascade and the rainbow e_2-cascade, with the sets of nodes denoted by A_j and B_j.

Let $j_1 = j_0 + k - 2$, where j_0 is the index guaranteed by Claim 3.2 for the rainbow e_1-cascade. Then by Claim 3.3, with sufficiently large n, using Bernoulli inequality, we have

$$|A_{j_1} \backslash S_{j_1}| \geq (n - 2\gamma_0 n)^{k-2} > (1 - 2k\gamma_0)n^{k-2}.$$

On the other hand by inequality (3.6) for $j = k-2$, we have $|B_{k-2}| > 2^{2-k}n^{k-2}$,

$$|B_{k-2} \cap (A_{j_1} \backslash S_{j_1})| > (2^{2-k} - 2k\gamma_0)n^{k-2} \geq \left(\frac{n}{2}\right)^{k-2}.$$

Hence, there is a not small node $g = (u_1, \ldots, u_{k-2}) \in A_{j_1}$ such that $g \cap (e_1 \cup e_2) = \emptyset$ and $g' = (u_{k-2}, \ldots, u_1) \in B_{k-2}$.

Let $e_2 = (w_1, \ldots, w_{k-1})$, $S = \{u_1, \ldots, u_{k-2}, w_{k-1}\}$ and V_0 be the set of prefixes v of the neighbors $f \in A_{j_1-1}$ of g. Since $g' = (u_{k-2}, \ldots, u_1) \in B_{k-2}$, we obtain that $w_1 \cdots w_{k-1} u_{k-2} \cdots u_1$ is an ***H***-rainbow path. By Fact 3.1, we have $|N_{H_{c_{j_1}}}(S) \cap V_0| > \gamma n$, and thus, there is at least one vertex $v_0 \notin e_2$ such that $\{v_0, u_1, \ldots, u_{k-2}, w_{k-1}\} \in H_{c_{j_1}}$.

Let $P_1 = e_1 \cdots v_0 u_1 \cdots u_{k-2}$ be an ***H***-rainbow path of length j_1 which avoids the vertices of e_2. The existence of P_1 follows from Claim 3.1 with $W = e_2$. The path P obtained from P_1 by adding the segment $(w_{k-1},\dots,w_1)$ and the "hook-up" edge $\{v_0,u_1,\dots,u_{k-2},w_{k-1}\}$, is the ***H***-rainbow path connecting e_1 and e_2 as desired.

By the bound on j_0 established in Claim 3.2 and since $\gamma \le 1/2$,

$$|V(P)| = j_1 + 2(k-1) = j_0 + 3k - 4 \le \frac{k-1}{\gamma^2} + 4k - 5 \le \frac{2k}{\gamma^2}.$$

$\square$

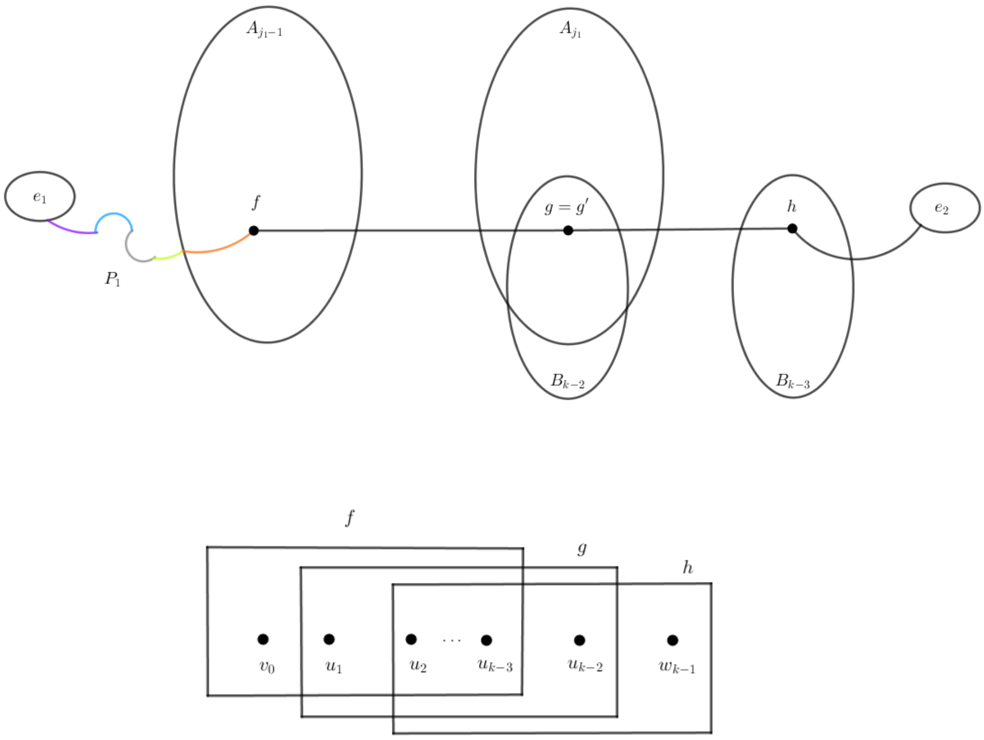


Figure 3.5 – An ***H***-rainbow path connecting two $(k-1)$-tuples e_1 and e_2

3.5 . Concluding remarks

Inspired by a series of recent successes on rainbow settings of matchings [110, 109, 112, 113] and Hamilton cycles [77] , we suspect the threshold for rainbow Hamilton cycle in a k-graph system is the same with the threshold for Hamilton cycle in a single k-graph.

Conjecture 3.1 *Suppose* ***H*** $= \{H_i\}_{i\in[n]}$ *is an* n*-vertex* k*-graph system on* V, $n \ge k+1 \ge 4$, *such that* $\delta_{k-1}(H_i) \ge \lfloor (n-k+3)/2 \rfloor$, *then there is an* ***H****-rainbow Hamilton cycle.*

On the other hand, the problem of giving the sufficient condition for the rainbow Hamilton ℓ-cycles, $\ell \in [k-2]$, is still open.

4 - Rainbow Hamilton cycles in hypergraph systems with minimum $(k-2)$-degree

Lang and Sanhueza-Matamala [105], Polcyn, Reiher, Rödl and Schülke [129] independently proved that for any $\gamma > 0$ and sufficiently large $n \in N$, every n-vertex k-graph with $\delta_{k-2}(H) \geq (5/9+\gamma)\binom{n}{2}$ contains a Hamilton cycle. Gupta, Hamann, Müyesser, Parczyk, and Sgueglia [63] mentioned the following problem as "there is a well-known (uncolored) Dirac-type result whose rainbow version is missing" and "it would be an interesting challenge to obtain this result" : Given a 3-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$ with minimum vertex degree condition of each H_i, does $\boldsymbol{H}$ admit a rainbow Hamilton cycle? In this chapter, we develop the sequentially Hamilton framework, which generalized the Hamilton framework in [105], and give a general result as follows.

Theorem 4.1 *For every $k \geq 3, \gamma > 0$, there exists n_0 such that the following holds for $n \geq n_0$. Given a k-graph system $\boldsymbol{H} = \{H_i\}_{i\in[n]}$, if $\delta_{k-2}(H_i) \geq (5/9+\gamma)\binom{n}{2}$ for $i \in [n]$, then there exists an $\boldsymbol{H}$-rainbow Hamilton cycle.*

4.1 . Notation and preliminaries

We call a hypergraph H a $(1,k)$-graph if $V(H)$ can be partitioned into V_1 and V_2 such that every edge contains exactly one vertex of V_1 and k vertices of V_2. Given a partition $V(H) = V_1 \cup V_2$, a $(1,d)$-subset S of $V(H)$ contains one vertex in V_1 and d vertices in V_2. Let $\delta_{1,d}(H) := \min\{\deg_H(S) : S \text{ is a } (1,d)\text{-subset of } V(H)\}$ for $d \in [k-1]$. The *relative degree* $\overline{\deg}(S)$ to be $\deg(S)/\binom{n-d}{k-d}$. The *minimum relative* $(1,d)$*-degree* of a $(1,k)$-graph H, written by $\overline{\delta}_{1,d}(H)$, is the minimum of $\overline{\deg}(S)$ over all $(1,d)$-subsets S of $V(H)$.

A k-graph H is k-partite if $V(H)$ can be partitioned into k parts $V_1, \dots, V_k$ such that every edge consists of exactly one vertex from each class. Given a $(k+1)$-partite $(k+1)$-graph H with $V(H) = V_0 \cup V_1 \cup \dots \cup V_k$. A $(k+1)$-uniform sequential path P of *length* t in H is a $(k+1)$-graph with vertex set $V(P) = C(P) \cup I(P)$ where $C(P) = \{c_1, \dots, c_{t-k+1}\} \subseteq V_0$, $I(P) = \{v_1, \dots, v_t\} \subseteq V_1 \cup \dots \cup V_k$ and edge set $\{e_1, \dots, e_{t-k+1}\}$ such that $e_i = \{c_i, v_i, \dots, v_{i+k-1}\}$ for $i \in [t-k+1]$. Denote the length of P by $\ell(P)$. We call $c_1, \dots, c_{t-k+1}$ the *colors* of P and $v_1, \dots, v_t$ the *points* of P. Furthermore, if $(v_1, \dots, v_t)$ is a cyclically ordered set, then we call this sequential path a *sequential cycle*. A $(k+1)$-uniform sequential walk is an ordered set of points with an ordered set of colors such that the set of the i_{th} k consecutive points along with the i_{th} color forms an edge. In particular, if the order is cyclical, then we call it *sequentially closed walk*. Note that the points, edges and colors in a sequential walk are allowed to be repeated. The *length* of a sequential walk is its number of points.

Before we give the proof of Theorem 4.1, we use the following similar definitions with [105].

Definition 4.1 (Sequentially Hamilton cycle threshold) *The minimum $(1, k-2)$-degree threshold for sequentially Hamilton cycles, denoted by $thc_{k-2}(k)$, is the smallest number $\delta > 0$ such that, for every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that every $(1, k)$-graph H on $[n] \cup V$ with minimum degree $\delta_{1,k-2}(H) \geq (\delta + \varepsilon)\binom{n}{2}$ contains a sequentially Hamilton cycle where $|V| = n \geq n_0$.*

Definition 4.2 (Sequentially tight connectivity) *A subgraph H' of a $(1, k)$-graph H is sequentially tightly connected, if any two edges of H' can be connected by a sequential walk. A sequentially tight component of H is an edge maximal sequentially tightly connected subgraph.*

Given $\mathbf{b} : V(H) \to [0, 1]$, we define the $\mathbf{b}$-*fractional matching* to be a function $\mathbf{w} : E(H) \to [0, 1]$ such that $\sum_{e:v\in e} \mathbf{w}(e) \leq \mathbf{b}(v)$ for every vertex $v \in V(H)$. Moreover, if the equality holds, then we call $\mathbf{w}$ *perfect*. Denote the maximum size of a $\mathbf{b}$-fractional matching by $\nu(H, \mathbf{b}) = \max_{\mathbf{w}} \sum_{e\in E(H)} \mathbf{w}(e)$ where $\mathbf{w}$ is a $\mathbf{b}$-fractional matching. It is well-known that perfect matchings are closely related to its fractional counterpart. In particular, when $\mathbf{b}(v) = 1$ for every vertex $v \in V(H)$, the $\mathbf{b}$-*fractional matching* is called *fractional matching*. The *density* of a $\mathbf{b}$-fractional matching is $\sum_{e\in E(H)} \mathbf{w}(e)/|V(H)|$. Besides, we require the following characterization. Given a k-graph H, we say that H is γ-*robustly matchable* if the following holds. For every vertex weight $\mathbf{b} : V(H) \to [1 - \gamma, 1]$, there is an edge weight $\mathbf{w} : E(H) \to [0, 1]$ with $\sum_{e:v\in e} \mathbf{w}(e) = \mathbf{b}(v)/(k - 1)$ for every vertex $v \in V(H)$. Note that a γ-robustly matchable k-graph H admits a $\mathbf{b}$-fractional matching of size $\sum_{v\in V(H)} \mathbf{b}(v)/k(k - 1)$ for every vertex weighting $\mathbf{b} : V(H) \to [1 - \gamma, 1]$. The following definition plays an important role in our proof.

Definition 4.3 (Link graph) *Given $\ell \in [0, k - 1]$, a $(1, k)$-graph H on $V(H) = [n] \cup V$ where $|V| = n$ and a set S of $(1, \ell)$-subset of $V(H)$, we define the link $(k - \ell)$-graph of S in H as the graph $L_H(S)$ with vertex set V and edge set $\{X : X \cup S \in E(H)\}$. If H is clear, then we simply write $L(S)$.*

Let $H = (V, E)$ be a k-graph, $V' \subseteq V$, an *induced subgraph* $H[V']$ of a k-graph H is a k-graph with vertex set V' and edge set E' where each edge is precisely the edge of H consisting of k vertices in V'. We usually denote H' by $H[V']$.

Definition 4.4 (Sequentially Hamilton framework) *Let α, γ, δ be positive constants. Suppose R is a $(1, k)$-graph on $[t] \cup V$ where $|V| = t$, we call a subgraph H of R an (α, γ, δ)-sequentially Hamilton framework, if H has the following properties.*

(F1) $H_i := H[\{i\} \cup V]$ *is sequentially tightly connected for* $i \in [t]$*,*

(F2) H_i *contains a sequentially closed walk of length 1 mod* k *for* $i \in [t]$*,*

(F3) $H_{W_i} := H[[t(i-1)/k + 1, ti/k] \cup V]$ *is* γ*-robustly matchable for* $i \in [k]$*,*

(F4) For every color $i \in [t]$*, there are at least* $(1-\alpha)t$ *points* $v \in V$ *such that* $\{i, v\}$ *has relative* $(1,1)$*-degree at least* $1 - \delta + \gamma$*,*

(F5) $L_H(\{i\})$ *and* $L_H(\{j\})$ *intersect in an edge for each* $i, j \in [t]$*.*

We write $x \ll y$ to mean that for any $y \in (0,1]$, there exists an $x_0 \in (0,1)$ such that for all $x \leq x_0$, the subsequent statements hold. Hierarchies with more constants are defined similarly to be read from right to left.

Definition 4.5 (Sequentially Hamilton framework threshold) *The minimum* $(1, k-2)$*-degree threshold for* $(1,k)$*-uniform sequentially Hamilton framework, denoted by* $rhf_{k-2}(k)$*, is the smallest value of* δ *such that the following holds.*

Suppose $\varepsilon, \alpha, \gamma, \mu > 0$ *and* $t \in \mathbb{N}$ *with* $1/t \ll \varepsilon \ll \alpha \ll \gamma \ll \mu$*. If* R *is a* $(1,k)$*-graph on* $[t] \cup V$ *where* $|V| = t$*, with minimum relative* $(1, k-2)$*-degree at least* $\delta + \mu$ *and a set* $I \subseteq E(R)$ *of at most* $\varepsilon t \binom{t}{k}$ *perturbed edges, then* R *contains an* (α, γ, δ)*-sequentially Hamilton framework* H *that avoids the edges of* I*.*

We transform the problem of bounding the sequentially Hamilton cycle threshold to bound the sequentially Hamilton framework threshold.

Theorem 4.2 (Framework Theorem) *For* $k \geq 3$*, we have* $thc_{k-2}(k) \leq rhf_{k-2}(k)$*.*

For any $j \in [k]$, let the *shadow graph* $\partial_j(H)$ of $(1,k)$-graph H at level j be the $(1,j)$-graph on $[n] \cup V$ whose edges are $(1,j)$-sets contained in the edges of H.

Definition 4.6 (Vicinity) *Given a* $(1,k)$*-graph* R *on* $[t] \cup V$*, we say that* $\mathcal{C}_i = \{C_S \subseteq L(S) : S \in \partial_{k-2}(R) \text{ and } i \in S\}$ *for each* $i \in [t]$ *is a* $(k-2)$*-vicinity. We define the* $(1,k)$*-graph* $H_{\mathcal{C}_i}$ *generated by* $\mathcal{C}_i$ *as the subgraph of* R *with vertex set* $V(H) = \{i\} \cup V$ *and edge set*

$$E(H) = \bigcup_{i \in S, S \in \partial_{k-2}(R)} \{A \cup S : A \in C_S\}.$$

Besides, we need the following structures.

Definition 4.7 (Switcher) *A switcher in a graph* G *is an edge* ab *such that* a *and* b *shares a common neighbor in* G*.*

Note that a switcher together with its common neighbor generates a triangle.

Definition 4.8 (Arc) *Let* R_i *be a* $(1,k)$*-graph on* $\{i\} \cup V$ *with* $(k-2)$*-vicinity* $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R_i)\}$*. We say that a* $(1, k+1)$*-tuple* $(i, v_1, \ldots, v_{k+1})$ *is an arc for* $\mathcal{C}_i$ *if the following holds.*

- $\{i, v_1, \ldots, v_{k-2}\} \in \partial_{k-2}(R_i)$ *with* $\{v_{k-1}, v_k\} \in C_{\{i,v_1,\ldots,v_{k-2}\}}$.
- $\{i, v_2, \ldots, v_{k-1}\} \in \partial_{k-2}(R_i)$ *with* $\{v_k, v_{k+1}\} \in C_{\{i,v_2,\ldots,v_{k-1}\}}$.

Definition 4.9 (Sequentially Hamilton vicinity) *Let $\gamma, \delta > 0$. Suppose that R is a $(1,k)$-graph on $[t] \cup V$, let $R_i := R[\{i\} \cup V]$. We say that a family $\mathcal{C} = \{\mathcal{C}_i : i \in [t]\}$ of $(k-2)$-vicinities where $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R_i)\}$ is (γ, δ)-sequentially Hamilton if for any $S, S' \in \partial_{k-2}(R_i)$ and $T \in \partial_{k-2}(R_j)$ where $i \neq j$, the followings hold,*

(V1) C_S is tightly connected,
(V2) C_S and $C_{S'}$ intersect in an edge,
(V3) C_S has a switcher and the vicinity $\mathcal{C}_i$ has an arc for $i \in [t]$,
(V4) C_S has a fractional matching of density $(1 + 1/k)(1/(k+1) + \gamma)$,
(V5) C_S has edge density at least $1 - \delta + \gamma$,
(V6) C_S and C_T intersect in an edge.

Definition 4.10 (Perturbed degree) *Let $\alpha, \delta > 0$. We say that a $(1,k)$-graph R has α-perturbed minimum relative $(1, k-2)$-degree at least δ if the followings hold for $j \in [k-2]$.*

(P1) every edge of $\partial_j(R)$ has relative degree at least δ in R,
(P2) $\overline{\partial_j(R)}$ has edge density at most α, where $\overline{\partial_j(R)}$ denotes the complement of $\partial_j(R)$,
(P3) each $(1, j-1)$-tuple of $\partial_{j-1}(R)$ has relative degree less than α in $\overline{\partial_j(R)}$.

Definition 4.11 (Sequentially Hamilton vicinity threshold) *The minimum $(1, k-2)$-degree threshold for $(1,k)$-uniform sequentially Hamilton vicinities, denoted by $rhv_{k-2}(k)$, is the smallest value $\delta > 0$ such that the following holds. Let $\alpha, \gamma, \mu > 0$, $t \in \mathbb{N}$ with $1/t \ll \alpha \ll \gamma \ll \mu$ and R be a $(1,k)$-graph on $[t] \cup V$. If each $R_i := R[\{i\} \cup V]$ has α-perturbed minimum relative $(1, k-2)$-degree at least $\delta + \mu$ for $i \in [t]$, then R admits a family of (γ, δ)-sequentially Hamilton $(k-2)$-vicinities.*

Theorem 4.3 (Vicinity Theorem) *For $k \geq 3$, $rhf_{k-2}(k) \leq rhv_{k-2}(k)$.*

Combining Theorem 4.3 with Theorem 4.2, we just need to prove the following theorem, and we can obtain Theorem 4.1.

Theorem 4.4 *For $k \geq 3$, $rhv_{k-2}(k) \leq 5/9$.*

We use the following concentration inequalities.

A hypergraph $\mathcal{H} = (V, E)$ is a *complex* if its edge set is down-closed, meaning that whenever $e \in E$ and $e' \subseteq e$, we have $e' \in E$. A k-complex is a complex where all edges have size at most k. Given a complex $\mathcal{H}$, we use $\mathcal{H}^{(i)}$ to denote the i-graph obtained by taking all vertices of $\mathcal{H}$ and edges of size i. Denote the number of edges of size i in $\mathcal{H}$ by $e_i(\mathcal{H})$.

Let $\mathcal{P}$ partition a vertex set V into parts $V_1, \dots, V_s$. Then we say that a subset $S \subseteq V$ is *$\mathcal{P}$-partite* if $|S \cap V_i| \leq 1$ for every $i \in [s]$. Similarly, we say that hypergraph $\mathcal{H}$ is *$\mathcal{P}$-partite* if all of its edges are $\mathcal{P}$-partite. In this case we refer to the parts of $\mathcal{P}$ as the *vertex class* of $\mathcal{H}$. We say that a hypergraph $\mathcal{H}$ is *s-partite* if there is some partition $\mathcal{P}$ of $V(\mathcal{H})$ into s parts for which $\mathcal{H}$ is $\mathcal{P}$-partite.

Let $\mathcal{H}$ be a $\mathcal{P}$-partite complex. Then for any $A \subseteq [s]$ we write V_A for $\bigcup_{i \in A} V_i$. The *index* of a $\mathcal{P}$-partite set $S \subseteq V$ is $i(S) := \{i \in [s] : |S \cap V_i| = 1\}$. We write $\mathcal{H}_A$ to denote the collection of edges in $\mathcal{H}$ with index A, that is, $\mathcal{H}_A$ can be regarded as an $|A|$-partite $|A|$-graph on vertex set V_A. Similarly, if X is a j-set of indexes of vertex classes of $\mathcal{H}$ we write $\mathcal{H}_X$ for the j-partite j-uniform subgraph of $\mathcal{H}^{(j)}$ induced by $\bigcup_{i \in X} V_i$. We write $\mathcal{H}_{X<}$ for the j-partite hypergraph with vertex set $\bigcup_{i \in V_X} V_i$ and edge set $\bigcup_{X' \subset X} \mathcal{H}_{X'}$.

Let H_i be any i-partite i-graph and H_{i-1} be any i-partite $(i-1)$-graph on a common vertex set V partitioned into i common vertex classes. Denote $K_i(H_{i-1})$ by the i-partite i-graph on V whose edges are all i-sets which are supported on H_{i-1}(i.e. induce a copy of complete $(i-1)$-graph K_i^{i-1} on i vertices in H_{i-1}). The *density of H_i with respect to H_{i-1}* is defined to be

$$d(H_i|H_{i-1}) := \frac{|K_i(H_{i-1}) \cap H_i|}{|K_i(H_{i-1})|}$$

if $|K_i(H_{i-1})| > 0$. For convenience, we take $d(H_i|H_{i-1}) := 0$ if $|K_i(H_{i-1})| = 0$. When H_{i-1} is clear from the context, we simply refer $d(H_i|H_{i-1})$ as the *relative density of H_i*. More generally, if $\mathbf{Q} := (Q_1, \dots, Q_r)$ is a collection of r not necessarily disjoint subgraphs of H_{i-1}, we define

$$K_i(\mathbf{Q}) := \bigcup_{j=1}^{r} K_i(Q_j)$$

and

$$d(H_i|\mathbf{Q}) := \frac{|K_i(\mathbf{Q}) \cap H_i|}{|K_i(\mathbf{Q})|}$$

if $|K_i(\mathbf{Q})| > 0$. Similarly, we take $d(H_i|\mathbf{Q}) := 0$ if $|K_i(\mathbf{Q})| = 0$. We say that H_i is *(d_i, ε, r)-regular with respect to H_{i-1}* if we have $d(H_i|\mathbf{Q}) = d_i \pm \varepsilon$ for every r-set $\mathbf{Q}$ of subgraphs of H_{i-1} such that $|K_i(\mathbf{Q})| > \varepsilon |K_i(H_{i-1})|$. We refer to $(d_i, \varepsilon, 1)$-regularity simply as *(d_i, ε)-regularity*. We say that H_i is (ε, r)-regular with respect to H_{i-1} to mean that there exists some d_i for which H_i is (d_i, ε, r)-regular with respect to H_{i-1}. Given an i-graph G whose vertex set contains that of H_{i-1}, we say that G is *(d_i, ε, r)-regular with respect to H_{i-1}* if the i-partite subgraph of G induced by the vertex classes of H_{i-1} is (d_i, ε, r)-regular with respect to H_{i-1}. Similarly, when H_{i-1} is clear from the context, we refer to the relative density of this i-partite subgraph of G with respect to H_{i-1} as the *relative density of G*.

Now let $\mathcal{H}$ be an s-partite k-complex on vertex classes $V_1, \ldots, V_s$, where $s \geq k \geq 3$. Since $\mathcal{H}$ is a complex, if $e \in \mathcal{H}^{(i)}$ for some $i \in [2,k]$, then the vertices of e induce a copy of K_i^{i-1} in $\mathcal{H}^{(i-1)}$. This means that for any index $A \in \binom{[s]}{i}$, the density $d(\mathcal{H}^{(i)}[V_A]|\mathcal{H}^{(i-1)}[V_A])$ can be regarded as the proportion of 'possible edges' of $\mathcal{H}^{(i)}[V_A]$ which are indeed edges. We say that $\mathcal{H}$ is $(d_2, \ldots, d_k, \varepsilon_k, \varepsilon, r)$-*regular* if

1. for $i \in [2, k-1]$ and $A \in \binom{[s]}{i}$, the induced subgraph $\mathcal{H}^{(i)}[V_A]$ is (d_i, ε)-regular with respect to $\mathcal{H}^{(i-1)}[V_A]$ and
2. for any $A \in \binom{[s]}{k}$, the induced subgraph $\mathcal{H}^{(k)}[V_A]$ is (d_k, ε_k, r)-regular with respect to $\mathcal{H}^{(k-1)}[V_A]$.

The Regular Slice Lemma says that any k-graph G admits a regular slice. Informally speaking, a regular slice of G is a partite $(k-1)$-complex $\mathcal{J}$ whose vertex classes have equal size, whose subgraphs $\mathcal{J}^{(2)}, \ldots, \mathcal{J}^{(k-1)}$ satisfy certain regularity properties and which moreover has the property that G is regular with respect to $\mathcal{J}^{(k-1)}$. The first two of these conditions are formalised in the following definition : we say that a $(k-1)$-complex $\mathcal{J}$ is (t_0, t_1, ε)-*equitable*, if it has the following properties.

1. $\mathcal{J}$ is $\mathcal{P}$-partite for a $\mathcal{P}$ which partitions $V(\mathcal{J})$ into t parts of equal size, where $t_0 \leq t \leq t_1$. We refer to $\mathcal{P}$ as the *ground partition* of $\mathcal{J}$, and to the parts of $\mathcal{P}$ as the *clusters* of $\mathcal{J}$.
2. There exists a *density vector* $\mathbf{d} = (d_2, \ldots, d_{k-1})$ such that for $i \in [2, k-1]$ we have $d_i \geq 1/t_1$ and $1/d_i \in \mathbb{N}$ and for each $A \subseteq \mathcal{P}$ of size i, the i-graph $\mathcal{J}^{(i)}[V_A]$ induced on V_A is (d_i, ε)-regular with respect to $\mathcal{J}^{(i-1)}[V_A]$.

If $\mathcal{J}$ has density vector $\mathbf{d} = (d_2, \ldots, d_{k-1})$, then we will say that $\mathcal{J}$ is $(d_2, \ldots, d_{k-1}, \varepsilon)$-regular, or $(\mathbf{d}, \varepsilon)$-*regular*, for short. For any k-set X of clusters of $\mathcal{J}$, we write $\hat{\mathcal{J}}_X$ for the k-partite $(k-1)$-graph $\mathcal{J}_{X_<}^{(k-1)}$. Given a (t_0, t_1, ε)-equitable $(k-1)$-complex $\mathcal{J}$, a k-set X of clusters of $\mathcal{J}$ and a k-graph G on $V(\mathcal{J})$, we say that G is (d, ε_k, r)-*regular with respect to* X if G is (d, ε_k, r)-regular with respect to $\hat{\mathcal{J}}_X$. We will also say that G is (ε_k, r)-*regular with respect to* X if there exists a d such that G is (d, ε_k, r)-regular with respect to X. We write $d^*_{\mathcal{J},G}(X)$ for the relative density of G with respect to $\hat{\mathcal{J}}_X$, or simply $d^*(X)$ if $\mathcal{J}$ and G are clear from the context, which will always be the case in applications.

We now give the key definition of the Regular Slice Lemma.

Definition 4.12 (Regular slice) *Given $\varepsilon, \varepsilon_k > 0$, $r, t_0, t_1 \in \mathbb{N}$, a k-graph G and a $(k-1)$-complex $\mathcal{J}$ on $V(G)$, we call $\mathcal{J}$ a $(t_0, t_1, \varepsilon, \varepsilon_k, r)$-regular slice for G if $\mathcal{J}$ is (t_0, t_1, ε)-equitable and G is (ε_k, r)-regular with respect to all but at most $\varepsilon_k \binom{t}{k}$ of the k-sets of clusters of $\mathcal{J}$, where t is the number of clusters of $\mathcal{J}$.*

It will sometimes be convenient not to specify all parameters, we may write that $\mathcal{J}$ is $(\cdot, \cdot, \varepsilon)$-equitable or is a $(\cdot, \cdot, \varepsilon, \varepsilon_k, r)$-slice for G, if we do not wish to specify t_0 and t_1.

Given a regular slice $\mathcal{J}$ for a k-graph G, it will be important to know the relative densities $d^*(X)$ for k-sets X of clusters of $\mathcal{J}$. To keep track of these we make the following definition.

Definition 4.13 (Weighted reduced k-graph) *Let G be a $(1,k)$-graph and let $\mathcal{J}$ be a $(t_0, t_1, \varepsilon, \varepsilon_{k+1}, r)$-regular slice for G. We define the* weighted reduced $(1,k)$-graph *of G, denoted by $R(G)$, to be the complete weighted $(1,k)$-graph whose vertices are the clusters of $\mathcal{J}$ and where each edge X is given weight $d^*(X)$.*

Similarly, for $d_{k+1} > 0$, we define the d_{k+1}-reduced $(1,k)$-graph $R_{d_{k+1}}(G)$ to be the (unweighted) $(1,k)$-graph whose vertices are the clusters of $\mathcal{J}$ and whose edges are all $(1,k)$-sets X of clusters of $\mathcal{J}$ such that G is (ε_{k+1}, r)-regular with respect to X and $d^*(X) \geq d_{k+1}$.

Given a $(1,k)$-graph G on $[n] \cup V$, a vertex $v \in V$ and a color $c \in [n]$, recall that $\deg_G(c,v)$ is the number of edges of G containing c and v and $\overline{\deg}_G(c,v) = \deg_G(c,v)/\binom{n-1}{k-1}$ is the relative degree of $\{c,v\}$ in G. Given a (t_0,t_1,ε)-equitable $(k-1)$-complex $\mathcal{J}$ with $V(\mathcal{J}) \subseteq V(G)$, the *rooted degree* of (c,v) *supported by* $\mathcal{J}$, written by $\deg_G((c,v),\mathcal{J})$, is defined as the number of $(k-1)$-sets T in $\mathcal{J}^{(k-1)}$ such that $T \cup \{c,v\}$ forms an edge in G. Then the relative degree $\overline{\deg}_G((c,v);\mathcal{J})$ of (c,v) in G supported by $\mathcal{J}$ is defined as $\overline{\deg}_G((c,v);\mathcal{J}) = \deg_G((c,v);\mathcal{J})/e(\mathcal{J}^{(k-1)})$.

