Platonic solids and symmetric solutions of the N-vortex problem on the sphere

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Abstract

We consider the N-vortex problem on the sphere assuming that all vortices have equal strength. We develop a theoretical framework to analyse solutions of the equations of motion with prescribed symmetries. Our construction relies on the discrete reduction of the system by twisted subgroups of the full symmetry group that rotates and permutes the vortices. Our approach formalises and extends ideas outlined previously by Tokieda (*C. R. Acad. Sci., Paris I* 333 (2001)) and Soulière and Tokieda (*J. Fluid Mech.* 460 (2002)) and allows us to prove the existence of several 1-parameter families of periodic orbits. These families either emanate from equilibria or converge to collisions possessing a specific symmetry. Our results are applied to show existence of families of small nonlinear oscillations emanating from the platonic solid equilibria.

Dedicated to J. Ize

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1 Introduction

In the last three decades, a growing number of publications have considered the N-vortex problem on the sphere. A (necessarily incomplete) list of references is [40, 21, 8, 9, 32, 28, 43, 24, 41, 5, 11, 35, 45, 15, 46]. The first reference to the problem seems to go back to Zermelo [48] while the equations of motion were presented by Gromeka [18] and Bogomolov [7]. The importance of the model is usually associated with geophysical fluid dynamics since it describes the interaction of hurricanes on the Earth. Interestingly, similar models of vortices are also relevant in the study of Bose-Einstein condensates [20], while the steady solutions of the problem have applications in semiconductors [4] and reaction-diffusion models [44]. We refer the reader to the papers by Aref et al [1], Aref [2] and the book by Newton [38] for a general overview of the N-vortex problem and for an extensive bibliography on the subject.

The investigation of the N-vortex problem on the sphere is interesting from a mathematical point of view since the system is Hamiltonian and invariant under rotations by the group $\mathrm{SO}(3)$. Symplectic reduction leads to the conclusion that the problem is integrable if $N \leq 3$, and also for $N = 4$ if the *centre of vorticity* (momentum map) vanishes [8, 21, 33]. A strong indication that the problem is non-integrable for $N = 4$ for a general centre of vorticity is given by Bagrets & Bagrets [3], and hence it is natural to expect that the system is fully chaotic for $N > 4$. The dynamics of the problem in the integrable case $N = 3$ was considered in [8, 21, 22, 12].

Much effort has been devoted to the investigation of particular solutions of the problem for $N \geq 4$. Following the pioneering work of Lim, Montaldi & Roberts [28], several publications have considered the existence, stability and bifurcations of fixed equilibria and relative equilibria e.g. [24, 5, 27]. On the other hand, relative periodic solutions have been found by Laurent-Polz [25] and García-Azpeitia [15]. Periodic solutions with prescribed symmetry are determined in Tokieda [43] and Soulière & Tokieda [41]. Choreographies were found by Borisov, Mamaev & Kilin [11] for $N = 4$ and by García-Azpeitia [15] for general N. Some of these choreographies were computed numerically by Calleja, Doedel & García-Azpeitia [13].

In this paper we consider the case in which all vortices have equal strength for $N \geq 4$. With the appropriate normalisations, the governing equations for the motion of the vortices become

$$\dot{v}_j = \sum_{i=1 (i \neq j)}^{N} \frac{v_i \times v_j}{\left|v_j - v_i\right|^2}, \qquad j = 1, \ldots, N, \tag{1.1}$$

where $v_j(t)$ belongs to the unit sphere S^2 in $\mathbb{R}^3$ and denotes the position of the j^{th} vortex and $\times$ denotes the vector product. The derivation of the equations may be found in Newton's book [38]. The assumption that the vortices have equal strength results in the invariance of the system under the action of the permutation group S_N on the vortices and this extra symmetry is essential in our analysis. The Hamiltonian structure of Eqs. (1.1) is described in Section 2 below. An interesting Lagrangian interpretation of the system is given in Vankerschaver & Leok [45].

The most fundamental solutions of Eqs. (1.1) are the equilibria, and in particular, the ground states of the Hamiltonian, but their determination is difficult for large N. Actually, determining the ground state of the Hamiltonian is a special case of one of Smale's open problems, generalising the Thomson problem from the Coulomb potential to more general ones. The ground states for different number of vortices exhibit many symmetries that have been established rigorously only for $N \leq 5$. On the other hand, the five platonic solids are natural equilibrium solutions for $N = 4, 6, 8, 12, 20$. It was proved by Kurakin [23] that the tetrahedron, octahedron and icosahedron are nonlinearly stable, while the cube and the dodecahedron are unstable.

Goals of the paper

The original motivation of this paper was to prove the existence of small nonlinear oscillations near the platonic solid equilibria. This is a non-trivial task since the Liapunov Centre Theorem [29] and its extensions obtained by Weinstein [47] and Moser [36] do not apply. The reason is that the equilibria in question are not isolated, but rather form $SO(3)$ orbits on the phase space. The symmetric extension of these theorems by Montaldi, Roberts & Stewart [34] does not apply for the exact same reason. Other extensions relying on topological methods developed by Ize & Vignoli [19] and Strzelecki [42] also do not apply because they require the phase space to be a euclidean space and the action to be linear.

Our results are summarised below. We succeeded in finding several (but not all) families of periodic solutions emanating from the platonic solids by developing a general framework used to analyse solutions of Eqs. (1.1) with prescribed symmetries. Apart from the nonlinear oscillations around the platonic solids, our construction proves the existence of many other families of periodic solutions of Eqs. (1.1) which either emanate from an equilibrium or converge to collisions possessing a specific symmetry.

Summary of results

Let $\mathcal{P}$ be one of the platonic solids and N be the number of its vertices. Our approach to prove the existence of periodic solutions of (1.1) in which the vortices oscillate around the vertices of $\mathcal{P}$ is to restrict our attention to the family of solutions that are K-symmetric, where $K < \mathrm{SO}(3)$ is a discrete subgroup that leaves $\mathcal{P}$ invariant. The crucial point is to choose K in such a way that the aforementioned family of solutions forms a 1-degree of freedom integrable Hamiltonian subsystem of (1.1), and hence its generic orbits are periodic. The idea of the method goes back to Tokieda [43] and Soulière & Tokieda [41]. A similar approach is applied by Fusco, Gronchi & Negrini [14] to the N-body problem.

In this paper we proceed with a degree of generality beyond the case of the platonic solids described above. Our main contribution is to develop a general framework, valid for arbitrary $N \geq 4$, for the analysis of the K-symmetric solutions of (1.1) that form a 1-degree of freedom integrable Hamiltonian subsystem of (1.1), where $K < \mathrm{SO}(3)$ is one of the following groups: the dihedral group $\mathbb{D}_n$, tetrahedral group $\mathbb{T}$, octahedral group $\mathbb{O}$ or icosahedral group $\mathbb{I}$. Our construction relies on the concept of (K,F)***-symmetric solutions***, that are solutions of Eqs. (1.1) of the form

$$(v_1(t),\dots,v_N(t)) = (u(t), g_2u(t),\dots,g_mu(t), f_{m+1},\dots,f_N), \tag{1.2}$$

where $K_o = (g_1 = e, g_2,\dots,g_m)$ is an ordering of K (e is the identity element in $\mathrm{SO}(3)$) and $F_o = (f_{m+1},\dots,f_N)$ is an ordering of a certain set $F \subset \mathcal{F}[K]$, which is assumed to be K-invariant. Here $\mathcal{F}[K]$ denotes the set of points in S^2 having non-trivial K-isotropy.

In Theorem 3.5 we prove that (1.2) is a solution of Eqs. (1.1) if and only if $u(t)$ is a solution of the ***reduced system***

$$\begin{aligned} \dot{u} &= -\frac{1}{m} u \times \nabla_u h_{(K,F)}(u), \\ h_{(K,F)}(u) &= -\frac{m}{4}\sum_{j=2}^{m} \ln\left|u - g_j u\right|^2 - \frac{m}{2}\sum_{j=m+1}^{N} \ln\left|u - f_j\right|^2 . \end{aligned} \tag{1.3}$$

We call (1.3) the reduced system since it is obtained by the discrete reduction (see e.g. Marsden [30]) of Eqs. (1.1) by an appropriate twisted subgroup $\hat{K} < S_N \times \mathrm{SO}(3)$ which is isomorphic to K. The smooth function $h_{(K,F)} : S^2 \setminus \mathcal{F}[K] \to \mathbb{R}$ is the ***reduced Hamiltonian***. In Theorem 3.5 we also specify the symmetries of the reduced system in terms of the normaliser group $N(K)$ of K in $\mathrm{SO}(3)$ and the invariance properties of F. Furthermore, we show that the centre of vorticity of every (K,F)-symmetric solution vanishes.

After proving Theorem 3.5, we systematically analyse the properties of the reduced system (1.3) and determine the implications about the corresponding (K,F)-symmetric solutions of (1.1). We first work with general K and F. The reduced system (1.3) is a smooth, 1-degree of freedom, integrable Hamiltonian system on $S^2 \setminus \mathcal{F}[K]$. We show that points in $\mathcal{F}[K]$ are in one-to-one correspondence with (K,F)-symmetric collisions of Eqs. (1.1) and we propose a smooth regularisation of the reduced system to all of S^2. This is done in terms of the ***regularised Hamiltonian*** $\tilde{h}_{(K,F)}$ that is the smooth function on S^2 given by

$$\tilde{h}_{(K,F)}(u) = \exp(-2h_{(K,F)}(u)).$$

The resulting regularised system is a 1-degree of freedom, integrable Hamiltonian system on the compact manifold S^2 whose dynamics consists of equilibrium points, periodic solutions and heteroclinic/homoclinic orbits. In particular, we conclude that all regular level sets of $\tilde{h}_{(K,F)}$ (and hence also of $h_{(K,F)}$) are periodic solutions. This allows us to prove the existence of 1-parameter families of periodic solutions near the extrema of $h_{(K,F)}$ and the (K,F)-symmetric collisions of Eqs. (1.1) (Corollary 3.17).

We then proceed to analyse the reduced system (1.3) in detail for specific choices of K and F. Our choices include all possibilities for which the corresponding (K,F)-symmetric solutions contain the platonic solids as equilibria. In Theorems 4.1 and 5.1 we classify all the equilibria and collisions for $K = \mathbb{D}_n$ for the cases in which the set F is, respectively, empty and consists of the north and south poles. The corresponding (K,F)-symmetric equilibria and collisions of Eqs. (1.1) are respectively illustrated in Figures 4.1 and 5.1. The equilibria are equatorial polygons, prisms and anti-prisms (with and without a pair of perpendicular antipodal vortices) for which we give the explicit dimensions for arbitrary even N. The collisions are polygonal (a binary collision occurring at each vertex) and a polar collision in which half of the vortices occupy the north and south poles. Using this classification, and applying our theoretical framework, we establish the existence of the 1-parameter families of periodic orbits emanating from the stable equilibria of the reduced system and the collisions (Corollaries 4.2 and 5.2). These periodic solutions are respectively illustrated in Figures 4.2 and 5.2. For particular values of n, the periodic solutions emanating from the stable equilibria of the reduced system prove the existence of the following 1-parameter families of periodic solutions of (1.1) near the platonic solid equilibria:

(i) a $\mathbb{D}_2$-symmetric family of 4 vortices emanating from the tetrahedron;

(ii) a $\mathbb{D}_2$-symmetric family of 6 vortices emanating from the octahedron in which two antipodal vortices remain fixed;

(iii) a $\mathbb{D}_3$-symmetric family of 6 vortices emanating from the octahedron;

(iv) a $\mathbb{D}_3$-symmetric family of 8 vortices emanating from the cube in which two antipodal vortices remain fixed;

(v) a $\mathbb{D}_5$-symmetric family of 12 vortices emanating from the icosahedron in which two antipodal vortices remain fixed.

Next we consider the case $K = \mathbb{T}$. Theorems 6.1 and 7.1 respectively classify the equilibria and collisions of the reduced system for F empty and F consisting of two antipodal tetrahedra that make up a cube. These equilibria and collisions are respectively illustrated in Figures 6.1 and 7.1. Our theoretical framework applied to this classification proves the existence of the 1-parameter families of periodic solutions described in Corollaries 6.4 and 7.3 and respectively illustrated in Figures 6.2 and 7.2. In particular we determine the existence of:

(vi) a $\mathbb{T}$-symmetric family of periodic solutions of 12 vortices emanating from the icosahedron;

(vii) a $\mathbb{T}$-symmetric family of periodic solutions of 20 vortices emanating from the dodecahedron in which eight vortices remain fixed at the vertices of a cube.

We also present the phase portrait of the reduced system (1.3) obtained numerically for all choices of K and F described above that have a platonic solid as a (K,F)-symmetric equilibria. These are given in Figures 4.3, 5.3, 6.3 and 7.3.

Future work

A natural continuation of this work is to apply the theoretical framework of Section 3 to different choices of the group K and the K-invariant set $F \subset \mathcal{F}[K]$. The cases treated in Sections 4 through 7 are only a few possibilities that we chose to work with because they allowed us to prove the existence of nonlinear small oscillations around the platonic solids. It turns out that for the subgroups $K = \mathbb{D}_n, \mathbb{T}, \mathbb{O}, \mathbb{I}$, the set $\mathcal{F}[K]$ is finite and has been classified in [28, Table 1]. Based upon this classification one concludes that for each dihedral group $\mathbb{D}_n$ and for the tetrahedral group $\mathbb{T}$ there are 6 distinct choices of F, whereas for $\mathbb{O}$ and icosahedral group $\mathbb{I}$ there are 8 such possibilities. Some interesting cases are:

(i) 24 vortices with octahedral symmetry $\mathbb{O}$ (this case contains a truncated octahedron as equilibrium).

(ii) 60 vortices with icosahedral symmetry $\mathbb{I}$ (this case contains a truncated icosahedron or Fullerene as equilibrium).

It is also of interest to investigate the persistence of the periodic, equilibrium and heteroclinic/homoclinic solutions that we found, and of the invariant sets $M_{(K_o,F_o)}$, under perturbations. Such perturbations will in general destroy the $S_N \times \mathrm{SO}(3)$ equivariance of the system and the fate of these objects is unclear. Possible sources of this perturbation may be:

(i) a variation of the strength of some of the vortices. The "twisters" of Soulière & Tokieda [41] are an indication that persistence may indeed be expected in some cases.

(ii) a variation of the underlying Riemannian metric on S^2 as considered by Boatto & Koiller [6]. It was recently found by Wang [46] that the system has infinitely many periodic orbits.

Another interesting extension of this work is to generalise Theorem 3.5 considering larger values of N such that the reduced system is no longer integrable for a subgroup $K < \mathrm{SO}(3)$. For instance, one could look for (K,F)-symmetric solutions generated by two vortices replacing the ansatz (1.2) with

$$(v_1(t),\dots,v_N(t)) = (u(t),g_2u(t),\dots,g_mu(t),w(t),g_2w(t),\dots,g_mw(t),f_{2m+1},\dots,f_N),$$

where, as usual, $K_o = \{g_1 = e, g_2, \dots g_m\}$ and $F_o = \{f_{2m+1},\dots,f_N\}$ are orderings of K and F. The corresponding reduced system for $(u(t),w(t))$ is a 2-degree of freedom Hamiltonian system on (an open dense set of) $S^2 \times S^2$. Although we expect the reduced dynamics to be non-integrable, one may apply the Lyapunov Centre Theorem or KAM techniques to prove the existence of periodic and quasi-periodic solutions of the system. We plan to pursue this research direction in a future publication. Some interesting cases of the above setup which contain platonic solids as equilibria are:

(i) 12 vortices with symmetry $K = \mathbb{D}_3$ (the icosahedron is an equilibrium).

(ii) 20 vortices with symmetry $K = \mathbb{D}_5$ (the dodecahedron is an equilibrium).

One could also apply the techniques followed by García-Azpeitia [15] to prove the existence of relative periodic solutions near the SO(3)-orbit of a platonic solid. In such approach one looks for periodic solutions in a rotating frame of reference and performs a stereographic projection. The solutions in question then correspond to critical points of an $\mathrm{SO}(2) \times S^1$ equivariant gradient map on $\mathbb{R}^{2N}$, where the SO(2) action is linear. Given that SO(2) is abelian, one may apply the equivariant degree theory of Ize & Vignoli [19] to prove the existence of a global family of such periodic solutions in the rotating frame which are the sought relative periodic solutions of the system. Alternatively, the local existence of the family of relative periodic solutions may be established using equivariant Conley index as in [42] or Poincaré maps as in [37]. It is important to notice that these solutions have a non-vanishing centre of vorticity and, therefore, in contrast to the solutions found in this paper, do not remain close to a platonic solid configuration but only to its $\mathrm{SO}(3)$-orbit. Finally, we mention that the approach of introducing a rotating frame of reference is of interest because the resulting equations coincide with those describing the motion of N-vortices on a rotating sphere which is a problem with a natural physical relevance. Existence of relative equilibria and quasi-periodic solutions for this system has been respectively considered by Laurent-Polz [26] and Newton & Shokraneh [39].

Structure of the paper

We begin by introducing some preliminary material in Section 2. All of this material is known except perhaps for Proposition 2.1 that states that, under our hypothesis that all vortices have equal strength, the system cannot evolve into collision. Our theoretical framework for the analysis of (K,F)-symmetric solutions is developed in Section 3. We begin by giving some basic definitions in Subsection 3.1 and then formulate and prove our main Theorem 3.5 on the reduction of the dynamics in Subsection 3.2. The regularisation of the collisions is treated in Subsection 3.3 and the qualitative properties of (K,F)-symmetric solutions is described in Subsection 3.4. In Sections 4 through 7 we apply the results of Section 3 to analyse (K,F)-symmetric solutions for specific choices of K and F as described above. Section 4 deals with $K=\mathbb{D}_n$ and $F=\varnothing$. Section 5 with $K=\mathbb{D}_n$ and F consisting of the north and south poles. In Section 6 we take $K=\mathbb{T}$ and $F=\varnothing$ and in Section 7 we consider $K=\mathbb{T}$ and F consisting of two antipodal tetrahedra that make up a cube.

2 Preliminaries: the equations of motion and their symmetries

Let $M=S^2\times\cdots\times S^2$, the product of N copies of the unit sphere S^2 on $\mathbb{R}^3$. The motion of N vortices on the sphere is described by the Hamiltonian system on M where the Hamilton function H and symplectic form Ω are given by

$$H(v)=-\frac{\Gamma_i\Gamma_j}{4\pi}\sum_{i<j}\ln\left(\left|v_j-v_i\right|^2\right),\qquad \Omega=\sum_{i=1}^{N}\Gamma_j\pi_i^*\omega_{S^2}.$$

Here $v=(v_1,\dots,v_N)\in M$ and v_j is the position of the jth vortex whose vorticity is assumed to be Γ_j, and π_i is the Cartesian projection on to the ith factor with ω_{S^2} denoting the usual area form on S^2.

