

Upper and Lower Bounds for Competitive Group Testing

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Abstract

We consider competitive algorithms for adaptive group testing problems. In the first part of the paper, we develop an algorithm with competitive constant $c < 1.452$ thus improving the up to now best known algorithms with constants $1.5 + \epsilon$ from 2003. In the second part, we prove the first nontrivial lower bound for competitive constants, namely that c is always larger than 1.31.

1 Introduction

Assume that we are given some finite set S , $|S| = n$, of *items* which can be *good* or *defective*. Our task is to identify the set D of defectives by (successively) choosing subsets W_1, W_2, \dots of S and testing whether D is disjoint from W_i or not. In choosing W_i , we may assume that the results of the first $i - 1$ tests are already known (*adaptive search*). We want to minimize the number of tests needed to identify the set D in the worst case.

This problem is called the *adaptive group testing* problem and dates back to 1943 when Dorfman [1] in his seminal paper discussed possibilities to reduce the number of blood tests for syphilitic antigene of American inductees. Although group testing was not used in this context, many applications in different areas such as quality control, information theory, computer networks and clone library were found in the following years. The most complete account on group testing can be found in [2]. The books [3] and [4] treat search problems in more generality but also cover the subject while [5] focusses on nonadaptive group testing with applications in the life sciences. In this paper, we deal with adaptive group testing only.

Most papers on this subject considered the case that the number d of defectives is known in advance, the so-called (d, n) -group testing problem. We write $M_A(d, n)$ for the maximum number of tests required by a group testing algorithm A to identify all items and denote by $M(d, n) := \min_A M_A(d, n)$ the worst

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case number of tests for the problem. By general considerations, we have the *information theoretic lower bound*

$$M(d, n) \geq \left\lceil \log \binom{n}{d} \right\rceil \geq d \log \frac{n}{d},$$

where the base of log is two throughout this paper.

There is a famous and beautiful conjecture on the so-called cutoff point for adaptive group testing due to Hu, Hwang and Wang: If the ratio $r := n/d$ is at most 3, it is optimal to test each item individually.

Conjecture 1.1. [6] For $0 < d < n$ and $r \leq 3$,

$$M(d, n) = n - 1.$$

The equation $M(d, n) = n - 1$ was proved if $r \leq \frac{21}{8} = 2.625$ by Du and Hwang [7]. Leu, Lin and Weng extended this to $r \leq \frac{43}{16} = 2.6875$ for $d \geq 193$ [8] and Riccio and Colbourn to $r < \log_{\frac{3}{2}} 3 \approx 2.709$ for sufficiently large d depending on r [9]. Wegener, Fischer and Klasner proved the conjecture if the cardinality of the test sets is at most 2 [10]. For later use, we record the result of Leu, Lin and Weng

Theorem 1.2. [8] For $\frac{16}{43}n \leq d < n$, $d \geq 193$,

$$M(d, n) = n - 1.$$

For $r > \frac{43}{16}$, the best group testing algorithms we know about are due to A. Allemann [11]:

Theorem 1.3. [11] There is an algorithm which, for all $0 < d \leq \frac{n}{2}$, finds d defectives by at most

$$\log \binom{n}{d} + 0.255d + \frac{1}{2} \log d + 6.5$$

tests.

In practice, the prior knowledge of d is often an unrealistic assumption. However, if nothing is known about d , the worst case number of tests is clearly n and can be realized by testing all elements individually. Suppose the number of defectives turns out to be one. Then those n singleton tests are very many as compared to the $\lceil \log n \rceil$ tests needed if $d = 1$ is known in advance. Du and Hwang thus asked for an algorithm A with the following property: Algorithm A successfully identifies D without prior knowledge of d for all n . Denote by $M_A(d|n)$ the worst case number of tests used by A if the number of defectives turns out to be d . Then the ratio $M_A(d|n)/M(d, n)$ should be bounded by some constant c . More precisely:

Definition 1.4. [12] Let c be some real number, $c \geq 1$. Algorithm A is called *c-competitive* if there is some constant $a \in \mathbb{N}$ such that

$$M_A(d|n) \leq cM(d, n) + a,$$

for all $0 \leq d < n$. The number c is called a *competitive ratio* for A .

