Convergence of discrete time option pricing models under stochastic interest rates

Lesne, Jean-Philippe; Prigent, Jean-Luc; Scaillet, Olivier

Finance Stochast. 4, 81–93 (2000)

Finance and Stochastics

Convergence of discrete time option pricing models under stochastic interest rates

J.-P. Lesne[1], **J.-L. Prigent**[1], **O. Scaillet**[2]

[1] THEMA, Université de Cergy-Pontoise, 33 bd du Port, F-95011 Cergy-Pontoise, France
(e-mail: prigent@u-cergy.fr)
[2] Institut d'Administration et de Gestion and Département des Sciences Economiques, Université Catholique de Louvain, 3 Place Montesquieu, B-1348 Louvain-la-Neuve, Belgique
(e-mail: scaillet@ires.ucl.ac.be)

Abstract. We analyze the joint convergence of sequences of discounted stock prices and Radon-Nicodym derivatives of the minimal martingale measure when interest rates are stochastic. Therefrom we deduce the convergence of option values in either complete or incomplete markets. We illustrate the general result by two main examples: a discrete time i.i.d. approximation of a Merton type pricing model for options on stocks and the trinomial tree of Hull and White for interest rate derivatives.

Key words: Weak convergence, incomplete market, option pricing, minimal martingale measure, stochastic interest rate, trinomial tree

JEL classification: D52, E43, G13

Mathematics Subject Classification (1991): 60F05, 90A09

1 Introduction

In this paper, we consider the convergence of discrete time pricing models to continuous time models. The markets may be either complete or incomplete, and are made of a financial asset and a money market account. The money market account is obtained from a money unit invested at the initial date and rolled over at the successive instantaneous interest rates. Its price is used as discount factor.

We would like to thank J. Mémin for useful comments. We also thank the associate editor and the referee for constructive criticism. We have received fruitful comments from R. Anderson, O. Renault, participants at AFFI meeting Lille 1998 and EFA meeting Fontainebleau 1998. The third author gratefully acknowledges financial support by the grant "Actions de Recherche Concertée" 93/98-162 from the Ministry of Scientific Research (Belgian French Speaking Community). Part of this research was done when he was visiting THEMA.
Manuscript received: January 1998; final version received: February 1999

The continuous time dynamics of the asset price and the instantaneous interest rate are characterized under the historical probability by diffusion processes driven by a multidimensional Brownian motion. Under no arbitrage we know that there exists an equivalent martingale measure under which discounted prices are martingales. Unfortunately, as soon as the martingale multiplicity is larger than the number of nonredundant assets, markets are incomplete (see Duffie 1986). When markets are incomplete, the equivalent martingale measure is not unique. A suitable choice is the minimal martingale measure (see Föllmer and Schweizer 1991; Schweizer 1991, 1992, 1993). It corresponds to setting some risk premia to zero and it is also related to risk-minimizing strategies (see Föllmer and Sondermann 1986). Furthermore when markets are complete the minimal martingale measure will be the unique equivalent martingale measure.

The discrete time models correspond to suitable discretizations of the asset price and interest rate dynamics. From these discretizations, we deduce discretizations of the Radon-Nicodym derivatives of the minimal martingale measure. The main study concerns the joint convergence of the sequence of stock prices, instantaneous interest rates, discounted stock prices, and Radon-Nicodym derivatives of the minimal martingale measure (see Runggaldier and Schweizer 1995; Prigent 1995, for similar results in incomplete stock markets with a constant interest rate). This joint convergence will lead to the convergence of contingent claim prices. Thus we propose an approach to achieve a consistent approximation of prices of the underlying asset and derivative securities in an incomplete market framework. When the limiting model is complete, related results are given in Hubalek and Schachermayer (1998) where the emphasis is put on the appropriate contiguity properties to impose on the sequences of discretized measures in order to get convergence of contingent claim prices (see also their Remark 4.3 for an example of convergence problems arising in the context of incomplete limiting models).

Two practically relevant examples illustrate the approach. The first example particularizes the results to a Merton type option pricing model (Merton 1973) where the asset is a stock and the discretizations correspond to the i.i.d. case (Euler discretizations or binomial approximations). The second example deals with trinomial approximations of discount bond price processes often used in term structure modelling to build recombining trees. As a by-product of this second example we establish the convergence of trinomial trees (Hull and White 1994a) using results for martingale difference arrays. For these examples we will essentially use convergence results given in Jacod and Shiryaev (1987).

In Sect. 2 we present the continuous time framework and set some notations. Section 3 is devoted to the discretizations schemes and to the study of convergence of discrete time models to continuous time models in the general case of incomplete markets. In Sect. 4 we illustrate the results by two examples. Section 5 concludes the paper. Technical details and proofs can be found in an appendix.

2 The continuous time model

We consider a market with a financial asset and a money market account. The price at date t, $t \in [0, T]$, of the financial asset is denoted by S_t (S_t can be either a stock price or a discount bond price), while the price of the money market account (or accumulation factor) is denoted by $\beta_t = \exp \int_0^t r_s ds$, where r_t is the instantaneous interest rate. The discounted price is equal to $Z_t = S_t/\beta_t$.