In this work we assume that all vortices have the same vorticity. After suitable re-scalings, the system is described by the Hamiltonian system on M with

$$H(v)=-\frac{1}{2}\sum_{i<j}\ln\left(\left|v_j-v_i\right|^2\right),\qquad \Omega=\sum_{i=1}^{N}\pi_i^*\omega_{S^2}. \tag{2.1}$$

The corresponding equations of motion take the form

$$\dot v_j=-v_j\times\nabla_{v_j}H(v)=\sum_{i=1\,(i\neq j)}^{N}\frac{v_i\times v_j}{\left|v_j-v_i\right|^2},\qquad j=1,\dots,N, \tag{2.2}$$

where here, and throughout '×' denotes the vector product in $\mathbb{R}^3$. One may check that the above equations indeed define a vector field X on M as follows: consider them as a system on $(\mathbb{R}^3)^N$ and notice that $|v_j|^2$ is a first integral for $j = 1,\dots,N$. Then the equations may be restricted to the level set M where all these integrals take the value 1. The vector field X satisfies $\Omega(X,\cdot) = dH$.

Collisions Both H and the equations of motion are undefined at the *collision set* $\Delta \subset M$ where at least two vortices occupy the same position, i.e.

$$\Delta = \{(v_1,\dots,v_n) \in M : v_i = v_j \quad \text{for some} \quad i \neq j\}.$$

It is usual to remove these points from the phase space to work with smooth objects. In our approach we will often find it convenient not to do this (in fact we work with a regularisation of the equations of motion ahead). In any case, it is convenient to have in mind that any collision-free configuration $v \in M \smallsetminus \Delta$ *cannot* evolve into a collision as we show in the following proposition.

Proposition 2.1. *The flow of* (2.2) *on* $M \smallsetminus \Delta$ *is complete. Namely, if* $v_0 \in M \smallsetminus \Delta$ *and* $t \mapsto v(t)$ *denotes the solution of* (2.2) *with initial condition* v_0*, then* $v(t)$ *is defined for all time* t*. (In particular* $v(t) \notin \Delta$ *for all* $t \in \mathbb{R}$*.)*

Proof. Since the sphere S^2 is a bounded set, for any i, j we have $|v_i - v_j| \leq 2$, so $-\ln|v_i - v_j|^2$ is bounded from below. As a consequence, if $\{v^{(k)}\}_{k\in\mathbb{N}}$ is a sequence in $M \smallsetminus \Delta$ with $v^{(k)} \to \Delta$ as $k \to \infty$, then necessarily $H(v^{(k)}) \to \infty$. Considering that $H(v(t)) = H(v_0) < \infty$ we conclude that $v(t)$ stays away from Δ at all time at which it is defined. However, since $M \smallsetminus \Delta$ is a bounded set, standard theorems on extensibility of of solutions of differential equations imply that $v(t)$ can only cease to exist if it approaches the boundary of $M \smallsetminus \Delta$. But this boundary is precisely Δ. Therefore $v(t)$ is defined for all time $t \in \mathbb{R}$. □

Remark 2.2. Note that if the vortex strengths are not identical and have different signs, collisions may indeed occur in finite time [21, 22]. On the other hand, the above property of the N vortex problem on the sphere is a fundamental difference with the N-body problem on the sphere [10] where collisions may indeed take place. This is due to dependence of the Hamiltonian on the velocities in the latter problem. When going to collision, the kinetic energy approaches infinity and the potential energy approaches minus infinity while their sum remains constant.

2.1 Symmetries

Rotational symmetries. The group $\mathrm{SO}(3)$ acts diagonally on $M = S^2 \times \cdots \times S^2$ and is easy to check that the action is symplectic and the Hamiltonian H is invariant. As a consequence, the equations of motion (2.1) are $\mathrm{SO}(3)$-equivariant. Moreover, Noether's theorem applies and we have a conservation law: the quantity

$$J : M \to \mathbb{R}^3, \qquad J(v_1,\dots,v_N) = \sum_{i=1}^{N} v_i,$$

is constant along the motion. This statement may be verified directly from the equations of motion (2.2). In geometric terms, J is the momentum map of the $\mathrm{SO}(3)$ action on M, with the usual identification of $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$. We will refer to $J(v)$ as the *centre of vorticity* of the configuration $v = (v_1,\dots,v_N)$.

Vortex relabelling symmetry. Since all the vortices have the same strength, the system is also invariant under relabelling of the vortices. This may be represented by the action of the permutation group S_N on M,

$$\sigma : (v_1, \dots, v_N) \mapsto (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(N)}), \qquad \text{for} \quad \sigma \in S_N.$$

One may check that this action is symplectic, that the Hamiltonian H is invariant and the equations of motion (2.1) are S_N-equivariant.

Remark 2.3. In our convention, the product of two permutations $\sigma_1, \sigma_2 \in S_N$ is $\sigma_1\sigma_2 := \sigma_1 \circ \sigma_2$. The action above is defined with σ^{-1} in order to have a *left* action with respect to this product. Other papers on the subject (e.g. [15] and references therein) consider instead the action $\sigma : (v_1, \dots, v_N) \mapsto (v_{\sigma(1)}, \dots, v_{\sigma(N)})$ which is a left action only if the product on S_N is defined according to the opposite convention, $\sigma_1\sigma_2 := \sigma_2 \circ \sigma_1$.

Full symmetries. The two symmetries described above may be combined into a symplectic action of the direct product group $\hat{G} := S_N \times \mathrm{SO}(3)$ on M. Throughout the paper this action will be denoted by a *centre dot* '$\cdot$' as follows:

$$(\sigma, g) \cdot (v_1, \dots, v_N) = (gv_{\sigma^{-1}(1)}, \dots, gv_{\sigma^{-1}(N)}), \qquad (\sigma, g) \in \hat{G}, \quad (v_1, \dots, v_N) \in M.$$

Apart from the simplecticity of this action, the Hamiltonian H is invariant and the equations of motion (2.2) are $\hat{G}$-equivariant. A key feature of this action is that it is not free and this will allow us to extract valuable information about the dynamics of (2.2).

Twisted subgroups. Suppose that $K < \mathrm{SO}(3)$ is a discrete subgroup and $\tau : K \to S_N$ is a group morphism. Then

$$\hat{K}_\tau := \{(\tau(g), g) : g \in K\} \tag{2.3}$$

is a discrete subgroup of $\hat{G} = S_N \times \mathrm{SO}(3)$ which is often called a ***twisted subgroup***. In this work a special role is played by the twisted subgroups of $\hat{G}$ corresponding to one-to-one group morphisms τ.

3 Symmetric solutions of the N-vortex problem on the sphere

3.1 Symmetric configurations: definitions

Let $K < \mathrm{SO}(3)$ be any subgroup. Then K acts on S^2 as usual. The set of points in S^2 having non-trivial K-isotropy will be denoted as

$$\mathcal{F}[K] = \left\{u \in S^2 : K_u \neq \{e\}\right\}.$$

Here, and in what follows, e denotes the identity element in K and K_u is the isotropy group of $u \in S^2$, namely, $K_u = \{g \in K : gu = u\}$.

Remark 3.1. For the (finite) subgroups $K = \mathbb{Z}_n, \mathbb{D}_n, \mathbb{T}, \mathbb{O}, \mathbb{I}$ the set $\mathcal{F}[K]$ is finite. Actually, the sets $\mathcal{F}[K]$ are completely classified in Table 1 and the appendix of [28] (that also gives a brief description of these groups).

The following definitions are essential in our work.

Definition 3.2. Let $K < \mathrm{SO}(3)$ be a discrete subgroup of order $m \leq N$ and $F \subset \mathcal{F}[K]$ be a K-invariant subset of order $N-m$. Let $K_o = (g_1 = e, g_2, \dots, g_m)$ be an ordering of K and $F_o = (f_{m+1}, ..., f_N)$ an ordering of F. We define $M_{(K_o,F_o)} \subset M$ as the set of configurations $(v_1, \dots, v_N) \in M$ that satisfy

$$\begin{aligned} v_j &= g_j v_1, \qquad j = 1, \dots, m, \\ v_j &= f_j, \qquad j = m+1, \dots, N. \end{aligned}$$

Definition 3.3. Let K and F be as in Definition 3.2, we will say that a configuration $v \in M$ is ***(K,F)-symmetric*** if $v \in M_{(K_o,F_o)}$ for certain orderings K_o and F_o. If F is empty and $m = N$ we will simply say that the corresponding configuration is ***K-symmetric***.

Remark 3.4. The above definitions require that the first m entries of a (K,F)-symmetric configuration to be described in terms of the elements of K and the remaining $N-m$ in terms of the elements of F. This constraint in the ordering is artificial and could be removed in view of the relabelling symmetry, but we keep it for clarity of the presentation. The same observation holds for our requirement that $g_1 = e$.

For the rest of the section, the symbol K will always denote one of the groups $\mathbb{D}_n, \mathbb{T}, \mathbb{O}, \mathbb{I} < \mathrm{SO}(3)$, and the symbol F will always denote a K-invariant subset of $\mathcal{F}[K]$. Moreover, we will continue to denote $m = |K| > 0$ and $N - m = |F| \geq 0$.

3.2 Reduction of the dynamics of symmetric configurations

Suppose that $v = (v_1, \dots, v_N)$ is a (K,F)-symmetric configuration so that $v \in M_{(K_o,F_o)}$ for certain orderings $K_o = (g_1 = e, \dots, g_m)$ of K and $F_o = (f_{m+1}, \dots, f_N)$ of F. Let

$$\rho_{(K_o,F_o)} : S^2 \to M, \qquad \rho_{(K_o,F_o)}(u) = (u, g_2 u, \dots, g_m u, f_{m+1}, \dots, f_N). \tag{3.1}$$

In this section we will prove that the solution of equations (2.2) with initial condition v is given by $t \mapsto \rho_{(K_o,F_o)}(u(t))$, where $u(t)$ is the solution to the ***reduced system*** on S^2,

$$\dot{u} = -\frac{1}{m} u \times \nabla_u h_{(K,F)}(u), \tag{3.2}$$

with initial condition $u(0) = v_1$. Here, $h_{(K,F)} : S^2 \to \mathbb{R}$ is the ***reduced Hamiltonian***, that is defined in terms of the Hamiltonian (2.1) by

$$h_{(K,F)}(u) := H(\rho_{(K_o,F_o)}(u)). \tag{3.3}$$

In particular, this shows that the evolution of a (K,F)-symmetric configuration remains a (K,F)-symmetric configuration at all time, and, therefore, we may speak of ***(K,F)-symmetric solutions***. Note that along these solutions, the vortices located at $f_{m+1}, \dots, f_N$ remain fixed.

We will also show that the reduced system (3.2) possesses a symmetry, and we will describe it in detail. Note that, in virtue of the invariance of H under the relabelling of the vortices, the reduced Hamiltonian $h_{(K,F)}$ is well defined independently of the specific orderings K_o and F_o of K and F, and this is reflected in our notation.

The properties described above follow from the first three items of the following theorem whose proof relies on the concept of discrete reduction (see e.g. [30]). Applications of discrete reduction to the study of the N-vortex problem on the sphere already appear in [28].

Theorem 3.5. *Let K be any of the groups $\mathbb{D}_n, \mathbb{T}, \mathbb{O}, \mathbb{I}$ and let $F \subset \mathcal{F}[K]$ be a K-invariant set. Suppose that $|K| = m \leq N$ and $|F| = N - m \geq 0$. Let $K_o = (g_1 = e, g_2, \dots, g_m)$ and $F_o = (f_{m+1}, \dots, f_N)$ be orderings of K and F. The following statements hold.*

(i) *The set $M_{(K_o,F_o)}$ is an embedded submanifold of M, diffeomorphic to S^2, and invariant under the flow of the equations of motion* (2.2).

(ii) *The restriction of the flow of* (2.2) *to $M_{(K_o,F_o)}$ is conjugated by $\rho_{(K_o,F_o)}$ to the flow of the integrable Hamiltonian system on $(S^2, m\omega_{S^2})$, with (reduced) Hamiltonian $h_{(K,F)} : S^2 \to \mathbb{R}$ defined by* (3.3). *That is, $t \to u(t)$ is a solution of* (3.2) *if and only if $t \mapsto \rho_{(K_o,F_o)}(u(t))$ is a solution of* (2.2).

(iii) *Let $N(K)$ be the normaliser of K in $\mathrm{SO}(3)$ and suppose that the subgroup $K_1 < \mathrm{SO}(3)$ satisfies $K \leq K_1 \leq N(K)$. If F is invariant with respect to the K_1-action on S^2, then the reduced Hamiltonian $h_{(K,F)}$ is K_1-invariant and the reduced system* (3.2) *is K_1-equivariant. In particular, these conclusions always hold for $K_1 = N(K)$ if $F = \varnothing$ and for $K_1 = K$ for general F.*

(iv) *Up to the addition of a constant term, the reduced Hamiltonian $h_{(K,F)} : S^2 \to \mathbb{R}$ satisfies*

$$h_{(K,F)}(u) = -\frac{m}{4} \sum_{j=2}^{m} \ln |u - g_j u|^2 - \frac{m}{2} \sum_{j=m+1}^{N} \ln |u - f_j|^2 . \tag{3.4}$$

(v) *The centre of vorticity of elements in $M_{(K_o,F_o)}$ is $0 \in \mathbb{R}^3$, i.e. $M_{(K_o,F_o)} \subset J^{-1}(0)$.*

Remark 3.6. In trying to understand which are the symmetries of the reduced system (3.2) it will be useful to keep in mind the following relations between the groups $\mathbb{D}_n, \mathbb{T}, \mathbb{O}$ and $\mathbb{I}$, and their normalisers in $\mathrm{SO}(3)$:

K	$\mathbb{D}_2$	$\mathbb{D}_n, n \geq 3$	$\mathbb{T}$	$\mathbb{O}$	$\mathbb{I}$
$N(K)$	$\mathbb{O}$	$\mathbb{D}_{2n}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{I}$

(3.5)

Remark 3.7. To be precise, at this point of the paper, all statements about the flow of (2.2) in items (i)-(v) of the theorem only make sense away from collisions. In fact, the reduced system (3.2) is only defined at those points of S^2 at which $h_{(K,F)}$ is smooth. In Section 3.3 ahead will show that the reduced system is well defined away from (finitely many) points in $\mathcal{F}[K]$ which are in one-to-one correspondence with the collision configurations within $M_{(K_o,F_o)}$. Moreover, we will introduce a regularisation that extends the reduced system (3.2) to all of S^2, the flow of (2.2) to all of $M_{(K_o,F_o)}$, and the conclusions of the theorem are valid for this regularisation without any restriction. We have decided to oversee this detail in the statement of the theorem and in its proof to simplify the presentation.

The proof of the theorem that we present relies on the following three lemmas whose proof is postponed until the end of the section.

Lemma 3.8. *Let K, F, K_o and F_o be as in the statement of Theorem* 3.5. *There exists a one-to-one group morphism $\tau : K \to S_N$ such that $M_{(K_o,F_o)}$ is a connected component of $\mathrm{Fix}(\hat{K}_\tau) \subset M$ where $\hat{K}_\tau < \hat{G}$ is the twisted subgroup* (2.3), *and where*

$$\mathrm{Fix}(\hat{K}_\tau) := \{ v \in M : \hat{g} \cdot v = v \quad \textit{for all} \quad \hat{g} \in \hat{K}_\tau \}.$$

Lemma 3.9. *Let K, F, K_o and F_o be as in the statement of Theorem* 3.5, *then $\rho^*_{(K_o,F_o)} \Omega = m\omega_{S^2}$.*

Lemma 3.10. *For the groups $K = \mathbb{D}_n, \mathbb{T}, \mathbb{O}, \mathbb{I} < \mathrm{SO}(3)$, we have $\sum_{g\in K} g = 0$.*

Proof of Theorem 3.5. (i) The set $M_{(K_o,F_o)}$ is clearly an embedded submanifold of M isomorphic to S^2 with the embedding given by (3.1). Indeed, we have $M_{(K_o,F_o)} = \rho_{(K_o,F_o)}(S^2)$. The invariance of $M_{(K_o,F_o)}$ under the flow of (2.2) is immediate in virtue of Lemma 3.8: since the system (2.2) is $\hat{G}$-equivariant then $\mathrm{Fix}(\hat{K}_\tau)$ is invariant by its flow and so are each of its connected components.

(ii) First note that for $\varphi : S^2 \to \mathbb{R}$, the associated Hamiltonian vector field X_φ on S^2, determined by $\omega_{S^2}(X_\varphi, \cdot) = d\varphi$, defines the equations of motion $\dot{u} = -u \times \nabla_u \varphi(u)$. If the symplectic form ω_{S^2} is scaled by a factor of m, then the corresponding Hamiltonian vector field X_φ inherits a rescaling by $1/m$ which leads to an appearance of this factor on the right hand side of the equations of motion. This shows that the Hamiltonian system on $(S^2, m\omega_{S^2})$ with Hamiltonian $h_{(K,F)}$ defines the equations (3.2), as required. Moreover, this system is trivially integrable in the Arnold-Liouville sense since S^2 has dimension 2 and $h_{(K,F)}$ is a first integral.

Next, since $\hat{G}$ acts symplectically on (M, Ω) and H is $\hat{G}$-invariant, it is known (e.g. [30]) that $\mathrm{Fix}(\hat{K}_\tau)$ is a symplectic submanifold of M and that the restriction of the flow of X to $\mathrm{Fix}(\hat{K}_\tau)$ is Hamiltonian with respect to the restricted Hamiltonian and symplectic form. The same is true about each of its connected components. In particular, in view of Lemma 3.8, this implies that $M_{(K_o,F_o)}$ is a symplectic manifold equipped with the restriction $\Omega_0 := \Omega|_{M_{(K_o,F_o)}}$ of the symplectic form Ω, and that the restriction of the flow of (2.2) to $M_{(K_o,F_o)}$ is Hamiltonian with respect to Ω_0 and the Hamilton function $H_0 := H|_{M_{(K_o,F_o)}}$.

The key point of the proof is to observe that $\rho_{(K_o,F_o)}$ defined by (3.1) is in fact a symplectomorphism between $(S^2, m\omega_{S^2})$ and $(M_{(K_o,F_o)}, \Omega_0)$. This is an immediate consequence of Lemma 3.9 together with the observation that $M_{(K_o,F_o)} = \rho_{(K_o,F_o)}(S^2)$. As any symplectomorphism, $\rho_{(K_o,F_o)}$ takes Hamiltonian vector fields into Hamiltonian vector fields (see e.g. [31]). Considering that the reduced Hamiltonian (3.3) and the restricted Hamiltonian H_0 are related by $h_{(K,F)} = H_0 \circ \rho_{(K_o,F_o)} = \rho^*_{(K_o,F_o)} H_0$, it follows that $\rho_{(K_o,F_o)}$ pulls back the vector field on $M_{(K_o,F_o)}$ defined by the restriction of (2.2) onto the vector field on S^2 defined by (3.2). In particular, $\rho_{(K_o,F_o)}$ maps solutions of (3.2) into solutions of (2.2) that are contained in $M_{(K_o,F_o)}$. This correspondence between solutions is one-to-one since $\rho_{(K_o,F_o)} : S^2 \to M_{(K_o,F_o)}$ is invertible.