Du and Hwang gave the first competitive algorithm with $c = 2.75$ in 1993 [12]. Several improvements were given by Bar-Noy, Hwang, Kessler and Kutten ($c = 2$) in 1994 [13], by Du, Xue, Sun and Cheng ($c = 1.65$) in 1994 [14] and by Schlaghoff and Triesch ($c = 1.5 + \epsilon$) in 2005 [15].

In [2, p. 77] we read the following remark: “*However, it certainly requires a new technique in order to push the competitive ratio down under 1.5.*”

No nontrivial *lower bound* on the competitive ratio has been proved so far. For an interesting related paper, see [16].

In the next section, we are going to present an algorithm with competitive ratio $c < 1.452$. We then go on to prove that for each c -competitive algorithm we have $c > 1.31$.

2 A 1.452-competitive algorithm

In this section, we present a new algorithm A with competitive ratio $c < 1.452$. Before we start explaining the ideas, we note two useful inequalities. The first is an estimate of $\log \binom{n}{d}$ by Stirling’s formula $k! = \sqrt{2\pi k} \left(\frac{k}{e}\right)^k \theta_k$ with $1 < \theta_k \leq e^{\frac{1}{12k}}$. Recall that r denotes the quotient n/d .

Theorem 2.1. [17] For $0 < d \leq \frac{n}{2}$ we have:

$$M(d, n) \geq d \cdot \left(\log r + (r - 1) \log \left(\frac{r}{r - 1} \right) \right) - \frac{1}{2} \log d - \frac{3}{2}.$$

For $r \geq 3$ we have $(r - 1) \log \left(\frac{r}{r - 1} \right) > 1.1699$.

For a detailed proof, we refer the reader to [17].

The second tool we need is the following standard inequality which is implied by the *concavity of the logarithm* and *Jensen’s inequality*:

Theorem 2.2. Let $0 < d_i \leq n_i$ for $1 \leq i \leq k$, $d := \sum_{i=1}^k d_i$, $n := \sum_{i=1}^k n_i$. Then the following inequality holds:

$$\sum_{i=1}^k d_i \log \frac{n_i}{d_i} \leq d \log \frac{n}{d}.$$

The algorithm we present naturally decomposes into “phases” where some good and some defective items are identified. If in phase i we detect g_i good and d_i defective items, we will show that the corresponding number of tests t_i is at most $1.45198 d_i (\log \frac{g_i + d_i}{d_i} + 1.1699)$. Summing over i and estimating by Theorem 2.2 we obtain an upper bound of $1.45198 d (\log \frac{n}{d} + 1.1699)$ for the total number of tests.

For a formal description compare Algorithm 1 which uses some subalgorithms described below. The variable S denotes the set of still unclassified elements through the execution of the algorithm. The command TEST(X) means that we test whether X contains at least one defective element or not. In the first case, the test is called positive and negative in the second.

Procedure DIG (Algorithm 3) is applied to a defective set X and finds a defective element by binary splitting. The number of tests is of course $\lceil \log |X| \rceil$. More interesting is Algorithm 2 called FOURTEST. It is also applied to some defective set X where we assume that $|X| = 2^l$ is a power of two, $|X| \geq 4$. It first splits X into four disjoint subsets T, U, V, W of equal size and tests one of them, say T . If it is defective, DIG is applied to find a defective element by $\log |T| = l - 2$ additional tests. Otherwise, the sets U and V are tested. If one or both of them are defective, DIG is applied to find one or two defective elements. If none of them is defective, we know that W is defective and, again find a defective by applying DIG. Summarizing, FOURTEST produces one the following results:

- one defective is identified using $l - 1$ tests.
- one defective and 2^{l-1} good elements are detected by $3 + l - 2 = l + 1$ tests.
- two defectives and 2^{l-2} good elements are detected by $3 + 2(l - 2) = 2l - 1$ tests.
- one defective and $3 \cdot 2^{l-2}$ good elements are found by $3 + l - 2 = l + 1$ tests.

