Annals of Mathematics and Artificial Intelligence (2024) 92:601–629
https://doi.org/10.1007/s10472-023-09881-y

Altruism in coalition formation games

Anna Maria Kerkmann[1] · **Simon Cramer**[1] · **Jörg Rothe**[1]

Accepted: 9 June 2023 / Published online: 31 July 2023

Abstract
Nguyen et al. (2016) introduced altruistic hedonic games in which agents' utilities depend not only on their own preferences but also on those of their friends in the same coalition. We propose to extend their model to coalition formation games in general, considering also the friends in other coalitions. Comparing our model to altruistic hedonic games, we argue that excluding some friends from the altruistic behavior of an agent is a major disadvantage that comes with the restriction to hedonic games. After introducing our model and showing some desirable properties, we additionally study some common stability notions and provide a computational analysis of the associated verification and existence problems.

Keywords Coalition formation · Hedonic game · Altruism · Cooperative game theory

1 Introduction

We consider coalition formation games where agents have to form coalitions based on their preferences. Among other compact representations of hedonic coalition formation games, Dimitrov et al. [2] in particular proposed the *friends-and-enemies encoding with friend-oriented preferences* which involves a *network of friends*: a (simple) undirected graph whose vertices are the players and where two players are connected by an edge exactly if they are friends of each other. Players not connected by an edge consider each other as enemies. Under friend-oriented preferences, player i prefers a coalition C to a coalition D if C contains more of i's friends than D, or C and D have the same number of i's friends but C contains fewer enemies of i's than D. This is a special case of the *additive encoding* [3]. For more

Missing Open Access funding information has been added in the Funding Note.

✉ Anna Maria Kerkmann
anna.kerkmann@uni-duesseldorf.de

Simon Cramer
simon.cramer@hhu.de

Jörg Rothe
rothe@hhu.de

1 Institut für Informatik, Heinrich-Heine-Universität Düsseldorf, Universitätsstr. 1, Düsseldorf 40225, Germany

background on these two compact representations, see Section 2 and the book chapter by Aziz and Savani [4].

Based on friend-oriented preferences, Nguyen et al. [1] introduced *altruistic hedonic games* (AHGs) where agents gain utility not only from their own satisfaction but also from their friends' satisfaction. However, Nguyen et al. [1] specifically considered hedonic games only, which require that an agent's utility only depends on her own coalition. In their interpretation of altruism, the utility of an agent is composed of the agent's own valuation of her coalition and the valuation of all this agent's friends *in this coalition*. While Nguyen et al. [1] used the average when aggregating some agents' valuations, Wiechers and Rothe [5] proposed a variant of altruistic hedonic games where some agents' valuations are aggregated by taking the minimum.

Inspired by the idea of altruism, we extend the model of altruism in hedonic games to coalition formation games in general. That is, we propose a model where agents behave altruistically to *all their friends*, not only to the friends in the same coalition. Not restricting to hedonic games, we aim to capture a more natural notion of altruism where none of an agent's friends is excluded from her altruistic behavior.

Example 1 To become acquainted with this idea of altruism, consider the coalition formation game that is represented by the *network of friends* in Fig. 1. For the coalition structures $\Gamma = \{\{1, 2, 3\}, \{4\}\}$ and $\Delta = \{\{1, 2, 4\}, \{3\}\}$, it is clear that player 1 is indifferent between coalitions $\{1, 2, 3\}$ and $\{1, 2, 4\}$ under friend-oriented preferences, as both coalitions contain 1's only friend (player 2) and one of 1's enemies (either 3 or 4). Under altruistic hedonic preferences [1], however, player 1 behaves altruistically to her friend 2 (who is friends with 3 but not with 4) and therefore prefers $\{1, 2, 3\}$ to $\{1, 2, 4\}$. Now, consider the slightly modified coalition structures $\Gamma' = \{\{1\}, \{2, 3\}, \{4\}\}$ and $\Delta' = \{\{1\}, \{2, 4\}, \{3\}\}$. Intuitively, one would still expect 1 to behave altruistically to her friend 2. However, under any *hedonic* preference (which requires the players' preferences to depend *only* on their own coalitions), player 1 (being in the same coalition for both Γ' and Δ') must be indifferent between Γ' and Δ'.

In order to model altruism globally, we release the restriction to hedonic games and introduce *altruistic coalition formation games* where agents behave altruistically to all their friends, independently of their current coalition.

1.1 Related work

Coalition formation games, as considered here, are closely related to the subclass of *hedonic games* which has been broadly studied in the literature, addressing the issue of compactly representing preferences, conducting axiomatic analyses, dealing with different notions of stability, and investigating the computational complexity of the associated problems (see, e.g., the book chapter by Aziz and Savani [4]) and the survey of Woeginger [6]).

Closest related to our work are the *altruistic hedonic games* by Nguyen et al. [1] (see also the related minimization-based variant by Wiechers and Rothe [5] and the recent journal version of these two papers [7]), which we modify to obtain our more general models of altruism. Kerkmann and Rothe [8] continued the study of altruistic hedonic games, considering the notions of popularity and strict popularity. Based on the model due to Nguyen et al. [1],

1 —— 2 —— 3 —— 4

Fig. 1 Network of friends for Example 1

Schlueter and Goldsmith [9] defined *super altruistic hedonic games* where friends have a different impact on an agent based on their distances in the underlying network of friends. More recently, Bullinger and Kober [10] introduced *loyalty in cardinal hedonic games* where agents are loyal to all agents in their so-called loyalty set. In their model, the utilities of the agents in the loyalty set are aggregated by taking the minimum. They then study the loyal variants of common classes of cardinal hedonic games such as additively separable and friend-oriented hedonic games.[1]

Altruism has also been studied for *noncooperative* games. Most prominently, Ashlagi et al. [11] introduced *social context games* where a social context is applied to a strategic game and the costs in the resulting game depend on the original costs and a graph of neighborhood. Their so-called *Min-Max collaborations* (where players seek to minimize the maximal cost of their own and their neighbors) are related to our minimization-based equal-treatment model. Still, the model of Ashlagi et al. [11] differs from ours in that they consider *non*cooperative games. While Ashlagi et al. [11] focus on *resource selection games* as underlying strategic games, Bilò et al. [12] study social context games for *linear congestion games* and *Shapley cost sharing games*. Hoefer et al. [13] study *considerate equilibria* in strategic games, where players perform group deviations in a *considerate* way to not decrease the utility of their neighbors. Anagnostopoulos et al. [14] study *altruism* and *spite* in strategic games. In these games, every agent assigns a real value to any other agent expressing the (altruistic or spiteful) attitude that she has towards the other agent. In the same context, Bilò [15] studies the existence and inefficiency of pure Nash equilibria in linear congestion games.

There is further work modeling altruism in noncooperative games without social networks but with some altruistic component that is incorporated into the agents' payoff functions. For example, Hoefer and Skopalik [16] study altruism in *atomic congestion games* and Chen et al. [17] study altruism in *nonatomic congestion games*. Both use so-called *altruistic levels*, real values between zero and one that indicate the degree to which the agents are altruistic. Apt and Schäfer [18] introduce *selfishness levels* for strategic games. Their model is equivalent to the model by Chen et al. [17] and a model by Caragiannis et al. [19]. Kleer and Schäfer [20] generalize these settings to a more general framework that can also model spiteful agents. They study the price of anarchy and the price of stability for this generalized setting; Schröder [21] continues their study. Furthermore, Rahn and Schäfer [22] introduce *social contribution games* which take account of the costs that agents induce on society. For more background on altruism in game theory, see, e.g., the survey by Rothe [23].

Concerning the concepts of stability that we consider here, there exists a lot of related work in the context of coalition formation games, including hedonic games.[2] For example, Bogomolnaia and Jackson [25] introduced some stability notions for hedonic games that deal with single player deviations. These notions include Nash stability and individual stability.

[1] Note that their loyal variant of symmetric friend-oriented hedonic games is equivalent to the minimization-based altruistic hedonic games under equal treatment introduced by Wiechers and Rothe [5].

[2] In fact, stability in coalition formation has also been studied for *nonhedonic* cooperative games. For example, Magaña and Carreras [24] studied the stability of coalition formation in cooperative games with transferrable utility (unlike hedonic games, which are cooperative games with nontransferrable utility) by combining the Shapley value with notions such as strong Nash equilibrium from noncooperative game theory. However, our model of altruistic coalition formation games is very close to the model of altruism originally proposed by Nguyen et al. [1] (see also [7]) for hedonic games. Therefore, the stability notions we consider are essentially those common in the study of hedonic games, like (strict) core stability, Nash stability, perfectness, etc. Still, as we drop the requirement of hedonism that players only care about the members of their own coalition, we study *nonhedonic* coalition formation, even if very closely related to its hedonic counterpart.

Also, core stability has been studied extensively in the literature, e.g., by Banerjee et al. [26], Dimitrov et al. [2], Alcade and Romero-Medina [27], Woeginger [28], Peters [29], and Ota et al. [30]. Karakaya [31] and Aziz and Brandl [32] defined some more complicated notions of group stability that they call *strong Nash stability*, *strictly strong Nash stability*, and *strong individually stability*. Popularity was first introduced by Gärdenfors [33] in the context of marriage games and was later formulated by Aziz et al. [3] and Lang et al. [34] for hedonic games.

1.2 Our contribution

Conceptually, we extend the models of altruism proposed by Nguyen et al. [1] and Wiechers and Rothe [5] from hedonic games to general coalition formation games. We argue how this captures a more global notion of altruism and show that our models fulfill some desirable properties that are violated by the previous models. We then study the common stability concepts in this model and analyze the associated verification and existence problems in terms of their computational complexity.

This work extends a preliminary version that appeared in the proceedings of the *29th International Joint Conference on Artificial Intelligence* (IJCAI'20) [35]. Parts of this work were also presented at the 16th and 17th International Symposium on Artificial Intelligence and Mathematics (ISAIM'20 and ISAIM'22) and at the *8th International Workshop on Computational Social Choice* (COMSOC'21), each with nonarchival proceedings.

2 The model

In *coalition formation games*, players divide into groups based on their preferences. Before introducing altruism, we now give some foundations of such games.

2.1 Coalition formation games

Let $N = \{1, \ldots, n\}$ be a set of *agents* (or *players*). Each subset of N is called a *coalition*. A *coalition structure* Γ is a partition of N, and we denote the set of all possible coalition structures for N by $\mathscr{C}_N$. For a player $i \in N$ and a coalition structure $\Gamma \in \mathscr{C}_N$, $\Gamma(i)$ denotes the unique coalition in Γ containing i. Now, a *coalition formation game (CFG)* is a pair $(N, \succeq)$, where $N = \{1, \ldots, n\}$ is a set of agents, $\succeq = (\succeq_1, \ldots, \succeq_n)$ is a profile of preferences, and every preference $\succeq_i \in \mathscr{C}_N \times \mathscr{C}_N$ is a complete weak order over all possible coalition structures. Given two coalition structures $\Gamma,\ \Delta \in \mathscr{C}_N$, we say that i *weakly prefers* Γ to Δ if $\Gamma \succeq_i \Delta$. When $\Gamma \succeq_i \Delta$ but not $\Delta \succeq_i \Gamma$, we say that i *prefers* Γ to Δ (denoted by $\Gamma \succ_i \Delta$), and we say that i is *indifferent* between Γ and Δ (denoted by $\Gamma \sim_i \Delta$) if $\Gamma \succeq_i \Delta$ and $\Delta \succeq_i \Gamma$.

Note that *hedonic games* are a special case of coalition formation games where the agents' preference relations only depend on the coalitions containing themselves. In a hedonic game $(N, \succeq)$, agent $i \in N$ is indifferent between any two coalition structures Γ and Δ as long as her coalition is the same, i.e., $\Gamma(i) = \Delta(i) \Longrightarrow \Gamma \sim_i \Delta$. Therefore, the preference order of any agent $i \in N$ in a hedonic game $(N, \succeq)$ is usually represented by a complete weak order over the set of coalitions containing i.

2.2 The "friends and enemies" encoding

Since $|\mathscr{C}_N|$, the number of all possible coalition structures, is super-exponential in the number of agents,[3] it is not reasonable to ask every agent for her complete preference over $\mathscr{C}_N$. Instead, we are looking for a way to compactly represent the agents' preferences. In the literature, many such representations have been proposed for hedonic games, such as the *additive encoding* due to Bogomolnaia and Jackson [25], the *singleton encoding* due to Cechlárová and Romero-Medina [38] which was further studied by Cechlárová and Hajduková [39], the *friends-and-enemies encoding* due to Dimitrov et al. [2], and FEN-hedonic games due to Kerkmann et al. [40]. Here, we use the *friends-and-enemies encoding* due to Dimitrov et al. [2]. We focus on their friend-oriented model and will later adapt it to our altruistic model.

In the friend-oriented model, the preferences of the agents in N are given by a network of friends, i.e., a (simple) undirected graph $G = (N, A)$ whose vertices are the players and where two players $i, j \in N$ are connected by an edge $\{i, j\} \in A$ exactly if they are each other's friends. Agents not connected by an edge consider each other as enemies. For an agent $i \in N$, we denote the set of i's friends by $F_i = \{j \in N \mid \{i, j\} \in A\}$ and the set of i's enemies by $E_i = N \setminus (F_i \cup \{i\})$.Under *friend-oriented preferences* as defined by Dimitrov et al. [2], between any two coalitions players prefer the coalition with more friends, and if there are equally many friends in both coalitions, they prefer the coalition with fewer enemies:

$$C \succeq_i^F D \iff |C \cap F_i| > |D \cap F_i| \text{ or } (|C \cap F_i| = |D \cap F_i| \text{ and } |C \cap E_i| \leq |D \cap E_i|).$$

This can also be represented additively. Assigning a value of n to each friend and a value of -1 to each enemy, agent $i \in N$ values coalition C containing herself with $v_i(C) = n|C \cap F_i| - |C \cap E_i|$. Note that $-(n-1) \leq v_i(C) \leq n(n-1)$, and $v_i(C) > 0$ if and only if there is at least one friend of i's in C. For a given coalition structure $\Gamma \in \mathscr{C}_N$, we also write $v_i(\Gamma)$ for player i's value of $\Gamma(i)$.

