# Clique minors in double-critical graphs 

Martin Rolek* and Zi-Xia Song ${ }^{\dagger}$<br>Department of Mathematics<br>University of Central Florida<br>Orlando, FL 32816

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#### Abstract

A connected $t$-chromatic graph $G$ is double-critical if $G-\{u, v\}$ is $(t-2)$-colorable for each edge $u v \in E(G)$. A long-standing conjecture of Erdős and Lovász that the complete graphs are the only double-critical $t$-chromatic graphs remains open for all $t \geq 6$. Given the difficulty in settling Erdős and Lovász's conjecture and motivated by the well-known Hadwiger's conjecture, Kawarabayashi, Pedersen and Toft proposed a weaker conjecture that every doublecritical $t$-chromatic graph contains a $K_{t}$ minor and verified their conjecture for $t \leq 7$. Albar and Gonçalves recently proved that every double-critical 8-chromatic graph contains a $K_{8}$ minor, and their proof is computer-assisted. In this paper we prove that every double-critical $t$-chromatic graph contains a $K_{t}$ minor for all $t \leq 9$. Our proof for $t \leq 8$ is shorter and computer-free.


## 1 Introduction

All graphs in this paper are finite and simple. For a graph $G$ we use $|G|, e(G), \delta(G)$ to denote the number of vertices, number of edges and minimum degree of $G$, respectively. The degree of a vertex $v$ in a graph is denoted by $d_{G}(v)$ or simply $d(v)$. For a subset $S$ of $V(G)$, the subgraph induced by $S$ is denoted by $G[S]$ and $G-S=G[V(G) \backslash S]$. If $G$ is a graph and $K$ is a subgraph of $G$, then by $N(K)$ we denote the set of vertices of $V(G) \backslash V(K)$ that are adjacent to a vertex of $K$. If $V(K)=\{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x]=N(x) \cup\{x\}$, and similarly will use the same symbol for the graph induced by that set. If $u, v$ are distinct nonadjacent vertices of a graph $G$, then by $G+u v$ we denote the graph obtained from $G$ by adding an edge with ends $u$ and $v$. If $u, v$ are adjacent or equal, then we define $G+u v$ to be $G$.

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from a subgraph of $G$ by contracting edges. We write $G \geq H$ if $H$ is a minor of $G$. In those circumstances we also say that $G$ has an $H$ minor. A connected graph $G$ is called double-critical if for any edge $u v \in E(G)$, we have $\chi(G-\{u, v\})=\chi(G)-2$. The following long-standing Double-Critical Graph Conjecture is due to Erdős and Lovász [3].

[^0]Conjecture 1.1 Double-Critical Graph Conjecture (Erdős and Lovász [3]) For every integer $t \geq 1$, the only double-critical $t$-chromatic graph is $K_{t}$.

Conjecture 1.1 is a special case of the so-called Erdős-Lovász Tihany Conjecture [3]. It is trivially true for $t \leq 3$ and reasonably easy for $t=4$. Mozhan [8] and Stiebitz [10] independently proved Conjecture 1.1 for $t=5$.

Theorem 1.2 (Mozhan [8]; Stiebitz [10]) The only double-critical 5-chromatic graph is $K_{5}$.
Conjecture 1.1 remains open for all $t \geq 6$. Given the difficulty in settling Conjecture 1.1 and motivated by the well-known Hadwiger's conjecture [4], Kawarabayashi, Pedersen and Toft proposed a weaker conjecture.

Conjecture 1.3 (Kawarabayashi, Pedersen and Toft [6]) For every integer $t \geq 1$, every doublecritical $t$-chromatic graph contains a $K_{t}$ minor.

Conjecture 1.3 is a weaker version of Hadwiger's conjecture [4], which states that for every integer $t \geq 1$, every $t$-chromatic graph contains a $K_{t}$ minor. Conjecture 1.3 is true for $t \leq 5$ by Theorem 1.2, In the same paper [6], Kawarabayashi, Pedersen and Toft verified their conjecture for $t \in\{6,7\}$.

Theorem 1.4 (Kawarabayashi, Pedersen and Toft [6]) For every integer $t \leq 7$, every double-critical $t$-chromatic graph contains a $K_{t}$ minor.

Recently, Albar and Gonçalves [1] announced a proof for the case $t=8$.
Theorem 1.5 (Albar and Gonçalves [1]) Every double-critical 8-chromatic graph has a $K_{8}$ minor.
Our main result is the following next step.

Theorem 1.6 For integers $k, t$ with $1 \leq k \leq 9$ and $t \geq k$, every double-critical $t$-chromatic graph contains a $K_{k}$ minor.

We actually prove a much stronger result, the following.
Theorem 1.7 For $k \in\{6,7,8,9\}$, let $G$ be a $(k-3)$-connected graph with $k+1 \leq \delta(G) \leq 2 k-5$. If every edge of $G$ is contained in at least $k-2$ triangles and for any minimal separating set $S$ of $G$ and any $x \in S, G[S \backslash\{x\}]$ is not a clique, then $G \geq K_{k}$.

