

A graph-theoretical axiomatization of oriented matroids

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Abstract

We characterize which systems of sign vectors are the cocircuits of an oriented matroid in terms of the cocircuit graph.

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1 Introduction

The cocircuit graph is a natural combinatorial object associated with an oriented matroid. In the case of spherical pseudoline-arrangements, i.e., rank 3

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oriented matroids, its vertices are the intersection points of the lines and two points share an edge if they are adjacent on a line. More generally, the Topological Representation Theorem of Folkman and Lawrence [9] says that every oriented matroid can be represented as an arrangement of pseudospheres. The cocircuit graph is the 1-skeleton of this arrangement. Cordovil, Fukuda and Guedes de Oliveira [5] show that a cocircuit graph does not uniquely determine an oriented matroid. But Babson, Finschi and Fukuda [1] show that *uniform* oriented matroids are determined up to isomorphism by their cocircuit graph. Moreover they provide a polynomial time recognition algorithm for cocircuit graphs of uniform oriented matroids. In [11], Montellano-Ballesteros and Strausz give a characterization of uniform oriented matroids in view of sign labeled cocircuit graphs. This characterization is strengthened by Felsner, Gómez, Knauer, Montellano-Ballesteros and Strausz [8] and used to improve the recognition algorithm of [1].

In this paper we present a generalization and strengthening of the characterization of sign labeled cocircuit graphs of uniform oriented matroids of [8] to general oriented matroids. After introducing the necessary preliminaries in the next section, we prove the main theorem in the last section.

2 Preliminaries

Here we will only introduce the terminology necessary for proving our result, for a more general introduction, see [3]. A *signed set* X on a ground set E is pair $X = (X^+, X^-)$ of disjoint subsets of E . For $e \in E$ we write $X(e) = +$ and $X(e) = -$ if $e \in X^+$ and $e \in X^-$, respectively, and $X(e) = 0$, otherwise. The *support* \underline{X} of a signed set X is the set $X^+ \cup X^-$. The *zero-support* of X is $X^0 := E \setminus \underline{X}$. By $-X$ we denote the signed set (X^-, X^+) . Given signed sets X, Y their *separator* is defined as $S(X, Y) := (X^+ \cap Y^-) \cup (X^- \cap Y^+)$.

Definition 2.1 A pair $\mathcal{M} = (E, \mathcal{C}^*)$ is called *oriented matroid* with *cocircuits* \mathcal{C}^* if \mathcal{C}^* is a system of signed sets with ground set E , satisfying the following axioms:

- (C0) $\emptyset \notin \mathcal{C}^*$
- (C1) $\mathcal{C}^* = -\mathcal{C}^*$
- (C2) if $X, Y \in \mathcal{C}^*$ and $\underline{X} \subseteq \underline{Y}$ then $X = \pm Y$
- (C3) for all $X, Y \in \mathcal{C}^*$ with $X \neq \pm Y$ and $e \in S(X, Y)$ exists $Z \in \mathcal{C}^*$ with $Z(e) = 0$, $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$

The *composition* of signed sets X, Y is the signed set $X \circ Y := (X^+ \cup$

$(Y^+ \setminus X^-, X^- \cup (Y^- \setminus X^+))$. Given a system \mathcal{C}^* of signed sets we denote by $\mathcal{L}(\mathcal{C}^*) := \{X_1 \circ \dots \circ X_k \mid X_1, \dots, X_k \in \mathcal{C}^*\}$ the set of all (finite) compositions of \mathcal{C}^* . Note that the empty set is considered as the empty composition of cocircuits, and so $\emptyset \in \mathcal{L}(\mathcal{C}^*)$. If \mathcal{C}^* are the cocircuits of an oriented matroid \mathcal{M} , then the elements of $\mathcal{L}(\mathcal{C}^*)$ are called the *covectors* of \mathcal{M} . One can endow $\mathcal{L}(\mathcal{C}^*)$ with a partial order relation where $Y \leq X$ if and only if $S(X, Y) = \emptyset$ and $\underline{Y} \subseteq \underline{X}$. Adding a global maximum $\hat{\mathcal{L}} := \mathcal{L}(\mathcal{C}^*) \cup \hat{1}$ it is easy to see that one obtains a lattice $\mathcal{F}_{\text{big}}(\mathcal{L}) := (\hat{\mathcal{L}}, \leq)$. If \mathcal{C}^* is the set of cocircuits of an oriented matroid, then $\mathcal{F}_{\text{big}}(\mathcal{L})$ is graded lattice with rank function h . In this case $\mathcal{F}_{\text{big}}(\mathcal{L})$ is called *the big face lattice* of \mathcal{M} . The *rank* $r(\mathcal{M})$ of \mathcal{M} is $h(\hat{1}) - 1$, i.e., one less than the rank of $\mathcal{F}_{\text{big}}(\mathcal{L})$.