We assume that the dynamics of S_t and r_t are described under the historical probability $\mathbb{P}$ by the following SDE's :

$$\begin{cases} dS_t = S_t[\mu^S(t, r_t, S_t)dt + \sigma^S(t, r_t, S_t)' dW_t], \\ dr_t = \mu^r(t, r_t)dt + \sigma^r(t, r_t)' dW_t, \end{cases} \tag{1}$$

with : $\sigma^l(t, r_t, S_t) = (\sigma_1^l(t, r_t, S_t), \ldots, \sigma_d^l(t, r_t, S_t))'$, $l = S, r$, and W_t a d-dimensional standard Brownian motion, and where $'$ denotes the transpose operator.

We assume that there exists a unique solution to (1). Note that model (1) is general enough to cover both stock and discount bond prices.

Let $\tilde{\mu}^S$ be the function $\tilde{\mu}^S(t, r, s) = s\ \mu^S(t, r, s)$ and $\tilde{\sigma}^S$ the function $\tilde{\sigma}^S(t, r, s) = s\ \sigma^S(t, r, s)$. We require that $\tilde{\mu}^S$, μ^r, $\tilde{\sigma}^S$ and σ^r satisfy a continuity and linear growth property.

Property (CLG). *A function f defined on $\mathbb{R}^+ \times \mathbb{R}^d$ with values in $\mathbb{R}$ is said to satisfy the Property CLG iff f is continuous and $|f(t,x)| \leq K(t)(1+ \| x \|)$ for all t in $[0, T]$ and for all x in $\mathbb{R}^d$ with the condition:* $\sup_{t \in [0,T]} K(t) < \infty$.

For notations and definitions, we adopt the conventions of Jacod and Shiryaev (1987): $\alpha.X$ for the stochastic integral of the predictable process α w.r.t. the semimartingale X, $\mathscr{E}(X)$ for the Dade-Doléans exponential of X, and $\langle M, M \rangle$ for the predictable quadratic variation of the locally square-integrable martingale M.

The convergence analysis of the next section will be based on Proposition 3.1 p. 24 in Ansel and Stricker (1993). This proposition states that, under no arbitrage and an integrability condition, the discounted price process Z is a special semimartingale and has the particular form:

$$Z_t = Z_0 + \alpha.\langle M^Z, M^Z \rangle_t + M_t^Z,$$

for its canonical decomposition. Furthermore the Radon-Nicodym derivative process $\hat{\eta}$ of the minimal martingale measure $\hat{\mathbb{P}}$ is equal to:

$$\hat{\eta} = \mathscr{E}(-\alpha.M^Z).$$

The computation of α is obtained by applying the integration by part formula (see Jacod and Shiryaev 1987, p. 52) for semimartingales to $Z = S/\beta$ and by identifying the finite variation part of the semimartingale Z and $\alpha.\langle M^Z, M^Z \rangle$:

$$\alpha_t = \frac{1}{Z_t} \frac{\mu^S(t, r_t, S_t) - r_t}{\sigma^S(t, r_t, S_t)'\sigma^S(t, r_t, S_t)}.$$

The process α has the interpretation of a scaled Sharpe ratio.

In the next section we consider discretizations of model (1) and will compute the counterpart of $\hat{\eta}$ in discrete time.

3 Convergence of discrete time models

Let us consider for the discretization of model (1):

$$\left\{\begin{array}{rcl} S_{n,k} - S_{n,k-1} & = & S_{n,k-1}\left[\mu_n^S\left(\frac{(k-1)T}{n}, r_{n,k-1}, S_{n,k-1}\right)\frac{T}{n}\right. \\ & + & \left.\sigma_n^S\left(\frac{(k-1)T}{n}, r_{n,k-1}, S_{n,k-1}\right)'\epsilon_{n,k}\right], \\ r_{n,k} - r_{n,k-1} & = & \mu_n^r\left(\frac{(k-1)T}{n}, r_{n,k-1}\right)\frac{T}{n} \\ & + & \sigma_n^r\left(\frac{(k-1)T}{n}, r_{n,k-1}\right)'\epsilon_{n,k}, \end{array}\right. \tag{2}$$

where k is the integer part of tn/T, and $(\epsilon_{n,k})$ is a d-dimensional martingale difference triangular array chosen to converge to the d-dimensional Brownian motion (see Jacod and Shiryaev 1987, p. 424 and p. 436).