Theorem 1.6 follows directly from Proposition 2.1 (see below) and Theorem 1.7. Our proof of Theorem 1.7 closely follows the proof of the extremal function for $K_{9}$ minors by Song and Thomas 99 (see Theorem 1.10 below). Note that the proof of Theorem 1.4 for $k=7$ is about ten pages long and the proof of Theorem 1.5 is computer-assisted. Our proof of Theorem 1.6 is much shorter and computer-free for $k \leq 8$. For $k=9$, our proof is computer-assisted as it applies a computer-assisted lemma from [9] (see Lemma 1.13 below). Note that a computer-assisted proof of

Theorem 1.7 for all $k \leq 8$ (and hence computer-assisted proofs of Theorem 1.4 and Theorem (1.5) follows directly from Theorem 1.7 for $k=9$. (To see that, let $G$ and $k \leq 8$ be as in Theorem 1.7, and let $H$ be obtained from $G$ by adding $9-k$ vertices, each adjacent to every other vertex of the graph. Then $H$ is 6 -connected and satisfies all the other conditions as stated in Theorem 1.7. Thus $H \geq K_{9}$ and so $G \geq K_{k}$.) Conjecture 1.3 remains open for all $t \geq 10$. It seems hard to generalize Theorem 1.6

We need some known results to prove our main results. Before doing so, we need to define $(H, k)$-cockade. For a graph $H$ and an integer $k$, let us define an $(H, k)$-cockade recursively as follows. Any graph isomorphic to $H$ is an $(H, k)$-cockade. Now let $G_{1}, G_{2}$ be $(H, k)$-cockades and let $G$ be obtained from the disjoint union of $G_{1}$ and $G_{2}$ by identifying a clique of size $k$ in $G_{1}$ with a clique of the same size in $G_{2}$. Then the graph $G$ is also an $(H, k)$-cockade, and every $(H, k)$-cockade can be constructed this way. We are now ready to state some known results. The following theorem is a result of Dirac [2] for $p \leq 5$ and Mader [7] for $p \in\{6,7\}$.

Theorem 1.8 (Dirac [2]; Mader [7]) For every integer $p \in\{1,2, \ldots, 7\}$, a graph on $n \geq p$ vertices and at least $(p-2) n-\binom{p-1}{2}+1$ edges has a $K_{p}$ minor.

Jørgensen [5] and later Song and Thomas [9] generalized Theorem 1.8 to $p=8$ and $p=9$, respectively, as follows.

Theorem 1.9 (Jørgensen [5]) Every graph on $n \geq 8$ vertices with at least $6 n-20$ edges either contains a $K_{8}$-minor or is isomorphic to a ( $K_{2,2,2,2,2}, 5$ )-cockade.

Theorem 1.10 (Song and Thomas [9]) Every graph on $n \geq 9$ vertices with at least $7 n-27$ edges either contains a $K_{9}$-minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a ( $K_{1,2,2,2,2,2}, 6$ )cockade.

In our proof of Theorem 1.7, we need to examine graphs $G$ such that $k+1 \leq|G| \leq 2 k-5$, $\delta(G) \geq k-2$ and $G \nsupseteq K_{k} \cup K_{1}$. We shall use the following results. Lemma 1.11 is a result of Jørgensen [5].

Lemma 1.11 (Jørgensen [5]) Let $G$ be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex $x$ in $G, G-x$ is not contractible to $K_{6}$. Then $G$ is one of the graphs $K_{2,2,2,2}, K_{3,3,3}$ or the complement of the Petersen graph.

Lemma 1.11 implies Lemma 1.12 below. To see that, let $G$ be a graph satisfying the conditions given in Lemma 1.12, By applying Lemma 1.11 to the graph obtained from $G$ by adding $6-t$ vertices, each adjacent to every other vertex of the graph, we see that $G \geq K_{t} \cup K_{1}$.

Lemma 1.12 For $t \in\{1,2,3,4,5\}$, let $G$ be a graph with $n \leq 2 t-1$ vertices and $\delta(G) \geq t$. Then $G \geq K_{t} \cup K_{1}$.

Lemma 1.13 is a result of Song and Thomas [9. Note that the proof of Lemma 1.13 is computerassisted.

Lemma 1.13 (Song and Thomas [9]) Let $G$ be a graph with $|G| \in\{9,10,11,12,13\}$ such that $\delta(G) \geq 7$. Then either $G \geq K_{7} \cup K_{1}$, or $G$ satisfies the following
(A) either $G$ is isomorphic to $K_{1,2,2,2,2}$, or $G$ has four distinct vertices $a_{1}, b_{1}, a_{2}, b_{2}$ such that $a_{1} a_{2}, b_{1} b_{2} \notin E(G)$ and for $i=1,2$ the vertex $a_{i}$ is adjacent to $b_{i}$, the vertices $a_{i}, b_{i}$ have at most four common neighbors, and $G+a_{1} a_{2}+b_{1} b_{2} \geq K_{8}$,
(B) for any two sets $A, B \subseteq V(G)$ of cardinality at least five such that neither is complete and $A \cup B$ includes all vertices of $G$ of degree at most $|G|-2$, either
(B1) there exist $a \in A$ and $b \in B$ such that $G^{\prime} \geq K_{8}$, where $G^{\prime}$ is obtained from $G$ by adding all edges $a a^{\prime}$ and $b b^{\prime}$ for $a^{\prime} \in A-\{a\}$ and $b^{\prime} \in B-\{b\}$, or
(B2) there exist $a \in A-B$ and $b \in B-A$ such that $a b \in E(G)$ and the vertices $a$ and $b$ have at most five common neighbors in G , or
(B3) one of $A$ and $B$ contains the other and $G+a b \geq K_{7} \cup K_{1}$ for all distinct nonadjacent vertices $a, b \in A \cap B$.

## 2 Basic properties of non-complete double-critical graphs

We begin with basic properties of non-complete double-critical $k$-chromatic graphs established in [6]. We only list those that will be used in our proofs.

