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Lie point symmetries and conservation laws for a class of BBM-KdV systems

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Abstract

We determine the Lie point symmetries of a class of BBM-KdV systems and establish its nonlinear self-adjointness. We then construct conservation laws via Ibragimov's Theorem.

Keywords: BBM-KdV system, Lie point symmetries, nonlinear self-adjointness, conservation laws.

1 Introduction

Motivated by the works [4, 5], we introduce the following class of systems

$$\begin{cases} F_1 \equiv u_t + (a+b)vu_x + (au+c)v_x + \epsilon u_{txx} + \kappa v_{xxx} = 0 \\ F_2 \equiv v_t + (bu+c)u_x + (a+b)vv_x + \lambda u_{xxx} + \sigma v_{txx} = 0 \end{cases}, \tag{1}$$

henceforth simply referred to as BBM-KdV system, a two-component generalization[1] of the classic equations [2]

$$BBM\colon\ u_t + (u+1)u_x - u_{txx} = 0 \tag{2}$$

and

$$KdV\colon\ u_t + (u+1)u_x + u_{xxx} = 0, \tag{3}$$

with the objective of studying it from the point of view of the group analysis.

[1] If $u = v$, the system (1) is reduced to equations $u_t + [(2a+b)u + c]u_x + \epsilon u_{txx} + \kappa u_{xxx} = 0$ and $u_t + [(a+2b)u + c]u_x + \sigma u_{txx} + \lambda u_{xxx} = 0$. Both contain (2) and (3) as special cases.

In (1), the constants are such that $(a+b)c \neq 0$ and $\{\epsilon, \kappa, \lambda, \sigma\} \neq \{0\}$. Particularly when $a = c = 1$ and $b = 0$, we obtain the already widely investigated systems of Boussinesq ($\epsilon = \kappa = \lambda = 0$, $\sigma = -1/3$), Kaup ($\epsilon = \lambda = \sigma = 0$, $\kappa = 1/3$) and Bona-Smith ($\epsilon = \sigma = \lambda/2 - 1/6$, $\kappa = 0$, $\lambda < 0$), all of them first-order approximations to the Euler equations in the framework of hydrodynamics. Useful in situations where dissipative effects are not significant, these models provide a good description for the two-dimensional motion of small-amplitude long waves on the surface of an ideal fluid. In this context, the independent variable x represents the distance traveled along a fixed depth channel and t the time. The quantities $u(t,x)$ and $v(t,x)$ are related to the deviation of the surface from its undisturbed level and to the horizontal velocity of the fluid, respectively. For more information, see [4, 5] and references therein. Relevant results, including exact solutions, can be found in [1, 6, 7].

It's well known that evolution equations don't possess an usual Lagrangian. Therefore this paper is thus organized: first we determine the Lie point symmetries (Section 2) of the BBM-KdV system and establish its nonlinear self-adjointness (Section 3); we then construct conservation laws via Ibragimov's Theorem (Section 4), an extension of the celebrated Noether's Theorem to problems with no variational structure. In the next sections, unless otherwise stated, c_i's are arbitrary constants. All functions are smooth.

We consider that the reader is familiar with the fundamental concepts of group analysis. The basic literature used is [3, 8, 9, 10, 11, 12, 13, 14].

2 Lie Point Symmetries Classification

Without many details, applying the standard algorithm presented in [3] and [14], a differential operator

$$X = \mathcal{T}(t,x,u,v)\frac{\partial}{\partial t} + \mathcal{X}(t,x,u,v)\frac{\partial}{\partial x} + \mathcal{U}(t,x,u,v)\frac{\partial}{\partial u} + \mathcal{V}(t,x,u,v)\frac{\partial}{\partial v}$$

generates the Lie point symmetries of the system (1) if the conditions of invariance (the so-called determining equations)