(iv) By definition of $h_{(K,F)}$ we have

$$h_{(K,F)}(u) = -\frac{1}{2} \sum_{1\le i<j\le m} \ln\left|g_i u - g_j u\right|^2 - \frac{1}{2} \sum_{j=m+1}^{N} \sum_{i=1}^{m} \ln\left|g_i u - f_j\right|^2 - \frac{1}{2} \sum_{m+1\le i<j\le N} \ln\left|f_i - f_j\right|^2. \tag{3.6}$$

Now, on the one hand we have

$$\sum_{1\le i<j\le m} \ln\left|g_i u - g_j u\right|^2 = \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{m} \ln\left|g_i u - g_j u\right|^2 = \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{m} \ln\left|u - g_i^{-1} g_j u\right|^2 = \frac{m}{2} \sum_{j=2}^{m} \ln\left|u - g_j u\right|^2. \tag{3.7}$$

On the other hand, fix $f \in F$ and let $I \subset \{m+1, \ldots, N\}$ be such that the K-orbit of f satisfies $Kf = \{f_i : i \in I\}$. For each $i \in I$ let $h_i \in K$ such that $h_i f = f_i$. Since the orbit Kf is isomorphic to K/K_f we have $Kf = \{h_i f : i \in I\}$ and $K = \cup_{i\in I} h_i K_f$. Thus

$$\sum_{i=1}^{m} \ln|g_i u - f|^2 = \sum_{g\in K} \ln|u - gf|^2 = \sum_{i\in I} \sum_{h\in K_f} \ln|u - h_i h f|^2 = |K_f| \sum_{i\in I} \ln|u - f_i|^2.$$

Considering that the above formula holds when setting $f = f_j$ for all $j \in I$, and that $|K_{f_j}| = m/|I|$ is constant for $j \in I$, we have

$$\sum_{j\in I}\sum_{i=1}^{m}\ln\left|g_i u - f_j\right|^2 = |I|\frac{m}{|I|}\sum_{i\in I}\ln|u-f_i|^2 = m\sum_{j\in I}\ln\left|u-f_j\right|^2.$$

Therefore, breaking up the set of indices $\{m+1,\dots,N\}$ into the disjoint subsets I_k, each containing the indices of a K orbit of F, we have

$$\sum_{j=m+1}^{N}\sum_{i=1}^{m}\ln\left|g_i u - f_j\right|^2 = \sum_{k}\sum_{j\in I_k}\sum_{i=1}^{m}\ln\left|g_i u - f_j\right|^2 = m\sum_{k}\sum_{j\in I_k}\ln\left|u-f_j\right|^2 = m\sum_{j=m+1}^{N}\ln\left|u-f_j\right|^2. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) yields (3.4) since the third sum in (3.6) is a constant independent of u.

(iii) Let $g \in K_1$. Starting from (3.4) and using the SO(3)-invariance of the euclidean norm we have

$$h_{(K,F)}(gu) = -\frac{m}{4}\sum_{j=2}^{m}\ln\left|u - g^{-1}g_j g u\right|^2 - \frac{m}{2}\sum_{j=m+1}^{N}\ln\left|u-g^{-1}f_j\right|^2.$$

Now, for any $g \in K_1 \le N(K)$ the map $k \in K \mapsto g^{-1}kg \in K$ is bijective, and hence $(g'_1 = e, g'_2, \dots, g'_m)$ with $g'_j = g^{-1}g_j g$ is a new ordering of K. Moreover, since, by hypothesis, F is K_1-invariant, then $(f'_{m+1},\dots,f'_N)$ with $f'_j = g^{-1}f_j$ is a new ordering of F. Therefore $h_{(K,F)}(gu) = h_{(K,F)}(u)$ showing that $h_{(K,F)}$ is indeed K_1-invariant. Since the K_1 action on $(S^2, m\omega_{S^2})$ is symplectic, it follows that the flow of the reduced system (3.2) is K_1-equivariant.

(v) In view of Lemma 3.10, if $v = (v_1,\dots v_N) \in M_{(K_o,F_o)}$, we have

$$J(v) = \sum_{j=1}^{N} v_j = \sum_{j=1}^{m} g_j v_1 + \sum_{j=m+1}^{N} f_j = \left(\sum_{g\in K} g\right) v_1 + \sum_{j=m+1}^{N} f_j = \sum_{j=m+1}^{N} f_j.$$

To show that the remaining sum on the right also vanishes, we rely on the K-invariance of F. Proceeding as in the proof of item (iv) above, fix $f_k \in F$ and let $I_k \subset \{m+1,\dots,N\}$ be such that the K-orbit of f_k satisfies $Kf_k = \{f_j : j \in I_k\}$. For each $j \in I_k$ let $h_j \in K$ such that $h_j f_k = f_j$. Since the orbit Kf_k is isomorphic to K/K_{f_k} we have $Kf_k = \{h_j f_k : j \in I_k\}$ and $K = \cup_{j\in I_k} h_j K_{f_k}$. Thus

$$0 = \left(\sum_{g\in K} g\right) f_k = \sum_{j\in I_k}\sum_{h\in K_{f_k}} h_j h f_k = \left|K_{f_k}\right|\sum_{j\in I_k} h_j f_k = \left|K_{f_k}\right|\sum_{f\in Kf_k} f.$$

This shows that the sum of the elements of the orbit of f_k is zero. Since F is K-invariant, then $\sum_{j=m+1}^{N} f_j$ is the sum of elements in disjoint orbits, each of which vanishes. Therefore, $\sum_{j=m+1}^{N} f_j = 0$ and $v \in J^{-1}(0)$.

□

We finish this section with the proofs of Lemmas 3.8, 3.9 and 3.10.

Proof of Lemma 3.8. Let Ψ and Λ denote the *index* mappings associated to the given orderings $K_o = (g_1 = e, g_2, \dots, g_m)$ and $F_o = (f_{m+1},\dots,f_N)$ of K and F:

$$\Psi : K \to \{1,\dots,m\}, \qquad g_i \mapsto i,$$
$$\Lambda : F \to \{m+1,\dots,N\}, \qquad f_j \mapsto j.$$

Then Ψ and Λ are well defined bijections that satisfy $g_{\Psi(\tilde{g})} = \tilde{g}$, $f_{\Lambda(\tilde{f})} = \tilde{f}$ for any $\tilde{g} \in K$ and $\tilde{f} \in F$.

We define $\tau : K \to S_N$ by

$$\tau(g_i)(j) = \begin{cases} \Psi(g_i g_j) & \text{if} \quad j \in \{1, \ldots, m\}, \\ \Lambda(g_i f_j) & \text{if} \quad j \in \{m+1, \ldots, N\}. \end{cases}$$

It is a simple exercise to show that τ as defined above is indeed a one-to-one group morphism with respect to our product convention in S_N (see Remark 2.3).

We now show that $M_{(K_o,F_o)} \subset \mathrm{Fix}(\hat{K}_\tau)$. Let $(g_1 v_1, \ldots, g_m v_1, f_{m+1}, \ldots, f_N) \in M_{(K_o,F_o)}$ and for $i \in \{1, \ldots, m\}$ denote $w_i = g_i v_1$. Using the definition of the action of $\hat{K}_\tau$ on M, we have, for any $j \in \{1, \ldots, m\}$, that

$$(\tau(g_j), g_j) \cdot (g_1 v_1, \ldots, g_m v_1, f_{m+1}, \ldots, f_N) = \\ (g_j w_{\tau(g_j^{-1})(1)}, \ldots, g_j w_{\tau(g_j^{-1})(m)}, g_j f_{\tau(g_j^{-1})(m+1)}, \ldots, g_j f_{\tau(g_j^{-1})(N)}).$$

However, using the definition of τ, we find

$$g_j w_{\tau(g_j^{-1})(i)} = g_j w_{\Psi(g_j^{-1} g_i)} = g_j g_{\Psi(g_j^{-1} g_i)} v_1 = g_j g_j^{-1} g_i v_1 = g_i v_1, \quad i = 1, \ldots, m,$$

and

$$g_j f_{\tau(g_j^{-1})(k)} = g_j f_{\Lambda(g_j^{-1} f_k)} = g_j g_j^{-1} f_k = f_k, \quad k = m+1, \ldots, N,$$

which shows that

$$(\tau(g_j), g_j) \cdot (g_1 v_1, \ldots, g_m v_1, f_{m+1}, \ldots, f_N) = (g_1 v_1, \ldots, g_m v_1, f_{m+1}, \ldots, f_N),$$

and indeed $M_{(K_o,F_o)} \subset \mathrm{Fix}(\hat{K}_\tau)$.

Now let $v = (v_1, \ldots, v_N) \in \mathrm{Fix}(\hat{K}_\tau)$. The condition that $(\tau(g_j), g_j) \cdot v = v$ in particular implies that

$$v_j = g_j v_{\tau(g_j^{-1})(j)} = g_j v_{\Psi(g_j^{-1} g_j)} = g_j v_{\Psi(e)} = g_j v_1, \qquad j = 1, \ldots, m,$$

where the last identity uses that $g_1 = e$ in the ordering K_o. Thus $v_i = g_i v_1$ for all $i \in \{1, \ldots, m\}$. Below we show that for any $g \in K$ and $i, j \in \{m+1, \ldots N\}$ we have $f_j = g f_i$ if and only if $v_j = g v_i$. This implies that $\{v_{m+1}, \ldots, v_N\}$ is a K-invariant subset of S^2 and, moreover, that the K-isotropy of f_j coincides with the K-isotropy of v_j. Thus $\{v_{m+1}, \ldots, v_N\}$ is a K-invariant subset of $\mathcal{F}[K]$. In particular, considering that $\mathcal{F}[K]$ is finite (Remark 3.1), we conclude that there are finitely many possibilities for the last $N - m$ entries of of $v \in \mathrm{Fix}(\hat{K}_\tau)$. It is not hard to see that each of these possibilities for $\{v_{m+1}, \ldots, v_N\}$ defines a connected component of $\mathrm{Fix}(\hat{K}_\tau)$. In particular, we conclude that $M_{(K_o,F_o)}$ is indeed a connected component of $\mathrm{Fix}(\hat{K}_\tau)$ as required.

Let $i, j \in \{m+1, \ldots N\}$. We now show that indeed, for any $g \in K$ we have $f_j = g f_i$ if and only if $v_j = g v_i$. Suppose first that $f_j = g f_i$. Using that $(\tau(g), g) \cdot v = v$ and the definition of τ, we find

$$v_j = g v_{\tau(g^{-1})(j)} = g v_{\Lambda(g^{-1} f_j)} = g v_{\Lambda(f_i)} = g v_i.$$

Conversely, suppose that $v_j = g v_i$. Using again that $(\tau(g), g) \cdot v = v$ we get

$$v_j = g v_{\tau(g^{-1})(j)},$$

and we conclude that $v_i = v_{\tau(g^{-1})(j)}$. Hence $\tau(g^{-1})(j) = i$ which, in view of the definition of τ, implies that $f_j = g f_i$. □

Remark 3.11. The above proof, together with the observation that $M_{(K_o,F_o)}$ is diffeomorphic to S^2, shows that, in fact, each of the finitely many connected components of $\mathrm{Fix}(\hat{K}_\tau)$ is diffeomorphic to S^2.

Proof of Lemma 3.9. Let $v = (v_1,\dots,v_N) \in M$. The tangent space T_vM is given by $T_vM = T_{v_1}S^2 \times \dots \times T_{v_N}S^2$. If $\alpha = (a_1,\dots,a_N)$ and $\beta = (b_1,\dots,b_N) \in T_vM$, then, by the definition of Ω in (2.1), we have $\Omega(v)(\alpha,\beta) = \sum_{j=1}^N \omega_{S^2}(v_j)(a_j,b_j)$. Now, let $u \in S^2$ and $a,b \in T_uS^2$. It is not difficult to compute

$$T_u\rho_{(K_o,F_o)}(a) = (a, g_2a, \dots, g_ma, 0, \dots 0), \qquad T_u\rho_{(K_o,F_o)}(b) = (b, g_2b, \dots, g_mb, 0, \dots 0).$$

Therefore,

$$\begin{aligned}\Omega\big(\rho_{(K_o,F_o)}(u)\big)\big(T_u\rho_{(K_o,F_o)}(a), T_u\rho_{(K_o,F_o)}(b)\big) &= \sum_{j=1}^m \omega_{S^2}(g_ju)(g_ja,g_jb) + \sum_{j=m+1}^N \omega_{S^2}(f_j)(0,0)\\ &= m\omega_{S^2}(u)(a,b),\end{aligned}$$

where the last identity uses $\omega_{S^2}(g_ju)(g_ja,g_jb) = \omega_{S^2}(u)(a,b)$, which follows from the fact that the $\mathrm{SO}(3)$ action on S^2 preserves the area form ω_{S^2}. The above calculation shows that $\rho^*_{(K_o,F_o)}\Omega = m\omega_{S^2}$ as required. □

Proof of Lemma 3.10. For the subgroup $K = \mathbb{D}_n < \mathrm{SO}(3)$, $n \geq 2$, we consider the generator matrices $A = e^{J\zeta} \oplus 1$ and $B = 1 \oplus -1 \oplus -1$, where J is the symplectic 2×2 matrix and $\zeta = 2\pi/n$. Then we have

$$\begin{aligned}\sum_{j=1}^n A^j &= \sum_{j=1}^n \left(e^{jJ\zeta} \oplus 1\right) = 0 \oplus 0 \oplus n,\\ \sum_{j=1}^n BA^j &= \sum_{j=1}^n \left(e^{-jJ\zeta} \oplus -1\right) = 0 \oplus 0 \oplus -n.\end{aligned}$$

Thus $\sum_{g\in\mathbb{D}_n} g = 0$. The groups $K = \mathbb{T}, \mathbb{O}, \mathbb{I}$ contain $\mathbb{D}_2$ as a subgroup, and since $K = h_1\mathbb{D}_2 \cup \dots \cup h_L\mathbb{D}_2$, then

$$\sum_{g\in K} g = \sum_{l=1}^L \sum_{g\in\mathbb{D}_2} h_l g = \sum_{l=1}^L h_l \left(\sum_{g\in\mathbb{D}_2} g\right) = 0.$$

□

3.3 Regularisation of collisions of symmetric configurations

We now consider in more detail the collisions of (K,F)-symmetric configurations. We begin with the following propositions that clarify the role of $\mathcal{F}[K]$.

Proposition 3.12. *Let K, F, K_o and F_o be as in the statement of Theorem 3.5. The following statements hold:*

- *(i) There is a one-to-one correspondence between $\mathcal{F}[K]$ and the collision configurations within $M_{(K_o,F_o)}$. In particular, $M_{(K_o,F_o)}$ contains finitely many collision points.*
- *(ii) If $u \in \mathcal{F}[K]$ then the point $\rho_{(K_o,F_o)}(u)$ is a (K,F)-symmetric collision configuration whose only collisions occur at the points of the orbit Ku. Moreover, these are all k-tuple collisions where $k = |K_u|$ if $u \notin F$ and $k = |K_u| + 1$ if $u \in F$.*

Proof. (i) We will prove that

$$M_{(K_o,F_o)} \cap \Delta = \rho_{(K_o,F_o)}(\mathcal{F}[K]), \tag{3.9}$$

where $\rho_{(K_o,F_o)} : S^2 \to M_{(K_o,F_o)}$ is defined by (3.1). This completes the proof since, with this specified range, $\rho_{(K_o,F_o)}$ is a bijection. Let $u \in \mathcal{F}[K]$. Then, by definition of $\mathcal{F}[K]$, there exists $g_j \neq e$ such that $g_j u = u$. This implies that the first and jth entries of $\rho_{(K_o,F_o)}(u)$ coincide and hence $\rho_{(K_o,F_o)}(u) \in \Delta$. Considering that $\rho_{(K_o,F_o)}(S^2) = M_{(K_o,F_o)}$, it follows that $\rho_{(K_o,F_o)}(\mathcal{F}[K]) \subset M_{(K_o,F_o)} \cap \Delta$. Now let $v = (v_1, g_2 v_1, \dots, g_m v_1, f_{m+1}, \dots, f_N) \in M_{(K_o,F_o)} \cap \Delta$. Then one of the two following possibilities necessarily holds:

(a) $g_i v_1 = g_j v_1$ for some $i \neq j \in \{1, \dots, m\}$. In this case we have $v_1 = g_i^{-1} g_j v_1$ implying that $v_1 \in \mathcal{F}[K]$.

(b) $g_i v_1 = f_k$ for some $i \in \{1, \dots, m\}$, $k \in \{m+1, \dots, N\}$. Then we may write $v_1 = g_i^{-1} f_k$. Since $F \subset \mathcal{F}[K]$ is K-invariant, this implies that $v_1 \in \mathcal{F}[K]$.

Thus, in any case, if $v \in M_{(K_o,F_o)} \cap \Delta$ we conclude that $v_1 \in \mathcal{F}[K]$. Considering that we may write $v = \rho_{(K_o,F_o)}(v_1)$ we conclude that $v \in \rho_{(K_o,F_o)}(\mathcal{F}[K])$ and hence $M_{(K_o,F_o)} \subset \Delta = \rho_{(K_o,F_o)}(\mathcal{F}[K])$.

(ii) The first m entries of $v = (u, g_2 u, \dots, g_m u, f_{m+1}, \dots, f_N)$ belong to the orbit Ku, so it is clear that collisions can only occur at points in this orbit. Since Ku is isomorphic to K/K_u, it follows that there are only $m/|K_u|$ distinct points among the first m entries of v, and that each of them is repeated exactly $|K_u|$ times. Now, if $u \notin F$ then, since K is F-invariant, the last $m+1$ entries of v are distinct from the first m entries of v and we indeed have $|K_u|$-tuple collisions. On the other hand, if $u \in F$, then, again by K-invariance of F, each point in the orbit Ku appears exactly once within the list $(f_{m+1}, \dots, f_N)$ and we have $(|K_u| + 1)$-tuple collisions. □

Proposition 3.13. *The reduced Hamiltonian $h_{(K,F)}$ given by* (3.3) *and the reduced system* (3.2) *are well-defined and smooth away from the finite set $\mathcal{F}[K]$. Moreover, the reduced system* (3.2) *is complete on $S^2 \smallsetminus \mathcal{F}[K]$.*

Proof. Equation (3.9) implies that away from $\mathcal{F}[K]$ we may write $h_{(K,F)}$ as a composition of smooth maps: $h_{(K,F)} = H \circ \rho_{(K_o,F_o)}$. So $h_{(K,F)}$ is smooth on $S^2 \smallsetminus \mathcal{F}[K]$, and, therefore, so is the reduced system (3.2). The completeness of the reduced flow on $S^2 \smallsetminus \mathcal{F}[K]$ follows from Proposition 2.1 and item (ii) of Theorem 3.5. □

In view of Proposition 3.13, the reduced system (3.2) is smooth away from the finite set $\mathcal{F}[K]$. We wish to define a regularisation that extends the reduced system (3.2) to the points in $\mathcal{F}[K]$ and yields a complete flow on S^2. Since, again by Proposition 3.13, the flow of (3.2) is complete on $S^2 \smallsetminus \mathcal{F}[K]$, then the points in $\mathcal{F}[K]$ have to be added as equilibrium points.

