Partitioning a graph into cycles with a specified number of chords

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Abstract

For a graph G, let $\sigma_2(G)$ be the minimum degree sum of two non-adjacent vertices in G. A chord of a cycle in a graph G is an edge of G joining two non-consecutive vertices of the cycle. In this paper, we prove the following result, which is an extension of a result of Brandt et al. (J. Graph Theory 24 (1997) 165–173) for large graphs: For positive integers k and c, there exists an integer $f(k, c)$ such that, if G is a graph of order $n \geq f(k, c)$ and $\sigma_2(G) \geq n$, then G can be partitioned into k vertex-disjoint cycles, each of which has at least c chords.

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the readers to [3]. Let G be a graph. For a vertex v of G, we denote by $d_G(v)$ and $N_G(v)$ the degree and the neighborhood of v in G. Let $\delta(G)$ be the minimum degree of G and let $\sigma_2(G)$ be the minimum degree sum of two non-adjacent vertices in G, i.e., if G is non-complete, then $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G), u \neq v, uv \notin E(G)\}$; otherwise, let $\sigma_2(G) = +\infty$. If the graph G is clear from the context, we often omit the graph parameter G in the graph invariant. We denote by K_t the complete graph of order t. In this paper, "partition" and "disjoint" always mean "vertex-partition" and "vertex-disjoint", respectively.

A graph is *hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all the vertices of the graph. It is well known that determining whether a given graph is hamiltonian or not, is NP-complete. Therefore, it is natural to study sufficient conditions for hamiltonicity of graphs.

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In particular, since the approval of the following two theorems, various studies have considered degree conditions.

Theorem A (Dirac [7]) *Let G be a graph of order $n \geq 3$. If $\delta \geq \frac{n}{2}$, then G is hamiltonian.*

Theorem B (Ore [16]) *Let G be a graph of order $n \geq 3$. If $\sigma_2 \geq n$, then G is hamiltonian.*

In 1997, Brandt et al. generalized the above theorems by showing that the Ore condition, i.e., the σ_2 condition in Theorem B, guarantees the existence of a partition of a graph into a prescribed number of cycles.

Theorem C (Brandt et al. [4]) *Let k be a positive integer, and let G be a graph of order $n \geq 4k - 1$. If $\sigma_2 \geq n$, then G can be partitioned into k cycles, i.e., G contains k disjoint cycles $C_1, \ldots, C_k$ such that $\bigcup_{1 \leq p \leq k} V(C_p) = V(G)$.*

In order to generalize results on Hamilton cycles, degree conditions for partitioning graphs into a prescribed number of cycles with some additional conditions, have been extensively studied. See a survey paper [6].

On the other hand, Hajnal and Szemerédi (1970) gave the following minimum degree condition for graphs to be partitioned into k complete graphs of order t.

Theorem D (Hajnal and Szemerédi [12]) *Let k and t be integers with $k \geq 1$ and $t \geq 3$, and let G be a graph of order $n = tk$. If $\delta \geq \frac{t-1}{t}n$, then G can be partitioned into k subgraphs, each of which is isomorphic to K_t.*

In 2008, Kierstead and Kostochka improved the δ condition into the following σ_2 condition.

Theorem E (Kierstead and Kostochka [13]) *Let k and t be integers with $k \geq 1$ and $t \geq 3$, and let G be a graph of order $n = tk$. If $\sigma_2 \geq \frac{2(t-1)}{t}n - 1$, then G can be partitioned into k subgraphs, each of which is isomorphic to K_t.*

The above two theorems concern with the existence of an equitable (vertex-)coloring in graphs. In fact, Theorem D implies that a conjecture of Erdős [9] ("every graph of maximum degree at most $k - 1$ has an equitable k-coloring") is true. Motivated by this conjecture, Seymour [17] proposed a more general conjecture, which states that every graph of order $n \geq 3$ and of minimum degree at least $\frac{t-1}{t}n$ contains $(t-1)$-th power of a Hamilton cycle. It is also a generalization of Theorem A by including the case $t = 2$. In [15], Komlós et al. proved the Seymour's conjecture for sufficiently large graphs by using the Regularity Lemma. For other related results, see a survey paper [14].

In this paper, we focus on a relaxed structure of a complete subgraph in graphs as follows. For an integer $c \geq 1$, a cycle C in a graph G is called a *c-chorded cycle* if there are at least c edges between the vertices on the cycle C that are not edges of C, i.e., $\left|E(G[V(C)]) \setminus E(C)\right| \geq c$, where for a vertex subset X of G, $G[X]$ denotes the subgraph of G induced by X. We call each edge of $E(G[V(C)]) \setminus E(C)$ a *chord* of C. Since a Hamilton cycle of K_t has exactly $\frac{t(t-3)}{2}$ chords, we can regard a c-chorded cycle as a relaxed structure of K_t for $c = \frac{t(t-3)}{2}$. Concerning the existence of a partition into such structures, we give the following result. Here, for positive integers k and c, we define $f(k,c) = 8k^2c + 10kc - 4k + 2c + 1$.