Furthermore, we denote the sum of the values of i's friends by $\text{sum}_i^F(\Gamma) = \sum_{f \in F_i} v_f(\Gamma)$. Analogously, we also define $\text{sum}_i^{F+}(\Gamma) = \sum_{f \in F_i \cup \{i\}} v_f(\Gamma)$, $\min_i^F(\Gamma) = \min_{f \in F_i} v_f(\Gamma)$, and $\min_i^{F+}(\Gamma) = \min_{f \in F_i \cup \{i\}} v_f(\Gamma)$.

2.3 Three degrees of altruism

When we now define altruistic coalition formation games based on the friend-oriented preference model, we consider the same three degrees of altruism that Nguyen et al. [1] introduced for altruistic hedonic games. However, we adapt them to our model, extending the agents' altruism to *all* their friends, not only to their friends in the same coalition.

- **Selfish First (SF):** Agents first rank coalition structures based on their own valuations. Only in the case of a tie between two coalition structures, their friends' valuations are considered as well.
- **Equal Treatment (EQ):** Agents treat themselves and their friends the same. That means that an agent $i \in N$ and all of i's friends have the same impact on i's utility for a coalition structure.
- **Altruistic Treatment (AL):** Agents first rank coalition structures based on their friends' valuations. They only consider their own valuations in the case of a tie.

[3] The number of possible partitions of a set with n elements equals the n-th Bell number [36, 37], defined as $B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$ with $B_0 = B_1 = 1$. For example, for $n = 10$ agents, we have $B_{10} = 115,975$ possible coalition structures.

We further distinguish between a sum-based and a min-based aggregation of some agents' valuations. Formally, for an agent $i \in N$ and a coalition structure $\Gamma \in \mathscr{C}_N$, we denote i's sum-based utility for Γ under SF by $u_i^{sumSF}(\Gamma)$, under EQ by $u_i^{sumEQ}(\Gamma)$, and under AL by $u_i^{sumAL}(\Gamma)$, and her min-based utility for Γ under SF by $u_i^{minSF}(\Gamma)$, under EQ by $u_i^{minEQ}(\Gamma)$, and under AL by $u_i^{minAL}(\Gamma)$.For a constant $M \geq n^3$, they are defined as

$$\begin{aligned} u_i^{sumSF}(\Gamma) &= M \cdot v_i(\Gamma) + \mathrm{sum}_i^F(\Gamma); & u_i^{minSF}(\Gamma) &= M \cdot v_i(\Gamma) + \min_i^F(\Gamma); \\ u_i^{sumEQ}(\Gamma) &= \mathrm{sum}_i^{F+}(\Gamma); & u_i^{minEQ}(\Gamma) &= \min_i^{F+}(\Gamma); \\ u_i^{sumAL}(\Gamma) &= v_i(\Gamma) + M \cdot \mathrm{sum}_i^F(\Gamma); & u_i^{minAL}(\Gamma) &= v_i(\Gamma) + M \cdot \min_i^F(\Gamma). \end{aligned}$$

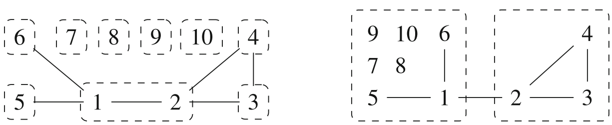
In the case of $F_i = \emptyset$, we define the minimum of the empty set to be zero.

For any coalition structures $\Gamma, \Delta \in \mathscr{C}_N$, agent i's sum-based SF preference is then defined by $\Gamma \succeq_i^{sumSF} \Delta \iff u_i^{sumSF}(\Gamma) \geq u_i^{sumSF}(\Delta)$. Her other altruistic preferences ($\succeq_i^{sumEQ}$; $\succeq_i^{sumAL}$; $\succeq_i^{minSF}$; $\succeq_i^{minEQ}$; and $\succeq_i^{minAL}$) are defined analogously, using the respective utility functions. The factor M, which is used for the SF and AL models, ensures that an agent's utility is first determined by the agent's own valuation in the SF model and first determined by the friends' valuations in the AL model. Similarly as Nguyen et al. [1] prove the corresponding properties in hedonic games, we can show that for $M \geq n^3$, $v_i(\Gamma) > v_i(\Delta)$ implies $\Gamma \succ_i^{sumSF} \Delta$ and $\Gamma \succ_i^{minSF} \Delta$, and for $M \geq n^2$, $\mathrm{sum}_i^F(\Gamma) > \mathrm{sum}_i^F(\Delta)$ implies $\Gamma \succ_i^{sumAL} \Delta$ while $\min_i^F(\Gamma) > \min_i^F(\Delta)$ implies $\Gamma \succ_i^{minAL} \Delta$. An *altruistic coalition formation game (ACFG)* is a coalition formation game where the agents' preferences were obtained by a network of friends via one of these cases of altruism. Hence, we distinguish between sum-based SF, sum-based EQ, sum-based AL, min-based SF, min-based EQ, and min-based AL ACFGs. For any ACFG, the players' utilities can obviously be computed in polynomial time.

3 Axiomatic analysis of ACFGs

Nguyen et al. [1] focus on altruism in hedonic games where an agent's utility only depends on her own coalition. As we have already seen in Example 1, there are some aspects of altruistic behavior that cannot be realized by hedonic games. The following example shows that our model crucially differs from the models due to Nguyen et al. [1] and Wiechers and Rothe [5].

Example 2 Consider an ACFG $(N, \succeq)$ with the network of friends shown in Fig. 2 with the the coalition structures $\Gamma = \{\{1, 2\}, \{3\}, \{4\}, \ldots, \{10\}\}$ (shown in Fig. 2a) and $\Delta = \{\{1, 5, \ldots, 10\}, \{2, 3, 4\}\}$ (shown in Fig. 2b). We will now compare agent 1's preferences for these two coalition structures under our altruistic models to 1's preferences under the altruistic hedonic models [1, 5]. Table 1 shows all relevant values that are needed to compute the utilities of agent 1.



(a) Coalition structure Γ (b) Coalition structure Δ

Fig. 2 Network of friends for Example 2 with the two coalition structures Γ and Δ marked by dashed lines

Table 1 Friend-oriented valuations of agent 1 and her friends and her aggregated valuations for the game in Example 2 with the network of friends in Fig. 2

	v_1	v_2	v_5	v_6	sum_1^F	sum_1^{F+}	$\min_1^F$	$\min_1^{F+}$
Γ	10	10	0	0	10	20	0	0
Δ	16	20	5	5	30	46	5	5

One can observe that agent 1 and all her friends assign a greater value to Δ than to Γ. Consequently, also the aggregations of the friends' values (sum_1^F, sum_1^{F+}, $\min_1^F$, $\min_1^{F+}$) are greater for Δ. Hence, 1 prefers Δ to Γ under all our sum-based and min-based altruistic preferences.

The hedonic models due to Nguyen et al. [1] and Wiechers and Rothe [5], however, are blind to the fact that agent 1 and all her friends are better off in Δ than in Γ. Under their altruistic hedonic preferences, player 1 compares the two coalition structures Γ and Δ only based on her own coalitions $\Gamma(1) = \{1, 2\}$ and $\Delta(1) = \{1, 5, \dots, 10\}$. She then only considers her friends that are in the same coalition, i.e., player 2 for Γ and players 5 and 6 for Δ. This leads to 1 preferring $\Gamma(1)$ to $\Delta(1)$ under altruistic hedonic EQ and AL preferences. In particular, the average (and minimum) valuation of 1's friends in $\Gamma(1)$ is 10 while the average (and minimum) valuation of 1's friends in $\Delta(1)$ is 5. Also considering 1's own value for EQ, the average (and minimum) in $\Gamma(1)$ is 10 while the average (respectively, minimum) value in $\Delta(1)$ is $8.\overline{6}$ (respectively, 5).

3.1 Some basic properties

As we have seen in Example 2, altruistic *hedonic* games [1, 5] allow for players that prefer coalition structures that make themselves and all their friends worse off. To avoid this kind of unreasonable behavior, we focus on general coalition formation games. In fact, all our altruistic *coalition-formation* preferences fulfill unanimity: For an ACFG $(N, \succeq)$ and a player $i \in N$, we say that $\succeq_i$ is *unanimous* if, for any two coalition structures $\Gamma, \Delta \in \mathscr{C}_N$, $v_a(\Gamma) > v_a(\Delta)$ for each $a \in F_i \cup \{i\}$ implies $\Gamma \succ_i \Delta$. Informally, this means that if i and all her friends assign a higher value to Γ than to Δ in an altruistic coalition formation game, then i will prefer Γ to Δ. This intuitively makes sense since the unanimous opinion of all agents that i cares about (namely herself and her friends) is then represented by her preference.

This property crucially distinguishes our preference models from the corresponding altruistic *hedonic* preferences, which are not unanimous under EQ or AL preferences, as Example 2 shows. Note that Nguyen et al. [1] define a restricted version of unanimity in altruistic hedonic games by considering only the agents' own coalitions. Other desirable properties that were studied by Nguyen et al. [1] for altruistic hedonic preferences can be generalized to coalition formation games. We show that these desirable properties also hold for our models and summarize our findings in Table 2. First, we collect some basic observations:

Observation 1 Consider any ACFG $(N, \succeq)$ with an underlying network of friends G.

1 All preferences $\succeq_i$, $i \in N$, are reflexive and transitive.
2 For any player $i \in N$ and any two coalition structures $\Gamma, \Delta \in \mathcal{C}_N$, it can be decided in polynomial time (in the number of agents) whether $\Gamma \succeq_i \Delta$.
3 The preferences $\succeq_i$, $i \in N$, only depend on the structure of G.

Note that the third statement of Observation 1 implies that the properties that Nguyen et al. [1] call *anonymity* and *symmetry* are both satisfied in ACFGs. Another desirable prop-

Table 2 Axiomatic properties of our altruistic preferences in comparison to the altruistic hedonic preferences by Nguyen et al. [1]

Property	ACFGs	AHGs
Reflexivity & Transitivity	✓	✓
Relations are decidable in polynomial time	✓	✓
Relations only dependent on graph structure	✓	✓
Unanimity	✓	✗ (EQ or AL)
Sovereignty of players	✓	✓
Type-I-monotonicity	✗ (min-based)	✗ (min-based or sum-based EQ or AL)
Type-II-monotonicity	✓	✗ (EQ or AL)

"✓" means that the property is fulfilled for all degrees of altruism and both aggregation methods."✗ (...)" means that the property is not satisfied for the models listed in brackets but for all other models considered here

erty they consider is called *sovereignty of players* and inspired by the axiom of *"citizens' sovereignty"* from social choice theory:[4] Given a set of agents N, a coalition structure $\Gamma \in \mathscr{C}_N$, and an agent $i \in N$, we say that an altruistic preference extension satisfies *sovereignty of players* if there is a network of friends G on N such that Γ is i's most preferred coalition structure in the ACFG induced by G under this extension.

Proposition 1 *ACFGs satisfy sovereignty of players under all sum-based and min-based SF, EQ, and AL altruistic preferences.*

Proof Sovereignty of players in ACFGs can be shown with an analogous construction as in the proof of Nguyen et al. [1, Theorem 5]: For a given set of players N, player $i \in N$, and coalition structure $\Gamma \in \mathscr{C}_N$, we construct a network of friends where all players in $\Gamma(i)$ are friends of each other while there are no other friendship relations. Then Γ is i's (nonunique) most preferred coalition structure under all sum-based and min-based SF, EQ, and AL altruistic preferences. □

3.2 Monotonicity

The next property describes the monotonicity of preferences and further distinguishes our models from altruistic hedonic games. In fact, Nguyen et al. [1] define two types of monotonicity, which we here adapt to our setting. Intuitively, we say that a preference relation of an agent i is monotonic if her preference between two coalition structures cannot be altered by turning an enemy (who has the same opinion as i about the two coalition structures) into a friend. The two types of monotonicity differ in whether the enemy is part of i's coalition in both coalition structures (type-I) or only in the more preferred one (type-II). These notions are motivated by the idea that adding a friend that agrees with i's preference should only strengthen i's preference and should not reverse it.

Definition 1 Consider any ACFG $(N, \succeq)$, agents $i, j \in N$ with $j \in E_i$, and coalition structures $\Gamma, \Delta \in \mathscr{C}_N$. Let further $\succeq_i'$ be the preference relation resulting from $\succeq_i$ when j turns from being i's enemy to being i's friend (all else remaining equal). We say that $\succeq_i$ is

[4] Informally stated, a voting rule satisfies *citizens' sovereignty* if every candidate can be made a winner of an election for a suitably chosen preference profile.

- *type-I-monotonic* if (1) $\Gamma \succ_i \Delta$, $j \in \Gamma(i) \cap \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$ implies $\Gamma \succ'_i \Delta$, and (2) $\Gamma \sim_i \Delta$, $j \in \Gamma(i) \cap \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$ implies $\Gamma \succeq'_i \Delta$;
- *type-II-monotonic* if (1) $\Gamma \succ_i \Delta$, $j \in \Gamma(i) \setminus \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$ implies $\Gamma \succ'_i \Delta$, and (2) $\Gamma \sim_i \Delta$, $j \in \Gamma(i) \setminus \Delta(i)$, and $v_j(\Gamma) \geq v_j(\Delta)$ implies $\Gamma \succeq'_i \Delta$.

