EXTREMAL AND OPTIMAL PROPERTIES OF B-BASES COLLOCATION MATRICES *

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Abstract. Totally positive matrices are related with the shape preserving representations of a space of functions. The normalized B-basis of the space has optimal shape preserving properties. B-splines and rational Bernstein bases are examples of normalized B-bases. Some results on the optimal conditioning and on extremal properties of the minimal eigenvalue and singular value of the collocation matrices of normalized B-bases are proved. Numerical examples confirm the theoretical results and answer related questions.

Key words. totally positive matrices, stochastic matrices, eigenvalues, singular values, conditioning, B-basis

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1. Introduction.

Totally positive matrices, which are also called totally nonnegative in the literature, play an important role in many fields, such as approximation theory, computer aided geometric design (CAGD), mechanics, differential or integral equations, statistics, combinatorics, economics and biology (see [1], [10], [12], [14] or [20]). A matrix is *totally positive* (TP) if all its minors are nonnegative. Relevant properties of TP matrices about algebraic computations with high relative accuracy have been found recently (cf. [9, 15]). In fact, for some classes of TP matrices adequately parameterized, one can compute their eigenvalues, singular values, inverses or the solutions of some linear systems with high relative accuracy independently of their conditioning (see [9], [16] and [8]). This holds for many popular matrices, such as positive Vandermonde matrices or Hilbert matrices, which are TP. An important source of examples of TP matrices comes from the collocation matrices of systems of functions. Let $\mathcal{U}$ be a vector space of real functions defined on a real interval I and $(u_0(t),\ldots,u_n(t))$ $(t \in I)$ be a basis of $\mathcal{U}$. The *collocation matrix* of $(u_0(t),\ldots,u_n(t))$ at $t_0 < \cdots < t_m$ in I is given by

$$M\begin{pmatrix} u_0,\ldots,u_n \\ t_0,\ldots,t_m \end{pmatrix} := (u_j(t_i))_{i=0,\ldots,m;j=0,\ldots,n}. \tag{1.1}$$

The collocation matrices of a given basis are the coefficient matrices of the linear systems associated with Lagrange interpolation problems in that basis.

A system of functions is TP when all its collocation matrices (1.1) are TP. In CAGD, the functions $u_0,\ldots,u_n$ also satisfy that $\sum_{i=0}^n u_i(t) = 1\ \forall\, t \in I$ (i.e., the system $(u_0,\ldots,u_n)$ is *normalized*), and a normalized TP system is denoted by NTP. It is known that shape preserving representations are associated with NTP bases (see [19] or [2]). Clearly, the collocation matrices of NTP bases are stochastic TP matrices. By Theorem 4.2 (ii) of [3] (see also [4, 19]), given a space with an NTP basis, there

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exists a unique NTP basis of the space with optimal shape preserving properties, which is called the *normalized B-basis* of the space. An important normalized B-basis is the Bernstein basis $(b_0^n, \dots, b_n^n)$ of the space $\mathcal{P}_n([0,1])$ of polynomials of degree less than or equal to n on $[0,1]$, given by

$$b_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \dots, n \tag{1.2}$$

(see [2], [3]). Other examples of normalized B-bases are presented at the end of Section 3 and include the important examples of B-splines and rational Bernstein bases.

In this paper, we prove that the minimal eigenvalue (and singular value) of a collocation matrix of an NTP basis is always bounded above by the minimal eigenvalue (and singular value, respectively) of the corresponding collocation matrix of the normalized B-basis of the space. The information on the minimal eigenvalue and singular value has important potential applications. For instance, here we extend the optimal conditioning for the ∞-norm of the Bernstein basis proved in [7] to any normalized B-basis. On the other hand, similar results for the maximal singular value of the corresponding collocation matrices do not hold, as shown in Section 4.

The paper is organized as follows. Section 2 presents basic concepts and notations, as well as some auxiliary results for TP matrices. In particular, it recalls the characterization of stochastic TP matrices as a product of matrices associated with elementary corner cuttings. In Section 3, we prove that multiplying a nonsingular TP matrix by a matrix associated with an elementary corner cutting decreases the minimal eigenvalue and singular value and increases the ∞-norm condition number. This result is a key tool to prove the mentioned result on the extremal and optimal properties of the collocation matrices of a normalized B-basis. In Section 4, we include numerical examples confirming our theoretical results and counterexamples answering other related questions.