Proposition 2.1 (Kawarabayashi, Pedersen and Toft [6]) If $G$ is a non-complete double-critical $k$-chromatic graph, then the following hold:
(a) $\delta(G) \geq k+1$.
(b) Every edge $x y \in E(G)$ belongs to at least $k-2$ triangles.
(c) $G$ is 6 -connected and no minimal separating set of $G$ can be partitioned into two sets $A$ and $B$ such that $G[A]$ and $G[B]$ are edge-empty and complete, respectively.

Two proper vertex-colorings $c_{1}$ and $c_{2}$ of a graph $G$ are equivalent if, for all $x, y \in V(G)$, $c_{1}(x)=c_{1}(y)$ iff $c_{2}(x)=c_{2}(y)$. Two vertex-colorings $c_{1}$ and $c_{2}$ of a graph $G$ are equivalent on a set $A \subseteq V(G)$ if the restrictions $c_{1 \mid A}$ and $c_{2 \mid A}$ to $A$ are equivalent on the subgraph $G[A]$. Let $S$ be a separating set of $G$, and let $G_{1}, G_{2}$ be connected subgraphs of $G$ such that $G_{1} \cup G_{2}=G$ and $G_{1} \cap G_{2}=G[S]$. If $c_{1}$ is a $k$-coloring of $G_{1}$ and $c_{2}$ is a $k$-coloring of $G_{2}$ such that $c_{1}$ and $c_{2}$ are equivalent on $S$, then it is clear that $c_{1}$ and $c_{2}$ can be combined to a $k$-coloring of $G$ by a suitable permutation of the color classes of, say $c_{1}$. The main technique in the proof of Proposition 2.1(c) involves reassigning and permuting the colors on a separating set $S$ of a non-complete doublecritical $k$-chromatic graph $G$ so that $c_{1}$ and $c_{2}$ are equivalent on $S$ to obtain a contradiction, where
$c_{1}$ is a $(k-1)$-coloring of $G_{1}$ and $c_{2}$ is a $(k-1)$-coloring of $G_{2}$. It seems hard to use this idea to prove that every non-complete double-critical $k$-chromatic graph is 7 -connected, but we can use it to say a bit more about minimal separating sets of size 6 in non-complete double-critical graphs.

Lemma 2.2 Suppose $G$ is a non-complete double-critical $k$-chromatic graph. If $S$ is a minimal separating set of $G$ with $|S|=6$, then either $G[S] \subseteq K_{3,3}$ or $G[S] \subseteq K_{2,2,2}$.

Proof. By Propostion [2.1(c), $G$ is 6 -connected. Let $S=\left\{v_{1}, \ldots, v_{6}\right\} \subset V(G)$ be a minimal separating set of $G$ such that neither $G[S] \subseteq K_{3,3}$ nor $G[S] \subseteq K_{2,2,2}$. Let $G_{1}$ and $G_{2}$ be subgraphs of $G$ such that $G_{1} \cup G_{2}=G, G_{1} \cap G_{2}=S$, and there are no edges from $G_{1}-S$ to $G_{2}-S$. Since $k \geq 6$ by Theorem 1.2, we have $\delta(G) \geq 7$ by Propostion 2.1(囵). In particular, since $|S|=6$, there must exist at least one edge $y_{i} z_{i}$ in $G_{i}-S$ for $i \in\{1,2\}$. It follows then that $G_{i}$ is $(k-2)$-colorable since it is a subgraph of $G-\left\{y_{3-i}, z_{3-i}\right\}$. Let $c_{1}, c_{2}$ be $(k-2)$-colorings of $G_{1}$ and $G_{2}$, respectively. For $i=1,2$, define $\left|c_{i}(A)\right|$ to be the number of distinct colors assigned to the vertices of $A$ by $c_{i}$ for any $A \subseteq S$. Clearly $c_{1}$ and $c_{2}$ are not equivalent on $S$, otherwise $c_{1}$ and $c_{2}$, after a suitable permutation of the colors of $c_{2}$, can be combined to a $(k-2)$-coloring of $G$, a contradiction. By Proposition 2.1(c), $\alpha(G[S]) \leq 4$ and so neither $c_{1}$ nor $c_{2}$ applies the same color to more than four vertices of $S$. Utilizing a new color, say $\beta$, we next redefine the colorings $c_{1}$ and $c_{2}$ so that $c_{1}$ and $c_{2}$ are $(k-1)$-colorings of $G_{1}$ and $G_{2}$, respectively, and are equivalent on $S$. This yields a contradiction, as $c_{1}$ and $c_{2}$, after a suitable permutation of the colors of $c_{2}$, can be combined to a ( $k-1$ )-coloring of $G$.

Suppose that one of the colorings $c_{1}$ and $c_{2}$, say $c_{1}$, assigns the same color to four vertices of $S$, say $c_{1}\left(v_{3}\right)=c_{1}\left(v_{4}\right)=c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)$. Then $\left\{v_{3}, v_{4}, v_{5}, v_{6}\right\}$ is an independent set in $G$. By Proposition 2.1(c), we must have $v_{1} v_{2} \notin E(G)$. But then $G[S] \subseteq K_{2,2,2}$, a contradiction. Thus neither $c_{1}$ nor $c_{2}$ assigns the same color to four distinct vertices of $S$.