Definition 4.14 (Representative rooted degree) *Let $\eta > 0$, G be a $(1,k)$-graph on $[n] \cup V$ and $\mathcal{J}$ be a $(t_0,t_1,\varepsilon,\varepsilon_{k+1})$-regular slice for G. We say that $\mathcal{J}$ is η-rooted-degree-representative if for any vertex $v \in V$ and any color $c \in [n]$, we have*

$$|\overline{\deg}_G((c,v);\mathcal{J}) - \overline{\deg}_G(c,v)| < \eta.$$

Definition 4.15 (Regular setup) *Let $k, m, r, t \in \mathbb{N}$ and $\varepsilon, \varepsilon_{k+1}, d_2, \ldots, d_{k+1} > 0$. We say that $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ is a $(k, m, t, \varepsilon, \varepsilon_{k+1}, r, d_2, \ldots, d_{k+1})$-regular setup, if*

(RS1) G is a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$ and $G_{\mathcal{J}} \subseteq G$,

*(RS2) $\mathcal{J}$ is a $(\cdot,\cdot,\varepsilon,\varepsilon_{k+1},r)$-regular slice for G with density vector **d** $= (d_2, \ldots, d_k)$,*

(RS3) $\mathcal{P}$ is the ground partition of $\mathcal{J}$ with initial partition of $[n] \cup V$ and $2t$ clusters, each of size m,

(RS4) R is a subgraph of $R_{d_{k+1}}(G)$,

(RS5) for each $X \in E(R)$, $G_{\mathcal{J}}$ is $(d_{k+1}, \varepsilon_{k+1}, r)$-regular with respect to X.

We further say that $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ is representative if

(RS6) $\mathcal{J}$ is ε_{k+1}-rooted-degree-representative.

The Regular Slice Lemma of [7] ensures that every sufficiently large k-graph has a representative regular slice. Given the existence of a regular slice, it is easy to derive the existence of a regular setup. In [105], it is stated directly in terms of regular setups. And it is an easy corollary of giving a sufficiently large $(1,k)$-graph.

Lemma 4.1 (Regular Setup Lemma [7]) *Let k, t_0 be positive integers, $\delta, \mu, \alpha, \varepsilon_{k+1}, d_{k+1}$ be positive and $r : \mathbb{N} \to \mathbb{N}$ and $\varepsilon : \mathbb{N} \to (0,1]$ be functions. Suppose that*

$$k \geq 3, \varepsilon_{k+1} \ll \alpha, d_{k+1} \ll \mu.$$

*Then there exists t_1 and m_0 such that the following holds for all $n \geq 2t_1m_0$. Let G be a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$ and suppose that G has minimum relative $(1,k-2)$-degree $\overline{\delta}_{1,k-2}(G) \geq \delta + \mu$. There exists **d** $= (d_2, \ldots, d_{k+1})$ and a representative $(k, m, 2t, \varepsilon(t_1), \varepsilon_{k+1}, r(t_1),$ **d**$)$-regular setup $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R_{d_{k+1}})$ with $t \in [t_0, t_1]$, $m_0 \leq m$ and $n \leq (1+\alpha)mt$. Moreover, there is a $(1,k)$-graph I on $\mathcal{P}$ of edge density at most ε_{k+1} such that $R = R_{d_{k+1}} \cup I$ has minimum relative $(1, k-2)$-degree at least $\delta + \mu/2$.*

Let $\mathcal{G}$ be a $\mathcal{P}$-partite k-complex and $X_1, \ldots, X_s \in \mathcal{P}$(possibly with repetition), and let $\mathcal{H}$ be a k-complex on vertices $[s]$. We say that an embedding of $\mathcal{H}$ in $\mathcal{G}$ is *partition-respecting*, if i is embedded in X_i for $i \in [s]$. Note that this notion depends on the labeling of $V(\mathcal{H})$ and the clusters $X_1, \ldots, X_s$, but these will be clear in the paper. Denote the set of labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$ by $\mathcal{H}_{\mathcal{G}}[\bigcup_{i \in S} X_i]$. When $X_1, \ldots, X_s$ are clear, we denote it by $\mathcal{H}_{\mathcal{G}}$ for short. Recall that $e_i(\mathcal{H})$ denotes the number of edges of size i in $\mathcal{H}$.

The following lemma states that the number of copies of a given small k-graph inside a regular slice is roughly what we expect if the edges inside a regular slice were chosen randomly. There are many different versions in [7, 33, 61, 140] and we use the following version in [33].

Lemma 4.2 (Counting Lemma [33]) *Let k, s, r, m be positive integers and let $\beta, d_2, \ldots, d_k, \varepsilon, \varepsilon_k$ be positive constants such that $1/d_i \in \mathbb{N}$ for $i \in [2, k-1]$ and such that*

$$1/m \ll 1/r, \varepsilon \ll \varepsilon_k, d_2, \ldots, d_{k-1},$$

$$\varepsilon_k \ll \beta, d_k, 1/s.$$

*Let H be a k-graph on $[s]$ and let $\mathcal{H}$ be the k-complex generated by the down-closure of H. Let **d** $= (d_2, \cdots, d_k)$, let $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ be a $(k, m, \cdot, \varepsilon, \varepsilon_k, r,$ **d**$)$-regular setup and $\mathcal{G} = \mathcal{J} \cup G_{\mathcal{J}}$. Suppose $X_1, \ldots, X_s$ are such that $i \mapsto X_i$ is a homomorphism from H into R, then the number of labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$ satisfies*

$$|\mathcal{H}_{\mathcal{G}}| = (1 \pm \beta) \left(\prod_{i=2}^{k} d_i^{e_i(\mathcal{H})} \right) m^s.$$

The following tool allows us to extend small subgraphs into a regular slice. It was given by Cooley, Fountoulakis, Kühn and Osthus [33].

Lemma 4.3 (Extension Lemma [33]) *Let k, s, s', r, m be positive integers, where $s' < s$ and let $\beta, d_2, \dots, d_k, \varepsilon, \varepsilon_k$ be positive constants such that $1/d_i \in \mathbb{N}$ for $i \in [2, k-1]$ and such that*

$$1/m \ll 1/r, \varepsilon \ll \varepsilon_k, d_2, \dots, d_{k-1},$$
$$\varepsilon_k \ll \beta, d_k, 1/s.$$

*Suppose H is a k-graph on $[s]$. Let $\mathcal{H}$ be the k-complex generated by the down-closure of H and $\mathcal{H}'$ be an induced subcomplex of $\mathcal{H}$ on s' vertices. Let **d** $= (d_2, \dots, d_k)$ and $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ be a $(k, m, \cdot, \varepsilon, \varepsilon_k, r,$ **d**$)$-regular setup and $\mathcal{G} = \mathcal{J} \cup G_{\mathcal{J}}$. Suppose $X_1, \dots, X_s$ are such that $i \mapsto X_i$ is a homomorphism from H into R. Then all but at most $\beta|\mathcal{H}'_{\mathcal{G}}|$ labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{G}$ extend to*

$$(1 \pm \beta) \left(\prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H}) - e_i(\mathcal{H}')} \right) m^{s-s'}$$

labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$.

In some certain situation, we look for structures whose edges lie entirely in the $(k-1)$-complex $\mathcal{J}$ of a regular setup. We can no longer use the above lemmas whose input is a regular setup rather than an equitable complex. Also, the above lemmas requires r to be large enough with respect to ε_k while the $(k-1)$-th level of $\mathcal{J}$ will only need to be (d_{k-1}, ε)-regular with respect to the lower level. We can use a Dense Counting Lemma as proved by Kohayakawa, Rödl and Skokan [92]. We state the following version given by Cooley, Fountoulakis, Kühn and Osthus [33].

Lemma 4.4 (Dense Counting Lemma [33]) *Let k, s, m be positive integers and $\varepsilon, d_2, \dots, d_{k-1}, \beta$ be positive constants such that*

$$1/m \ll \varepsilon \ll \beta \le d_2, \dots, d_{k-1}, 1/s.$$

*Suppose H is a $(k-1)$-graph on $[s]$ and $\mathcal{H}$ is the $(k-1)$-complex generated by the down-closure of H. Let **d** $= (d_2, \dots, d_{k-1})$ and $\mathcal{J}$ be a $($**d**$, \varepsilon)$-regular $(k-1)$-complex with ground partition $\mathcal{P}$, each size of whose vertex class is m. If $X_1, \dots, X_s \in \mathcal{P}$, then*

$$|\mathcal{H}_{\mathcal{J}}| = (1 \pm \beta) \prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H})} m^s.$$

The following lemma gives the number of edges in each layer of a regular slice.

Lemma 4.5 ([7]) *Suppose that $1/m \ll \varepsilon \ll \beta \ll d_2, \dots, d_{k-1}, 1/k$ and that $\mathcal{J}$ is a $(\cdot, \cdot, \varepsilon)$-equitable $(k-1)$-complex with density vector $(d_2, \dots, d_{k-1})$ and clusters of size m. Let X be a set of at most $k-1$ clusters of $\mathcal{J}$. Then*

$$|\mathcal{J}_X| = (1 \pm \beta) \left(\prod_{i=2}^{|X|} d_i^{\binom{|X|}{i}} \right) m^{|X|}.$$

Analogously, we have a dense version of Extension Lemma [33].

Lemma 4.6 (Dense Extension Lemma [33]) *Let k, s, s', m be positive integers, where $s' < s$ and $\varepsilon, \beta, d_2, \ldots, d_{k-1}$ be positive constants such that $1/m \ll \varepsilon \ll \beta \ll d_2, \ldots, d_{k-1}, 1/s$. Let H be a $(k-1)$-graph on $[s]$. Let $\mathcal{H}$ be the $(k-1)$-complex generated by the down-closure of H and $\mathcal{H}'$ be an induced subcomplex of $\mathcal{H}$ on s' vertices. Let $\boldsymbol{d} = (d_2, \ldots, d_{k-1})$ and let $\mathcal{J}$ be a $(\boldsymbol{d}, \varepsilon)$-regular $(k-1)$-complex, with ground partition $\mathcal{P}$ with vertex classes of size m each. If $X_1, \ldots, X_s \in \mathcal{P}$, then all but at most $\beta|\mathcal{H}'_{\mathcal{J}}|$ labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{J}$ extend to*

$$(1 \pm \beta) \left(\prod_{i=2}^{k-1} d_i^{e_i(\mathcal{H}) - e_i(\mathcal{H}')} \right) m^{s-s'}$$

labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{J}$.

The restriction of a regular complex to a large subset of its vertex is also a regular complex, with slightly altered constants.

Lemma 4.7 (Regular Restriction Lemma [7]) *Let k, r, m, s be integers and α, $\varepsilon, \varepsilon_k, d_2, \ldots, d_k$ be positive constants such that $1/d_i \in \mathbb{N}$ for $\in [2, k]$ and*

$$1/m \ll \varepsilon \ll \varepsilon_k, d_2, \ldots, d_{k-1},$$

and

$$\varepsilon_k \ll \alpha.$$

Let $\mathcal{G}$ be an s-partite k-complex on vertex classes $V_1, \ldots, V_s$, each of size m and which is $(\boldsymbol{d}, \varepsilon_k, \varepsilon, r)$-regular where $\boldsymbol{d} = (d_2, \ldots, d_k)$. Choose any $V'_i \subseteq V_i$ with $|V'_i| \geq \alpha m$ for $i \in [s]$. Then the induced subcomplex $\mathcal{G}[V'_1 \cup \cdots \cup V'_s]$ is $(\boldsymbol{d}, \sqrt{\varepsilon_k}, \sqrt{\varepsilon}, r)$-regular.

The chapter is organised as follows. In Section 4.2, we show the minimum degree condition guarantees a sequentially Hamilton vicinity. In Section 4.3, we show that how a sequentially Hamilton vicinity deduce a sequentially Hamilton framework. In Section 4.4, we show that how a sequentially Hamilton framework deduce a sequentially Hamilton cycle. The sequentially Hamilton cycle in a $(1, k)$-graph is a rainbow Hamilton cycle in a k-graph system, as desired.

4.2 . Obtaining sequentially Hamilton vicinity with degree condition

In this section, we determine the $(k-2)$-vicinity threshold of $(1, k)$-graphs. Lovász's formulation of the Kruskal-Katona theorem states that, for any $x > 0$, if G is a k-graph with $e(G) \geq \binom{x}{k}$ edges, then $e_j(G) \geq \binom{x}{j}$ for every $j \in [k]$ (Theorem 2.14 in [57]). By approximating the binomial coefficients, they [105] deduce the following variant.

Lemma 4.8 (Kruskal-Katona theorem [105]) *Let $1/t \ll \varepsilon \ll 1/k$ and G be a graph on t vertices and edge density δ, then $\partial(G)$ has at least $(\delta^{1/2} - \varepsilon)t$ vertices.*

Proposition 4.1 *Let $t \in N$ and $\gamma, \delta', \delta > 0$ with $1/t \ll \varepsilon \ll \delta$ and $\delta + \delta^{1/2} > 1 + \varepsilon$. Let R_i be a $(1, k)$-graph on $\{i\} \cup V$ where $|V| = t$ with a subgraph that is generated by a $(k-2)$-vicinity $\mathcal{C}_i$. Suppose that each $C_S \in \mathcal{C}_i$ has edge density at least $\delta + \mu$, then $\mathcal{C}_i$ admits an arc.*

Proof. Consider an arbitrary set $S = \{i, v_1, \ldots, v_{k-2}\} \in \partial_{k-2}(R_i)$. By averaging, there is a vertex v_{k-1} with relative vertex degree at least δ in C_S. Set $S' = \{i, v_2, \ldots, v_{k-1}\}$, we have $S' \in \partial_{k-2}(R_i)$. Thus, $C_{S'} - \{v_1\}$ has edge density at least $\delta + \mu/2$. By Lemma 4.8, $\partial(C_{S'} - \{v_1\})$ has at least $(\delta^{1/2} - \varepsilon)t$ vertices.

By the choice of v_{k-1} and the pigeonhole principle, $\partial(C_{S'} - \{v_1\})$ and $L(\{i, v_1, \ldots, v_{k-1}\})$ must share a common vertex v_k. Since $v_k \in \partial(C_{S'} - \{v_1\})$, there is another vertex v_{k+1} such that $\{v_k, v_{k+1}\} \in C_{S'} - \{v_1\}$. Thus, $\{i, v_1, \ldots, v_{k+1}\}$ is an arc. □

We use the following result of [105].

Lemma 4.9 ([105]) *Let $1/t \ll \gamma \ll \mu$, suppose that L_1 and L_2 are graphs on a common vertex set of size t such that L_1, L_2 has edge density at least $5/9 + \mu$. For $i \in [2]$, let C_i be a tight component of L_i with a maximum number of edges. We have*

(i) C_1 and C_2 has an edge in common,

(ii) C_i has a switcher for $i \in [2]$,

(iii) C_i has a fractional matching of density $1/3 + \gamma$ for $i \in [2]$,

(iv) C_i has edge density at least $4/9 + \gamma$ for $i \in [2]$.

Proof. [The proof of Theorem 4.4] Let $\alpha, \gamma, \mu > 0$ with

$$1/t \ll \alpha \ll \delta \ll \mu \ll 5/9.$$

Consider a $(1, k)$-graph R on $[t] \cup V$ where $|V| = t$ and each $R_i := R[\{i\} \cup V]$ has α-perturbed minimum relative $(1, k-2)$-degree at least $5/9 + \mu$. For every $S \in \partial_{k-2}(R)$, let C_S be a tight component of $L(S)$ with a maximum number of edges and $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R) \text{ and } i \in S\}$. By the choice of C_S, (V1) holds obviously. By Lemma 4.9, $\mathcal{C}_i$ satisfies (V2), (V4), (V5) and (V6). Every $C_S \in \mathcal{C}_i$ contains a switcher. By Proposition 4.1, $\mathcal{C}_i$ contains an arc since $4/9 + (4/9)^{1-1/2} = 1 + 1/9$, thus $\mathcal{C} = \{\mathcal{C}_i : i \in [t]\}$ satisfies (V3), as desired. □

4.3 . From sequentially Hamilton vicinity to sequentially Hamilton framework

Our goal is to prove Theorem 4.3 in this part. We need the followings lemmas.

Lemma 4.10 *Let R_i be a $(1,k)$-graph on $\{i\} \cup V$ with a $(k-2)$-vicinity $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R_i)\}$ for $i \in [t]$. For every $S, S' \in \partial_{k-2}(R_i)$, if the vicinity $\mathcal{C}_i$ has an arc for $i \in [t]$, C_S and $C_{S'}$ intersect, C_S is tightly connected and has a switcher, then the vertex spanning subgraph $H_{\mathcal{C}_i}$ of R_i generated by $\mathcal{C}_i$ is sequentially tightly connected and contains a sequentially closed walk of length 1 mod k.*

Lemma 4.11 *Let $\gamma, \alpha, \delta > 0$ such that $1/t \ll \alpha, \gamma \ll 1/k$. Let R be a $(1,k)$-graph on $[t] \cup V$ where $|V| = t$ and each R_i has α-perturbed minimum relative $(1, k-2)$-degree at least δ. Let $\mathcal{C} = \{\mathcal{C}_i : i \in [t]\}$ be a family of $(k-2)$-vicinities where $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R_i)\}$. If for every $S \in \partial_{k-2}(R)$, C_S has a fractional matching of density $(1+1/k)(1/(k+1)+\gamma)$, then the graph $H_{\mathcal{C}_{W_i}} \subseteq R$ generated by $\mathcal{C}_{W_i} := \{\mathcal{C}_j : j \in [t(i-1)/k+1, ti/k]\}$ is γ-robustly matchable for each $i \in [k]$.*

Lemma 4.12 *Let $t, k \in \mathbb{N}, i \in [t]$ and $\delta, \alpha, \varepsilon > 0$ with $1/t \ll \varepsilon \ll \alpha \ll \delta, 1/k$. Let R_i be a $(1,k)$-graph on $\{i\} \cup V$ with minimum relative $(1, k-2)$-degree at least δ where $|V| = t$. Let I be a subgraph of R_i with edge density at most ε, there exists a vertex spanning subgraph $R_i' \subseteq R_i - I$ of α-perturbed minimum relative $(1, k-2)$-degree at least $\delta - \alpha$.*

Proof. [Proof of Theorem 4.3] Let $\delta = rhv_{k-2}(k)$ and $\varepsilon, \alpha, \gamma > 0$ such that

$$1/t \ll \varepsilon \ll \alpha \ll \alpha' \ll \gamma \ll \mu \ll \delta, 1/k.$$

Moreover,the constants $t, \varepsilon, \alpha, \mu$ are compatible with the constant hierarchy given by Definition 4.11, $t, \varepsilon, 2\alpha, \mu$ satisfy the conditions of Lemma 4.11 and $t, \varepsilon, \alpha, \delta$ satisfy the conditions of Lemma 4.12.

Given a $(1,k)$-graph R_i on $\{i\} \cup V$ with minimum relative $(1, k-2)$-degree at least $\delta + 2\mu$ and a set I of at most $\varepsilon\binom{t}{k}$ perturbed edges. We start by selecting a subgraph of R_i. By Lemma 4.12, we obtain a vertex spanning subgraph $R_i' \subseteq R_i - I$ of α-perturbed minimum relative $(1, k-2)$-degree at least $\delta + \mu$.

By the definition of 4.11, $R' := \bigcup_{i \in [t]} R_i'$ has a family of $(2\gamma, \delta)$-sequentially Hamilton $(k-2)$-vicinities $\mathcal{C} = \{\mathcal{C}_i : i \in [t]\}$ where $\mathcal{C}_i = \{C_S : S \in \partial_{k-2}(R_i')\}$. Each $\mathcal{C}_i$ generates a $(1,k)$-graph G_i. Let $H = \bigcup_{i \in [t]} G_i$. Note that G_i does not contain the edges of I and $V(G_i) = V(R_i')$. By Lemma 4.10 and 4.11, H also satisfies (F1)-(F3). For $k \geq 4$, by repeatedly applying Definition 4.10, we deduce that for all but at most αt $(1,1)$-sets of $V(R_i')$ is contained in at least $(1-2\alpha)^{k-3}\binom{|V'|-1}{k-3} \geq (1-2(k-3)\alpha)\binom{|V'|-1}{k-3}$ many $(1, k-2)$-sets in $\partial_{k-2}(R_i')$. Note that $\partial_{k-2}(R_i') = \partial_{k-2}(G_i)$. This implies that for all but at most αt $(1,1)$-sets of $V(G_i)$ has relative degree at least $1 - 2(k-3)\alpha$ in $\partial_{k-2}(G_i)$. Moreover, every $(1, k-2)$-set in $\partial_{k-2}(G_i)$ has relative degree at least $1 - \delta + 2\gamma$ in G_i, since G_i is generated from $(2\gamma, \delta)$-sequentially Hamilton $(k-2)$-vicinity and Definition 4.9. Thus, we obtain that for each color $i \in [t]$, there are at least $(1-\alpha)t$ points $v \in V$ such that $\{i, v\}$ has relative $(1,1)$-degree at least $1 - \delta + \gamma$, which implies

(F4) for $k \geq 4$. While for $k = 3$, by Definition 4.9, we have every $(1, 1)$-set has relative degree at least $1 - \delta + 2\gamma$ in G_i, which implies (F4) for $k = 3$.

Besides, it is obvious that (V6) implies (F5), we obtain an (α, γ, δ)-framework, as desired. $\square$

We define a *directed edge* in a k-graph to be a k-tuple whose vertices correspond to an underlying edge. Note that the directed edges $(a, b, c), (b, c, a)$ corresponds to the same underlying edge $\{a, b, c\}$. Given a k-graph system $\boldsymbol{H} = \{H_i\}_{i \in [n]}$ on vertex set V, we consider the hypergraph H with vertex set $[n] \cup V$ and edge set $\{\{i\} \cup e : e \in E(H_i), i \in [n]\}$. Define a directed edge to be a $(1, k)$-tuple $(i, v_1, \ldots, v_k)$ with k points corresponding to an underlying edge $\{v_1, \ldots, v_k\}$ in H_i. Given a k-tuple $\overrightarrow{S} = (v_1, \ldots, v_k)$, abbreviated as $v_1 \cdots v_k$, we use $\overrightarrow{S} \subseteq V$ to mean that the corresponding k-set of $\overrightarrow{S}$ is a subset of V. Similarly, given a family F of k-sets and a k-tuple $\overrightarrow{S}$, we use $\overrightarrow{S} \in F$ to denote that the corresponding k-set of $\overrightarrow{S}$ is an element of F. Let $\overrightarrow{S} = (v_1, \ldots, v_k)$, $\overrightarrow{S} \setminus \{v_i\}$ is the $(k-1)$-tuple $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ for $i \in [k]$, $\{v_i'\} \cup \overrightarrow{S} \setminus \{v_i\}$ is the k-tuple $(v_1, \ldots, v_{i-1}, v_i', v_{i+1}, \ldots, v_k)$.

Definition 4.16 (Strong connectivity) *A hypergraph is called strongly connected, if every two directed edges lie on a sequential walk.*

Claim 4.1 *If G is a tightly connected graph, then G is strongly connected.*

Proof. Let ab be a switcher in G, by Definition 4.7, we obtain that a and b share a neighbor c. If we can prove that (a, b) and (b, a) are on a walk W, then we can obtain that G is strongly connected. Since we consider any two directed edges D_1 and D_2 of G, there are walks W_1 and W_2 starting from D_1 and D_2 respectively and ending with $\{a, b\}$, $W_1 W W_2$ is a tight walk starting from D_1 and ending with D_2. While it is easy to see that $abcaba$ is a tight walk from (a, b) to (b, a) containing a closed walk of length 3, as desired. $\square$

Next, we want to show that switchers can control the length of sequential walks. Note that a triangle is a closed walk of odd length in a tightly connected graph containing a switcher and we obtain the following proposition.

Proposition 4.2 *If G is a tightly connected graph containing a switcher, then G has a closed walk of odd length.*

Proposition 4.3 *Let R be a $(1, k)$-graph with a subgraph $H_{\mathcal{C}_i}$ which is generated by $\mathcal{C}_i$. Suppose that $\mathcal{C}_i$ satisfies the conditions of Lemma 4.10, for any $(1, k-2)$-tuple $\overrightarrow{S} \in \partial_{k-2}(H_{\mathcal{C}_i})$ and two directed edges $D_1, D_2 \in C_{\overrightarrow{S}}$, there exists a sequential walk W of length 0 mod k in $H_{\mathcal{C}_i}$ starting from $\overrightarrow{S} D_1$ and ending with $\overrightarrow{S} D_2$.*

Proof. Let $\mathcal{C}_i = \{C_{\overrightarrow{S}} : \overrightarrow{S} \in \partial_{k-2}(R) \text{ and } i \in \overrightarrow{S}\}$ and $\overrightarrow{S} = \{i\} \cup \overrightarrow{S}'$ where $\overrightarrow{S}'$ is a $(k-2)$-tuple. By Proposition 4.2, there is a closed walk W_1 of odd length in $C_{\overrightarrow{S}}$. By Claim 4.1, there is a tight walk W_2 starting from D_1, ending with D_2

and containing W_1 as a subwalk. Let $\ell(W_2) = p$. We obtain W_3 from W_2 by replacing W_1 with the concatenation of $p+1$ mod 2 copies of W_1. Hence, W_3 is a tight walk of even length in $C_{\overrightarrow{S}}$ starting from D_1 and ending with D_2.

Suppose that $W_3 = (a_1, a_2, \ldots, a_{2m})$, we have $D_1 = (a_1, a_2)$ and $D_2 = (a_{2m-1}, a_{2m})$. Note that $(i \ldots i, \overrightarrow{S}' a_1 a_2 \overrightarrow{S}' a_3 a_4 \cdots \overrightarrow{S}' a_{2m-1} a_{2m})$ is a sequential walk in $H_{\mathcal{C}_i}$. Moreover, it has length 0 mod k, as desired. $\square$

Proposition 4.4 *Let R be a $(1,k)$-graph with a subgraph $H_{\mathcal{C}_i}$ that is generated by $\mathcal{C}_i$. Suppose $\mathcal{C}_i$ satisfies the conditions of Lemma 4.10, we consider directed edges $\overrightarrow{S}, \overrightarrow{T} \in \partial_{k-2}(H_{\mathcal{C}_i})$ and $D_1 \in C_{\overrightarrow{S}}$, $D_2 \in C_{\overrightarrow{T}}$. If $\overrightarrow{S}$ and $\overrightarrow{T}$ differ in exactly one coordinate, then there is sequential walk of length 0 mod k in $H_{\mathcal{C}_i}$ starting from $\overrightarrow{S} D_1$ and ending with $\overrightarrow{T} D_2$.*

Proof. Let $\overrightarrow{S} = (i, v_1 \ldots v_i \ldots v_{k-2})$ and $\overrightarrow{T} = (i, v_1 \ldots u_i \ldots v_{k-2})$ where $u_i \neq v_i$. By Definition 4.9, there is a directed edge D_3 in $C_{\overrightarrow{S}} \cap C_{\overrightarrow{T}}$, thus $(ii, \overrightarrow{S} \setminus \{i\} D_3 \overrightarrow{T} \setminus \{i\})$ is a sequential walk in $H_{\mathcal{C}_i}$. By Proposition 4.3, there is a sequential walk W_1 of length 0 mod k starting from $\overrightarrow{S} D_1$ and ending with $\overrightarrow{S} D_3$, W_2 of length 0 mod k starting from $\overrightarrow{T} D_3$ and ending with $\overrightarrow{T} D_2$, $(C(W_1)C(W_2), I(W_1)I(W_2))$ is the desired walk.

$\square$

Proposition 4.5 *Let R be a $(1,k)$-graph with a subgraph $H_{\mathcal{C}_i}$ that is generated by $\mathcal{C}_i$. Suppose $\mathcal{C}_i$ satisfies the conditions of Lemma 4.10, we consider directed edges $\overrightarrow{S}, \overrightarrow{T} \in \partial_{k-2}(H_{\mathcal{C}_i})$ and $D_1 \in C_{\overrightarrow{S}}$, $D_2 \in C_{\overrightarrow{T}}$. There is a sequential walk of length 0 mod k in $H_{\mathcal{C}_i}$ starting from $\overrightarrow{S} D_1$ and ending with $\overrightarrow{T} D_2$.*

Proof. Let $r \in [k-2]$ be the number of indices where $\overrightarrow{S}$ and $\overrightarrow{T}$ differ. If $r = 1$, the result follows from Proposition 4.4. Suppose the result is known for $r-1$. By Definition 4.9, there exists an edge pq in $C_{\overrightarrow{S}} \cap C_{\overrightarrow{T}}$.

Suppose that the ith coordinate vertex of $\overrightarrow{S}$ and $\overrightarrow{T}$ are different, which are replaced with p, we obtain $\overrightarrow{S}'$ and $\overrightarrow{T}'$. Note that $\overrightarrow{S}', \overrightarrow{T}' \in \partial_{k-2}(H_{\mathcal{C}_i})$. Choose $D_1' \in C_{\overrightarrow{S}'}$. By Proposition 4.4, there is a sequential walk W_1 of length 0 mod k from $\overrightarrow{S} D_1$ to $\overrightarrow{S}' D_1'$, similarly, there is a sequential walk W_3 of length 0 mod k from $\overrightarrow{T}' D_2'$ to $\overrightarrow{T} D_2$ where $D_2' \in C_{\overrightarrow{T}'}$. By induction, there is a sequential walk W_2 from $\overrightarrow{S}' D_1'$ to $\overrightarrow{T}' D_2'$ of length 0 mod k. Thus, $(C(W_1)C(W_2)C(W_3), I(W_1)I(W_2)I(W_3))$ is the desired walk. $\square$

Proof. [The proof of Lemma 4.10] Consider any two edges X and Y of $H_{\mathcal{C}_i}$. Let $X = S \cup A$ and $Y = T \cup B$ where $A \in C_S$ and $B \in C_T$. The desired walk can be obtained from Proposition 4.5.

Next, we need to show that $H_{\mathcal{C}_i}$ contains a closed walk of length 1 mod k. Since $\mathcal{C}_i$ admits an arc $\{i, v_1, \ldots, v_{k+1}\}$, by Proposition 4.5, there is a sequential walk W of length 0 mod k from $\{i, v_2, \ldots, v_{k+1}\}$ to $\{i, v_1, \ldots, v_k\}$. Thus, $(C(W)i, I(W)v_{k+1})$ is a closed walk of length 1 mod k. $\square$

The following claim can be seen in [105], we use a corollary of the claim in this paper.

Claim 4.2 ([105]) *Let H be a k-graph and $\mathbf{b} : V(H) \to [0,1]$. Suppose that there exists $m \le \sum_{v\in V(H)} \mathbf{b}(v)/k$ such that for every $v \in V(H)$, the link graph $L_H(\{v\})$ has a $\mathbf{b}$-fractional matching of size m, then H has a $\mathbf{b}$-fractional matching of size m.*

Corollary 4.1 *Let H be a k-graph, $\alpha \in [0,1)$ and $\mathbf{b} : V(H) \to [0,1]$. Suppose that there exists $m \le \sum_{v\in V(H)} \mathbf{b}(v)/k$ such that for all but at most $\alpha|V(H)|$ isolated vertices v, the link graph $L_H(\{v\})$ has a $\mathbf{b}$-fractional matching of size m, then H has a $\mathbf{b}$-fractional matching of size m.*

Proof. We first delete the isolated vertices of H and obtain a subgraph H' of H. Thus, $L_{H'}(\{v\})$ has a $\mathbf{b}$-fractional matching of size m. By Claim 4.1, we obtain that H' has a $\mathbf{b}$-fractional matching $\mathbf{w}$ of size m. Assign a weight $\mathbf{b}'(u) \in [0,1]$ to each isolated vertex u of H, and $\mathbf{b}'(v) = \mathbf{b}(v)$ for each non-isolated vertex v of H, it is obvious that H has a $\mathbf{b}'$-fractional matching $\mathbf{w}$ of size m since $\sum_{e\ni u} \mathbf{w}(e) = 0$ for any isolated vertex u and $E(H') = E(H)$. $\square$

Proposition 4.6 *Let R be a $(1,k)$-graph on $[n/k] \cup V$ where $|V| = n$, $\gamma > 0$, $\alpha \in [0,1)$, $\mathbf{b} : [n/k] \cup V \to [1-\gamma, 1]$. Suppose that there exists $m \le \sum_{v\in V(R)} \mathbf{b}(v)/(k+1)$ such that given $c \in [n/k]$, for all but at most αn vertices $v \in V$, the link graph $L_R(\{c,v\})$ has a $\mathbf{b}$-fractional matching of size m, then R has a $\mathbf{b}$-fractional matching of size m/k.*

Proof. By Corollary 4.1 with H being $L_R(\{c\})$ for $c \in [n/k]$, we obtain that $L_R(\{c\})$ has a $\mathbf{b}$-fractional matching of size m for $c \in [n/k]$.