2. Basic notations and auxiliary results. By Theorem 2.6 of [19] (or by Theorem 4.5 of [13]) we have the following characterization of a nonsingular stochastic TP matrix.

THEOREM 2.1. *A nonsingular $n \times n$ matrix A is stochastic and TP if and only if it can be factorized in the form*

$$A = F_{n-1} F_{n-2} \cdots F_1 G_1 \cdots G_{n-2} G_{n-1},$$

with

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{i+1,1} & 1-\alpha_{i+1,1} & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{n,n-i} & 1-\alpha_{n,n-i} \end{pmatrix}$$

and

$$G_i = \begin{pmatrix} 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & 1-\alpha_{1,i+1} & \alpha_{1,i+1} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1-\alpha_{n-i,n} & \alpha_{n-i,n} \\ & & & & & & 1 \end{pmatrix},$$

where, $\forall\,(i,j)$, $0 \le \alpha_{i,j} < 1$.

The following remark provides a new factorization in terms of elementary bidiagonal matrices.

REMARK 2.2. *If we denote by* $U_i(\lambda)$ *the bidiagonal, nonsingular and upper triangular matrix with at most one nonzero off-diagonal element in the entry* $(i-1,i)$

$$U_i(\lambda) = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & 1-\lambda & \lambda & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix}, \quad 0 \le \lambda < 1 \tag{2.1}$$

and by $L_i(\lambda)$ *the bidiagonal, nonsingular and lower triangular matrix with at most one nonzero off-diagonal element in the entry* $(i,i-1)$

$$L_i(\lambda) = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \lambda & 1-\lambda & & & \\ & & & \ddots & \ddots & & \\ & & & & & 1 & \\ & & & & & 0 & 1 \end{pmatrix}, \quad 0 \le \lambda < 1, \tag{2.2}$$

then we can write

$$F_i = L_{i+1}(\alpha_{i+1,1}) \cdots L_n(\alpha_{n,n-i}) \quad \textit{and} \quad G_i = U_n(\alpha_{n-i,n}) \cdots U_{i+1}(\alpha_{1,i+1}).$$

In Section 2 of [19], it is shown that the elementary matrices (2.1) and (2.2) have a geometric interpretation as elementary corner cutting transformations.

Now let us recall some notations, concepts and results of Linear Algebra that will be used later in order to get a paper as self-contained as possible. Given two square matrices $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$, we denote $A \le B$ if $a_{ij} \le b_{ij}$ for all i,j. We say that A is *nonnegative* if $a_{ij} \ge 0$ for all i,j. If $C = (c_{ij})_{1\le i,j\le n}$ is a complex matrix and $A = (a_{ij})_{1\le i,j\le n}$ is a nonnegative matrix such that $|c_{ij}| \le a_{ij}$ for all i, j, then A is said to *dominate* C, and so $|C| := (|c_{ij}|)_{1\le i,j\le n} \le A$. The following result is due to Wienlandt (see Corollary 2.1 of Chapter II of [17]):

THEOREM 2.3. *Let M be a nonnegative matrix with maximal eigenvalue r, and let C be a complex matrix dominated by M. Then $r = \rho(M) \ge \rho(C)$.*

The following result collects two properties of TP matrices which will be used in the proofs of the main results. The first part corresponds to Corollary 6.6 of [1] and the second part to Theorem 3.3 of [1].

THEOREM 2.4. *Let A be a nonsingular TP $n \times n$ matrix. Then:*

(i) All the eigenvalues of A are positive.

(ii) Given the $n \times n$ diagonal matrix

$$J := diag(1, -1, 1, \dots, (-1)^{n-1}), \tag{2.3}$$

the matrix $JA^{-1}J$ is TP.

Given a nonsingular matrix A, for $p = 1, 2, \infty$ we shall use the condition numbers $\kappa_p(A) := \|A\|_p \, \|A^{-1}\|_p$.