Next suppose that one of the colorings $c_{1}$ and $c_{2}$, say $c_{1}$, assigns the same color to three vertices of $S$, say $c_{1}\left(v_{4}\right)=c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)$. Then $\left\{v_{4}, v_{5}, v_{6}\right\}$ is an independent set in $G$. Since $G[S] \nsubseteq K_{3,3}$, we have $\left|c_{2}\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right| \geq 2$. If $\left|c_{2}\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right|=2$, we may assume that $c_{2}\left(v_{2}\right)=c_{2}\left(v_{3}\right)$. Then $\left\{v_{2}, v_{3}\right\}$ is an independent set. Then redefining $c_{2}\left(v_{4}\right)=c_{2}\left(v_{5}\right)=c_{2}\left(v_{6}\right)=\beta$ and $c_{1}\left(v_{2}\right)=c_{1}\left(v_{3}\right)=$ $\beta$ will make $c_{1}$ and $c_{2}$ equivalent on $S$, a contradiction. Thus $\left|c_{2}\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right|=3$ and so $c_{2}$ assigns distinct colors to each of $v_{1}, v_{2}, v_{3}$. We redefine $c_{2}\left(v_{4}\right)=c_{2}\left(v_{5}\right)=c_{2}\left(v_{6}\right)=\beta$. Clearly $c_{1}$ and $c_{2}$ are equivalent on $S$ if $c_{1}$ assigns distinct colors to each of $v_{1}, v_{2}, v_{3}$. Thus $\left|c_{1}\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right| \leq 2$. Since $G[S] \nsubseteq K_{3,3}$, we have $\left|c_{1}\left(\left\{v_{1}, v_{2}, v_{3}\right\}\right)\right|=2$. We may assume that $c_{1}\left(v_{2}\right)=c_{1}\left(v_{3}\right)$. Now redefining $c_{1}\left(v_{3}\right)=\beta$ yields that $c_{1}$ and $c_{2}$ are equivalent on $S$. This proves that neither $c_{1}$ nor $c_{2}$ assigns the same color to three distinct vertices of $S$. Thus $6 \geq\left|c_{i}(S)\right| \geq 3(i=1,2)$. Since $G[S] \nsubseteq K_{2,2,2}$, we have $\left|c_{i}(S)\right| \geq 4(i=1,2)$. We may assume that $\left|c_{1}(S)\right| \geq\left|c_{2}(S)\right|$. Then $\left|c_{2}(S)\right| \leq 5$, for otherwise $c_{1}$ and $c_{2}$ are equivalent on $S$. Thus $5 \geq\left|c_{2}(S)\right| \geq 4$.

Suppose that $\left|c_{2}(S)\right|=5$. Then $\left|c_{1}(S)\right|=5$ or $\left|c_{1}(S)\right|=6$. We can make $c_{1}$ and $c_{2}$ equivalent on $S$ by assigning color $\beta$ to one of the two vertices that are colored the same color by $c_{1}$ (if $\left.\left|c_{1}(S)\right|=5\right)$ and $c_{2}$. Thus $\left|c_{2}(S)\right|=4$. Since neither $c_{1}$ nor $c_{2}$ assigns the same color to more than two distinct vertices of $S$, we may assume that $c_{2}\left(v_{3}\right)=c_{2}\left(v_{4}\right)$ and $c_{2}\left(v_{5}\right)=c_{2}\left(v_{6}\right)$. Then $v_{3} v_{4} \notin E(G)$ and $v_{5} v_{6} \notin E(G)$. Since $G[S] \nsubseteq K_{2,2,2}$, we have $v_{1} v_{2} \in E(G)$. Thus $c_{1}\left(v_{1}\right) \neq c_{1}\left(v_{2}\right)$. We may assume that $c_{1}\left(v_{3}\right) \neq c_{1}\left(v_{4}\right)$ as $c_{1}$ and $c_{2}$ are not equivalent on $S$. If $\left|c_{1}(S)\right|=6$, then redefining $c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)=\beta$ and $c_{2}\left(v_{3}\right)=\beta$ will make $c_{1}$ and $c_{2}$ equivalent. If $\left|c_{1}(S)\right|=5$, then at least one of $v_{3}, v_{4}, v_{5}, v_{6}$ shares a color with another vertex of $S$, say $c_{1}\left(v_{6}\right)=c_{1}\left(v_{i}\right)$ for some $i \in\{1, \ldots, 5\}$. Then redefining $c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)=\beta$ and $c_{2}\left(v_{3}\right)=\beta$ will again make $c_{1}$ and $c_{2}$ equivalent. Thus $\left|c_{1}(S)\right|=4$. Suppose that one of $v_{1}$ or $v_{2}$ shares a color with another vertex of $S$. Since $v_{1} v_{2} \in E(G)$, we may assume by symmetry that $c_{1}\left(v_{1}\right)=c_{1}\left(v_{3}\right)$. If $c_{1}\left(v_{5}\right)$ and $c_{1}\left(v_{6}\right)$ are the two colors each assigned to only a single vertex of $S$ by $c_{1}$, then we also have $c_{1}\left(v_{2}\right)=c_{2}\left(v_{4}\right)$. Now redefining $c_{1}\left(v_{3}\right)=c_{1}\left(v_{4}\right)=\beta$ and $c_{2}\left(v_{5}\right)=\beta$ will make $c_{1}$ and $c_{2}$ equivalent. Hence one of the colors $c_{1}\left(v_{5}\right)$ and $c_{1}\left(v_{6}\right)$ is assigned to two vertices of $S$, say $c_{1}\left(v_{6}\right)=c_{1}\left(v_{i}\right)$ for some $i \in\{2,4,5\}$. If $i=2$ then redefine $c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)=\beta$ and $c_{2}\left(v_{1}\right)=c_{2}\left(v_{3}\right)=\beta$, if $i=4$ then redefine $c_{1}\left(v_{3}\right)=c_{1}\left(v_{4}\right)=\beta$ and $c_{2}\left(v_{6}\right)=\beta$, and if $i=5$ then redefine $c_{1}\left(v_{3}\right)=\beta$ and $c_{2}\left(v_{3}\right)=\beta$, and in each case $c_{1}$ is equivalent to $c_{2}$. Therefore $c_{1}\left(v_{1}\right)$ and $c_{1}\left(v_{2}\right)$ are the two colors assigned to only a single vertex of $S$ by $c_{1}$. Since $c_{1}$ and $c_{2}$ are not equivalent, we must have, say $c_{1}\left(v_{3}\right)=c_{1}\left(v_{5}\right)$ and $c_{1}\left(v_{4}\right)=c_{1}\left(v_{6}\right)$. Now redefining $c_{1}\left(v_{5}\right)=c_{1}\left(v_{6}\right)=\beta$ and $c_{2}\left(v_{3}\right)=\beta$ will make $c_{1}$ and $c_{2}$ equivalent.