Next, we want to construct a $\mathbf{b}$-fractional matching of size m/k for R. Let $\mathbf{w}_c : E(L_R(\{c\})) \to [0,1]$ such that $\sum_{v\in e, e\in L_R(\{c\})} \mathbf{w}_c(e) \le \mathbf{b}(v)$ where $\sum_{e\in L_R(\{c\})} \mathbf{w}_c(e) = m$. Let $\mathbf{w}(f) = \frac{1}{n}\mathbf{w}_c(e)$ for $e \in L_R(\{c\})$ and $f = e \cup \{c\}$, $c \in [n/k]$. Thus, we have $\sum_{f\in E(R)} \mathbf{w}(f) = \sum_{c\in[n/k]} \sum_{e\in L_R(\{c\})} \frac{1}{n}\mathbf{w}_c(e) = \frac{m}{k}$. It is easy to see that $\sum_{c\in f} \mathbf{w}(f) = \sum_{e\in L_R(\{c\})} \frac{1}{n}\mathbf{w}_c(e) = \frac{m}{n} \le \frac{1}{k} \le \mathbf{b}(c)$. And $\sum_{v\in f} \mathbf{w}(f) = \sum_{c\in[n/k]} \sum_{v\in e, e\in L_R(\{c\})} \frac{k}{n}\mathbf{w}_c(e) \le \sum_{c\in[n/k]} \frac{k}{n}\mathbf{b}(v) = \mathbf{b}(v)$ for $v \in V$. As desired. $\square$

We use the following results of [105] directly.

Proposition 4.7 ([105]) *Let H be a k-graph and $m \le v(H)/k$. If for every vertex v of $V(H)$, $L_H(\{v\})$ has a fractional matching of size m, then H has a fractional matching of size m.*

Proposition 4.8 *Let $d \in [k-2]$ and $\alpha, \gamma, \delta > 0, k \ge 3$ such that $\alpha, \gamma \ll 1/k$. Let R be a $(1,k)$-graph on $[t] \cup V$ with α-perturbed minimum $(1, k-2)$-degree δ where $|V| = t$. If for every $S \in \partial_d(R)$, the link graph $L(S)$ contains a fractional matching of size at least $(1+1/k)(1/(k+1)+\gamma)t$, then for every edge $S' \in \partial_1(R)$, the link graph $L(S')$ contains a fractional matching of size at least $(1+1/k)(1/(k+1)+\gamma)t$.*

Proof. We prove it by induction on d. Note that the base case when $d = 1$ is obvious. Suppose that given $d \in [2, k-2]$, we obtain the conclusion for $d' < d$. Let $S \subseteq V(R)$ be a $(1, d-1)$-set in $\partial_{d-1}(R)$. Consider any vertex s' in $\partial_1(L_R(S))$, $S \cup \{s'\}$ is an edge in $\partial_d(R)$. By assumption, $L_R(S \cup \{s'\})$ has a fractional matching of size at least $(1+1/k)(1/(k+1)+\gamma)t$, thus, we have $L_{R'}(\{s'\})$ contains a fractional matching of size at least $(1+1/k)(1/(k+1)+\gamma)t$ for any vertex s' of V where R' is the subgraph of $L_R(S)$ induced on the non-isolated vertices of $L_R(S)$.

By Definition 4.10, S has at most αt neighbors in $\overline{\partial_d(R)}$. It follows that $v(R') = \partial_1(L_R(S)) \geq (1-\alpha)t$ and $(1+1/k)(1/(k+1)+\gamma)t \leq v(R')/(k-d+1)$ since $\alpha, \gamma \ll 1/k$. By Proposition 4.7 with the condition that $L_{R'}(\{s'\})$ contains a fractional matching of size at least $(1+1/k)(1/(k+1)+\gamma)t$ for any vertex s' of V, we obtain R'(and thus $L_R(S)$) contains a fractional matching of size $(1+1/k)(1/(k+1)+\gamma)t$. Since S is arbitrary, for any $S \in \partial_{d-1}(R)$, $L_R(S)$ contains a fractional matching of size $(1+1/k)(1/(k+1)+\gamma)t$. Hence, we are done by the induction hypothesis.

□

Proof. [The proof of Lemma 4.11] Suppose that $V(H) = [t/k] \cup V'$ where $|V'| = t$. By assumption, C_S contains a fractional matching of size $(1+1/k)(1/(k+1)+\gamma)t$ for every $S \in \partial_{k-2}(H)$ and C_S is a subgraph of $L_H(S)$. By Proposition 4.8, we have $L_H(\{i, v\})$ contains a fractional matching of size $(1+1/k)(1/(k+1)+\gamma)t$ for every $\{i, v\} \in \partial_1(H)$.

We want to show that H is γ-robustly matchable. Given a vertex weight $\mathbf{b} : [t/k] \cup V' \to [1-\gamma, 1]$, we have to find a $\mathbf{b}$-fractional matching $\mathbf{w}$ such that $\sum_{e \ni v} \mathbf{w}(e) = \mathbf{b}(v)/k$ for any vertex $v \in V(H)$. That is, we need to find a $\mathbf{b}$-fractional matching with size $\sum_{v \in V(H)} \mathbf{b}(v)/k(k+1)$. Given $i \in [t/k]$, there are at most αt isolated $(1,1)$-tuples by Definition 4.10. For any non-isolated $(1,1)$-tuple (i, v) of $V(H)$, let $\mathbf{x}$ be a fractional matching in $L_H(\{i, v\})$ of size at least $(1+1/k)(1/(k+1)+\gamma)t$ and let $\mathbf{w}' = (1-\gamma)\mathbf{x}$, since $1-\gamma \leq \mathbf{b}(v)$ for any $v \in V(H)$, thus $\mathbf{w}'$ is a $\mathbf{b}$-fractional matching in $L_H(\{i, v\})$. Moreover $\mathbf{w}'$ has size at least $(1-\gamma)(1+1/k)(1/(k+1)+\gamma)t \geq (1+1/k)t/(k+1) \geq \sum_{v \in V(H)} \mathbf{b}(v)/(k+1)$ since $1/t \ll \gamma \ll 1/k$. We can assume that $\mathbf{w}'$ has size exactly $\sum_{v \in V(H)} \mathbf{b}(v)/(k+1)$. By Proposition 4.6, we obtain that H has a $\mathbf{b}$-fractional matching of size $\sum_{v \in V(H)} \mathbf{b}(v)/k(k+1)$, as desired. □

We use the following claim directly, which can be seen in [105].

Claim 4.3 ([105]) *Let t, d, k be integers with $d \in [k-1]$ and $\delta, \varepsilon, \alpha > 0$ with $1/t \ll \varepsilon \ll \alpha \leq \delta, 1/k$. Let R be a k-graph on t vertices with minimum relative d-degree $\overline{\delta_d(R)} \geq \delta$. Let I be a subgraph of R of edge density at most ε. Then there exists a vertex spanning subgraph $R' \subseteq R - I$ of α-perturbed minimum relative d-degree at least $\delta - \alpha$.*

The $(1, k)$-graph R_i on $\{i\} \cup V$ with minimum relative $(1, k-2)$-degree at least δ is equivalent to a k-graph R_i' on V with minimum relative $(k-2)$-degree at least δ. Thus, by Claim 4.3, we obtain Lemma 4.12.

4.4 . From sequentially Hamilton framework to sequentially Hamilton cycle

In this section, we use the following absorption lemma and almost cover lemma to prove Theorem 4.2. The proof of these two lemmas will be found in Section 8 and 9. Before we give these two lemmas, we need some definition.

Definition 4.17 (Extensible paths) *Let $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ be a $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup, G be a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$ and $c, \nu > 0$. A $(k-1)$-tuple A in V^{k-1} is said to be (c,ν)-extensible rightwards to an ordered edge $Y = (Y_0, Y_1, \dots, Y_k)$ in R if there exists a connection $S \subseteq [n] \cup V$ and a target set $T \subseteq \mathcal{J}_{(Y_2,\dots,Y_k)}$ with the following properties.*

- $|T| \geq \nu |\mathcal{J}_{(Y_2,\dots,Y_k)}|$,
- *for every $(v_2, \dots, v_k) \in T$, there are at least cm^{3k+1} many $(3k+1)$-tuples $(c_1, \dots, c_{2k}, w_1, \dots w_k, v_1)$ with $v_1 \in S \cap Y_1$, $w_i \in S \cap Y_i$ and $c_j \in Y_0$ for $i \in [k]$ and $j \in [2k]$ such that $(c_1 \dots c_{2k}, Aw_1 \dots w_k v_1 \dots v_k)$ is a sequential path in G.*

Given a sequential path P in a $(1,k)$-graph G and an ordered edge X in R, we say that P is (c,ν)-*extensible rightwards* to X if the $(k-1)$-tuple corresponding P's last $k-1$ vertices is (c,ν)-extensible rightwards to X. We call X as the *right extension*. We can define leftwards path extensions for $(k-1)$-tuples and for sequential paths in an analogous way (this time corresponding to the first $k-1$ vertices of P). A *connection set* of a sequential path is the union of the connection set of the initial $(k-1)$-tuple and the connection set of the end $(k-1)$-tuple.

Given that $X = (a,b,c)$ and $Y = (a,c,b)$, there is no guarantee that H contains a walk from X to Y. While if Y is a cyclic shift of X, that is, (b,c,a) or (c,a,b), then a walk from X to Y does exist. More generally, a *cyclic shift* of a k-tuple $(v_1, \dots, v_k)$ is any k-tuple of the form $(v_i, \dots, v_k, v_1, \dots, v_{i-1})$ for $i \in [k]$.

An orientation of a $(1,k)$-graph G on $[n] \cup V$ is a family of ordered $(1,k)$-tuples $\{\overrightarrow{e} \in [n] \times V^k : e \in E(G)\}$. We say that a family $\overrightarrow{G}$ of ordered $(1,k)$-tuples is an *oriented* $(1,k)$-*graph* if there exists a $(1,k)$-graph G such that $\overrightarrow{G} = \{\overrightarrow{e} \in [n] \times V^k : e \in E(G)\}$. Given an oriented $(1,k)$-graph $\overrightarrow{R}$, we say that $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{R})$ is an *oriented* $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-*regular setup* if $\overrightarrow{R}$ is an orientation of R and $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ is a $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup. Consider a $(1,k)$-graph G with an orientation $\overrightarrow{G}$ and vertex set $[n] \cup V$. Given an ordered k-tuple Y of distinct vertices in V and $c \in [n]$, we say that $\{c\} \cup Y$ is *consistent with* $\overrightarrow{G}$ if there exists an oriented edge $\{c\} \cup \overrightarrow{e} \in \overrightarrow{G}$ such that

$\overrightarrow{e}$ is a cyclic shift of Y. We say that an extensible path is *consistent with* $\overrightarrow{G}$ if its left and right extensions are consistent with $\overrightarrow{G}$. Finally, when considering multiple paths, we refer to the union of their connection sets as their *joint connection set*.

Let $\overrightarrow{G}$ be an orientation of a $(1,k)$-graph G. A sequential walk W in G is said to be *compatible* with $\overrightarrow{G}$ if each oriented edge of $\overrightarrow{G}$ appears at least once in W as a sequence of k consecutive vertices.

Let G be a $(1,k)$-graph on $[n]\cup V$ where $|V|=n$, and $S\subseteq V, O\subseteq [n], |O|=|S|=k$, P be a sequential path. Recall that $(C(P),I(P))$ is used to denote a sequential path where $C(P)$ is the color set of P and $I(P)$ is the point set of P. We say that P is (S,O)-*absorbing* in G if there exits a sequential path P' in G with the same initial $(k-1)$-tuple and the same terminal $(k-1)$-tuple with P, $I(P')=I(P)\cup S$ and $C(P')=C(P)\cup O$. We say that P is η-*absorbing* in G if it is (S,O)-absorbing in G for every S of size at most ηn divisible by k, any O of size $|S|$, and $S\cap I(P)=\emptyset, O\cap C(P)=\emptyset$.

Lemma 4.13 (Absorption lemma) *Let $k,r,m,t\in\mathbb{N}$ and $d_2,\dots,d_{k+1},\varepsilon,\varepsilon_{k+1}$, $\eta,\mu,\delta,\alpha,c,\nu,\lambda$ be such that*

$$\begin{aligned}1/m\ll 1/r,\varepsilon &\ll 1/t,c,\varepsilon_{k+1},d_2,\dots,d_k,\\ c&\ll d_2,\dots,d_k,\\ 1/t\ll \varepsilon_{k+1}&\ll d_{k+1},\nu\le 1/k,\\ c\ll \varepsilon_{k+1}&\ll\alpha\ll\eta\ll\lambda\ll\nu\ll\mu\ll\delta,1/k.\end{aligned}$$

Let $\mathbf{d}=(d_2,\dots,d_{k+1})$ and let $\mathfrak{S}=(G,G_{\mathcal{J}},\mathcal{J},\mathcal{P},\overrightarrow{H})$ be an oriented representative $(k,m,2t,\varepsilon,\varepsilon_{k+1},r,\mathbf{d})$-regular setup. Let G be $(1,k)$-graph on $[n]\cup V$ with minimum relative $(1,1)$-degree being at least $\delta+\mu$ where $|V|=n$, $n\le(1+\alpha)mt$. Suppose that there exists a sequentially closed walk which is compatible with the orientation $\overrightarrow{H}$ of H and

- *(F1) H_i is sequentially tightly connected,*
- *(F4) For every color $i\in[t]$, there are at least $(1-\alpha)t$ points $v\in V$ such that $\{i,v\}$ has relative $(1,1)$-degree at least $1-\delta+\gamma$.*

Then there exists a sequential path P in G such that the following holds.

- *(1) P is (c,ν)-extensible and consistent with $\overrightarrow{H}$,*
- *(2) $V(P)$ is λ-sparse in $\mathcal{P}$ and $V(P)\cap S=\emptyset$, where S denotes the connection set of P,*
- *(3) P is η-absorbing in G.*

Lemma 4.14 (Almost cover lemma) *Let $k,r,m,t\in\mathbb{N}$, $d_2,\dots,d_{k+1},\varepsilon,\varepsilon_{k+1}$, $\alpha,\gamma,c,\nu,\lambda$ be such that*

$$1/m\ll 1/r,\varepsilon\ll 1/t,c,\varepsilon_{k+1},d_2,\dots,d_k,$$

$$c\ll d_2,\dots,d_k,$$

$$1/t \ll \varepsilon_{k+1} \ll d_{k+1}, \nu, \alpha \leq 1/k,$$

$$\alpha \ll \eta \ll \lambda \ll \nu \ll \gamma.$$

*Let **d** $= (d_2, \ldots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r,$ **d**$)$-regular setup. Suppose that G is a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$ and $n \leq (1+\alpha)mt$, H is a $(1,k)$-graph on $[t] \cup V'$ where $|V'| = t$ and*

- *(F1) H_i is sequentially tightly connected,*
- *(F2) H_i contains a sequentially closed walk W compatible with $\overrightarrow{H}$ whose length is 1 mod k,*
- *(F3) H_{W_i} is γ-robustly matchable for $i \in [k]$,*
- *(F5) $L_H(\{i\})$ and $L_H(\{j\})$ intersect in an edge for each $i, j \in [t]$.*

Suppose that P is a sequential path in G such that

- *(1) P is (c, ν)-extensible and consistent with $\overrightarrow{H}$,*
- *(2) $V(P)$ is λ-sparse in $\mathcal{P}$ and $V(P) \cap S = \emptyset$ where S is the connection set of P,*

then there exists a sequential cycle C of length at least $(1-\eta)n$ which contains P as a subpath. Moreover, the number of uncovered points of V is divisible by k and the number of uncovered colors of $[n]$ has the same size with the number of uncovered points.

Proof. [The proof of Theorem 4.2] Let $\delta = rhf_{k-2}(k)$, $\mu > 0$ and

$$\varepsilon_{k+1} \ll \alpha \ll \eta \ll \lambda \ll \nu \ll \gamma \ll \mu,$$

$$1/t_0 \ll \varepsilon_{k+1} \ll d_{k+1} \ll \mu.$$

We apply Lemma 4.1 with input $\varepsilon_{k+1}, 1/t_0, r, \varepsilon$ to obtain t_1, m_0. Choose $c \ll 1/t_1$ and $1/n_0 \ll 1/t_1, 1/m_0, c, 1/r, \varepsilon$. Let G be a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$ and $2n \geq n_0$ vertices with $\overline{\delta}_{1,k-2}(G) \geq \delta + \mu$. Our goal is to prove that G contains a sequentially Hamilton cycle. By Lemma 4.1, there exists a representative $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, d_2, \ldots, d_{k+1})$-regular setup $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R_{d_{k+1}})$ with $t_0 \leq t \leq t_1$ and $n \leq (1+\alpha)mt$. Moreover, there is a $(1,k)$-graph I of edge density at most ε_{k+1} such that $R = R_{d_{k+1}} \cup I$ has minimum relative $(1, k-2)$-degree at least $\delta + \mu/2$. By Definition 4.5 and $\delta = rhf_{k-2}(k)$, we obtain that R contains an (α, γ, δ)-sequentially Hamilton framework H that avoids edges of I. Thus, $H \subseteq R_{d_{k+1}}$.

Next, we want to fix an orientation $\overrightarrow{H}$ and a compatible walk W. Since H is an (α, γ, δ)-sequentially Hamilton framework, H_i is sequentially tightly connected and has a sequentially closed walk of length 1 mod k, $L_H(\{i\})$ and $L_H(\{j\})$ intersect in an edge for each $i, j \in [t]$. We obtain a sequentially closed walk of length 1 mod k visiting all edges of H. Define an orientation $\overrightarrow{H} = \{\overrightarrow{e} \in V(H)^k : e \in H\}$ by choosing for every edge e of H, a k-tuple (or subpath) $\overrightarrow{e}$ in W which contains the vertices of e. Note that W is compatible with $\overrightarrow{H}$.

Firstly, we select a sequentially absorbing path P. Note that $1/t_1 \le d_2, \dots, d_k$, since $\mathcal{J}$ is a (t_0, t_1)-equitable complex. Since H is an (α, γ, δ)-sequentially Hamilton framework, it follows that there exists a sequential path P in G by Lemma 4.13 such that

1. P is (c, ν)-extensible and consistent with $\overrightarrow{H}$,
2. $V(P)$ is λ-sparse in $\mathcal{P}$ and $V(P) \cap T = \emptyset$, where T denotes the connection set of P,
3. P is η-absorbing in G.

Next, by Lemma 4.14, there is a sequential cycle A of length at least $(1-\eta)n$ which contains P as a subpath. Moreover, the number of uncovered points $|V \setminus I(A)|$ is divisible by k and the number of uncovered colors is of size $|[n] \setminus C(A)| = |V \setminus I(A)|$.

Finally, we absorb the uncovered points and colors into A. Note that $|V \setminus I(A)| \le \eta n$. Thus, there is a sequential path P' with point set $I(P) \cup (V \setminus I(A))$ and color set $C(P) \cup ([n] \setminus C(A))$, which has the same endpoints as P, as desired. □

Embedding sequential paths Given sequential walks W and W' with the property that the terminal $(k-1)$-tuple of W is identical to the initial $(k-1)$-tuple of W', we may *concatenate* W and W' to form a new sequential walk with color set $C(W) + C(W')$, which we denote $W + W'$.

Lemma 4.15 *Let k, r, n_0, t, B be positive integers and $\psi, d_2, \dots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, \nu$ be positive constants such that $1/d_i \in \mathbb{N}$ for $i \in [2, k]$ and such that $1/n_0 \ll 1/t$,*

$$\frac{1}{n_0}, \frac{1}{B} \ll \frac{1}{r}, \varepsilon \ll \varepsilon_{k+1}, d_2, \dots, d_k,$$

$$\varepsilon_{k+1} \ll \psi, d_{k+1}, \nu, \frac{1}{k}.$$

Then the following holds for all integers $n \ge n_0$.

Let G be a $(1, k)$-graph on $[n] \cup V$ where $|V| = n$, $\mathcal{J}$ be a $(\cdot, \cdot, \varepsilon, \varepsilon_{k+1}, r)$-regular slice for G on $[t] \cup V'$ where $|V'| = t$ with density vector $\mathbf{d} = (d_2, \dots, d_k)$. Let $\mathcal{J}_{W_i}$ be the induced subcomplex of $\mathcal{J}$ on $[t(i-1)/k+1, ti/k] \cup V'$ for $i \in [k]$. We call $[t]$ the family of color clusters and V' the family of point clusters. Let $R_{W_i} := R\left[[t(i-1)/k+1, ti/k] \cup V'\right]$ be the induced subgraph of $R := R_{d_{k+1}}(G)$. Let R_{W_i} be sequentially tightly connected for $i \in [k]$ and $\mathbf{w}_i$ be a fractional matching of size $\mu_i = \sum_{e \in E(R_{W_i})} \mathbf{w}_i(e)$ for $i \in [k]$ and $\mu_i(Z) = \sum_{Z \in e, e \in E(R_{W_i})} \mathbf{w}_i(e) \le 1/k$ for each cluster Z. Also, let X and Y be $(k-1)$-tuples of point clusters, S_X and S_Y be the subsets of $\mathcal{J}_X$ and $\mathcal{J}_Y$ of sizes at least $\nu|\mathcal{J}_X|$ and $\nu|\mathcal{J}_Y|$ respectively. Finally, let W be a sequential walk from X to Y of length at most t^{2k+1} in R_{W_i} and denote $\ell(W)$ by p. For $i \in [k]$, we have *(i) for any ℓ divisible by k with $4k \leq \ell \leq (1-\psi)\mu_i kn/t$, there is a sequential path P in G of length $\ell - 1 + \ell(W)(k+1)$ whose initial $(k-1)$-tuple belongs to S_X and whose terminal $(k-1)$-tuple belongs to S_Y,*

(ii) P uses at most $\mu_i(Z)n/t + B$ vertices from any point cluster $Z \in V'$ and at most $k\mu_i(C)n/t + B$ vertices from any color cluster $C \in [t(i-1)/k+1, ti/k]$ where $\mu_i(Z') = \sum_{Z' \in e, e \in R_{W_i}} \mathbf{w}_i(e)$ for any cluster Z'.

Proof. Let $\alpha = \psi/5$ and $\beta = 1/200$. When using Lemma 4.3, we require that $\varepsilon \ll c^2$ and choose m_0 to be large enough so that $m \geq \alpha m_0$ is acceptable for all these applications. Given t, let

$$n_0 = t \cdot \max(m_0, \frac{200k^2}{\varepsilon}, \frac{8k^2}{\alpha\sqrt{\varepsilon}}, \frac{10k(k+1)t^{2k+1}}{\alpha}). \tag{4.1}$$

We write $\mathcal{G}$ for the $(k+1)$-complex obtained from $\mathcal{J}_{W_i}$ by adding all edges of G supported on $\mathcal{J}_{W_i}^{(k)}$ as the '$(k+1)$th level' of $\mathcal{G}$. So for any edge $X = (X_0, X_1, \ldots, X_k) \in R_{W_i}$, $\mathcal{G}[\bigcup_{i \in [0,k]} X_i]$ is a $(d_2, \ldots, d_k, d^*(X), \varepsilon, \varepsilon_{k+1}, r)$-regular $(k+1)$-partite $(k+1)$-complex with $d^*(X) \geq d_{k+1}$.

Since $\mathcal{J}$ is a regular slice for G, for any $(1,k)$-set of clusters $X = \{X_0, X_1, \ldots, X_k\}$ in $\mathcal{J}_{W_i}$, the $(k+1)$-partite k-complex $\mathcal{J}_{W_i}[\bigcup_{j \in [0,k]} X_j]$ is $(\mathbf{d}, \varepsilon)$-regular. By adding all $(k+1)$-sets supported on $\hat{\mathcal{J}}_{W_i X}$ as the '$(k+1)$th level', we may obtain a $(d_2, \ldots, d_k, 1, \varepsilon, \varepsilon_{k+1}, r)$-regular $(k+1)$-partite $(k+1)$-complex, whose vertex clusters are subsets $Y_j \subseteq X_j$ for $j \in [0,k]$ of size $|Y_1| = \cdots = |Y_k| = \alpha m/k$ and $|Y_0| = \alpha m$. Y_0 can be seen as $\bigcup_{i \in [k]} Y_{0,i}$ where $|Y_{0,i}| = \alpha m/k$ for $i \in [k]$ and we obtain a $(d_2, \ldots, d_k, 1, \sqrt{\varepsilon}, \sqrt{\varepsilon_{k+1}}, r)$-regular by Lemma 4.7. We conclude by Lemma 4.5 that for any subset Y_i, $i \in [k-1]$ of distinct clusters of $\mathcal{J}$, each of size $\alpha m/k$, we have

$$|\mathcal{G}(Y_1, \ldots, Y_{k-1})| \geq \varepsilon m^{k-1}. \tag{4.2}$$

The following claim plays an important role in Lemma 4.15.

Claim 4.4 *Let $\{X_0, X_1, \ldots, X_k\}$ be an edge of R and choose any $Y_j \subseteq X_j$ for each $j \in [0,k]$ so that $|Y_0| = k|Y_1| = \cdots = k|Y_k| = \alpha m$. Let $\mathcal{P}$ be a collection of at least $\frac{1}{2}|\mathcal{G}(Y_1, \ldots, Y_{k-1})|$ sequential paths in G(not necessarily contained in $\bigcup_{j \in [k]} Y_j$) each of length at most $3k$ and whose terminal $(k-1)$-tuples are distinct members of $\mathcal{G}(Y_1, \ldots, Y_{k-1})$. Then for each $\sigma \in \{0,1\}$ there is a path $P \in \mathcal{P}$ and a collection $\mathcal{P}'$ of $\frac{9}{10}e(\mathcal{G}(Y_{\sigma+1}, \ldots, Y_{\sigma+k-1}))$ sequential paths in G, each of length $2k-1+\sigma$, all of whose initial $(k-1)$-tuples are the same (terminal $(k-1)$-tuple of P). Furthermore, the terminal $(k-1)$-tuples of paths in $\mathcal{P}'$ are distinct members of $\mathcal{G}(Y_{\sigma+1}, \ldots, Y_{\sigma+k-1})$. If $j \leq k-1$, then the jth vertex x of each path in $\mathcal{P}'$ lies in Y_j, if $j \geq k$, then x is not contained in P, and k new colors are not contained in P.*

Proof. Let $\sigma \in \{0,1\}$ be fixed, we take $\mathcal{H}$ to be the $(k+1)$-complex generated by the down-closure of a sequential path of length $2k-1+\sigma$ with vertex set $\{c_1,\dots,c_{k+\sigma}\} \cup \{v_1,\dots,v_{2k-1+\sigma}\}$ and consider its $(k+1)$-partition $V_0 \cup V_1 \cup \dots \cup V_k$ where $\{c_1,\dots,c_{k+\sigma}\} \subseteq V_0$ and the ith vertex of the path lies in the vertex class V_j with $j = i \bmod k$. We take $\mathcal{H}'$ to be the subcomplex of $\mathcal{H}$ induced by $\{v_1,\dots,v_{k-1},v_{k+1+\sigma},\dots,v_{2k-1+\sigma}\}$. Consider the pair (e,f), where e is an ordered $(k-1)$-tuple of $\mathcal{G}(Y_1,\dots,Y_{k-1})$ and f is an ordered $(k-1)$-tuple of $\mathcal{G}(Y_{\sigma+1},\dots,Y_{\sigma+k-1})$. For any such ordered $(k-1)$-tuple e, there are at most km^{k-2} such ordered $(k-1)$-tuples f which intersect e, thus there are at most $1/200$-proportion of the pairs (e,f) are not disjoint. On the other hand, if e and f are disjoint, then the down-closure of the pair (e,f) forms a labelled copy of $\mathcal{H}'$ in $\mathcal{G}[\bigcup_{j\in[0,k]} Y_j]$, so by Lemma 4.3 with $s = 3k+2\sigma-1$ and $s' = 2k-2$, for all but at most $1/200$-proportion of the disjoint pairs (e,f), there are at least $c(\alpha m/k)^{k+2\sigma+1} \geq \sqrt{\varepsilon}(\alpha m/k)^{k+2\sigma+1}$ extensions to copies of $\mathcal{H}$ in $\mathcal{G}[\bigcup_{j\in[0,k]} Y_j]$. Each such copy of $\mathcal{H}$ corresponds to a sequential path in G of length $2k-1+\sigma$ with all vertices in the desired clusters. We conclude that at least $99/100$-proportion of all pairs (e,f) of ordered $(k-1)$-tuples are disjoint and are linked by at least $\sqrt{\varepsilon}(\alpha m/k)^{k+2\sigma+1}$ sequential paths in G of length $2k-1+\sigma$, where $c_i \in V_0$ for $i \in [k+\sigma]$ and $v_\ell \in V_j$ with $j = \ell \bmod k$. We call these pairs *extensible*.

We call an ordered $(k-1)$-tuple $e \in \mathcal{G}(Y_1,\dots,Y_{k-1})$ *good* if at most $1/20$ of the ordered edges $f \in \mathcal{G}(Y_{\sigma+1},\dots,Y_{\sigma+k-1})$ do not make an extensible pair with e. Then at most $1/5$ of the ordered $(k-1)$-tuples in $\mathcal{G}(Y_1,\dots,Y_{k-1})$ are not good. Thus, there exists a path $P \in \mathcal{P}$ whose terminal $(k-1)$-tuple is a good ordered $(k-1)$-tuple e. Fix such a P and e, and any ordered $(k-1)$-tuple f in $\mathcal{G}(Y_{\sigma+1},\dots,Y_{\sigma+k-1})$ which is disjoint from P, suppose that (e,f) is an extensible pair, there are at least $\sqrt{\varepsilon}(\alpha m/k)^{k+2\sigma+1}$ sequential paths in G from e to f. We claim that at least one of these paths has the further property that if $j \geq k$, then the jth vertex is not contained in P and the $k+\sigma$ new colors are not contained in P, we can therefore put it in $\mathcal{P}'$. Indeed as f is disjoint from P, if $\sigma = 0$, then it suffices to show that one of these paths has the property that $v_k \in Y_k \setminus V(P)$ and $c_i \in Y_0 \setminus V(P)$ for $i \in [k]$. This is true because there are only at most $(2k+1)(\alpha m)^k + k(2k+1)(\alpha m/k)^k < \sqrt{\varepsilon}(\alpha m/k)^{k+1}$ paths which do not have this property by (4.1). If $\sigma = 1$, then we need a path whose kth and $(k+1)$st vertices are not in $V(P)$ and $c_i \in Y_0 \setminus V(P)$ for $i \in [k+1]$, which is possible since $2(2k+1)(\alpha m/k)^{k+2} + (k+1)(2k+1)(\alpha m/k)^{k+2} < \sqrt{\varepsilon}(\alpha m/k)^{k+3}$ by (4.1).

Finally, considering the ordered $(k-1)$-tuple $f \in \mathcal{G}(Y_{\sigma+1},\dots,Y_{\sigma+k-1})$, we have $20|V(P)|(k-1)(\alpha m/k)^{k-2} \leq \varepsilon m^{k-1} \leq e(\mathcal{G}(Y_{\sigma+1},\dots,Y_{\sigma+k-1}))$ by (4.1) and (4.2), at most $1/20$ of these $(k-1)$-tuples f intersect P and by the choice of e, at most $1/20$ of these $(k-1)$-tuples f are such that (e,f) is not extensible. This leaves at least $9/10$ of $(k-1)$-tuples f remaining, and choose a sequential path for each such f as described above gives the desired set $\mathcal{P}'$. $\square$

Let $X = (X_1, \ldots, X_{k-1})$, $Y = (Y_1, \ldots, Y_{k-1})$, X_k be the cluster following X in W and Y_k be the cluster preceding Y in W. Without loss of generality, we may assume that $\{X_0, X_1, \ldots, X_k\}$ is an edge of R_1 and $\{Y_0, Y_1, \ldots, Y_k\}$ is an edge of R_k. By the condition, we have S_X constitutes at least a ν proportion of $\mathcal{G}(X_1, \ldots, X_{k-1})$ and S_Y constitutes at least a ν proportion of $\mathcal{G}(Y_1, \ldots, Y_{k-1})$. Given any subsets $X'_j \subseteq X_j$ of size $\alpha m/k$ for $j \in [k]$ and $X'_0 \subseteq X_0$ of size αm, we say that a $(k-1)$-tuple $e \in \mathcal{G}(X_1, \ldots, X_{k-1})$ is *well-connected to* $(X'_1, \ldots, X'_{k-1})$ via X'_k and X'_0 if for at least $9/10$ of the $(k-1)$-tuples f in $\mathcal{G}(X'_1, \ldots, X'_{k-1})$, there exist distinct k-subsets $\{c_1, \ldots, c_k\}$, $\{f_1, \ldots, f_k\}$ of X'_0 and distinct $u, v \in X'_k$ such that $(c_1 \cdots c_k, e(u)f)$ and $(f_1 \cdots f_k, e(v)f)$ are sequential paths in G of length $2k-1$.