We assume that all the functions $\tilde{\mu}_n^S$, $\tilde{\sigma}_n^S$, μ_n^r, σ_n^r satisfy Property CLG. For the discretization of the discount factor, we take:

$$\beta_{n,k} = \prod_{l \le k}\left(1 + r_{n,l}\frac{T}{n}\right).$$

Then we get for the discounted price:

$$Z_{n,k} = \frac{S_{n,k}}{\beta_{n,k}} = Z_{n,k-1}(1 + R_{n,k})$$

with

$$R_{n,k} = \frac{Y_{n,k} - r_{n,k}\frac{T}{n}}{1 + r_{n,k}\frac{T}{n}}$$

and

$$Y_{n,k} = \mu_n^S\left(\frac{(k-1)T}{n}, r_{n,k-1}, S_{n,k-1}\right)\frac{T}{n} + \sigma_n^S\left(\frac{(k-1)T}{n}, r_{n,k-1}, S_{n,k-1}\right)'\epsilon_{n,k}.$$

To any triangular arrays $(X_{n,k})$ we associate as usual a sequence of continuous time processes corresponding to the step processes $(X_{n,t})$ such that $X_{n,t} = X_{n,k}$ when k is the integer part of tn/T. In particular we have $S_{n,t} = S_{n,k} = \mathscr{E}(\sum_{l \le k} Y_{n,l})$, $r_{n,t} = r_{n,k}$, and $Z_{n,t} = Z_{n,k} = \mathscr{E}(\sum_{l \le k} R_{n,l})$.

We denote by M_n^Z the martingale part of Z_n and by analogy with $\hat{\eta}$, we get from Ansel and Stricker (1993, p. 24):

$$\hat{\eta}_n = \mathscr{E}(-\alpha_n.M_n^Z).$$

From Schweizer (1993), we know that:

$$\alpha_{n,k} = \frac{1}{Z_{n,k-1}} \frac{E(R_{n,k}|\mathscr{F}_{n,k-1})}{E(R_{n,k}^2|\mathscr{F}_{n,k-1})},$$

where $\mathscr{F}_{n,k}$ denotes the σ-algebra generated by $\epsilon_{n,l}$, $l = 1, \ldots, k$.

Let $\mathbb{D}^d[0,T]$ be the set of cadlag functions on $[0,T]$ with values in $\mathbb{R}^d$ endowed with the Skorokhod topology, and $M_{n,t} = \sum_{l \le k} \epsilon_{n,l}$. We write $X_n \overset{\mathscr{L}(\mathbb{D}^d[0,T])}{\Longrightarrow} X$ when the sequence X_n converges in distribution to X on the space $\mathbb{D}^d[0,T]$. To prove the main result of the paper, we need to assume that M_n satisfies the property of Uniform Tightness (UT or "Uniforme Tension"). This property is used to insure the convergence of stochastic integrals (see Stricker 1985; Jakubowski et al. 1989).

Recall that this condition is the following: for all t the family of distributions $\left\{\mathbb{P}_{H_n.X_{n,t}}, H_n \in \mathscr{H}_{n,t}\right\}$ is tight, where $\mathscr{H}_{n,t}$ is the set of all elementary predictable processes H_n such that $H_{n,s} = Y_{n,0} + \sum_{i=0}^{k} Y_{n,t_i} \mathbb{1}_{]t_i,t_{i+1}]}(s)$ with Y_{n,t_i} any $\mathscr{F}_{n,t_i}$-measurable variable satisfying $|Y_{n,t_i}| \le 1$ and $\{t_0, \ldots, t_k\}$ any subdivision of $[0,t]$. This property will essentially guarantee the existence of the limit integrals $H.X$. Details can be found in Mémin and Slominsky (1991).

In our examples, in order to verify that M_n satisfies UT, we will use Remark 2.1 in Mémin and Slominsky (1991) which concerns sequence of local martingales built from martingale difference arrays. This remark states that $M_{n,t} = \sum_{l \le k} \epsilon_{n,l}$ will satisfy UT when for all t, there exists at least a strictly positive a such that:

$$\left\{ \sum_{l \le k} \left| E[\epsilon_{n,l} \mathbb{1}_{|\epsilon_{n,l}|>a} | \mathscr{F}_{n,l-1}] \right|, n \in \mathbb{N} \right\}$$

is stochastically bounded.

The local consistency conditions which can be found in the Markov chain approximation literature (see Kushner 1997; Kushner and Dupuis 1992) will not be sufficient here since they only guarantee that the chain has the local properties (mean and variance) of the diffusion process. Such local requirements are used to prove the tightness of the approximation. To establish the next proposition, we need more, namely stability under integration, than this minimal behavior which only leads to convergence in distribution (see also Kurtz and Protter (1991) for further discussion). In fact the additional requirements are the CLG conditions on the drift and diffusion coefficients (Theorem 3.5 in Mémin and Slominsky 1991).

Proposition 1 *If the functions* $\tilde{\mu}_n^S$, $\tilde{\sigma}_n^S$, μ_n^r, σ_n^r *converge uniformly on every compact of* $[0,T]\times\mathbb{R}\times\mathbb{R}$ *to* $\tilde{\mu}^S$, $\tilde{\sigma}^S$, μ^r, σ^r, *respectively, and if* M_n *satisfies the property of Uniform Tightness and* $(M_n,\alpha_n)\overset{\mathscr{L}(\mathbb{D}^{d+1}[0,T])}{\Rightarrow}(W,\alpha)$ *we have:*

$$(S_n,r_n,Z_n,\hat{\eta}_n)\overset{\mathscr{L}(\mathbb{D}^4[0,T])}{\Longrightarrow}(S,r,Z,\hat{\eta}).$$

Proof. See Appendix.