We now begin discussing our main procedure, i.e., Algorithm 1:

Main Procedure

At the beginning, we check whether the number of elements in S is larger than $576 = 8 + 8 \cdot 64$. If less elements are unclassified, we use the command INDVTEST(S) which stands for individual tests of all elements of S . The constant a in the definition of competitiveness takes care of this constant number of tests. (A similar remark applies to the subalgorithms A_k , $2 \leq k \leq 6$, compare Algorithm 4.) If enough elements are unclassified, we test a subset of cardinality 64 and test it. Let us suppose first that this group of 64 elements is good. Then the algorithm enters an *expansion phase* which we explain and analyse first:

Expansion Phase

With each negative test we expand the size of the test set by a factor of 8. That means our test sets have sizes $64 \cdot 8, 64 \cdot 8^2, 64 \cdot 8^3, \dots$ until one of the following two cases occurs: Either we get a contaminated test set X of size $64 \cdot 8^k, k \geq 1$ or the remaining set S has less than $64 \cdot 8^k, k \geq 1$ elements. Note that after two negative tests, we test the whole remaining set. If it is good, the algorithm stops. This means that all remaining elements are good and we needed 3 tests for verifying this. Of course, this can happen only in one expansion phase and the additive constant a takes again care of this situation.

Assume first that we get a positive test with $|X| = 64 \cdot 8^k, k \geq 1$. By now, we have detected $64 \frac{8^k - 1}{7}$ good elements using $k + 1$ or $k + 2$ tests, depending on whether $k = 1$ or not. Now this defective group X runs the subroutine FOURTEST(X) with $|X| = 64 \cdot 8^k = 2^l, l = 3k + 6$. According to the discussion above, the expansion phase ends with one of the following cases:

Algorithm 1: MAIN (set S)

```
1 global  $S$ 
2 global  $D = \emptyset$ 
3 global  $G = \emptyset$ 
4  $k = 0$ 
5 while  $|S| > 576$  do
6    $X = \min(64 \cdot 8^k, |S|)$  elements from  $S$ 
7   if  $|X| = 64 \cdot 8^k$  then
8     TEST( $X$ )
9     if  $X$  is good then
10       $G = G \cup X$ 
11       $S = S \setminus X$ 
12      if  $k = 1$  then
13        TEST( $S$ )
14        if  $S$  is good then
15           $G = G \cup S$ 
16           $S = \emptyset$ 
17        end if
18      end if
19       $k = k + 1$ 
20    else
21      if  $k = 0$  then
22         $A_6(X, S \setminus X)$ 
23      else
24        FOURTEST( $X$ )
25         $k = 0$ 
26      end if
27    end if
28  else
29    DIG( $X$ )
30  end if
31 end while
32 INDVTEST( $S$ )
33 return  $D, G$ 
```

Algorithm 2: FOURTEST (set X)

```
1  $T, U, V, W =$  a partition of  $X$  into 4 sets of equal cardinality
2 TEST( $T$ )
3 if  $T$  is defective then
4   | DIG( $T$ )
5 else
6   |  $G = G \cup T$ 
7   |  $S = S \setminus T$ 
8   | TEST( $U$ )
9   | TEST( $V$ )
10  if  $U$  is defective then
11    | DIG( $U$ )
12  else
13    |  $G = G \cup U$ 
14    |  $S = S \setminus U$ 
15  end if
16  if  $V$  is defective then
17    | DIG( $V$ )
18  else
19    |  $G = G \cup V$ 
20    |  $S = S \setminus V$ 
21  end if
22  if  $U$  and  $V$  are good then
23    | DIG( $W$ )
24  end if
25 end if
```

Algorithm 3: DIG (set X) (compare [2])

```
1 while  $|X| \geq 1$  do
2   |  $Y = \lceil |X|/2 \rceil$  elements from  $X$ 
3   | TEST( $Y$ )
4   | if  $Y$  is good then
5     |  $G = G \cup Y$ 
6     |  $S = S \setminus Y$ 
7     |  $X = X \setminus Y$ 
8   | else
9     |  $X = Y$ 
10  | end if
11 end while
12  $S = S \setminus X$ 
13  $D = D \cup X$ 
```