Theorem 1 *Let k and c be positive integers, and let G be a graph of order $n \geq f(k,c)$. If $\sigma_2 \geq n$, then G can be partitioned into k c-chorded cycles.*

This theorem says that for a sufficiently large graph, the Ore condition also guarantees the existence of a partition into k subgraphs, each of which is a relaxed structure of a complete graph. The complete bipartite graph $K_{\frac{n-1}{2},\frac{n+1}{2}}$ (n is odd) shows the sharpness of the lower bound on the degree condition. But we do not know whether the order condition (the function $f(k,c)$) is sharp or not. It comes from our proof techniques.

Related results can be found in [1, 2, 5, 10, 11]. In these papers, degree conditions for *packing cycles with many chords* in a graph, i.e., finding a prescribed number of disjoint cycles with many chords (it may not form a partition of a graph), are given and some of the results are also generalizations of Theorem D.

In Section 2, we give lemmas which are obtained from arguments for hamiltonian problems. By using the lemmas, in Section 3, we first show that the collection of disjoint c-chorded cycles in a graph G satisfying the conditions of Theorem 1, can be transformed into a partition of G (Theorem 2 in Section 3). Then we show that Theorem 2 and a result on packing cycles lead to Theorem 1 as a corollary (see the last of Section 3). In Section 4, we give some remarks on the order condition and show that the order condition in Theorem 1 can be improved for the case of the Dirac condition.

2 Lemmas

We prepare terminology and notations which will be used in our proofs. Let G be a graph. For $v \in V(G)$ and $X \subseteq V(G)$, we let $N_X(v) = N_G(v) \cap X$ and $d_X(v) = |N_X(v)|$. For $V, X \subseteq V(G)$, let $N_X(V) = \bigcup_{v \in V} N_X(v)$. For a subgraph F of G, we define $\overline{E_G}(F) = E(G[V(F)]) \setminus E(F)$. A (u,v)-path in G is a path from a vertex u to a vertex v in G. We write a cycle (or a path) C with a given orientation by $\overrightarrow{C}$. If there exists no fear of confusion, we abbreviate $\overrightarrow{C}$ by C. Let C be an oriented cycle (or path). We denote by $\overleftarrow{C}$ the cycle (or the path) C with the reverse orientation. For $v \in V(C)$, we denote by v^+ and v^- the successor and the predecessor of v on $\overrightarrow{C}$, respectively. For $X \subseteq V(C)$, we define $X^+ = \{v^+ : v \in X\}$ and $X^- = \{v^- : v \in X\}$. For $u, v \in V(C)$, we denote by $C[u,v]$ the (u,v)-path on $\overrightarrow{C}$. The reverse sequence of $C[u,v]$ is denoted by $\overleftarrow{C}[v,u]$. In the rest of this paper, we often identify a subgraph F of G with its vertex set $V(F)$.

We next prepare some lemmas. In the proof, we use the technique for proofs concerning hamiltonian properties of graphs. To do that, in the rest of this section, we fix the following. Let k and c be positive integers, and let G be a graph of order n and L a fixed vertex subset of G. Let $C_1, \dots, C_k$ be k disjoint c-chorded cycles each with a fixed orientation in G, and suppose that $C^* := \bigcup_{1 \leq p \leq k} C_p$ is not a spanning subgraph of G. Let $H^* = G - C^*$ and H be a component of H^*. Assume that $C_1, \dots, C_k$ are chosen so that

(A1) $|V(C^*) \cap L|$ is as large as possible, and

(A2) $|C^*|$ is as large as possible, subject to (A1).

Then the choices lead to the following.

Lemma 1 *Let $C = C_p$ with $1 \le p \le k$, and let $v \in N_C(H)$ and $x \in V(H)$. Then (i) $v^+x \notin E(G)$, and (ii) $d_{H^*\cup C}(v^+) + d_{H^*\cup C}(x) \le |H^* \cup C| - 1$.*

Proof of Lemma 1. We let $\overrightarrow{P}$ be a (v^+, x)-path consisting of the path $C[v^+, v]$ and a (v, x)-path in $G[V(H) \cup \{v\}]$.

Suppose first that there exists a vertex a in $(N_P(v^+))^- \cap N_P(x)$, where the superscript $^-$ refers to the orientation of $\overrightarrow{P}$ (see Figure 1). Consider the cycle $D := v^+P[a^+, x]\overleftarrow{P}[a, v^+]$. Then by the definitions of P and D, we have $(\overline{E_G}(C)\setminus\{v^+a^+\})\cup\{vv^+\} \subseteq \overline{E_G}(D)$ or $\overline{E_G}(C)\cup\{aa^+\} \subseteq \overline{E_G}(D)$, and hence D is a c-chorded cycle in $G[V(H^* \cup C)]$. Moreover we also have $V(C) \subset V(P) = V(D)$. Therefore, by replacing C with D, this contradicts (A1) or (A2). Thus

$$(N_P(v^+))^- \cap N_P(x) = \emptyset. \tag{1}$$

This in particular implies that $v^+x \notin E(G)$. Thus (i) holds.