This proposition establishes the joint convergence of the sequence of stock prices, instantaneous interest rates, discounted stock prices, and Radon-Nicodym derivatives of the minimal measure. From Proposition 1 and the continuous mapping theorem, we deduce that if $g:\mathbb{D}^3[0,T]\to\mathbb{R}$ is a continuous function:

$$(g(S_n,r_n,Z_n),\hat{\eta}_n)\overset{\mathscr{L}(\mathbb{D}^2[0,T])}{\Longrightarrow}(g(S,r,Z),\hat{\eta}).$$

Furthermore if $g(S_n,r_n,Z_n)\hat{\eta}_n$ is equiintegrable we get the convergence of the expectations $E[g(S_n,r_n,Z_n)\hat{\eta}_n]$ to $E[g(S,r,Z)\hat{\eta}]$, i.e. the contingent claim prices.

Note that it may happen that $\hat{\eta}_n$ assumes negative values for all n although $\hat{\eta}$ is positive, so the approximating minimal measures exist only as signed measures. Nevertheless the convergence result remains true (see Prigent 1995). Then some care has to be taken for the price interpretation: a better wording would be values (see Runggaldier and Schweizer 1995).

Before examining some examples, let us remark that, when the sequence of martingales M_n does not converge to W (as in the pathological example developped in Jacod and Shiryaev (1987, p. 435): M_n has the property UT but does not converge to a martingale), we loose the convergence of $\hat{\eta}_n$ to $\hat{\eta}$. However sensible approximating schemes are not designed so.

4 Examples

The result of the previous section is quite general. To enlighten its usefulness, we examine now two examples. The first example deals with a continuous time stock market and the second with a discount bond market.

4.1 Stock and money market account

We particularize model (1) to:

$$\begin{cases} dS_t &= S_t[\mu^S(t,S_t)dt+\sigma^S(t,S_t)dW_t^{(1)}],\\ dr_t &= \mu^r(t,r_t)dt+\sigma^r(t,r_t)(\rho dW_t^{(1)}+\sqrt{1-\rho^2}dW_t^{(2)}). \end{cases} \tag{3}$$

This Merton type market allows for very general dynamics for the stock price and the instantaneous interest rate.

The discrete time market is obtained by a straightforward discretization using i.i.d. processes, i.e. we set $\epsilon_{n,k} = (\epsilon^{(1)}_{n,k}, \epsilon^{(2)}_{n,k})'$ where $(\epsilon^{(1)}_{n,k})$ and $(\epsilon^{(2)}_{n,k})$ are two independent rowwise triangular arrays, and $(\epsilon^{(1)}_{n,k})$, $(\epsilon^{(2)}_{n,k})$ are independent.

To ensure the convergence of both discrete time processes to Brownian motions, we assume that the triangular arrays satisfy the assumptions of the Lindeberg-Feller Theorem (Jacod and Shiryaev 1987, p. 404), and take: $E[\epsilon^{(1)}_{n,k}] = E[\epsilon^{(2)}_{n,k}] = 0$, and $E[(\epsilon^{(1)}_{n,k})^2] = E[(\epsilon^{(2)}_{n,k})^2] = \frac{T}{n}$. This discretization covers either Euler schemes where $\epsilon^{(1)}_{n,k}$, $\epsilon^{(2)}_{n,k}$ are centered Gaussian variables with variance T/n or binomial trials where the values $\sqrt{T/n}$ and $-\sqrt{T/n}$ are taken with probabilities $1/2$. The discrete time market is obviously incomplete while the continuous time market is complete,

To obtain the convergence of $(S_n, r_n, Z_n, \hat{\eta}_n)$ to $(S, r, Z, \hat{\eta})$, we impose similar conditions on the functions μ^S_n, μ^r_n, σ^S_n, and σ^r_n as in Proposition 1, and we assume that $M^{(1)}_{n,t} = \sum_{l\leq k} \epsilon^{(1)}_{n,l}$, $M^{(2)}_{n,t} = \sum_{l\leq k} \epsilon^{(2)}_{n,l}$ share the property of Uniform Tightness. For the Euler schemes and binomial trials, it can be checked that it is indeed the case from Remark 2.1 of Mémin and Slominsky (1991, p. 167). Furthermore we can verify that α_n converges to α by using Taylor expansions of $R_{n,k}$ and the first and second order moments of $\epsilon^{(1)}_{n,k}$ and $\epsilon^{(2)}_{n,k}$.