$$\begin{gathered}
\mathcal{T}_x = \mathcal{T}_u = \mathcal{T}_v = \mathcal{X}_u = \mathcal{X}_v = 0,\\
\epsilon\mathcal{X}_t = \epsilon\mathcal{X}_x = \sigma\mathcal{X}_t = \sigma\mathcal{X}_x = 0,\\
\mathcal{U}_t = \mathcal{U}_x = \mathcal{U}_v = \mathcal{V}_t = \mathcal{V}_x = a\mathcal{U} - (au+c)\mathcal{U}_u = 0,\\
b\mathcal{U} + (bu+c)[\mathcal{U}_u + 2(\mathcal{T}_t - \mathcal{X}_x)] = 0,\\
\kappa(\mathcal{U}_u + 2\mathcal{X}_x) = \lambda[\mathcal{U}_u + 2(\mathcal{T}_t - 2\mathcal{X}_x)] = 0,\\
(a+b)[\mathcal{V} + (\mathcal{T}_t - \mathcal{X}_x)v] - \mathcal{X}_t = 0
\end{gathered} \tag{4}$$

are satisfied. From (4), it's easy to see that

$$\begin{gathered}
\mathcal{T} = (a+b)c_1 t + c_2,\ \ \mathcal{X} = (a+b)(c_3 x + c_4 t) + c_5,\\
\mathcal{U} = 2(c_3 - c_1)(au+c),\ \ \mathcal{V} = (a+b)(c_3 - c_1)v + c_4
\end{gathered}$$

with

$$b(a-b)(c_1-c_3)=0,\ \epsilon c_3=\epsilon c_4=\kappa[ac_1-(2a+b)c_3]=0,$$
$$\lambda[bc_1-(a+2b)c_3]=\sigma c_3=\sigma c_4=0.$$

Proposition 1. *The Lie point symmetries of the BBM-KdV system are summarized in Table 1, where*

$$X_1=(a+b)\left(t\frac{\partial}{\partial t}-v\frac{\partial}{\partial v}\right)-2(au+c)\frac{\partial}{\partial u},$$
$$X_2=\frac{\partial}{\partial t},\ X_3=(a+b)\left(x\frac{\partial}{\partial x}+v\frac{\partial}{\partial v}\right)+2(au+c)\frac{\partial}{\partial u},$$
$$X_4=(a+b)t\frac{\partial}{\partial x}+\frac{\partial}{\partial v},\ X_5=\frac{\partial}{\partial x}.$$

	$b=0$	$a=b$	$b(a-b)\neq 0$
$\{\epsilon,\sigma\}=\{0\}$	X_1 $(\kappa=0)$ $2X_1+X_3$ $(\lambda=0)$ $X_2,\ X_4,\ X_5$	$3X_1+X_3$ $X_2,\ X_4,\ X_5$	$X_2,\ X_4,\ X_5$
$\{\epsilon,\sigma\}\neq\{0\}$	X_1 $(\kappa=0)$, $X_2,\ X_5$	X_1 $(\kappa=\lambda=0)$ $X_2,\ X_5$	$X_2,\ X_5$

Table 1.

3 Self-Adjointness Classification

To begin with, let $\bar{u}$ and $\bar{v}$ be the new dependent variables. The formal Lagrangian of the system (1) is

$$\mathcal{L}=\bar{u}F_1+\bar{v}F_2.$$

Calculated the adjoint equations

$$\begin{cases} F_1^*\equiv-\dfrac{\delta\mathcal{L}}{\delta u}=\bar{u}_t+(a+b)v\bar{u}_x+\ (bu+c)\bar{v}_x+\ b\bar{u}v_x+\epsilon\bar{u}_{txx}+\lambda\bar{v}_{xxx}=0 \\ F_2^*\equiv-\dfrac{\delta\mathcal{L}}{\delta v}=\bar{v}_t+(au+c)\bar{u}_x+(a+b)v\bar{v}_x-b\bar{u}u_x+\kappa\bar{u}_{xxx}+\sigma\bar{v}_{txx}=0 \end{cases},$$

where $\delta/\delta u$ and $\delta/\delta v$ are Euler-Lagrange operators, we assume that

$$F_1^*|_{(\bar{u},\bar{v})=(\varphi,\psi)}=MF_1+NF_2,\quad F_2^*|_{(\bar{u},\bar{v})=(\varphi,\psi)}=PF_1+QF_2. \tag{5}$$