Suppose next that there exists a vertex b in $N_G(v^+) \cap N_G(x) \cap (V(H^* \cup C) \setminus V(P))$. Consider the cycle $D' := P[v^+, x]bv^+$. Then by the similar argument as above, replacing C with D' would violate (A1) or (A2). Thus $N_G(v^+) \cap N_G(x) \cap (V(H^* \cup C) \setminus V(P)) = \emptyset$. Combining this with (1), we get $d_{H^*\cup C}(v^+) + d_{H^*\cup C}(x) \le |H^* \cup C| - 1$. Thus (ii) holds. □

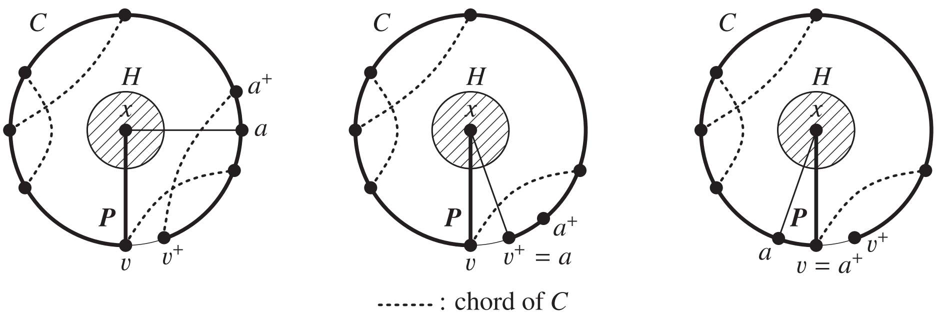

Figure 1: Lemma 1

Lemma 2 *Let $C = C_p$ with $1 \le p \le k$, and let $u^*, v^* \in (N_C(H))^- \cup (N_C(H))^+$ and $x \in V(H)$. If $\sigma_2(G) \ge n$, then the following hold.*

(i) $d_{C_q}(u^*) + d_{C_q}(x) \ge |C_q| + 1$ *for some q with $q \ne p$.*

(ii) $d_{C_{q'}}(u^*) + d_{C_{q'}}(v^*) \ge |C_{q'}| + 1$ *for some q' with $q' \ne p$.*

Proof of Lemma 2. Note that by Lemma 1-(i) [1], $u^*x, v^*x \notin E(G)$. Since $\sigma_2(G) \ge n$, it follows from Lemma 1-(ii) [1] that

$$d_{C^*-C}(u^*) + d_{C^*-C}(x) \ge n - (|H^* \cup C| - 1) = |C^* - C| + 1, \tag{2}$$

$$d_{C^*-C}(v^*) + d_{C^*-C}(x) \ge n - (|H^* \cup C| - 1) = |C^* - C| + 1. \tag{3}$$

[1]We use the symmetry of $\overrightarrow{C}$ and $\overleftarrow{C}$ for a vertex of $(N_C(H))^-$.

Then (2) and the Pigeonhole Principle yield that (i) holds. Since $d_{C_r}(x) \le |C_r|/2$ for $1 \le r \le k$ by Lemma 1-(i), combining (2) and (3), and the Pigeonhole Principle yield that (ii) holds. □

3 Proof of Theorem 1

In order to show Theorem 1, we first prove the following theorem.

Theorem 2 *Let k, c and G be the same as the ones in Theorem 1. Suppose that G contains k disjoint c-chorded cycles. If $\sigma_2 \ge n$, then G can be partitioned into k c-chorded cycles.*

In the proof of Theorem 2, we use the following lemma.

Lemma A (see Lemma 2.3 in [8]) *Let d be an integer, and let G be a 2-connected graph of order n and $a \in V(G)$. If $d_G(u) + d_G(v) \ge d$ for any two distinct non-adjacent vertices u, v of $V(G) \setminus \{a\}$, then G contains a cycle of order at least $\min\{d, n\}$.*

Proof of Theorem 2. Let L, $C_1, \dots, C_k$, C^* and H^* be the same ones as in the paragraph preceding Lemma 1 in Section 2.

Claim 1 *If H is a component of H^*, then $|N_{C_p}(H)| \le 2c$ for $1 \le p \le k$.*

Proof. Let H be a component of H^*. It suffices to consider the case $p = 1$. Suppose that $|N_{C_1}(H)| \ge 2c + 1$. Let $e_1, \dots, e_c$ be c distinct edges in $\overline{E_G}(C_1)$. Note that by Lemma 1-(i), $N_{C_1}(H) \cap (N_{C_1}(H))^+ = \emptyset$. Then, since $|N_{C_1}(H)| \ge 2c + 1$, we can take two distinct vertices v_1, v_2 in $N_{C_1}(H)$ such that

the end vertices of $e_1, \dots, e_c$ do not appear in $C_1[v_1^+, v_2^-]$, i.e., $e_1, \dots, e_c \in \overline{E_G}(C_1[v_2, v_1])$. (4)