Claim 4.5 *For any subsets $X'_j \subseteq X_j$ of size $\alpha m/k$, $Z_j \subseteq X_j$ of size $\alpha m/k$ for $j \in [k]$ and $X'_0 \subseteq X_0$, $Z_0 \subseteq X_0$ of size αm such that each X'_j is disjoint from Z_j, the following statements hold.*

(1) At least $9/10$ of the $(k-1)$-tuples e in $\mathcal{G}(Z_1, \ldots, Z_{k-1})$ are well-connected to $(Z_1, \ldots, Z_{k-1})$ via Z_k and Z_0.

(2) At least $9/10$ of the $(k-1)$-tuples e in $\mathcal{G}(Z_1, \ldots, Z_{k-1})$ are well-connected to $(X'_1, \ldots, X'_{k-1})$ via X'_k and X'_0.

(3) At least $9/10$ of the $(k-1)$-tuples e in $\mathcal{G}(X'_1, \ldots, X'_{k-1})$ are well-connected to $(Z_1, \ldots, Z_{k-1})$ via X'_k and X'_0.

Proof. *From the proof of Claim 4.4, we know that all but at most $1/100$-proportion of pairs (e, f), where $e, f \in \mathcal{G}(Z_1, \ldots, Z_{k-1})$, are disjoint and are linked by at least $\sqrt{\varepsilon}(\alpha m/k)^{k+1}$ sequentially tight paths in G of length $2k-1$. It is obvious that at least $9/10$-proportion $(k-1)$-tuples of $\mathcal{G}(Z_1, \ldots, Z_{k-1})$ can be extended to at least $9/10$-proportion $(k-1)$-tuples of $\mathcal{G}(Z_1, \ldots, Z_{k-1})$ by at least $\sqrt{\varepsilon}(\alpha m/k)^{k+1}$ sequential paths. To prove (2), we apply Lemma 4.6 with $\mathcal{H}$ being the $(k+1)$-complex generated by the down-closure of a sequential path of length $2k-1$ and $\mathcal{H}'$ being the subcomplex induced by its initial and terminal $(k-1)$-tuples. We regard $\mathcal{H}$ as a $(2k)$-partite $(k+1)$-complex with k colors in the color cluster and one point in each point cluster. The role of $\mathcal{G}$ in Lemma 4.3 is the $(2k)$-partite subcomplex of $\mathcal{G}$ with vertex classes $X'_0, Z_1, \ldots, Z_{k-1}, X'_k, X'_1, \ldots, X'_{k-1}$, the colors of $\mathcal{H}$ are embedded in X'_0, the first point $\mathcal{H}$ is to be embedded in Z_1, the second one in Z_2, and so forth. By Lemmas 4.7 and 4.3, the proportion of pairs (e, f) for which there is no path as in (2) is at most $1/200$, and the remainder of the argument can be followed in (1). (3) can be proved similarly.* $\square$

We are ready to construct our path. Arbitrarily choose a subset $X_0^{(0)} \subseteq X_0$, $Z_0 \subseteq Y_0$ of size αm and $X_j^{(0)} \subseteq X_j$, $Z_j \subseteq Y_j$ of size $\alpha m/k$ for $j \in [k]$. By Theorem 4.2, Theorem 4.3, Theorem 4.5, there are at least $|S_X||\mathcal{G}(X_1^{(0)}, \ldots, X_{k-1}^{(0)})|/2$ pairs (e, f), where $e \in S_X$ and $f \in \mathcal{G}(X_1^{(0)}, \ldots, X_{k-1}^{(0)})$, can be extended to $\sqrt{\varepsilon}(\alpha m/k)^{k+1}$ sequential paths whose remaining point lies in $X_k^{(0)}$ and colors lie in $X_0^{(0)}$. Thus, we choose a $(k-1)$-tuple $P^{(0)}$ of S_X such that the following holds, there is a set $\mathcal{P}^{(0)}$ of sequential paths of the form $(c_1\cdots c_k, P^{(0)}(v)f)$ for $v \in X_k^{(0)}$, $c_1,\ldots,c_k \in X_0^{(0)}$ and $f \in \mathcal{G}(X_1^{(0)},\ldots,X_{k-1}^{(0)})$ for which the terminal $(k-1)$-tuples of paths in $\mathcal{P}^{(0)}$ are all distinct and constitute at least half of the ordered $(k-1)$-tuples of $\mathcal{G}(X_1^{(0)},\ldots,X_{k-1}^{(0)})$. Similarly, we can choose $e \in S_Y$ such that for at least half the members e' of $\mathcal{G}(Z_1,\ldots,Z_{k-1})$, there is a sequential path of length $2k-1$ in G from e' to e whose remaining point lies in Z_k and colors lie in Z_0.

We now construct the desired path. Since H_{W_i} is sequentially tightly connected, we can obtain $W = e_1\cdots e_s$ passing all edges of H_{W_i}. For each $i \in [s]$, let n_i be any integer with $0 \leq n_i \leq (1-3\alpha)\mathbf{w}(e_i)m$. Set the initial state to be 'filling the edge e_1', we proceed for $j \geq 1$ as follows,

★ The terminal $(k-1)$-tuple of the path family $\mathcal{P}^{(j)}$ constitute at least half of the ordered $(k-1)$-tuples $\mathcal{G}(X_1^{(j)},\ldots,X_{k-1}^{(j)})$.

Suppose that our current state is 'filling the edge e_i' for some i, if we have previously completed n_i steps in this state, then we do nothing and change the state to 'position 1 in traversing the walk W'. Otherwise, since ★ holds for $j-1$, we apply Claim 4.4 with $\sigma = 0$ to obtain a path $P \in \mathcal{P}^{(j-1)}$ and a collection $\mathcal{P}^{(j)}$ of $\frac{9}{10}e(\mathcal{G}(X_1^{(j-1)},\ldots,X_{k-1}^{(j-1)}))$ sequential paths of length $2k-1$, all of whose initial $(k-1)$-tuples are the same (the terminal $(k-1)$-tuple of P) and whose terminal $(k-1)$-tuples are distinct numbers of $\mathcal{G}(X_1^{(j-1)},\ldots,X_{k-1}^{(j-1)})$ and are disjoint from $V(P)$, whose colors lie in $X_0^{(j-1)} \setminus C(P)$, and whose remaining vertex lies in $X_k^{(j-1)} \setminus V(P)$. We define $P^{(j)}$ to be the concatenation $P^{(j-1)} + P$ with color classes $C(P^{(j-1)}) \cup C(P)$. For $p \in [0,k]$, we generate $X_p^{(j)}$ from $X_p^{(j-1)}$ by removing the vertices of $P^{(j)}$ in $X_p^{(j)}$ and replacing them by vertices from the same cluster which do not lie in Z or in $P^{(j)}$. We will prove that this is possible in Claim 4.6.

Now suppose that our current state is 'position q in traversing the walk W'. Since ★ holds for $j-1$, applying Claim 4.4 with $\sigma = 1$ to obtain a path $P \in \mathcal{P}^{(j-1)}$ and a collection $\mathcal{P}^{(j)}$ of $\frac{9}{10}e(\mathcal{G}(X_1^{(j-1)},\ldots,X_{k-1}^{(j-1)}))$ sequential paths of length $2k$, all of whose initial $(k-1)$-tuples are the same (the terminal $(k-1)$-tuple of P) and whose terminal $(k-1)$-tuples are distinct numbers of $\mathcal{G}(X_2^{(j-1)},\ldots,X_k^{(j-1)})$ and are disjoint from $V(P)$, and whose two remaining vertices lie in $X_k^{(j-1)} \setminus V(P)$ and $X_1^{(j-1)} \setminus V(P)$ respectively with colors in $X_0^{(j-1)} \setminus C(P)$. Exactly as before we define $P^{(j)}$ to be the concatenation $P^{(j-1)} + P$. We generate $X_p^{(j)}$ from $X_{p+1}^{(j-1)}$ for $p \in [0,k-1]$ by removing the vertices of $P^{(j-1)}$ in $X_{p+1}^{(j-1)}$ and replacing them by vertices from the same cluster do not lie in Z or $P^{(j)}$. If we have not reached the end of W, we choose $X_k^{(j)}$ to be a subset of the cluster at position $q+k$ in the sequence of W such that $X_k^{(j)}$ is disjoint from $P^{(j)} \cup Z$. In this case, we change our state to 'position $q+1$

in traversing W'. Alternatively, if we have reached the end of W, meaning that the $(k-1)$-tuple of clusters containing $X_1^{(j)}, \ldots, X_{k-1}^{(j)}$ is $(Y_1 \ldots, Y_{k-1})$, then we choose $X_k^{(j)}$ to be a subset of Y_k which has size $\alpha m/k$ and is disjoint from $P^{(j)} \cup Z$. We may choose a path $P \in \mathcal{P}^{(j-1)}$ such that the terminal $(k-1)$-tuple $f \in G(X_1^{(j)}, \ldots, X_{k-1}^{(j)})$ of P is well-connected to $(Z_1, \ldots, Z_{k-1})$ via Z_k and Z_0. This implies that we may choose a $(k-1)$-tuple e' in $\mathcal{G}(Z_1, \ldots, Z_{k-1}), v, v'$ in Z_k with new colors C^*, C^{**} in Z_0 with $|C^*| = |C^{**}| = k$ such that $(C^*, f(v')e')$ is a sequential path Q' and $(C^{**}, e'(v)e)$ is a sequential path Q. Return $P^{(j)}+Q'+Q$ as the output sequential path in G. Note that an edge may appear multiple times. When it first appears in the walk, the process executes 'filling the edge'. When it appears later, 'filling the edge' is no longer needed. Again we prove Claim 4.6 that these choices are all possible.

Claim 4.6 *The algorithm described above is well-defined(that is, it is always possible to construct the sets $X_p^{(j)}$), maintains ★ and returns a sequential path of length*

$$4k - 1 + \left(\sum_{i \in [s]} n_i\right) \cdot k + \ell(W) \cdot (k+1).$$

Proof.

We prove that ★ is maintained, recall that $e(\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)})) \geq \varepsilon m^{k-1}$ for each j. Fixing some j, for either $A_p := X_p^{(j-1)}$ or $A_p := X_{p+1}^{(j-1)}$, we obtain sets $A_1, \ldots, A_{k-1}$, each with size αm such that the terminal $(k-1)$-tuples of $\mathcal{P}^{(j)}$ constitute at least $9/10$ of the ordered edges of $\mathcal{G}(A_1, \ldots, A_{k-1})$ and for each $i \in [k-1]$, $X_i^{(j)}$ is formed from A_i by removing at most two vertices and replacing them with the same number of vertices. Since each vertex is in at most m^{k-2} ordered $(k-1)$-tuples of either $\mathcal{G}(A_1, \ldots, A_{k-1})$ or $\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)})$, we conclude that the fraction of ordered $(k-1)$-tuples of $\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)})$ which are the terminal $(k-1)$-tuples of paths in $\mathcal{P}^{(j)}$ is at least

$$\begin{aligned}
&\frac{\frac{9}{10}e(\mathcal{G}(A_1, \ldots, A_{k-1})) - 2(k-1)m^{k-2}}{e(\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)}))} \\
&\geq \frac{\frac{9}{10}(e(\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)})) - 2(k-1)m^{k-2}) - 2(k-1)m^{k-2}}{e(\mathcal{G}(X_1^{(j)}, \ldots, X_{k-1}^{(j)}))} \\
&\geq \frac{9}{10} - \frac{4(k-1)m^{k-2}}{\varepsilon m^{k-1}} \geq \frac{1}{2},
\end{aligned} \tag{4.3}$$

where the last equality holds since $m \geq m_0 \geq 16(k-1)/\varepsilon$. Thus, we obtain ★.

To prove that we can always construct the set $X_p^{(j)}$, observe that it is enough to check that at termination every cluster still have at least $2\alpha m$ vertices not in $P^{(j)}$, as then there are at least αm vertices outside Z. In each walk-traversing step, each path in $\mathcal{P}^{(j)}$ contains precisely $k+1$ new points and $k+1$ new colors and the total number of walk-traversing steps is precisely $\ell(W)$. Recall that this number is at most t^{2k+1}, we have $(k+1)t^{2k+1} < \frac{\alpha m}{2k}$ and $(k+1)^2 t^{2k+1} < \frac{\alpha m}{2}$ by (4.1). When we are in the state 'filling the edge e_i', we have n_i steps and in each step, each path in $\mathcal{P}^{(j)}$ contains k new points, one from each cluster of $e_i \setminus C(e_i)$ and k new colors from $C(e_i)$. So for any color cluster C, the number of whose vertices which are added to $P^{(j)}$ is at most $\sum_{i:C\in e_i} kn_i \leq \sum_{i:C\in e_i}(1-3\alpha)k\mathbf{w}(e_i)m \leq (1-3\alpha)m$. And for any point cluster X, the number of whose vertices which are added to $P^{(j)}$ is at most $\sum_{i:X\in e_i} n_i \leq \sum_{i:X\in e_i}(1-3\alpha)\mathbf{w}(e_i)m \leq (1-3\alpha)m/k$. Together with e and the k vertices of the chosen path in $\mathcal{P}^{(0)}$, we conclude that there are at most $(1-2\alpha)m$ vertices of any color cluster and at most $(1-2\alpha)m/k$ vertices of any point cluster contained in $P^{(j)}$ at termination.

Finally, the length of the path is equal to the number of points. Recall that $P^{(0)}$ contains $k-1$ points. Next, k points and k colors are added from $P^{(0)}$ to form $P^{(1)}$. Each of the $\sum_{i\in[s]} n_i$ edge-filling steps resulted in k new points and k new colors being added to $P^{(j)}$ and each of the $\ell(W)$ walk-traversing steps resulted in $k+1$ new points and $k+1$ new colors being added to $P^{(j)}$. When completing the path, we need $2k$ points which are not in the final paths $P^{(j)}$ (v, v', e and e'). Thus, the final path has length

$$(k-1)+k+\left(\sum_{i\in[s]} n_i\right)\cdot k+\ell(W)\cdot(k+1)+2k.$$

□

We obtain the shortest sequential path by never entering the state 'filling an edge', in which case we can obtain a sequential path of length $4k-1+\ell(W)(k+1)$. On the other hand, by extending W to include all edges of R_{W_i}, we take n_i to be $(1-\psi)\mathbf{w}(e_i)m$ for each $i\in[s]$. We can obtain a sequential path of length at least $(1-\psi)\mu_i kn/t$, with using at most $k\mu_i(C)n/t+B$ vertices from any color cluster C in R_{W_i} and at most $\mu_i(X)n/t+B$ where $\mu_i(Z) = \sum_{Z\in e, e\in R_{W_i}} \mathbf{w}_i(e)$ for $i\in[k]$ and $B = B(t,k)$. By choosing n_i appropriately, we can obtain tight cycles of certain length between two extremes. □

Similarly with Lemma 4.15, we can obtain the following lemma.

Lemma 4.16 *Let k, r, n_0, t, B be positive integers and $\psi, d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, \nu$ be positive constants such that $1/d_i \in \mathbb{N}$ for $i\in[2,k]$ and such that $1/n_0 \ll 1/t$,*

$$\frac{1}{n_0} \ll \frac{1}{t} \ll \frac{1}{B} \ll \frac{1}{r}, \varepsilon \ll \varepsilon_{k+1}, d_2, \ldots, d_k,$$

$$\varepsilon_{k+1} \ll \psi, d_{k+1}, \nu, \frac{1}{k}.$$

Then the following holds for all integers $n \geq n_0$.

Let G be a $(1,k)$-graph on $[n] \cup V$ where $|V| = n$, $\mathcal{J}$ be a $(\cdot,\cdot,\varepsilon,\varepsilon_{k+1},r)$-regular slice for G on $[t] \cup V'$ where $|V'| = t$ with density vector $\boldsymbol{d} = (d_2,\dots,d_k)$. Let $\mathcal{J}_{W_i}$ be the induced subcomplex of $\mathcal{J}$ on $[t(i-1)/k+1, ti/k] \cup V'$ for $i \in [k]$. Let $R_{W_i} := R\left[[t(i-1)/k+1, ti/k] \cup V'\right]$ be the induced subgraph of $R := R_{d_{k+1}}(G)$. Let R_{W_i} be sequentially tightly connected for $i \in [k]$ and $\boldsymbol{w}_i$ be a fractional matching of size $\mu_i = \sum_{e \in E(R_{W_i})} \boldsymbol{w}_i(e)$ for $i \in [k]$ with $\mu_i(Z) \leq 1/k$ for each cluster Z and $i \in [k]$. Also, let X and Y be $(k-1)$-tuples of point clusters, S_X and S_Y be the subsets of $\mathcal{J}_X$ and $\mathcal{J}_Y$ of sizes at least $\nu|\mathcal{J}_X|$ and $\nu|\mathcal{J}_Y|$ respectively. Finally, let W be a sequential walk traversing all edges of each H_{W_i} from X to Y of length at most t^{2k+1} and denote $\ell(W)$ by p. For $i \in [k]$, we have

1. *for any ℓ divisible by k with $4k \leq \ell \leq (1-\psi)\sum_{i \in [k]} \mu_i kn/t$, there is a sequential path P in G of length $\ell - 1 + \ell(W)(k+1)$ whose initial $(k-1)$-tuple belongs to S_X and whose terminal $(k-1)$-tuple belongs to S_Y,*
2. *P uses at most $\sum_{i \in [k]} \mu_i(Z)n/t + B$ vertices from any point cluster $Z \in V'$ and at most $k\mu_i(C)n/t + B$ vertices from any color cluster $C \in [t]$ where $\mu_i(Z') = \sum_{Z' \in e, e \in R_{W_i}} \boldsymbol{w}_i(e)$ for any cluster Z'.*

Connecting Let us begin with the existence of extensible paths. The following proposition states that most tuples in the complex induced by an edge of the reduced graph of a regular slice also extend to that edge.

Proposition 4.9 *Let $k,m,t,r \in \mathbb{N}$ and $\varepsilon, \varepsilon_{k+1}, d_2, \dots, d_{k+1}, \beta, c, \nu$ be such that*

$$1/m \ll 1/r, \varepsilon \ll c \ll \varepsilon_{k+1}, d_2, \dots, d_k,$$

$$\varepsilon_{k+1} \ll \beta \ll d_{k+1}, \nu.$$

Let $\boldsymbol{d} = (d_2,\dots,d_{k+1})$ and let $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ be a $(k,m,2t,\varepsilon,\varepsilon_{k+1},r,\boldsymbol{d})$-regular setup. Let $Y = (Y_0, Y_1, \dots, Y_k)$ be an ordered edge in R, then all but at most $\beta|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|$ many tuples $(v_1,\dots,v_{k-1}) \in \mathcal{J}_{(Y_1,\dots,Y_{k-1})}$ are (c,ν)-extensible both left and rightwards to Y.

Proof. Let $P = (c_1,\dots,c_{2k},v_1,\dots,v_{3k-1})$ be a sequential path. Partition its vertex set in $k+1$ clusters $X_0, X_1, \dots, X_k$ such that $X_0 = \{c_1,\dots,c_{2k}\}$, and $X_i = \{v_j : j = i \bmod k\}$ for $i \in [k]$. Thus, P is a $(k+1)$-partite $(k+1)$-graph.

Let $\mathcal{H}$ be the down-closure of the path P, which is a $(k+1)$-partite $(k+1)$-complex. Let $V_1 = \{v_1,\dots,v_{k-1}\}$ and $V_2 = \{v_{2k+1},\dots,v_{3k-1}\}$. Let $\mathcal{H}'$ be the induced subcomplex of $\mathcal{H}$ on $V_1 \cup V_2$. Thus, $\mathcal{H}'$ is a k-partite $(k-1)$-complex on $2k-2$ points. Let $\mathcal{G} = \mathcal{J} \cup G_{\mathcal{J}}$.

Let $\mathcal{H}'_{\mathcal{G}}$ be the set of labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{G}$. It follows that

$$|\mathcal{H}'_{\mathcal{G}}| = (1 \pm \varepsilon_{k+1})|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|^2, \tag{4.4}$$

where the error term accounts for the fact that we do not count the intersecting pairs of $(k-1)$-tuples in $\mathcal{J}_{(Y_1,\dots,Y_{k-1})}$. Since Y is an edge of R, any function $\phi : V(P) \to V(R)$ such that $\phi(X_i) \subseteq Y_i$ is a homomorphism. By Lemma 4.3 with β^2 playing the role of β, we deduce that all but at most $\beta^2|\mathcal{H}'_{\mathcal{G}}|$ of labelled partition-respecting copies of $\mathcal{H}'$ in $\mathcal{G}$ extend to at least cm^{3k+1} labelled partition-respecting copies of $\mathcal{H}$ in $\mathcal{G}$, since $c \ll d_2, \dots, d_{k-1}$. For each $e \in \mathcal{J}_{(Y_1,\dots,Y_{k-1})}$, let $T(e)$ be the number of tuples e' in $\mathcal{J}_{(Y_1,\dots,Y_{k-1})}$ such that $e \cup e'$ can be extended to at least cm^{3k+1} copies of $\mathcal{H}$ in $\mathcal{G}$, We have

$$\sum_{e \in \mathcal{J}_{(Y_1,\dots,Y_{k-1})}} T(e) \geq (1-2\beta^2)|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|^2. \tag{4.5}$$

Let $S \subseteq \mathcal{J}_{(Y_1,\dots,Y_{k-1})}$ be the set of $(k-1)$-tuples e which is not (c,ν)-extensible leftwards to Y, that is $T(e) < \nu|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|$. Combining with (4.5) and $\beta \ll \nu$, we have

$$\sum_{e \in \mathcal{J}_{(Y_1,\dots,Y_{k-1})}} T(e) \leq |S| \cdot \nu|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}| + (|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}| - |S|)|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|,$$

furthermore, we have

$$|S| \leq \frac{2\beta^2}{1-\nu}|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}| \leq \frac{\beta}{2}|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|.$$

A symmetric fact shows that all but at most $\frac{\beta}{2}|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|$ $(k-1)$-tuples in $\mathcal{J}_{(Y_1,\dots,Y_{k-1})}$ are not (c,ν)-extensible rightwards to Y. Thus, all but at most $\beta|\mathcal{J}_{(Y_1,\dots,Y_{k-1})}|$ pairs in $\mathcal{J}_{(Y_1,\dots,Y_{k-1})}$ are not (c,ν)-extensible both left and rightwards to Y. □

In Proposition 4.9, we know that most tuples in the complex induced by an edge of the reduced graph of a regular slice also extend to that edge. The following lemma allows us to connect up two extensible paths using either very few or quite a lot of vertices.

Lemma 4.17 *Let $k, r, m, t \in \mathbb{N}$, and $d_2, \dots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \lambda$ be such that*

$$1/m \ll 1/r, \varepsilon \ll c \ll \varepsilon_{k+1}, d_2, \dots, d_k,$$

$$\lambda \ll \nu \ll 1/k,$$

$$\varepsilon_{k+1} \ll d_{k+1}.$$

Let $\boldsymbol{d} = (d_2, \dots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, H)$ be a $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$-regular setup where $\mathcal{P}$ has an initial partition of $[n] \cup V$ and H is a $(1,k)$-graph on $[t] \cup V'$. Suppose that $H_{W_i} = H[[t(i-1)/k, ti/k] \cup V']$ and H_{W_i} is sequentially tightly connected for $i \in [k]$. Let P_1, $P_2 \subseteq G$ be (c,ν)-extensible paths such that P_1 extends rightwards to X and P_2 extends leftwards to Y. Suppose that P_1 and P_2 are either identical or disjoint, let W be a sequential walk traversing each H_{W_i}

of length at most t^{2k+1} that starts from X and ends with Y. Let T be the joint connection set of P_1 and P_2. Suppose that T and $S \subseteq V(G)$ are λ-sparse in $\mathcal{P}$, $V(P_1) \cup V(P_2) \subseteq S$ and $T \cap S = \emptyset$, then

(1) there is a sequential path Q of length $4k-1+(\ell(W)+2)(k+1)$ in $G[V(\mathcal{P})]$ such that P_1QP_2 is a sequential path, containing no vertices of S and exactly $6k+2$ vertices of T,

*(2) consider ψ with $\varepsilon_{k+1} \ll \psi$, let **w** be a fractional matching of size $\mu = \sum_{i\in[k]} \sum_{e\in E(H_{W_i})} \mathbf{w}_i(e) \geq 5/m$ such that $\sum_{Z\in e, e\in H_{W_i}} \mathbf{w}_i(e) \leq (1-2\lambda)/k$ for each $Z \in \mathcal{P}$. There is a sequential path Q of length $\ell(W)+1$ mod k in $G[V(\mathcal{P})]$ such that P_1QP_2 is a sequential path, containing no vertices of S and exactly $6k+2$ vertices of T. Moreover, there is a set $U \subseteq V(\mathcal{P})$ of size at most ψmt such that $U \cup V(Q)$ has exactly $\lceil \sum_{i\in[k]} \sum_{Z\in e, e\in H_{W_i}} \mathbf{w}_i(e) m\rceil + B$ vertices in each point cluster Z.*

Proof. Let $X = (X_0, X_1, \dots, X_k)$, since P_1 extends rightwards to X, thus there exists a target set $T_1 \subseteq \mathcal{J}_{(X_2,\dots,X_k)}$ of size $|T_1| \geq \nu|\mathcal{J}_{(X_2,\dots,X_k)}|$ such that for every $(v_2, \dots, v_k) \in T_1$, there are at least cm^{3k+1} many $(3k+1)$-tuples $(c_1, \dots, c_{2k}, w_1, \dots, w_k, v_1)$ with $c_i \in T \cap X_0$ for $i \in [2k]$, $w_i \in T \cap X_i$ for $i \in [k]$ and $v_1 \in T \cap X_1$ such that $((c_1, \dots, c_{2k}), P_1(w_1, \dots, w_k, v_1, \dots, v_k))$ is a sequential path. Let $Y = (Y_0, Y_1, \dots, Y_k)$, P_2 extends leftwards to Y with target set $T_2 \subseteq \mathcal{J}_{(Y_2,\dots,Y_k)}$.

For each $Z \in \mathcal{P}$, let $Z' \subseteq Z \setminus (S \cup T)$ of size $m' = (1-2\lambda)m$ since S and T are λ-sparse. Let $\mathcal{P}' = \{Z'\}_{Z\in\mathcal{P}}$, $G' = G[V(\mathcal{P}')]$ and $\mathcal{J}' = \mathcal{J}[V(\mathcal{P}')]$. By lemma 4.7, $\mathfrak{S}' := (G', G'_{\mathcal{J}}, \mathcal{J}', \mathcal{P}', H)$ is a $(k, m', 2t, \sqrt{\varepsilon}, \sqrt{\varepsilon_{k+1}}, r, \mathbf{d})$-regular setup.

For (2), let $\mu' = \mu/(1-2\lambda)$ be the scaled size of **w** and $B \in \mathbb{N}$ such that $1/B \ll 1/r, \varepsilon$. Let ℓ be the largest integer divisible by k with $4k \leq \ell \leq (1-\psi/4)\mu' m' k$. Note that such an ℓ exists since $(1-\psi/4)\mu' m' \geq 4$, where the latter inequality follows from $\mu \geq 5/m$. Applying Lemma 4.16 with $G', \mathcal{J}', W, \ell, \mathbf{w}, \mu'$ and T_1, T_2, we obtain a sequential path Q' whose initial $(k-1)$-tuple belongs to T_1 and whose terminal $(k-1)$-tuple belongs to T_2. Furthermore, Q' has length $\ell - 1 + \ell(W)(k+1)$ and uses at most $\sum_{i\in[k]} \mu_i(Z)m + B$ vertices from any point cluster Z where $\mu_i(Z) = \sum_{Z\in e, e\in H_{W_i}} \mathbf{w}_i(e)$ and $B \ll \psi\mu mk$. Note that $\ell \geq (1-\psi/4)\mu km - k$, it follows that

$$\begin{aligned}
&\sum_{Z\in V'} \sum_{i\in[k]} \mu_i(Z)m - \sum_{Z\in V'} |V(Q') \cap Z| \\
&\leq \mu km - (1-\frac{\psi}{4})\mu km + k + 1 - \ell(W)(k+1) \\
&\leq \frac{\psi}{4}\mu km + k + 1 \\
&\leq \frac{\psi}{4}(1-2\lambda)tm + k + 1 \leq \frac{\psi}{2}mt.
\end{aligned}$$

Hence, there is a set $U \subseteq V(\mathcal{P})$ of size at most ψmt such that $U \cup V(Q')$ has $\lceil \sum_{i\in[k]} \mu_i(Z)m\rceil + B$ vertices from any point cluster $Z \in V'$.

For (1), we can choose a path Q' in the same way. The only difference is that in this case **w** is a single edge of weight 1 and $\ell = 4k$. Hence, Q' is a path of length $4k - 1 + \ell(W)(k+1)$.

Finally, we use the above extensible paths to choose $c_1, \dots, c_{k+1}, w_1, \dots, w_k, v_1$ and $f_1, \dots, f_{k+1}, v'_k, w'_1, \dots, w'_k$ in T such that for

$$Q = ((c_1, \dots, c_{k+1})C(Q')(f_1, \dots, f_{k+1}), (w_1, \dots, w_k, v_1)Q'(v'_k, w'_1, \dots, w'_k)),$$

the concatenation $P_1 Q P_2$ is a sequential path and Q is disjoint from S, since $V(S) \cap T = \emptyset\, T \cap V(Q') = \emptyset$. It is obvious that the length of Q in (1) is $4k - 1 + (\ell(W) + 2)(k+1)$ and the length of Q in (2) is $\ell(W) + 1$ mod k.

□

Proposition 4.10 *Let W be a sequential walk in a $(1,k)$-graph H on $[t] \cup V'$ which starts from $(1,k)$-tuple X and ends with $(1,k)$-tuple Y where $|V'| = t$. There exists a sequential walk W' of length at most kt^{k+1}, which starts from X and ends with Y. Moreover, $\ell(W') = \ell(W)$ mod k.*

Proof. Suppose that $\ell(W) = j$ mod k for a $j \in [0, k-1]$. Let W' be a vertex-minimal sequentially tightly walk from X to Y of size j mod k. Our goal is to show that every $(1,k)$-tuple repeats at most k times in W'.

Assume that W' contains $k+1$ copies of the same $(1,k)$-tuple Z and denote by n_j the position in W' where the jth repetition Z begins. It is obvious that $n_j - n_1 \not\equiv 0$ mod k, otherwise it is contrary to the minimal of W'. By the pigeonhole principle, there exist two indices j, j' such that $n_j - n_1 \equiv n_{j'} - n_1$ mod k for $1 \le j < j' \le k+1$. That is, $n_j - n_{j'} \equiv 0$ mod k. We can also reduce the length of W' by deleting the vertices between n_j and $n_{j'} - 1$, a contradiction. □

Proposition 4.11 *Let $j, k, t \in \mathbb{N}$ with $j \in [k]$. Let W be a sequentially closed walk that is compatible with respect to an orientation $\overrightarrow{H}$ of a $(1,k)$-graph H on $[t] \cup V'$ where $|V'| = t$. Let X_1 and X_2 be consistent with $\overrightarrow{H}$. There exists a sequential walk W' of length at most kt^{k+1}, which starts from X_1 and ends with X_2. Moreover, if W has length 1 $\mathrm{mod}\ k$, then W' has length j $\mathrm{mod}\ k$.*

Proof. For the first part, by Proposition 4.10, it suffices to show that there is a sequential walk starting from X_1 and ending with X_2. Since X_1 is consistent with $\overrightarrow{H}$, there is a sequential path W_{X_1} of length at most $k-1$ from X_1 to X'_1 in H where X'_1 is an oriented edge in $\overrightarrow{H}$ which is a cyclic shift of X_1. Similarly, there is a sequential path W_{X_2} of length at most $k-1$ from X_2 to X'_2 in H where X'_2 is an oriented edge in $\overrightarrow{H}$ which is a cyclic shift of X_2. Since W is compatible with respect to an orientation $\overrightarrow{H}$, there is a subwalk $W_{X'_1 X'_2} \subseteq W$ starting from X'_1 and ending with X'_2, hence $(C(X_1)C(W_{X_1})C(W_{X'_1 X'_2})C(W_{X_2})$ $C(X_2), I(X_1)I(W_{X_1})I(W_{X'_1 X'_2})I(W_{X_2})I(X_2))$ is the desired W'.