Theorem 1 *Let $(N, \succeq)$ be an ACFG.*

1. *If $(N, \succeq)$ is sum-based, its preferences satisfy type-I- and type-II-monotonicity.*
2. *If $(N, \succeq)$ is min-based, its preferences satisfy type-II- but not type-I-monotonicity.*

Proof Let $(N, \succeq)$ be an ACFG with an underlying network of friends $G = (N, H)$. Consider $i \in N$, $\Gamma, \Delta \in \mathscr{C}_N$, and $j \in E_i$ and denote with $G' = (N, H \cup \{\{i, j\}\})$ the network of friends resulting from G when j turns from being i's enemy to being i's friend (all else being equal). Let $(N, \succeq')$ be the ACFG induced by G'. For any agent $a \in N$ and coalition structure $\Gamma \in \mathscr{C}_N$, denote a's value for Γ in G' by $v'_a(\Gamma)$, a's preference relation in $(N, \succeq')$ by $\succeq'_a$, and a's friends and enemies in $(N, \succeq')$ by F'_a and E'_a, respectively. That is, we have $F'_i = F_i \cup \{j\}$, $E'_i = E_i \setminus \{j\}$, $F'_j = F_j \cup \{i\}$, and $E'_j = E_j \setminus \{i\}$. Further, v'_i, v'_j, and $\succeq'_i$ might differ from v_i, v_j, and $\succeq_i$, while the friends, enemies, and values of all other players stay the same, i.e., $F'_a = F_a$, $E'_a = E_a$, and $v'_a = v_a$ for all $a \in N \setminus \{i, j\}$.
Type-I-monotonicity under sum-based preferences.
Let $j \in \Gamma(i) \cap \Delta(i)$ and $v_j(\Gamma) \geq v_j(\Delta)$. It then holds that

$$v'_i(\Gamma) = n|\Gamma(i) \cap F'_i| - |\Gamma(i) \cap E'_i| = n|\Gamma(i) \cap F_i| + n - |\Gamma(i) \cap E_i| + 1 = v_i(\Gamma) + n + 1.$$

Equivalently, $v'_i(\Delta) = v_i(\Delta) + n + 1$, $v'_j(\Gamma) = v_j(\Gamma) + n + 1$, and $v'_j(\Delta) = v_j(\Delta) + n + 1$. Furthermore,

$$\begin{aligned} \text{sum}_i^{F'}(\Gamma) &= \sum_{a \in F'_i} v'_a(\Gamma) = \sum_{a \in F_i \cup \{j\}} v'_a(\Gamma) = \sum_{a \in F_i} v_a(\Gamma) + v'_j(\Gamma) \\ &= \text{sum}_i^F(\Gamma) + v_j(\Gamma) + n + 1 \quad \text{and} \qquad (1) \\ \text{sum}_i^{F'}(\Delta) &= \text{sum}_i^F(\Delta) + v_j(\Delta) + n + 1. \qquad (2) \end{aligned}$$

(1) sumSF: If $\Gamma \succ_i^{sumSF} \Delta$ then either (i) $v_i(\Gamma) = v_i(\Delta)$ and $\text{sum}_i^F(\Gamma) > \text{sum}_i^F(\Delta)$, or (ii) $v_i(\Gamma) > v_i(\Delta)$.

In case (i), $v_i(\Gamma) = v_i(\Delta)$ implies $v'_i(\Gamma) = v'_i(\Delta)$. Applying $\text{sum}_i^F(\Gamma) > \text{sum}_i^F(\Delta)$ and $v_j(\Gamma) \geq v_j(\Delta)$ to (1) and (2), we get $\text{sum}_i^{F'}(\Gamma) > \text{sum}_i^{F'}(\Delta)$. This together with $v'_i(\Gamma) = v'_i(\Delta)$ implies $\Gamma \succ_i^{sumSF'} \Delta$.

In case (ii), $v_i(\Gamma) > v_i(\Delta)$ implies $v'_i(\Gamma) > v'_i(\Delta)$. Hence, $\Gamma \succ_i^{sumSF'} \Delta$.

If $\Gamma \sim_i^{sumSF} \Delta$ then $v_i(\Gamma) = v_i(\Delta)$ and $\text{sum}_i^F(\Gamma) = \text{sum}_i^F(\Delta)$. $v_i(\Gamma) = v_i(\Delta)$ implies $v'_i(\Gamma) = v'_i(\Delta)$. Applying $\text{sum}_i^F(\Gamma) = \text{sum}_i^F(\Delta)$ and $v_j(\Gamma) \geq v_j(\Delta)$ to (1) and (2), we get $\text{sum}_i^{F'}(\Gamma) \geq \text{sum}_i^{F'}(\Delta)$. This together with $v'_i(\Gamma) = v'_i(\Delta)$ implies $\Gamma \succeq_i^{sumSF'} \Delta$.

(2) sumEQ: If $\Gamma \succ_i^{sumEQ} \Delta$ then $\text{sum}_i^F(\Gamma) + v_i(\Gamma) > \text{sum}_i^F(\Delta) + v_i(\Delta)$. Using (1), (2), $v'_i(\Gamma) = v_i(\Gamma) + n + 1$, $v'_i(\Delta) = v_i(\Delta) + n + 1$, and $v_j(\Gamma) \geq v_j(\Delta)$, this implies $\text{sum}_i^{F'}(\Gamma) + v'_i(\Gamma) > \text{sum}_i^{F'}(\Delta) + v'_i(\Delta)$. Hence, $\Gamma \succ_i^{sumEQ'} \Delta$.

If $\Gamma \sim_i^{sumEQ} \Delta$, using the same equations, $\Gamma \succeq_i^{sumEQ'} \Delta$ is implied.

(3) sumAL: If $\Gamma \succ_i^{sumAL} \Delta$ then either (i) $\text{sum}_i^F(\Gamma) = \text{sum}_i^F(\Delta)$ and $v_i(\Gamma) > v_i(\Delta)$, or (ii) $\text{sum}_i^F(\Gamma) > \text{sum}_i^F(\Delta)$.

In case (i), $\text{sum}_i^F(\Gamma) = \text{sum}_i^F(\Delta)$ together with (1), (2), and $v_j(\Gamma) \geq v_j(\Delta)$ implies $\text{sum}_i^{F'}(\Gamma) \geq \text{sum}_i^{F'}(\Delta)$. Further, $v_i(\Gamma) > v_i(\Delta)$ together with $v_i'(\Gamma) = v_i(\Gamma) + n + 1$ and $v_i'(\Delta) = v_i(\Delta) + n + 1$ implies $v_i'(\Gamma) > v_i'(\Delta)$. Altogether, this implies $\Gamma \succ_i^{sumAL'} \Delta$.

In case (ii), $\text{sum}_i^{F'}(\Gamma) > \text{sum}_i^{F'}(\Delta)$ is implied and $\Gamma \succ_i^{sumAL'} \Delta$ follows.

If $\Gamma \sim_i^{sumAL} \Delta$ then $\text{sum}_i^F(\Gamma) = \text{sum}_i^F(\Delta)$ and $v_i(\Gamma) = v_i(\Delta)$. Using the same equations as before, $\Gamma \succeq_i^{sumAL'} \Delta$ is implied.

Type-II-monotonicity under sum-based and min-based preferences.
Let $j \in \Gamma(i) \setminus \Delta(i)$ and $v_j(\Gamma) \geq v_j(\Delta)$. It follows that $v_i'(\Gamma) = v_i(\Gamma)+n+1$, $v_i'(\Delta) = v_i(\Delta)$, $v_j'(\Gamma) = v_j(\Gamma) + n + 1$, and $v_j'(\Delta) = v_j(\Delta)$. Furthermore,

$$\text{sum}_i^{F'}(\Gamma) = \text{sum}_i^F(\Gamma) + v_j(\Gamma) + n + 1, \tag{3}$$

$$\text{sum}_i^{F'}(\Delta) = \text{sum}_i^F(\Delta) + v_j(\Delta), \tag{4}$$

$$\min_i^{F'}(\Gamma) = \min\big(\min_i^F(\Gamma), v_j(\Gamma) + n + 1\big), \tag{5}$$

$$\min_i^{F'}(\Delta) = \min\big(\min_i^F(\Delta), v_j(\Delta)\big), \tag{6}$$

$$\min_i^{F+'}(\Gamma) = \min\big(\min_i^F(\Gamma), v_j(\Gamma) + n + 1, v_i(\Gamma) + n + 1\big), \quad \text{and} \tag{7}$$

$$\min_i^{F+'}(\Delta) = \min\big(\min_i^F(\Delta), v_j(\Delta), v_i(\Delta)\big). \tag{8}$$

(1) sumSF and minSF: If $\Gamma \succeq_i^{SF} \Delta$ then $v_i(\Gamma) \geq v_i(\Delta)$. Hence, $v_i'(\Gamma) = v_i(\Gamma)+n+1 \geq v_i(\Delta) + n + 1 > v_i(\Delta) = v_i'(\Delta)$, which implies $\Gamma \succ_i^{SF'} \Delta$.

(2) sumEQ: If $\Gamma \succeq_i^{sumEQ} \Delta$ then $\text{sum}_i^F(\Gamma) + v_i(\Gamma) \geq \text{sum}_i^F(\Delta) + v_i(\Delta)$. Together with (3), (4), and $v_j(\Gamma) \geq v_j(\Delta)$ this implies $\text{sum}_i^{F'}(\Gamma) + v_j'(\Gamma) > \text{sum}_i^{F'}(\Delta) + v_j'(\Delta)$. Hence, $\Gamma \succ_i^{sumEQ'} \Delta$.

(3) sumAL: If $\Gamma \succeq_i^{sumAL} \Delta$ then $\text{sum}_i^F(\Gamma) \geq \text{sum}_i^F(\Delta)$. Together with (3), (4), and $v_j(\Gamma) \geq v_j(\Delta)$ this implies $\text{sum}_i^{F'}(\Gamma) > \text{sum}_i^{F'}(\Delta)$, so $\Gamma \succ_i^{sumAL'} \Delta$.

(4) minEQ: First, assume that $\Gamma \succ_i^{minEQ} \Delta$. We then have $\min\big(\min_i^F(\Gamma), v_i(\Gamma)\big) > \min\big(\min_i^F(\Delta), v_i(\Delta)\big)$. It follows that $\Gamma \succ_i^{minEQ'} \Delta$ because

$$\begin{aligned}\min_i^{F+'}(\Gamma) &= \min\big(\min_i^F(\Gamma), v_j(\Gamma) + n + 1, v_i(\Gamma) + n + 1\big) \\ &> \min\big(\min_i^F(\Delta), v_j(\Gamma), v_i(\Delta)\big) \geq \min\big(\min_i^F(\Delta), v_j(\Delta), v_i(\Delta)\big) = \min_i^{F+'}(\Delta).\end{aligned} \tag{9}$$

Second, assume $\Gamma \sim_i^{minEQ} \Delta$. Then $\min\big(\min_i^F(\Gamma), v_i(\Gamma)\big) = \min\big(\min_i^F(\Delta), v_i(\Delta)\big)$. Similarly as in (9), it follows that $\min_i^{F+'}(\Gamma) \geq \min_i^{F+'}(\Delta)$. Hence, $\Gamma \succeq_i^{minEQ'} \Delta$.

(5) minAL: First, assume $\Gamma \succ_i^{minAL} \Delta$. Then either (i) $\min_i^F(\Gamma) > \min_i^F(\Delta)$, or (ii) $\min_i^F(\Gamma) = \min_i^F(\Delta)$ and $v_i(\Gamma) > v_i(\Delta)$.

In case of (i), we get $\Gamma \succ_i^{minAL'} \Delta$ because

$$\begin{aligned}\min_i^{F'}(\Gamma) &= \min\big(\min_i^F(\Gamma), v_j(\Gamma) + n + 1\big) \geq \min\big(\min_i^F(\Gamma), v_j(\Delta) + n + 1\big) \\ &> \min\big(\min_i^F(\Delta), v_j(\Delta)\big) = \min_i^{F'}(\Delta).\end{aligned} \tag{10}$$

In case of (ii), similarly as in (10), we get $\min_i^{F'}(\Gamma) \geq \min_i^{F'}(\Delta)$. Furthermore, $v_i(\Gamma) > v_i(\Delta)$ implies $v_i'(\Gamma) > v_i'(\Delta)$. Hence, $\Gamma \succ_i^{minAL'} \Delta$.

Second, assume that $\Gamma \sim_i^{minAL} \Delta$. Then $\min_i^F(\Gamma) = \min_i^F(\Delta)$ and $v_i(\Gamma) = v_i(\Delta)$. Similarly as in (10), we get $\min_i^{F'}(\Gamma) \geq \min_i^{F'}(\Delta)$. Furthermore, $v_i(\Gamma) = v_i(\Delta)$ implies $v_i'(\Gamma) > v_i'(\Delta)$. Hence, $\Gamma \succ_i^{minAL'} \Delta$.

Type-I-monotonicity under min-based preferences.
To see that $\succeq^{minSF}$, $\succeq^{minEQ}$, and $\succeq^{minAL}$ are not type-I-monotonic, consider the game $\mathscr{G}$ with the network of friends in Fig. 3a. Furthermore, consider the coalition structures $\Gamma = \{\{1,2,3,4,5\},\{6\}\}$ and $\Delta = \{\{1,2,3,4,6\},\{5\}\}$ and players $i = 1$ and $j = 2$ with $2 \in \Gamma(1) \cap \Delta(1)$ and $v_2(\Gamma) = -4 = v_2(\Delta)$. It holds that $v_1(\Gamma) = v_1(\Delta) = 10$,$\min_1^F(\Gamma) = \min_1^{F+}(\Gamma) = 10$, and $\min_1^F(\Delta) = \min_1^{F+}(\Delta) = 3$. Hence, $\Gamma \succ_1^{minSF} \Delta$, $\Gamma \succ_1^{minEQ} \Delta$, and $\Gamma \succ_1^{minAL} \Delta$.

Now, making 2 a friend of 1's leads to the game $\mathscr{G}'$ with the network of friends in Fig. 3b. For this game, we have $v_1'(\Gamma) = v_1'(\Delta) = 17$ and $\min_1^{F'}(\Gamma) = \min_1^{F+'}(\Gamma) = \min_1^{F'}(\Delta) = \min_1^{F+'}(\Delta) = 3$. This implies $\Gamma \sim_1^{minSF'} \Delta$, $\Gamma \sim_1^{minEQ'} \Delta$, and $\Gamma \sim_1^{minAL'} \Delta$, which contradicts type-I-monotonicity. This completes the proof. □

Note that the hedonic models of altruism [1, 5] violate both type-I- and type-II-monotonicity for EQ and AL. Hence, it is quite remarkable that all three degrees of our extended sum-based model of altruism satisfy both types of monotonicity.