Here M, N, P and Q is a set of coefficients to be determined and

$$\varphi=\varphi(t,x,u,v),\quad \psi=\psi(t,x,u,v) \tag{6}$$

two functions that not vanish simultaneously. As

$$F_1^*|_{(\bar{u},\bar{v})=(\varphi,\psi)} = D_t\varphi + (a+b)vD_x\varphi + (bu+c)D_x\psi + b\varphi v_x + \epsilon D_t D_x^2\varphi + \lambda D_x^3\psi$$

and

$$F_2^*|_{(\bar{u},\bar{v})=(\varphi,\psi)} = D_t\psi + (au+c)D_x\varphi + (a+b)vD_x\psi - b\varphi u_x + \kappa D_x^3\varphi + \sigma D_t D_x^2\psi,$$

from (5) it's possible to conclude that $M=\varphi_u$, $N=\varphi_v$, $P=\psi_u$, $Q=\psi_v$ and

$$\begin{gathered}\varphi_t + (a+b)v\varphi_x = \epsilon\varphi_x = 0,\\ \psi_t + (au+c)\varphi_x = \psi_x = 0,\\ b\varphi = (au+c)\varphi_u - (bu+c)\psi_v,\\ \varphi_v - \psi_u = (\epsilon-\sigma)\varphi_v = \kappa\varphi_u - \lambda\psi_v = 0,\\ \epsilon\varphi_{uu} = \varphi_{uv} = \varphi_{vv} = 0.\end{gathered}$$

Hence

$$\varphi = (c_1t+c_2)av + f(u) - c_1x, \quad \psi = c_3v + (c_1t+c_2)au + c_1ct + c_4$$

with

$$\begin{gathered}bc_1 = bc_2 = \epsilon c_1 = \sigma c_1 = (\epsilon-\sigma)c_2 = 0,\\ \epsilon f''(u) = \kappa f'(u) - \lambda c_3 = 0,\\ bf(u) = (au+c)f'(u) - (bu+c)c_3.\end{gathered}$$

Proposition 2. *The BBM-KdV system is nonlinearly self-adjoint. The substitutions* (6) *are as follows.*

i) *If* $b=0$*,*

$$\varphi = (c_1t+c_2)av + c_3c\ln(au+c) - c_1x + c_4, \quad \psi = c_3av + (c_1t+c_2)(au+c) + c_5$$

where

$$\begin{cases} c_1 = 0, \text{ to } \{\epsilon,\sigma\} \neq \{0\},\\ c_2 = 0, \text{ to } \epsilon \neq \sigma,\\ c_3 = 0, \text{ to } \{\epsilon,\kappa,\lambda\} \neq \{0\}.\end{cases}$$

ii) *If* $a=b$*,*

$$\varphi = (au+c)[c_1\ln(au+c)+c_2], \quad \psi = c_1av + c_3$$

where

$$\begin{cases} c_1 = 0, \text{ to } \{\epsilon,\kappa,\lambda\} \neq \{0\},\\ c_2 = 0, \text{ to } \kappa \neq 0.\end{cases}$$

iii) *Let* $b(a-b)\neq 0$*.*

iii.a) *If* $a = 0$,

$$\varphi = c_1 e^{bu/c} - c_2(bu + 2c), \quad \psi = c_2 bv + c_3$$

where

$$\begin{cases} c_1 = 0, \text{ to } \{\epsilon, \kappa\} \neq \{0\}, \\ c_2 = 0, \text{ to } \kappa \neq -\lambda. \end{cases}$$

iii.b) *If* $a \neq 0$,

$$\varphi = c_1(au + c)^{b/a} + c_2[b^2 u + (2b - a)c], \quad \psi = c_2(a - b)bv + c_3$$

where

$$\begin{cases} c_1 = 0, \text{ to } \{\epsilon, \kappa\} \neq \{0\}, \\ c_2 = 0, \text{ to } \lambda a \neq (\kappa + \lambda)b. \end{cases}$$

Remark. Actually, the system (1) is quasi self-adjoint. It becomes strictly self-adjoint in only two circumstances: $a = 2b$ and $\kappa = \lambda$; or $b = 0$ and $\epsilon = \sigma$.