Note that we choose $W_{X_1'X_2'}$ such that W' has length j mod k by extending $W_{X_1'X_2'}$ along the same $(1,k)$-tuple with copies of W, for an appropriate number of times. This is possible since any number coprime to k is a generator for the finite cyclic group $\mathbb{Z}/k\mathbb{Z}$. □

Lemma 4.18 (Connecting lemma) *Let $k, m, r, t \in \mathbb{N}$, $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, p, \nu, \lambda, \zeta$ be such that*

$$1/m \ll 1/r, \varepsilon \ll 1/t, \zeta, \varepsilon_{k+1}, d_2, \ldots, d_k,$$

$$\zeta \ll p \ll d_2, \ldots, d_k,$$

$$1/t \ll \varepsilon_{k+1} \ll d_{k+1}, \nu \leq 1/k,$$

$$\lambda \ll \nu \ll 1/k.$$

Let $\mathbf{d} = (d_2, \ldots, d_{k+1})$ and let $(G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, H)$ be a $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup with H being sequentially tightly connected. Let $\overrightarrow{H}$ be an orientation of H with a compatible closed walk W. Suppose that $\mathcal{C}$ is a collection of pairwise disjoint (p, ν)-extensible paths consistent with $\overrightarrow{H}$ and with joint connection set T. Assume that

(1) $|\mathcal{C}| \leq \zeta m$,
(2) $V(\mathcal{C})$ is λ-sparse in $\mathcal{P}$,
(3) $V(\mathcal{C}) \cap T = \emptyset$.

Consider any two elements P_1, P_2 of $\mathcal{C}$, there is a sequential path P in G such that

(a) P connects every path of $\mathcal{C}$,
(b) P starts from P_1 and ends with P_2,
(c) $V(P) \setminus V(\mathcal{C}) \subseteq V(\mathcal{P})$,
(d) $V(P) \setminus V(\mathcal{C})$ intersects in at most $10k^2\mathcal{C}_Z + t^{2t+3k+2}$ vertices with each cluster $Z \in \mathcal{P}$, where $\mathcal{C}_Z$ denotes the number of paths of $\mathcal{C}$ intersecting with Z.

Proof. Choose a set T' from $V(G)$ by including each vertex of $V(\mathcal{P})$ independently at random with probability p. By Lemma 1.1 and the union bound, we obtain that the set T' is $(2p)$-sparse with probability $1 - 2t\exp(-\Omega(m))$. By Lemma 1.5, we obtain that the set T' is a connection set of a fixed $(p^{3k+2}/2, \nu)$-extensible path in $\mathcal{C}$ with probability $1 - 2m^{k-1}\exp(-\Omega(m))$. Since $|\mathcal{C}| \leq \zeta m$, with positive probability, we get a set T' satisfying all these properties.

Initiate $S = V(\mathcal{C})$. While there are two paths $Q_1, Q_2 \in \mathcal{C}$ such that the extension to the right of Q_1 equals to the left of Q_2, apply Lemma 4.17 (1) with $\ell(W) = kp^{k+4}/2$ to obtain a path Q of length $10k^2$ which avoids S and has exactly $6k+2$ vertices in T'. Add $V(Q)$ to S, replace Q_1, Q_2 with Q in $\mathcal{C}$ and delete the $6k+2$ vertices used by Q in T'. Denote the set of paths after the procedure by $\mathcal{C}'$.

Note that the size of S grows by at most $10k^2|\mathcal{C}| \leq 10k^2\zeta m \leq \lambda m$, we delete at most $(6k+2)|\mathcal{C}| \leq (6k+2)\zeta m \leq p^{3k+2}m/4$ vertices from T throughout this process since $\zeta \ll p$. This implies that every path of $\mathcal{C}$ remains $(p^{3k+2}/4, \nu)$-extensible with connection set T'. Hence the conditions of Lemma 4.17 (1) are satisfied in every step and $\mathcal{C}'$ is well-defined.

Note that when the procedure ends, $\mathcal{C}'$ has size at most t^{2t}. Moreover, the paths of $\mathcal{C}'$ inherit the property of being consistent with $\overrightarrow{H}$. We continue by connecting up the paths of $\mathcal{C}'$ to the desired path P along the orientation. As the paths of $\mathcal{C}'$ are consistent with $\overrightarrow{H}$, the left and right extensions of each path in $\mathcal{C}'$ are contained in the walk W. Since W is compatible with $\overrightarrow{H}$, we can apply Proposition 4.11 to obtain a sequential walk in H of length of at most t^{2k+1} between the left and right end of each path in $\mathcal{C}'$. Use Lemma 4.7 and Lemma 4.17 (1), we can connect up the paths of $\mathcal{C}'$ using at most $t^{2t+3k+2}$ further vertices of $V(\mathcal{P})$.

Thus, P contains every path in $\mathcal{C}$ as a subpath and $V(P) \setminus V(\mathcal{C}) \subseteq V(\mathcal{P})$. Moreover, note that $V(\mathcal{C}') \setminus \mathcal{C}$ intersects in at most $10k^2\mathcal{C}_Z$ vertices for each $Z \in \mathcal{P}$, where $\mathcal{C}_Z$ denotes the number of paths of $\mathcal{C}$ that intersects with Z. It is obvious that P can start and end with any two paths of $\mathcal{C}$. □

Proof. [Proof of Lemma 4.14] Let $P_1 = P$. Suppose that P_1 extends rightwards to X and leftwards to Y, there exists a path P_2 of length $k-1$ which (c, ν)-extends both leftwards and rightwards to Y by Proposition 4.9. Moreover, we can assume that $V(P_1)$ is disjoint from $V(P_2)$ and T_2, where T_2 is the connection set of P_2. By Lemma 1.1 and Lemma 1.5, we can choose a λ-sparse vertex set T' such that P_1, P_2 are $(c^{3k+2}/2, \nu)$-extensible paths with connection set T'.

Firstly, let $S_1 = V(P_1) \cup V(P_2)$, and we choose κ such that $\lambda \ll \kappa \ll \gamma$. For each $Z \in \mathcal{P}$, we can select a subset Z' of Z of size $m' = \kappa m$ such that $Z \cap S_1 \subseteq Z'$ since S_1 is 2λ-sparse, $1/m \ll 1/t \ll \alpha \ll \lambda$ and $2\lambda \ll \kappa$. Let $\mathcal{P}' = \{Z'\}_{Z\in\mathcal{P}}$, $V(\mathcal{P}') = \bigcup_{Z\in\mathcal{P}} Z'$, $G' = G[V(\mathcal{P}')]$, $G'_{\mathcal{J}'} = G_{\mathcal{J}}[V(\mathcal{P}')]$ be the corresponding induced subgraphs and $\mathcal{J}' = \mathcal{J}[V(\mathcal{P}')]$ be the induced subcomplex. By Lemma 4.7, $\mathfrak{S}' = (G', G'_{\mathcal{J}'}, \mathcal{J}', \mathcal{P}', H)$ is a $(k, m', 2t, \sqrt{\varepsilon}, \sqrt{\varepsilon_{k+1}}, r, d_2, \dots, d_{k+1})$-regular setup.

Now we define a fractional matching that complements the discrepancy of S_1 in the clusters of $\mathcal{P}$. Consider $\mathbf{b}_i \in \mathbb{R}^{V(H_{W_i})}$ by setting $\mathbf{b}_i(Z') = |Z' \setminus S_1|/|Z'|$ for every $Z \in V(H_{W_i})$. Recall that $|S_1 \cap Z| \leq 2\lambda m$, $|Z'| = \kappa m$ and $\lambda \ll \kappa, \gamma$. It follows that

$$1 - \gamma \leq 1 - \frac{2\lambda}{\kappa} \leq 1 - \frac{|S_1|}{|Z'|} \leq \mathbf{b}_i \leq 1.$$

Since H_{W_i} is γ-robustly matchable, there is a fractional matching $\mathbf{w}_i$ such that $\sum_{Z\in e, e\in H_{W_i}} \mathbf{w}_i(e) = \mathbf{b}_i(Z')/k$ for every cluster $Z' \in \mathcal{P}'$ of H_{W_i} where $i \in [k]$. Consider $\psi > 0$ with $\varepsilon_{k+1} \ll \psi \ll \alpha$, there exists a sequential path Q_1 in G' such that $P_2Q_1P_1$ is a sequential path in G which contains no vertices of S_1

and $4k+2$ vertices of T' by Lemma 4.17. Moreover, there is a set $U \subseteq V(\mathcal{P})$ of size at most ψmt such that $U \cup V(Q_1)$ has $\lceil \sum_{i\in[k]} \sum_{Z\in e, e\in H_{W_i}} \mathbf{w}_i(e)\kappa m\rceil + B$ vertices in each point cluster Z. In other words, $V(P_2Q_1P_1) \cup U$ has $\kappa m + B$ vertices in each point cluster of $V(H)$ and uses $(\kappa m + B)(1-\alpha)t$ vertices of V since $|V(L_H(i))| \geq (1-\alpha)t$ for $i \in [t]$.

We now choose the second path Q_2. Note that $P_2Q_1P_1$ has right extension X and left extension Y, which are consistent with $\overrightarrow{H}$. Since W is compatible with $\overrightarrow{H}$, we can apply Proposition 4.11 to obtain a sequential walk W' in H of length $p \leq t^{2k+1}$ starting from X and ending with Y. Moreover, since W has length coprime to k, we can choose W' such that

$$p + 1 = |V(G) \setminus V(P_2Q_1P_1)| \bmod k.$$

Let $S_2 = V(P_2Q_1P_1)$ and $T'' = T' \setminus S_2$. Define $\mathbf{c}_i \in \mathbb{R}^{V(H_{W_i})}$ by setting $\mathbf{c}_i(Z) = (m - |Z \cap S_2|)/m$ for every $Z \in V(H_{W_i})$. Note that $1-\gamma \leq 1-\kappa-\psi \leq \mathbf{c}_i \leq 1$. Since H_{W_i} is robustly matchable, there is a fractional matching $\mathbf{z}_i$ such that $\sum_{Z\in e, e\in H_{W_i}} \mathbf{z}_i(e) = \mathbf{c}_i(Z)/k$ for every $Z \in \mathcal{P}$ of H_{W_i}. By Lemma 4.17, there exists a sequential path Q_2 in G of length $p+1$ mod k which contains no vertices of S_2 and $4k+2$ vertices of T'' such that $P_2Q_1P_1Q_2$ is a sequential cycle. Besides, there is a set $U' \subseteq V(\mathcal{P})$ of size at most ψmt such that $U' \cup V(Q_2)$ has $\lceil \sum_{i\in[k]} \sum_{Z\in e, e\in H_{W_i}} \mathbf{z}_i(e)m\rceil + B$ vertices in each point cluster Z. Thus, $U' \cup V(Q_2)$ uses at least $((1-\kappa)m - B + B)(1-\alpha)t = (1-\kappa)m(1-\alpha)t$ vertices of V. Denote the set of uncovered vertices in all clusters of $\mathcal{P}$ by M.

Note that $P_2Q_1P_1Q_2$ contains all vertices of $V(G)$ but M, U and U'. We know that $|M| \leq \alpha mt$, $|U| \leq \psi mt$, $|U'| \leq \psi mt$. Thus $P_2Q_1P_1Q_2$ covers all but at most $\alpha mt + 2\psi mt \leq 3\alpha n \leq \eta n$ vertices. Since the length of Q_2 is $p+1$ mod k, it follows that $|V \setminus V(P_2Q_1P_1Q_2)|$ is divisible by k. $\square$

Absorption Next, we will give the proof of Lemma 4.13. The method can be sketched as follows. We define absorbing gadget to absorb a set T of k vertices and a set O of k colors. For each (T, O), the absorbing gadgets are numerous. Based on the above properties, we can choose a small family of vertex-disjoint gadgets such that for every (T, O), there are many absorbing gadgets. Such a family is obtained by probabilistic method. Connecting all these gadgets yields the desired absorbing path.

In this part, we will obtain some results to help us attach vertices to regular complexes. Let H be a $(1,k)$-graph with vertex set $[n] \cup V$, $\mathcal{J}$ be a regular slice with cluster set $\mathcal{P}$. Given a $(0, k-1)$-subset $X \subseteq \mathcal{P}$, $\mathcal{J}_X$ is an $|X|$-partite $|X|$-graph containing all edges of $|X|$-level of $\mathcal{J}$. For any $v \in V$, $\delta > 0$ and any color cluster C, let

$$N_{\mathcal{J}}((v, C), \delta) = \{X \subseteq \mathcal{P} : |X| = k-1, \text{for any } c \in C, |N_H((v, c); \mathcal{J}_X)| > \delta|\mathcal{J}_X|\}.$$

Lemma 4.19 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, \mu, \delta$ be such that*

$$1/m \ll 1/r, \varepsilon \ll \varepsilon_{k+1}, d_2, \ldots, d_k,$$

$$\varepsilon_{k+1} \ll d_{k+1} \leq 1/k,$$

and

$$\varepsilon_{k+1} \ll \mu \ll \delta.$$

Let $\boldsymbol{d} = (d_2, \ldots, d_{k+1})$ *and let* $(H, H_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ *be a representative* $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$ *-regular setup. Suppose that* H *has minimum relative* $(1,1)$*-degree at least* $\delta + \mu$ *with vertex set* $[n] \cup V$*. Then for any* $v \in V$ *and any color cluster* C*, we have*

$$|N_{\mathcal{J}}((v,C), \frac{\mu}{3})| \geq (\delta + \frac{\mu}{4})\binom{t}{k-1}.$$

For any $c \in [n]$ *and any point cluster* Z*, we have*

$$|N_{\mathcal{J}}((c,Z), \frac{\mu}{3})| \geq (\delta + \frac{\mu}{4})\binom{t}{k-1}.$$

Proof. *Let* $v \in V$ *and* $c \in C$ *be arbitrary. The minimum relative degree condition implies that* $\overline{\deg}_H(v,c) \geq \delta + \mu$*. Since the regular setup is representative and* $\varepsilon_{k+1} \ll \mu$*, we have* $|\overline{\deg}_H(v,c) - \overline{\deg}_H((v,c); \mathcal{J})| < \varepsilon_{k+1}$ *and*

$$\deg_H((v,c), \mathcal{J}^{(k-1)}) \geq (\delta + \mu - \varepsilon_{k+1})|\mathcal{J}^{(k-1)}| \geq (\delta + \frac{2}{3}\mu)|\mathcal{J}^{(k-1)}|.$$

For any $(0, k-1)$*-subset* X *of* $\mathcal{P}$*,* $\mathcal{J}_X$ *corresponds to the* $(k-1)$*-edges of* $\mathcal{J}^{(k-1)}$ *which are* X*-partite. Define* $d_X = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{i}}$*. By Lemma 4.5, we have* $|\mathcal{J}_X| = (1 \pm \varepsilon_{k+1}) d_X m^{k-1}$*. By summing over all the* $(0, k-1)$*-subsets of* $\mathcal{P}$*, we have*

$$|\mathcal{J}^{(k-1)}| \geq (1 - \varepsilon_{k+1})\binom{t}{k-1} d_X m^{k-1}.$$

Moreover, let X *range over all* $(0, k-1)$*-subsets of* $\mathcal{P}$*, we have*

$$\sum_X |N_H((v,c); \mathcal{J}_X)| = \deg_H((v,c); \mathcal{J}^{(k-1)}) \geq (\delta + \frac{2}{3}\mu)|\mathcal{J}^{(k-1)}|.$$

Finally, we obtain

$$\begin{aligned}
&(\delta + \frac{2}{3}\mu)|\mathcal{J}^{(k-1)}| \\
&\leq \sum_X |N_H((v,c); \mathcal{J}_X)| \leq \sum_{X \in N_{\mathcal{J}}((v,c),\mu/3)} |\mathcal{J}_X| + \sum_{X \notin N_{\mathcal{J}}((v,c),\mu/3)} \frac{\mu}{3}|\mathcal{J}_X| \\
&\leq \left(|N_{\mathcal{J}}((v,c),\mu/3)| + \frac{\mu}{3}\left(\binom{t}{k-1} - |N_{\mathcal{J}}((v,c),\mu/3)|\right)\right)(1+\varepsilon_{k+1}) d_X m^{k-1} \\
&\leq \left((1 - \frac{\mu}{3})|N_{\mathcal{J}}((v,c),\mu/3)| + \frac{\mu}{3}\binom{t}{k-1}\right) \frac{1+\varepsilon_{k+1}}{1-\varepsilon_{k+1}} \frac{|\mathcal{J}^{(k-1)}|}{\binom{t}{k-1}} \\
&\leq \left(|N_{\mathcal{J}}((v,c),\mu/3)| + \frac{\mu}{3}\binom{t}{k-1}\right)(1 + 2\varepsilon_{k+1}) \frac{|\mathcal{J}^{(k-1)}|}{\binom{t}{k-1}}.
\end{aligned}$$

Thus, for any $v \in V$ and $c \in C$, we have

$$|N_{\mathcal{J}}((v,c),\mu/3)| \geq (\delta + \frac{\mu}{4})\binom{t}{k-1},$$

and by definition, the following holds for any $v \in V$ and color cluster C,

$$|N_{\mathcal{J}}((v,C),\mu/3)| \geq (\delta + \frac{\mu}{4})\binom{t}{k-1}.$$

Similarly, we can obtain the following result holds for any $c \in [n]$ and point cluster Z,

$$|N_{\mathcal{J}}((c,Z),\mu/3)| \geq (\delta + \frac{\mu}{4})\binom{t}{k-1}.$$

□

Lemma 4.20 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, \mu, \lambda$ be such that*

$$1/m \ll 1/r, \varepsilon \ll \varepsilon_{k+1}, d_2, \ldots, d_k,$$

$$\varepsilon_{k+1} \ll d_{k+1} \leq 1/k,$$

and

$$\varepsilon_{k+1} \ll \lambda \ll \mu.$$

Let $\boldsymbol{d} = (d_2, \ldots, d_{k+1})$ and let $(H, H_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, R)$ be a $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$-regular setup. Let $T \subseteq V(H)$ such that $|Z_1 \cap T| = |Z_2 \cap T| \leq \lambda m$ for every $Z_1, Z_2 \in \mathcal{P}$. Let $Z' = Z \setminus T$ for each $Z \in \mathcal{P}$, and let $\mathcal{J}' = \mathcal{J}[\bigcup Z']$ be the induced subcomplex. For every $v \in V$ and color cluster C, we have

$$|N_{\mathcal{J}}((v,C),2\mu)| \leq |N_{\mathcal{J}'}((v,C),\mu)|,$$

and for every $c \in [n]$ and point cluster Z, we have

$$|N_{\mathcal{J}}((c,Z),2\mu)| \leq |N_{\mathcal{J}'}((c,Z),\mu)|,$$

Proof. For any $v \in V$, color cluster C and a $(0, k-1)$-set $X \in N_{\mathcal{J}}((v,C),2\mu)$. By the definition, we have $|N_H((v,c);\mathcal{J}_X)| > 2\mu|\mathcal{J}_X|$ for any $c \in C$. Let $X = \{X_1, \ldots, X_{k-1}\}$ and $X' = \{X'_1, \ldots, X'_{k-1}\}$ be the corresponding clusters in the complex $\mathcal{J}'$. Our goal is to prove that $X' \in N_{\mathcal{J}'}((v,C),\mu)$.

Let $\varepsilon \ll \beta \ll \varepsilon_{k+1}$ and $d_X = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{i}}$. By Lemma 4.5, we have

$$|\mathcal{J}_X| = (1 \pm \beta) d_X m^{k-1}$$

and

$$|N_H((v,c);\mathcal{J}_X)| > 2\mu|\mathcal{J}_X| \geq 2\mu(1-\beta) d_X m^{k-1}.$$

Let $m' = |X_1 \setminus T|$, we have $|Z'| = m'$ for each $Z \in \mathcal{P}$, note that $m' \geq (1-\lambda)m$. By Lemma 4.7, $\mathcal{J}'$ is a $(\cdot, \cdot, \sqrt{\varepsilon}, \sqrt{\varepsilon_{k+1}}, r)$-regular slice. By Lemma 4.5, we have

$$(1+\beta)d_X(m')^{k-1} \geq |\mathcal{J}'_{X'}| \geq (1-\beta)d_X(m')^{k-1} \geq (1-\beta)(1-\lambda)^{k-1}d_X m^{k-1}.$$

Since $\beta \ll \varepsilon_{k+1} \ll \lambda \ll \mu$, we have

$$\begin{aligned}|N_H((v,c);\mathcal{J}'_{X'})| &\geq |N_H((v,c);\mathcal{J}_X)| - (|\mathcal{J}_X| - |\mathcal{J}'_{X'}|)\\ &\geq (1-\beta)(2\mu - (1-(1-\lambda)^{k-1}))d_X m^{k-1}\\ &\geq \mu(1+\beta)d_X m^{k-1} \geq \mu|\mathcal{J}'_{X'}|.\end{aligned}$$

Thus, we obtain that $X \in N_{\mathcal{J}'}((v,C),\mu)$.

Similarly, for every $c \in [n]$ and point cluster Z, we have

$$|N_{\mathcal{J}}((c,Z),2\mu)| \leq |N_{\mathcal{J}'}((c,Z),\mu)|.$$

□

In a $(k+1)$-uniform sequential cycle, the link graph of a point corresponds to a k-uniform sequential path. Thus, we will look for sequential paths in the neighbors of vertices inside a regular complex. The following lemma states that by looking at a μ-fraction of $(1,k-1)$-edges of a regular complex, we will find lots of sequential paths.

Lemma 4.21 *Let $1/m \ll \varepsilon \ll d_2,\dots,d_k,1/k,\mu$ and $k \geq 3$. Suppose that $\mathcal{J}$ is a $(\cdot,\cdot,\varepsilon)$-equitable complex with density vector **d** $= (d_2,\dots,d_k)$ and ground partition $\mathcal{P}$, the size of each vertex class is m. Let $W = \{W_0, W_1, \dots, W_{k-1}\} \subseteq \mathcal{P}$. Let $S \subseteq \mathcal{J}_W$ be with size at least $\mu|\mathcal{J}_W|$ and Q be a k-uniform sequential path $(c_1 \cdots c_k, v_1 \cdots v_{2k-2})$ with vertex classes $\{X_0, X_1, \dots, X_{k-1}\}$ such that $v_i, v_{i+k-1} \in X_i$ for $i \in [k-1]$ and $c_j \in X_0$ for $j \in [k]$. Let $\mathcal{Q}$ be the down-closed k-complex generated by Q and $\mathcal{Q}_S \subseteq \mathcal{Q}_{\mathcal{J}}$ be the copies of $\mathcal{Q}$ whose edges in the k-th level are in S. We have*

$$|\mathcal{Q}_S| \geq \frac{1}{2}\left(\frac{\mu}{8k}\right)^{k+1}|\mathcal{Q}_{\mathcal{J}}|.$$

Proof. The proof consists of three steps. Firstly, we use the dense version of the counting and extension lemma to count the number of various hypergraphs in $\mathcal{J}$. Secondly, we remove some $(1,k-1)$-tuples without good properties. Finally, we use an iterative procedure to return sequential paths using good $(1,k-1)$-tuples, as desired.

Firstly, let β be such that $\varepsilon \ll \beta \ll d_2,\dots,d_k,1/k,\mu$. Define

$$d_a = \prod_{i=2}^{k-2} d_i^{\binom{k-2}{i}}, d_b = \prod_{i=2}^{k-2} d_i^{\binom{k}{i}-\binom{k-2}{i}} \cdot \prod_{i=k-1}^{k} d_i^{\binom{k}{i}}.$$

Let $W' = W \setminus \{W_0, W_{k-1}\}$. By Lemmas 4.4 and 4.5, we have

$$|\mathcal{J}_W| = (1 \pm \beta)d_a d_b m^k, \tag{4.6}$$
$$|\mathcal{J}_{W'}| = (1 \pm \beta)d_a m^{k-2},$$
$$|\mathcal{Q}_{\mathcal{J}}| = (1 \pm \beta)d_a d_b^k m^{3k-2}.$$

Since $S \subseteq \mathcal{J}_W$ with $|S| \geq \mu|\mathcal{J}_W|$, with (4.6), we have

$$|S| \geq (1-\beta)\mu d_a d_b m^k.$$

Let $B_{W'} \subseteq \mathcal{J}_{W'}$ be the $(k-2)$-edges which are not extensible to $(1 \pm \beta)d_b m^2$ copies of a k-edge in $\mathcal{J}_W$. By Lemma 4.6, we have

$$|B_{W'}| \leq \beta|\mathcal{J}_{W'}|.$$

Secondly, we delete from S the edges which contain a $(k-2)$-set from $B_{W'}$ to obtain S', the number of edges deleted is at most

$$|B_{W'}|m^2 \leq \beta|\mathcal{J}_{W'}|m^2 \leq \beta(1+\beta)d_a m^k \leq |S|/3,$$

since $\beta \ll \mu, d_2, \dots, d_k$. Thus, we have $|S'| \geq 2|S|/3$. Furthermore, if there is any partite $(k-2)$-set T in $\mathcal{J}$ which lies in less than $\mu d_b m^2/(4k)$ edges of S', then we delete all edges in S' containing T to obtain S'' and iterate this until no further deletions are possible. Note that the number of partite $(k-2)$-sets supported in the clusters of $W \setminus \{W_0\}$ is $(k-1)(1 \pm \beta)d_a m^{k-2}$. Thus the number of edges deleted is at most

$$(k-1)(1+\beta)d_a m^{k-2}\frac{\mu d_b m^2}{4k} \leq (1+\beta)\frac{\mu d_a d_b m^k}{4} \leq \frac{|S|}{3}.$$

Thus, $|S''| \geq |S|/3$. Each partite $(k-2)$-set in $W_1, \dots, W_{k-1}$ is either contained in zero edges of S'' or in at least $\mu d_b m^2/(4k)$ edges in S''.

Finally, we use the properties of S'' to construct many labelled partition-respecting paths in $\mathcal{Q}_S$.

Step 1. Select $T = \{x_1, \dots, x_{k-2}\} \in \mathcal{J}_{W'}$ which is contained in at least $\mu d_b m^2/4$ edges in S''.

Step 2. Choose (c_1, x_{k-1}) such that $\{c_1, x_1, x_2, \dots, x_{k-1}\} \in S''$ and c_1, x_{k-1} are not in T.

Step 3. For $i \in [k, 2k-2]$, choose (c_{i-k+2}, x_i) such that $\{c_{i-k+2}, x_{i-k+2}, \dots, x_i\} \in S''$ and c_{i-k+2}, x_i are not used before.

This constructs a sequential path $\mathcal{Q}_S$ on $3k-2$ vertices such that each edge in the k-th level is in S'', thus in S. Next, we count the size of $\mathcal{Q}_S$.

In Step 1, let $G \subseteq \mathcal{J}_{W'}$ be the set of $(k-2)$-sets which are contained in less than $\mu d_b m^2/4$ edges in S'', we have

$$\frac{|S|}{3} \leq |S''| = \sum_{T \in \mathcal{J}_{W'}} \deg_{S''}(T) \leq |G|\frac{\mu}{4}d_b m^2 + (|\mathcal{J}_{W'}| - |G|)d_b m^2(1+\beta),$$

it gives that $|G| \leq (1-\beta)(1-\mu/12)d_a m^{k-2}$, thus, the choices for T is at least $|\mathcal{J}_{W'}| - |G| \geq \mu/13 d_a m^{k-2}$. In Step 2, we have at least $\mu d_b m^2/4$ choices for (c_1, x_{k-1}). In Step 3, $\{x_{i-k+2}, \ldots, x_{i-1}\}$ is a $(k-2)$-set contained in S'', by the construction of S'', there are at least $\mu d_b m^2/(4k)$ choices for (c_{i-k+2}, x_i), furthermore, at least $\mu d_b m^2/(8k)$ are different from the previous choices.

Thus, the number of paths in $\mathcal{Q}_S$ is at least

$$\left(\frac{\mu}{13} d_a m^{k-2}\right)\left(\frac{\mu}{4} d_b m^2\right)\left(\frac{\mu}{8k} d_b m^2\right)^{k-1} \geq (\frac{\mu}{8k})^{k+1} d_a d_b^k m^{3k-2} \geq \frac{1}{2}(\frac{\mu}{8k})^{k+1}|\mathcal{Q}_{\mathcal{J}}|,$$

since $\beta \ll \mu, 1/k$. □

Lemma 4.22 *Let $1/m \ll \varepsilon \ll d_2, \ldots, d_k, 1/k, \mu$ and $k \geq 3$. Suppose that $\mathcal{J}$ is a $(\cdot,\cdot,\varepsilon)$-equitable complex with density vector **d** $= (d_2, \ldots, d_k)$ and ground partition $\mathcal{P}$, the size of each vertex class is m. Let $W = \{W_1, \ldots, W_{k-1}, W_k\} \subseteq \mathcal{P}$. Let $S \subseteq \mathcal{J}_W$ be with size at least $\mu|\mathcal{J}_W|$ and Q be a k-uniform tight path $v_1, \ldots, v_{k-1}, b, v_k, \ldots, v_{2k-2}$ with vertex classes $\{X_1, \ldots, X_{k-1}, X_k\}$ such that v_i, $v_{i+k-1} \in X_i$ for $i \in [k-1]$ and $b \in X_k$. Let $\mathcal{Q}$ be the down-closed k-complex generated by Q and $\mathcal{Q}_S \subseteq \mathcal{Q}_{\mathcal{J}}$ be the copies of $\mathcal{Q}$ whose edges in the k-th level are in S. We have*

$$|\mathcal{Q}_S| \geq \frac{1}{2}\left(\frac{\mu}{8k}\right)^{k+1}|\mathcal{Q}_{\mathcal{J}}|.$$

Proof. The proof consists of three steps. Firstly, we use the dense version of the counting and extension lemma to count the number of various hypergraphs in $\mathcal{J}$. Secondly, we remove some k-tuples without good properties. Finally, we use an iterative procedure to return a tight path using good k-tuples, as desired.

Firstly, let β be such that $\varepsilon \ll \beta \ll d_2, \ldots, d_k, 1/k, \mu$. Define

$$d_a = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{i}}, d_b = \prod_{i=2}^{k} d_i^{\binom{k-1}{i-1}}.$$

Let $W' = W \setminus \{W_k\}$. By Lemma 4.4 and 4.5, we have

$$|\mathcal{J}_W| = (1 \pm \beta) d_a d_b m^k, \tag{4.7}$$

$$|\mathcal{J}_{W'}| = (1 \pm \beta) d_a m^{k-1},$$

$$|\mathcal{Q}_{\mathcal{J}}| = (1 \pm \beta) d_a d_b^k m^{2k-1}.$$

Since $S \subseteq \mathcal{J}_W$ with $|S| \geq \mu|\mathcal{J}_W|$, with (4.7), we have

$$|S| \geq (1-\beta)\mu d_a d_b m^k.$$

Let $B_{W'} \subseteq \mathcal{J}_{W'}$ be the $(k-1)$-edges which are not extensible to $(1 \pm \beta) d_b m$ copies of a k-edge in $\mathcal{J}_W$. By Lemma 4.6, we have

$$|B_{W'}| \leq \beta |\mathcal{J}_{W'}|.$$

Secondly, we delete from S the edges which contain a $(k-1)$-set from $B_{W'}$ to obtain S', the number of edges deleted is at most

$$|B_{W'}|m \leq \beta|\mathcal{J}_{W'}|m \leq \beta(1+\beta)d_a m^k \leq |S|/3,$$

since $\beta \ll \mu, d_2, \ldots, d_k$. Thus, we have $|S'| \geq 2|S|/3$. Furthermore, if there is any partite $(k-1)$-set T in $\mathcal{J}$ which lies in less than $\mu d_b m/(4k)$ edges of S', then we delete all edges in S' containing T to obtain S'' and iterate this until no further deletions are possible. Note that the number of partite $(k-1)$-sets supported in the clusters of W is $k(1 \pm \beta)d_a m^{k-1}$. Thus the number of edges deleted is at most

$$k(1+\beta)d_a m^{k-1}\frac{\mu d_b m}{4k} \leq (1+\beta)\frac{\mu d_a d_b m^k}{4} \leq \frac{|S|}{3}.$$

Thus, $|S''| \geq |S|/3$. Each partite $(k-1)$-set in $W_1, \ldots, W_k$ is either contained in zero edges of S'' or in at least $\mu d_b m/(4k)$ edges in S''.