3. Main results. The following theorem shows that the elementary matrices corresponding to elementary corner cuttings decrease the minimal singular value and the minimal eigenvalue and increase some condition numbers when they multiply a TP matrix to its right or when their transposes multiply a TP matrix to its left.

THEOREM 3.1. *Let M be a nonsingular TP matrix, $A := ME$ and $C := E^TM$ with $E = U_i(\lambda)$ or $E = L_i(\lambda)$ an elementary matrix given by* (2.1) *and* (2.2)*, respectively, for $0 \le \lambda < 1$. Then the following properties hold:*

(i) $|A^{-1}|$ and $|C^{-1}|$ dominate M^{-1}.

(ii) The minimal eigenvalue of A and C are bounded above by the minimal eigenvalue of M.

(iii) The minimal singular value of A and C are bounded above by the minimal singular value of M.

(iv) $\kappa_\infty(M) \le \kappa_\infty(A)$ and $\kappa_1(M) \le \kappa_1(C)$

Proof. Since E is obviously TP and M is also TP, we deduce from Theorem 3.1 of [1] that the products $A = ME$ and $C = E^TM$ are also TP, and they also inherit the nonsingularity of M and E. If J is the diagonal matrix given by (2.3), since A, C and M are TP nonsingular, by Theorem 2.4 (ii), $JA^{-1}J$, $JC^{-1}J$ and $JM^{-1}J$ are TP and so, in particular, nonnegative and

$$JA^{-1}J = |A^{-1}|, \quad JC^{-1}J = |C^{-1}|. \tag{3.1}$$

Besides, $JA^{-1}J$, $JC^{-1}J$ and $JM^{-1}J$ are similar to A^{-1}, C^{-1} and M^{-1}, respectively.

(i) Taking into account that $J = J^{-1}$ we derive

$$JA^{-1}J = J(ME)^{-1}J = (JE^{-1}J)(JM^{-1}J).$$

So, in order to prove that $|A^{-1}|$ dominates M^{-1}, it is sufficient by (3.1) to see that

$$JM^{-1}J \le JA^{-1}J = (JE^{-1}J)(JM^{-1}J). \tag{3.2}$$

We can observe that the matrix $JE^{-1}J$ is also nonnegative. In addition, $JE^{-1}J$ has one of the two following forms:

$$\begin{pmatrix} 1 & 0 & & & & & & \\ & 1 & 0 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \frac{1}{1-\lambda} & \frac{\lambda}{1-\lambda} & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1 & 0 \\ & & & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \frac{\lambda}{1-\lambda} & \frac{1}{1-\lambda} & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & & 1 & \\ & & & & & & 0 & 1 \end{pmatrix}, \tag{3.3}$$

with $0 \leq \lambda < 1$. Taking into account the previous formula, that $JM^{-1}J$ is nonnegative and that $1/(1-\lambda) \geq 1$, it can be deduced that $(JE^{-1}J)(JM^{-1}J) \geq JM^{-1}J$ and formula (3.2) holds, and so $|A^{-1}|$ dominates M^{-1}. Since $C^T = M^T E$, we can deduce that $|(C^T)^{-1}| = |(C^{-1})^T|$ dominates $(M^T)^{-1} = (M^{-1})^T$ and so $|C^{-1}|$ dominates M^{-1}, and (i) holds.

(ii) By Theorem 2.4 (i), the eigenvalues of A are positive. By (3.1), (i) and Theorem 2.3, we derive

$$\rho(JA^{-1}J) \geq \rho(JM^{-1}J) \tag{3.4}$$

and, since $JA^{-1}J$ and $JM^{-1}J$ are similar to A^{-1} and M^{-1} (respectively), the minimal eigenvalue of A is bounded above by the minimal eigenvalue of M. Using again that $C^T = M^T E$ and that the eigenvalues do not change when transposing a matrix, we also conclude that the minimal eigenvalue of C is bounded above by the minimal eigenvalue of M, and (ii) holds.