Algorithm 4: $A_k(\text{set } X_k, \text{set } T_k)$ for $3 \leq k \leq 6$

```
1 global  $X_k$ 
2 global  $T_k$ 
3 if  $|T_k| < 2^{k-1}$  then
4   | INDVTEST( $T_k$ )
5 else
6   |  $Y = 2^{k-1}$  items from  $T_k$ 
7   | TEST( $Y$ )
8   | if  $Y$  is good then
9     |  $G = G \cup Y$ 
10    |  $S = S \setminus Y$ 
11    | FOURTEST( $X_k$ )
12    | if  $3 \leq k \leq 5$  then
13      |  $A_{k+1}(X_{k+1}, T_k \setminus Y)$ 
14    | end if
15  | else
16    | if  $k > 3$  then
17      |  $A_{k-1}(Y, T_k \setminus Y)$ 
18    | else
19      |  $A_2(Y)$ 
20    | end if
21  | end if
22 end if
```

Algorithm 5: $A_2(\text{set } \{x, y, z, w\})$

```
1 for  $p \in \{x, y, z\}$  do
2   | TEST( $\{p\}$ )
3   | if  $p$  is defective then
4     |  $D = D \cup \{p\}$ 
5   | else
6     |  $G = G \cup \{p\}$ 
7   | end if
8   |  $S = S \setminus \{p\}$ 
9 end for
10 if  $\{x, y, z\}$  is good then
11   |  $S = S \setminus \{w\}$ 
12   |  $D = D \cup \{w\}$ 
13 end if
14  $T_3 = T_3 \cap S$ 
15  $A_3(X_3, T_3)$ 
```

- We detected $d_1 = 1$ defective element and $g_1 = 64 \frac{8^k - 1}{7}$ good elements using t_1 tests, where $t_1 = k + 2 + 1 + 3k + 4 = 4k + 7$ if $k > 1$ and $t_1 = 4k + 6 = 10$ if $k = 1$.
- We found $g_2 := 64 \frac{8^k - 1}{7} + 32 \cdot 8^k$ good elements and $d_2 = 1$ defective ones using at most $t_2 = 4k + 9$ tests
- $g_3 := 64 \frac{8^k - 1}{7} + 16 \cdot 8^k$ good elements and $d_3 = 2$ defective ones using at most $t_3 = 7k + 13$ tests.
- We found $g_4 := 64 \frac{8^k - 1}{7} + 48 \cdot 8^k$ good items and $d_4 = 1$ defectives by at most $t_4 = 4k + 9$ tests.

It is easy to check that

$$1.45198 d_i \left(\log \frac{g_i + d_i}{d_i} + 1.1699 \right) > t_i$$

for $1 \leq i \leq 4$.

In the second case we have less than $64 \cdot 8^k$ elements left. Since this phase of the algorithm started with more than $64 + 8 \cdot 64$ elements, we have $k \geq 2$ and we know that the remaining set is defective, hence we can extract a defective element by binary splitting. The number of tests is thus at most $k + 2 + 3k + 6 = 4k + 8$ which is smaller than $1.45198(\log(64 \frac{8^k - 1}{7} + 1) + 1.1699)$ for $k \geq 2$.

First Set Contaminated

Now we consider the case that the first set X of 64 elements is contaminated. If we would use FOURTEST, it might happen that one defective and no good element are detected by 6 tests which is too much. Instead, we resort to procedure A_6 (compare Algorithm 4) which uses $X_6 = X$ and $T_6 = S \setminus X$ as a source of new elements. We first test a set Y of 32 new elements from T_6 . If it is negative, we use FOURTEST on X_6 and proceed with MAIN, hence choosing 64 new elements and so on. If it is positive, procedure A_5 (compare Algorithm 4) is called with $X_5 = Y$. In general, for each $k \geq 3$, the procedure A_k , formally described in Algorithm 4, is applied to a defective subset X_k of size 2^k .

The algorithm first tests a set Y of 2^{k-1} elements from T_k which is disjoint from X_k . If Y is defective, algorithm A_{k-1} is applied to it.

If Y is good, FOURTEST(X_k) is executed and A_{k+1} is called again if $k < 6$.