The regularisation that we propose is built with the ***regularised reduced Hamiltonian*** that is the smooth function $\tilde{h}_{(K,F)} : S^2 \to \mathbb{R}$ given by

$$\tilde{h}_{(K,F)}(u) = \exp(-2h_{(K,F)}(u)) = \prod_{j=2}^{m} |u - g_j u|^m \prod_{j=m+1}^{N} |u - f_j|^{2m}. \tag{3.10}$$

Finally, the ***regularised reduced system*** is the (smooth) Hamiltonian vector field on $(S^2, m\omega_{S^2})$ with Hamilton function $\tilde{h}_{(K,F)}$, i.e.,

$$\dot{u} = -\frac{1}{m} u \times \nabla_u \tilde{h}_{(K,F)}(u). \tag{3.11}$$

The relationship between the the reduced system (3.2) and its regularisation (3.11) is given next.

Proposition 3.14. *The following statements hold.*

(i) The curve $t \mapsto u(t)$ is a solution of the reduced system (3.2) if and only if $t \mapsto u(at)$ is a solution of the regularised reduced system (3.11) not contained in $\mathcal{F}[K]$, where $a = -2\exp(-2h_{(K,F)}(u(0)))$.

(ii) The points in $\mathcal{F}[K]$ are stable equilibria of the regularised reduced system (3.11).

Proof. (i) For $u \in S^2 \setminus \mathcal{F}[K]$ one computes

$$\nabla_u \tilde{h}_{(K,F)}(u) = -2\exp(-2h_{(K,F)}(u))\, \nabla_u h_{(K,F)}(u).$$

A simple calculation that uses conservation of energy verifies the result. (ii) This follows from the fact that 0 is the minimum value of $\tilde{h}_{(K,F)}$ and $\mathcal{F}[K]$ is the corresponding level set.

□

Based on the above proposition, the points in $\mathcal{F}[K]$ will be called ***collision equilibria*** of the reduced system (3.2) and its regularisation (3.11). It is important to remember that these are always stable. Other equilibrium points of these systems will be called ***non-collision equilibria***.

Remark 3.15. To finish this section, we note that one may also define a regularisation of the unreduced system (2.2) by considering the Hamiltonian system on (M,Ω) with ***regularised Hamiltonian*** $\tilde{H}: M \to \mathbb{R}$ defined by

$$\tilde{H}(v) := \exp(-2H(v)) = \prod_{i<j} |v_i - v_j|,$$

with $v = (v_1, \ldots, v_N) \in M$ (recall that Ω is defined by (2.1)). This leads to the regularised equations of motion on M

$$\dot{v}_j = -v_j \times \nabla_{v_j} \tilde{H}(v) = 2 \sum_{i=1 (i \neq j)}^{N} v_i \times v_j, \qquad j = 1, \ldots, N.$$

Since the regularised Hamiltonian $\tilde{H}$ is also $\hat{G}$-invariant, a version of Theorem 3.5 about the (discrete) reduction of the above system to the regularised reduced system (3.11) holds, and such result is valid also at the collision configurations (compare with Remark 3.7).

3.4 Qualitative properties of (K,F)-symmetric solutions

We are now ready to state several facts about the qualitative properties of the reduced system (3.2).

Proposition 3.16. *The following statements hold about the dynamics of the reduced system (3.2).*

(i) The non-collision equilibrium points are in one-to-one correspondence with the critical points of $h_{(K,F)}: S^2 \setminus \mathcal{F}[K] \to \mathbb{R}$. Moreover, local maxima and minima are (Lyapunov) stable equilibrium points surrounded by a 1-parameter family of periodic orbits that may be parametrised by their energy, and saddle points are unstable equilibrium points.

(ii) All regular level sets of the reduced Hamiltonian $h_{(K,F)}$ are periodic orbits.

(iii) There exists a 1-parameter family of periodic orbits, parametrised by their energy, around each collision equilibrium point $u_0 \in \mathcal{F}[K]$. The energy of these periodic orbits approaches ∞ and the period approaches 0 as the orbits approach u_0.

The above proposition exhausts the possibilities of motion of the reduced system except for the possible existence of heteroclinic/homoclinic solutions emanating from the unstable non-collision equilibrium points.

Proof. (i) Since the reduced Hamiltonian $h_{(K,F)}$ is a first integral, the result is standard for Hamiltonian systems on a symplectic manifold of dimension 2.

(ii) The regularised reduced system (3.11) is an integrable, 1-degree of freedom, Hamiltonian system on the compact symplectic manifold S^2. By the Arnold-Liouville Theorem, all regular level sets of the regularised reduced Hamiltonian $\tilde{h}_{(K,F)}$ are periodic orbits. However, it is a simple exercise to show that the regular level sets of $\tilde{h}_{(K,F)}$ are in one-to-one correspondence with the regular level sets of $h_{(K,F)}$.

(iii) This follows from Proposition 3.14 and the fact that $h_{(K,F)}(u_k) \to \infty$ for any sequence $\{u_k\}_{k\in\mathbb{N}}$ of points in $S^2 \setminus \mathcal{F}[K]$ that approaches $\mathcal{F}[K]$ as as $k \to \infty$. □

Proposition 3.16 may be combined with Theorem 3.5 to prove the existence of several periodic solutions of the system (2.2) describing the full dynamics of the N-vortex problem on the sphere. The following corollary gives two particular instances. The first of these will be used in the sections ahead to prove the existence of nonlinear oscillations in the vicinity of the platonic solid equilibrium configurations.

Corollary 3.17. *Let K_o and F_o be orderings of K and F.*

(i) If $u_0 \in S^2 \setminus \mathcal{F}[K]$ is a local maximum or minimum of the reduced Hamiltonian $h_{(K,F)}$ given by (3.3)*, then $v_0 = \rho_{(K_o,F_o)}(u_0)$ is an equilibrium of* (2.2)*, and there exists a 1-parameter family of periodic solutions $v_h(t)$ of* (2.2)*, emanating from v_0, and parametrised by their energy h. Moreover, these solutions are of the form $v_h(t) = \rho_{(K_o,F_o)}(u_h(t))$, where $u_h(t)$ is the family of periodic solutions of the reduced system* (3.2) *emanating from u_0 described in item (i) of Proposition 3.16.*

(ii) If $u_0 \in \mathcal{F}[K]$, then $v_0 = \rho_{(K_o,F_o)}(u_0)$ is a collision configuration (described in detail in Proposition 3.12) and there exists a 1-parameter family of periodic solutions of (2.2)*, which may be parametrised by their energy h, which approaches v_0 as $h \to \infty$, and whose period tends to zero in this limit. These solutions have the form $v_h(t) = \rho_{(K_o,F_o)}(u_h(t))$, where $u_h(t)$ is the 1-parameter family of periodic solutions of* (3.2) *described in item (iii) of Proposition 3.16.*

4 $\mathbb{D}_n$-symmetric solutions of $N = 2n$ vortices (with no fixed vortices)

We consider the dihedral subgroup $K = \mathbb{D}_n < \mathrm{SO}(3)$, $n \geq 2$, generated by the matrices

$$A = \begin{pmatrix} \cos\zeta & -\sin\zeta & 0 \\ \sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.1}$$

where here, and throughout, we denote $\zeta := 2\pi/n$. The set $\mathcal{F}[\mathbb{D}_n]$ is given by

$$\mathcal{F}[\mathbb{D}_n] = \left\{ \left(\cos\left((j-1)\frac{\zeta}{2} \right), \sin\left((j-1)\frac{\zeta}{2} \right), 0 \right) \,:\, j = 1, \ldots, 2n \right\} \cup \{(0,0,\pm 1)\}. \tag{4.2}$$

4.1 Classification of $\mathbb{D}_n$-symmetric equilibrium configurations of $N = 2n$ vortices

We consider $K = \mathbb{D}_n$ and $F = \varnothing$ so $N = 2n$ and analyse the reduced system (3.2) in detail. We start by noting that, in view of item (iii) of Theorem 3.5 and Table (3.5), the system is $\mathbb{O}$-equivariant if $n = 2$ and $\mathbb{D}_{2n}$-equivariant for $n \geq 3$. The following theorem gives the full classification of the collision and non-collision equilibria of the reduced system (3.2) and describes their stability. It also indicates the correspondence of these equilibria with the equilibrium configurations of the equations of motion (2.2).

In the statement of the theorem, and for the rest of the paper, T_k and U_k respectively denote the Chebyshev polynomials of the first and second kind of degree k. To simplify the presentation, the proof is postponed to Section 4.3 that is devoted to it.

Theorem 4.1. *Let $K = \mathbb{D}_n$, $F = \varnothing$, $n \geq 2$ and $N = 2n$. The classification and stability of the equilibrium points of the reduced system (3.2) is as follows.*

(i) The only non-collision equilibria of (3.2) are:

*(a) The **anti-prism equilibrium configurations** at the $4n$ points given by:*

$$A_j^{\pm} := \left(\sqrt{1-z_a^2}\cos\left((2j-1)\zeta/4\right), \sqrt{1-z_a^2}\sin\left((2j-1)\zeta/4\right), \pm z_a\right), \qquad j = 1, \dots, 2n,$$

where $z_a = z_a(n) \in (0,1)$ is uniquely determined by $z_a^2 = 1 - 1/\lambda_a^2$ where $\lambda_a = \lambda_a(n)$ is the unique root greater than 1 of the polynomial

$$\mathcal{P}_a(\lambda) := (3n-1)T_{2n}(\lambda) - nU_{2n}(\lambda) + 2n - 1.$$

These are stable *equilibria of (3.2) which correspond to equilibrium configurations of (2.2) where the $N = 2n$ vortices occupy the vertices of the S^2-inscribed n-gon anti-prism of height $2z_a$ (see Fig.4.1a).*

*(b) The **prism equilibrium configurations** at the $4n$ points given by:*

$$P_j^{\pm} := \left(\sqrt{1-z_p^2}\cos\left((j-1)\zeta/2\right), \sqrt{1-z_p^2}\sin\left((j-1)\zeta/2\right), \pm z_p\right), \qquad j = 1, \dots, 2n,$$

where $z_p = z_p(n) \in (0,1)$ is uniquely determined by $z_p^2 = 1 - 1/\lambda_p^2$ where $\lambda_p = \lambda_p(n)$ is the unique root greater than 1 of the polynomial

$$\mathcal{P}_p(\lambda) := (3n-1)T_{2n}(\lambda) - nU_{2n}(\lambda) - 2n + 1.$$

These are unstable *equilibria (saddle points) of (3.2) which correspond to equilibrium configurations of (2.2) where the $N = 2n$ vortices occupy the vertices of the S^2-inscribed n-gon prism of height $2z_p$ (see Fig.4.1b).*

*(c) The **polygon equilibrium configurations** at the $2n$ points given by:*

$$Q_j := \left(\cos\left((2j-1)\zeta/4\right), \sin\left((2j-1)\zeta/4\right), 0\right), \qquad j = 1, \dots, 2n.$$

These are unstable *equilibria (saddle points) of (3.2) which correspond to equilibrium configurations of (2.2) where the $N = 2n$ vortices occupy the vertices of a regular $2n$-gon at the equator (see Fig.4.1c).*

(ii) The only collision equilibria of (the regularisation of) (3.2) are:

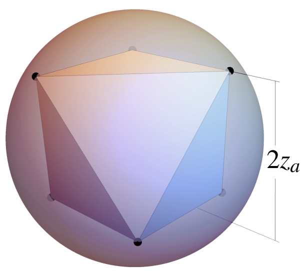
(a) The ***polar collisions*** *at the north and south poles* $(0,0,\pm 1)$*. These correspond to collision configurations of (2.2) having two simultaneous n-tuple collisions at antipodal points (see Fig.5.1d).*

(b) The ***polygonal collisions*** *at the* $2n$ *points given by:*

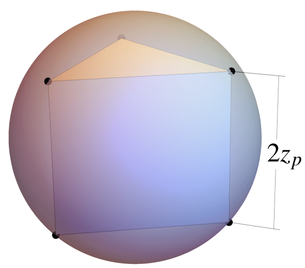
$$C_j := (\cos((j-1)\zeta/2), \sin((j-1)\zeta/2), 0), \qquad j = 1, \ldots, 2n.$$

These correspond to collision configurations of (2.2) having n simultaneous binary collisions at a regular n-gon at the equator (see Fig.5.1e).

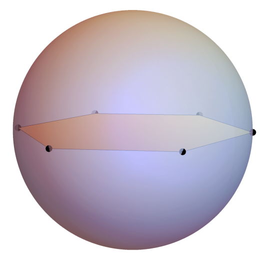
All collision configurations are stable equilibria of (the regularisation of) (3.2).



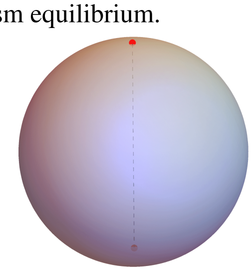
(a) Anti-prism equilibrium.



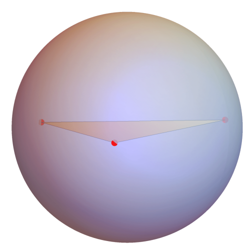
(b) Prism equilibrium.



(c) Polygon equilibrium.



(d) Polar collision (n-tuple collision at antipodal points).



(e) Polygonal collision (binary collisions at vertices of a regular n-gon).

Figure 4.1: Non-collision and collision equilibrium configurations described in Theorem 4.1 for $n = 3$ and $N = 2n = 6$.

In Tables (4.3) and (4.4) below we give explicit expressions for the polynomials $\mathcal{P}_a(\lambda)$, $\mathcal{P}_p(\lambda)$ and the numbers λ_a, z_a, λ_p and z_p, appearing in the statement of the theorem for $n = 2, \ldots, 5$.

n	$\mathcal{P}_a(\lambda)$	λ_a	z_a
2	$8\lambda^4 - 16\lambda^2 + 6$	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{3}}$
3	$64\lambda^6 - 144\lambda^4 + 72\lambda^2$	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{3}}$
4	$384\lambda^8 - 1024\lambda^6 + 800\lambda^4 - 192\lambda^2 + 14$	$\frac{1}{2}\sqrt{\frac{1}{3}\left(10+\sqrt{58}\right)}$	$\sqrt{\frac{1}{7}\left(2\sqrt{58}-13\right)}$
5	$2048\lambda^{10} - 6400\lambda^8 + 6720\lambda^6 - 2800\lambda^4 + 400\lambda^2$	$\frac{1}{4}\sqrt{15+\sqrt{65}}$	$\sqrt{\frac{1}{10}\left(\sqrt{65}-5\right)}$

(4.3)

n	$\mathcal{P}_p(\lambda)$	λ_p	z_p
2	$8\lambda^4 - 16\lambda^2$	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$
3	$64\lambda^6 - 144\lambda^4 + 72\lambda^2 - 10$	$\frac{\sqrt{4+\sqrt{6}}}{2}$	$\sqrt{\frac{1}{5}\left(2\sqrt{6}-3\right)}$
4	$384\lambda^8 - 1024\lambda^6 + 800\lambda^4 - 192\lambda^2$	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{3}}$
5	$2048\lambda^{10} - 6400\lambda^8 + 6720\lambda^6 - 2800\lambda^4 + 400\lambda^2 - 18$	$\approx 1.20467\ldots$	$\approx 0.557613\ldots$

(4.4)

4.2 Dynamics of $\mathbb{D}_n$-symmetric configurations of $N = 2n$ vortices

Combining Theorem 4.1 with Corollary 3.17 we may establish the existence of three families of periodic orbits of the equations of motion (2.2) for N even, $N \geq 4$.

Corollary 4.2. *Let $n \geq 2$ and $N = 2n$.*

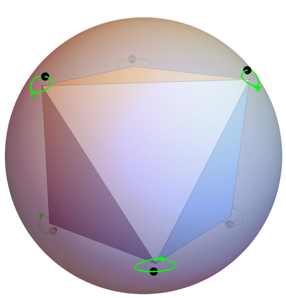
(i) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) emanating from the anti-prism equilibrium configurations described in Theorem 4.1. Along these solutions, each vortex travels around a small closed loop around a vertex of the n-gon anti-prism of height $2z_a(n)$ (see Fig. 4.2a).

(ii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the polar collision described in Theorem 4.1. Along these solutions, n vortices travel along a closed loop around the north pole and the remaining n vortices travel along a closed loop around the south pole in the opposite direction (see Fig. 4.2b).

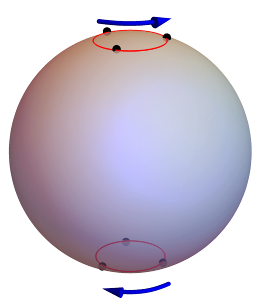
(iii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the polygonal collisions described in Theorem 4.1. Along these solutions, there is a pair of vortices that travels along a small closed loop around each of the vertices of the regular n-gon at the equator (see Fig. 4.2c).

Each of these families may be parametrised by the energy h. In cases (ii) and (iii) we have $h \to \infty$ as the solutions approach collision, and the period approaches zero in this limit.

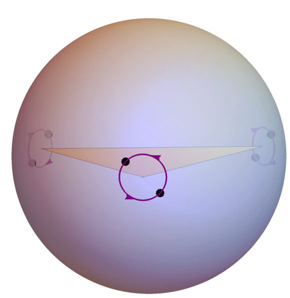
For each solution described above, the distinct closed loops traversed by the vortices, and the position the vortices within the loop at each instant, may be obtained from a single one by the action of $\mathbb{D}_n$.



(a) Periodic solution near the anti-prism equilibrium.



(b) Periodic solution near the polar-collision.



(c) Periodic solution near the polygonal-collision.

Figure 4.2: Periodic solutions described in Corollary 4.2 for $n = 3$, $N = 6$.

We now specialise our discussion to the cases $n = 2,3,4$ which lead to appearance of platonic solids as either prism or anti-prism equilibria.

Case $n = 2$**,** $N = 4$**. Nonlinear small oscillations around the tetrahedron.**

As we may read from Table (4.3), the height of the anti-prism for $n = 2$ is $2/\sqrt{3}$ and it is elementary to verify that the anti-prism is in fact a tetrahedron whose edges have length $2\sqrt{2/3}$. These configurations are known [23] to be stable equilibria of the unreduced dynamics (2.2) and in fact global minimisers of the Hamiltonian H. Item (i) of Corollary 4.2 shows the existence of small nonlinear oscillations of (2.2) around these equilibria.

On the other hand, Table (4.4) indicates that the prism configurations have height $\sqrt{2}$. These (degenerate) prisms are in fact squares of length $\sqrt{2}$. So, for $n = 2$, the distinction between the prism and the polygonal equilibria is artificial. Similarly, since a 2-gon on the equator degenerates to a diameter of the sphere, the distinction between the polar and the polygonal collisions is artificial.

The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 4.3a below. The anti-prism equilibrium points $A_j^{\pm}$ are indicated in green, the prism and polygonal equilibrium points, $P_j^{\pm}$ and Q_j, in black, and the polar and collision configurations C_j in red. We note that the different families of periodic orbits are separated by heteroclinic orbits connecting the unstable equilibria. Also, as predicted by item (i) of Theorem 4.1, we observe octahedral symmetry in the reduced dynamics.