4.2 Discount bond and money market account

Here we examine the case of a discount bond, i.e. an asset which delivers one money unit at maturity. The price at date t of a discount bond with maturity H is denoted $B(t,H)$. We take the particular case of Gaussian one factor affine models also known as extended Vasicek models (see Hull and White 1990; Frachot and Lesne 1995; Duffie and Kan 1996), and the tree building procedure proposed by Hull and White (1994a) to compute contingent claim prices (see also Hull and White 1993, 1994b, 1996a or Part V on the implementation of term structure models in Hull and White 1996b). Let us remark that our framework is rich enough to allow for other approximations and more complex discount bond price dynamics.

We particularize model (1) to:

$$\begin{cases} dB(t,H) = B(t,H)[(r_t + \varphi(t)\sigma^B(t,H))dt + \sigma^B(t,H)dW_t], \\ dr_t = (\theta(t) - ar_t)dt + \sigma dW_t, \end{cases} \tag{4}$$

where $\sigma^B(t,H) = \sigma\frac{1-e^{-a(H-t)}}{a}$. The time varying parameter $\theta(t)$ is often chosen to adjust the initial term structure: $\theta(t) - \varphi(t)\sigma^2 = \partial_2 f(0,t) + af(0,t) + \sigma^2\frac{1-e^{-2at}}{2a}$ (see Hull and White 1990), where $f(0,t)$ denotes the forward instantaneous interest rate at the initial date and with maturity t, and $\varphi(t)$ is a deterministic function corresponding to the market price of risk.

Note that by using a standard constant variation method the solution r_t of

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t,$$

can be expressed in terms of the solution r_t^* of the same SDE with $\theta(t) = 0$:

$$r_t = r_t^* + \int_0^t e^{-a(t-s)}\theta(s)ds.$$

Therefore the approximation of r_t can be made in two steps: first approximate separately r_t^* and $\int_0^t e^{-a(t-s)}\theta_s ds$, and then add the two components.

This corresponds to the tree building procedure of Hull and White (1994a) where the first stage consists at using a recombining trinomial tree to approximate r_t^* and then shift the nodes of this tree to get a new recombining tree for the approximation of r_t.

Hereafter we analyze the convergence of this trinomial procedure. More precisely, to approximate model (1), we use:

$$\left\{\begin{array}{lcl} B_{n,k} - B_{n,k-1} & = & B_{n,k-1}\left[\left(r_{n,k-1} + \varphi_n\left(\frac{(k-1)T}{n}\right)\right.\right. \\ & \times & \left.\left.\sigma_n^B\left(\frac{(k-1)T}{n}, H\right)\right)\frac{T}{n}\right. \\ & + & \left.\sigma_n^B(\frac{(k-1)T}{n}, H)\epsilon_{n,k}\right], \\ r_{n,k}^* - r_{n,k-1}^* & = & -ar_{n,k-1}^*\frac{T}{n} + \sigma\epsilon_{n,k}, \\ r_{n,k} & = & r_{n,k}^* + \frac{T}{n}\left(1 - a\frac{T}{n}\right)^k \sum_{l=1}^{k} \frac{\theta_n\left(\frac{(l-1)T}{n}\right)}{\left(1 - a\frac{T}{n}\right)^l}, \end{array}\right. \tag{5}$$

where θ_n and φ_n converge uniformly on every compact of $[0, T]$ to θ and φ, respectively.

The discretization $\frac{T}{n}(1-a\frac{T}{n})^k \sum_{l=1}^k \theta_n(\frac{(l-1)T}{n})/(1-a\frac{T}{n})^l$ of $\int_0^t e^{-a(t-s)}\theta(s)ds$ comes from applying the constant variation method in discrete time and ensures:

$$r_{n,k} - r_{n,k-1} = \left(\theta_n\left(\frac{(k-1)T}{n}\right) - ar_{n,k-1}\right)\frac{T}{n} + \sigma\epsilon_{n,k}.$$

The trinomial tree used to approximate r_t^* is built as follows (see Hull and White 1994a). $r_{n,k}$ is set equal to $r_{n,k} = r_0 + J_{n,k}\Delta_n r$ with $\Delta_n r = \sigma\sqrt{3T/n}$,

$$\begin{array}{lll} J_{n,k} & = J_{n,k-1} + 1 & \text{with probability} \quad p_{n,k}^u, \\ & = J_{n,k-1} & \text{with probability} \quad p_{n,k}^m, \\ & = J_{n,k-1} - 1 & \text{with probability} \quad p_{n,k}^d, \end{array} \tag{6}$$

and $J_{n,0} = 0$. The conditional probabilities $p_{n,k}^u$, $p_{n,k}^m$, $p_{n,k}^d$, are the probabilities associated respectively with the highest, middle and lowest branches emanating from a node. They are chosen in order to match the conditional expected change and variance of the change in the instantaneous interest rate:

$$E[r^*_{n,k} - r^*_{n,k-1}|\mathscr{F}_{n,k-1}] = -ar^*_{n,k-1}\frac{T}{n}, \tag{7}$$

$$Var[r^*_{n,k} - r^*_{n,k-1}|\mathscr{F}_{n,k-1}] = \sigma^2\frac{T}{n}. \tag{8}$$