We apply Lemma 2-(ii) with $(p, u^*, v^*) = (1, v_1^+, v_2^-)$. Then there exists another cycle C_q with $q \ne 1$, say $q = 2$, such that $d_{C_2}(v_1^+) + d_{C_2}(v_2^-) \ge |C_2| + 1$. This inequality implies that [2] there exists an edge $w^- w$ in $E(\overrightarrow{C_2})$ such that $v_1^+ w^-, v_2^- w \in E(G)$. We consider two cycles

$$D_1 := C_1[v_2, v_1]P[v_1, v_2] \text{ and } D_2 := C_1[v_1^+, v_2^-]C_2[w, w^-]v_1^+,$$

where $P[v_1, v_2]$ denotes a (v_1, v_2)-path in $G[V(H) \cup \{v_1, v_2\}]$ such that $V(P) \cap V(H) \ne \emptyset$. Then by (4), D_1 is a c-chorded cycle. Since $\overline{E_G}(C_2) \subseteq \overline{E_G}(D_2)$, D_2 is also a c-chorded cycle. Moreover, $V(D_1) \cap V(D_2) = \emptyset$ and $V(D_1) \cup V(D_2) = V(C_1) \cup V(C_2) \cup V(P)$. Hence, replacing C_1 and C_2 with D_1 and D_2, this contradicts (A1) or (A2). □

Now we define the fixed vertex subset L of G as follows:

$$L = \Big\{v \in V(G) : d_G(v) < \frac{n}{2}\Big\}.$$

Case 1. $|H^*| \ge \frac{n}{2} - 2kc + 1$.

[2] Change the orientation of C_2 if necessary.

Since G is connected, there exists a vertex $x \in V(H^*)$ and a cycle C_p, say $p = 1$, such that $N_{C_1}(x) \neq \emptyset$. Let $H^{**} = H^* - \{x\}$.

In this case, we show that the following claim holds.

Claim 2 *H^{**} contains a c-chorded cycle.*

Proof. We first define the following real number $\omega(c)$. Let $\omega(c)$ be the positive root of the equation $\frac{t(t-3)}{2} - c = 0$, i.e., $\omega(c) = \frac{\sqrt{8c+9}+3}{2}$. Since $|E(K_t)| - t = \frac{t(t-3)}{2}$, it follows that a Hamilton cycle of a complete graph of order at least $\lceil \omega(c) \rceil$ has at least c chords.

If $V(H^{**}) \subseteq L$, then by the definition of L, H^{**} is a complete graph, and hence a Hamilton cycle of H^{**} has at least c chords since $|H^{**}| \geq \frac{n}{2} - 2kc \geq \omega(c)$. Thus we may assume that $V(H^{**}) \setminus L \neq \emptyset$. Let H' be a component of H^{**} such that $V(H') \setminus L \neq \emptyset$. Note that by Claim 1, for $x' \in V(H') \setminus L$, $|H'| \geq d_{H'}(x') + 1 \geq \left(\frac{n}{2} - d_{C^*}(x') - |\{x\}|\right) + 1 \geq \frac{n}{2} - 2kc \geq 3$.

We define an induced subgraph B of H' as follows: If H' is not 2-connected, let B be an end block with a single cut vertex a such that $V(B) \setminus (\{a\} \cup L) \neq \emptyset$ (note that we can take such a block B because $|H'| \geq 3$ and hence H' has at least two end blocks); If H' is 2-connected, then let $B = H'$ and a be a vertex of H' such that $V(B) \setminus (\{a\} \cup L) \neq \emptyset$ (recall that $V(H') \setminus L \neq \emptyset$). Then by Claim 1, the definitions of B and a, it follows that for $b \in V(B) \setminus (\{a\} \cup L)$,

$$|B| \geq d_B(b) + 1 \geq \left(\frac{n}{2} - d_{C^*}(b) - |\{x\}|\right) + 1 \geq \frac{n}{2} - 2kc.$$

In particular, B is 2-connected since $\frac{n}{2} - 2kc \geq 3$. Moreover, we also see that

$$d_B(u) + d_B(v) \geq n - 4kc - 2|\{x\}| = n - 4kc - 2 \text{ for } u, v \in V(B) \setminus \{a\} \text{ with } u \neq v \text{ and } uv \notin E(G).$$

Hence, by Lemma A,

$$B \text{ contains a cycle } C \text{ of order at least } \min\{n - 4kc - 2, |B|\}.$$

To complete the proof of the claim, we show that the cycle C is a c-chorded cycle.

Suppose $G[V(C) \setminus \{a\}]$ is complete. Since $n \geq f(k, c)$ and $|B| \geq \frac{n}{2} - 2kc$, we have $|V(C) \setminus \{a\}| \geq \min\{n - 4kc - 3, |B| - 1\} \geq \omega(c)$, and hence it follows that C has at least c chords. Thus we may assume that there exist two distinct non-adjacent vertices u, v of $V(C) \setminus \{a\}$. Then by Claim 1, the definitions of B and a, we have

$$\begin{aligned}
d_C(u) + d_C(v) &\geq n - \left(d_{C^*}(u) + d_{C^*}(v)\right) - \left(d_{B-C}(u) + d_{B-C}(v)\right) - 2|\{x\}| \\
&\geq n - 4kc - 2(|B| - |C|) - 2 \\
&\geq n - 4kc - 2\Big(|B| - \min\{n - 4kc - 2, |B|\}\Big) - 2 \\
&= n - 4kc - 2 + 2 \cdot \min\{n - 4kc - 2 - |B|, 0\}.
\end{aligned}$$

Note that each C_i has order at least $\omega(c)$ because C_i has at least c chords, and hence

$$|B| \leq |H^{**}| = |H^* - \{x\}| = n - 1 - |C^*| \leq n - 1 - k \cdot \omega(c).$$