(iii) The minimal singular values of M and $A = ME$ are the minimal eigenvalues of $M^T M$ and $E^T M^T ME$, respectively. By Theorem 3.1 of [1], the product $M^T M$ is TP. Then, by (ii), the minimal eigenvalue of $M^T M$ is greater than or equal to the minimal eigenvalue of $M^T ME$. Applying (i) again, the minimal eigenvalue of $M^T ME$ is greater than or equal to the minimal eigenvalue of $E^T M^T ME$. In conclusion, the minimal singular value of $A = ME$ is bounded above by the minimal singular value of M. Taking into account that $C^T = M^T E$ and that the singular values do not change when transposing a matrix, we also have that the minimal singular value of C is bounded above by the minimal singular value of M, and (iii) holds.

(iv) From (i), we derive $\|M^{-1}\|_\infty \leq \|A^{-1}\|_\infty$. Since A and M are TP, they are nonnegative. Since E is stochastic, if we denote $e := (1, \ldots, 1)^T$, then we have $\|A\|_\infty = \|Ae\|_\infty = \|MEe\|_\infty = \|Me\|_\infty = \|M\|_\infty$. Therefore $\kappa_\infty(M) \leq \kappa_\infty(A)$. Finally, we deduce that $\kappa_1(C) = \kappa_\infty(C^T) = \kappa_\infty(M^T E) \geq \kappa_\infty(M^T) = \kappa_1(M)$, and the result follows. $\square$

The following corollary shows that any nonsingular stochastic TP matrix produces the same effects as those described in Theorem 3.1 for the elementary matrices corresponding to elementary corner cuttings when multiplying TP matrices.

Corollary 3.2. *Let M be a nonsingular TP matrix, K a nonsingular stochastic TP matrix, $A := MK$ and $C := K^T M$. Then the following properties hold:*

(i) $|A^{-1}|$ and $|C^{-1}|$ dominate M^{-1}.

(ii) The minimal eigenvalue of A and C are bounded above by the minimal eigenvalue of M.

(iii) The minimal singular value of A and C are bounded above by the minimal singular value of M.

(iv) $\kappa_\infty(M) \le \kappa_\infty(A)$ *and* $\kappa_1(M) \le \kappa_1(C)$

Proof. By Theorem 2.1 and Remark 2.2 we deduce that $K = \prod_{i=1}^r E_i$, where r is a positive integer and each E_i is equal to $U_j(\lambda_i)$ or $L_j(\lambda_i)$ given by (2.1) and (2.2), respectively, for $0 \le \lambda_i < 1$. Therefore, we get that

$$A = M\left(\prod_{i=1}^r E_i\right),$$

with $E_i = U_j(\lambda_i)$ or $L_j(\lambda_i)$ for $0 \le \lambda_i < 1$ and $i \in \{1, \ldots, r\}$. So, applying in an iterative way Theorem 3.1 to the previous formula, the result follows for A.

Analogously, since $C = (E_r^T \cdots E_1^T)M$ with each E_i a matrix of the form (2.1) or (2.2), we can apply Theorem 3.1 in an iterative way to prove the result for C. □

The next corollary applies previous results to deduce some extremal and optimal properties of the collocation matrices of the normalized B-basis of a space.

COROLLARY 3.3. *Let $u = (u_0, \ldots, u_n)$ be an NTP basis on $[a, b]$ of a space of functions $\mathcal{U}$ and let $v = (b_0, \ldots, b_n)$ the normalized B-basis of $\mathcal{U}$. If we consider an increasing sequence of nodes $\mathbf{t} = (t_i)_{i=0}^n$ on $[a, b]$, let us denote by A to the collocation matrix of u at $\mathbf{t}$ and by M to the collocation matrix of v at $\mathbf{t}$. Then the minimal eigenvalue and singular value of M are greater than or equal to the minimal eigenvalue and singular value of A, respectively. Moreover, if A and M are nonsingular, then $\kappa_1(M^T) = \kappa_\infty(M) \le \kappa_\infty(A) = \kappa_1(A^T)$.*

Proof. Since v is the normalized B-basis of $\mathcal{U}$ and u an NTP basis, by Theorem 4.2 (ii) of [3], we have that there exists a nonsingular TP stochastic matrix K such that

$$(u_0, \ldots, u_n) = (b_0, \ldots, b_n)K.$$

Taking collocation matrices in the previous expression at $\mathbf{t}$ we have that

$$A = MK. \tag{3.5}$$

Since the bases u and v are NTP, A and M are stochastic and TP. If A (or equivalently M) is singular, then the minimal eigenvalue and singular value of both matrices are equal to 0. Otherwise, the result follows from (3.5) and from (ii), (iii) and (iv) of Corollary 3.2. □

We now give a list of examples of important normalized B-bases. By the previous result, their collocation matrices satisfy the mentioned extremal and optimal properties.