4 Conservation Laws

In view of Proposition 2, the components of the conserved vector $C = (C^t, C^x)$ associated to X, a Lie point symmetry admitted by the system (1), are according to Ibragimov's Theorem given by

$$C^t = (\varphi - \epsilon D_x \varphi D_x)W^u + (\psi - \sigma D_x \psi D_x)W^v$$

and

$$\begin{aligned} C^x = {} & [(a + b)v\varphi + (bu + c)\psi + \epsilon(\varphi D_t D_x + D_t D_x \varphi) + \lambda(\psi D_x^2 - D_x \psi D_x + D_x^2 \psi)]W^u + \\ & + [(au + c)\varphi + (a + b)v\psi + \sigma(\psi D_t D_x + D_t D_x \psi) + \kappa(\varphi D_x^2 - D_x \varphi D_x + D_x^2 \varphi)]W^v, \end{aligned}$$

with

$$W^u = \mathcal{U} - \mathcal{T} u_t - \mathcal{X} u_x, \quad W^v = \mathcal{V} - \mathcal{T} v_t - \mathcal{X} v_x.$$

We find the conservation laws corresponding to each generator of Table 1. In most cases, however, we are led to trivial vectors or the vectors

$$C^t = u + \epsilon u_{xx}, \quad C^x = (au + c)v + \kappa v_{xx}$$

and

$$C^t = 2(v + \sigma v_{xx}), \quad C^x = (a + b)v^2 + (bu + 2c)u + 2\lambda u_{xx}$$

that can be obtained from the first (when b = 0) and second equation of the BBM-KdV system by simple integration (obvious conservation laws). The really interesting cases we list below.

Proposition 3. **i)** *Let $b = 0$.*

i.a) *From X_1, $2X_1 + X_3$ and X_2, we obtain*

$$C^t = 2(uv - \epsilon u_x v_x),$$
$$C^x = cu^2 + (2au + c)v^2 - (\lambda u_x^2 + \kappa v_x^2) + 2[u(\lambda u_x + \epsilon v_t)_x + v(\epsilon u_t + \kappa v_x)_x]$$

when $\epsilon = \sigma$.

i.b) *For $\epsilon = \kappa = 0$, X_1 also provides*

$$C^t = \frac{1}{a}(au + c)\ln(au + c) + \frac{a}{2c}(v^2 - \sigma v_x^2),$$
$$C^x = (au + c)[\ln(au + c) + 1]v + \frac{av}{c}\left(\frac{av^2}{3} + \sigma v_{tx}\right)$$

when $\lambda = 0$ and

$$C^t = 2[t(au + c)v - xu],$$
$$C^x = t[c(au + 2c)u - a\lambda u_x^2] + 2(au + c)[(atv - x)v + \lambda t u_{xx}]$$

when $\sigma = 0$.

ii) *Let $a = b$.*

ii.a) *From X_1 and $3X_1 + X_3$, we obtain*

$$C^t = (au + 2c)u - a\epsilon u_x^2,$$
$$C^x = 2(au + c)[(au + c)v + \epsilon u_{tx}]$$

when $\kappa = 0$.

ii.b) *X_1 also provides*

$$C^t = \frac{1}{a}(au + c)^2\ln(au + c) + a(v^2 - \sigma v_x^2),$$
$$C^x = (au + c)^2[2\ln(au + c) + 1]v + 2av\left(\frac{2av^2}{3} + \sigma v_{tx}\right)$$

when $\epsilon = \kappa = \lambda = 0$.

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