4 Stability in ACFGs

The main question in coalition formation games is which coalition structures might form. There are several stability concepts that are well-studied for hedonic games, each indicating whether a given coalition structure would be accepted by the agents or if there are other coalition structures that are more likely to form. Although we consider more general coalition formation games, we can easily adapt these definitions to our framework.

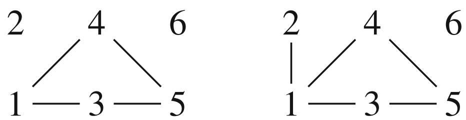
Let $(N, \succeq)$ be an ACFG with preferences $\succeq = (\succeq_1, \dots, \succeq_n)$ obtained from a network of friends via one of the three degrees of altruism and with either sum-based or min-based aggregation of the agents' valuations. We use the following notation. For a coalition structure $\Gamma \in \mathscr{C}_N$, a player $i \in N$, and a coalition $C \in \Gamma \cup \{\emptyset\}$, $\Gamma_{i \to C}$ denotes the coalition structure that arises from Γ when moving i to C, i.e.,

$$\Gamma_{i \to C} = \Gamma \setminus \{\Gamma(i), C\} \cup \{\Gamma(i) \setminus \{i\}, C \cup \{i\}\}.$$

In addition, we use $\Gamma_{C \to \emptyset}$, with $C \subseteq N$, to denote the coalition structure that arises from Γ when all players in C leave their respective coalition and form a new one, i.e.,

$$\Gamma_{C \to \emptyset} = \Gamma \setminus \{\Gamma(j) \mid j \in C\} \cup \{\Gamma(j) \setminus C \mid j \in C\} \cup \{C\}.$$

Finally, for any two coalition structures $\Gamma, \Delta \in \mathscr{C}_N$, let $\#_{\Gamma \succ \Delta} = |\{i \in N \mid \Gamma \succ_i \Delta\}|$ be the number of players that prefer Γ to Δ. Now, we are ready to define the common stability notions.



(a) Network of $\mathscr{G}$ (b) Network of $\mathscr{G}'$

Fig. 3 Networks of friends in the proof of Theorem 1

Definition 2 A coalition structure Γ is said to be

- *Nash stable* (NS) if no player prefers moving to another coalition:

$$(\forall i \in N)(\forall C \in \Gamma \cup \{\emptyset\})[\Gamma \succeq_i \Gamma_{i \to C}];$$

- *individually rational* (IR) if no player would prefer being alone:

$$(\forall i \in N)[\Gamma \succeq_i \Gamma_{i \to \emptyset}];$$

- *core stable* (CS) if no nonempty coalition blocks Γ:

$$(\forall C \subseteq N, C \neq \emptyset)(\exists i \in C)[\Gamma \succeq_i \Gamma_{C \to \emptyset}];$$

- *strictly core stable* (SCS) if no coalition weakly blocks Γ:

$$(\forall C \subseteq N)\big[(\exists i \in C)[\Gamma \succ_i \Gamma_{C \to \emptyset}] \vee (\forall i \in C)[\Gamma \sim_i \Gamma_{C \to \emptyset}]\big];$$

- *popular* (POP) if for every other coalition structure Δ, at least as many players prefer Γ to Δ as there are players who prefer Δ to Γ:

$$(\forall \Delta \in \mathscr{C}_N, \Delta \neq \Gamma)\big[\#_{\Gamma \succ \Delta} \geq \#_{\Delta \succ \Gamma}\big];$$

- *strictly popular* (SPOP) if for every other coalition structure Δ, more players prefer Γ to Δ than there are players who prefer Δ to Γ:

$$(\forall \Delta \in \mathscr{C}_N, \Delta \neq \Gamma)\big[\#_{\Gamma \succ \Delta} > \#_{\Delta \succ \Gamma}\big];$$

- *perfect* (PF) if no player prefers any coalition structure to Γ:

$$(\forall i \in N)(\forall \Delta \in \mathscr{C}_N)[\Gamma \succeq_i \Delta].$$

We now study the associated *verification* and *existence problems* in terms of their computational complexity. We assume the reader to be familiar with the complexity classes P (deterministic polynomial time), NP (nondeterministic polynomial time) and coNP (the class of complements of NP sets). For more background on computational complexity, we refer to, e.g., the textbooks by Garey and Johnson [41] and Rothe [42]. Given a stability concept α, we define:

- α- Verification: Given an ACFG $(N, \succeq)$ and a coalition structure $\Gamma \in \mathscr{C}_N$, does Γ satisfy α?
- α- Existence: Given an ACFG $(N, \succeq)$, does there exist a coalition structure $\Gamma \in \mathscr{C}_N$ that satisfies α?

Table 3 summarizes our results for these problems.

4.1 Individual rationality

Verifying individual rationality is easy: We just need to iterate over all agents and compare two coalition structures in each iteration. Since players' utilities can be computed in polynomial time, individual rationality can be verified in time polynomial in the number of agents. The existence problem is trivial, since $\Gamma = \{\{1\}, \dots, \{n\}\}$ is always individually rational. Furthermore, we give the following characterizations.

Table 3 Overview of complexity results

	α-Verification						α-Existence					
	sum-based			min-based			sum-based			min-based		
α	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*	*SF*	*EQ*	*AL*
IR	P	P	P	P	P	P	✓	✓	✓	✓	✓	✓
NS	P	P	P	P	P	P	✓	✓	✓	✓	✓	✓
CS	coNP-c	coNP	coNP	coNP-c	coNP	coNP	✓	Σ_2^p	Σ_2^p	✓	Σ_2^p	Σ_2^p
SCS	coNP	coNP	coNP	coNP	coNP	coNP	✓	Σ_2^p	Σ_2^p	✓	Σ_2^p	Σ_2^p
POP	coNP-c	coNP	coNP	coNP-c	coNP	coNP	✗	✗	✗	✗	✗	✗
SPOP	coNP-c	coNP	coNP	coNP-c	coNP	coNP	coNP-h	✗	✗	coNP-h	✗	✗
PF	P	coNP	coNP	P	coNP	coNP	P	coNP	✗	P	✗	✗

"P" means that the problem is in P; "coNP" means that the problem is in coNP; "coNP-h" means that the problem is coNP-hard; "coNP-c" means that the problem is coNP-complete; "✓" means that the answer to the problem is always yes, so the problem is trivially in P; "✗" means that the problem is provably nontrivial (but, of course, in Σ_2^p); and "Σ_2^p" means that the problem is in Σ_2^p (and it is open whether it is trivial or not). Gray entries indicate open questions

Theorem 2 *Given an ACFG $(N, \succeq)$, a coalition structure $\Gamma \in \mathscr{C}_N$ is individually rational*

1. *under sum-based SF, sum-based EQ, sum-based AL, min-based SF, or min-based AL preferences if and only if it holds for all players $i \in N$ that $\Gamma(i)$ contains a friend of i's or i is alone, formally: $(\forall i \in N)[\Gamma(i) \cap F_i \neq \emptyset \vee \Gamma(i) = \{i\}]$;*
2. *under min-based EQ preferences if and only if for all players $i \in N$, $\Gamma(i)$ contains a friend of i's or i is alone or there is a friend of i's whose valuation of Γ is less than or equal to i's valuation of Γ, formally: $(\forall i \in N)[\Gamma(i) \cap F_i \neq \emptyset \vee \Gamma(i) = \{i\} \vee (\exists j \in F_i)[v_j(\Gamma) \leq v_i(\Gamma)]]$.*[5]

Proof 1. To show the implication from left to right, if Γ is individually rational, we assume for the sake of contradiction that $\Gamma(i) \cap F_i = \emptyset$ and $\Gamma(i) \neq \{i\}$for some player $i \in N$. First, we observe that for all $j \in F_i$ we have $v_j(\Gamma) = v_j(\Gamma_{i \to \emptyset})$, as their respective coalition is not affected by i's move. It directly follows that, for all considered models of altruism, player i's utilities for Γ and $\Gamma_{i \to \emptyset}$ only depend on her own valuation, which is greater for $\Gamma_{i \to \emptyset}$ than for Γ (since there are enemies in $\Gamma(i)$ but not in $\Gamma_{i \to \emptyset}(i)$). Hence, i prefers $\Gamma_{i \to \emptyset}$ to Γ, so Γ is not individually rational. This is a contradiction.
From right to left, assume that $\Gamma(i) \cap F_i \neq \emptyset$ or $\Gamma(i) = \{i\}$ for all players $i \in N$. We will now show that all players weakly prefer Γ to $\Gamma_{i \to \emptyset}$ which means that Γ is individually rational. For all players $i \in N$ with $\Gamma(i) = \{i\}$, it holds that $\Gamma = \Gamma_{i \to \emptyset}$ which obviously implies that i is indifferent between Γ and $\Gamma_{i \to \emptyset}$. For all players $i \in N$ with $\Gamma(i) \cap F_i \neq \emptyset$, we know that i and all her friends assign to Γ a value greater than or equal to the valuethey assign to $\Gamma_{i \to \emptyset}$: Specifically, $v_i(\Gamma_{i \to \emptyset}) = 0 < v_i(\Gamma)$ because i has a friend in $\Gamma(i)$; for all $f \in \Gamma(i) \cap F_i$, $v_f(\Gamma_{i \to \emptyset}) = v_f(\Gamma) - n < v_f(\Gamma)$ since they lose i as a friend; and for all $f \in F_i \setminus \Gamma(i)$, $v_f(\Gamma_{i \to \emptyset}) = v_f(\Gamma)$ since their coalitions stay the same if i deviates. This directly implies that i's utility for Γ is greater than or equal to her utility for $\Gamma_{i \to \emptyset}$, and she thus weakly prefers Γ to $\Gamma_{i \to \emptyset}$. In total, we have shown that all players weakly prefer Γ to $\Gamma_{i \to \emptyset}$. Hence, Γ is individually rational.

2. From left to right, we have that Γ is individually rational and, for the sake of contradiction, we assume that there is a player $i \in N$ with $\Gamma(i) \cap F_i = \emptyset$ and $\Gamma(i) \neq \{i\}$ and for all $j \in F_i$ we have $v_j(\Gamma) > v_i(\Gamma)$. Since i is the least satisfied player in $F_i \cup \{i\}$, we have $u_i^{minEQ}(\Gamma) = v_i(\Gamma)$. With $v_j(\Gamma_{i \to \emptyset}) = v_j(\Gamma) > v_i(\Gamma)$ for all $j \in F_i$ and $v_i(\Gamma_{i \to \emptyset}) = 0 > v_i(\Gamma)$, we immediately obtain $u_i^{minEQ}(\Gamma_{i \to \emptyset}) > u_i^{minEQ}(\Gamma)$ and thus $\Gamma_{i \to \emptyset} \succ_i^{minEQ} \Gamma$. This is a contradiction to Γ being individually rational.
From right to left, we have to consider two cases. First, if $\Gamma(i) \cap F_i \neq \emptyset$ or $\Gamma(i) = \{i\}$ for some $i \in N$, we obviously have $\Gamma \succeq_i^{minEQ} \Gamma_{i \to \emptyset}$. Second, if $\Gamma(i) \cap F_i = \emptyset$ and $\Gamma(i) \neq \{i\}$, we know that there is at least one $j \in F_i$ with $v_j(\Gamma) \leq v_i(\Gamma) < 0$. Let j' denote a least satisfied friend of i's in Γ (pick one randomly if there are more than one). Since $\Gamma(i) \cap F_i = \emptyset$, it holds that $\Gamma(j) = \Gamma_{i \to \emptyset}(j)$ for all $j \in F_i$. Consequently, j' is i's least satisfied friend in both coalition structures and we have $u_i^{minEQ}(\Gamma) = v_{j'}(\Gamma) = v_{j'}(\Gamma_{i \to \emptyset}) = u_i^{minEQ}(\Gamma_{i \to \emptyset})$. Hence, $\Gamma \sim_i^{minEQ} \Gamma_{i \to \emptyset}$, so Γ is individually rational. □

[5] Note that the last part of the disjunction is only necessary for min-based EQ preferences and not for the other preference models: In the other models, increasing the valuation of player i while not altering her friends' valuations will certainly increase i's altruistic utility. Yet, under min-based EQ, it might be the case that the increase of i's valuation has no impact on her utility since there can be another friend who represents the minimum in her utility function. Thus it can happen that i has no inventive to deviate to a singleton coalition when one of her friends has a valuation that is less than or equal to hers.

4.2 Nash stability

Since there are at most $|N|$ coalitions in a coalition structure $\Gamma \in \mathcal{C}_N$, we can verify Nash stability in polynomial time: We just iterate over all agents $i \in N$ and all the (at most $|N| + 1$) coalitions $C \in \Gamma \cup \{\emptyset\}$ and check whether $\Gamma \succeq_i \Gamma_{i \to C}$. Since we can check a player's altruistic preferences over any two coalition structures in polynomial time and since we have at most a quadratic number ofiterations $(|N| \cdot (|N| + 1))$, Nash stability verification is in P for any ACFG.

Nash stability existence is trivially in P for any ACFG; indeed, the same example that Nguyen et al. [1] gave for altruistic hedonic games works here as well. Specifically, for $C = \{i \in N \mid F_i = \emptyset\} = \{c_1, \dots, c_k\}$, the coalition structure $\{\{c_1\}, \dots, \{c_k\}, N \setminus C\}$ is Nash stable as no player has an incentive to deviate to another coalition. Interestingly, the same coalition structure is also Nash stable under friend-oriented preferences. This intuitively makes sense because, under all these preference models, no agent wants to leave some friends and deviate to a coalition that only contains enemies.

Note that Nash stability implies several other common stability notions such as *individual stability* [25], *contractually individual stability* [25], and *contractual Nash stability* [43]. The above existence result naturally transfers to these weaker notions. Also, similar to Nash stability, these notions can be verified in polynomial time for any ACFG.

4.3 Core stability and strict core stability

We now turn to core stability and state some results for sum-based and min-based SF ACFGs. We first show that (strict) core stability existence is trivial for SF ACFGs.