The result of this execution of A_k is one of the following:

- $d_1 = 1$ defective and $g_1 = 2^{k-1}$ good elements by $t_1 = k + 1$ tests.
- $d_2 = 1$ and $g_2 = 2^k$ good elements by $t_2 = k + 3$ tests.
- $d_3 = 2$ defectives and $g_3 = 3 \cdot 2^{k-2}$ good elements by $t_3 = 2k + 1$ tests.
- $d_4 = 1$ defective and $g_4 = 5 \cdot 2^{k-2}$ good elements by $t_4 = k + 3$ tests.

$t < 1.45198 d \left(\log \frac{g+d}{d} + 1.1699 \right)$ which includes the tests of X_k and Y .

It may happen that A_3 generates a defective set Y of cardinality 4 and calls procedure $A_2(Y)$ formally described in Algorithm 5.

At this point, we test three elements of Y individually and identify at least 3 elements with 4 tests. Hence, we are left with the following cases:

- one defective and two good elements:
 $1.45198 \cdot 1 \cdot (\log 3 + 1.1699) > 4 \Leftrightarrow 1.45198 > \frac{4}{\log 3 + 1.1699} \approx 1.451978$
- one defective and three good elements: $1.45198 \cdot 1 \cdot (2 + 1.1699) > 4$
- two defectives and one good element: $1.45198 \cdot 2 \cdot (\log 3 - 1 + 1.1699) > 5$
- three defectives and no good elements: $1.45198 \cdot 3 \cdot (\log 1 + 1.1699) > 5$

The worst over-all case is the first one and so we get $c < 1.45198$ as an upper bound. Summarizing, we get two estimates for the number of tests used by our algorithm A :

Theorem 2.3. *There is some constant a such that for all $0 \leq d < n$ we have*

$$i) M_A(d | n) \leq 1.45198d \left(\log \frac{n}{d} + 1.1699 \right) + a =: f(n, d) ,$$

$$ii) M_A(d | n) \leq \frac{4}{3}n .$$

Proof:

- i) The proof is by induction on d and just summarizes the previous discussion. If $d = 0$, we either need at most $|S|$ individual tests for $|S| \leq 576$ or just 3 tests.

So assume that $d > 0$. If the algorithm starts with a negative test, it enters the expansion phase and identifies g' good and $d' > 0$ defective elements by t tests with

$$t < 1.45198d' \left(\log \frac{g' + d'}{d'} + 1.1699 \right)$$

as described above. By induction, the total number of tests is at most

$$1.45198d' \left(\log \frac{g' + d'}{d'} + 1.1699 \right) + f(n - (g' + d'), d - d') \leq f(n, d)$$

by Theorem 2.2.

Otherwise, the algorithm calls procedure A_6 . By the discussion above and similar reasoning, inequality *i)* holds in this case as well.

- ii) This follows by induction on n and is trivial for small n or if the algorithm first enters the expansion phase. If algorithm A_k , $k > 2$, chooses some good set Y with $|Y| = 2^{k-1}$, then $\text{FOURTEST}(X)$ is executed. It is easy to check that the number of tests of procedure FOURTEST plus the tests for X and Y are less than $|Y|$ plus the number of elements identified by FOURTEST . Finally, if A_2 is executed, at least three elements are identified by four tests, the case which is responsible for the ratio $4/3$ in inequality *ii)*. □

This leads to:

Theorem 2.4. *Algorithm A is 1.452-competitive.*

Proof:

We may assume that $d > 0$ and n is sufficiently large. Moreover, we consider three different ranges of $r = n/d$. If $r < 2$, we use Theorem 2.3 ii) and Theorem 1.1:

$$\frac{M_A(d|n)}{M(d,n)} \leq \frac{\frac{4}{3}n}{n-1} \longrightarrow \frac{4}{3} < 1.452 .$$

For $3 \geq r \geq 2$, we apply Theorem 2.3 ii) and Theorem 2.1:

$$\frac{M_A(d|n)}{M(d,n)} \leq \frac{\frac{4}{3}n}{\frac{n}{3}(\log 3 + 2(\log 3 - \log 2)) + O(\log n)} \longrightarrow \frac{4}{3 \log 3 - 2} < 1.452 .$$