Since $n \geq f(k, c)$, it follows that

$$d_C(u) + d_C(v) \geq n - 4kc - 2 + 2 \cdot \min\{n - 4kc - 2 - (n - 1 - k \cdot \omega(c)), 0\}$$
$$= n - 12kc + 2k \cdot \omega(c) - 4 \geq c + 4.$$

This implies that C has at least c chords. □

Now let D_1 be a c-chorded cycle in H^{**} $(= H^* - \{x\})$. Recall that $N_{C_1}(x) \neq \emptyset$. Let $v \in N_{C_1}(x)$. Then by Lemma 2-(i), there exists a cycle C_q with $q \neq 1$, say $q = 2$, such that $d_{C_2}(v^+) + d_{C_2}(x) \geq |C_2| + 1$. This inequality implies that there exists an edge w^-w in $E(\overrightarrow{C_2})$ such that $v^+w^-, xw \in E(G)$. Let $D_2 = C_1[v^+, v]\, x\, C_2[w, w^-]v^+$. Then, since $\overline{E_G}(C_2) \subseteq \overline{E_G}(D_2)$, D_2 is a c-chorded cycle. Moreover, $V(D_1) \cap V(D_2) = \emptyset$ and $V(C_1) \cup V(C_2) \subset V(D_1) \cup V(D_2) \subseteq V(C_1) \cup V(C_2) \cup V(H^*)$. Hence, replacing C_1 and C_2 with D_1 and D_2 would violate (A1) or (A2), a contradiction.

This completes the proof of Case 1.

Case 2. $|H^*| < \frac{n}{2} - 2kc + 1$.

The following two claims are essential parts in this case.

Claim 3 (i) *$V(H^*) \subseteq L$ (in particular, H^* is complete) and $|H^*| \leq 2c + 1$.*

(ii) *$(V(C^*) \setminus N_{C^*}(H^*)) \cap L = \emptyset$.*

(iii) *$d_{C_p}(v) \geq |C_p| - 2kc + 1$ for $1 \leq p \leq k$ and $v \in V(C_p) \setminus N_{C_p}(H^*)$.*

Proof. We first show (i) and (ii). If there exists a vertex x of $V(H^*)$ such that $x \notin L$, then by Claim 1, $|H^*| \geq d_{H^*}(x) + |\{x\}| \geq (\frac{n}{2} - 2kc) + 1$, which contradicts the assumption of Case 2. Thus

$$V(H^*) \subseteq L.$$

In particular, H^* is a complete graph. Then by the definition of L, we have

$$(V(C^*) \setminus N_{C^*}(H^*)) \cap L = \emptyset.$$

This together with Claim 1 implies that $|V(C_p) \cap L| \leq 2c$ for $1 \leq p \leq k$. Therefore, if $|H^*| \geq 2c + 2$, then by replacing the cycle C_1 with a Hamilton cycle of H^*, this contradicts (A1). Thus we have [3]

$$|H^*| \leq 2c + 1.$$

We finally show (iii). Let $1 \leq p \leq k$ and $v \in V(C_p) \setminus N_{C_p}(H^*)$. We may assume that $p = 1$. Let x be an arbitrary vertex of H^*. Then by Claim 1, and since $v \notin N_{C_p}(H^*)$, we get

$$d_{C^*}(v) \geq n - d_{C^*}(x) - d_{H^*}(x) \geq n - 2kc - (|H^*| - 1) = |C^*| - 2kc + 1.$$

Since $d_{C_q}(v) \leq |C_q|$ for $2 \leq q \leq k$, we have $d_{C_1}(v) \geq |C_1| - 2kc + 1$. Thus (iii) holds. □

[3] This argument actually implies that $|H^*| \leq \max\{2c, 3\}$. But we make no attempt to optimize the upper bound on $|H^*|$ since it does not lead to a significant improvement of the condition on n.

Claim 4 *Let* $C = C_p$ *with* $1 \leq p \leq k$, *and* $w^- w \in E(\overrightarrow{C})$ *and* $S = N_C(H^*)$. *If* $|C| \geq 8kc + 10c - 4$, *then there exist two distinct chords* u_1v_1, u_2v_2 *of* C *satisfying the following conditions (A)–(C).*

(A) u_1, u_2, v_2, v_1 *are appear in the order along* $\overrightarrow{C}$,

(B) $w^-, w \in C[v_1, u_1]$ *and* $S \subseteq C[v_1, u_1] \cup C[u_2, v_2]$,

(C) $d_{C[v_1,u_1]}(u_1) \geq c + 2$ *and* $d_{C[u_2,v_2]}(u_2) \geq c + 2$.

Proof. Note that by Claim 3-(i), H^* consists of exactly one component, and hence Claim 1 yields that $|S| \leq 2c$. Note also that by Claim 3-(iii), $d_C(v) \geq |C| - 2kc + 1$ for $v \in V(C) \setminus S$.

We first define four vertices u_1, u_2, x, y of $V(C)$ by the following procedure (I)–(III) (the vertices u_1, u_2 will be the end vertices of the desired chords, and the vertices x, y will be candidates of the end vertices of the desired chords). See also Figure 2.