Theorem 3 *Let $(N, \succeq^{SF})$ be a (sum-based or min-based) SF ACFG with the underlying network of friends G. Let further $C_1, \dots, C_k$ be the vertex sets of the connected components of G. Then $\Gamma = \{C_1, \dots, C_k\}$ is strictly core stable (and thus core stable).*

Proof For the sake of contradiction, assume that Γ were not strictly core stable, i.e., that there is a coalition $D \neq \emptyset$ that weakly blocks Γ. Consider some player $i \in D$. Since i weakly prefers deviating from $\Gamma(i)$ to D, there have to be at least as many friends of i's in D as in $\Gamma(i)$. Since $\Gamma(i)$ contains all of i's friends, D also has to contain all friends of i's. Then all these friends of i's also have all their friends in D for the same reason, and so on. Consequently, D contains all players from the connected component $\Gamma(i)$, i.e., $\Gamma(i) \subseteq D$.

Since D weakly blocks Γ, D cannot be equal to $\Gamma(i)$ and thus needs to contain some $\ell \notin \Gamma(i)$. Yet, this is a contradiction, as ℓ is an enemy of i's and i would prefer Γ to $\Gamma_{D \to \emptyset}$ if D contains the same number of friends as $\Gamma(i)$ but more enemies than $\Gamma(i)$. □

However, the coalition structure from Theorem 3 is not necessarily core stable under EQ and AL preferences.

Example 3 Let $N = \{1, \dots, 10\}$ and consider the network of friends G shown in Fig. 4. Consider the coalition structure consisting of the connected component of G (i.e., of only the grand coalition: $\Gamma = \{N\}$) and the coalition $C = \{8, 9, 10\}$. C blocks Γ under sum-based

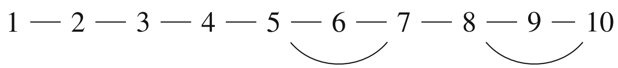


Fig. 4 Networks of friends for Example 3

and min-based EQ and AL preferences. To see this, consider how players 7, 8, 9, and 10 value Γ and $\Gamma_{C\to\emptyset}$:

$$v_7(\Gamma) = v_8(\Gamma) = 30 - 6 = 24, \qquad v_7(\Gamma_{C\to\emptyset}) = 20 - 4 = 16,$$
$$v_9(\Gamma) = v_{10}(\Gamma) = 20 - 7 = 13, \qquad v_8(\Gamma_{C\to\emptyset}) = v_9(\Gamma_{C\to\emptyset}) = v_{10}(\Gamma_{C\to\emptyset}) = 20.$$

We then obtain

- $\mathrm{sum}_8^{F+}(\Gamma) = 74 < 76 = \mathrm{sum}_8^{F+}(\Gamma_{C\to\emptyset})$ and $\mathrm{sum}_9^{F+}(\Gamma) = \mathrm{sum}_{10}^{F+}(\Gamma) = 50 < 60 = \mathrm{sum}_9^{F+}(\Gamma_{C\to\emptyset}) = \mathrm{sum}_{10}^{F+}(\Gamma_{C\to\emptyset})$, so $\Gamma_{C\to\emptyset} \succ_i^{sumEQ} \Gamma$ for all $i \in C$;
- $\mathrm{sum}_8^{F}(\Gamma) = 50 < 56 = \mathrm{sum}_8^{F}(\Gamma_{C\to\emptyset})$ and $\mathrm{sum}_9^{F}(\Gamma) = \mathrm{sum}_{10}^{F}(\Gamma) = 37 < 40 = \mathrm{sum}_9^{F}(\Gamma_{C\to\emptyset}) = \mathrm{sum}_{10}^{F}(\Gamma_{C\to\emptyset})$,so $\Gamma_{C\to\emptyset} \succ_i^{sumAL} \Gamma$ for all $i \in C$;
- $\min_8^{F+}(\Gamma) = \min_8^{F}(\Gamma) = 13 < 16 = \min_8^{F+}(\Gamma_{C\to\emptyset}) = \min_8^{F}(\Gamma_{C\to\emptyset})$ and $\min_9^{F+}(\Gamma) = \min_9^{F}(\Gamma) = \min_{10}^{F+}(\Gamma) = \min_{10}^{F}(\Gamma) = 13 < 20 = \min_9^{F+}(\Gamma_{C\to\emptyset}) = \min_9^{F}(\Gamma_{C\to\emptyset}) = \min_{10}^{F+}(\Gamma_{C\to\emptyset}) = \min_{10}^{F}(\Gamma_{C\to\emptyset})$, which implies $\Gamma_{C\to\emptyset} \succ_i^{minEQ} \Gamma$ and $\Gamma_{C\to\emptyset} \succ_i^{minAL} \Gamma$ for all $i \in C$.

Thus C blocks Γ under sum-based and min-based EQ and AL preferences.

Turning to core stability verification, we can show that this problem is hard under SF preferences, and we strongly suspect that this hardness also extends to EQ and AL. Note that even though we have shown in Theorem 3 that the 'connected components coalition structure' is core stable for any SF ACFG, verifying core stability in general (for other coalition structures) is a coNP-complete problem.

Proposition 2 *Strict core stability and core stability verification are in* coNP *for any ACFG.*

Proof To see that strict core stability verification and core stability verification are in coNP, consider any coalition structure $\Gamma \in \mathscr{C}_N$ in an ACFG $(N, \succeq)$. Γ is not (strictly) core stable if there is a coalition $C \subseteq N$ that (weakly) blocks Γ. Hence, we nondeterministically guess a coalition $C \subseteq N$ and check whether C (weakly) blocks Γ. This can be done in polynomial time since the preferences of the agents in C for the coalition structures Γ and $\Gamma_{C\to\emptyset}$ can be verified in polynomial time for all our altruistic models. □

We start with showing that core stability verification is coNP-complete for min-based SF ACFGs and will then continue with sum-based SF ACFGs in Theorem 5.

Theorem 4 *For min-based SF ACFGs, core stability verification is* coNP*-complete.*

Proof To show coNP-hardness of core stability verification under min-based SF ACFGs, we use RX3C, which is a restricted variant of Exact Cover by 3- Sets and known to be NP-complete [41, 44]. We provide a polynomial-time many-one reduction from RX3C to the complement of our verification problem. Let $(B, \mathscr{S})$ be an instance of RX3C, consisting of a set $B = \{1, \ldots, 3k\}$ and a collection $\mathscr{S} = \{S_1, \ldots, S_{3k}\}$ of 3-element subsets of B, where each element of B occurs in exactly three sets in $\mathscr{S}$. The question is whether there exists an exact cover for B in $\mathscr{S}$, i.e., a subset $\mathscr{S}' \subseteq \mathscr{S}$ with $|\mathscr{S}'| = k$ and $\bigcup_{S\in\mathscr{S}'} S = B$. We assume that $k > 4$.

From $(B, \mathscr{S})$ we now construct the following ACFG. The set of players is $N = \{\beta_b \mid b \in B\} \cup \{\zeta_S, \alpha_{S,1}, \alpha_{S,2}, \alpha_{S,3}, \delta_{S,1}, \ldots, \delta_{S,4k-3} \mid S \in \mathscr{S}\}$ and we define the sets

$$Beta = \{\beta_b \mid b \in B\},$$
$$Zeta = \{\zeta_S \mid S \in \mathscr{S}\}, \text{ and}$$
$$Q_S = \{\zeta_S, \alpha_{S,1}, \alpha_{S,2}, \alpha_{S,3}, \delta_{S,1}, \ldots, \delta_{S,4k-3}\} \text{ for each } S \in \mathscr{S}.$$

Figure 5 shows the network of friends, where a dashed rectangle around a group of players means that all these players are friends of each other:

- All players in *Beta* are friends of each other.
- For every $S \in \mathscr{S}$, ζ_S is friend with every β_b with $b \in S$ and with $\alpha_{S,1}$, $\alpha_{S,2}$, and $\alpha_{S,3}$.
- For every $S \in \mathscr{S}$, $\alpha_{S,1}, \alpha_{S,2}, \alpha_{S,3}$, and $\delta_{S,1}$ are friends of each other.
- For every $S \in \mathscr{S}$, all players in $\{\delta_{S,1}, \dots, \delta_{S,4k-3}\}$ are friends of each other.

Furthermore, consider the coalition structure $\Gamma = \{Beta, Q_{S_1}, \dots, Q_{S_{3k}}\}$. We will now show that $\mathscr{S}$ contains an exact cover for B if and only if Γ is not core stable under the min-based SF model. The idea of this proof is that a group of players from *Zeta* who represents an exact cover for B will always have an incentive to deviate together with the players from *Beta*. But this is the only possible deviating group: For any group of players from *Zeta* who do not represent an exact cover (because there are more than k players or they do not completely cover B), there will always be a player not preferring the deviation.

Only if: Assume that there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ for B. Then $|\mathscr{S}'| = k$. Consider coalition $C = Beta \cup \{\zeta_S \mid S \in \mathscr{S}'\}$. C blocks Γ, i.e., $\Gamma_{C\to\emptyset} \succ_i^{minSF} \Gamma$ for all $i \in C$, because (a) every $\beta_b \in Beta$ has $3k$ friends in C but only $3k-1$ friends in *Beta* and (b) every ζ_S with $S \in \mathscr{S}'$ has 3 friends and $4k-4$ enemies in C but 3 friends and $4k-3$ enemies in Q_S.

If: Assume that Γ is not core stable and let $C \subseteq N$ be a coalition that blocks Γ. Then $\Gamma_{C\to\emptyset} \succ_i^{minSF} \Gamma$ for all $i \in C$. First, observe that every $i \in C$ needs to have at least as many friends in C as in $\Gamma(i)$. So, if any $\alpha_{S,j}$ or $\delta_{S,j}$ is in C, it follows quite directly that $Q_S \subseteq C$. However, since Q_S is a coalition in Γ and since every other player (from $N \setminus Q_S$) is an enemy of all δ-players, any coalition C with $Q_S \subseteq C$ cannot be a blocking coalition for Γ. This contradiction implies that no $\alpha_{S,j}$ or $\delta_{S,j}$ is in C.

We now have $C \subseteq Beta \cup Zeta$. Since any $\beta_b \in C$ has $3k-1$ friends and no enemies in $\Gamma(\beta_b)$ and prefers $\Gamma_{C\to\emptyset}$ to Γ, one of the following holds: (a) β_b has at least $3k$ friends in C or (b) β_b has $3k-1$ friends and no enemies in C and β_b's friends assign a higher value to $\Gamma_{C\to\emptyset}$ than to Γ. For a contradiction, assume that (b) holds for some $\beta_b \in C$. First, observe that there are exactly $3k$ players in C (namely,β_b and β_b's $3k-1$ friends). We now distinguish two cases:

Case 1: *All the* $3k-1$ *friends of* β_b*'s are* β*-players.* Then C consists of all β-players, i.e., $C = Beta$. This is a contradiction, as *Beta* is already a coalition in Γ.

Case 2: *There are some* ζ*-players in C that are* β_b*'s friends.* Since β_b has three ζ-friends in total and no enemies in C, there are between one and three ζ-players in C. Hence, there are between $3k-3$ and $3k-1$ β-players in C. Then one of the β-players has no ζ-friend in C. (The at most three ζ-players are friends with at most nine β-players, but $3k-3 > 9$ for

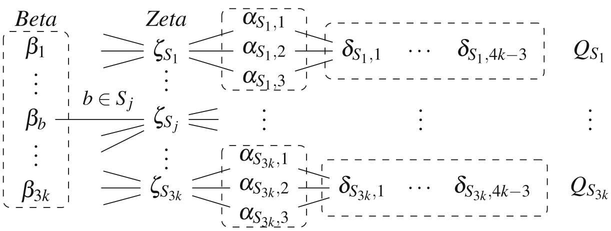


Fig. 5 Network of friends in the proof of Theorem 4 that is used to show coNP-hardness of core stability verification in min-based SF ACFGs. A dashed rectangle around a group of players indicates that all these players are friends of each other

$k > 4$.) Consequently, this β-player has only the other (at most $3k-2$) β-players as friends in C and does not prefer $\Gamma_{C\to\emptyset}$ to Γ. This is a contradiction.

Hence, option (a) holds for each $\beta_b \in C$. In total, each β_b has exactly three ζ-friends and $3k-1$ β-friends. Thus at least $3k-3$ of β_b's friends in C are β-players and at least one of β_b's friends in C is a ζ-player. Also counting β_b herself, there are at least $3k-2$ β-players in C. Since all of these $3k-2$ β-players have at least one ζ-friend in C, there are at least k ζ-players in C. (Note that $k-1$ ζ-players are friends with at most $3k-3$ β-players.)

Consider some $\zeta_S \in C$. Since ζ_S has three friends and $4k-3$ enemies in Q_S, at most three friends in C, and prefers $\Gamma_{C\to\emptyset}$ to Γ, ζ_S has exactly three friends and at most $4k-3$ enemies in C. Hence, C contains at most $4k-3+3+1 = 4k+1$ players.

So far we know that there are at least $3k-2$ β-players in C. If C contains exactly $3k-2$ (or $3k-1$) β-players then each of this players has only $3k-3$ (or $3k-2$) β-friends in C and additionally needs at least three (or two) ζ-friends in C. Hence, we have at least $(3k-2)\cdot 3 = 9k-6$ (or $6k-2$) edges between the β- and ζ-players in C. Then there are at least $3k-2$ (or $2k$) ζ-players in C. Thus there are at least $(3k-2)+(3k-2) = 6k-4$ (or $5k-1$) players in C which is a contradiction because there are at most $4k+1$ players in C. Hence, there are exactly $3k$ β-players in C.