EXAMPLES 3.4.

(a) *The space of polynomials of degree at most n on a compact interval $[a, b]$, $\mathcal{P}_n([a, b])$, has the normalized B-basis given by $(b_0^n \ \ldots, b_n^n)$ with*

$$b_i^n(t; a, b) = \binom{n}{i}\frac{(b-t)^{n-i}(t-a)^i}{(b-a)^n}, \quad i = 0, 1 \ldots, n$$

(see [3, 11] and Section 4 of [2]).

(b) *Let us consider a sequence $(w_i)_{0 \le i \le n}$ of positive weights. Then the system of functions $(r_0^n, \ldots, r_n^n)$ defined on the compact interval $[a, b]$ by*

$$r_i^n(t) = \frac{w_i b_i^n(t; a, b)}{\sum_{j=0}^n w_j b_j^n(t; a, b)}, \quad i = 0, 1, \ldots, n,$$

is the normalized B-basis of the corresponding spanned space of functions (see Example 4.14 of [19]), and is called the rational Bernstein basis of its space. Observe that, if all weights $w_i = 1$ *for all* i*, then* $(r_0^n, \ldots, r_n^n) = (b_0^n \ldots, b_n^n)$ *is the Bernstein basis on* $[a, b]$.

(c) *The space of even trigonometric functions given by*

$$\mathcal{C}_n = span\{1, \cos t, \cos 2t, \ldots, \cos nt\}$$

on the compact interval $[0, \pi]$ *has the normalized B-basis* $(u_0^n, \ldots, u_n^n)$ *given by*

$$u_i^n(t) = \binom{n}{i} \cos^{2(n-i)}(t/2) \sin^{2i}(t/2), \quad i = 0, 1, \ldots, n$$

(see [18]).

(d) *The space of trigonometric polynomials*

$$\mathcal{T}_n = \{1, \cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos nt, \sin nt\}$$

on $I = [-A, A]$ *with* $A < \frac{\pi}{2}$ *has the normalized B-basis* $(v_0, \ldots, v_m)$*,* $m = 2n$*, defined by*

$$v_i(t) = d_i \left(\frac{\sin\left(\frac{A+t}{2}\right)}{\sin A} \right)^i \left(\frac{\sin\left(\frac{A-t}{2}\right)}{\sin A} \right)^{m-i}, \quad i = 0, 1, \ldots, m$$

with

$$d_i = \sum_{k=0}^{[i/2]} \binom{m/2}{i-k} \binom{i-k}{k} (2 \cos A)^{i-2k}, \quad i = 0, 1, \ldots, m$$

(see Section 3 of [21]).

(e) *A very important example is the case of B-spline bases (see [22]) and NURBS. Let us consider a sequence of positive weights* $(w_i)_{0 \le i \le n}$ *and a knots vector* $(t_0, \ldots, t_{n+d})$ *with* $t_i \le t_{i+1}$ *for all* $i = 0, 1, \ldots, n + d - 1$*. Then the B-spline basis* $(N_{0,d}, N_{1,d}, \ldots, N_{n,d})$ *defined over the previous knots vector by*

$$N_{i,0}(t) = \begin{cases} 1, & \textit{if } t_i \le t < t_{i+1}, \\ 0, & \textit{otherwise}, \end{cases}$$
$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t), \quad k = 1, \ldots, d,$$

is the normalized B-basis of the corresponding splines space (see [3]). The basis $(r_0, \ldots, r_n)$ *defined by*

$$r_i(t) = \frac{w_i N_{i,d}(t)}{\sum_{j=0}^n w_j N_{j,d}(t)}, \quad i = 0, 1 \ldots, n,$$

is the normalized B-basis of the corresponding NURBS space (see Section 4 of [3]).