(I) Let u_1, u_2 be vertices of $V(C)$ such that

$$u_1 = u_2^-, \quad \text{(I-1)} \qquad \text{and} \quad u_1, u_2 \notin \{w^-\} \cup S. \quad \text{(I-2)}$$

Note that we can take such two vertices because $|C| \geq 8kc + 10c - 4$ and $|\{w^-\} \cup S| \leq 2c + 1$. Choose u_1, u_2 so that $|C[w, u_1]|$ is as small as possible. Then by the choice,

$$|C[w, u_1]| \leq 2|S| + |\{u_1\}| \leq 4c + 1. \quad \text{(I-3)}$$

(II) Since $d_C(u_1) \geq |C| - 2kc + 1 \geq c + 2$ and $u_1u_2 \in E(G)$, and by (I-1), (I-2), we can take a vertex x of $N_C(u_1)$ such that

$$w^- \in C[x, u_1], \quad \text{(II-1)} \qquad \text{and} \quad d_{C[x,u_1]}(u_1) \geq c + 2. \quad \text{(II-2)}$$

In fact, the vertex u_2 can be such a vertex x. Choose x so that $d_{C[x,u_1]}(u_1)$ is as small as possible, subject to (II-1) and (II-2). Then by the choice,

if $d_{C[w,u_1]}(u_1) \leq c + 1$,
then $d_{C[x,w^-]}(u_1) = c + 2 - d_{C[w,u_1]}(u_1)$, that is, $d_{C[x,u_1]}(u_1) = c + 2$;
if $d_{C[w,u_1]}(u_1) \geq c + 2$,
then $d_{C[x,w^-]}(u_1) = |\{x\}| = 1$, that is, $d_{C[x,u_1]}(u_1) \leq |V(C[w, u_1]) \setminus \{u_1\}| + 1$.

In either case, by (I-3),

$$d_{C[x,u_1]}(u_1) \leq 4c + 1. \quad \text{(II-3)}$$

Moreover, since $|V(C) \setminus N_C(u_1)| \leq |C| - (|C| - 2kc + 1) = 2kc - 1$, we have

$$\begin{aligned} |C[x, u_1]| &= \left|N_{C[x,u_1]}(u_1)\right| + \left|V(C[x, u_1]) \setminus N_{C[x,u_1]}(u_1)\right| \\ &\leq (4c + 1) + (2kc - 1) = 2kc + 4c. \end{aligned} \quad \text{(II-4)}$$

(III) Let y be the vertex of $N_C(u_2)$ such that

$$d_{C[u_2,y]}(u_2) = c + 2. \tag{III-1}$$

By the similar argument as in (II-4), we have

$$|C[u_2, y]| \leq (c + 2) + (2kc - 1) = 2kc + c + 1. \tag{III-2}$$

Recall that $|C| \geq 8kc + 10c - 4$. Hence by the definitions of x, y and, (I-1), (II-4) and (III-2),

$$y \text{ and } x \text{ appear in the order along } C[u_2^+, u_1^-], \text{ and } y^+ \neq x. \tag{III-3}$$

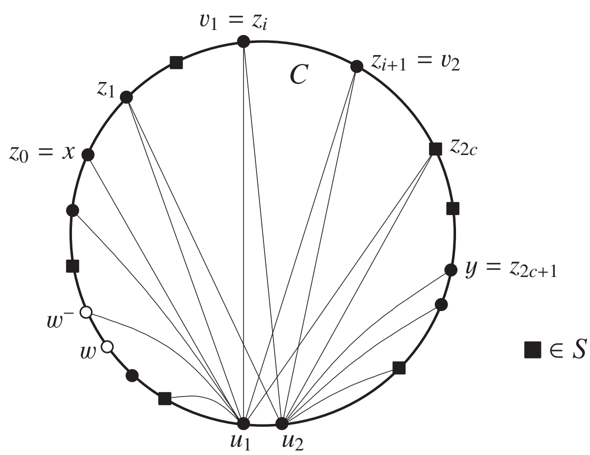

Figure 2: The vertices u_1, u_2, v_1, v_2

To complete the proof of the claim, we next define two vertices v_1, v_2 of $V(C)$ as follows.

(IV) We first show that

$$\left|N_{C[y^+,x^-]}(u_1) \cap N_{C[y^+,x^-]}(u_2)\right| \geq 2c. \tag{IV-1}$$

Assume not. Then for some i with $i \in \{1, 2\}$,

$$\begin{aligned} d_{C[y^+,x^-]}(u_i) &\leq \frac{1}{2}\left(\left|V(C[y^+, x^-]) \setminus \left(N_{C[y^+,x^-]}(u_1) \cap N_{C[y^+,x^-]}(u_2)\right)\right|\right) \\ &\quad + \left|N_{C[y^+,x^-]}(u_1) \cap N_{C[y^+,x^-]}(u_2)\right| \\ &\leq \frac{1}{2}\left(|C[y^+, x^-]| - (2c - 1)\right) + (2c - 1). \end{aligned}$$