Summing up, there are exactly $3k$ β-players, at least k ζ-players, and at most $4k+1$ players in C. Hence, there are k or $k+1$ ζ-players in C. For the sake of contradiction, assume that there are $k+1$ ζ-players in C. Then each $\zeta_S \in C$ has $4k-3$ enemies in C. Since ζ_S prefers $\Gamma_{C\to\emptyset}$ to Γ, this implies that ζ_S has exactly three friends and $4k-3$ enemies in C and the minimal value assigned to $\Gamma_{C\to\emptyset}$ by ζ_S's friends is higher than the minimal value assigned to Γ by ζ_S's friends. In both coalition structures, the minimal value is given by ζ_S's α-friends. However, since these α-players lose ζ_S as a friend when ζ_S deviates to C, the minimal value assigned to Γ is higher than for $\Gamma_{C\to\emptyset}$. This is a contradiction. Hence, there are exactly k ζ-players in C. Finally, since every of the $3k$ $\beta_b \in C$ has one of the k $\zeta_S \in C$ as a friend, it holds that $\{S \mid \zeta_S \in C\}$ is an exact cover for B. This completes the proof. □

Theorem 5 *For sum-based SF ACFGs, core stability verification is* coNP*-complete.*

Proof For sum-based SF ACFGs, coNP-hardness of core stability verification can be shown by a similar construction as in the proof of Theorem 4. Again, given an instance $(B, \mathscr{S})$ of RX3C, with $B = \{1,\dots,3k\}$, $\mathscr{S} = \{S_1,\dots,S_{3k}\}$, and $k > 8$, we construct the following ACFG. The set of players is $N = \{\beta_b \mid b \in B\} \cup \{\zeta_S, \alpha_{S,1}, \alpha_{S,2}, \alpha_{S,3}, \delta_{S,1},\dots,\delta_{S,4k-3} \mid S \in \mathscr{S}\}$. We define the sets $Beta = \{\beta_b \mid b \in B\}$ and $Q_S = \{\zeta_S, \alpha_{S,1}, \alpha_{S,2}, \alpha_{S,3}, \delta_{S,1},\dots,\delta_{S,4k-3}\}$ for each $S \in \mathscr{S}$. The network of friends is given in Fig. 6, where a dashed rectangle around a group of players means that all these players are friends of each other:

- All players in $Beta$ are friends of each other.
- For every $S \in \mathscr{S}$, all players in Q_S are friends of each other.
- For every $S \in \mathscr{S}$, ζ_S is friend with $\alpha_{S,1}$, $\alpha_{S,2}$, and $\alpha_{S,3}$ and with every β_b with $b \in S$.

Similar arguments as in the proof of Theorem 4 show that the coalition structure $\Gamma = \{Beta\} \cup \{\{\zeta_S\} \cup Q_S \mid S \in \mathscr{S}\}$ is not core stable under sum-based SF preferences if and only if $\mathscr{S}$ contains an exact cover for B. □

4.4 Popularity and strict popularity

Now we take a look at popularity and strict popularity. For all considered models of altruism, there are games for which no (strictly) popular coalition structure exists.

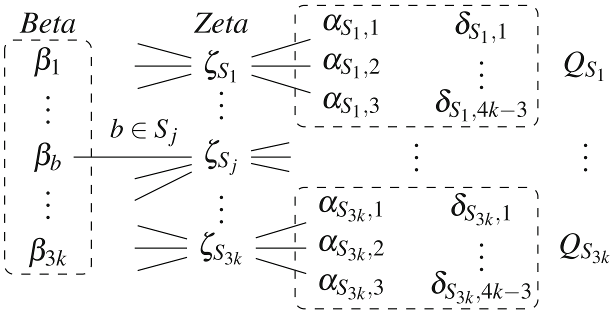


Fig. 6 Network of friends in the proof of Theorem 4 that is used to show coNP-hardness of core stability verification in sum-based SF ACFGs. A dashed rectangle around a group of players indicates that all these players are friends of each other

Example 4 Let $N = \{1, \ldots, 10\}$ and consider the network of friends shown in Fig. 7. Then there is no strictly popular and no popular coalition structure for any of the sum-based or min-based degrees of altruism. Since perfectness implies popularity, there is also no perfect coalition structure for this ACFG. Recall from Footnote 3 that there are 115, 975 possible coalition structures for this game with ten players. We checked this example using brute force by iterating over all these coalition structures and checking whether one of them is popular.

As Example 4 shows, the (strict) popularity and perfectness existence problems are not trivial. In Theorem 9, we will give a simple characterization of when perfect coalition structures exist for SF ACFGs. But before turning to perfectness, we will show that deciding whether there exists a strictly popular coalition structure for a given SF ACFG is coNP-hard. Also, it is coNP-complete to verify if a given coalition structure is (strictly) popular for SF ACFGs.

First, we establish the following upper bound for (strict) popularity verification.

Proposition 3 *For any ACFG, (strict) popularity verification is in* coNP.

Proof We observe that the verification problems are in coNP: To verify that a given coalition structure Γ is not (strictly) popular, we can nondeterministically guess a coalition structure Δ, compare both coalition structures in polynomial time, and accept exactly if Δ is more popular than (or at least as popular as) Γ. □

In Theorem 6, we show that strict popularity verification is also coNP-hard for min-based SF ACFGs. For this, we use a polynomial-time many-one reduction from EXACT COVER BY 3- SETS. In Theorem 7, we will show (by means of a similar construction) that the same result also holds for sum-based SF ACFGs.

Theorem 6 *For min-based SF ACFGs, strict popularity verification is* coNP*-complete.*

Proof To show coNP-hardness of strict popularity verification for min-based SF ACFGs, we again employ a polynomial-time many-one reduction from RX3C. Let $(B, \mathscr{S})$ be an

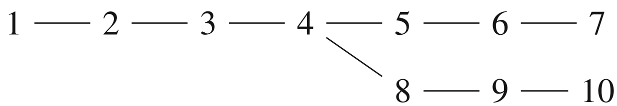


Fig. 7 Network of friends for Example 4

instance of RX3C, consisting of a set $B = \{1, \dots, 3k\}$ and a collection $\mathscr{S} = \{S_1, \cdots, S_{3k}\}$ of 3-element subsets of B. Recall that every element of B occurs in exactly three sets in $\mathscr{S}$ and the question is whether there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ of B.

We now construct a network of friends based on this instance. The set of players is given by $N = \{\alpha_1, \dots, \alpha_{2k}\} \cup \{\beta_b \mid b \in B\} \cup \{\zeta_S, \eta_{S,1}, \eta_{S,2} \mid S \in \mathscr{S}\}$, so in total we have $n = 14k$ players. For convenience, we define $Alpha = \{\alpha_1, \dots, \alpha_{2k}\}$, $Beta = \{\beta_b \mid b \in B\}$, and $Q_S = \{\zeta_S, \eta_{S,1}, \eta_{S,2} \mid S \in \mathscr{S}\}$ for $S \in \mathscr{S}$. The network of friends is shown in Fig. 8, where a dashed square around a group of players means that all these players are friends of each other: All players in $Alpha \cup Beta$ are friends of each other; for every $S \in \mathscr{S}$, all players in Q_S are friends of each other; and ζ_S is a friend of every β_b with $b \in S$.

We consider the coalition structure $\Gamma = \{Alpha \cup Beta\} \cup \{Q_S \mid S \in \mathscr{S}\}$ and will now show that $\mathscr{S}$ contains an exact cover for B if and only if Γ is not strictly popular under min-based SF preferences.

Only if: Assuming that there is an exact cover $\mathscr{S}' \subset \mathscr{S}$ for B, we define the coalition structure $\Delta = \{Alpha \cup Beta \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\}$. We will now show that Δ is as popular as Γ under min-based SF preferences.

First, all $2k$ α-players prefer Γ to Δ, since they only add enemies to their coalition in Δ. Second, the $3k$ β-players prefer Δ to Γ, as each β-player gains a ζ-friend and then has $5k$ friends instead of $5k - 1$. Next, we consider the Q_S-groups for $S \in \mathscr{S}'$, i.e., the groups that were added to $Alpha \cup Beta$ in Δ. We observe that every ζ_S-player in these Q_S-groups prefers Δ to Γ, since ζ_S gains three additional β-friends. For the η-players, on the other hand, the new coalition only contains more enemies, so the η-players prefer Γ to Δ. Since we have $|\mathscr{S}'| = k$, this means k ζ-players prefer Δ to Γ, and $2k$ η-players prefer Γ to Δ. Finally, we consider the remaining Q_S-groups with $S \in \mathscr{S} \setminus \mathscr{S}'$. Here, the coalition containing these players is the same in Γ and Δ. Hence, for each player $p \in Q_S$, we have $v_p(\Gamma) = v_p(\Delta)$. Thus the players have to ask their friends for their valuations. For $\zeta_S \in Q_S$ with $S \in \mathscr{S} \setminus \mathscr{S}'$, the minimum value of her friends is in both structures given by an η-friend, since $\eta_{S,1}$ and $\eta_{S,2}$ value Γ and Δ both with $n \cdot 2$, while the β-friends of ζ_S assign values $n \cdot (5k - 1)$ to Γ and $n \cdot 5k - (3k - 1)$ to Δ. So we have $u_{\zeta_S}^{minSF}(\Gamma) = u_{\zeta_S}^{minSF}(\Delta)$ and, therefore, $2k$ ζ-players that are indifferent. The η-players in Q_S, $S \in \mathscr{S} \setminus \mathscr{S}'$, are also indifferent, as all their friends value Γ and Δ the same. In total, $\#_{\Delta \succ \Gamma} = |Beta \cup \{\zeta_S \mid S \in \mathscr{S}'\}| = 4k = |Alpha \cup \{\eta_{S,1}, \eta_{S,2} \mid S \in \mathscr{S}'\}| = \#_{\Gamma \succ \Delta}$ and, therefore, Δ is exactly as popular as Γ, so Γ is not strictly popular.

If: Assuming that Γ is not strictly popular, there is some coalition structure $\Delta \in \mathscr{C}_N$ with $\Delta \neq \Gamma$ such that Δ is at least as popular as Γ under min-based SF preferences. We will now show that this implies the existence of an exact cover for B in $\mathscr{S}$.

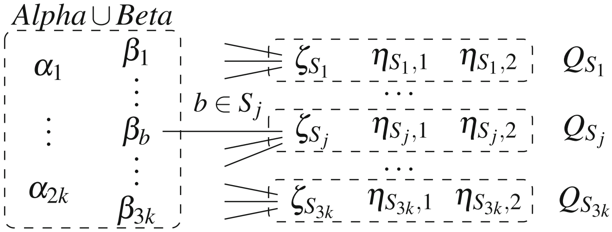


Fig. 8 Network of friends in the proof of Theorem 6 that is used to show coNP-hardness of strict popularity verification in min-based SF ACFGs. A dashed rectangle around a group of players indicates that all these players are friends of each other

First of all, we observe that all α-players' most preferred coalition is $Alpha \cup Beta$, as it contains all their friends and no enemies. Thus we have $\Gamma \succ_{\alpha}^{minSF} \Delta$ if $Alpha \cup Beta \notin \Delta$ and $\Gamma \sim_{\alpha}^{minSF} \Delta$ if $Alpha \cup Beta \in \Delta$.

For the sake of contradiction, we assume that $Alpha \cup Beta \in \Delta$. As $\Delta \neq \Gamma$, the players in the Q_S-groups have to be partitioned differently. However, that would not increase any player's valuation since every player in Q_S can only lose friends and gain enemies. That means that no β-player prefers Δ to Γ, as they are in the same coalition as in Γ and their friends are not more satisfied. We also have at least three players of a Q_S-group that are no longer in the same coalition, so they prefer Γ to Δ. This is a contradiction, as we assumed that Δ is at least as popular as Γ. Thus we have $Alpha \cup Beta \notin \Delta$.

Now consider the η-players. For every $S \in \mathscr{S}$, we know that Q_S is the best valued coalition for $\eta_{S,1}$ and $\eta_{S,2}$. So again, $\eta_{S,1}$ and $\eta_{S,2}$ prefer Γ to Δ if and only if $Q_S \notin \Delta$, and they are indifferent otherwise. Define $k' = |\{S \in \mathscr{S} \mid Q_S \notin \Delta\}|$. So $2k'$ is the number of η-players that prefer Γ to Δ, and the remaining $6k - 2k'$ η-players are indifferent between Γ and Δ. We first collect some observations:

1. All $2k$ α-players prefer Γ to Δ.
2. $2k'$ η-players prefer Γ to Δ, and $6k - 2k'$ η-players are indifferent.
3. $3k - k'$ ζ-players are in the same coalition in both coalition structures, so their utilities depend on their friends' valuations. In Γ, the minimum value of their friends is given by an η-player. Since this η-player is also in the same coalition in Δ and thus assigns the same value, it is not possible that the minimum value of the friends is higher in Δ than in Γ. So $3k - k'$ ζ-players are indifferent or prefer Γ to Δ.
4. We have $14k$ players in total, so we can have at most $14k - 2k - 2k' - (6k - 2k') - (3k - k') = 3k + k'$ players that prefer Δ to Γ.

Next, we show that $k' = k$. First, assume that $k' > k$: We have $\#_{\Gamma \succ \Delta} \geq 2k + 2k'$, and since $k' > k$, we have $2k + 2k' > 3k + k' \geq \#_{\Delta \succ \Gamma}$. This is a contradiction to $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$, so we obtain $k' \leq k$.

Second, let us assume $k' < k$: Since every ζ-player has three β-friends and there are k' ζ-players that are not in their respective Q_S coalition in Δ, there are at most $3k'$ β-players that gain a ζ-friend in Δ. The $3k - 3k'$ other β-players have at most $5k - 1$ friends in Δ, namely all other α- and β-players. But as $Alpha \cup Beta \notin \Delta$, they would also gain at least one enemy, so we have $3k - 3k'$ β-players that prefer Γ. That means we have $\#_{\Gamma \succ \Delta} \geq 2k + 2k' + 3k - 3k' = 5k - k'$ and $\#_{\Delta \succ \Gamma} \leq 3k + k' - (3k - 3k') = 4k'$. Since $k' < k$, we have $5k - k' > 5k - k = 4k > 4k'$, and therefore, $\#_{\Gamma \succ \Delta} > \#_{\Delta \succ \Gamma}$, which again is a contradiction. Thus we conclude that $k' \geq k$ and, in total, $k' = k$.