4. Numerical experiments and further questions. For the construction of the numerical examples we shall consider three different TP bases $u=(u_0^n,\dots,u_n^n)$ of $\mathcal{P}_n([0,1])$. For each of the bases, given a sequence of positive weights $(w_i)_{i=0}^n$, we can construct a rational NTP basis $(r_0^n,\dots,r_n^n)$ defined by

$$r_i^n(t)=\frac{w_i u_i^n(t)}{\sum_{j=0}^n w_j u_j^n(t)},\quad i=0,1,\dots,n. \tag{4.1}$$

In fact, it is straightforward to check that, if u is TP, then $(r_0^n,\dots,r_n^n)$ is NTP. In the case that $u=(b_0^n,\dots,b_n^n)$ is the normalized B-basis of the space $\mathcal{P}_n([0,1])$ given in Examples 3.4 (a) for $a=0$ and $b=1$ (see (1.2)), then it is well known that the corresponding rational Bernstein basis $r_B=(r_0^n,\dots,r_n^n)$ is the normalized B-basis of its spanned space $\langle r_B\rangle$ (see Examples 3.4 (b)).

Now, let us consider the Said-Ball basis $s=(s_0^n,\dots,s_n^n)$ (for more details see [6] and the references therein) given by

$$s_i^n(t)=\binom{\lfloor n/2\rfloor+i}{i}t^i(1-t)^{\lfloor n/2\rfloor+1},\quad 0\le i\le\lfloor (n-1)/2\rfloor,$$
$$s_i^n(t)=\binom{\lfloor n/2\rfloor+n-i}{n-i}t^{\lfloor n/2\rfloor+1}(1-t)^{n-i},\quad \lfloor n/2\rfloor+1\le i\le n,$$

and, if n is even

$$s_{n/2}^n(t)=\binom{n}{n/2}t^{n/2}(1-t)^{n/2},$$

where $\lfloor m\rfloor$ is the greatest integer less than or equal to m. In [6] it was proved that the Said-Ball basis is NTP. In the case that $u=(s_0^n,\dots,s_n^n)$, the corresponding NTP basis $r_{SB}=(r_0^n,\dots,r_n^n)$, constructed as in (4.1), will be called rational Said-Ball basis.

Finally, let us consider the DP basis $c=(c_0^n,\dots,c_n^n)$ of $\mathcal{P}_n([0,1])$ given by (see [5])

$$\begin{aligned}
c_0^n(t)&=(1-t)^n,\\
c_i^n(t)&=t(1-t)^{n-i},\quad 1\le i\le\lfloor n/2\rfloor-1,\\
c_i^n(t)&=t^i(1-t),\quad \lfloor (n+1)/2\rfloor+1\le i\le n-1,\\
c_n^n(t)&=t^n,
\end{aligned}$$

and, if n is even

$$c_{\frac{n}{2}}^n(t)=1-t^{\frac{n}{2}+1}-(1-t)^{\frac{n}{2}+1},$$

and, if n is odd,

$$\begin{aligned}
c_{\frac{n-1}{2}}^n(t)&=t(1-t)^{\frac{n+1}{2}}+\frac{1}{2}\left[1-t^{\frac{n+1}{2}+1}-(1-t)^{\frac{n+1}{2}+1}\right],\\
c_{\frac{n+1}{2}}^n(t)&=\frac{1}{2}\left[1-t^{\frac{n+1}{2}+1}-(1-t)^{\frac{n+1}{2}+1}\right]+t^{\frac{n+1}{2}}(1-t).
\end{aligned}$$

In [5] it was also proved that the DP basis is also NTP. In the case that $u=(c_0^n,\dots,c_n^n)$, the corresponding basis $r_{DP}=(r_0^n,\dots,r_n^n)$, constructed as in (4.1), will be called rational DP basis.

As commented above, the rational Said-Ball and DP bases are also NTP.