If this inequality holds for $i = 1$, then by (I-1), (II-2), (II-3) and (III-1)–(III-3),

$$\begin{aligned}|C| - 2kc + 1 &\le d_C(u_1)\\ &\le d_{C[x,u_1]}(u_1) + d_{C[u_2,y]}(u_1) + d_{C[y^+,x^-]}(u_1)\\ &\le (4c+1) + (2kc + c + 1) + \frac{1}{2}(|C[y^+, x^-]| - (2c-1)) + (2c-1)\\ &= \frac{|C|}{2} - \frac{1}{2}(|C[x, u_1]| + |C[u_2, y]|) + 2kc + 6c + \frac{3}{2}\\ &\le \frac{|C|}{2} - \frac{1}{2}(c+3+c+3) + 2kc + 6c + \frac{3}{2}\\ &= \frac{|C|}{2} + 2kc + 5c - \frac{3}{2}.\end{aligned}$$

This implies that $|C| \le 8kc + 10c - 5$, a contradiction. Similarly, for the case $i = 2$, it follows from (I-1), (II-2), (II-4), (III-1) and (III-3) that $|C| \le 8kc + 10c - 5$, a contradiction again. Thus (IV-1) is proved.

By (IV-1), we can take $2c$ distinct vertices $z_1, \dots, z_{2c}$ in $N_{C[y^+,x^-]}(u_1) \cap N_{C[y^+,x^-]}(u_2)$. We may assume that $z_{2c}, \dots, z_2, z_1$ appear in the order along $C[y^+, x^-]$. Let $z_0 = x$ and $z_{2c+1} = y$. Then

$$z_{2c+1}, z_{2c}, \dots, z_2, z_1, z_0 \text{ appear in the order along } C[u_2^+, u_1^-]. \quad \text{(IV-2)}$$

Note also that

$$z_i \in N_G(u_1) \text{ for } 0 \le i \le 2c \text{ and } z_i \in N_G(u_2) \text{ for } 1 \le i \le 2c+1. \quad \text{(IV-3)}$$

Moreover, since $|S| \le 2c$, it follows that there exists an index i with $0 \le i \le 2c$ such that

$$z_i = z_{i+1}^+, \text{ or } z_i \ne z_{i+1}^+ \text{ and } C[z_{i+1}^+, z_i^-] \cap S = \emptyset. \quad \text{(IV-4)}$$

Then we define

$$v_1 = z_i \text{ and } v_2 = z_{i+1}. \quad \text{(IV-5)}$$

Now let u_1, u_2, v_1, v_2 be the vertices defined as in the above (I)–(IV). By (IV-3) and (IV-5), u_1v_1 and u_2v_2 are chords of C. By (IV-2) and (IV-5), we also see that u_1, u_2, v_2, v_1 appear in the order along $\overrightarrow{C}$. Thus (A) holds. By (I-1), (I-2), (II-1), (IV-4) and (IV-5), we have $w^-, w \in C[v_1, u_1]$ and $S \subseteq C[v_1, v_2] = C[v_1, u_1] \cup C[u_2, v_2]$. Thus (B) holds. By (II-2), (III-1) and (IV-5), we have $d_{C[v_1,u_1]}(u_1) \ge c + 2$ and $d_{C[u_2,v_2]}(u_2) \ge c + 2$. Thus (C) also holds.

This completes the proof of Claim 4. □

Let $x \in V(H^*)$ and C_p be a cycle with $1 \le p \le k$ such that $N_{C_p}(x) \ne \emptyset$. Let $v \in N_{C_p}(x)$. We may assume that $p = 1$. Then by Lemma 2-(i), there exists a cycle C_q with $q \ne 1$, say $q = 2$, such that $d_{C_2}(v^+) + d_{C_2}(x) \ge |C_2| + 1$. This inequality implies that there exists an edge w^-w in $E(\overrightarrow{C_2})$ such that $v^+w^-, xw \in E(G)$. On the other hand, since $|C^*| = n - |H^*| \ge n - 2c - 1$ by Claim 3-(i), there exists a cycle C_r with $1 \le r \le k$ such that $|C_r| \ge \frac{1}{k}(n - 2c - 1)$.

Suppose that $r \geq 3$, say $r = 3$. Then, since $|C_3| \geq \frac{1}{k}(n - 2c - 1) \geq \frac{1}{k}(f(k, c) - 2c - 1) \geq 8kc + 10c - 4$, we can apply Claim 4 to C_3 with $S = N_{C_3}(H^*)$ [4], i.e., C_3 has two chords u_1v_1, u_2v_2 satisfying the conditions (A)–(C). Let

$$D_1 := C_1[v^+, v]\ x\ C_2[w, w^-]v^+,\ D_2 := u_1C_3[v_1, u_1] \text{ and } D_3 := C_3[u_2, v_2]u_2.$$

Since $\overline{E_G}(C_1) \subseteq \overline{E_G}(D_1)$, D_1 is a c-chorded cycle. By the condition (C), D_2 and D_3 are also c-chorded cycles. By the definitions of D_1, D_2, D_3, the condition (B) and Claim 3-(ii), we have $V(C_1) \cup V(C_2) \cup (V(C_3) \cap L) \cup \{x\} \subseteq \bigcup_{1 \leq s \leq 3} V(D_s) \subseteq \bigcup_{1 \leq s \leq 3} V(C_s) \cup \{x\}$. Moreover, by the condition (A), D_1, D_2 and D_3 are disjoint. Since $x \in L$ by Claim 3-(i), replacing C_1, C_2 and C_3 with D_1, D_2 and D_3 would violate (A1), a contradiction.