Finally, we use the properties of S'' to construct many labelled partition-respecting paths in $\mathcal{Q}_S$.

Step 1. Select $T = \{x_1, \ldots, x_{k-1}\} \in \mathcal{J}_{W'}$ which is contained in at least $\mu d_b m/4$ edges in S''.

Step 2. Choose b such that $\{x_1, x_2, \ldots, x_{k-1}, b\} \in S''$ and $b \notin T$.

Step 3. For $i \in [k, 2k-2]$, choose x_i such that $\{x_{i-k+2}, \ldots, x_{k-1}, b, x_k, \ldots, x_i\} \in S''$ and x_i is not used before.

This constructs a sequential path $\mathcal{Q}_S$ on $2k-1$ vertices such that each edge in the k-th level is in S'', thus in S. Next, we count the size of $\mathcal{Q}_S$.

In Step 1, let $G \subseteq \mathcal{J}_{W'}$ be the set of $(k-1)$-sets which are contained in less than $\mu d_b m/4$ edges in S'', we have

$$\frac{|S|}{3} \leq |S''| = \sum_{T \in \mathcal{J}_{W'}} \deg_{S''}(T) \leq |G|\frac{\mu}{4}d_b m + (|\mathcal{J}_{W'}| - |G|)d_b m(1+\beta),$$

it gives that $|G| \leq (1-\beta)(1-\mu/12)d_a m^{k-1}$, thus, the choices for T is at least $|\mathcal{J}_{W'}| - |G| \geq \mu/13 d_a m^{k-1}$. In Step 2, we have at least $\mu d_b m/4$ choices for b. In Step 3, $\{x_{i-k+2}, \ldots, x_{k-1}, b, x_k, \ldots, x_{i-1}\}$ is a $(k-1)$-set contained in S'', by the construction of S'', there are at least $\mu d_b m/(4k)$ choices for x_i, furthermore, at least $\mu d_b m/(8k)$ are different from the previous choices.

Thus, the number of paths in $\mathcal{Q}_S$ is at least

$$\left(\frac{\mu}{13}d_a m^{k-1}\right)\left(\frac{\mu}{4}d_b m\right)\left(\frac{\mu}{8k}d_b m\right)^{k-1} \geq (\frac{\mu}{8k})^{k+1}d_a d_b^k m^{2k-1} \geq \frac{1}{2}(\frac{\mu}{8k})^{k+1}|\mathcal{Q}_{\mathcal{J}}|,$$

since $\beta \ll \mu, 1/k$. $\square$

Before we build the absorbing path, we need to define absorbing gadget, which is useful to absorb a particular set T of k vertices and a particular set O of k colors.

Definition 4.18 (Absorbing gadget) *Let $T = \{t_1, \dots, t_k\}$ be a k-set of points of G and $O = \{o_1, \dots, o_k\}$ be a k-set of colors of G. We say that $F \subseteq G$ is an absorbing gadget for (T, O) if $F = F_1 \cup F_2$ where $F_1 = A \cup B \cup E \cup \bigcup_{i=1}^{k}(P_i \cup Q_i) \cup C \cup \bigcup_{i=1}^{k} C_k$ and $F_2 = A' \cup B' \cup E' \cup \bigcup_{i=1}^{k}(P'_i \cup Q'_i) \cup C' \cup \bigcup_{i=1}^{k} C'_k$ such that*

(1) $A, B, E, P_1, Q_1, \dots, P_k, Q_k, A', B', E', P'_1, Q'_1, \dots, P'_k, Q'_k$ are pairwise disjoint and also disjoint from T. $C, C_1, \dots, C_k, C', C'_1, \dots, C'_k$ are pairwise disjoint and also disjoint from O,

(2) $C_i = (c_{i,1}, \dots, c_{i,k-1})$ and $C'_i = (c'_{i,1}, \dots, c'_{i,k-1})$ for $i \in [k]$,

(3) A, B, E, A', B', E' are k-tuples of points of G, C and C' are $(k+1)$-tuples of colors of G, (C, AE), $(C', A'E')$ and $(C'(c_{1,1}, \dots, c_{k,1}), A'B'E')$ are sequential paths,

(4) for $B = (b_1, \dots, b_k)$, each of P_i, Q_i has $k-1$ vertices for $i \in [k]$, both $(C_i, P_i b_i Q_i)$ and $(\{o_i\} \cup C_i \setminus \{c_{i,1}\}, P_i b_i Q_i)$ are sequential paths of length $2k-1$ for $i \in [k]$,

(5) for $B' = (b'_1, \dots, b'_k)$, each of P'_i, Q'_i has $k-1$ vertices for $i \in [k]$, both $(C'_i, P'_i b'_i Q'_i)$ and $(C'_i, P'_i t_i Q'_i)$ are sequential paths of length $2k-1$ for $i \in [k]$.

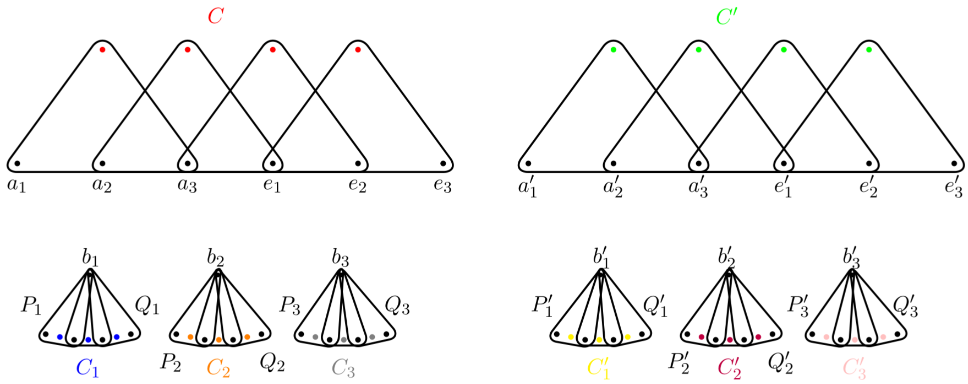


Figure 4.1 – Before the Absorption ($k = 3$), the paths $(C, AE), (C', A'E')$, $(C_1, P_1 b_1 Q_1), (C'_1, P'_1 b'_1 Q'_1), (C_2, P_2 b_2 Q_2), (C'_2, P'_2 b'_2 Q'_2), (C_3, P_3 b_3 Q_3), (C'_3, P'_3 b'_3 Q'_3)$ are sequential paths, the black dots represent the points while other dots with the same color come from the same color set.

Note that an absorbing gadget F spans $4k^2 + 2k$ points together with $2k^2 + 2k + 2$ colors.

Definition 4.19 ($\mathfrak{S}$-gadget) *Suppose $F = F_1 \cup F_2$ is an absorbing gadget where $F_1 = A \cup B \cup E \cup \bigcup_{i=1}^{k}(P_i \cup Q_i) \cup C \cup \bigcup_{i=1}^{k} C_k$ and $F_2 = A' \cup B' \cup E' \cup \bigcup_{i=1}^{k}(P'_i \cup Q'_i) \cup C' \cup \bigcup_{i=1}^{k} C'_k$ with $A = (a_1, \dots, a_k)$, $B = (b_1, \dots, b_k)$, $E = (e_1, \dots, e_k)$, $C = (c_1, \dots, c_{k+1})$, $C_i = (c_{i,1}, \dots, c_{i,k})$, $P_i = (p_{i,1}, \dots, p_{i,k-1})$ and*

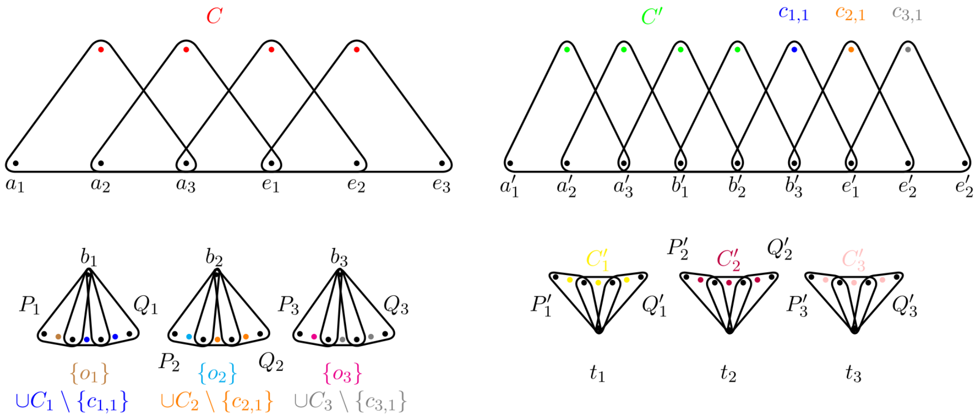


Figure 4.2 – After the Absorption ($k = 3$), the paths $(\{o_1\} \cup C_1 \setminus \{c_{1,1}\}, P_1 b_1 Q_1)$, $(\{o_2\} \cup C_2 \setminus \{c_{2,1}\}, P_2 b_2 Q_2)$, $(\{o_3\} \cup C_3 \setminus \{c_{3,1}\}, P_3 b_3 Q_3)$, $(C'_1, P'_1 t_1 Q'_1)$, $(C'_2, P'_2 t_2 Q'_2)$,$(C'_3, P'_3 t_3 Q'_3)$, (C, AE), $(C'(c_{1,1}, c_{2,1}, c_{3,1}), A'B'E')$ are sequential paths.

$Q_i = (q_{i,1}, \ldots, q_{i,k-1})$ for $i \in [k]$, $A' = (a'_1, \ldots, a'_k)$, $B' = (b'_1, \ldots, b'_k)$, $E' = (e'_1, \ldots, e'_k)$, $C' = (c'_1, \ldots, c'_{k+1})$, $C'_i = (c'_{i,1}, \ldots, c'_{i,k})$, $P'_i = (p'_{i,1}, \ldots, p'_{i,k-1})$ and $Q'_i = (q'_{i,1}, \ldots, q'_{i,k-1})$ for $i \in [k]$. Suppose that $\varepsilon, \varepsilon_{k+1}, d_2, \ldots, d_{k+1}, c, \nu > 0$. Let $\mathbf{d} = (d_2, \ldots, d_{k+1})$ and suppose that $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ is an oriented $(k+1, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup. We say that F is an $\mathfrak{S}$-gadget if

- *(G1) there exists an oriented edge $Y' = (Y_0, Z_1, \ldots, Z_k) \in \overrightarrow{H}$ and a color cluster Z_0, such that $C \cup C' \cup \bigcup_{i \in [k]} C_i \subseteq Y_0$, $\bigcup_{i \in [k]} C'_i \subseteq Z_0$, $a_i, b_i, e_i \in Z_i$ for $i \in [k]$,*
- *(G2) there exists an oriented edge $Y = (Y_0, Y_1, \ldots, Y_k) \in \overrightarrow{H}$, such that $a'_i, b'_i, e'_i \in Y_i$ for $i \in [k]$,*
- *(G3) there exists an ordered k-tuple of clusters $W_i = (W_{i,1}, \ldots, W_{i,k-1})$ such that $W_i \cup \{Y_0, Z_i\}$ is an edge in H and $(Y_0, W_{i,1}, \ldots, W_{i,k-1}, Z_i)$ is consistent with $\overrightarrow{H}$, $p_{i,j}, q_{i,j} \in W_{i,j}$ for $i \in [k], j \in [k-1]$,*
- *(G4) there exists an ordered k-tuple of clusters $W'_i = (W'_{i,1}, \ldots, W'_{i,k-1})$ such that $W'_i \cup \{Z_0, Y_i\}$ is an edge in H and $(Z_0, W'_{i,1}, \ldots, W'_{i,k-1}, Y_i)$ is consistent with $\overrightarrow{H}$, $p'_{i,j}, q'_{i,j} \in W'_{i,j}$ for $i \in [k], j \in [k-1]$,*
- *(G5) $F \subseteq G_{\mathcal{J}}$,*

We will further say that F is (c, ν)-extensible if the following also holds :

- *(G6) The path (C, AE) is (c, ν)-extensible both left- and rightwards to the ordered tuple $Y' = (Y_0, Z_1, \ldots, Z_k)$ and the path $(C_i, P_i b_i Q_i)$ is (c, ν)-extensible leftwards to $(Y_0, W_{i,1}, \ldots, W_{i,k-1}, Z_i)$ and rightwards to $(Y_0, Z_i, W_{i,1}, \ldots, W_{i,k-1})$ for $i \in [k]$.*
- *(G7) The path $(C', A'E')$ is (c, ν)-extensible both left- and rightwards to the ordered tuple $Y = (Y_0, Y_1, \ldots, Y_k)$ and the path $(C'_i, P'_i b'_i Q'_i)$ is (c, ν)-extensible*

leftwards to $(Z_0, W'_{i,1}, \dots, W'_{i,k-1}, Y_i)$ and rightwards to $(Z_0, Y_i, W'_{i,1}, \dots, W'_{i,k-1})$ for $i \in [k]$.

Definition 4.20 (Reduced gadget) *A reduced gadget is a $(1,k)$-graph L consisting of $Y \cup W_1 \cup \dots \cup W_k \cup Z_0 \cup Z_1 \cup \dots \cup Z_k \cup W'_1 \cup \dots \cup W'_k$ where $Y = \{Y_0, Y_1, \dots, Y_k\}$, $W_i = \{W_{i,1}, \dots, W_{i,k-1}\}$ for $i \in [k]$, $W'_i = \{W'_{i,1}, \dots, W'_{i,k-1}\}$ for $i \in [k]$ and $2(k+1)$ edges given by $Y, Y' = \{Y_0, Z_1, \dots, Z_k\}$, $W_i \cup \{Y_0, Z_i\}$ for $i \in [k]$ and $W'_i \cup \{Z_0, Y_i\}$ for $i \in [k]$. We refer to Y and Y' as the core edges of L and $W_i, W'_i, i \in [k]$ as the peripheral sets of L.*

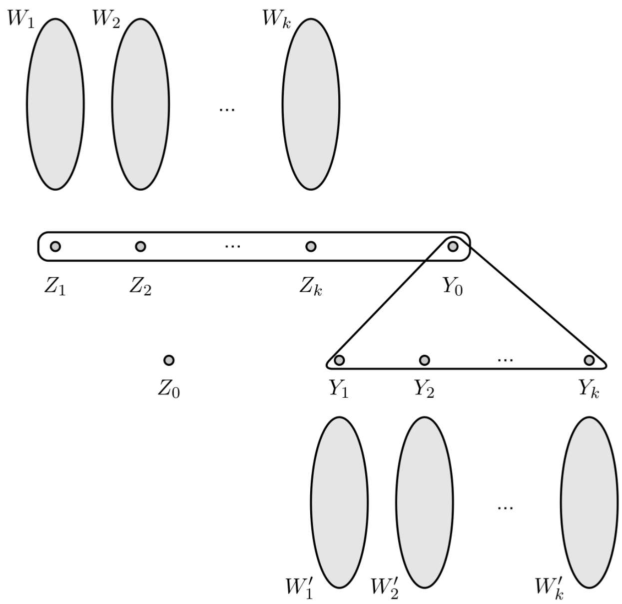


Figure 4.3 – Reduced Gadget

Given an oriented $(1,k)$-graph $\overrightarrow{H}$, a reduced gadget in $\overrightarrow{H}$ is a copy of L such that Y coincides with the orientation of that edge in $\overrightarrow{H}$ and such that $(Z_0, W_{i,1}, \dots, W_{i,k-1}, Y_i)$ is consistent with that edge in $\overrightarrow{H}$.

Let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented regular setup. Let $c, \nu > 0$, $T = \{t_1, \dots, t_k\}$ be a k-set of V and $O = \{o_1, \dots, o_k\}$ be a k-set of $[n]$, and L be a reduced gadget in $\overrightarrow{H}$. We define the following sets :

1. Denote the set of all reduced gadgets in $\overrightarrow{H}$ by $\mathfrak{L}_{\overrightarrow{H}}$,
2. Denote the set of $\mathfrak{S}$-gadgets which use precisely the clusters of L as in Definition 4.20 by $\mathfrak{F}_L$,
3. Denote the set of $\mathfrak{S}$-gadgets in $\mathfrak{F}_L$ which are $(c, \nu, V(G))$-extensible by $\mathfrak{F}_L^{\text{ext}}$,

4. Denote the set of all $\mathfrak{S}$-gadgets by $\mathfrak{F}$,
5. Denote the set of all $(c, \nu, V(G))$-extensible $\mathfrak{S}$-gadgets by $\mathfrak{F}^{\text{ext}} \subseteq \mathfrak{F}$,
6. For any k-subset T of V and any k-subset O of $[n]$, let $\mathfrak{F}_{(T,O)} \subseteq \mathfrak{F}$ be the set of absorbing $\mathfrak{S}$-gadgets for (T, O),
7. Denote the set of $\mathfrak{S}$-gadgets absorbing (T, O) which are (c, ν)-extensible by $\mathfrak{F}^{\text{ext}}_{(T,O)} = \mathfrak{F}_{(T,O)} \cap \mathfrak{F}^{\text{ext}}$.

Lemma 4.23 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \beta$ be such that*

$$\begin{aligned} 1/m &\ll 1/r, \varepsilon \ll 1/t, c, \varepsilon_{k+1}, d_2, \ldots, d_k, \\ c &\ll d_2, \ldots, d_k, \\ 1/t &\ll \varepsilon_{k+1} \ll \beta, d_{k+1} \leq 1/k, \\ \varepsilon_{k+1} &\ll \nu. \end{aligned}$$

Let $\boldsymbol{d} = (d_2, \ldots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$-regular setup and $L \in \mathcal{L}_{\overrightarrow{H}}$ be a reduced gadget in $\overrightarrow{H}$. Let $\mathcal{F}$ be the $(k+1)$-complex corresponding to the down-closure of $(1, k)$-graph F as in Definition 4.19. Then

$$|\mathfrak{F}_L| = (1 \pm \beta) \left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})} \right) m^{6k^2+4k+2}, \tag{4.8}$$

$$|\mathfrak{F}_L \setminus \mathfrak{F}_L^{\text{ext}}| \leq \beta |\mathfrak{F}_L|.$$

Proof. Let $Y = (Y_0, Y_1, \ldots, Y_k), Y' = (Y_0, Z_1, \ldots, Z_k) \in \overrightarrow{H}$ be the ordered core edge of L and $W_i = \{W_{i,1}, \ldots, W_{i,k-1}\}$, $W_i' = \{W_{i,1}', \ldots, W_{i,k-1}'\}$ for $i \in [k]$, be the peripheral sets, ordered such that $(Y_0, W_{i,1}, \ldots, W_{i,k-1}, Z_i)$ and $(Z_0, W_{i,1}', \ldots, W_{i,k-1}', Y_i)$ are consistent with $\overrightarrow{H}$. Note that $|V(F)| = 6k^2 + 4k + 2$. The bounds on $|\mathfrak{F}_L|$ are given by Lemma 4.2 directly.

Let $Y^* = (Y_1, \ldots, Y_{k-1})$ and denote the ordered tuples in the $(k-1)$-th level of $\mathcal{J}$ in the clusters $\{Y_1, \ldots, Y_{k-1}\}$ by $\mathcal{J}_{Y^*}$. Let $d_{Y^*} = \prod_{i=2}^{k-1} d_i^{\binom{k-1}{i}}$. By Lemma 4.5 we have

$$|\mathcal{J}_{Y^*}| = (1 \pm \beta) d_{Y^*} m^{k-1}.$$

Let β_1 be such that $\varepsilon_{k+1} \ll \beta_1 \ll \beta, d_k, d_{k+1}, 1/k$. Let $B_1 \subseteq \mathcal{J}_{Y^*}$ be the set of $(k-1)$-tuples which are not (c, ν)-extensible leftwards to $(Y_0, Y_1, \ldots, Y_k)$. By Proposition 4.9 with β_1 playing the role of β, we deduce that

$$|B_1| \leq \beta_1 |\mathcal{J}_{Y^*}|.$$

Let β_2 be such that $\varepsilon \ll \beta_2 \ll \varepsilon_{k+1}, d_2, \ldots, d_{k-1}$. Let $\phi : V(F) \to L$ be the homomorphism and $Z \subseteq V(F)$ corresponds to the first $k-1$ points $\{a_1, \ldots, a_{k-1}\}$ of path AE. Let $\mathcal{F}^-$ be the $(k-1)$-complex generated by removing the $(k+1)$-st and k-th layer from the down-closure $\mathcal{F}$ of F. Let $\mathcal{Z} = \mathcal{F}^-[Z]$

be the induced subcomplex of $\mathcal{F}^-$ in Z. Note that $\phi(a_i) = Y_i$ for $i \in [k-1]$. Thus the labelled partition-respecting copies of $\mathcal{Z}$ in $\mathcal{J}$ correspond exactly to $\mathcal{J}_{Y^*}$. Define

$$d_{\mathcal{F}^- \setminus \mathcal{Z}} = \prod_{i=2}^{k-1} d_i^{e_i(\mathcal{F}^-) - e_i(\mathcal{Z})}.$$

Let $B_2 \subseteq \mathcal{J}_{Y^*}$ be the set of $(k-1)$-tuples which are not extensible to $(1 \pm \beta_2) d_{\mathcal{F}^- \setminus \mathcal{Z}} m^{6k^2+3k+3}$ labelled partition-respecting copies of $\mathcal{F}^-$ in $\mathcal{J}$. By Lemma 4.6 with β_2 playing the role of β, we have

$$|B_2| \leq \beta_2 |\mathcal{J}_{Y^*}|.$$

By (4.8), we have

$$|\mathfrak{F}_L| = (1 \pm \beta) d_{k+1}^{e_{k+1}(\mathcal{F})} d_k^{e_k(\mathcal{F})} d_{\mathcal{F}^- \setminus \mathcal{Z}} d_{Y^*} m^{6k^2+4k+2}.$$

Let $\mathcal{G} = \mathcal{J} \cup G_{\mathcal{J}}$. Say that a labelled partition-respecting copy of $\mathcal{F}$ in $\mathcal{G}$ is *nice* if the vertices of $\{a_1, \ldots, a_{k-1}\}$ are not in $B_1 \cup B_2$. For every $Z \in \mathcal{J}_{Y^*}$, let $N^*(Z)$ be the number of labelled partition-respecting copies of $\mathcal{F}$ in $\mathcal{G}$ which extend Z. We have

$$\begin{aligned}
\sum_{Z \in B_1 \cup B_2} N^*(Z) &= \sum_{Z \in B_1 \setminus B_2} N^*(Z) + \sum_{Z \in B_2} N^*(Z) \\
&\leq [|B_1|(1+\beta_2) d_{\mathcal{F}^- \setminus \mathcal{Z}} + |B_2|] m^{6k^2+3k+3} \\
&\leq [\beta_1(1+\beta_2) d_{\mathcal{F}^- \setminus \mathcal{Z}} + \beta_2] |\mathcal{J}_{Y'}| m^{6k^2+3k+3} \\
&\leq 3\beta_1 d_{\mathcal{F}^- \setminus \mathcal{Z}} |\mathcal{J}_{Y^*}| m^{6k^2+3k+3} \\
&\leq 3\beta_1 (1+\beta) d_{\mathcal{F}^- \setminus \mathcal{Z}} d_{Y^*} m^{6k^2+4k+2} \\
&\leq \frac{3\beta_1(1+\beta)}{(1-\beta) d_{k+1}^{e_{k+1}(\mathcal{F})} d_k^{e_k(\mathcal{F})}} |\mathcal{F}_L| \\
&\leq \frac{\beta}{4k+4} |\mathcal{F}_L|,
\end{aligned}$$

since $0 \leq N^*(Z) \leq m^{6k^2+3k+3}$ and $\beta_1 \ll \beta, d_k, d_{k+1}, 1/k$ and $\beta_2 \ll d_2, \ldots, d_{k-1}, \varepsilon_{k+1}$.

The same analysis shows that we define nice tuples for any $(k-1)$-set of vertices of F, the number of copies of F which are not nice with respect to that $(k-1)$-set is at most $\beta|\mathcal{F}_L|/(4k+4)$. Note that $F \in \mathfrak{F}_L$ is extensible if and only if paths (C, AE), $(C', A'E')$, $(C_i, P_i b_i Q_i)$ and $(C_i', P_i' b_i' Q_i')$ for $i \in [k]$ contained in F are extensible with certain edges of the reduced graph. This means that $4(k+1)$ many $(k-1)$-tuples are extensible with certain edges of the reduced graph. Thus, $F \in \mathfrak{F}_L \setminus \mathfrak{F}_L^{\text{ext}}$ implies that F is not nice with one of $4k+4$ many $(k-1)$-sets. Thus,

$$|\mathfrak{F}_L \setminus \mathfrak{F}_L^{\text{ext}}| \leq (4k+4) \frac{\beta}{4k+4} |\mathcal{F}_L| = \beta |\mathcal{F}_L|.$$

$\square$

Lemma 4.24 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \dots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \beta, \mu$ be such that*

$$\begin{aligned} 1/m \ll 1/r, \varepsilon &\ll 1/t, c, \varepsilon_{k+1}, d_2, \dots, d_k, \\ c &\ll d_2, \dots, d_k, \\ 1/t \ll \varepsilon_{k+1} &\ll \beta, d_{k+1} \leq 1/k, \\ \varepsilon_{k+1} &\ll \nu, \mu, \\ \alpha &\ll \mu. \end{aligned}$$

Let $\boldsymbol{d} = (d_2, \dots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$-regular setup. Suppose that for each color cluster C, there are at least $(1-\alpha)t$ point clusters Z such that $\{C, Z\}$ has relative $(1,1)$-degree at least μ in H, then

$$\frac{\mu^{2k+2}}{8}\binom{t}{k}^2\binom{t}{k-1}^{2k} t(t-1) \leq |\mathfrak{L}_{\overrightarrow{H}}| \leq \binom{t}{k}^2\binom{t}{k-1}^{2k} t(t-1).$$

Let $\mathcal{F}$ be the $(k+1)$-complex corresponding to the down-closure of the $(1,k)$-graph F. For each reduced gadget $L \in \mathfrak{L}_{\overrightarrow{H}}$ in $\overrightarrow{H}$, we have

$$|\mathfrak{F}_L^{ext}| = (1 \pm \beta)\left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2}$$

and

$$|\mathfrak{F}^{ext}| = (1 \pm \beta)\left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2} |\mathfrak{L}_{\overrightarrow{H}}|.$$

Proof. The lower bound of $\mathfrak{L}_{\overrightarrow{H}}$ can be done as follows. Let $Y = (Y_0, Y_1, \dots, Y_k)$, $Y' = (Y_0, Z_1, \dots, Z_k) \in \overrightarrow{H}$ be the ordered core edge of L and $W_i = \{W_{i,1}, \dots, W_{i,k-1}\}$, $W_i' = \{W_{i,1}', \dots, W_{i,k-1}'\}$ for $i \in [k]$, be the peripheral sets, ordered such that $(Z_0, W_{i,1}', \dots, W_{i,k-1}', Y_i)$ and $(Y_0, W_{i,1}, \dots, W_{i,k-1}, Z_i)$ are consistent with $\overrightarrow{H}$. We first choose Y_0, Z_0 arbitrarily, there are at least $t(t-1)$ choices. For $(Y_1, \dots, Y_k)$, there are at least $\mu\binom{t}{k} - \alpha t\binom{t}{k-1} \geq \mu\binom{t}{k}/2$ choices. Similarly, for $(Z_1, \dots, Z_k)$, there are at least $\mu\binom{t}{k}/2$ choices. Furthermore, W_i' and W_i for $i \in [k]$ can be chosen in at least $\mu\binom{t}{k-1}$ ways for $i \in [k]$, but we need to delete the possible choices of intersecting reduced gadgets, whose number is at most $t(t-1)(2k^2)^2 t^{2k^2-2} \leq (2k^2)^2 t^{2k^2}$. We have

$$\begin{aligned} |\mathfrak{L}_{\overrightarrow{H}}| &\geq \frac{\mu^{2k+2}}{4}\binom{t}{k}^2\binom{t}{k-1}^{2k} t(t-1) - (2k^2)^2 t^{2k^2} \\ &\geq \frac{\mu^{2k+2}}{8}\binom{t}{k}^2\binom{t}{k-1}^{2k} t(t-1), \end{aligned}$$

since $1/t \ll \mu, 1/k$.

While the upper bound is obvious.

We choose β' such that $\varepsilon_{k+1} \ll \beta' \ll \beta, d_k, d_{k+1}, 1/k$. By Lemma 4.23 (with β' in place of β), we obtain that

$$(1-\beta)\left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2} \le (1-\beta')^2 \left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2}$$

$$\le (1-\beta')|\mathfrak{F}_L| \le |\mathfrak{F}_L^{\text{ext}}|,$$

$$|\mathfrak{F}_L^{\text{ext}}| \le |\mathfrak{F}_L| \le (1+\beta')\left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2}$$

$$\le (1+\beta)\left(\prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}\right) m^{6k^2+4k+2}.$$

Note that

$$\mathfrak{F}^{\text{ext}} = \bigcup_{L \in \mathfrak{L}_{\overrightarrow{H}}} \mathfrak{F}_L^{\text{ext}},$$

and the union is disjoint, the bounds of $|\mathfrak{F}^{\text{ext}}|$ are easy to see. $\square$

Lemma 4.25 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \theta, \mu$ be such that*

$$\begin{aligned} 1/m &\ll 1/r, \varepsilon \ll 1/t, c, \varepsilon_{k+1}, d_2, \ldots, d_k, \\ c &\ll d_2, \ldots, d_k, \\ 1/t &\ll \varepsilon_{k+1} \ll d_{k+1} \le 1/k, \\ \varepsilon_{k+1} &\ll \nu \ll \theta \ll \mu \ll 1/k. \end{aligned}$$

*Let **d** $= (d_2, \ldots, d_{k+1})$ and let $\mathfrak{G} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon,$ $\varepsilon_{k+1}, r,$ **d**$)$-regular setup. Suppose that for each color cluster C, there are at least $(1-\alpha)t$ point clusters Z such that $\{C, Z\}$ has relative $(1,1)$-degree at least μ in H. For any point v of G, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((v,C),\mu) \cap N_H(Z,C)| \ge \mu\binom{t}{k-1}$. And for every $c \in [n]$, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((c,Z),\mu) \cap N_H(C,Z)| \ge \mu\binom{t}{k-1}$. Let $T \subseteq V$ be a k-set and $O \subseteq [n]$ be a k-set, we have*

$$|\mathfrak{F}_{(T,O)}^{\text{ext}}| \ge \theta |\mathfrak{F}^{\text{ext}}|.$$

Given a k-subset $T = \{t_1, \ldots, t_k\}$ of V and a k-subset $O = (o_1, \ldots, o_k)$ of $[n]$, the family $\mathfrak{L}_{\overrightarrow{H}}$ and $\mu > 0$, we define $\mathfrak{L}_{\overrightarrow{H},(T,O),\mu}$ of *reduced $((T,O),\mu)$-absorbers* as the set of (T,O)-absorbers $Y \cup W_1 \cup \cdots \cup W_k \cup Z_0 \cup Z_1 \cup \ldots \cup Z_k \cup W_1' \cup \cdots \cup W_{k'}'$, where $W_i \subseteq N_{\mathcal{J}}((c_i, Z_i), \mu)$ and $W_i' \subseteq N_{\mathcal{J}}((t_i, Z_0), \mu)$ for $i \in [k]$.