## 3 Proofs of Theorem 1.7 and Theorem 1.6

In this section we first prove Theorem 1.7.

Proof. Let $G$ be a graph as in the statement with $n$ vertices. By assumption, we have
(1) $k+1 \leq \delta(G) \leq 2 k-5$ and $\delta(N(x)) \geq k-2$ for any $x$ in $G$; and
(2) $G$ is $(k-3)$-connected and for any minimal separating set $S$ of $G$ and any $x \in S, G[S \backslash\{x\}]$ is not a complete subgraph.

We first show that the statement is true for $k=6$. Then $G$ is 3 -connected with $\delta(G)=7$. The statement is trivially true if $G$ is complete, so we may assume $G$ is not complete. Let $x \in V(G)$ be a vertex of degree 7. By (1), $\delta(N(x)) \geq 4$, and so $e(N(x)) \geq 14$. If $e(N(x)) \geq 16$, then by Theorem [1.8, $N(x) \geq K_{5}$ and so $G \geq N[x] \geq K_{6}$. If $e(N(x))=15$, then let $K$ be a component of $G-N[x]$ with $|N(K)|$ minimum. By (2), $|N(K)| \geq 3$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be non-adjacent in $N(x)$ and let $P$ be a ( $y, z$ )-path with interior vertices in $K$. We see that $G \geq K_{6}$ by contracting all but one of the edges of $P$. So we may assume that $e(N(x))=14$, and so $N(x)$ is 4-regular and $\overline{N(x)}$ is 2-regular. Thus $\overline{N(x)}$ is then either isomorphic to $C_{7}$ or to $C_{4} \cup C_{3}$, and
in both cases it is easy to see that $N(x) \geq K_{5}$ and thus $G \geq K_{6}$, as desired. Hence we may assume $7 \leq k \leq 9$.

Suppose for a contradiction that $G \nsupseteq K_{k}$. We next prove the following.
(3) Let $x \in V(G)$ be such that $k+1 \leq d(x) \leq 2 k-5$. Then there is no component $K$ of $G-N[x]$ such that $N\left(K^{\prime}\right) \cap M \subseteq N(K)$ for every component $K^{\prime}$ of $G-N[x]$, where $M$ is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.

Proof. Suppose such a component $K$ exists. Among all vertices $x$ with $k+1 \leq d(x) \leq 2 k-5$ for which such a component exists, choose $x$ to be of minimal degree, and among all such components $K$ of $G-N[x]$, choose $K$ such that $|N(K)|$ is minimum. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M-N(K) \neq \emptyset$, and let $y \in M \backslash N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y)<d(x)$. Let $J$ be the component of $G-N[y]$ containing $K$. Since $d(y)<d(x)$ the choice of $x$ implies that $N(x) \backslash N[y] \nsubseteq V(J)$. Let $H=N(x) \backslash(N[y] \cup N(K))$. We have $d_{G}(z) \geq d_{G}(y)$ for all $z \in V(H)$ by the choice of $y$. Let $t=|V(H)|$. Then $t \geq 2$, for otherwise the vertex $y$ and component $H$ contradict the choice of $x$. On the other hand $t \leq d(x)-d(y) \leq(2 k-5)-(k+1)=k-6 \leq 3$ and so $k \geq 8$. Notice that $t=2$ when $k=8$. From (1) applied to $y$ we deduce that $N(y) \cap N(x)$ has minimum degree at least $k-3$. Let $L$ be the subgraph of $G$ induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of $L$ consists of edges of $N(x) \cap N(y)$, edges incident with $y$, and edges incident with $V(H)$. Clearly, $e(L-V(H), H)=\sum_{z \in V(H)}(d(z)-1)-2 e(H) \geq t(d(y)-1)-2 e(H)$. Thus

$$
\begin{aligned}
e(L) & \geq \frac{(k-3)(d(y)-1)}{2}+d(y)-1+e(L-V(H), H)+e(H) \\
& \geq \frac{(k-3)(d(y)-1)}{2}+d(y)-1+t(d(y)-1)-e(H) \\
& \geq \frac{(k-3)(d(y)-1)}{2}+d(y)-1+t(d(y)-1)-\frac{1}{2} t(t-1) \\
& \geq\left\{\begin{array}{cr}
5(d(y)+2)+\frac{d(y)}{2}-\frac{33}{2} & \text { if } \quad k=8 \\
6(d(y)+t)+(t-2) d(y)-4-7 t-\frac{1}{2} t(t-1) \quad \text { if } & k=9
\end{array}\right. \\
& \geq(k-3)|V(L)|-\binom{k-2}{2}+1,
\end{aligned}
$$