As the probabilities must sum up to one, this leads to a system of three equations in three unknowns. Unfortunately when $J_{n,k}$ becomes large, the probabilities $p^u_{n,k}$, $p^m_{n,k}$, $p^d_{n,k}$, may be negative, and the branching process has to be switched. Let us denote by J^{max}_n and J^{min}_n the levels at which this switching has to be made for large and small values of $J_{n,k}$, respectively. We get either:

$$\begin{aligned} J_{n,k} &= J_{n,k-1} && \text{with probability} \quad p^{u,max}_{n,k}, \\ &= J_{n,k-1} - 1 && \text{with probability} \quad p^{m,max}_{n,k}, \\ &= J_{n,k-1} - 2 && \text{with probability} \quad p^{d,max}_{n,k}, \end{aligned} \tag{9}$$

when $J_{n,k-1} = J^{max}_n$, or

$$\begin{aligned} J_{n,k} &= J_{n,k-1} + 2 && \text{with probability} \quad p^{u,min}_{n,k}, \\ &= J_{n,k-1} + 1 && \text{with probability} \quad p^{m,min}_{n,k}, \\ &= J_{n,k-1} && \text{with probability} \quad p^{d,min}_{n,k}, \end{aligned} \tag{10}$$

when $J_{n,k-1} = J^{min}_n$. The conditional probabilities of the branching processes (9) and (10) are determined in the same way as for the first process (6) (see Hull and White 1994a for the values). Hence by construction the discretized process of the interest rate is a Markov process allowing for a recombining tree.

Let us now examine the convergence. By identifying:

$$r^*_{n,k} - r^*_{n,k-1} = -ar^*_{n,k-1}\frac{T}{n} + \sigma\epsilon_{n,k},$$

with:

$$r^*_{n,k} - r^*_{n,k-1} = \Delta_n r\ (J_{n,k} - J_{n,k-1}),$$

we deduce:

$$\begin{aligned} \epsilon_{n,k} &= \frac{1}{\sigma}\Delta_n r \left(J_{n,k} - J_{n,k-1} + a\frac{T}{n}J_{n,k-1}\right) \\ &= \sqrt{3T/n}\left(J_{n,k} - J_{n,k-1} + a\frac{T}{n}J_{n,k-1}\right). \end{aligned}$$

Hence from (7), we have:

$$E[\epsilon_{n,k}|\mathscr{F}_{n,k-1}] = 0,$$

which means that $(\epsilon_{n,k})$ is a martingale difference array (see Jacod and Shiryaev 1987, p. 436). From (8), the second order moment is equal to:

$$E\left[\epsilon^2_{n,k}|\mathscr{F}_{n,k-1}\right] = \frac{T}{n}.$$

Therefore $M_{n,t} = \sum_{l \le k} \epsilon_{n,l}$ will converge to a Brownian motion (Jacod and Shiryaev 1987, Theorem 3.32, p. 437). It also satisfies the property of Uniform Tightness from Remark 2.1 of Mémin and Slominsky (1991, p. 167). Finally as in the first example by using Taylor expansions of $R_{n,k}$ and the first and second order moments of $\epsilon_{n,k}$ we can verify that α_n converges to α.

5 Conclusion

The results of this paper deal with the convergence of option values given by discrete time models to their continuous time counterparts in complete or incomplete markets. They are established in a stochastic interest rate environment, and are general enough to cover the main discretization schemes such as Euler approximations, binomial or trinomial trees. They are of practical relevance since these approximations are often used in applied work. Indeed when no closed form exists in continuous time, there is an urge for approximating option prices.

Appendix
Proof of Proposition 1

The proof is based on weak convergence and properties of Uniform Tightness (condition UT) of stochastic differential equations (SDE) given in Jakubowski et al. (1989) and Mémin and Slominsky (1991). As indicated in Jakubowski et al. (1989, p. 113), all results can be immediately extended to the d-dimensional case.[1] The proof will made in three steps: (a) establish the convergence of (S_n, r_n, Z_n, M_n) to (S, r, Z, W), (b) show that the martingale part M_n^Z of the process Z_n has property UT, (c) deduce the convergence of $\hat{\eta}_n = \mathscr{E}(-\alpha_n.M_n^Z)$ to $\hat{\eta} = \mathscr{E}(-\alpha.M^Z)$.

(a) First consider the processes S_n and r_n which are solutions of the SDE associated with equations (2). Recall that we assume that SDE (1) has a unique solution. Under the conditions on the functions $\tilde{\mu}_n^S$, $\tilde{\sigma}_n^S$, μ_n^r, σ_n^r (in particular Property CLG) and the properties of M_n, we can apply Theorem 3.5 of Mémin and Slominsky (1991, p. 174) and get the joint convergence of (S_n, r_n, M_n) to (S, r, W). Then, from the definition of the processes Z_n and Z, we can deduce the joint convergence of (S_n, r_n, Z_n, M_n) to (S, r, Z, W).