Suppose next that $r \in \{1, 2\}$, say [5] $r = 2$. We apply Claim 4 to C_2 so that the edge w^-w of C_2 is the same one as in Claim 4 and $S = N_{C_2}(H^*)$, i.e., C_2 has two chords u_1v_1, u_2v_2 satisfying the conditions (A)–(C). Let

$$D_1 := C_1[v^+, v]\ x\ C_2[w, u_1]C_2[v_1, w^-]v^+ \text{ and } D_2 := C_2[u_2, v_2]u_2.$$

Since $\overline{E_G}(C_1) \subseteq \overline{E_G}(D_1)$, D_1 is a c-chorded cycle. By the condition (C), D_2 is also a c-chorded cycle. By the definitions of D_1, D_2, the condition (B) and Claim 3-(ii), we have $V(C_1) \cup (V(C_2) \cap L) \cup \{x\} \subseteq V(D_1) \cup V(D_2) \subseteq V(C_1) \cup V(C_2) \cup \{x\}$. Moreover, by the condition (A), D_1 and D_2 are disjoint. Since $x \in L$, replacing C_1 and C_2 with D_1 and D_2 would violate (A1), a contradiction again.

This completes the proof of Theorem 2. □

We finally prove Theorem 1. In 2009, Babu and Diwan gave the following result concerning the existence of k disjoint c-chorded cycles in graphs. (They actually proved a stronger result, see [1] for the detail. See also [6, Theorem 3.4.16].)

Theorem F (Babu and Diwan [1]) *Let k and c be positive integers, and let G be a graph of order at least $k(c + 3)$. If $\sigma_2 \geq 2k(c + 2) - 1$, then G contains k disjoint c-chorded cycles.*

Combining this with Theorem 2, we get Theorem 1 as follows.

Proof of Theorem 1. Let k, c and G be the same ones as in Theorem 1, and suppose $\sigma_2(G) \geq n$. Since $\sigma_2(G) \geq n \geq f(k, c) \geq \max\{k(c + 3), 2k(c + 2) - 1\}$, Theorem F yields that G contains k disjoint c-chorded cycles. Then by Theorem 2, G can be partitioned into k c-chorded cycles. □

4 Concluding remarks

In this paper, we have shown that for a sufficiently large graph G, the Ore condition for partitioning the graph G into k cycles (Theorem C), also guarantees the existence of a partition of G into

[4]We do not use w^-w in Claim 4.

[5]Since (C_1, v, v^+) and $(\overleftarrow{C_2}, w, w^-)$ are symmetric, we may assume that $r = 2$.

k cycles with c chords which are relaxed structures of a complete graph (see Theorem 1). But, as mentioned in Section 1, we do not know whether the order condition (the function $f(k,c)$) is sharp or not. Perhaps, a weaker order condition may suffice to guarantee the existence.

For the case of the Dirac condition, it follows from our arguments that the order condition can be improved as follows. If we assume $\delta(G) \geq \frac{n}{2}$, then we have $L = \emptyset$ in the proof of Theorem 2, i.e., Case 2 does not occur (see Claim 3-(i)). On the other hand, in the proof of Case 1 of Theorem 2, we have used the order condition in the following parts:

- $\frac{n}{2} - 2kc \geq \omega(c)$ $(= \frac{\sqrt{8c+9}+3}{2} \geq 3)$,
- $\min\{n - 4kc - 3, |B| - 1\} \geq \min\{n - 4kc - 3, \frac{n}{2} - 2kc - 1\} \geq \omega(c)$,
- $n - 12kc + 2k \cdot \omega(c) - 4 \geq c + 4$.

In the proof of Theorem 1, we have also used the order condition in the following part:

- $n \geq \max\{k(c + 3), 2k(c + 2) - 1\}$.

Therefore, as a corollary of our arguments, we get the following.

Theorem 3 *Let k and c be positive integers, and let G be a graph of order $n \geq 12kc - 2k \cdot \omega(c) + c + 8$, where $\omega(c) = \frac{\sqrt{8c+9}+3}{2}$. If $\delta \geq \frac{n}{2}$, then G can be partitioned into k c-chorded cycles.*

We finally remark about the necessary order condition. Let c be a positive integer, and let $\psi(c)$ be the positive root of the equation $t(t-2) - c = 0$, i.e., $\psi(c) = \sqrt{c+1} + 1$. Note that $|E(K_{t,t})| - 2t = t(t-2)$. If a bipartite graph contains a c-chorded cycle, then by the definition of $\psi(c)$, it follows that the order of the bipartite graph is at least $2\lceil\psi(c)\rceil$. Therefore, the complete bipartite graph $G \cong K_{k\lceil\psi(c)\rceil-1, k\lceil\psi(c)\rceil-1}$ satisfies $\delta(G) = |G|/2$ and $\sigma_2(G) = |G|$, but G cannot be partitioned into k c-chorded cycles. Thus the order at least $2k\lceil\psi(c)\rceil - 1$ is necessary, and the order conditions in Theorems 1 and 3 might be improved into $n \geq 2k\lceil\psi(c)\rceil - 1$. Theorem C supports it by including the case $c = 0$, since $\psi(c) = 2$ for the case $c = 0$. For the case $c = 1$, related results can be also found in [6, Corollary 3.4.7].

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