because $d(y) \geq k+1$ and $2 \leq t \leq k-6$. If $k=9$, since $12 \leq|V(L)| \leq 13$ the graph $L$ is not a ( $K_{2,2,2,2,2}, 5$-cockade. By Theorem 1.8 and Theorem 1.9, $N(x) \geq L \geq K_{k-1}$. Thus $G \geq N[x] \geq K_{k}$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) \geq K_{k-2} \cup K_{1}$, then $N(x)$ has a vertex $y$ such that $N(x)-y \geq K_{k-2}$. If $y \notin M$, then $N(x) \geq K_{k-1}$. Otherwise, by contracting the connected set $V(K) \cup\{y\}$ we can contract $K_{k-1}$ onto $N(x)$. Thus in either case $G \geq K_{k}$, a contradiction. Thus $N(x) \nsupseteq K_{k-2} \cup K_{1}$. If $k \leq 8$, by Lemma 1.11 and Lemma 1.12, we have $k=8$ and $N(x)$ is either $K_{3,3,3}$ or $\bar{P}$, where $\bar{P}$ is the complement of the Petersen graph. If $N(x)=\bar{P}$, it can be easily checked that $\bar{P}+y z \geq K_{7}$ for any $y z \in E(P)$. By (2), $|N(K)| \geq 5$ and $N(K)$ is not complete. Let $y, z \in N(K)$ be non-adjacent in $N(x)$ and let $Q$ be a $(y, z)$-path with interior vertices in $K$. We see that $G \geq K_{8}$ by contracting
all but one of the edges of $Q$, a contradiction. Thus $N(x)=K_{3,3,3}$, and so $M=N(x)$. Let $\left\{a_{1}, a_{2}, a_{3}\right\}$ and $\left\{b_{1}, b_{2}, b_{3}\right\}$ be the vertex sets of two disjoint triangles of $\overline{N(x)}$. Suppose $G-N[x]$ is 2 -connected or has at most two vertices. By Proposition 2.1(b), the vertices $a_{i}, b_{i}(\mathrm{i}=1,2)$ have at least two common neighbors in $G-N[x]$. Let $u_{1}, u_{2}$ (resp. $w_{1}, w_{2}$ ) be two distinct common neighbors of $a_{1}$ and $b_{1}$ (resp. $a_{2}$ and $b_{2}$ ) in $G-N[x]$. By Menger's Theorem, $G-N[x]$ contains two disjoint paths from $\left\{u_{1}, u_{2}\right\}$ to $\left\{w_{1}, w_{2}\right\}$ and so $G \geq N[x]+a_{1} a_{2}+b_{1} b_{2} \geq K_{8}$, a contradiction. Thus $G-N[x]$ has at least three vertices and is not 2-connected. If $G-N[x]$ is disconnected, let $H_{1}=K$ and $H_{2}$ be another connected component of $G-N[x]$. If $G-N[x]$ has a cut-vertex, say $w$, let $H_{1}$ be a connected component of $G-N[x]-w$ and let $H_{2}=G-N[x]-V\left(H_{1}\right)$. In either case, $H_{1}$ and $H_{2}$ are disjoint connected subgraphs of $G-N[x]$ such that $M \subseteq N\left(H_{1}\right) \cup N\left(H_{2}\right)$ (because we have shown that $M \subseteq N(K)$ ). Thus $N\left(H_{1}\right) \cup N\left(H_{2}\right)=N(x)$ because $M=N(x)$. By (2), $N\left(H_{i}\right)$ is not complete and $\left|N\left(H_{i}\right)\right| \geq 4$ since $k=8$. Thus each of $N\left(H_{1}\right)$ and $N\left(H_{2}\right)$ must contain at least one edge of $\overline{N(x)}$. Since $N(x)=K_{3,3,3}$ and $N\left(H_{1}\right) \cup N\left(H_{2}\right)=N(x)$, we may thus assume that $a_{1} a_{2} \in \overline{N\left(H_{1}\right)}$ and $b_{1} b_{2} \in \overline{N\left(H_{2}\right)}$. By contracting $H_{1}$ onto $a_{1}$ and $H_{2}$ onto $b_{1}$ we see that $G \geq N[x]+a_{1} a_{2}+b_{1} b_{2} \geq K_{8}$, a contradiction. This proves that $k=9$ and so by Lemma 1.13, we may assume that $N(x)$ satisfies properties (A) and (B).