Claim 4.7 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \theta, \mu$ be such that*

$$\begin{aligned}
1/m \ll 1/r, \varepsilon &\ll 1/t, c, \varepsilon_{k+1}, d_2, \ldots, d_k, \\
c &\ll d_2, \ldots, d_{k+1}, \\
1/t &\ll \varepsilon_{k+1} \ll d_{k+1} \leq 1/k, \\
\varepsilon_{k+1} &\ll \nu \ll \theta \ll \mu \ll 1/k, \\
\alpha &\ll \mu.
\end{aligned}$$

Let $\mathbf{d} = (d_2, \ldots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup. Suppose that for each color cluster C, there are at least $(1-\alpha)t$ point clusters Z such that $\{C, Z\}$ has relative $(1,1)$-degree at least μ in H. For any point v of G, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((v, C), \mu) \cap N_H(Z, C)| \geq \mu\binom{t}{k-1}$. And for every $c \in [n]$, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((c, Z), \mu) \cap N_H(C, Z)| \geq \mu\binom{t}{k-1}$. Let $T \subseteq V$ be a k-set and $O \subseteq [n]$ be a k-set, we have

$$|\mathfrak{L}_{\overrightarrow{H},(T,O),\mu}| \geq \theta |\mathfrak{L}_{\overrightarrow{H}}|.$$

Proof. Let $T = \{t_1, \ldots, t_k\}$ and $O = (o_1, \ldots, o_k)$. Since H has minimum relative $(1,1)$-degree at least μ, there are at least $\mu t\binom{t}{k} - t\alpha t\binom{t}{k-1} \geq \mu t\binom{t}{k}/2$ choices for Y. Besides, there are at least $t-1$ choices for Z_0. For $(Z_1, \ldots, Z_k)$, there are at least $\mu\binom{t}{k}/2 - k^2\binom{t}{k-1} \geq \mu\binom{t}{k}/3$ choices. Each W_i is chosen from $N_{\mathcal{J}}((o_i, Z_i), \mu) \cap N_H(Y_0, Z_i)$ for $i \in [k]$, thus, W_i can be chosen in at least $\mu\binom{t}{k-1} - (k-1)((i-1)(k-1)+2k)\binom{t}{k-2} \geq \mu\binom{t}{k-1}/2$ ways for $i \in [k]$, since there are at most $(k-1)((i-1)(k-1)+2k)\binom{t}{k-2}$ choices for W_i which intersects with $Y \setminus \{Y_0\}, Z_1, \ldots, Z_k, W_1, \ldots, W_{i-1}$.

And each W'_i is chosen from $N_{\mathcal{J}}((t_i, Z_0), \mu) \cap N_H(Y_i, Z_0)$ for $i \in [k]$. Similarly, there are at least $(\mu/2)\binom{t}{k-1}$ possible choices for each W'_i for $i \in [k]$. Thus, the number of reduced $((T, O), \mu)$-absorbers is at least

$$\frac{\mu t}{2}\binom{t}{k}(t-1)\frac{\mu}{3}\binom{t}{k}\left(\frac{\mu}{2}\binom{t}{k-1}\right)^{2k} \geq \theta\binom{t}{k}^2\binom{t}{k-1}^{2k} t(t-1) \geq \theta|\mathfrak{L}_{\overrightarrow{H}}|$$

since $\theta \ll \mu$. □

Claim 4.8 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \theta, \mu$ be such that*

$$\begin{aligned}
1/m \ll 1/r, \varepsilon &\ll 1/t, c, \varepsilon_{k+1}, d_2, \ldots, d_k, \\
c &\ll d_2, \ldots, d_k, \\
1/t &\ll \varepsilon_{k+1} \ll d_{k+1} \leq 1/k, \\
\varepsilon_{k+1} &\ll \nu \ll \theta \ll \mu \ll 1/k.
\end{aligned}$$

Let $\mathbf{d} = (d_2, \ldots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \mathbf{d})$-regular setup. Let $T \subseteq V$ and $O \subseteq [n]$ be k-sets and let $L \in \mathfrak{L}_{\overrightarrow{H}}$ be a reduced $((T, O), \mu)$-gadget in $\overrightarrow{H}$. We have

$$|\mathfrak{F}_L \cap \mathfrak{F}_{(T,O)}| \geq \theta|\mathfrak{F}_L|.$$

Proof. Let $T = \{t_1, \dots, t_k\}$ and $O = \{o_1, \dots, o_k\}$, $L = Y \cup W_1 \cup \dots \cup W_k \cup Z_0 \cup Z_1 \cup \ldots \cup Z_k \cup W'_1 \cup \dots \cup W'_k$ where $W_i = \{W_{i,1}, \dots, W_{i,k-1}\}$ and $W'_i = \{W'_{i,1}, \dots, W'_{i,k-1}\}$. Choose P_i, Q_i in W_i and P'_i, Q'_i in W'_i, let $\mathcal{Q}_{Z_i,W_i}$ be the set of k-uniform tight paths $(b_i, v_1, \dots, v_{2k-2})$ such that $b_i \in Z_i$, $v_\ell, v_{\ell+k-1} \in W_{i,\ell}$ for $i, j \in [k]$, $\ell \in [k-1]$ and its down-closure is in $\mathcal{J}$. Let $\mathcal{Q}_{o_i,(Z_i,W_i)} \subseteq \mathcal{Q}_{Z_i,W_i}$ be the set of those paths whose edges in the k-th level are in $N_G(o_i)$. Note that F is the absorbing gadget for (T, O). Let $\mathcal{F}$ be the down-closure of F. Since L is a reduced (T,μ)-gadget, we have $W_i \in N_H(Y_0, Z_i) \cap N_{\mathcal{J}}((o_i, Z_i), \mu)$, thus $|N_G((o_i, Z_i), \mathcal{J}_{W_i})| \geq \mu |\mathcal{J}_{W_i}|$. By Lemma 4.22 with S being the set of k-sets where each k-set consists of $k-1$ points from $N_G((o_i, Z_i), \mathcal{J}_{W_i})$ and one point from Z_i, we have

$$|\mathcal{Q}_{o_i,(Z_i,W_i)}| \geq \frac{1}{2}\left(\frac{\mu}{8k}\right)^{k+1} |\mathcal{Q}_{Z_i,W_i}|.$$

Let $\mathcal{Q}_{Z_0,W'_i}$ be the set of k-uniform sequential paths $(c'_1, \dots, c'_k, v'_1, \dots, v'_{2k-2})$ such that $c'_j \in Z_0$, $v'_\ell, v'_{\ell+k-1} \in W'_{i,\ell}$ for $i, j \in [k]$, $\ell \in [k-1]$ and its down-closure is in $\mathcal{J}$. Let $\mathcal{Q}_{t_i,(Z_0,W'_i)} \subseteq \mathcal{Q}_{Z_0,W'_i}$ be the set of those paths whose edges in the k-th level are in $N_G(t_i)$. Since L is a reduced $((T,O),\mu)$-gadget, we have $W'_i \in N_H(Z_0, Y_i) \cap N_{\mathcal{J}}((t_i, Z_0), \mu)$, thus $|N_G((t_i, Z_0), \mathcal{J}_{W'_i})| \geq \mu |\mathcal{J}_{W'_i}|$. By Lemma 4.21 with S being the set of k-sets where each k-set consists $k-1$ points from $N_G((t_i, Z_0), \mathcal{J}_{W'_i})$ and one color from Z_0, we have

$$|\mathcal{Q}_{t_i,(Z_0,W'_i)}| \geq \frac{1}{2}\left(\frac{\mu}{8k}\right)^{k+1} |\mathcal{Q}_{Z_0,W'_i}|.$$

Let $\phi : V(F) \to V(L)$ be the homomorphism which labels the copies of F in $\mathfrak{F}_L$. Set $Z = \{b_1, \dots, b_k\} \cup \bigcup_{i=1}^k (V(P_i) \cup V(Q_i)) \cup \bigcup_{i=1}^k (C'_i \cup V(P'_i) \cup V(Q'_i))$. Thus, $|Z| = 5k^2 - 3k$. Let $\mathcal{Z} = \mathcal{F}[Z]$ be the induced subcomplex of $\mathcal{F}$ in Z. Note that $\mathcal{Z}$ consists of k vertex-disjoint k-uniform tight paths of length $2k-1$ where the i-th path lies in $\mathcal{Q}_{o_i,(Z_i,W_i)}$ and k vertex-disjoint k-uniform sequential paths of length $2k-2$ where the i-th path lies in $\mathcal{Q}_{t_i,(Z_0,W'_i)}$. Let $\mathcal{G} = \mathcal{J} \cup G_{\mathcal{J}}$ and $\mathcal{Z}_{\mathcal{G}}$ be the set of labelled partition-respecting copies of $\mathcal{Z}$ in $\mathcal{G}$. Let β_1 be such that $\varepsilon \ll \beta_1 \ll d_2, \dots, d_k, \varepsilon_{k+1}$ and define $d_{\mathcal{Z}} = \prod_{i=2}^k d_i^{e_i(\mathcal{Z})}$. By Lemma 4.4, we have

$$|\mathcal{Z}_{\mathcal{G}}| = \prod_{i=1}^k |\mathcal{Q}_{Z_i,W_i}||\mathcal{Q}_{Z_0,W'_i}| = (1 \pm \beta_1) d_{\mathcal{Z}} m^{5k^2-3k}.$$

Let $\mathcal{Z}_{(T,O),\mathcal{G}} \subseteq \mathcal{Z}_{\mathcal{G}}$ be the labelled partition-respecting copies of $\mathcal{Z}$ absorbing (T, O), thus we have

$$\begin{aligned}|\mathcal{Z}_{(T,O),\mathcal{G}}| &\geq \prod_{i=1}^k |\mathcal{Q}_{o_i,(Z_i,W_i)}||\mathcal{Q}_{t_i,(Z_0,W'_i)}| \geq \left(\frac{1}{2}\left(\frac{\mu}{8k}\right)^{k+1}\right)^{2k} \prod_{i=1}^k |\mathcal{Q}_{Z_i,W_i}||\mathcal{Q}_{Z_0,W'_i}| \\ &\geq 3\theta |\mathcal{Z}_{\mathcal{G}}|,\end{aligned}$$

since $\theta \ll \mu, 1/k$.

Let β_2 be such that $\varepsilon_{k+1} \ll \beta_2 \ll \theta, d_{k+1}, 1/k$ and $d_{\mathcal{F}-\mathcal{Z}} = \prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})-e_i(\mathcal{Z})}$. Let $I \subseteq \mathcal{Z}_{\mathcal{G}}$ be the set of labelled partition-respecting copies of $\mathcal{Z}$ which are not extensible to $(1 \pm \beta_2)d_{\mathcal{F}-\mathcal{Z}}m^{k^2+7k+2}$ labelled partition-respecting copies of $\mathcal{F}$ in $\mathcal{G}$. By Lemma 4.3, we have

$$|I| \leq \beta_2|\mathcal{Z}_{\mathcal{G}}| \leq \theta|\mathcal{Z}_{\mathcal{G}}|,$$

since $\beta_2 \ll \theta$. By Lemma 4.23, we have

$$|\mathfrak{F}_L| = (1 \pm \beta_2)d_{\mathcal{F}-\mathcal{Z}}d_{\mathcal{Z}}m^{6k^2+4k+2},$$

since $\varepsilon_{k+1} \ll \beta_2 \ll \theta, d_{k+1}, 1/k$.

Note that a labelled partition-respecting copy of $\mathcal{F}$ in $\mathcal{G}$ containing a $Z \in \mathcal{Z}_{(T,O),\mathcal{G}}$ yields exactly one gadget in $\mathfrak{F}_L \cap \mathfrak{F}_{(T,O)}$, we have

$$\begin{aligned}
|\mathfrak{F}_L \cap \mathfrak{F}_{(T,O)}| &\geq |\mathcal{Z}_{(T,O),\mathcal{G}} \setminus I|(1-\beta_2)d_{\mathcal{F}-\mathcal{Z}}m^{k^2+7k+2} \\
&\geq (|\mathcal{Z}_{(T,O),\mathcal{G}}| - |I|)(1-\beta_2)d_{\mathcal{F}-\mathcal{Z}}m^{k^2+7k+2} \\
&\geq 2\theta|\mathcal{Z}_{\mathcal{G}}|(1-\beta_2)d_{\mathcal{F}-\mathcal{Z}}m^{k^2+7k+2} \\
&\geq 2\theta(1-\beta_2)(1-\beta_1)d_{\mathcal{Z}}m^{5k^2-3k}d_{\mathcal{F}-\mathcal{Z}}m^{k^2+7k+2} \\
&\geq 2\theta(1-2\beta_2)d_{\mathcal{Z}}d_{\mathcal{F}-\mathcal{Z}}m^{6k^2+4k+2} \\
&\geq 2\theta\frac{1-2\beta_2}{1+\beta_2}|\mathfrak{F}_L| \\
&\geq \theta|\mathfrak{F}_L|,
\end{aligned}$$

since $\beta_2 \ll \theta$. $\square$

Proof. [Proof of Lemma 4.25] Let $\theta \ll \theta' \ll \mu$. By Claim 4.8 with θ', we have for every reduced $((T,O),\mu)$-gadget $L \in \mathfrak{L}_{\overrightarrow{H}}$,

$$|\mathfrak{F}_L \cap \mathfrak{F}_{(T,O)}| \geq \theta'|\mathfrak{F}_L|.$$

Let β be such that $\varepsilon_{k+1} \ll \beta \ll d_{k+1}, \theta'$, by Lemma 4.23 with θ', we have $|\mathfrak{F}_L \setminus \mathfrak{F}_L^{\mathrm{ext}}| \leq \beta|\mathfrak{F}_L| \leq \theta'|\mathfrak{F}_L|/2$. Thus,

$$|\mathfrak{F}_{(T,O)}^{\mathrm{ext}} \cap \mathfrak{F}_L| \geq |\mathfrak{F}_L \cap \mathfrak{F}_{(T,O)}| - |\mathfrak{F}_L \setminus \mathfrak{F}_L^{\mathrm{ext}}| \geq \frac{\theta'}{2}|\mathfrak{F}_L|.$$

By Claim 4.7 with θ' and Lemma 4.24, we have $|\mathfrak{L}_{\overrightarrow{H},(T,O),\mu}| \geq \theta'|\mathfrak{L}_{\overrightarrow{H}}|$ and

$$|\mathfrak{F}_{(T,O)}^{\mathrm{ext}}| \geq \sum_{L \in \mathfrak{L}_{\overrightarrow{H},(T,O),\mu}} |\mathfrak{F}_{(T,O)}^{\mathrm{ext}} \cap \mathfrak{F}_L| \geq \frac{\theta'}{2} \sum_{L \in \mathfrak{L}_{\overrightarrow{H},(T,O),\mu}} |\mathfrak{F}_L| \geq \theta|\mathfrak{F}^{\mathrm{ext}}|.$$

$\square$

Lemma 4.26 *Let $k, r, m, t \in \mathbb{N}$ and $d_2, \ldots, d_{k+1}, \varepsilon, \varepsilon_{k+1}, c, \nu, \theta, \mu, \alpha, \zeta$ be such that*

$$\begin{aligned} 1/m \ll 1/r, \varepsilon &\ll 1/t, \zeta, \varepsilon_{k+1}, d_2, \ldots, d_k, \\ \zeta \ll c &\ll d_2, \ldots, d_k, \\ 1/t \ll \varepsilon_{k+1} &\ll d_{k+1}, \nu \le 1/k, \\ c \ll \varepsilon_{k+1} &\ll \alpha \ll \theta \ll \mu \ll 1/k. \end{aligned}$$

Let $\boldsymbol{d} = (d_2, \ldots, d_{k+1})$ and let $\mathfrak{S} = (G, G_{\mathcal{J}}, \mathcal{J}, \mathcal{P}, \overrightarrow{H})$ be an oriented $(k, m, 2t, \varepsilon, \varepsilon_{k+1}, r, \boldsymbol{d})$-regular setup. Suppose that $V(G) = [n] \cup V$ where $|V| = n \le (1+\alpha)mt$ and $V(H) = [t] \cup V'$ where $|V'| = t$. Suppose that for each color cluster C, there are at least $(1-\alpha)t$ point clusters Z such that $\{C, Z\}$ has relative $(1,1)$-degree at least μ in H. For any point v of G, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((v, C), \mu) \cap N_H(Z, C)| \ge \mu\binom{t}{k-1}$. And for every $c \in [n]$, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$ such that $|N_{\mathcal{J}}((c, Z), \mu) \cap N_H(C, Z)| \ge \mu\binom{t}{k-1}$. Then there exists a family $\mathfrak{F}''$ of pairwise disjoint $\mathfrak{S}$-gadgets which are (c, ν)-extensible with the following properties.

(1) $|\mathfrak{F}''| \le \zeta m$,

(2) $|\mathfrak{F}'' \cap \mathfrak{F}^{\text{ext}}_{(T,O)}| \ge \zeta\theta m$ *for any k-subset T of V and k-subset O of $[n]$,*

(3) $V(\mathfrak{F}'')$ *is $(2(k+1)\zeta/t)$-sparse in $\mathcal{P}$.*

Proof. Let $\beta > 0$ be such that $\varepsilon_{k+1} \ll \beta \ll d_{k+1}$. Let F be the $(1,k)$-graph as in Definition 4.19 and let $\mathcal{F}$ be the $(k+1)$-complex generated by its down-closure. Let $d_F = \prod_{i=2}^{k+1} d_i^{e_i(\mathcal{F})}$. By Lemma 4.24, we have

$$|\mathfrak{F}^{\text{ext}}| \le (1+\beta) d_F m^{6k^2+4k+2} \binom{t}{k}^2 \binom{t}{k-1}^{2k} t(t-1) \le d_F m^{6k^2+4k+2} t^{2k^2+2},$$

$$\begin{aligned} |\mathfrak{F}^{\text{ext}}| &\ge \frac{\mu^{k+1}}{2}(1-\beta) d_F m^{6k^2+4k+2} \binom{t}{k}^2 \binom{t}{k-1}^{2k} t(t-1) \\ &\ge \frac{\mu^{k+1}}{2^{k+2} k^{2k} (k-1)^{2k^2}} d_F m^{6k^2+4k+2} t^{2k^2+2} \\ &\ge 6\theta^{1/2} d_F m^{6k^2+4k+2} t^{2k^2+2}, \end{aligned}$$

since $1/t \ll \varepsilon_{k+1} \ll \beta \ll d_{k+1} \ll 1/k$ and $\theta \ll \mu, 1/k$. By Lemma 4.24, for each reduced gadget $L \in \mathfrak{L}_{\overrightarrow{H}}$ in $\overrightarrow{H}$, we have

$$|\mathfrak{F}_L^{ext}| \le 2 d_F m^{6k^2+4k+2}.$$

By Lemma 4.25 with $\theta^{1/2}$, for any k-set $T \subseteq V$ and any k-set $O \subseteq [n]$, we have

$$|\mathfrak{F}^{\text{ext}}_{(T,O)}| \ge \theta^{1/2} |\mathfrak{F}^{\text{ext}}| \ge 6\theta d_F m^{6k^2+4k+2} t^{2k^2+2}.$$

Choose a family $\mathfrak{F}'$ from $\mathfrak{F}^{\text{ext}}$ by including each $\mathfrak{S}$-gadget independently at random with probability

$$p = \frac{\zeta m}{2 d_F m^{6k^2+4k+2} t^{2k^2+2}}.$$

Note that $|\mathfrak{F}'|$, $|\mathfrak{F}' \cap \mathfrak{F}^{\mathrm{ext}}_{(T,O)}|$ are binomial random variables, for any k-set $T \subseteq V$ and any k-set $O \subseteq [n]$, we have

$$\mathbb{E}[|\mathfrak{F}'|] = p|\mathfrak{F}^{\mathrm{ext}}| \leq \frac{\zeta m}{2},$$

$$\mathbb{E}[|\mathfrak{F}' \cap \mathfrak{F}^{\mathrm{ext}}_{(T,O)}|] = p|\mathfrak{F}^{\mathrm{ext}}_{(T,O)}| \geq 3\theta\zeta m.$$

For each $Z \in \mathcal{P}$, note that Z exists in at most t^{2k^2+1} reduced gadgets, thus, there are at most $2d_F m^{6k^2+4k+2} t^{2k^2+1}$ $\mathfrak{S}$-gadgets with vertices in Z. Note that each $\mathfrak{S}$-gadget contains at most $k^2 + 2k + 2$ vertices in a cluster. Hence, for each cluster $Z \in \mathcal{P}$, we have

$$\mathbb{E}[|V(\mathfrak{F}') \cap Z|] \leq 2(k^2 + 2k + 2) d_F m^{6k^2+4k+2} t^{2k^2+1} p = \frac{(k^2 + 2k + 2)\zeta m}{t}.$$

By Lemma 1.1, with probability $1 - o(1)$, the family $\mathfrak{F}'$ satisfies the following properties.

$$|\mathfrak{F}'| \leq 2\mathbb{E}[|\mathfrak{F}'|] \leq \zeta m,$$

$$|\mathfrak{F}' \cap \mathfrak{F}^{\mathrm{ext}}_{(T,O)}| \geq 2\theta\zeta m,$$

$$|V(\mathfrak{F}') \cap Z| \leq \frac{2(k^2 + k + 1)\zeta m}{t}$$

for any k-set $T \subseteq V$, k-set $O \subseteq [n]$ and cluster $Z \in \mathcal{P}$. We say that two $\mathfrak{S}$-gadgets are *intersecting* if they share at least one vertex. Note that there at most $(2k^2 + 2)^2 t^{4k^2+3}$ pairs of intersecting reduced gadgets. Hence, there are at most $(6k^2 + 4k + 2)^2 m^{12k^2+8k+1} (2k^2 + 2)^2 t^{4k^2+3}$ pairs of intersecting $\mathfrak{S}$-gadgets. We can bound the expected number of pairs of intersecting $\mathfrak{S}$-gadgets by

$$\begin{aligned}
&(6k^2 + 4k + 2)^2 m^{12k^2+8k+3} (2k^2 + 2)^2 t^{4k^2+3} p^2 \\
&= \frac{\zeta^2 (6k^2 + 4k + 2)^2 (2k^2 + 2)^2 m}{4 d_F^2 t} \leq \frac{\zeta\theta m}{2},
\end{aligned}$$

since $\zeta \ll d_2, \ldots, d_{k+1}, \theta, 1/k$. Using Markov's inequality, we derive that with probability at least $1/2$, $\mathfrak{F}'$ contains at most $\zeta\theta m$ pairs intersecting $\mathfrak{S}$-gadgets. Remove one gadget from each intersecting pair in such a family and remove gadgets that are not absorbing for any (T, O) where $T \subseteq V$, $O \subseteq [n]$ and $|T| = |O|$. We obtain a subfamily $\mathfrak{F}''$, satisfying the following properties.

(1) $|\mathfrak{F}''| \leq \zeta m$,
(2) $|\mathfrak{F}'' \cap \mathfrak{F}^{\mathrm{ext}}_{(T,O)}| \geq \theta\zeta m$,
(3) $V(\mathfrak{F}'')$ is $(2(k^2 + k + 1)\zeta/t)$-sparse in $\mathcal{P}$,

as desired. □

Proof. [The proof of Lemma 4.13] Since G has minimum relative $(1,1)$-degree at least $\delta+\mu$ and $\mathfrak{S}$ is a representative setup. For any $v \in V$ and any color cluster C, we have

$$|N_{\mathcal{J}}((v,C),\frac{\mu}{3})| \geq (\delta+\frac{\mu}{4})\binom{t}{k-1}.$$

For any $c \in [n]$ and any point cluster Z, we have

$$|N_{\mathcal{J}}((c,Z),\frac{\mu}{3})| \geq (\delta+\frac{\mu}{4})\binom{t}{k-1}.$$

by Lemma 4.19. Let $\zeta > 0$ with $1/r, \varepsilon \ll \zeta \ll c$ and let $\theta > 0$ with $\eta \ll \theta \ll \mu, 1/k$ and $M := \lceil \eta t/(\theta\zeta) \rceil$. Firstly, we need the following claim.

Claim 4.9 *For each $j \in [0,M]$, and any $S \subseteq V$ of size at most $j\theta\zeta n/t$ divisible by k and any $O \subseteq [n]$ of size $|S|$, there is a sequential path $P_j \subseteq G$ such that the following holds.*

(i) P_j is (S,O)-absorbing in G,
(ii) P_j is (c,ν)-extensible and consistent with $\overrightarrow{H}$,
(iii) $V(P_j)$ is $(100k^3 j\zeta/t)$-sparse in $\mathcal{P}$ and $V(P_j) \cap T_j = \emptyset$, where T_j denotes the connection set of P_j.

Proof. [Proof of the claim] Take P_0 to be the empty path and P_j satisfy the above conditions for $j \in [0,M)$.

Select a subset $Z' \subseteq Z \backslash V(P_j)$ of size $m' = (1-\lambda)m$, this can be done since $100k^3 j\zeta/t \leq (2\eta t/(\zeta\theta))(100k^3\zeta/t) \leq \lambda$ which follows from $\zeta \ll c \ll \eta \ll \lambda, \theta$. Also, since $n \leq (1+\alpha)mt$, we have $m' \geq n/(2t)$. Let $\mathcal{P}' = \{Z'\}_{Z\in\mathcal{P}}$, $\mathcal{J}' = \mathcal{J}[V(\mathcal{P}')]$ and $G'_{\mathcal{J}'} = G_{\mathcal{J}}[V(\mathcal{P}')]$. By lemma 4.7, $\mathfrak{S}' := (G', G'_{\mathcal{J}'}, \mathcal{J}', \mathcal{P}', H)$ is a $(k, m', 2t, \sqrt{\varepsilon}, \sqrt{\varepsilon_{k+1}}, r, \mathbf{d})$-regular setup.

By Lemma 4.20, for every $v \in V$ and color cluster C, we have

$$|N_{\mathcal{J}'}((v,C),\mu/6)| \geq |N_{\mathcal{J}}((v,C),\mu/3)| \geq (\delta+\mu/4)\binom{t}{k-1},$$

and for every $o \in [n]$ and point cluster Z, we have

$$|N_{\mathcal{J}'}((o,Z),\mu/6)| \geq |N_{\mathcal{J}}((o,Z),\mu/3)| \geq (\delta+\mu/4)\binom{t}{k-1},$$

Thus, we obtain that for every $v \in V$, $o \in [n]$, color cluster C, there are at least $(1-\alpha)t$ point clusters $Z \in \mathcal{P}$, we have

$$|N_{\mathcal{J}}((v,C),\mu/6) \cap N_H(Z,C)| \geq \frac{\mu}{5}\binom{t}{k-1},$$

and

$$|N_{\mathcal{J}}((o,Z),\mu/6) \cap N_H(C,Z)| \geq \frac{\mu}{5}\binom{t}{k-1}.$$

By Lemma 4.26 with $4c$ instead of c, 2ζ instead of ζ, we obtain a set $\mathcal{A}'$ of pairwise-disjoint $\mathfrak{S}'$-gadgets which are $(4c,\nu)$-extensible and such that

(1) $|\mathcal{A}'| \leq 2\zeta m'$,

(2) $|\mathcal{A}' \cap \mathfrak{F}_{(T,O)}| \geq 2\zeta\theta m'$ for any k-subset of V,

(3) $V(\mathcal{A}')$ is $(4(k^2+k+1)\zeta/t)$-sparse in $\mathcal{P}'$.

Next, we would connect all paths of absorbing gadgets in $\mathcal{A}'$ and P_j to obtain P_{j+1}. By Definition 4.19, there are $2(k+1)$ pairwise disjoint sequential paths in each $\mathfrak{S}'$-gadget in $\mathcal{A}'$ which are $(4c,\nu)$-extensible in $\mathfrak{S}'$. Let $\mathcal{A}$ be the union of all such sequential paths of all gadgets of $\mathcal{A}'$ and P_j. Set $T_{j+1} = V(G) \setminus V(\mathcal{A})$, it is obvious that $\mathcal{A}$ is a set of pairwise disjoint sequential paths in G such that

(1') $|\mathcal{A}| \leq 4(k+1)\zeta m' + 1$,

(2') $V(\mathcal{A})$ is $(100k^3 j\zeta/t + 4(k^2+k+1)\zeta/t)$-sparse in $\mathcal{P}$ and $V(\mathcal{A}) \cap T_{j+1} = \emptyset$,

(3') every path in $\mathcal{A} \setminus \{P_j\}$ is $(2c,\nu,T_{j+1})$-extensible in $\mathfrak{S}$ and consistent with $\overrightarrow{H}$. P_j is (c,ν,T_{j+1})-extensible in $\mathfrak{S}$ and consistent with $\overrightarrow{H}$.

Note that (1') follows from (1) and the addition of P_j. (2') follows from (iii), (3) and the definition of T_{j+1}. (3') follows from (ii) and (3) since $4(k^2+k+1)\zeta m/t \leq 2cm$. In particular, P_j is (c,ν)-extensible by (ii) while all other paths go from $(4c,\nu)$-extensible in $\mathfrak{S}'$ to $(2c,\nu)$-extensible in $\mathfrak{S}$. The consistency with $\overrightarrow{H}$ is given by the consistency of P_j and the definition of $\mathfrak{S}'$-gadgets.

By Lemma 4.18, we obtain a sequential path P_{j+1} with the following properties.

(A) P_{j+1} contains every path of $\mathcal{A}$,

(B) P_{j+1} starts and ends with two paths different from P_j,

(C) $V(P_{j+1}) \setminus V(\mathcal{A}) \subseteq V(\mathcal{P}')$,

(D) $V(P_{j+1}) \setminus V(\mathcal{A})$ intersects in at most $10k^2\mathcal{A}_Z + t^{2t+3k+2}$ vertices with each cluster $Z \in \mathcal{P}$, where $\mathcal{A}_Z$ denotes the number of paths of $\mathcal{A}$ that intersect with Z.

We claim that P_{j+1} satisfies (i)-(iii). First, we prove (iii). Note that for every cluster $Z \in \mathcal{P}$, the number of paths of $\mathcal{A}$ that intersect with Z is bounded by $4(k+1)\zeta m/t + 1$. (D) implies that $V(P_{j+1}) \setminus V(\mathcal{A})$ intersects in at most $100k^3\zeta m/t$ vertices with each cluster $Z \in \mathcal{P}$. Together with (iii), it follows that $\mathcal{A}$ is $(100k^3(j+1)\zeta/t)$-sparse in $\mathcal{P}$.

Next, we want to prove (ii), $V(P_{j+1}) \setminus V(\mathcal{A})$ intersects in at most $100k^3\zeta m/t \leq cm/4$ vertices with each cluster $Z \in \mathcal{P}$, since $\zeta \ll c$. Also, we have $V(\mathcal{A}) \cap T_{j+1} = \emptyset$. Hence, we obtain (ii) after deleting the vertices of P_{j+1} from T_{j+1}. After the deletion, we go from $(2c,\nu)$-extensible in (3') to (c,ν)-extensible. It is crucial that P_{j+1} starts and ends with two paths different from P_j by (B).

Finally, we claim that P_{j+1} is (S,O)-absorbing in G for any $S \subseteq V$ of size divisible by k and at most $(j+1)\zeta\theta n/t$ and any $O \subseteq [n]$ of size $|S|$. Partition S into two sets S_1 and S_2 such that both $|S_1|, |S_2|$ are divisible by k and S_1 is maximal such that $|S_1| \leq j\zeta\theta n/t$. Partition O into two sets O_1 and O_2 such that $|O_1| = |S_1|$ and $|O_2| = |S_2|$. Since P_j is (S',O')-absorbing in G for any set $S' \subseteq V$ of size at most $(j\zeta\theta n/t)$ and $|O'| = |S'|$, there exists a path P_j' with the same endpoints as P_j such that $I(P_j') = S_1 \cup I(P_j)$ and $C(P_j') = O_1 \cup C(P_j)$, besides, P_j is a subpath of P_{j+1}. So it remains to absorb S_2. By the choice of S_1, we have $|S_2| \leq \zeta\theta n/t + k \leq 2\zeta^3 n/t \leq 2(1+\alpha)\zeta^3 m \leq 5\zeta^3 m/2$. Therefore, we can partition S_2 and O_2 into $\ell \leq 5\zeta^3 m/(2k) \leq 2\zeta\theta m'$ sets of size k each, let $D_1, \ldots, D_\ell$ and $R_1, \ldots, R_\ell$ be those sets. By (2), we have $|\mathfrak{F}_{(D_i,R_i)} \cap \mathcal{A}'| \geq \ell$. Thus, we can associate each (D_i, R_i) with a different gadget $F_i \in \mathcal{A}'$ for each $i \in [\ell]$. Each F_i yields a collection of $2(k+1)$ sequential paths $P_{i,1}, \ldots, P_{i,2(k+1)}$ and we can replace those paths with a collection of different paths with the same endpoints. Since P_j and each $P_{i,u}$, $i \in [\ell], u \in [2(k+1)]$, are subpaths of P_{j+1}, the sequential path P'_{j+1} has the same endpoints with P_{j+1}. Also, P'_{j+1} is exactly $(C(P_{j+1}) \cup O, I(P_{j+1}) \cup S)$. $\square$

To finish, note that P_M and C_M has the desired properties. By the choice of $M = \lceil \eta t/(\zeta\theta) \rceil$, we have $M\zeta\theta/t \geq \eta$, so P_M with C_M is η- absorbing in G. Moreover, since $M(100k^3\zeta/t) \leq 200k^2\eta/\theta \leq \lambda$ and $\eta \ll \lambda$, $V(P_M)$ is λ-sparse in $\mathcal{P}$. $\square$

5 - Long rainbow cycles in complete bipartite graphs

In 1989, Andersen [13] conjectured that all proper edge-colorings of K_n admit a rainbow path which omits only one vertex.