If we consider a sequence of positive weights $(w_i^n)_{i=0}^n$ and taking into account that $\sum_{j=0}^n w_j^n b_j^n(t) \in \mathcal{P}_n([0,1])$ and that s and c are bases of $\mathcal{P}_n([0,1])$, then there exist two sequences of weights $(\overline{w}_i^n)_{i=0}^n$ and $(\tilde{w}_i^n)_{i=0}^n$ satisfying

$$\sum_{j=0}^n w_j^n b_j^n(t) = \sum_{j=0}^n \overline{w}_j^n s_j^n(t) = \sum_{j=0}^n \tilde{w}_j^n c_j^n(t), \quad t \in [0,1]. \tag{4.2}$$

If $\overline{w}_i^n, \tilde{w}_i^n > 0$ for all $i = 0, \ldots, n$, then the rational Said-Ball basis r_{SB} formed with the weights $(\overline{w}_i^n)_{i=0}^n$ and the rational DP basis r_{DP} formed with the weights $(\tilde{w}_i^n)_{i=0}^n$ are both NTP bases of the space of rational functions $\langle r_B \rangle$, where r_B is the rational Bernstein basis formed with the weights $(w_i^n)_{i=0}^n$. So, sequences of positive weights $(w_i^n)_{i=0}^n$ have been randomly generated for each n in $\{3, \ldots, 8\}$, where each w_i^n is an integer in the interval $[1, 1000]$, until we have obtained a sequence such that there exists positive sequences $(\overline{w}_i^n)_{i=0}^n$ and $(\tilde{w}_i^n)_{i=0}^n$ satisfying (4.2). Then we have the normalized B-basis r_B, and the NTP bases r_{SB} and r_{DP} of $\langle r_B \rangle$.

Let $(t_i)_{i=1}^{n+1}$ be the sequence of points given by $t_i = i/(n+2)$ for $i = 1, \ldots, n+1$. Then we have considered the following collocation matrices:

$$M^n = \left(\frac{w_j^n b_j^n(t_i)}{\sum_{k=0}^n w_k^n b_k^n(t_i)} \right)_{1 \le i \le n+1}^{0 \le j \le n},$$

$$B_1^n = \left(\frac{\overline{w}_j^n s_j^n(t_i)}{\sum_{k=0}^n \overline{w}_k^n s_k^n(t_i)} \right)_{1 \le i \le n+1}^{0 \le j \le n} \quad \text{and} \quad B_2^n = \left(\frac{\tilde{w}_j^n c_j^n(t_i)}{\sum_{k=0}^n \tilde{w}_k^n c_k^n(t_i)} \right)_{1 \le i \le n+1}^{0 \le j \le n},$$

for $n = 3, \ldots, 8$. We have computed the eigenvalues and the singular values of M^n, B_1^n and B_2^n for $n = 3, \ldots, 8$ with Mathematica using a precision of 100 digits. We can see the corresponding minimal eigenvalues and singular values in Table 4.1. It can be observed that the minimal eigenvalue, resp. singular value, of M_n is higher than the minimal eigenvalue, resp. singular value, of B_1^n and B_2^n as Corollary 3.3 has proved.

n	M^n		B_1^n		B_2^n	
	λ_{min}	σ_{min}	λ_{min}	σ_{min}	λ_{min}	σ_{min}
3	$2.9940e-2$	$1.2267e-2$	$2.6333e-2$	$1.2097e-2$	$7.1114e-3$	$5.2420e-3$
4	$6.7992e-3$	$5.4745e-3$	$6.3025e-3$	$5.3558e-3$	$5.8627e-3$	$5.2003e-3$
5	$7.1826e-3$	$6.6451e-3$	$3.0020e-3$	$2.9674e-3$	$4.0691e-4$	$3.5263e-4$
6	$2.1129e-3$	$2.0654e-3$	$6.9654e-4$	$5.8389e-4$	$4.1580e-4$	$3.2558e-4$
7	$1.0044e-3$	$4.2778e-4$	$2.7894e-4$	$2.2178e-4$	$2.1500e-5$	$1.6099e-5$
8	$3.3227e-4$	$3.2780e-4$	$4.2257e-5$	$1.8605e-5$	$2.4263e-6$	$1.0410e-6$

TABLE 4.1
The minimal eigenvalue and singular value of M^n, B_1^n and B_2^n

We have also computed $\kappa_\infty(M^n)$, $\kappa_\infty(B_1^n)$ and $\kappa_\infty(B_2^n)$ for $n = 3, \ldots, 8$ with Mathematica. The results can be seen in Table 4.2. It can be observed that $\kappa_\infty(M^n) \le \kappa_\infty(B_i^n)$ for $i = 1, 2$, as it has been shown in Corollary 3.3.