Finally, if $r \geq 3$, we have with Theorem 2.3 i) and Theorem 2.1:

$$\frac{M_A(d|n)}{M(d,n)} \leq \frac{1.45198d(\log \frac{n}{d} + 1.1699) + a}{d(\log \frac{n}{d} + 1.1699) - \frac{1}{2} \log d - \frac{3}{2}} \longrightarrow 1.45198 < 1.452 .$$

□

3 A lower bound

To prove a lower bound for competitive constants c , we assume familiarity with the definition of the worst case search length as a game between two players as developed, e.g., in [3, section 1.10]: The first player (Algy) chooses a test set W and the second (the Strategist, his adversary) answers *good* or *defective*. Of course, his answers have to be consistent and his goal is to force Algy to ask as many questions as possible whereas Algy wants to minimize that number. Moreover, we can assume that Algy's test sets do not contain any classified elements. The worst case number of tests is just the number of tests in the game if both players play optimally from their point of view. In particular, if the adversary player follows some specific strategy and Algy plays optimally against this strategy, the number of tests needed is a lower bound for the general worst case number of tests. We are going to define and analyze such a strategy \mathcal{S} .

In order to precisely define the answers of the Strategist in the t -th round, we have to keep track of the results of the preceding tests. This is done by some hypergraph $H_t = (S_t, E_t)$, where S_t is the set of all items which have not been identified as good or defective after t tests, and E_t consists of all inclusion-minimal sets which are known to contain a defective element after t tests. H_t is defined inductively as follows:

- $H_0 := (S, \emptyset)$
- If the hypergraph H_{t-1} is defined (after $t-1$ tests and answers), assume that Algy chooses some test set W , w.l.o.g. $\emptyset \neq W \subset S_{t-1}$.
 - If the adversary answers *good*, all elements of W are removed from S_{t-1} . Furthermore, let U denote the set of items $x \in S_{t-1}$ such that $\{x\} = Y \setminus W$ for some $Y \in E_{t-1}$, i.e., all vertices which lie in edges of size one after the removal of W . The elements of U are identified as *defective* and are removed from S_{t-1} as well. Thus $S_t := S_{t-1} \setminus (U \cup W)$. Now let $E'_t := \{Y \setminus W \mid Y \in E_{t-1} \text{ and } Y \cap U = \emptyset\}$. E_t is defined as the set of all inclusion-minimal sets in E'_t .

- If the adversary answers *defective*, we distinguish two cases: If $|W| > 1$, we let $S_t := S_{t-1}$ and E_t consists of all inclusion-minimal sets in $E_{t-1} \cup \{W\}$. Otherwise, if $|W| = 1$, set $S_t := S_{t-1} \setminus W$ and $E'_t := \{Y | Y \in E_{t-1} \text{ and } Y \cap W = \emptyset\}$. E_t is defined as the set of all inclusion-minimal sets in E'_t .

Recall that a *vertex cover* of H_t is a set X with $X \cap Y \neq \emptyset$ for all $Y \in E_t$. After t tests, the possible sets of defectives are just the unions of the defective elements in $S \setminus S_t$ with the vertex covers of H_t and with elements which are not covered by any edge in H_t . For $|S_t| \geq 2$ all sets in E_t have cardinality at least two. Hence, subsets of S_t with $|S_t| - 1$ elements are vertex covers. Therefore, the set of defectives is not determined yet. The case $|S_t| = 1$ arises only if $E_t = \emptyset$. It follows that the set of defectives is determined iff $S_t = \emptyset$.

We can now define our strategy \mathcal{S} :

- If a set W of at least two elements is tested, the answer is “defective”.
- If a singleton is tested which is not contained in some $Y \in E_t$, the answer is “good”.
- If a singleton is tested which is contained in some $Y \in E_t$, the answer is “defective”.