(b) To deduce the weak convergence of stochastic integrals, we need to prove that the sequences of processes of interest satisfy the property UT. For this, we use the following result of ”stability” (Let us remark that the goodness property of Proposition 4.1 in Duffie and Protter (1992, p. 5) is analogue to this stability property):

[1] We thank J. Mémin for all useful discussions about this point.

If (X_n) satisfies UT and (H_n) is a sequence of adapted cadlag processes and if $(X_n, H_n) \overset{\mathscr{L}(\mathbb{D}^2[0,T])}{\Longrightarrow} (X, H)$ then the semimartingale $(\int H_{n-} dX_n)$ also satisfies property UT. If (H_n) is a sequence of predictable processes then, under the joint convergence of (X_n, H_n), $\int H_n dX_n$ also satisfies property UT. [See Mémin and Slominsky (1991, Lemma 1.6, p. 165) when (H_n) is predictable and Duffie and Protter (1989, Theorem 2, p. 6)].

From the above properties, we can deduce that the martingale part M_n^Z of Z_n satisfies UT. For this, note first that M_n^Z is equal to $Z_n.M_n^{\Phi}$ where M_n^{Φ} is the martingale part of the process Φ_n defined by the relations $\Phi_{n,k} = \sum_{l\leq k} R_{n,l}$. Besides, $M_{n,k}^{\Phi}$ is equal to $\sum_{l\leq k}(R_{n,l} - E[R_{n,l}|\mathscr{F}_{n,l-1}])$ and the term $R_{n,l}$ can be expressed as $(Y_{n,l} - r_{n,l}\frac{T}{n})(1 - r_{n,l}\frac{T}{n} + \Delta_{n,l})$ where $\Delta_{n,l}$ is equal to $\frac{1}{1+r_{n,l}\frac{T}{n}} - (1 - r_{n,l}\frac{T}{n})$. Therefore, using the expressions of $Y_{n,l}$ and $r_{n,l}$, we get:

$$\begin{aligned} M_{n,k}^{\Phi} &= \sum_{l\leq k}(Y_{n,l} - E[Y_{n,l}|\mathscr{F}_{n,l-1}]) - \frac{T}{n}\left[\sum_{l\leq k}(r_{n,l} - E[r_{n,l}|\mathscr{F}_{n,l-1}])\right] \\ &+ \sum_{l\leq k}\left(\left(Y_{n,l} - r_{n,l}\frac{T}{n}\right)\left(-r_{n,l}\frac{T}{n} + \Delta_{n,l}\right)\right. \\ &- \left. E\left[\left(Y_{n,l} - r_{n,l}\frac{T}{n}\right)\left(-r_{n,l}\frac{T}{n} + \Delta_{n,l}\right)|\mathscr{F}_{n,l-1}\right]\right). \end{aligned}$$

1) The first sum is equal to the stochastic integrals $\sigma_n^S.M_n$. By the weak convergence of (σ_n^S, M_n), the property UT of M_n and the "stability" property of UT, the processes $\sigma_n^S.M_n$ satisfy UT.
2) The second sum is equal to the stochastic integrals $-\frac{T}{n}\sigma_n^r.M_n$ and for the same reasons as above, they satisfy UT. Note that they in fact converge weakly to 0.
3) The last term has finite variation V_n (as for all discrete semimartingales) and for all t, the sequences $V_{n,t}$ converge to 0 (since (ϵ_n) is a martingale difference array, we have $E[\epsilon_{n,k}|\mathscr{F}_{n,k-1} = 0$ and by using a Taylor expansion for $\Delta_{n,k}$ and some straighforward calculations, this variation is dominated by a sequence which takes the form $\frac{T}{n} \times X_{n,t}$ where the sequence $(X_{n,t})_n$ converges in distribution).

Therefore, for all t, the set of distributions $\{\mathbb{P}_{V_{n,t}}\}$ is tight and so the last term satisfies UT (see Jakubowski et al. 1989, p. 112).

From 1), 2) and 3), we deduce the property UT of M_n^{Φ}.
Then, since (Z_n, M_n^{Φ}) converges to $(Z, \sigma^S.W)$, the stability property of UT implies that the martingale part M_n^Z of Z_n, equal to $Z_n.M_n^{\Phi}$, has also the property UT.

(c) Finally, from the assumption about the convergence of (α_n) and from the above results, we deduce that (α_n, M_n^Z) converges to (α, M^Z) and (M_n^Z) satisfies UT. So $(-\alpha_n.M_n^Z)$ converges to $(-\alpha.M^Z)$ and $(-\alpha_n.M_n^Z)$ satisfies UT. Consequently, using results about the convergence of the Dade-Doléans exponentials (see for example Corollary 4.3. and Remark 4.4. in Jakubowski et al. (1989, p. 131) or Theorem 3.5 in Mémin and Slominsky (1991) with the special case $f_n(x,t) = x$) we get the convergence of the sequence $\hat{\eta}_n = \mathscr{E}(-\alpha_n.M_n^Z)$ to $\hat{\eta} = \mathscr{E}(-\alpha.M^Z)$.