Consequently, we know that $4k$ players prefer Γ to Δ, namely all α-players and the $2k$ η-players that are not in Q_S anymore. Subtracting all the indifferent players, we observe that all other players have to prefer Δ to Γ in order to ensure $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$. These other players are the $3k$ β-players and the k ζ-players that are not in Q_S anymore. Finally, that is only possible if every β-player gains a ζ-friend in Δ. Hence each one of those k ζ-players has to be friends with three different β-players. Therefore, the set $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover for B. □

Theorem 7 *For sum-based SF ACFGs, strict popularity verification is* coNP*-complete.*

Proof To show coNP-hardness of strict popularity verification for sum-based SF ACFGs, we use a similar construction as in the proof of Theorem 6.

For an instance $(B, \mathscr{S})$ of RX3C with $B = \{1, \dots, 3k\}$ and $\mathscr{S} = \{S_1, \dots, S_{3k}\}$, where each element of B occurs in exactly three sets in $\mathscr{S}$, we construct the following ACFG.

The set of players is given by $N = \{\alpha_1, \ldots, \alpha_{5k}\} \cup \{\beta_b \mid b \in B\} \cup \{\zeta_S, \eta_S \mid S \in \mathscr{S}\}$. Let $Alpha = \{\alpha_1, \ldots, \alpha_{5k}\}$, $Beta = \{\beta_b \mid b \in B\}$, and $Q_S = \{\zeta_S, \eta_S\}$ for each $S \in \mathscr{S}$. The network of friends is given in Fig. 9, where a dashed rectangle around a group of players means that all these players are friends of each other: All players in $Alpha \cup Beta$ are friends of each other and, for every $S \in \mathscr{S}$, ζ_S is friends with η_S and every β_b with $b \in S$.

Consider the coalition structure $\Gamma = \{Alpha \cup Beta, Q_{S_1}, \ldots, Q_{S_{3k}}\}$. We show that $\mathscr{S}$ contains an exact cover for B if and only if Γ is not strictly popular.

Only if: Assuming that there is an exact cover $\mathscr{S}' \subseteq \mathscr{S}$ for B and considering coalition structure $\Delta = \{Alpha \cup Beta \cup \bigcup_{S \in \mathscr{S}'} Q_S\} \cup \{Q_S \mid S \in \mathscr{S} \setminus \mathscr{S}'\}$, it can be shown with similar arguments as before that $\#_{\Delta \succ \Gamma} = |\{\beta_1, \ldots, \beta_{3k}, \zeta_{S_1}, \ldots, \zeta_{S_{3k}}\}| = 6k = |\{\alpha_1, \ldots, \alpha_{5k}\} \cup \{\eta_S \mid S \in \mathscr{S}'\}| = \#_{\Gamma \succ \Delta}$. Hence, Δ and Γ are equally popular.

If: Assuming that Γ is not strictly popular, i.e., that there is a coalition structure $\Delta \in \mathcal{C}_N$, $\Delta \neq \Gamma$, with $\#_{\Gamma \succ \Delta} \leq \#_{\Delta \succ \Gamma}$, it can be shown similarly as before (as in the proof of Theorem 6) that the set $\{S \in \mathscr{S} \mid Q_S \notin \Delta\}$ is an exact cover for B. □

Extending the proofs of Theorems 6 and 7, we now also show that strict popularity existence and popularity verification are coNP-hard for min-based and sum-based SF ACFGs.

Theorem 8 *For (sum-based and min-based) SF ACFGs, strict popularity existence is* coNP-*hard and popularity verification is* coNP-*complete.*

Proof To show that strict popularity existence is coNP-hard for min-based and sum-based SF ACFGs, we consider the same two reductions as in the proofs of Theorems 6 and 7 but the coalition structures Γ are not given as a part of the problem instances. Then, there is an exact cover for B if and only if there is no strictly popular coalition structure. In particular, if there is an exact cover for B, Γ and Δ as defined in the proofs above are in a tie and every other coalition structure is beaten by Γ. And if there is no exact cover for B then Γ beats every other coalition structure and thus is strictly popular.

Popularity verification for min-based and sum-based SF ACFGs can be shown to be coNP-complete by using the same constructions as for strict popularity verification (see Figs. 8 and 9) but reducing the numbers of α-players to $2k-1$ and $5k-1$, respectively. Then there is an exact cover for B if and only if Γ, as defined above, is not popular. □

Summing up, we have shown that (strict) popularity verification is coNP-complete for both sum-based and min-based SF ACFGs. For EQ and AL ACFGs, the exact complexity of (strict) popularity verification remains an open problem but we are confident that the coNP-completeness also extends to these problems. We leave the (strict) popularity existence problems open and think that their study is an important direction for future research.

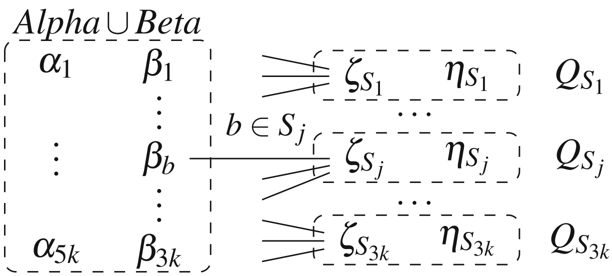


Fig. 9 Network of friends in the proof of Theorem 7 that is used to show coNP-hardness of strict popularity verification in sum-based SF ACFGs. A dashed rectangle around a group of players indicates that all these players are friends of each other

4.5 Perfectness

Turning now to perfectness, we start with the SF model. We give the following simple characterization of when a coalition structure is perfect.

Theorem 9 *For any sum-based or min-based SF ACFG* $(N, \succeq)$ *with an underlying network of friends G, a coalition structure* $\Gamma \in \mathcal{C}_N$ *is perfect if and only if it consists of the connected components of G and all of them are cliques.*

Proof From left to right, assume that the coalition structure $\Gamma \in \mathcal{C}_N$ is perfect. It then holds for all agents $i \in N$ and all coalition structures $\Delta \in \mathcal{C}_N$, $\Delta \neq \Gamma$, that i weakly prefers Γ to Δ. It follows that $v_i(\Gamma) \geq v_i(\Delta)$ for all $\Delta \in \mathcal{C}_N$, $\Delta \neq \Gamma$, and $i \in N$. Hence, every agent $i \in N$ has the maximal valuation $v_i(\Gamma) = n \cdot |F_i|$ and is together with all of her friends and none of her enemies. This implies that each coalition in Γ is a connected component and a clique.

The implication from right to left is obvious. □

Since it is easy to check this characterization, perfect coalition structures can be verified in polynomial time for sum-based and min-based SF ACFGs. It follows directly from Theorem 9 that the corresponding existence problem is also in P.

Corollary 1 *For any sum-based or min-based SF ACFG* $(N, \succeq)$ *with an underlying network of friends G, there exists a perfect coalition structure if and only if all connected components of G are cliques.*

Furthermore, we get the following upper bound for general ACFGs.

Proposition 4 *For any ACFG, perfectness verification is in* coNP.

Proof Consider any ACFG $(N, \succeq)$. A coalition structure $\Gamma \in \mathcal{C}_N$ is not perfect if and only if there is an agent $i \in N$ and a coalition structure $\Delta \in \mathcal{C}_N$ such that $\Delta \succ_i \Gamma$. Hence, we can nondeterministically guess an agent $i \in N$ and a coalition structure $\Delta \in \mathcal{C}_N$ and verify in polynomial time whether $\Delta \succ_i \Gamma$. □

Turning to EQ ACFGs, we initiate the characterization of perfectness with the following implication. (The *diameter* of a connected graph component is the greatest distance between any two of its vertices.)

Proposition 5 *For any sum-based EQ ACFG with an underlying network of friends G, it holds that if a coalition structure* Γ *is perfect for it, then* Γ *consists of the connected components of G and all these components have a diameter of at most two.*

Proof We first show that, in a perfect coalition structure, all agents have to be together with all their friends. For the sake of contradiction, assume that Γ is perfect but there are $i, j \in N$ with $j \in F_i$ and $j \notin \Gamma(i)$. We distinguish two cases.

Case 1: *All* $f \in F_i \cap \Gamma(i)$ *have a friend in* $\Gamma(j)$. Consider the coalition structure Δ that results from the union of $\Gamma(i)$ and $\Gamma(j)$, i.e., $\Delta = \Gamma \setminus \{\Gamma(i), \Gamma(j)\} \cup \{\Gamma(i) \cup \Gamma(j)\}$. It holds that i and all friends of i's either gain an additional friend in Δ or their coalition stays the same: First, i keeps all friends from $\Gamma(i)$ and gets j as an additional friend. Hence, i has at least one friend more in Δ than in Γ and we have $v_i(\Delta) > v_i(\Gamma)$. Second, all friends $f \in F_i \cap \Gamma(i)$ have a friend in $\Gamma(j)$ and therefore also gain at least one additional friend from the union of the two coalitions. Hence, $v_f(\Delta) > v_f(\Gamma)$ for all $f \in F_i \cap \Gamma(i)$. Third, all friends $f \in F_i \cap \Gamma(j)$ have i as friend. Hence, they also gain one friend from the union. Thus $v_f(\Delta) > v_f(\Gamma)$ for all $f \in F_i \cap \Gamma(j)$. Finally, all $f \in F_i$ who are not in $\Gamma(i)$ or $\Gamma(j)$ value Γ and Δ the same because their coalition is the same in both coalition structures. Hence, $v_f(\Delta) = v_f(\Gamma)$ for all $f \in F_i$ with $f \notin \Gamma(j)$ and $f \notin \Gamma(i)$. Summing up, we have $u_i^{sumEQ}(\Delta) > u_i^{sumEQ}(\Gamma)$, so i prefers Δ to Γ, which is a contradiction to Γ being perfect.

Case 2: *There is an* $f \in F_i \cap \Gamma(i)$ *who has no friends in* $\Gamma(j)$. Consider the coalition structure Δ that results from j moving to $\Gamma(i)$, i.e., $\Delta = \Gamma_{j \to \Gamma(i)}$. Let $k \in F_i \cap \Gamma(i)$ be one of the agents who have no friends in $\Gamma(j)$. Then $v_k(\Delta) = v_k(\Gamma) - 1$; $v_i(\Delta) = v_i(\Gamma) + n$; for all $f \in F_k \cap \Gamma(i)$, $f \neq i$, we have $v_f(\Delta) \geq v_f(\Gamma) - 1$; and for all $f \in F_k$, $f \notin \Gamma(i)$ (and $f \notin \Gamma(j)$), we have $v_f(\Delta) = v_f(\Gamma)$. Hence,

$$\begin{aligned}
u_k^{sumEQ}(\Delta) &= \sum_{a \in F_k \cup \{k\}} v_a(\Delta) = \sum_{a \in F_k \cap \Gamma(i), a \neq i} v_a(\Delta) + \sum_{a \in F_k \setminus \Gamma(i)} v_a(\Delta) + v_k(\Delta) + v_i(\Delta) \\
&\geq \sum_{a \in F_k \cap \Gamma(i), a \neq i} v_a(\Gamma) - 1 + \sum_{a \in F_k \setminus \Gamma(i)} v_a(\Gamma) + v_k(\Gamma) - 1 + v_i(\Gamma) + n \\
&= \sum_{a \in F_k \cup \{k\}} v_a(\Gamma) - (|F_k \cap \Gamma(i)| - 1) - 1 + n \\
&= u_k^{sumEQ}(\Gamma) - \underbrace{|F_k \cap \Gamma(i)|}_{<n} + n > u_k^{sumEQ}(\Gamma).
\end{aligned}$$

Therefore, k prefers Δ to Γ, which again is a contradiction to Γ being perfect.

Next, assume that Γ is perfect but there is a coalition C in Γ that has a diameter greater than two. Then there are agents $i, j \in C$ with a distance greater than two. Thus j is an enemy of i's and an enemy of all of i's friends. It follows that i prefers coalition structure $\Gamma_{j \to \emptyset}$ to Γ, which is a contradiction to Γ being perfect.

Summing up, in a perfect coalition structure Γ for a sum-based EQ ACFG every agent is together with all her friends and every coalition in Γ has a diameter of at most two. Together this implies that Γ consists of the connected components of G and all these components have a diameter of at most two. □

By Proposition 5, we know that there is only one candidate partition that might be perfect in a sum-based EQ ACFG: the partition consisting of the connected components of the underlying network of friends. Furthermore, we identified a necessary condition for its perfectness: All the components need to have a diameter of at most two. However, this is not a sufficient condition: The partition consisting of the connected components might be not perfect even if all components have a diameter of at most two. The following example shows that the condition is not sufficient and, thus, that Proposition 5 is not an equivalence.

Example 5 Consider the sum-based EQ ACFG $(N, \succeq^{sumEQ})$ with the network of friends G in Fig. 10. The coalition structure $\Gamma = \{N\}$ consists of the only connected component of

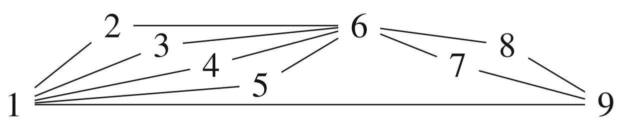


Fig. 10 Network of friends for Example 5

G, which has a diameter of two. However, agent 1 prefers $\Delta = \{\{1, \ldots, 6\}, \{7, 8, 9\}\}$ to Γ because

$$\begin{aligned} u_1^{sumEQ}(\Gamma) &= v_1(\Gamma) + \cdots + v_5(\Gamma) + v_9(\Gamma) = (9 \cdot 5 - 3) + 4 \cdot (9 \cdot 2 - 6) + (9 \cdot 3 - 5) = 112 \\ &< 113 = (9 \cdot 4 - 1) + 4 \cdot (9 \cdot 2 - 3) + (9 \cdot 2 - 0) = v_1(\Delta) + \cdots + v_5(\Delta) + v_9(\Delta) \\ &= u_1^{sumEQ}(\Delta). \end{aligned}$$

Hence, Γ is not perfect.

Still, from Propositions 4 and 5, we get the following corollary.

Corollary 2 *For sum-based EQ ACFGs, perfectness existence is in* coNP.

We believe that the further investigation of perfectness is important work for the future. So far, we can, in most cases, easily verify that a partition is not perfect or check that there is no perfect partition. But in the case that the connected components partition has diameters of at most two, we don't know yet how to verify the perfectness of this partition efficiently.