Conjecture 5.1 (Andersen [13]) *All proper edge-colorings of K_n admit a rainbow path of length $n-2$.*

It is best possible by a construction of Maamoun and Meyniel [115]. Akbari, Etesami, Mahini and Mahmoody [6] proved that every properly edge-colored K_n has a rainbow cycle of length at least $n/2-1$. Gyárfás and Mhalla [64] proved that if the set of edges with every used color forms a perfect matching in K_n, then there exists a rainbow path of length $(2n+1)/3$. Gyárfás, Ruszinkó, Sárközy and Schelp [65] showed that every properly colored K_n contains a rainbow cycle of length $(4/7-o(1))n$. Gebauer and Mousset [59] and Chen and Li [26], independently showed that every properly colored K_n contains a rainbow cycle of length $(3/4-o(1))n$. Alon, Pokrovskiy and Sudakov [9] proved that every properly edge-colored of K_n contains a rainbow path with length $n-O(n^{3/4})$, and the error bound has since been improved to $O(\sqrt{n}\cdot\log n)$ by Balogh and Molla [14].

In this chapter, we show that every properly edge-colored $K_{n,n}$ contains a long rainbow cycle as follows.

Theorem 5.1 *Every properly edge-colored $K_{n,n}$ contains a rainbow cycle of length at least $n-28n^{3/4}$ for sufficiently large n.*

The bound above is asymptotically optimal as each color class could be a perfect matching of $K_{n,n}$ and only n colors occur in $E(K_{n,n})$.

5.1 . Notation and preliminaries

For a bipartite graph G on vertex set $X\cup Y$ and (not necessarily distinct) vertex sets $A\subseteq X, B\subseteq Y$, we define $E_G(A,B)=\{ab: a\in A, b\in B, ab\in E(G)\}$. We often simply write $E(A,B)$ when G is clear from the context. Let $e_G(A,B)=|E_G(A,B)|$. A *path forest* $\mathcal{P}$ is a family of vertex-disjoint paths in a graph. Given a path forest $\mathcal{P}=\{P_1,\dots,P_t\}$, let $V(\mathcal{P})=V(P_1)\cup\cdots\cup V(P_t)$ and $E(\mathcal{P})=E(P_1)\cup\cdots\cup E(P_t)$. Given an edge-colored graph G and a subgraph H of G, let $C(H)$ be the set of colors appeared in $E(H)$. Furthermore, given a path forest $\mathcal{P}=\{P_1,\dots,P_t\}$ in an edge-colored graph, if each P_i is a rainbow path and $C(P_i)\cap C(P_j)=\emptyset$ for any $\{i,j\}\in\binom{t}{2}$, then we call $\mathcal{P}$ a *rainbow path forest*. For a natural number $n\in\mathbb{N}$, we define $[n]=\{1,2,\dots,n\}$. We write $a=(1\pm b)c$ to mean that the inequality $(1-b)c\le a\le(1+b)c$ holds.

Lemma 5.1 *For every $\varepsilon > 0$, there exists a constant C such that the following holds. Given a proper edge-coloring of a balanced bipartite graph H of $2n$ vertices on vertex set $X \cup Y$ such that $\delta(H) \geq (1-\eta)n$ for some $\eta = \eta(n)$, let G be the subgraph obtained by choosing every color class randomly and independently with probability p. Then, with high probability, every vertex v in G has degree $(1-\varepsilon)p \cdot d_G(v)$ and for every two disjoint subsets $A \subseteq X, B \subseteq Y$ with $|A| \geq C \log n/p, |B| \geq C(\log n/p)^2$, $e_G(A,B) \geq (1-\varepsilon)p|A||B|$.*

Lemma 5.2 *For all γ, δ, n with $\delta \geq \gamma$ and $3\gamma\delta - \gamma^2/2 > 2/n$ the following holds. Let G be a properly edge-colored bipartite graph with $2n$ vertices which is balanced and $\delta(G) \geq (1-\delta)n$. Then G contains a rainbow path forest $\mathcal{P}$ with at most γn paths and $|E(\mathcal{P})| \geq (1-4\delta)n$.*

Lemma 5.3 *For any $b, m, r > 0$ with $2mr \leq b$, the following holds. Let $\mathcal{P} = \{P_1, \ldots, P_r\}$ be a rainbow path forest in a properly edge-colored bipartite graph G on vertex set $X \cup Y$. Let H be a subgraph of G sharing no colors with $\mathcal{P}$ with $\delta(H) \geq 3b$ and $|E_H(A,B)| \geq b+1$ for any two sets of vertices $A \subseteq X, B \subseteq Y$ of size b. Then either $|P_1| \geq |V(\mathcal{P})| - 4b$ or there are two edges $e_1, e_2 \in H$ and a rainbow path forest $\mathcal{P}' = \{P'_1, \ldots, P'_r\}$ such that $E(\mathcal{P}') \subseteq E(\mathcal{P}) + e_1 + e_2$ and $|P'_1| \geq |P_1| + m$.*

Proof. [The proof of Theorem 5.1] Let H be the subgraph of $K_{n,n}$ obtained by choosing every color class randomly and independently with probability $p = 4.5b/n$ where $b = n^{3/4}$. By Lemma 5.1, with high probability, all vertices in H has degree $4b \leq (1-o(1))pn \leq 5b$ and $e_H(A,B) \geq (1-o(1))pb^2 > 4.4n^{1/2}b$ for any two disjoint sets $A \subseteq X, B \subseteq Y$ of size b. We choose such an H.

Let $G = K_{n,n} \setminus H$, then $\delta(G) \geq n - 5b = (1-5n^{-1/4})n$. By Lemma 5.2 with parameters $\delta = 5n^{-1/4}, \gamma = n^{-3/4}$, we obtain a rainbow path forest $\mathcal{P}$ in G with $n^{1/4}$ paths and $|E(\mathcal{P})| \geq n - 20n^{3/4}$.

Apply Lemma 5.3 in H repeatedly $2n^{1/2}$ times with parameters $b = n^{3/4}, r = n^{1/4}, m = n^{1/2}/2$. At each iteration, we delete all edges sharing a color with e_1 or e_2 to get a subgraph H'. We obtain that after i iterations, $\delta(H') \geq \delta(H) - 2i \geq 3b$ and for any $A \subseteq X, B \subseteq Y$ of size b, $|E_{H'}(A,B)| \geq 4.4n^{1/2}b - 2ib \geq b+1$. We either increase the length of P_1 by m or $|P_1| \geq |V(\mathcal{P})| - 4b$ at each iteration. Since $2n^{1/2}m = n > n - 4b$, $|P_1| \geq |V(\mathcal{P})| - 4b$ must occur at some step during the iterations. Thus, we obtain a rainbow path P of length at least $|V(\mathcal{P})| - 4b > |E(\mathcal{P})| - 4b \geq n - 24n^{3/4}$. Let S, T be the first $2b$ and the last $2b$ vertices of P respectively and $S' = S \cap A, T' = T \cap B$. Then the sizes of S' and T' are b and $e_{H'}(S', T') \geq b+1$. There must be edge between S' and T'. We add this edge to get a rainbow cycle of length at least $|P| - 4b \geq n - 28n^{3/4}$. □

We also need the following proposition and lemma.

Given a proper edge-coloring of $K_{n,n}$ on vertex set $X \cup Y$, we call a pair (A,B) of disjoint subsets $A \subseteq X, B \subseteq Y$ nearly-rainbow if the number of colors of edges between A and B is at least $(1-o(1))|A||B|$.

Lemma 5.4 *For any $\varepsilon > 0$, there exists a constant C such that the following holds. Given a proper edge-coloring of $K_{n,n}$ on vertex set $X \cup Y$, let G be a subgraph of $K_{n,n}$ obtained by choosing every color class with probability p. Then, with high probability, all nearly rainbow pairs (A, B) with $|A| = |B| = y \geq C \log n/p$, $A \subseteq X, B \subseteq Y$ satisfy $e_G(A, B) \geq (1-\varepsilon)py^2$.*

Proof. For any y, we choose ε such that every nearly-rainbow pair (A, B) has at least $(1-\varepsilon/2)|A||B|$ colors and $C = 40/\varepsilon^2$. Let (A, B) be a nearly-rainbow pair with $|A| = |B| = y$, then the number of colors m between A and B in $K_{n,n}$ is at least $(1-\varepsilon/2)y^2$. Thus, the number of colors between A and B in G is binomially distributed with parameters (m, p) where $m \geq (1-\varepsilon/2)y^2$. By Lemma 1.1, we have

$$\mathbb{P}[e_G(A, B) \leq (1-\varepsilon)py^2] \leq e^{-\varepsilon^2 py^2/14}.$$

The result follows by taking a union bound over all $\binom{n}{y}^2$ pairs of sets of size y since $\varepsilon^2 y \geq 40 \log n/p$. □

Lemma 5.5 *For every $\varepsilon > 0$, there exists C such that the following holds. Given a proper edge-coloring of $K_{n,n}$ on vertex set $X \cup Y$, let $A \subseteq X, B \subseteq Y$ be two sets of sizes a and b respectively with $a \leq b$, $b \geq Cy^2$. Then there are partitions of A and B into sets $\{A_i\}$ and $\{B_j\}$ of size y where $y \mid a$ and $y \mid b$ such that all but an ε-fraction of pairs (A_i, B_j) are nearly-rainbow.*

Proof. For any $\varepsilon > 0$, we choose $C \geq 1/\varepsilon^2$. Let E_c be the set of edges between A and B with color c. Note that $\sum_c |E_c| = |E(A, B)| = ab$ and $|E_c| \leq \min\{a, b\} = a$ since the edge-coloring of $K_{n,n}$ is proper.

Let S and T be selected uniformly at random from $\binom{A}{y}$ and $\binom{B}{y}$ respectively. Thus, for any two disjoint edges $e, e' \in E(A, B)$, we have $\mathbb{P}[e \in E(S, T)] = \frac{y^2}{ab}$ and $\mathbb{P}[e, e' \in E(S, T)] = \frac{y^2(y-1)^2}{ab(a-1)(b-1)}$. By the inclusion-exclusion formula, we have

$$\begin{aligned}
\mathbb{P}[c \text{ is present in } E(S,T)] &\geq \sum_{e \in E_c} \mathbb{P}[e \in E(S,T)] - \sum_{\{e,f\} \in \binom{E_c}{2}} \mathbb{P}[e, f \in E(S,T)] \\
&\geq \frac{y^2}{ab}|E_c| - \frac{y^2(y-1)^2}{ab(a-1)(b-1)}\binom{|E_c|}{2} \\
&= \frac{y^2}{ab}|E_c|(1 - \frac{(y-1)^2(|E_c|-1)}{2(a-1)(b-1)}) \\
&\geq \frac{y^2}{ab}|E_c|(1 - \frac{(y-1)^2}{2(b-1)}) \geq \frac{y^2}{ab}|E_c|(1-\varepsilon^2).
\end{aligned}$$

The last inequality holds since $\frac{(y-1)^2}{2(b-1)} \leq \frac{y^2}{b} \leq \frac{1}{C} \leq \varepsilon^2$. Let Z be the number of colors in $E(S, T)$. Thus, $\mathbb{E}[Z] = \sum_c \mathbb{P}[c \text{ is present in } E(S,T)] \geq \sum_c \frac{y^2}{ab}|E_c|$

$(1-\varepsilon^2) = y^2(1-\varepsilon^2)$. Since $y^2 = e(S,T) \geq Z$, we have $y^2 - Z$ is non-negative with expectation at most $\varepsilon^2 y^2$. By Markov's inequality, we have $\mathbb{P}[y^2 - Z \geq \varepsilon y^2] \leq \varepsilon$, which implies that with high probability, (S,T) is nearly rainbow.

Let $\{A_i\}$ and $\{B_j\}$ be random partitions of A and B into sets of size y. Let Y denote the number of pairs (A_i, B_j) which are not nearly rainbow, then we have $\mathbb{E}[Y] = \frac{ab}{y^2}\mathbb{P}[(A_i,B_j) \text{ is not nearly rainbow}] \leq \frac{ab}{y^2}\varepsilon$. By Markov's inequality, we have $\mathbb{P}[Y \geq \sqrt{\varepsilon}\frac{ab}{y^2}] \leq \sqrt{\varepsilon}$, which implies that with high probability, there exists a partition satisfying the lemma. □

5.2 . Obtaining long rainbow cycles

Proof. [The proof of Lemma 5.1] For any $\varepsilon > 0$, let $\varepsilon' = \varepsilon/3$, $C' \geq 1/\varepsilon$ and $C = 6C'/\varepsilon$. Let $y = \lceil C' \log n/p \rceil$, $|A| \geq C \log n/p, |B| \geq CC'^2(\log n/p)^2$. Assume that $\delta(H) \geq (1-\eta)n \geq C\frac{\log n}{p}$, let G be the subgraph of H obtained by choosing every color class with probability p. The degree of vertex v in G is binomially distributed with $(d_H(v), p)$. By Lemma 1.1, for each vertex v, $\mathbb{P}[|d_G(v) - p \cdot d_H(v)| \geq \varepsilon p \cdot d_H(v)] \leq 2e^{-\frac{\varepsilon^2 p \cdot d_H(v)}{3}} \leq 2/n$. By the union bound, with high probability, all vertices of G have degree $(1-\varepsilon)p \cdot d_H(v)$.

By Lemma 4.11 with ε' and C', we obtain that with high probability,

$$e_G(S,T) \geq (1-\varepsilon')y^2, \tag{5.1}$$

for every nearly rainbow pair (S,T) of sets of size y. Let b be the smallest integer larger that $C'y^2$ that is divisible by y and $A' \subseteq A, B' \subseteq B$ be subsets of sizes y and b respectively. By Lemma 4.12 with ε' and C', we can obtain that there exists a partition $\{B'_j\}$ of B' into parts of size y such that all but at most ε'-fraction of pairs (A', B'_j) are nearly rainbow. If we let $J = \{j : (A', B'_j) \text{ is nearly rainbow}\}$, then $|J| \geq (1-\varepsilon')b/y$. Therefore, we have

$$e_G(A',B') \geq \sum_{j \in J} e_G(A', B'_j) \geq \frac{(1-\varepsilon')b}{y} \cdot (1-\varepsilon')y^2 \geq (1-2\varepsilon')by.$$

Note that $\varepsilon'|A| \geq y$ and $\varepsilon'|B| \geq b$ since $\varepsilon' = 2C'/C$. Therefore, there exists a collection of at least $(1-\varepsilon')|A|/y$ disjoint subsets of A, each of size of y and a collection of at least $(1-\varepsilon')|B|/b$ disjoint subsets of B, each of size of b.

Then, with high probability,

$$e_G(A,B) \geq (1-\varepsilon')\frac{|A|}{y}(1-\varepsilon')\frac{|B|}{b}(1-2\varepsilon')yb \geq (1-\varepsilon)|A||B|.$$

□

Proof. [The proof of Lemma 5.2] Let $\mathcal{P} = \{P_1, \ldots, P_{\gamma n}\}$ be a rainbow path forest and $|E(\mathcal{P})|$ is as large as possible. Suppose that $|E(\mathcal{P})| < (1-4\delta)n$. Note that $|V(\mathcal{P})| \leq |E(\mathcal{P})| + \gamma n < n - \gamma n$. Thus, we can obtain γn non-empty paths. Since otherwise we can replace the empty path by a vertex outside $V(\mathcal{P})$. Let $P_i = v_{i,1} \cdots v_{i,|P_i|}$ for $i \in [\gamma n]$. For simplicity, we denote $c(v_{i,j-1}v_{i,j})$ by $c(v_{i,j})$ and $v_{i,j-1}v_{i,j}$ by $e(v_{i,j})$. Let C_0 be the set of colors not used on $E(\mathcal{P})$ and $C_i = \{c(x) : x \in N_{C_{i-1}}(v_{i,1}) \cap V(\mathcal{P}) \setminus \bigcup_{j\in[\gamma n]}\{v_{j,1}\}\} \cup C_{i-1}$ for $i \in [\gamma n]$. Note that for each $c \in C_i \setminus C_{i-1}$, there is a vertex $x \in V(\mathcal{P})$ such that $c(v_{i,1}x) \in C_{i-1}$ and $c(x) = c$. Firstly, we need to prove the following claim.

Claim 5.1 *$N_{C_{i-1}}(v_{i,1}) \subseteq V(\mathcal{P}) \setminus \bigcup_{j=i+1}^{\gamma n}\{v_{j,1}\}$ for $i \in [\gamma n]$.*

Proof. Suppose that there is an edge $v_{i,1}v_{j,1}$ with color from C_{i-1} for some $j \in [i+1, \gamma n]$. The case when there is an edge $v_{i,1}x$ for some $x \notin V(\mathcal{P})$ is identical. We proceed the following process.

Step1 Let $i_0 = i$ and $x_0 = v_{j,1}$.

Step2 We maintain that if $i_t \geq 1$, then $c(v_{i_t,1}x_t) \in C_{i_t-1}$.

Step3 Let $c_t = c(v_{i_{t-1},1}x_{t-1})$. Note that $c_t \in C_{i_{t-1}-1}$, $t \geq 1$.

Step4 Let i_t be the smallest number such that $c_t \in C_{i_t}$. Note that $c_t \in C_{i_t} \setminus C_{i_t-1}$ for $t \geq 1$.

Step5 For $t \geq 1$, if $i_t \geq 0$, then let x_t be the vertex of $V(\mathcal{P})$ with $c(x_t) = c_t$. Since $c_t \in C_{i_t} \setminus C_{i_t-1}$, by the definition, there is a vertex $x_t \in V(\mathcal{P})$ such that $c(v_{i_t,1}x_t) \in C_{i-1}$ and $c(x_t) = c_t$.

Step6 The iteration stops if $i_s = 0$.

Note that $i_0 > i_1 > \cdots > i_s$. Since $c_t \in C_{i_{t-1}-1}$ and i_t is the smallest number such that $c_t \in C_{i_t}$, we have $i_{t-1} - 1 \geq i_t$. We also have $x_t \neq x_{t'}$ for $t \neq t'$. Since $c(x_t) = c_t \in C_{i_t} \setminus C_{i_t-1}$ and $c(x_{t'}) = c_{t'} \in C_{i_{t'}} \setminus C_{i_{t'}-1}$, the case $c(x_t) = c(x_{t'})$ occurs only when $i_t = i_{t'}$, that is, $t = t'$. Our next goal is to find a larger rainbow path forest.

Claim 5.2 *$\mathcal{P}' = P_1 \cup \cdots \cup P_{\gamma n} \cup \{v_{i_0,1}x_0, v_{i_1,1}x_1, \ldots, v_{i_{s-1},1}x_{s-1}\} \setminus \{e(x_1), \ldots, e(x_{s-1})\}$ is a rainbow path forest.*

Proof. Our proof is divided into three steps, we prove that $\mathcal{P}'$ is rainbow in the first step, $\mathcal{P}'$ is a forest in the second step and $\mathcal{P}'$ is a path forest in the last step.

Note that $c(x_\ell) = c_\ell = c(v_{i_{\ell-1},1}x_{\ell-1})$, $c_s = c(v_{i_{s-1}}x_{s-1}) \in C_{i_s} = C_0$ for $\ell \in [s-2]$ and $C_0 \cap C(\mathcal{P}) = \emptyset$, thus $\mathcal{P}'$ is rainbow.

Suppose that there is a cycle. Let $v_{i_\ell,1}, x_\ell, u_1, u_2, \ldots, u_k, v_{i_\ell,1}$ be the cycle sequence. Since $x_\ell \in V(\mathcal{P})$, we may assume that $x_\ell = v_{t,j}$ for some t and j. Since $e(x_t)$ is absent in $\mathcal{P}'$, we have that $u_1 = v_{t,j+1}$. Let r be the smallest index for which $u_r \neq v_{t,j+r}$. By the definition of $\mathcal{P}'$, we have that $u_{r-1}u_r$ must be the form of $x_{\ell'}v_{\ell',1}$ for some $\ell \neq \ell'$ with $u_{r-1} = x_{\ell'}$ and $u_r = v_{\ell',1}$. However, $e(x_{\ell'}) = u_{r-2}u_{r-1}$ is absent in $\mathcal{P}'$, a contradiction.

Note that $\mathcal{P}'$ has maximum degree 2. Since the degrees of vertices $x_0, v_{i_0,1}, \ldots, v_{i_{s-1},1}$ increase from 1 to 2 and the degrees of other vertices do not increase from $\mathcal{P}$ to $\mathcal{P}'$. Thus, $\mathcal{P}'$ is a path forest. $\square$

From the above claim, we obtain a larger rainbow path forest $\mathcal{P}'$, contradicting the maximality of $\mathcal{P}$. $\square$

Let $m_i = |C_i| - |C_0|$ for $i \in [s]$. Note that $|C(G)| = |C_0| + e(\mathcal{P})$ and $e(\mathcal{P}) \le (1-4\delta)n$. We have $|N_{C_i}(v)| \ge |N(v)| - (|C(G)| - |C_i|) \ge (1-\delta)n - (|C_0| + e(\mathcal{P})) + (|C_0| + m_i) \ge 3\delta n + m_i$ for any vertex v.

From the definition of C_i, we have $|C_i| \ge |C_0| + |N_{C_{i-1}}(v_{i,1}) \cap \{v_{j,k} : k \ge 2, j \in [\gamma n]\}|$. From Claim 5.1, we have $|N_{C_{i-1}}(v_{i,1}) \cap \{v_{j,k} : k \ge 2, j \in [\gamma n]\}| \ge |N_{C_{i-1}}(v_{i,1})| - i$. Thus, $|C_i| \ge |C_0| + m_{i-1} + 3\delta n - i$ and $m_i \ge m_{i-1} + 3\delta n - i$. Iterating it, we obtain $m_i \ge 3i\delta n - \binom{i}{2}$. Let $i = \gamma n$, we have $2n \ge 3\gamma\delta n^2 - (\gamma n)^2/2$, which contradicts with $3\gamma\delta - \gamma^2/2 > 2/n$. $\square$

Proof. [The proof of Lemma 5.3] Suppose that $|P_1| < |V(\mathcal{P})| - 4b$, let $P_1 = v_1 v_2 \cdots v_k$ and let U be the set of vertices of these paths $P_2, \ldots, P_r$ which have length at least $2m$. Assume that $v_1 \in X$ without loss of generality. Notice that there are at most $2mr$ vertices of paths $P_2, \ldots, P_r$ which have length at most $2m$. Thus, $|U| \ge |V(\mathcal{P})| - |P_1| - 2mr \ge 3b$. Note that $||U \cap X| - |U \cap Y|| \le r \le b$. Thus, we can choose $U' \subseteq U \cap Y$ with $|U'| = b$ and $U^0 \subseteq U \cap X$ with $|U^0| = b$.

Suppose that there is an edge of H from v_1 to a vertex $x \in U'$ on some path P_i of length at least $2m$. Let $e_1 = e_2 = v_1 x$. Partition P_i into P_i^1 and P_i^2, we may assume that the length of P_i^1 is at least m without loss of generality. Let $P_1' = P_1 + e_1 + P_i^1$, $P_i' = P_i^2$ and $P_j' = P_j$ for all other j.

Suppose that $|N_H(v_1) \cap P_1| \ge b$. Let $W \subseteq N_H(v_1) \cap P_1$ with size b. $W^+ = \{v_{i-1} : v_i \in W\}$. Note that $W \subseteq Y$ and $W^+ \subseteq X$. Since $|W^+|, |U'| = b$, we have $|E_H(W^+, U')| \ge b+1$. There exists a vertex $v_j \in W^+$ such that $|N_H(v_j) \cap U'| \ge 2$. Thus, there exists some vertex $x \in N_H(v_j) \cap U'$ such that $c(v_1 v_{j+1}) \ne c(v_j x)$. Denote $v_1 v_{j+1}$ by e_1 and $v_j x$ by e_2. We may assume that $x \in P_i$ and the length of P_i is at least $2m$. Similarly, we can also partition P_i into P_i^1 and P_i^2. Without loss of generality, we may assume that the length of P_i^1 is at least m. Let $P_1' = P_1 + e_1 + e_2 - v_j v_{j+1} + P_i^1$, $P_i' = P_i^2$ and $P_j' = P_j$ for all other j.

Suppose that $|N_H(v_1) \cap P_1| < b$ and there is no edge from v_1 to U'. Since $N_H(v_1) \ge 3b$ and there are at most $2mr \le b$ vertices from v_1 to the vertices of paths P_i of length at most $2r$, there is a set $T \subseteq N_H(v_1) \setminus V(\mathcal{P})$ of size b and $T \subseteq Y$. Since $|U^0|, |T| \ge b$, there is an edge tx in H where $t \in T$ and $x \in U^0$. Since H is properly edge-colored, we have $c(v_1 t) \ne c(tx)$. Note that the vertex x is on some path P_i of length at least $2m$. Denote $v_1 t$ by e_1 and tx by e_2. Similarly, we can also partition P_i into P_i^1 and P_i^2. Without loss of generality, we may assume that the length of P_i^1 is at least m. Let $P_1' = P_1 + e_1 + e_2 + P_i^1$, $P_i' = P_i^2$ and $P_j' = P_j$ for all other j. $\square$

5.3 . Concluding remarks

The Andersen conjecture has not yet been proved, so the relevant conclusions about it can be further improved.

6 - Concluding remarks

6.1 . Rainbow Hamilton cycles in hypergraphs systems

Inspired by a series of very recent successes on rainbow matchings [110, 109, 112, 113], rainbow Hamilton cycles [28, 30, 77] and rainbow factors [27, 36, 120], we suspect the threshold for a rainbow spanning subgraph in (hyper)graph system is asymptotically same with the threshold for a spanning subgraph in a (hyper)graph.

Let $1 \leq d, \ell \leq k-1$. For $n \in (k-\ell)\mathbb{N}$, define $h_d^{\ell}(k,n)$ to be the smallest integer h such that every n-vertex k-graph H satisfying $\delta_d(H) \geq h$ contains a Hamilton ℓ-cycle. Han and Zhao [72] gave the result that

$$h_d^{k-1}(k,n) \geq \left(1 - \binom{t}{\lfloor t/2 \rfloor} \frac{\lceil t/2 \rceil^{\lceil t/2 \rceil}(\lfloor t/2 \rfloor + 1)^{\lfloor t/2 \rfloor}}{(t+1)^t} + o(1)\right)\binom{n}{t} \tag{6.1}$$

where $d \in [k-1]$ and $t = k-d$. In particular, $h_d^{k-1}(k,n) \geq (5/9+o(1))\binom{n}{2}, (5/8+o(1))\binom{n}{3}$ for $k-d = 2, 3$. Lang and Sanhueza-Matamala [105] conjectured that the minimum d-degree threshold for k-uniform tight Hamilton cycles coincides with the lower bounds in (6.1). This leads to the following conjecture.

Conjecture 6.1 *For every $k \geq 4, \mu > 0$, there exists n_0 such that the following holds for $n \geq n_0$. Given a k-graph system **G** $= \{G_i\}_{i\in[n]}$, if $\delta_{k-3}(G_i) \geq (5/8 + \mu)\binom{n}{3}$ for $i \in [n]$, then there is a **G**-rainbow Hamilton cycle.*

Furthermore, we believe the following holds.

Conjecture 6.2 *For every $k, d, \mu > 0$, there exists n_0 such that the following holds for $n \geq n_0$. Given a k-graph system **G** $= \{G_i\}_{i\in[n]}$, if $\delta_d(G_i) \geq h_d^{k-1}(k,n) + \mu\binom{n}{d}$ for $i \in [n]$, then there is a **G**-rainbow Hamilton cycle.*

6.2 . Exact results and the stability in graph and hypergraph systems

For rainbow Hamilton cycles in graph systems, the exact minimum degree threshold is known [77]. It is natural to ask whether exact results also hold for other structures in the rainbow setup, for example,

Question 6.1 *Given a graph system **G** $= \{G_i\}_{i\in[n]}$, if $\delta(G_i) \geq rn/(r+1)$, does there exist a **G**-rainbow K_r-factor?*

In the non-rainbow setup, exact results can typically be obtained by considering an extremal and non-extremal case separately, where the latter often gives stability, that is, even a smaller minimum degree condition is sufficient if the graph is far from any extremal construction. For graphs there are also more arguments that also work in the rainbow setup, for example, [111], thus, we can consider the stability in graph and hypergraph systems.

6.3 . Rainbow structures in random graph systems

Ferber, Han and Mao [53] gave the following results in random graph systems.

Theorem 6.1 ([53]) *For any $\varepsilon > 0$, $p = w(\log n/n)$ and a random graph system* ***G*** $= \{G_i\}_{i\in[n/2]}$ *on V where n is even, if each G_i is independent sample of $G(n,p)$ on the same vertex set $[n]$, then the following holds with high probability. For every spanning subgraphs H_i of G_i with $\delta(H_i) \geq (1/2+\varepsilon)np$, then there is an $\{H_i\}_{i\in[n/2]}$-rainbow perfect matching.*

Theorem 6.2 ([53]) *For any $\varepsilon > 0$, $p = w(\log n/n)$ and a random graph system* ***G*** $= \{G_i\}_{i\in[n]}$ *on V where n is even, if each G_i is independent sample of $G(n,p)$ on the same vertex set $[n]$, then the following holds with high probability. For every spanning subgraphs H_i of G_i with $\delta(H_i) \geq (1/2+\varepsilon)np$, then there is an $\{H_i\}_{i\in[n/2]}$-rainbow Hamilton cycle.*

Later, Anastos and Chakraborti [12] determined the threshold for the existence of a rainbow Hamilton cycle in a collection of random subgraphs of Dirac graphs in various settings.

Theorem 6.3 ([12]) *There exits a constant c such that the following holds. Suppose* ***G*** $= \{G_i\}_{i\in[n]}$ *is an n-vertex graph system and for every $i \in [n]$, $\delta(G_i) \geq n/2$, where $p = w(\log n/n)$. Then with high probability, there exist a* ***G*** $\cap\, G(n,p)$*-rainbow Hamilton cycle.*

Theorem 6.4 ([12]) *There exits a constant c such that the following holds. Suppose G is and n-vertex graph with $\delta(G) \geq n/2$, $p = w(\log n/n)$,* ***G*** $= \{G_i\}_{i\in[n]}$ *is an n-vertex random graph system, where each G_i is independently distributed as G_p, for $i \in [n]$. Then with high probability, there is a* ***G****-rainbow Hamilton cycle.*

Based on these results, we can consider the rainbow factors, rainbow trees in random graph system.

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Publications and manuscripts

(1) Xiaohan Cheng, **Bin Wang**, Jihui Wang, The adjacent vertex distinguishing edge choosability of planar graphs with maximum degree at least 11, *Discrete Applied Mathematics*, 313 (2022) 29-39.

(2) Yangyang Cheng, Jie Han, **Bin Wang**, Guanghui Wang, Rainbow spanning structures in graph and hypergraph systems, *Forum of Mathematics Sigma*, vol.11 : e95 (2023) 1-20.

(3) Yangyang Cheng, Jie Han, **Bin Wang**, Guanghui Wang, Donglei Yang, rainbow Hamilton cycle in hypergraph systems, submitted, 2021.

(4) Yucong Tang, **Bin Wang**, Guanghui Wang, Guiying Yan, Rainbow Hamilton cycle in hypergraph system, submitted, 2023.

(5) Hao Li, **Bin Wang**, Guanghui Wang, Long rainbow cycles in complete bipartite graphs, finished, 2023.