Case $n = 3$**,** $N = 6$**. Nonlinear small oscillations around the octahedron.**

For $n = 3$, Table (4.3) indicates that the height of the anti-prism is again $2/\sqrt{3}$ and it is easy to verify that the anti-prism is in fact an octahedron whose edges have length $\sqrt{2}$ (see Fig.4.1a). Again, these configurations are known [23] to be stable equilibria of the unreduced dynamics (2.2) and global minimisers of the Hamiltonian H. Item (i) of Corollary 4.2 shows the the existence of small nonlinear oscillations of (2.2) around these equilibria. Also, as predicted by item (i) of Theorem 4.1, we observe $\mathbb{D}_6$ symmetry in the reduced dynamics.

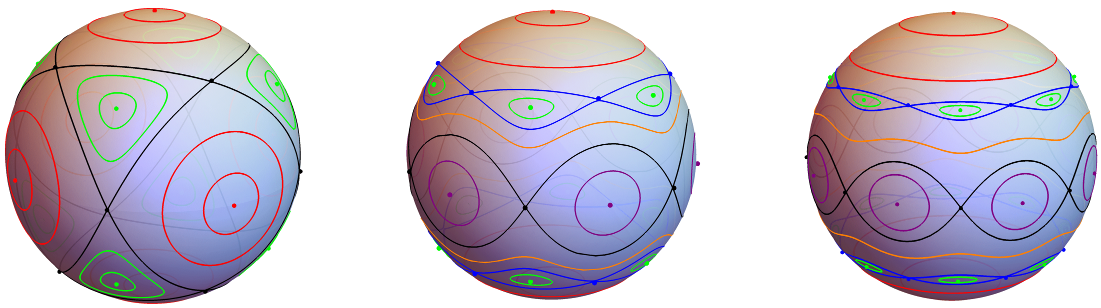
The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 4.3b below. The anti-prism equilibrium points $A_j^{\pm}$ are indicated in green, the prism equilibrium points $P_j^{\pm}$ in blue, polygonal equilibrium points Q_j in black, polar collisions in red and polygonal collisions C_j in purple. We have used the same colour code to indicate either periodic orbits near the stable equilibria or heteroclinic orbits emanating from the unstable equilibria. There is also a family of periodic orbits that do not approach an equilibria or a collision that we have indicated in orange.

Case $n = 4$**,** $N = 8$**. Instability of the cube.**

For $n = 4$, we read from Table (4.4) that the height of the prism configuration is $2/\sqrt{3}$ which corresponds to an inscribed cube whose edges have this length. In contrast with the cases $n = 2, 3$, treated above, our analysis does not lead to the existence of oscillations around a platonic solid, but rather to the conclusion that the cube is an unstable configuration of (2.2). The instability of the cube had been reported before [23].

On the other hand, we conclude from item (i) of Corollary 4.2 that there exists small nonlinear oscillations around the square anti-prism configuration of height $\sqrt{(8\sqrt{58} - 52)/7}$. These configurations are known [23] to be stable equilibria of the unreduced dynamics (2.2) and in fact global minimisers of the Hamiltonian H.

The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 4.3c below. The colour code is identical to the one followed in the case $n = 3$. This time we observe $\mathbb{D}_8$ symmetry in the reduced dynamics.



(a) $n=2$, $N=4$. (b) $n=3$, $N=6$. (c) $n=4$, $N=8$.

Figure 4.3: Phase space of the (regularised) reduced system (3.2) for $K=\mathbb{D}_n$, $F=\varnothing$ and $N=2n$ for $n=2,3,4$. See text for explanations and description of the colour code.

4.3 Proof of Theorem 4.1

Our proof of Theorem 4.1 relies on the following two lemmas that we state and prove first. The first lemma gives us a working expression of the reduced Hamiltonian $h_{(\mathbb{D}_n,\varnothing)}:S^2\to\mathbb{R}$ defined by (3.3), and the second one is a useful trigonometric identity. To simplify notation, for the rest of this section we denote $h_{(\mathbb{D}_n,\varnothing)}$ simply by h_n.

Lemma 4.3. *In cylindrical coordinates (z,θ) for S^2 defined by*

$$x=\sqrt{1-z^2}\cos\theta,\quad y=\sqrt{1-z^2}\sin\theta,\quad z=z, \tag{4.5}$$

we have, modulo the addition of constants:

$$h_n(z,\theta)=-\frac{n(n-1)}{2}\ln(1-z^2)-\frac{n}{2}\sum_{j=1}^{2n}\ln\left(1-\sqrt{1-z^2}\cos\left(\theta+\frac{j\zeta}{2}\right)\right), \tag{4.6}$$

and

$$h_n(z,\theta)=-\frac{n(n-1)}{2}\ln(1-z^2)-\frac{n}{2}\ln\Big(q_{2n}(\sqrt{1-z^2})-(1-z^2)^n\cos(2n\theta)\Big), \tag{4.7}$$

where $q_{2n}(\cdot)$ is the degree $2n$ polynomial defined by $q_{2n}(r)=r^{2n}T_{2n}(1/r)$. In particular we have

$$q_{2n}(\sqrt{1-z^2})-(1-z^2)^n\cos(2n\theta)>0 \tag{4.8}$$

for all (z,θ) corresponding to points on $S^2\setminus\mathcal{F}[\mathbb{D}_n]$.

Proof. We start by noticing that (3.4) yields

$$h_{(K,\varnothing)}(u)=-\frac{m}{4}\sum_{j=2}^{m}\ln|u-g_ju|^2. \tag{4.9}$$

Now we set $K=\mathbb{D}_n$ and work with the following ordering of $\mathbb{D}_n$

$$g_j=A^{j-1},\quad j=1,\dots,n,\qquad g_j=BA^{j-n-1},\quad j=n+1,\dots,2n,$$

where the matrices A and B are defined by (4.1). In view of (4.9) we have

$$h_n(u)=h_{(\mathbb{D}_n,\varnothing)}(u)=-\frac{n}{2}\left(\sum_{j=1}^{n-1}\ln\left|u-A^ju\right|^2+\sum_{j=1}^{n}\ln\left|u-BA^ju\right|^2\right).$$

Writing u in the (z,θ)-coordinates (4.5) we compute

$$\left|u - A^j u\right|^2 = \left(1 - z^2\right)\left|1 - e^{ij\zeta}\right|^2 = 4\left(1 - z^2\right)\sin^2\frac{j\zeta}{2},$$

and

$$\begin{aligned}\left|u - BA^j u\right|^2 &= \left(1 - z^2\right)\left|1 - e^{i(2\theta + j\zeta)}\right|^2 + 4z^2 \\ &= 4\left(1 - z^2\right)\sin^2(\theta + j\zeta/2) + 4z^2 = 4 - 4\left(1 - z^2\right)\cos^2(\theta + j\zeta/2).\end{aligned}$$

Therefore, modulo the addition of terms that are independent of (z,θ), we have

$$\ln\left|u - A^j u\right|^2 = \ln(1 - z^2), \qquad \ln\left|u - BA^j u\right|^2 = \ln\left(1 - \left(1 - z^2\right)\cos^2(\theta + j\zeta/2)\right),$$

and hence,

$$h_n(z,\theta) = -\frac{n(n-1)}{2}\ln(1 - z^2) - \frac{n}{2}\sum_{j=1}^{n}\ln\left(1 - \left(1 - z^2\right)\cos^2(\theta + j\zeta/2)\right).$$

The proof that (4.6) holds follows by noting that

$$\begin{aligned}&\sum_{j=1}^{n}\ln\left(1 - \left(1 - z^2\right)\cos^2(\theta + j\zeta/2)\right) \\ &\quad = \sum_{j=1}^{n}\ln\left(1 - \left(1 - z^2\right)^{1/2}\cos(\theta + j\zeta/2)\right) + \ln\left(1 + \left(1 - z^2\right)^{1/2}\cos(\theta + j\zeta/2)\right) \\ &\quad = \sum_{j=1}^{n}\ln\left(1 - \left(1 - z^2\right)^{1/2}\cos(\theta + j\zeta/2)\right) + \ln\left(1 - \left(1 - z^2\right)^{1/2}\cos(\theta + (n+j)\zeta/2)\right).\end{aligned}$$

In order to prove (4.7) we begin with the identity

$$\frac{1}{\lambda^k}\left(\cosh(k\mu) - \cos(k\theta)\right) = 2^{k-1}\prod_{j=1}^{k}\left(1 - \frac{1}{\lambda}\cos\left(\theta + \frac{2j\pi}{k}\right)\right),$$

where $k \in \mathbb{N}$ and $\lambda = \cosh\mu \ge 1$. This identity is a simple consequence of [17, Formula 1.395(2)]. Using the definition properties of the Chebyshev polynomials we may write $\cosh(k\mu) = T_k(\lambda)$, so, applying the above identity with $k = 2n$, we obtain

$$\frac{1}{2^{2n-1}}\left(q_{2n}(1/\lambda) - \frac{\cos 2n\theta}{\lambda^{2n}}\right) = \prod_{j=1}^{2n}\left(1 - \frac{1}{\lambda}\cos(\theta + j\zeta/2)\right). \tag{4.10}$$

Setting $\lambda = (1 - z^2)^{-1/2}$ and taking logarithms we obtain, modulo the addition of a constant,

$$\sum_{j=1}^{2n}\ln\left(1 - \left(1 - z^2\right)^{1/2}\cos^2(\theta + j\zeta/2)\right) = \ln\left(q_{2n}(\sqrt{1 - z^2}) - (1 - z^2)^n\cos 2n\theta\right),$$

which, in combination with (4.6), proves that (4.7) indeed holds. Finally, note that, since $\lambda \ge 1$, the right hand side of (4.10) is non-negative and can only vanish if $\lambda = 1$ and $\theta = 2\pi - j\zeta/2$, $j = 1,\ldots,2n$. This observation shows that inequality (4.8) holds away from the points $C_j \in \mathcal{F}[\mathbb{D}_n]$.

□

Lemma 4.4. *The following trigonometric identity holds*

$$\sum_{j=1}^{2n} \frac{\cos(\zeta/4 + j\zeta/2)}{1-\cos(\zeta/4 + j\zeta/2)} = 2n(n-1).$$

Proof. We begin by recalling the following identity from [16, Proposition 26]

$$\frac{1}{2}\sum_{j=1}^{l-1} \frac{\sin^2(kj\pi/l)}{\sin^2(j\pi/l)} = \frac{1}{2}k(l-k),$$

that holds for $l \in \mathbb{N}$ and $0 \le k \le l$. In particular, for l even and $k = l/2$, we obtain

$$l^2/8 = \frac{1}{2}\sum_{j=1}^{l-1} \frac{\sin^2(j\pi/2)}{\sin^2(j\pi/l)} = \frac{1}{2} \sum_{\substack{j=1 \\ (j \text{ odd})}}^{l-1} \frac{1}{\sin^2(j\pi/l)}. \tag{4.11}$$

On the other hand, we have

$$\sum_{j=1}^{2n} \frac{1}{1-\cos(j\pi/n + \pi/2n)} = \sum_{j=1}^{2n} \frac{1}{2\sin^2((2j+1)\pi/4n)} = \frac{1}{2} \sum_{\substack{j=1 \\ (j \text{ odd})}}^{4n-1} \frac{1}{\sin^2(j\pi/4n)} = 2n^2,$$

where we have used (4.11) in the last identity with $l = 4n$. The desired result is an immediate consequence of the above identity since we may write

$$\sum_{j=1}^{2n} \frac{\cos(\zeta/4 + j\zeta/2)}{1-\cos(\zeta/4 + j\zeta/2)} = \sum_{j=1}^{2n} \left(\frac{1}{1-\cos(j\pi/n + \pi/2n)} - 1 \right) = 2n^2 - 2n.$$

□

We are now ready to present:

Proof of Theorem 4.1. For item (ii), recall that the collision equilibrium configurations occur at the points in $\mathcal{F}[\mathbb{D}_n]$ and are always stable. The set $\mathcal{F}[\mathbb{D}_n]$ is described by (4.2) and consists of the north and south poles, and the points C_j. Moreover, one can verify that the isotropy group of each of the the poles has order n, and the isotropy group of C_j has order 2. Moreover, $\mathcal{F}[\mathbb{D}_n]$ contains three different $\mathbb{D}_n$-orbits which are $\{(0,0,\pm 1)\}$, $\{C_j,\ j \text{ odd}\}$ and $\{C_j,\ j \text{ even}\}$, and the latter ones determine a regular n-gon at the equator. These observations, together with item (ii) of Proposition 3.12, show that the collision equilibria described above indeed correspond to the collision configurations of (2.2) described in the statement of the theorem.

In order to prove item (i) about the non-collision equilibria, we rely on item (i) of Proposition 3.16, and determine the critical points of h_n. We will prove that these critical points are $A_j^{\pm}$, $P_j^{\pm}$ and Q_j, and that $A_j^{\pm}$ are local minima while $P_j^{\pm}$ and Q_j are saddle points. We will work with the coordinates (z,θ) defined by (4.5). These coordinates cover the whole sphere except for the north and south poles which are collision equilibria by item (ii)(a).

In view of item (iii) of Theorem 3.5 and Table (3.5), we know that h_n is $\mathbb{D}_{2n}$-invariant (for $n = 2$ the group $\mathbb{D}_4$ is a subgroup of the full symmetry group $\mathbb{O}$). This symmetry implies that h_n is $\zeta/2$-periodic in θ, i.e. $h_n(z,\theta) = h_n(z,\theta+\zeta/2)$, and also that $h_n(z,\theta) = h_n(-z,-\theta)$. Therefore, in our analysis of the critical points of h_n, we may restrict our attention to $(z,\theta) \in [0,1) \times [0,\zeta/2)$. Note that, out of the points $A_j^{\pm}$, $P_j^{\pm}$ and Q_j in the statement of the theorem, only A_1^{+}, P_1^{+} and Q_1 lie on this region, and the remaining ones may be obtained as $\mathbb{D}_{2n}$-orbits of A_1^+, P_1^+ and Q_1 respectively. Thus, we only need to prove that A_1^+, P_1^+ and Q_1 have the aforementioned properties and that h_n has no other (regular) critical points on $(z,\theta) \in [0,1) \times [0,\zeta/2)$. For the rest of the proof we write these latter points in terms of their (z,θ) coordinates, namely

$$A_1^+ = (z_a, \zeta/4), \qquad P_1^+ = (z_p, 0), \qquad Q_1 = (0, \zeta/4).$$

Using Eq. (4.7) from Lemma 4.3 we have $\partial_\theta h_n(z,\theta) = -G(z,\theta)\sin(2n\theta)$ where

$$G(\theta,z) := \frac{n^2(1-z^2)^n}{q_{2n}(\sqrt{1-z^2}) - (1-z^2)^n \cos(2n\theta)}.$$

The inequality (4.8) shows that G is a positive function away from the collision-equilibria. In particular, we conclude that $\partial_\theta h_n(z,\theta) = 0$ if $\theta = 0$ or $\theta = \zeta/4$ and that $\partial_\theta h_n(z,\theta) \neq 0$ for other values of $\theta \in [0,\zeta/4)$. Hence, equilibria of h_n in the region of interest can only occur if $\theta = 0$ or $\theta = \zeta/4$. Next we note from Lemma 4.3 that $h_n(z,\theta)$ is an even function of z and thus $\partial_z h_n(0,\theta) = 0$. Therefore, we have $\partial_z h_n(0,\zeta/4) = \partial_\theta h_n(0,\zeta/4) = 0$ which shows that Q_1 is indeed a critical point of h_n (the other critical point $(0,0)$ corresponds to the collision equilibrium C_1 at which h_n is undefined).

Now we prove that there is exactly one zero $z_a, z_p \in (0,1)$ of $\partial_z h_n(z,\zeta/4) = 0$ and $\partial_z h_n(z,0) = 0$, respectively. In order to simplify the proof we make the change of variables $r(z) = \sqrt{1-z^2} : (0,1) \to (0,1)$. Since $r'(z) \neq 0$, the existence of a unique critical point of $h_n(z,\theta)$ for $\theta = 0, \zeta/4$ is equivalent to the existence of a unique critical point of $h_n(r,\theta)$ for $\theta = 0, \zeta/4$. Using Eq. (4.6) from Lemma 4.3 we have

$$h_n(r,\theta) = -n(n-1)\ln(r) - \frac{n}{2}\sum_{j=1}^{2n} \ln\left(1 - r\cos\left(\theta + j\zeta/2\right)\right).$$

Since $\lim_{r\to 0} h_n(r,\theta) = \lim_{r\to 1} h_n(r,0) = +\infty$, there exists a minimum $r_p \in (0,1)$ of the function $r \mapsto h_n(r,0)$. On the other hand, differentiating the above expression and using Lemma 4.4 we find that for $\theta = \zeta/4$, we have

$$\partial_r h_n(1,\zeta/4) = -n(n-1) + \frac{n}{2}\sum_{j=1}^{2n} \frac{\cos\left(\zeta/4 + j\zeta/2\right)}{1 - \cos\left(\zeta/4 + j\zeta/2\right)} = n(n-1)^2 > 0. \tag{4.12}$$

Therefore, using again that $\lim_{r\to 0} h_n(r,\theta) = +\infty$, we conclude that there exists a minimum $r_a \in (0,1)$ of the function $r \mapsto h(r,\zeta/4)$. However, since

$$\partial_r^2 h_n(r,\theta) = n(n-1)\frac{1}{r^2} + \frac{n}{2}\sum_{j=1}^{2n} \frac{\cos^2\left(\theta + j\zeta/2\right)}{\left(1 + r\cos\left(\theta + j\zeta/2\right)\right)^2} > 0, \tag{4.13}$$

then $h_n(r,\theta)$ has at most one critical point for $r \in (0,1)$. We conclude that $z_p = \sqrt{1-r_p^2}$ and $z_a = \sqrt{1-r_a^2}$, are, respectively, the unique critical points of $h_n(z,0)$ and $h_n(z,\zeta/4)$ on the interval $z \in (0,1)$.

It remains to prove that z_a and z_p may indeed be determined in terms of the zeros of the polynomials $\mathcal{P}_a$ and $\mathcal{P}_p$ given in the statement of the theorem. For this purpose note that Eq. (4.6) and the condition $\partial_r h_n(r_a,\zeta/4) = 0$ yield

$$(2n-1)r_a^{2n} + (n-1)q_{2n}(r_a) + \frac{r_a}{2}q_{2n}'(r_a) = 0.$$

Using the definition of q_{2n}, and since $r_a > 0$, this is equivalent to

$$2n - 1 + (2n-1)T_{2n}(1/r_a) - \frac{1}{2r_a}T_{2n}'(1/r_a) = 0.$$

Therefore, $\lambda_a := 1/r_a$ satisfies

$$2n-1+(3n-1)T_{2n}(\lambda_a)-nU_{2n}(\lambda_a)=0,$$

where we have made use of the Chebyshev polynomial identities:

$$T'_{2n}(s)=2nU_{2n-1}(s), \qquad sU_{2n-1}(s)=U_{2n}(s)-T_{2n}(s).$$

This shows that z_a is indeed determined by a root $\lambda_a>1$ of $\mathcal{P}_a$ as explained in the theorem. The unicity of z_a as a critical point of $z\mapsto h_n(z,\zeta/4)$ shown above proves that such root of $\mathcal{P}_a$ is necessarily unique. The analogous conclusion for z_p is obtained *mutatis mutandis* starting from the condition $\partial_r h_n(r_p,0)=0$.