Since $d(x) \geq 10, N(x) \neq K_{1,2,2,2,2}$. If $G-N[x]$ is 2 -connected or has at most two vertices, then by property (A) and (2), the set $N(x)$ has four distinct vertices $a_{1}, b_{1}, a_{2}, b_{2}$ such that $a_{1} a_{2}, b_{1} b_{2} \notin$ $E(G), N(x)+a_{1} a_{2}+b_{1} b_{2} \geq K_{8}$ and for $i=1,2$ the vertex $a_{i}$ is adjacent to $b_{i}$, and the vertices $a_{i}, b_{i}$ have at least two common neighbors in $G-N[x]$. Let $u_{1}, u_{2}$ (resp. $w_{1}, w_{2}$ ) be two distinct common neighbors of $a_{1}$ and $b_{1}$ (resp. $a_{2}$ and $b_{2}$ ) in $G-N[x]$. By Menger's Theorem, $G-N[x]$ contains two disjoint paths from $\left\{u_{1}, u_{2}\right\}$ to $\left\{w_{1}, w_{2}\right\}$ and so $G \geq N[x]+a_{1} a_{2}+b_{1} b_{2} \geq K_{9}$, a contradiction. Thus $G-N[x]$ has at least three vertices and is not 2-connected. If $G-N[x]$ is disconnected, let $H_{1}=K$ and $H_{2}$ be another connected component of $G-N[x]$. If $G-N[x]$ has a cut-vertex, say $w$, let $H_{1}$ be a connected component of $G-N[x]-w$ and let $H_{2}=G-N[x]-V\left(H_{1}\right)$. In either case, $H_{1}$ and $H_{2}$ are disjoint connected subgraphs of $G-N[x]$ such that $M \subseteq N\left(H_{1}\right) \cup N\left(H_{2}\right)$ (because we have shown that $M \subseteq N(K)$ ). For $i=1,2$ let $A_{i}=N\left(H_{i}\right) \cap N(x)$. By (2), $A_{i}$ is not complete and $\left|A_{i}\right| \geq 5$ for $i=1,2$. By property (B), $A_{1}$ and $A_{2}$ satisfy properties (B1), (B2) or (B3).

Suppose first that $A_{1}$ and $A_{2}$ satisfy property (B1). Then there exist $a_{i} \in A_{i}$ such that $N(x)+\left\{a_{1} a: a \in A_{1} \backslash\left\{a_{1}\right\}\right\}+\left\{a_{2} a: a \in A_{2} \backslash\left\{a_{2}\right\}\right\} \geq K_{8}$. By contracting the connected sets $V\left(H_{1}\right) \cup\left\{a_{1}\right\}$ and $V\left(H_{2}\right) \cup\left\{a_{2}\right\}$ to single vertices, we see that $G \geq K_{9}$, a contradiction. Suppose next that $A_{1}$ and $A_{2}$ satisfy property (B2). Then there exist $a_{1} \in A_{1} \backslash A_{2}$ and $a_{2} \in A_{2} \backslash A_{1}$ such that $a_{1} a_{2} \in E(G)$ and the vertices $a_{1}$ and $a_{2}$ have at most five common neighbors in $N(x)$. Thus $a_{1}, a_{2} \in M$ by (1), and by another application of (1) there exists a common neighbor $u \in V(G) \backslash N[x]$ of $a_{1}$ and $a_{2}$. But $a_{1} \notin A_{2}$ and $a_{2} \notin A_{1}$, and hence $u \notin V\left(H_{1}\right) \cup V\left(H_{2}\right)$. Thus $G-N[x]$ is disconnected and $H_{1}=K$. But then $a_{2} \in M \subseteq N(K)=N\left(H_{1}\right)$, a contradiction. Thus we may assume that $A_{1}$ and $A_{2}$ satisfy (B3), and hence $A_{i} \subseteq A_{3-i}$ for some $i \in\{1,2\}$. As $M \subseteq A_{1} \cup A_{2}$, we have $M \subseteq N\left(H_{3-i}\right)$. Since $A_{i}$ is not complete, let $a, b \in A_{i}$ be distinct and not adjacent. By property
(B3), $N(x)+a b \geq K_{7} \cup K_{1}$. Let $P$ be an $(a, b)$-path with interior in $H_{i}$. By contracting all but one of the edges of the path $P$ and by contracting $H_{3-i}$ similarly as above, we see that $G \geq K_{9}$, a contradiction.
(4) $G-N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most $2 k-5$.

Proof. If $G-N[x]$ is not null, then it is disconnected by (3). Thus we may assume that $x$ is adjacent to every other vertex of $G$. Let $H=G-x$. Then $|H|=d(x)$ and $\delta(H) \geq k$. Thus $e(H) \geq \frac{k d(x)}{2}>(k-3) d(x)-\binom{k-2}{2}+1$ because $d(x) \leq 2 k-5$. By Theorem 1.8 and Theorem 1.9, $G-x$ has a $K_{k-1}$ minor and so the graph $G$ has a $K_{k}$ minor, a contradiction.
(5) Let $x \in V(G)$ be such that $k+1 \leq d(x) \leq 2 k-5$. Then there is no component $K$ of $G-N[x]$ such that $d_{G}(y) \geq 2 k-4$ for every vertex $y \in V(K)$.

Proof. Assume that such a component $K$ exists. Let $G_{1}=G-V(K)$ and $G_{2}=G[V(K) \cup N(K)]$. Let $d_{1}$ be the maximum number of edges that can be added to $G_{2}$ by contracting edges of $G$ with at least one end in $G_{1}$. More precisely, let $d_{1}$ be the largest integer so that $G_{1}$ contains disjoint sets of vertices $V_{1}, V_{2}, \ldots, V_{p}$ so that $G_{1}\left[V_{j}\right]$ is connected, $\left|N(K) \cap V_{j}\right|=1$ for $1 \leq j \leq p=|N(K)|$, and so that the graph obtained from $G_{1}$ by contracting $V_{1}, V_{2}, \ldots, V_{p}$ and deleting $V(G) \backslash\left(\bigcup_{j} V_{j}\right)$ has $e(N(K))+d_{1}$ edges. Let $G_{2}^{\prime}$ be a graph with $V\left(G_{2}^{\prime}\right)=V\left(G_{2}\right)$ and $e\left(G_{2}^{\prime}\right)=e\left(G_{2}\right)+d_{1}$ edges obtained from $G$ by contracting edges in $G_{1}$. By (1), $\left|G_{2}^{\prime}\right| \geq k+2$. If $e\left(G_{2}^{\prime}\right) \geq(k-2)\left|G_{2}^{\prime}\right|-\binom{k-1}{2}+2$, then by Theorem 1.8 and Theorem 1.9, $G \geq G_{2}^{\prime} \geq K_{k}$, a contradiction. Thus
$e\left(G_{2}\right)=e\left(G_{2}^{\prime}\right)-d_{1} \leq(k-2)\left|G_{2}\right|-\binom{k-1}{2}+1-d_{1}=(k-2)|N(K)|+(k-2)|K|-\binom{k-1}{2}+1-d_{1}$.
By contracting the edge $x z$, where $z \in N(K)$ has minimum degree $d$ in $N(K)$, we see that $d_{1} \geq$ $|N(K)|-d-1$ and hence