Theorem 3.1. *If answers are given according to strategy \mathcal{S} , the following assertions hold for each group testing algorithm A :*

- i) All elements will never be declared defective.*
- ii) If the number of defectives in the game is d , then A used at least $n + d$ tests.*
- iii) If A is c -competitive, we have for $n \rightarrow \infty$*

$$c \geq \frac{M_A(d|n)}{M(d,n)} - o(1) \geq \min \left\{ \frac{n+d}{M(d,n)} \mid 0 \leq d < n \right\} - o(1).$$

Proof:

i) Assume to the contrary that all elements are declared defective. Then the Strategist always answers *defective*. As long as $|S_t| \geq 2$, a new defective can only be identified if Algy chooses a singleton as test set. Hence, for some t , we have $|S_t| = 1$ and $E_t = \emptyset$. But then the next test has to be $W = S_t$ and the answer is *good*, a contradiction.

ii) It is clear that the $(n - d)$ good items are identified by testing $(n - d)$ singletons. If some item x is declared defective in the $(t+1)$ -th test, the corresponding test has to be $W = \{x\}$ and there has to be some prior test of a set $Y = Y(x)$ such that $x \in Y \in E_t$. Since $Y \notin E_{t+1}$, $Y(x) \neq Y(x')$ for all other defective elements $x' \neq x$. It follows that A needs at least two tests for each defective element. The total number of tests is thus at least $2d + (n - d) = n + d$.

iii) Obvious, by *i)* and *ii)*. □

In the range $d \geq \frac{16}{43}n$, individual tests are optimal by Theorem 1.2, hence

$$\frac{n+d}{M(d,n)} = \frac{n+d}{n-1} \geq \frac{59}{43} - o(1) > 1.37$$

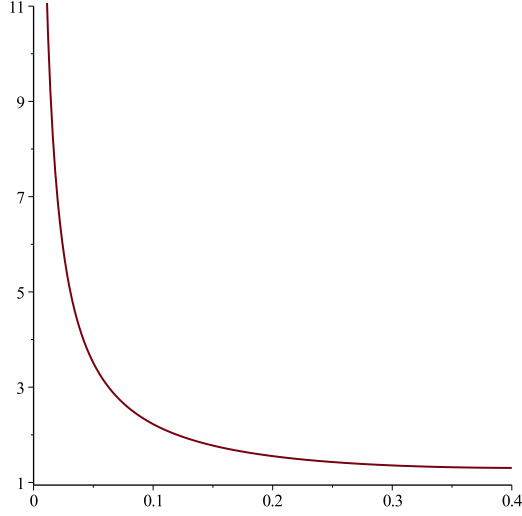


Figure 1: Function $f(p)$

For $d < \frac{16}{43}n$, we estimate

$$\frac{n+d}{M(d,n)} \geq \frac{n+d}{M_{A_d}(d,n)} .$$

where A_d is Allemann's group testing algorithm from Theorem 1.3, hence

$$M_{A_d}(d,n) \leq \log \left(\binom{n}{d} \right) + 0.255d + \frac{1}{2} \log(d) + 6.5$$

By using the estimate

$$\binom{n}{d} \leq \left(e \frac{n}{d} \right)^d ,$$

it is easy to see that for each constant B , there exists some $\epsilon > 0$ such that

$$\liminf_{n \rightarrow \infty, d \leq \epsilon n} \frac{n+d}{M_{A_d}(d,n)} \geq B :$$

Now let $p := p_n := \frac{d}{n}$ and $h(p) = -p \log p - (1-p) \log(1-p)$ denote the binary entropy function. It is well known that $\log \left(\binom{n}{d} \right) = n(h(p)) + o(1)$ for $\epsilon \leq p \leq 1/2$. We may thus write $n(h(p)) + 0.255p + o(1)$ for the number of tests in algorithm A_d , hence

$$\frac{n+d}{n(h(p)) + 0.255d + o(1)} = \frac{1+p}{h(p) + 0.255p + o(1)} .$$

The function $f(p) := \frac{1+p}{h(p)+0.255p}$ is non-increasing on $(0, 0.4)$ (see Figure 1). For $p = \frac{16}{43}$ we get $f(p) \approx 1.3103$. This shows our final result:

Theorem 3.2. *If an algorithm is c -competitive for sufficiently large n , then we have $c > 1.31$.*

4 Conclusion

We do not have a serious conjecture for the best possible competitive constant ($4/3$ would be nice) and guess that both bounds we developed can be improved significantly.

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