REMARK 4.1. *On the one hand, we have seen in Corollary 3.3 that the minimal eigenvalue and the minimal singular value of the collocation matrix of the normalized B-basis are always greater than the minimal eigenvalue and the minimal singular value, respectively, of the corresponding collocation matrix of the NTP bases of the corresponding space of functions. This fact has also been illustrated in the previous*

n	$\kappa_\infty(M^n)$	$\kappa_\infty(B_1^n)$	$\kappa_\infty(B_2^n)$
3	$1.4138e+2$	$1.4138e+2$	$2.9393e+2$
4	$3.4704e+2$	$3.4704e+2$	$3.4704e+2$
5	$1.6822e+2$	$4.3900e+2$	$4.3526e+3$
6	$7.5191e+2$	$3.1923e+3$	$4.2045e+3$
7	$4.7287e+3$	$5.5742e+3$	$8.1522e+4$
8	$4.2039e+3$	$1.2637e+5$	$1.6388e+6$

Table 4.2
Infinity conditions numbers of M^n, B_1^n and B_2^n

numerical experiments. On the other hand, the maximal eigenvalue of the collocation matrix of an NTP basis of a space of functions, including the corresponding normalized B-basis, is always equal to 1 *because all these collocation matrices are stochastic. So, an interesting question arises: does there exist any relation between the maximal singular value of the collocation matrices of the normalized B-basis of a space of functions and those of the corresponding collocation matrices of NTP bases of the same space? In order to answer this question Table 4.3 also shows the maximal singular value of M^n, B_1^n and B_2^n for $n = 3, \ldots, 8$. We can observe that in some cases the maximal singular value of M^n is lower than the maximal singular value of B_1^n and B_2^n, for example for $n = 5$. In other cases, the maximal singular value of M^n is higher than the maximal singular value of B_1^n and B_2^n, for example for $n = 4$. Hence, we can conclude that there is not a relation between the maximal singular value of the collocation matrix of a normalized B-basis and that of the corresponding collocation matrix of the NTP bases of the corresponding space of functions.*

By Corollary 3.3, we have that $\kappa_\infty(M^n) \leq \kappa_\infty(B_i^n)$ and that $\sigma_{min}(M^n) \geq \sigma_{min}(B_i^n)$ for $i = 1, 2$ and $n = 3, \ldots, 8$. Taking into account that $\kappa_2(A)$ is equal to $\sigma_{max}(A)/\sigma_{min}(A)$, another interesting question arises: does there exist an analogous relation with κ_2 instead of κ_∞ for the collocation matrices of normalized B-bases and NTP bases? From the data in Tables 4.1 and 4.3, we have that $\kappa_2(M^3) > \kappa_2(B_1^3)$ and $\kappa_2(M^5) < \kappa_2(B_i^5)$ for $i = 1, 2$. Hence, there is no any relation between the condition number κ_2 of the collocation matrices of the normalized B-basis and these of the corresponding collocation matrices of NTP bases.

n	$\sigma_{max}(M^n)$	$\sigma_{max}(B_1^n)$	$\sigma_{max}(B_2^n)$
3	$1.1934e+0$	$1.1215e+0$	$1.4619e+0$
4	$1.1074e+0$	$1.0977e+0$	$1.0542e+0$
5	$1.0608e+0$	$1.0764e+0$	$1.5601e+0$
6	$1.0709e+0$	$1.1728e+0$	$1.2136e+0$
7	$1.1237e+0$	$1.4003e+0$	$1.5872e+0$
8	$1.0968e+0$	$1.3374e+0$	$1.6461e+0$

Table 4.3
The maximal singular value of M^n, B_1^n and B_2^n

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