Thus, we have shown that indeed A_1^+, P_1^+ and Q_1 are the unique (non-collision) critical points of h_n on the region $(z,\theta)\in[0,1)\times[0,\zeta/2)$. We will now prove that A_1^+ is a local minimum, whereas P_1^+ and Q_1 are saddle points of h_n. Starting from the condition $\partial_\theta h_n(z,\theta)=-\sin(2n\theta)G(z,\theta)$ with G positive we have

$$\partial_\theta^2 h_n(z,\theta)=-2n\cos(2n\theta)G(z,\theta)-\sin(2n\theta)\partial_\theta G(z,\theta), \quad \partial_z\partial_\theta h_n(z,\theta)=-\sin(2n\theta)\partial_z G(z,\theta),$$

and therefore

$$\partial_\theta^2 h_n(z,0)<0, \quad \partial_\theta^2 h_n(z,\zeta/4)>0, \qquad \partial_z\partial_\theta h_n(z,0)=\partial_z\partial_\theta h_n(z,\zeta/4)=0. \tag{4.14}$$

On the other hand, we show below that

$$\partial_z^2 h_n(z_p,0)>0, \qquad \partial_z^2 h_n(z_a,\zeta/4)>0, \qquad \partial_z^2 h_n(0,\zeta/4)<0. \tag{4.15}$$

The relations in (4.14) and (4.15) prove that the Hessian matrix of h_n is positive definite at A_1^+ and indefinite at P_1^+ and Q_1.

To prove that the inequalities in (4.15) indeed hold we note that

$$\partial_z^2 h_n=\partial_z\left((\partial_r h_n)(\partial_z r)\right)=\left(\partial_r^2 h_n\right)(\partial_z r)^2+(\partial_r h_n)\left(\partial_z^2 r\right). \tag{4.16}$$

Evaluating at $(z_p,0)$ and $(z_a,\zeta/4)$ yields

$$\partial_z^2 h_n(z_p,0)=\partial_r^2 h_n(r_p,0)(\partial_z r(z_p))^2>0, \qquad \partial_z^2 h_n(z_a,\zeta/4)=\partial_r^2 h_n(r_a,\zeta/4)(\partial_z r(z_a))^2>0,$$

where we have used (4.13) and $\partial_r h_n(r_p,0)=\partial_r h_n(r_a,\zeta/4)=0$. On the other hand, taking the the limit as $z\to 0$ with $\theta=\zeta/4$ in (4.16), and considering that in this limit $r\to 1$, $\partial_z r\to 0$ and $\partial_z^2 r\to -1$, we obtain

$$\partial_z^2 h_n(0,\zeta/4)=-\partial_r h_n(1,\zeta/4)=-n(n-1)^2,$$

where the last identity follows from (4.12). In particular, this shows that the third inequality of (4.15) also holds.

Finally, we show that $A_j^\pm$, $P_j^\pm$ and Q_j respectively correspond to anti-prism, prism and polygonal equilibrium configurations of (2.2) with the stated properties. We begin by noting that the set $\{A_j^\pm : j=1,\dots,2n\}$ consists of two $\mathbb{D}_n$-orbits given by

$$\{A^+_{j\,\mathrm{odd}},A^-_{j\,\mathrm{even}}\} \quad \text{and} \quad \{A^+_{j\,\mathrm{even}},A^-_{j\,\mathrm{odd}}\}.$$

Each of these orbits has $2n$ points that lie on the vertices of an n-gon anti-prism as described in item (ii)(a) of the theorem. It follows that, for any ordering of $\mathbb{D}_n$, the mapping $\rho_{(\mathbb{D}_n,\varnothing)}$ defined by (3.1) maps each of the points $A_j^{\pm}$ into an anti-prism configuration with the given properties. Item (ii) of Theorem 3.5 implies that these are equilibrium configurations of (2.2). The analogous conclusion about the prism configurations is obtained by the same reasoning but noting this time that the set $\{P_j^{\pm} : j = 1,\ldots,2n\}$ consists of two $\mathbb{D}_n$-orbits given by $\{P^{\pm}_{j\,\mathrm{even}}\}$ and $\{P^{\pm}_{j\,\mathrm{odd}}\}$. The conclusion about Q_j is also analogous but it is reached at once since $\{Q_j\}$ consists of a single $\mathbb{D}_n$-orbit whose points lie on a regular n-gon at the equator.

□

5 $\mathbb{D}_n$-symmetric solutions of $N = 2n+2$ vortices (two antipodal vortices remain fixed)

We continue to consider $K = \mathbb{D}_n$ but now we take $F = \{(0,0,\pm 1)\}$ so $N = 2n+2$. Note that the set F satisfies both requirements in our setup since it is $\mathbb{D}_n$-invariant and is contained in $\mathcal{F}[\mathbb{D}_n]$ (see (4.2)). We analyse the reduced system (3.2) in detail. Since the set F is also $\mathbb{D}_{2n}$-invariant then, in view of item (iii) of Theorem 3.5 and Table (3.5), the system is $\mathbb{D}_{2n}$-equivariant for all $n \geq 2$ (note that, in contrast with the previous section, the set $F = \{(0,0,\pm 1)\}$ is *not* $\mathbb{O}$-invariant so we cannot expect that the reduced system is $\mathbb{O}$-equivariant for $n = 2$).

5.1 Classification of $\mathbb{D}_n$-symmetric equilibrium configurations of $N = 2n+2$ vortices

The analogous version of Theorem 4.1 on the classification and stability of the collision and non-collision equilibria of the reduced system (3.2) in this case is given next.

Theorem 5.1. *Let $K = \mathbb{D}_n$, $F = \{(0,0,\pm 1)\}$, $n \geq 2$ and $N = 2n+2$. The classification and stability of the equilibrium points of the reduced system (3.2) is as follows.*

(i) The only non-collision equilibria of (3.2) are:

(a) For $n \geq 3$, the ***anti-prism with poles equilibrium configurations*** *at the $4n$ points given by:*

$$\hat{A}_j^{\pm} := \left(\sqrt{1-\hat{z}_a^2}\cos\left((2j-1)\zeta/4\right), \sqrt{1-\hat{z}_a^2}\sin\left((2j-1)\zeta/4\right), \pm\hat{z}_a\right), \qquad j = 1,\ldots,2n,$$

where $\hat{z}_a = \hat{z}_a(n) \in (0,1)$ is uniquely determined as $\hat{z}_a^2 = 1 - 1/\hat{\lambda}_a^2$ where $\hat{\lambda}_a = \hat{\lambda}_a(n)$ is the unique root greater than 1 of the polynomial

$$\hat{\mathcal{P}}_a(\lambda) := (3n+1)T_{2n}(\lambda) - nU_{2n}(\lambda) + 2n + 1.$$

These are stable *equilibria of (3.2) which correspond to equilibrium configurations of (2.2) where $2n$ vortices occupy the vertices of the S^2-inscribed n-gon-anti-prism of height $2\hat{z}_a$, and the 2 remaining vortices are antipodal and determine the diameter that is perpendicular to the antiprism (see Fig.5.1a).*

(b) For all $n \geq 2$, the ***prism with poles equilibrium configurations*** *at the $4n$ points given by:*

$$\hat{P}_j^{\pm} := \left(\sqrt{1-\hat{z}_p^2}\cos\left((j-1)\zeta/2\right), \sqrt{1-\hat{z}_p^2}\sin\left((j-1)\zeta/2\right), \pm\hat{z}_p\right), \qquad j = 1,\ldots,2n,$$

where $\hat{z}_p = \hat{z}_p(n) \in (0,1)$ is uniquely determined as $\hat{z}_p^2 = 1 - 1/\hat{\lambda}_p^2$ where $\hat{\lambda}_p = \hat{\lambda}_p(n)$ is the unique root greater than 1 of the polynomial

$$\hat{\mathcal{P}}_p(\lambda) := (3n+1)T_{2n}(\lambda) - nU_{2n}(\lambda) - 2n - 1.$$

These are unstable *equilibria (saddle points) of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where* $2n$ *vortices occupy the vertices of the* S^2*-inscribed n-gon-prism of height* $2\hat{z}_p$*, and the 2 remaining vortices are antipodal and determine the diameter that is perpendicular to the prism (see Fig.5.1b).*

(c) *For all* $n \geq 2$*, the* ***polygon with poles equilibrium configurations*** *at the* $2n$ *points given by:*

$$\hat{Q}_j := (\cos((2j-1)/4), \sin((2j-1)\zeta/4), 0), \qquad j = 1, \dots, 2n.$$

These points are stable equilibria of (3.2) *if* $n = 2$ *and unstable (saddle points) if* $n \geq 3$*. Moreover, they correspond to equilibrium configurations of* (2.2) *where* $2n$ *vortices occupy the vertices of a regular* $2n$*-gon at the equator and the 2 remaining vortices are antipodal and determine the diameter that is perpendicular to the polygon (see Fig.5.1c).*

(ii) *The only collision equilibria of (the regularisation of)* (3.2) *are:*

(a) *The* ***polar collisions*** *at the north and south poles* $(0,0,\pm 1)$*. These correspond to collision configurations of* (2.2) *having two simultaneous* $(n+1)$*-tuple collisions at antipodal points (see Fig.5.1d).*

(b) *The* ***polygonal with poles collisions*** *at the* $2n$ *points given by:*

$$\hat{C}_j := (\cos((j-1)\zeta/2), \sin((j-1)\zeta/2), 0), \qquad j = 1, \dots, 2n.$$

These correspond to collision configurations of (2.2) *having n simultaneous binary collisions at a regular n-gon at the equator and the 2 remaining vortices are antipodal and determine the diameter that is perpendicular to the polygon (see Fig.5.1e).*

All collision configurations are stable equilibria of (the regularisation of) (3.2).

Proof of Theorem 5.1. In broad terms, the proof of the theorem is analogous to that of Theorem 4.1 so we only indicate the key differences. The main one is that the expressions for the reduced Hamiltonian in Lemma 4.3 have to be modified to account for the presence of the vortices at the poles. In view of (3.4), such correction is given by the addition of the term

$$-\frac{m}{2}\sum_{j=m+1}^{N} \ln\left|u - f_j\right|^2 = -n\left(\ln|u-(0,0,1)|^2 + \ln|u-(0,0,-1)|^2\right).$$

Writing u in the cylindrical coordinates (4.5) and performing elementary operations shows that, up to the addition of a constant, the above expression equals $-n\ln(1-z^2)$. Therefore, if we simplify the notation and denote the reduced Hamiltonian $h_{(\mathbb{D}_n,\{(0,0,\pm1)\})} : S^2 \to \mathbb{R}$ simply by $\hat{h}_n$, we conclude that

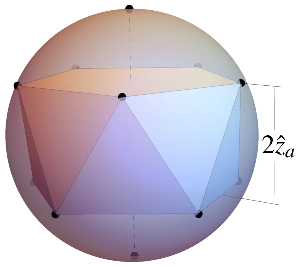
$$\hat{h}_n(z,\theta) = h_n(z,\theta) - n\ln(1-z^2), \tag{5.1}$$

where $h_n(z,\theta)$ is given by (4.6), (4.7).

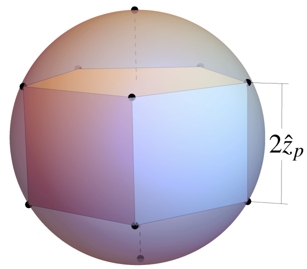
Using the expression (5.1), one may proceed in direct analogy with the proof of Theorem 4.1 to prove the result. One difference that is worth pointing out is the computation of

$$\partial_r \hat{h}_n(1,\zeta/4) = \partial_r h_n(1,\zeta/4) - 2n = n(n-1)^2 - 2n = n\left(n^2 - 2n - 1\right),$$

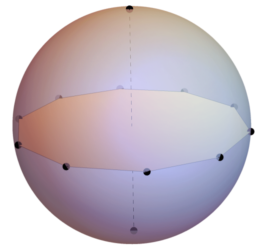
Thus $\partial_r \hat{h}_n(1,\zeta/4) = -2$ for $n = 2$ and $\partial_r \hat{h}_n(1,\zeta/4) > 0$ for $n \geq 3$. This implies that for $n \geq 3$ there is a unique $r_a \in (0,1)$ such that $\partial_r \hat{h}_n(r_a,\zeta/4) = 0$, while for $n = 2$ there are no solutions. Another difference is that the polygonal equilibrium points Q_j are local minima of $\hat{h}_n$ for $n = 2$ and saddle points of $\hat{h}_n$ for $n \geq 3$. We omit this and all other details. □



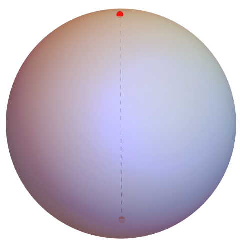
(a) Anti-prism with poles equilibrium.



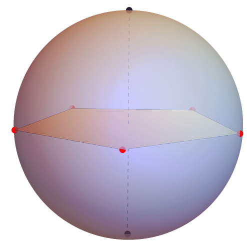
(b) Prism with poles equilibrium.



(c) Polygon with poles equilibrium.



(d) Polar collision (($n+1$)-tuple collision at antipodal points).



(e) Polygonal with poles collision (binary collisions at the vertices of a regular n-gon).

Figure 5.1: Non-collision and collision equilibrium configurations described in Theorem 5.1 for $n = 5$ and $N = 12$.

The Tables below provide explicit expressions for the polynomials $\hat{\mathcal{P}}_a(\lambda)$, $\hat{\mathcal{P}}_p(\lambda)$ and the numbers $\hat{\lambda}_a$, $\hat{z}_a$, $\hat{\lambda}_p$ and $\hat{z}_p$, in the statement of the theorem for $n = 2, \ldots, 5$.

n	$\hat{\mathcal{P}}_a(\lambda)$	$\hat{\lambda}_a$	$\hat{z}_a$
2	$24\lambda^4 - 32\lambda^2 + 10$	$-$	$-$
3	$128\lambda^6 - 240\lambda^4 + 108\lambda^2$	$\frac{3}{2\sqrt{2}}$	$\frac{1}{3}$
4	$640\lambda^8 - 1536\lambda^6 + 1120\lambda^4 - 256\lambda^2 + 18$	$\frac{1}{2}\sqrt{\frac{1}{5}\left(14+\sqrt{106}\right)}$	$\frac{1}{3}\sqrt{2\sqrt{106}-19}$
5	$3072\lambda^{10} - 8960\lambda^8 + 8960\lambda^6 - 3600\lambda^4 + 500\lambda^2$	$\frac{\sqrt{5}}{2}$	$\frac{1}{\sqrt{5}}$

(5.2)

n	$\hat{\mathcal{P}}_p(\lambda)$	$\hat{\lambda}_p$	$\hat{z}_p$
2	$24\lambda^4 - 32\lambda^2$	$\frac{2}{\sqrt{3}}$	$\frac{1}{2}$
3	$128\lambda^6 - 240\lambda^4 + 108\lambda^2 - 14$	$\frac{1}{4}\sqrt{13+\sqrt{57}}$	$\sqrt{\frac{1}{7}\left(\sqrt{57}-6\right)}$
4	$640\lambda^8 - 1536\lambda^6 + 1120\lambda^4 - 256\lambda^2$	$\frac{1}{2}\sqrt{\frac{1}{5}\left(19+\sqrt{41}\right)}$	$\frac{1}{4}\sqrt{\sqrt{41}-3}$
5	$3072\lambda^{10} - 8960\lambda^8 + 8960\lambda^6 - 3600\lambda^4 + 500\lambda^2 - 22$	$\approx 1.12677\ldots$	$\approx 0.460816\ldots$

(5.3)

5.2 Dynamics of $\mathbb{D}_n$-symmetric configurations of $N = 2n$ vortices

In analogy with Corollary 4.2, we may combine Theorem 5.1 with Corollary 3.17 to establish the existence of three families of periodic orbits of the equations of motion (2.2).

Corollary 5.2. *Let $N = 2n+2$.*

(i) For $n \geq 3$, there exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2), emanating from the anti-prism with poles equilibrium configurations described

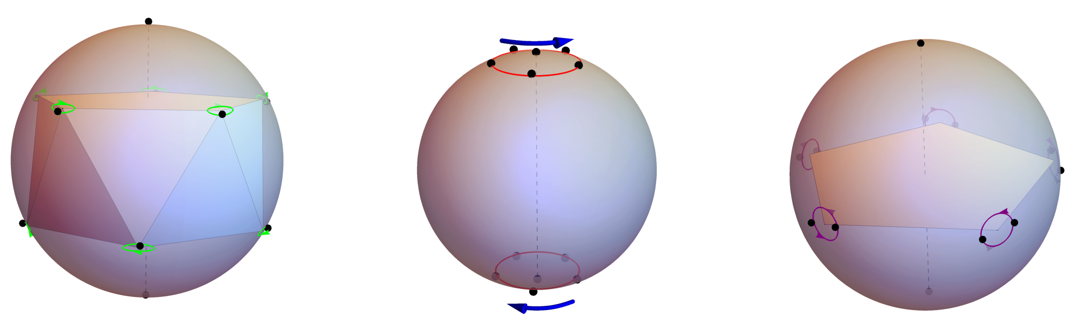
in Theorem 5.1. Along these solutions, two vortices remain fixed at the north and south poles and each remaining vortex travels around a small closed loop around a vertex of the n-gon anti-prism of height $2\hat{z}_a(n)$ *(see Fig. 5.2a).*

(ii) For $n \geq 2$*, there exists a 1-parameter family of periodic solutions* $v_h(t)$ *of the equations of motion* (2.2) *emanating from the polar collision described in Theorem 5.1. Along these solutions, two vortices remain fixed at the north and south poles, n vortices travel along a closed loop around the north pole and the remaining n vortices travel along a closed loop around the south pole in the opposite direction (see Fig. 5.2b).*

(iii) For $n \geq 2$*, there exists a 1-parameter family of periodic solutions* $v_h(t)$ *of the equations of motion* (2.2) *converging to the polygonal collisions with poles described in Theorem 5.1. Along these solutions, two vortices remain fixed at the north and south poles and there is a pair of vortices that travels along a small closed loop around each of the vertices of the regular n-gon at the equator (see Fig. 5.2c).*

Each of these families may be parametrised by the energy h. In cases (ii) and (iii) we have $h \to \infty$ *as the solutions approach collision, and the period approaches zero in this limit.*

For each solution described above, the distinct closed loops traversed by the vortices, and the position the vortices within the loop at each instant, may be obtained from a single one by the action of $\mathbb{D}_n$*.*



(a) Periodic solution near the anti-prism equilibrium with poles.

(b) Periodic solution near the polar-collision with poles.

(c) Periodic solution near the polygonal-collision with poles.

Figure 5.2: Periodic solutions described in Corollary 5.2 for $n = 5$, $N = 12$.

We now specialise our discussion to the cases $n = 2, 3, 5$ which lead to appearance of platonic solids as either polygon with poles equilibria or as anti-prism with poles equilibria.