To conclude, since all processes are defined on the same sequence (M_n) and from the previous results, the joint convergence of $(S_n, r_n, Z_n, \hat{\eta}_n)$ to the corresponding processes $(S, r, Z, \hat{\eta})$ is proved.

References

1. Ansel, J.P., Stricker, C.: Unicité et existence de la loi minimale. Lecture notes in math. Sem. Prob. 27, **1557**, 22–29. Berlin Heidelberg New York: Springer 1993
2. Duffie, D.: Stochastic equilibria: Existence, spanning number, and the no expected financial gain from trade hypothesis. Econometrica **54**, 1161–1183 (1986)
3. Duffie, D., Kan, R.: A yield-factor model of interest rates. Math. Finance **6**, 379–406 (1996)
4. Duffie, D., Protter, P.: From discrete to continuous time finance: Weak convergence of the financial gain process. Technical report 89-02. Purdue University 1989
5. Duffie, D., Protter, P.: From discrete to continuous time finance: Weak convergence of the financial gain process. Math. Finance **2**, 1–16 (1992)
6. Föllmer, H., Sondermann, D.: Hedging of non-redundant contingent claims. Contributions to Mathematical Economics in Honor of Gerard Debreu. Amsterdam: North-Holland 1986
7. Föllmer, H., Schweizer, M.: Hedging of contingent claims under incomplete information. In: Davis M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis, Stochastics Monographs. London New York: Gordon and Breach, 1991, pp. 389–414
8. Frachot, A., Lesne, J.P.: Modèles factoriels de la structure par termes des taux d'intérêt: théorie et application économétrique. Ann. Econom. Statist. **40**, 11–36 (1995)
9. Hubalek, F., Schachermayer, W.: When does convergence of asset price processes imply convergence of option prices? Math. Finance **8**, 385–403 (1998)
10. Hull, J., White, A.: Pricing interest rate derivative securities. Rev. Financial Stud. **3**, 573–592 (1990)
11. Hull, J., White, A.: Single factor interest rate models and the valuation of interest rate derivatives securities. J. Financial Quantitat. Anal. **28**, 235–254 (1993)
12. Hull, J., White, A.: Numerical procedures for implementing term structure models I: Single factor models. J. Derivat. **2**, 7–16 (1994a)
13. Hull, J., White, A.: Numerical procedures for implementing term structure models II: Two factor models. J. Derivat. **2**, 37–48 (1994b)
14. Hull, J., White, A.: Using Hull-White interest rate trees. J. Derivat. **3**, 26–36 (1996a)
15. Hull, J., White, A.: Hull-White on derivatives. London: Risk Publications 1996b
16. Jacod, J., Shiryaev, A.N.: Limit theorems for stochastic processes. Berlin Heidelberg New Vork: Springer 1987
17. Jakubowski, A., Mémin, J., Pagès, G.: Convergence en loi des suites d'intégrales stochastiques sur l'espace $\mathbb{D}^1$ de Skorokhod. Probab. Theory Relat. Fields **81**, 111–137 (1989)
18. Kushner, H.: Numerical methods for stochastic control problems in finance. In: Dempster, M.A., Pliska, S.R. (eds.) Mathematics of derivative securities. Publications of the Newton Institute, **15**, Cambridge: Cambridge University Press 1997, pp. 504–527
19. Kushner, H., Dupuis, P.G.: Numerical methods for stochastic control problems in continuous time. Berlin Heidelberg New York: Springer 1992
20. Kurtz, T., Protter, P.: Weak limit theorems for stochastic integrals and stochastic differential equations. Ann. Probab. **19**, 1035–1070 (1991)
21. Mémin, J., Slominsky, L.: Condition UT et stabilité en loi des solutions d'équations différentielles stochastiques. Lecture notes in math. Sem. Probab. 25, **1485**. Berlin Heidelberg New York: Springer 1991
22. Merton, R.: Theory of rational option pricing. Bell J. Econom. Manag. Sci. **4**, 141–183 (1973)
23. Prigent, J.L.: Incomplete markets: Convergence of options values under the minimal martingale measure. THEMA, preprint No. 9526. Forthcoming in Adv. Appl. Probab. **31** (1995)

24. Runggaldier, W.J., Schweizer, M.: Convergence of option values under incompleteness. In: Bolthausen, E., Dozzi, M., Russo, F. (eds.) Seminar on stochastic analysis, random fields and applications. Basel: Birkhäuser 1995, pp. 365–384
25. Schweizer, M.: Option hedging for semimartingales. Stochast. Processes Appl. **37**, 339–363 (1991)
26. Schweizer, M.: Mean-variance hedging for general claims. Ann. Appl. Probab. **2**, 171–179 (1992)
27. Schweizer, M.: Variance-optimal hedging in discrete time. Math. Oper. Res. **20**, 1–32 (1993)
28. Stricker, C.: Lois de semimartingales et critères de compacité. Lecture Notes in Math. Sem. Prob. 19, **1123**. Berlin Heidelberg New York: Springer 1985