5 Conclusions and open problems

We have proposed to extend the models of altruistic hedonic games due to Nguyen et al. [1] and Wiechers and Rothe [5] to coalition formation games in general. Our extension results in six types of altruistic coalition formation games (ACFGs): We distinguish between sum-based and min-based aggregation of the friends' valuations and between three degrees of altruism, namely selfish first (SF), equal treatment (EQ), and altruistic treatment (AL). We have compared our models to altruistic hedonic games and have motivated our work by removing some crucial disadvantages that come with the restriction to hedonic games. In particular, we have shown that all degrees of our altruistic preferences are unanimous while this is not the case for all altruistic hedonic preferences. Furthermore, all our sum-based degrees of altruism fulfill two types of monotonicity that are violated by the corresponding hedonic equal-treatment and altruistic-treatment preferences. For an overview of all properties that we studied, see Table 2.

Furthermore, we have investigated some common stability notions and have initiated a computational analysis of the associated verification and existence problems (see Table 3 for an overview of our results). First, we have studied individual rationality. As for any reasonable coalition formation game, the individual rationality existence problem is trivial for ACFGs since the coalition structure consisting of singletons is always individually rational. We have then provided simple characterizations of individual rationality for all our variants of ACFGs.

Turning to Nash stability, we have shown that Nash stable partitions do always exist for ACFGs and can be found efficiently. Interestingly, the same coalition structure that is Nash stable under friend-oriented preferences is also Nash stable under any degree of altruistic preference. Furthermore, verifying Nash stability is possible in polynomial time for any ACFG.

We have then studied the notions of core stability and strict core stability. For SF ACFGs, the coalition structure consisting of the connected components of the underlying network of friends is always strictly core stable. Yet, this existence result (which, again, also holds for friend-oriented preferences) does not carry over to EQ and AL ACFGs. Indeed, as some of the most interesting open challenges of our work, we propose to determine the complexity of the existence problems for core stability and strict core stability for EQ and AL ACFGs. We know that these problems are in Σ_2^p but it would be interesting to see whether they are even Σ_2^p-complete (as it is the case, e.g., for additively separable hedonic games [28–30]). Concerning the verification problems, we showed that verifying core stability is coNP-complete for SF ACFGs. We strongly suspect that these results also carry over to EQ and AL ACFGs. Potentially, our proofs can be modified to also work for these models but might require some more involved constructions.

We have also stated several hardness results for (strict) popularity verification and existence. While our results so far are limited to SF ACFGs, we assume that the coNP-hardness of (strict) popularity verification and existence also carries over to EQ and AL ACFGs. Wiechers and Rothe [5] have shown that strict popularity verification is coNP-complete for min-based AHGs and, recently, Kerkmann and Rothe [8] have shown that popularity and strict popularity verification are actually coNP-complete for all types of AHGs. They also show that strict popularity existence is coNP-hard for all types of AHGs. These results suggest that coNP-hardness could also hold for the corresponding verification and existence problems in EQ and AL ACFGs. We view it as important future work to find out if and to what extent these results carry over to ACFGs.

Finally, we have investigated perfectness in ACFGs. We have pinpointed the complexity of the verification and existence problem for SF ACFGs but only provided coNP upper bounds on the complexity of these problems for EQ and AL ACFGs. We have provided a characterization of perfect coalition structures in SF ACFGs that is based on graph-theoretical properties of the underlying network of friends. Furthermore, we have given some necessary (though not sufficient) conditions for perfect coalition structures to exist in sum-based EQ ACFGs. For future work, we propose to find characterizations of perfectness for all altruistic models. An interesting related result was established by Bullinger and Kober [10]: They have shown that it is an NP-complete problem to determine an agent's most preferred coalition in the loyal variant of a symmetric friend-oriented hedonic game. Since these games are equivalent to the min-based EQ AHGs by Wiechers and Rothe [5], this result directly transfers to these AHGs. As determining an agent's most preferred coalition is closely related to the notion of perfectness, it would be interesting to see whether the result by Bullinger and Kober [10] can be modified to also transfer to min-based EQ ACFGs (or to other models of ACFGs).

Summing up, we have introduced ACFGs and have pinpointed some axiomatic advantages in comparison to AHGs. We have initiated the study of several notions of stability, such as Nash stability, core stability, popularity, and perfectness. Some very interesting questions for future work are concerned with the existence of (strictly) core stable or (strictly) popular coalition structures in EQ and AL ACFGs.

Acknowledgements We thank the anonymous IJCAI'20 and AMAI reviewers for helpful comments. This work was supported in part by DFG grants RO 1202/14-2 and RO 1202/21-1. The first and third author have been supported in part by the research project "Online Participation" within the North Rhine-Westphalian funding scheme "Forschungskollegs."

Funding Open Access funding enabled and organized by Projekt DEAL. This work was supported in part by Deutsche Forschungsgemeinschaft under grants RO 1202/14-2 and RO 1202/21-1 and the NRW project "Online Participation."

Availability of data and materials Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Non-financial interests Author Jörg Rothe currently is on the following editorial boards of scientific journals:
• *Annals of Mathematics and Artificial Intelligence* (AMAI), Associate Editor, since 01/2020,
• *Journal of Artificial Intelligence Research* (JAIR), Associate Editor, since 09/2017, and
• *Journal of Universal Computer Science* (J.UCS), Editorial Board, since 01/2005.

Conflict of Interest The authors declare that they have no conflict of interest.

References

1. Nguyen, N., Rey, A., Rey, L., Rothe, J., Schend, L.: Altruistic hedonic games. In: Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems, pp. 251–259. IFAAMAS, Singapore (2016)
2. Dimitrov, D., Borm, P., Hendrickx, R., Sung, S.-C.: Simple priorities and core stability in hedonic games. Soc. Choice Welf. **26**(2), 421–433 (2006)
3. Aziz, H., Brandt, F., Seedig, H.G.: Computing desirable partitions in additively separable hedonic games. Artif. Intell. **195**, 316–334 (2013)
4. Aziz, H., Savani, R.: Hedonic games. In: Brandt, F., Conitzer, V., Endriss, U., Lang, J., Procaccia, A. (eds.) Handbook of Computational Social Choice, pp. 3560–376. Cambridge University Press, Cambridge, UK (2016) . Chap. 15
5. Wiechers, A., Rothe, J.: Stability in minimization-based altruistic hedonic games. In: Proceedings of the 9th European Starting AI Researchers' Symposium, vol. 2655. CEUR-WS.org, Santiago de Compostela, Spain (2020)
6. Woeginger, G.: Core stability in hedonic coalition formation. In: Proceedings of the 39th International Conference on Current Trends in Theory and Practice of Computer Science, pp. 33–50. Springer, Špindlerův Mlýn, Czech Republic (2013)
7. Kerkmann, A., Nguyen, N., Rey, A., Rey, L., Rothe, J., Schend, L., Wiechers, A.: Altruistic hedonic games. J. Artif. Intell. Res. **75**, 129–169 (2022)
8. Kerkmann, A., Rothe, J.: Popularity and strict popularity in altruistic hedonic games and minimum-based altruistic hedonic games. In: Proceedings of the 21st International Conference on Autonomous Agents and Multiagent Systems, pp. 1657–1659. IFAAMAS, Auckland, New Zealand (2022)
9. Schlueter, J., Goldsmith, J.: Super altruistic hedonic games. In: Proceedings of the 33rd International Florida Artificial Intelligence Research Society Conference, pp. 160–165. AAAI Press, Miami, FL, USA (2020)
10. Bullinger, M., Kober, S.: Loyalty in cardinal hedonic games. In: Proceedings of the 30th International Joint Conference on Artificial Intelligence, pp. 66–72. ijcai.org, Montréal, Canada (2021)
11. Ashlagi, I., Krysta, P., Tennenholtz, M.: Social context games. In: Proceedingsof the 4th International Workshop on Internet & Network Economics, pp. 675–683. Springer, Shanghai, China (2008)
12. Bilò, V., Celi, A., Flammini, M., Gallotti, V.: Social context congestion games. Theor. Comput. Sci. **514**, 21–35 (2013)
13. Hoefer, M., Penn, M., Polukarov, M., Skopalik, A., Vöcking, B.: Considerate equilibrium. In: Proceedings of the 22nd International Joint Conference on Artificial Intelligence, pp. 234–239. AAAI Press/IJCAI, Barcelona, Spain (2011)
14. Anagnostopoulos, A., Becchetti, L., de Keijzer, B., Schäfer, G.: Inefficiency of games with social context. In: Proceedings of the 6th International Symposium on Algorithmic Game Theory, pp. 219–230. Springer, Aachen, Germany (2013)

15. Bilò, V.: On linear congestion games with altruistic social context. In: Proceedings of the 20th International Conference on Computing and Combinatorics, pp. 547–558. Springer, Atlanta, GA, USA (2014)
16. Hoefer, M., Skopalik, A.: Altruism in atomic congestion games. In: Proceedings of the 17th Annual European Symposium on Algorithms, pp. 179–189. Springer, Copenhagen, Denmark (2009)
17. Chen, P., de Keijzer, B., Kempe, D., Schäfer, G.: The robust price of anarchy of altruistic games. In: Proceedings of the 7th International Workshop on Internet & Network Economics, pp. 383–390. Springer, Singapore (2011)
18. Apt, K., Schäfer, G.: Selfishness level of strategic games. J. Artif. Intell. Res. **49**, 207–240 (2014)
19. Caragiannis, I., Kaklamanis, C., Kanellopoulos, P., Kyropoulou, M., Papaioannou, E.: The impact of altruism on the efficiency of atomic congestion games. In: Proceedings of the 5th International Symposium on Trustworthy Global Computing, pp. 172–188. Springer, Munich, Germany (2010)
20. Kleer, P., Schäfer, G.: Tight inefficiency bounds for perception-parameterized affine congestion games. Theor. Comput. Sci. **754**, 65–87 (2019)
21. Schröder, M.: Price of anarchy in congestion games with altruistic/spiteful players. In: Proceedings of the 13th International Symposium on Algorithmic Game Theory, pp. 146–159. Springer, Augsburg, Germany (2020)
22. Rahn, M., Schäfer, G.: Bounding the inefficiency of altruism through social contribution games. In: Proceedings of the 9th International Workshop on Internet & Network Economics, pp. 391–404. Springer, Cambridge, UK (2013)
23. Rothe, J.: Thou shalt love thy neighbor as thyself when thou playest: Altruism in game theory. In: Proceedings of the 35th AAAI Conference on Artificial Intelligence, pp. 15070–15077. AAAI Press, Montréal, Canada (2021)
24. Magaña, A., Carreras, F.: Coalition formation and stability. Group Decis. Negot. **27**, 467–502 (2018)
25. Bogomolnaia, A., Jackson, M.: The stability of hedonic coalition structures. Games and Economic Behavior **38**(2), 201–230 (2002)
26. Banerjee, S., Konishi, H., Sönmez, T.: Core in a simple coalition formation game. Soc. Choice. Welf. **18**(1), 135–153 (2001)
27. Alcade, J., Romero-Medina, A.: Coalition formation and stability. Soc. Choice Welf. **27**(2), 365–375 (2006)
28. Woeginger, G.: A hardness result for core stability in additive hedonic games. Math. Soc. Sci. **65**(2), 101–104 (2013)
29. Peters, D.: Precise complexity of the core in dichotomous and additive hedonic games. In: Proceedings of the 5th International Conference on Algorithmic Decision Theory, pp. 214–227. Springer, Luxembourg (2017)
30. Ota, K., Barrot, N., Ismaili, A., Sakurai, Y., Yokoo, M.: Core stability in hedonic games among friends and enemies: Impact of neutrals. In: Proceedings of the 26th International Joint Conference on Artificial Intelligence, pp. 359–365. ijcai.org, Melbourne, Australia (2017)
31. Karakaya, M.: Hedonic coalition formation games: A new stability notion. Math. Soc. Sci. **61**(3), 157–165 (2011)
32. Aziz, H., Brandl, F.: Existence of stability in hedonic coalition formation games. In: Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systems, pp. 763–770. IFAAMAS, Valencia, Spain (2012)
33. Gärdenfors, P.: Match making: Assignments based on bilateral preferences. Behav. Sci. **20**(3), 166–173 (1975)
34. Lang, J., Rey, A., Rothe, J., Schadrack, H., Schend, L.: Representing and solving hedonic games with ordinal preferences and thresholds. In: Proceedings of the 14th International Conference on Autonomous Agents and Multiagent Systems, pp. 1229–1237. IFAAMAS, Istanbul, Turkey (2015)
35. Kerkmann, A., Rothe, J.: Altruism in coalition formation games. In: Proceedings of the 29th International Joint Conference on Artificial Intelligence, pp. 347–353. ijcai.org, Yokohama, Japan (2020)
36. Bell, E.T.: The iterated exponential integers. Ann. Math. **39**(3), 539–557 (1938)
37. Rota, G.-C.: The number of partitions of a set. The American Mathematical Monthly **71**(5), 498–504 (1964)
38. Cechlárová, K., Romero-Medina, A.: Stability in coalition formation games. Int. J. Game Theory **29**(4), 487–494 (2001)
39. Cechlárová, K., Hajduková, J.: Computational complexity of stable partitions with B-preferences. Int. J. Game Theory **31**(3), 353–364 (2003)
40. Kerkmann, A., Lang, J., Rey, A., Rothe, J., Schadrack, H., Schend, L.: Hedonic games with ordinal preferences and thresholds. J. Artif. Intell. Res. **67**, 705–756 (2020)
41. Garey, M., Johnson, D.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, New York, USA (1979)

42. Rothe, J.: Complexity Theory and Cryptology. An Introduction to Cryptocomplexity. EATCS Texts in Theor. Comput. Sci. Springer, Heidelberg and Berlin, Germany (2005)
43. Sung, S., Dimitrov, D.: On myopic stability concepts for hedonic games. Theor. Decis. **62**(1), 31–45 (2007)
44. Gonzalez, T.: Clustering to minimize the maximum intercluster distance. Theor. Comput. Sci. **38**, 293–306 (1985)