Case $n = 2$, $N = 6$. Nonlinear small oscillations around the octahedron.

In this case the polygonal with poles equilibrium configurations in Theorem 5.1 are stable and correspond to octahedral equilibrium configurations of (2.2). A 1-parameter family of periodic solutions emanating from this configuration is established from Corollary 3.17. Hence, we have a family of non-linear normal modes of oscillation with $\mathbb{D}_2$-symmetry around the octahedral equilibrium. We emphasise that this family is different from the one determined in the previous section by looking at the octahedron as the anti-prism equilibria with symmetry group $\mathbb{D}_3$ and $F = \varnothing$.

On the other hand, according to Table (5.3) the prism with poles equilibrium configurations have height 1. It is a simple exercise to verify that the prisms degenerate and, together with the poles, form hexagons which are contained on an equatorial plane.

The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 5.3a below. The polygonal with poles equilibrium points, $\hat{Q}_j$, corresponding to the octahedron configuration are indicated in green. The prism with poles equilibrium points, $\hat{P}_j^{\pm}$, corresponding to the hexagon configuration are illustrated in black. Finally, the polar collisions are red while the polygon with poles configurations are purple. We have used the same colour code to indicate either periodic orbits near the stable equilibria or heteroclinic orbits emanating from the unstable equilibria.

Finally, we note that the subset $F = \{(0,0,\pm 1)\}$ is not invariant under the action of $\mathbb{O} = N(\mathbb{D}_2)$. However, F is invariant under $\mathbb{D}_4$ and moreover, $\mathbb{D}_2 < \mathbb{D}_4 < \mathbb{O} = N(\mathbb{D}_2)$. So, as predicted by item (iii) of Theorem 3.5, we observe a $\mathbb{D}_4$ symmetry in the reduced dynamics.

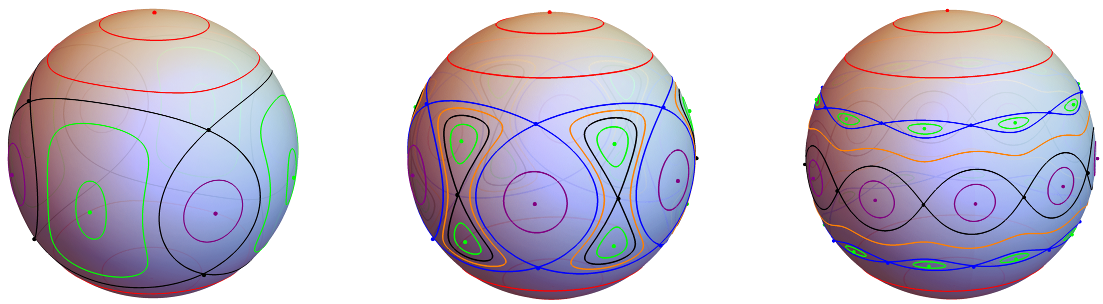
Case $n = 3$, $N = 8$. Nonlinear small oscillations around the cube.

For $n = 3$, Table (5.2) indicates that the height of the anti-prism is $2/3$. One may verify that the resulting anti-prism with poles is in fact a cube whose edges have length $2/\sqrt{3}$. Despite the instability of these configurations as equilibria of (2.2), we conclude from item (i) of Corollary 5.2 that there is a family of nonlinear small oscillations emanating from these configurations.

The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 5.3b below and the colour code is similar to the one used in Figures 4.3b and 4.3c. The anti-prism equilibrium points $\hat{A}_j^{\pm}$ are indicated in green, the prism equilibrium points $\hat{P}_j^{\pm}$ in blue, polygonal equilibrium points $\hat{Q}_j$ in black, polar collisions in red and polygonal collisions $\hat{C}_j$ in purple. The same colour is used to indicate either periodic orbits near the stable equilibria or heteroclinic/homoclinic orbits emanating from the unstable equilibria. There is also a family of periodic orbits that do not approach an equilibria or a collision that we have indicated in orange. Considering that the set $F = \{(0,0,\pm 1)\}$ is invariant under the action of $\mathbb{D}_6$, then, as predicted by item (iii) of Theorem 3.5, we observe a $\mathbb{D}_6$-symmetry in the reduced dynamics.

Case $n = 5$, $N = 12$. Nonlinear small oscillations around the icosahedron.

For $n = 5$, we read from Table (4.4) that the height of the anti-prism with poles configuration is $2/\sqrt{5}$ and one may show that this corresponds to an inscribed icosahedron whose edges have length $\sqrt{2 - \frac{2}{\sqrt{5}}}$. Figure 5.1a illustrates this. By connecting the north and south pole with each one of the vertices on the top and bottom faces of the anti-prism we get an icosahedron. These configurations are known [23] to be stable equilibria of (2.2) and item (i) of Corollary 5.2 shows the existence of a family of small oscillations emanating from them. The phase space of the (regularised) reduced dynamics obtained numerically is illustrated in Figure 5.3c below. The colour code is identical to the one followed in the case $n = 3$ described above. This time we observe $\mathbb{D}_{10}$-symmetry in the reduced dynamics.



(a) $n=2$, $N=6$. (b) $n=3$, $N=8$. (c) $n=5$, $N=12$.

Figure 5.3: Phase space of the (regularised) reduced system (3.2) for $K=\mathbb{D}_n$, $F=\{(0,0,\pm1)\}$ and $N=2n+2$ for $N=2,3,5$. See text for explanations and description of the colour code.

6 $\mathbb{T}$-symmetric solutions for $N=12$ vortices (with no fixed vortices)

We consider the tetrahedral subgroup $\mathbb{T}<\mathrm{SO}(3)$ generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $\mathbb{T}$ has order 12 and is isomorphic to the subgroup A_4 of even permutations of 4 elements. An explicit group isomorphism may be defined in terms of the above generators as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto (1,2,3)(4), \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto (1,2)(3,4),$$

where we have used the standard cyclic notation for permutations. The group $\mathbb{T}$ consists of the orientation preserving symmetries of the tetrahedra $\mathcal{T}_1,\mathcal{T}_2$ with vertices at

$$\begin{aligned} \mathcal{T}_1 &= \{c(1,1,1), c(-1,-1,1), c(-1,1,-1), c(1,-1,-1)\}, \\ \mathcal{T}_2 &= \{c(-1,-1,-1), c(1,1,-1), c(1,-1,1), c(-1,1,1)\}, \end{aligned} \tag{6.1}$$

where $c^{-1}=\sqrt{3}$. One may check that

$$\mathcal{F}[\mathbb{T}] = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}. \tag{6.2}$$

6.1 Classification of $\mathbb{T}$-symmetric equilibrium configurations of $N=12$ vortices

We now analyse the reduced system (3.2) in detail in the case $K=\mathbb{T}$ and $F=\varnothing$ so $N=12$. Item (iii) of Theorem 3.5 and Table (3.5) indicate that such system is $\mathbb{O}$-equivariant. The theorem below gives the full classification of the collision and non-collision equilibria.

Theorem 6.1. *Let $K=\mathbb{T}$, $F=\varnothing$ and $N=12$. The classification and stability of the equilibrium points of the reduced system* (3.2) *is as follows.*

(i) The only non-collision equilibria of (3.2) *are:*

(a) The ***icosahedron equilibrium configurations*** *occurring at all* 24 *points obtained by permuting the entries and considering all sign flips of*

$$\frac{1}{\sqrt{1+\phi^2}}\,(\pm\phi,\pm 1,0)\,,$$

where $\phi=\frac{1+\sqrt{5}}{2}$ *is the golden mean. These are* stable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of an* S^2*-inscribed regular icosahedron (see Fig.6.1a).*

(b) The ***truncated tetrahedron configurations*** *occurring at all* 24 *points obtained by permuting the entries and considering all sign flips of*

$$\left(\pm\alpha,\pm\alpha,\pm\sqrt{1-2\alpha^2}\right),$$

where $0<\alpha\approx 0.269484\dots$ *is characterised by the condition that* α^2 *is the unique zero of the polynomial* $p(\lambda)=1-13\lambda-13\lambda^2+33\lambda^3$ *between* 0 *and* $1/2$*. These are* unstable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of an irregular* S^2*-inscribed truncated tetrahedron (see Fig.6.1b and Remark 6.2).*

(c) The ***cub-octahedron configurations*** *occurring at all* 12 *points obtained by permuting the entries and considering all sign flips of*

$$\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},0\right).$$

These are unstable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of a regular* S^2*-inscribed cuboctahedron (see Fig.6.1c).*

(ii) The only collision equilibria of (the regularisation of) (3.2) *are:*

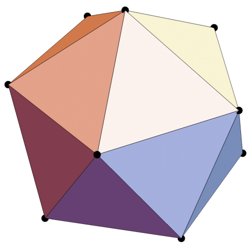
(a) The ***tetrahedral collisions*** *at the 8 points of* $\mathcal{T}_1\cup\mathcal{T}_2$*. These correspond to collision configurations of* (2.2) *having four simultaneous triple collisions at the vertices of a tetrahedron (see Fig.6.1d).*

(b) The ***octahedral collisions*** *at the* 6 *points* $(\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)$*. These correspond to collision configurations of* (2.2) *having* 6 *simultaneous binary collisions at the vertices of an octahedron (see Fig.6.1e).*

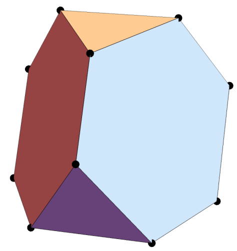
All collision configurations are stable equilibria of (the regularisation of) (3.2).

Remark 6.2. The polyhedron determined by the truncated tetrahedron equilibria consists of 4 irregular hexagonal faces and 4 four equilateral triangular faces. The distance between the vertices forming an edge between adjacent hexagonal faces is $2\sqrt{2}\alpha\approx 0.762215$ while the distance between vertices forming an edge of an equilateral triangular face is $\sqrt{2}(-\alpha+\sqrt{1-2\alpha^2})\approx 0.926377$.

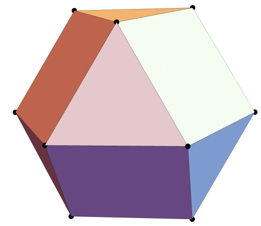
Before giving the proof of the theorem we present:



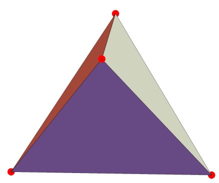
(a) Icosahedron equilibrium. The distance between adjacent vertices is $2/\sqrt{3}$.



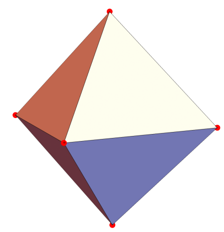
(b) Truncated tetrahedron equilibrium. See Remark 6.2 for details on the dimensions.



(c) Cuboctahedron equilibrium. The distance between adjacent vertices is 1.



(d) Tetrahedron collision (simultaneous triple collision at each vertex).



(e) Octahedron collision (simultaneous binary collision at each vertex).

Figure 6.1: Non-collision and collision equilibrium configurations described in Theorem 6.1.

Lemma 6.3. *Let*

$$\begin{aligned}
p_1^{\pm}(R,\Theta) &= 2+2R^2+R^2\sin 2\Theta \pm 2R(\cos\Theta-\sin\Theta),\\
p_2^{\pm}(R,\Theta) &= 2+2R^2-R^2\sin 2\Theta \pm 2R(\cos\Theta+\sin\Theta),\\
p_3(R,\Theta) &= 5R^8\cos 8\Theta+76\left(R^4+8R^2+8\right)R^4\cos 4\Theta+47R^8-864R^6-4320R^4-6912R^2-3456.
\end{aligned}$$

For $(R,\Theta)\in[0,\sqrt{2}]\times(0,\pi/4)$ *we have* $p_1^{\pm}(R,\Theta)>0$*,* $p_2^{\pm}(R,\Theta)>0$ *and* $p_3(R,\Theta)<0$*.*

Proof. For $\Theta\in(0,\pi/4)$ we have $\sin 2\Theta\in(0,1)$, $\cos\Theta-\sin\Theta\in(0,1)$ and $\cos\Theta+\sin\Theta\in(1,\sqrt{2})$. Therefore,

$$\begin{aligned}
p_1^{\pm}(R,\Theta) &> 2+2R^2-2R(\cos\Theta-\sin\Theta)>2(R^2-R+1)\geq 3/2,\\
p_2^{\pm}(R,\Theta) &> 2+R^2-2R(\cos\Theta+\sin\Theta)>R^2-2\sqrt{2}R+2\geq 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
p_3(R,\Theta) &\leq 5R^8+76\left(R^4+8R^2+8\right)R^4+47R^8-864R^6-4320R^4-6912R^2-3456\\
&= 128R^8-256R^6-3712R^4-6912R^2-3456\\
&\leq 128R^8-3456.
\end{aligned}$$

Therefore, for $R\in[0,\sqrt{2}]$ we may estimate $p_3(R,\Theta)\leq 128(2^4)-3456=-1408$. □

Proof of Theorem 6.1. Recall that the collision equilibria are always stable and occur at the points in $\mathcal{F}[\mathbb{T}]$. This set is given by (6.2) and consists of three $\mathbb{T}$-orbits: the tetrahedra $\mathcal{T}_1$ and $\mathcal{T}_2$, and $\{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$. The points on the latter orbit lie on the vertices of an octahedron. The proof of item (ii) in the theorem follows from these observations and item (ii) of Proposition 3.12.

In order to prove item (i), we will classify the critical points of the regularised reduced Hamiltonian $\tilde{h}_{(\mathbb{T},\varnothing)}$. Using (3.10) and writing $u=(x,y,z)$ we find $\tilde{h}_{(\mathbb{T},\varnothing)}(x,y,z)=2^{84}a(x,y,z)^6$, where $a:S^2\to\mathbb{R}$ is given by

$$a(x,y,z)=(x^2+y^2)(y^2+z^2)(x^2+z^2)(1-xy-xz-yz)^2(1+xy+xz-yz)^2 \\ \cdot(1-xy+xz+yz)^2(1+xy-xz+yz)^2.$$

The critical points of $\tilde{h}_{(\mathbb{T},\varnothing)}$ and a coincide and are of the same type, so, in what follows, we instead classify the critical points of a.

We begin by noting that the value of a does not change if x, y and z are permuted; and also if any of x, y or z are changed into $-x$, $-y$ or $-z$. This shows that a is invariant under the action of the group $\mathbb{O}_h$ consisting of all rotational and reflectional symmetries of a regular octahedron (the $\mathbb{O}$-symmetry of a was expected from item (iii) of Theorem 3.5 and the reflectional part is inherited from the invariance of the Hamiltonian $H:M\to\mathbb{R}$ under the diagonal action of $\mathrm{O}(3)$).

The group $\mathbb{O}_h$ has order 48 and a fundamental region $\mathcal{R}\subset S^2$ is determined by $0\le y\le x\le z\le 1$. Without loss of generality, we will restrict the analysis of the critical points of a to this region. Our strategy is to introduce local coordinates on S^2 tracking carefully the parametrisation of $\mathcal{R}$. First consider the gnomonic (or stereographic) projection from the origin to the tangent plane to the north pole. This defines the coordinates $(X,Y)\in\mathbb{R}^2$ on the northern hemisphere by

$$x=\frac{X}{\sqrt{X^2+Y^2+1}},\quad y=\frac{Y}{\sqrt{X^2+Y^2+1}},\quad z=\frac{1}{\sqrt{X^2+Y^2+1}},$$

and the fundamental region $\mathcal{R}$ corresponds to the triangle $0\le Y\le X\le 1$. Now pass to polar coordinates

$$X=R\cos\Theta,\qquad Y=R\sin\Theta.$$

The northern hemisphere is parametrised by $R\ge 0$ and $\Theta\in[0,2\pi)$, and the fundamental region $\mathcal{R}$ corresponds to

$$0\le R\le\frac{1}{\cos\Theta},\qquad 0\le\Theta\le\pi/4. \tag{6.3}$$

In particular, it will be convenient to notice that $\mathcal{R}$ is contained in the region parametrised by $(R,\Theta)\in[0,\sqrt{2}]\times[0,\pi/4]$. In these coordinates we have:

$$a(R,\Theta)=\frac{R^2\left(R^2+2-R^2\cos 2\Theta\right)\left(R^2+2+R^2\cos 2\Theta\right)\left(p_1^+(R,\Theta)p_1^-(R,\Theta)p_2^+(R,\Theta)p_2^-(R,\Theta)\right)^2}{1024\left(R^2+1\right)^{11}},$$

where $p_1^\pm$, and $p_2^\pm$ are defined in the statement of Lemma 6.3. With the help of a symbolic algebra software, one finds that the partial derivative $\partial_\Theta a(R,\Theta)$ may be written as

$$\partial_\Theta a(R,\Theta)=\frac{R^6\sin 4\Theta}{4096\left(R^2+1\right)^{11}}p_1^+(R,\Theta)p_1^-(R,\Theta)p_2^+(R,\Theta)p_2^-(R,\Theta)p_3(R,\Theta),$$

with p_3 given in the statement of Lemma 6.3. Because of this lemma we conclude that, when restricted to the fundamental region $\mathcal{R}$, the partial derivative $\partial_\Theta a(R,\Theta)$ can only vanish if $\Theta=0$ or $\Theta=\pi/4$.

Now, on the one hand one computes

$$\partial_R a(R,0)=-2R\left(R^2+1\right)^{-11}\left(R^4+R^2+1\right)^3\left(R^2-1\right)\left(R^4-3R^2+1\right),$$

whose real roots are $R = 0$, $R = \pm 1$ and $R = (\pm 1 \pm \sqrt{5})/2$. Hence, in view of (6.3), for $\Theta = 0$, the only critical points of a on the fundamental region $\mathcal{R}$ occur when $R = 0$, $R = (\sqrt{5}-1)/2$ and $R = 1$. These respectively correspond to the following points on S^2:

$$(0,0,1), \qquad \frac{1}{\sqrt{1+\phi^2}}(1,0,\phi), \qquad \frac{1}{\sqrt{2}}(1,0,1),$$

that are, respectively, representatives of the octahedral collisions of (ii)(b), of the icosahedral equilibria of (i)(a) and the cuboctahedron equilibria of (i)(c).

On the other hand, one finds

$$\partial_R a(R,\pi/4) = \frac{R}{512}\left(R^2+1\right)^{-12}\left(3R^2+2\right)^3\left(R^2-2\right)^2\left(R^4-4\right)q_3(R^2), \tag{6.4}$$

where q_3 is the cubic polynomial $q_3(\lambda) = 37\lambda^3 + 106\lambda^2 + 28\lambda - 8$. This polynomial has a unique positive root $\lambda = 2\alpha^2/(1-2\alpha^2)$ with α as defined in the statement of item (i)(b). Therefore, the only real roots of the right hand side of (6.4) are $R = 0$, $R = \pm\sqrt{2}$ and $R = \pm\sqrt{2}\alpha/(1-2\alpha^2)^{1/2}$, and, in view of (6.3), we conclude that for $\Theta = \pi/4$, the only critical points of a on the fundamental region $\mathcal{R}$ occur when $R = 0$, $R = \sqrt{2}\alpha/(1-2\alpha^2)^{1/2}$ and $R = \sqrt{2}$. These respectively correspond to the following points on S^2:

$$(0,0,1), \qquad \left(\alpha,\alpha,\sqrt{1-2\alpha^2}\right), \qquad \frac{1}{\sqrt{3}}(1,1,1),$$

that are, respectively, representatives of the octahedral collisions of (ii)(b), of the irregular truncated tetrahedron equilibria of (i)(b) and the tetrahedron collisions of (ii)(a).