$$
\begin{equation*}
e\left(G_{2}\right) \leq(k-3)|N(K)|+(k-2)|K|-\binom{k-1}{2}+2+d \tag{a}
\end{equation*}
$$

Let $t=e_{G}(N(K), K)$. We have $e\left(G_{2}\right)=e(K)+t+e(N(K))$ and

$$
\begin{equation*}
2 e(K) \geq(2 k-4)|K|-t \tag{b}
\end{equation*}
$$

and hence

$$
\begin{equation*}
e\left(G_{2}\right) \geq(k-2)|K|+t / 2+d|N(K)| / 2 \tag{c}
\end{equation*}
$$

Since $N(x)$ has minimum degree at least $k-2$, it follows that the subgraph $N(K)$ of $N(x)$ has minimum degree at least $(k-2)-(d(x)-|N(K)|)$. Thus $d \geq(k-2)-(d(x)-|N(K)|) \geq$ $|N(K)|-k+3$. From (a) and (c) we get

$$
-t / 2 \geq-(k-3)|N(K)|+d(|N(K)|-2) / 2+\binom{k-1}{2}-2 \geq\left\{\begin{array}{lll}
-8 & \text { if } & k=7  \tag{d}\\
-14 & \text { if } & k=8 \\
-18 & \text { if } & k=9
\end{array}\right.
$$

where the second inequality becomes $\frac{t}{2} \leq 11$ when $|N(K)|=2 k-6$ and $k=7,8$, and the second inequality holds with equality only when $|N(K)|=10$ and $k=9$. Since $G$ is not contractible to $K_{k}$, we deduce from (b) and Theorem [1.8, Theorem 1.9 and Theorem 1.10 that $|K|<8$. The inequalities $e(K) \geq 5|K|-8$ when $k=7, e(K) \geq 6|K|-14$ when $k=8$, and $e(K) \geq 7|K|-18$ when $k=9$ imply $|K| \leq 3$. But every vertex of $K$ has degree at least $2 k-4$ and $N(K)$ is a proper subgraph of $N(x)$, and hence $|K|=3,|N(K)|=2 k-6$ and $\frac{t}{2}=3(k-3) \geq 12$ when $k=7,8$, and (d) holds with equality for $|N(K)|=12$ when $k=9$, contrary to our earlier observation of (d) that $\frac{t}{2} \leq 11$ when $|N(K)|=2 k-6$ and $k=7,8$, and (d) holds with equality only when $|N(K)|=10$ and $k=9$.

By (1) there is a vertex $x$ of degree $k+1, k+2, \ldots$, or $2 k-5$ in $G$. Choose such a vertex $x$ so that $G-N[x]$ has a component $K$ of minimum order. Then choose a vertex $y \in V(K)$ of least degree in $G$. Thus $k+1 \leq d_{G}(y) \leq 2 k-5$ by (1) and (5). Let $L$ be the component of $G-N[y]$ containing $x$. We claim that $N(L)$ contains all vertices of $N(y)$ that are not adjacent to all other vertices of $N(y)$. Indeed, let $z \in N(y)$ be not adjacent to some vertex of $N(y) \backslash\{z\}$. We may assume that $z \notin N(x)$, for otherwise $z \in N(L)$. Thus $z \in V(K)$, and hence $d_{G}(z) \geq d_{G}(y)$ by the choice of $y$. Thus $z$ has a neighbor $z^{\prime} \in N[x] \cup V(K) \backslash N[y]$. Then $z^{\prime} \in V(L)$, for otherwise the component of $G-N[y]$ containing $z^{\prime}$ would be a proper subgraph of $K$. Thus $z \in N(L)$. This proves our claim that $N(L)$ contains all vertices $z$ as above, contrary to (3). This contradiction completes the proof of Theorem 1.7.

## We are now ready to prove Theorem 1.6,

Proof. Let $G$ be a double-critical $t$-chromatic graph with $t \geq k$. The assertion is trivially true if $G$ is complete. By Theorem [1.2, we may assume that $t \geq 6$. By Proposition [2.1(囯), $\delta(G) \geq k+1$. By Theorem [1.8, Theorem 1.9 and Theorem [1.10, we have $\delta(G) \leq 2 k-5$. By Proposition 2.1(b), every edge of $G$ is contained in at least $k-2$ triangles. By Proposition [2.1](c), $G$ is 6 -connected and no minimal separating set of $G$ can be partitioned into a clique and an independent set. By Theorem 1.7, $G \geq K_{k}$, as desired.

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[^0]:    * Current address: Department of Mathematics, College of William and Mary. E-mail address: msrolek@wm.edu.
    ${ }^{\dagger}$ Corresponding author. E-mail address: Zixia.Song@ucf.edu.