The analysis above proves that, indeed, the only equilibrium points of the (regularised) system are those described in the statement of the theorem. Now recall that the collision equilibria are always stable. To investigate the stability of the non-collision equilibria we compute the Hessian matrix of a at the representatives of these points. Using a symbolic algebra program one obtains:

$$\mathrm{Hess}(a)\left(\frac{\sqrt{5}-1}{2},0\right) = \frac{128}{3125}\begin{pmatrix} -3-\sqrt{5} & 0 \\ 0 & \frac{17}{5}(\sqrt{5}-5)\end{pmatrix}, \qquad \mathrm{Hess}(a)(1,0) = \frac{27}{512}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{37}{2}\end{pmatrix},$$

$$\mathrm{Hess}(a)\left(\frac{\sqrt{2}\alpha}{\sqrt{1-2\alpha^2}},\pi/4\right) \approx \begin{pmatrix} -1.42703 & 0 \\ 0 & 0.0859734\end{pmatrix}.$$

The first of these matrices is negative definite and the other two are indefinite. We conclude that icosahedral equilibria are local maxima of a, whereas cub-octahedral and truncated tetrahedral equilibria are saddle points of a. The same is true for the regularised reduced Hamiltonian $\tilde{h}_{(\mathbb{T},\varnothing)}$. On the other hand, this implies that the (non-regularised) reduced Hamiltonian $h_{(\mathbb{T},\varnothing)}$ has local minima at the icosahedral equilibria and saddle points at the cub-octahedral and truncated tetrahedral equilibria. In view of item (i) of Proposition 3.16, these observations imply that the stability properties described in the theorem hold.

It remains to show that the non-collision equilibrium points in items (i)(a)-(c) indeed correspond to the polyhedron equilibria of (2.2) described in the statement of the theorem. Let $\gamma = \phi/(1+\phi^2)^{1/2}$. The 24 points obtained by permuting the entries of $(\pm\gamma, \pm\sqrt{1-\gamma^2}, 0)$ lie at the vertices of a compound of two icosahedra. One of them corresponds to the even and the other to the odd permutations. Moreover, the vertices of each of these icosahedra lie on a $\mathbb{T}$-orbit. In particular, the embedding $\rho_{(\mathbb{T},\varnothing)}$ maps any of the 24 points into the vertices of a regular icosahedron. A similar scenario occurs for the set obtained by permuting the entries of $(\pm\alpha, \pm\alpha, \pm\sqrt{1-2\alpha^2})$. Such set has 24 elements and consists of two $\mathbb{T}$-orbits according to whether the product of the entries is positive or negative. Each of these orbits determine the vertices of a (irregular) truncated tetrahedron. The situation is simpler for the points obtained by permuting $1/\sqrt{2}(\pm 1, \pm 1, 0)$ since there are only 12 of them, they lie on a $\mathbb{T}$-orbit and lie on the vertices of a cuboctahedron. □

6.2 Dynamics of $\mathbb{T}$-symmetric configurations of $N = 12$ vortices

In view of Theorem 6.1 and Corollary 3.17 we deduce the existence of three families of periodic orbits of the equations of motion (2.2) for $N = 12$ that we describe in the following corollary. We note that the existence of the solutions described in items (ii) and (iii) had been already indicated by Soulière & Tokieda [41, Section 5].

Corollary 6.4. *Let $N = 12$.*

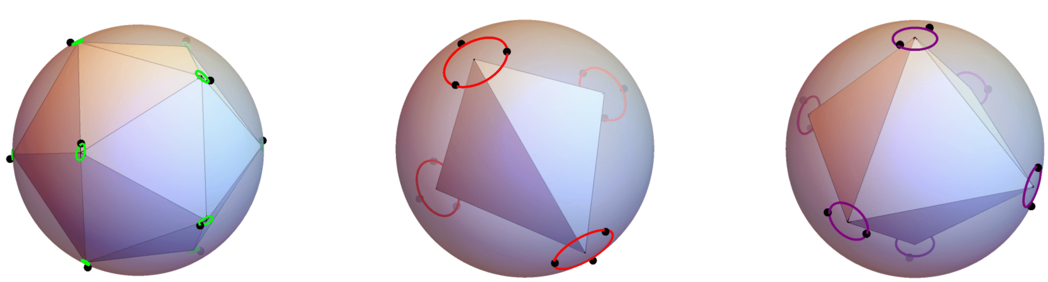
(i) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2), emanating from the icosahedral equilibrium configurations. Along these solutions, each vortex travels around a small closed loop around a vertex of the icosahedron (see Fig. 7.2a).

(ii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the tetrahedral collision described in Theorem 6.1. Along these solutions, three vortices travel along a closed loop around each of the 4 vertices of a tetrahedron (see Fig. 7.2b).

(iii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the octahedral collisions described in Theorem 6.1. Along these solutions, a pair of vortices travels along a small closed loop around each of the 6 vertices of an octahedron (see Fig. 7.2c).

Each of these families may be parametrised by the energy h. In cases (ii) and (iii) we have $h \to \infty$ as the solutions approach collision, and the period approaches zero in this limit.

For each solution described above, the distinct closed loops traversed by the vortices, and the position the vortices within the loop at each instant, may be obtained from a single one by the action of $\mathbb{T}$.



(a) Periodic solution near the icosahedron equilibrium.

(b) Periodic solution near the tetrahedral collision.

(c) Periodic solution near the octahedral collision.

Figure 6.2: Periodic solutions described in Corollary 6.4. (The view angle is different from the one in Fig 6.1.)

We emphasise that the family of periodic orbits emanating from the icosahedron configurations described in item (i) above is different than the one obtained from item (i) of Corollary 5.2 with $n = 5$.

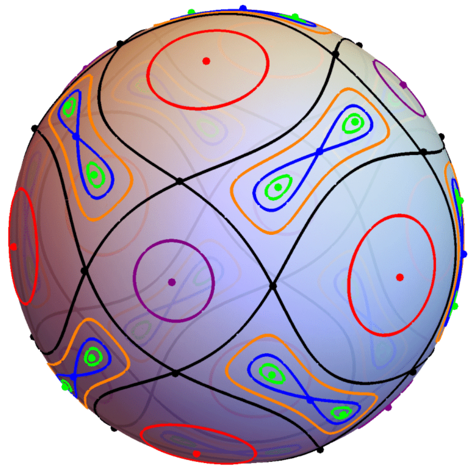


Figure 6.3: The reduced phase space for $\mathbb{T}$-symmetric solutions of the 12-vortex problem.

Figure 6.3 shows the phase space of the (regularised) reduced dynamics obtained numerically. The icosahedron equilibrium points are indicated in green, the truncated tetrahedron equilibrium points in blue, the cuboctahedron equilibrium points in black, tetrahedron collisions in red and octahedron collisions in purple. The same colour is used to indicate either periodic orbits near the stable equilibria or heteroclinic/homoclinic orbits emanating from the unstable equilibria. There is also a family of periodic orbits that do not approach an equilibria or a collision that we have indicated in orange.

7 $\mathbb{T}$-symmetric solutions of $N = 20$ vortices (8 vortices are fixed at the vertices of a cube)

We again consider $K = \mathbb{T}$ but now we take $F = \mathcal{T}_1 \cup \mathcal{T}_2$ (see Eq. (6.1)) so $N = 20$. The points in the set F lie on the vertices of a cube so all solutions of (2.2) treated in this section will have a fixed vortex at each vertex of this cube. We note that the set F is $\mathbb{T}$-invariant and, in view of (6.2), is contained in $\mathcal{F}[\mathbb{T}]$ so it satisfies both requirements in our setup. We analyse the reduced system (3.2) in detail. Since the set F is also $\mathbb{O}$-invariant then, in view of item (iii) of Theorem 3.5 and Table (3.5), the reduced system is $\mathbb{O}$-equivariant.

7.1 Classification of $\mathbb{T}$-symmetric equilibrium configurations of $N = 20$ vortices

The analogous version of Theorem 6.1 on the classification and stability of the collision and non-collision equilibria of the reduced system (3.2) in this case is given next.

Theorem 7.1. *Let $K = \mathbb{T}$, $F = \mathcal{T}_1 \cup \mathcal{T}_2$ (see Eq. (6.1)) and $N = 20$. The classification and stability of the equilibrium points of the reduced system (3.2) is as follows.*

(i) The only non-collision equilibria of (3.2) are:

*(a) The **dodecahedron equilibrium configurations** occurring at all 24 points obtained by permuting the entries and considering all sign flips of*

$$\frac{1}{\sqrt{3}}\left(\pm\phi, \pm\phi^{-1}, 0\right),$$

where $\phi = \frac{1+\sqrt{5}}{2}$ *is the golden mean. These are* stable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of an* S^2*-inscribed regular dodecahedron (see Fig.7.1a).*

(b) *The* ***truncated tetrahedron - cube configurations*** *occurring at all* 24 *points obtained by permuting the entries and considering all sign flips of*

$$\left(\pm\hat{\alpha}, \pm\hat{\alpha}, \pm\sqrt{1-2\hat{\alpha}^2}\right),$$

where $\hat{\alpha} \approx 0.21228\dots$ *is characterised by the condition that* $\hat{\alpha}^2$ *is the unique zero of the polynomial* $\hat{p}(\lambda) = 57\lambda^3 - 29\lambda^2 - 21\lambda + 1$ *between* 0 *and* 1/2*. These are* unstable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of the compound of an irregular* S^2*-inscribed truncated tetrahedron and a cube (see Fig.7.1b and Remark 7.2).*

(c) *The* ***cuboctahedron - cube configurations*** *occurring at all* 12 *points obtained by permuting the entries and considering all sign flips of*

$$\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right).$$

These are unstable *equilibria of* (3.2) *which correspond to equilibrium configurations of* (2.2) *where the vortices occupy the vertices of the compound of a regular* S^2*-inscribed cuboctahedron and a cube (see Fig.7.1c).*

(ii) *The only collision equilibria of (the regularisation of)* (3.2) *are:*

(a) *The* ***tetrahedral - cube collisions*** *at the 8 points of* $\mathcal{T}_1 \cup \mathcal{T}_2$*. These correspond to collision configurations of* (2.2) *where the 20 vortices lie on the vertices of a cube. There are four simultaneous quadruple collisions at the vertices of a tetrahedron and the other four vortices lie at each antipodal point (see Fig.7.1d).*

(b) *The* ***octahedral - cube collisions*** *at the* 6 *points* $(\pm1,0,0), (0,\pm1,0), (0,0,\pm1)$*. These correspond to collision configurations of* (2.2) *where the 20 vortices lie on the vertices of the compound of an octahedron and a cube and there are* 6 *simultaneous binary collisions at the vertices of the octahedron (see Fig.7.1e).*

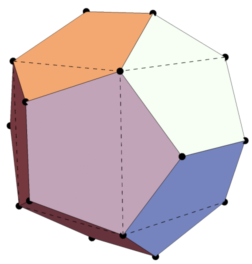
All collision configurations are stable equilibria of (the regularisation of) (3.2).

Remark 7.2. The truncated tetrahedron on the compound of item (i)(b) consists of 4 irregular hexagonal faces and 4 four equilateral triangular faces. The distance between vortices forming an edge between adjacent hexagonal faces is $2\sqrt{2}\hat{\alpha} \approx 0.600421$. The distance between vortices forming an edge of an equilateral triangular face is $\sqrt{2}(-\hat{\alpha} + \sqrt{1-2\hat{\alpha}^2}) \approx 1.04877$.

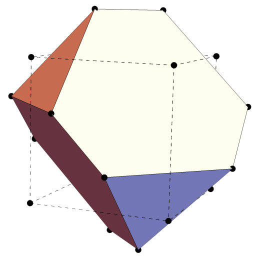
Proof. The conclusions of item (ii) about the collision equilibria follow from the description of $\mathcal{F}[\mathbb{T}]$ in (6.2) and Proposition 3.12 in analogy with the proof of Theorem 6.1.

In order to analyse the non-collision equilibria, let $f_{13}, \dots, f_{20}$ denote the points in $F = \mathcal{T}_1 \cup \mathcal{T}_2$. A direct calculation shows that for $u = (x,y,z) \in S^2$ we have

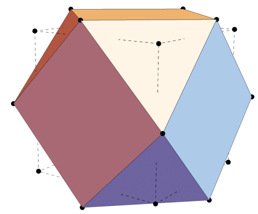
$$\prod_{j=m+1}^{N} \left|u - f_j\right|^2 = \prod_{j=13}^{20} \left|u - f_j\right|^2 = \left(\frac{8}{3}\right)^4 (1 - xy - xz - yz)(1 + xy + xz - yz)(1 - xy + xz + yz)(1 + xy - xz + yz).$$



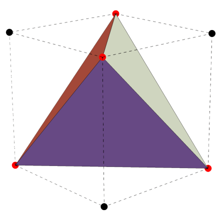
(a) Dodecahedron equilibrium. The distance between adjacent vertices is $\sqrt{2-\frac{2\sqrt{5}}{3}}$.



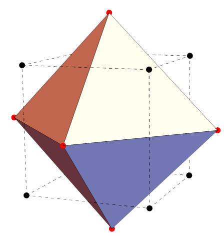
(b) Compound of a truncated tetrahedron and cube equilibrium configuration. See Remark 7.2 for details on the dimensions of the truncated tetrahedron.



(c) Compound of a regular cuboctahedron and a cube equilibrium configuration. The distance between adjacent vertices in the cuboctahedron is 1.



(d) Tetrahedron - cube collision. There is a quadruple collision at each vertex of the tetrahedron and no collisions at the other 4 vertices of the cube (marked in black).



(e) Octahedron -cube collision. There is a binary collision at each vertex of the octahedron and no collision at the 8 vertices of the cube.

Figure 7.1: Non-collision and collision equilibrium configurations described in Theorem 7.1. The dashed lines connect the elements in $F=\mathcal{T}_1\cup\mathcal{T}_2$ that lie on the vertices of a cube.

Therefore, in view of (3.10) and the proof of Theorem 6.1 we find $\tilde{h}_{(\mathbb{T},F)}(x,y,z)=2^{84}\left(\frac{8}{3}\right)^{48}\hat{a}(x,y,z)^6$, where $\hat{a}:S^2\to\mathbb{R}$ is given by

$$\begin{aligned}\hat{a}(x,y,z)=(x^2+y^2)(y^2+z^2)(x^2+z^2)(1-xy-xz-yz)^4(1+xy+xz-yz)^4\\ \cdot(1-xy+xz+yz)^4(1+xy-xz+yz)^4.\end{aligned}$$

The proof proceeds by finding the critical points of $\hat{a}$ on the fundamental region $\mathcal{R}$ described in the proof of Theorem 6.1 and is analogous to it. We omit the details. □

7.2 Dynamics of $\mathbb{T}$-symmetric configurations of $N=20$ vortices

By combining Theorem 7.1 with Corollary 3.17 we deduce the existence of three families of periodic orbits of the equations of motion (2.2) for $N=20$ that we describe in the following:

Corollary 7.3. *Let $N=20$.*

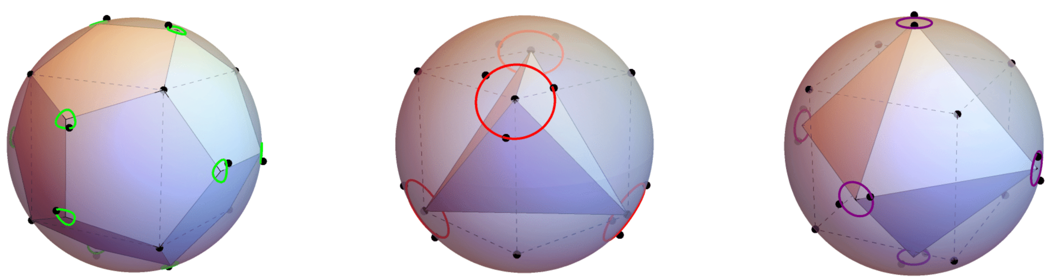
(i) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2), emanating from the dodecahedral equilibrium configurations. Along these solutions, 8 vortices are fixed at the vertices of a cube inscribed in the dodecahedron and the remaining 12 vortices travel around a small closed loop around the remaining vertices of the dodecahedron (see Fig. 7.2a).

(ii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the tetrahedral-cube collision described in Theorem 7.1. Along these solutions, 8 vortices are fixed on the vertices of a cube and there are 4 triples of vortices that travel along a closed loop around each of the 4 vertices of a tetrahedron inscribed in the cube (see Fig. 7.2b).

(iii) There exists a 1-parameter family of periodic solutions $v_h(t)$ of the equations of motion (2.2) converging to the octahedral collisions described in Theorem 6.1. Along these solutions, a pair of vortices travels along a small closed loop around each of the 6 vertices of an octahedron and the remaining 8 vortices are fixed at the vertices of the dual cube (see Fig. 7.2c) .

Each of these families may be parametrised by the energy h. In cases (ii) and (iii) we have $h \to \infty$ as the solutions approach collision, and the period approaches zero in this limit.

For each solution described above, the distinct closed loops traversed by the vortices, and the position the vortices within the loop at each instant, may be obtained from a single one by the action of $\mathbb{T}$.



(a) Periodic solution near the dodecahedron equilibrium.

(b) Periodic solution near the tetrahedral-cube collision.

(c) Periodic solution near the octahedral-cube collision.

Figure 7.2: Periodic solutions described in Corollary 7.3.

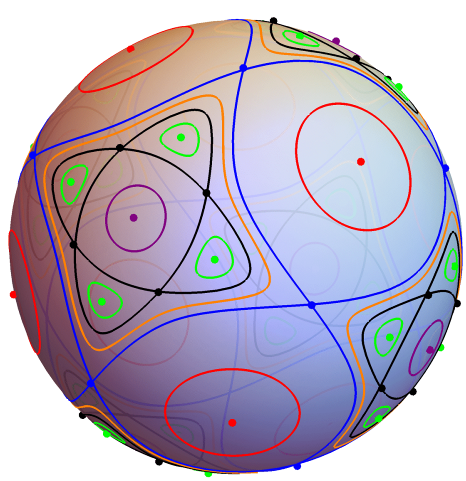


Figure 7.3: The reduced phase space for $\mathbb{T}$-symmetric solutions of the 20-vortex problem.

Figure 7.3 shows the phase space of the (regularised) reduced dynamics obtained numerically. The colour code is similar to the one used in the previous section. The dodecahedron equilibrium points are indicated in green, the truncated tetrahedron–cube equilibrium points in blue, the cuboctahedron–cube equilibrium points in black, tetrahedral–cube collisions in red and octahedral–cube collisions in purple. As usual, we use the same colour to indicate either periodic orbits near the stable equilibria or heteroclinic orbits emanating from the unstable equilibria and we indicate a family of periodic orbits that do not approach an equilibria in orange.

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