There are two important undirected graphs associated to $\mathcal{F}_{\text{big}}(\mathcal{L})$ – one on its atoms and one on its coatoms. So the first is a graph on \mathcal{C}^* . In the case of \mathcal{C}^* being the cocircuits of an oriented matroid \mathcal{M} it is called the *cocircuit graph* of \mathcal{M} . Define $G(\mathcal{C}^*)$ on \mathcal{C}^* such that two signed sets $X, Y \in \mathcal{C}^*$ are connected by an edge if and only if there is $Z \in \hat{\mathcal{L}}$ such that X, Y are the only elements of \mathcal{C}^* with $X, Y \leq Z$.

The other graph induced by $\mathcal{F}_{\text{big}}(\mathcal{L})$ is defined on the set \mathcal{T} of coatoms of $\mathcal{F}_{\text{big}}(\mathcal{L})$ the poset. Elements of \mathcal{T} are called *topes*. Topes $S, T \in \mathcal{T}$ are contained in an edge if and only if there is $Z \in \hat{\mathcal{L}}$ such that S, T are the only elements of \mathcal{T} with $X, Y \geq Z$. This graph called the *tope graph* is denoted by $G(\mathcal{T})$.

If G is a graph on a system \mathcal{S} of signed sets with ground set E . For $X_1, \dots, X_k \in \mathcal{S}$ we denote by $[X_1, \dots, X_k]$ the subgraph of G induced by $\{Z \in \mathcal{S} \mid Z(e) \in \{0, X_1(e), \dots, X_k(e)\} \text{ for all } e \in E\}$. We call $[X_1, \dots, X_k]$ the *crabbed hull* of X_1, \dots, X_k . An (X, Y) -path in G is called *crabbed* if it is contained in $[X, Y]$.

One important oriented matroid operation is the *contraction*. Let $A \subseteq E$, then \mathcal{M}/A is an oriented matroid on the ground set $E \setminus A$ with $\mathcal{C}^*/A := \{X \setminus A \mid X \in \mathcal{C}^* \text{ and } A \subseteq X^0\}$. The set $\mathcal{L}(\mathcal{C}^*/A)$ is easily seen to be $\{X \setminus A \mid X \in \mathcal{L}(\mathcal{C}^*) \text{ and } A \subseteq X^0\}$. It is easy to see that for $U \in \mathcal{L}(\mathcal{C}^*)$ we have $r(\mathcal{M}/U^0) = h(U)$, where $h(U)$ is the rank of U in $\mathcal{F}_{\text{big}}(\mathcal{L})$.

3 Result

In order to prove Theorem 3.3 we need two lemmas. The first one is about tope graphs of oriented matroids. Tope graphs of oriented matroids are a special class of *antipodal partial cubes* [10]. We will make use of a particular consequence of this:

Lemma 3.1 ([4]) *Let \mathcal{M} be an oriented matroid with topes $U, V \in \mathcal{T}$. For all $U, V \in \mathcal{T}$ there is a crabbed (U, V) -path in $G(\mathcal{T})$.*

The second lemma establishes a connection between tope graph and co-circuit graph. As an application of a theorem of Barnette [2] Cordovil and Fukuda prove:

Lemma 3.2 ([5]) *Let \mathcal{M} be an oriented matroid of rank r and $U \in \mathcal{T}$ a tope of \mathcal{M} . The graph $G(U)$ induced by $\{X \in \mathcal{C}^* \mid X \circ U = U\}$ in $G(\mathcal{C}^*)$ is $(r - 1)$ -connected.*

Together this enables us to prove a graph-theoretical axiomatization of oriented matroids:

Theorem 3.3 *Let \mathcal{C}^* be a set of sign vectors satisfying (C0)–(C2) then the following are equivalent*

- (i) \mathcal{C}^* is the set of cocircuits of an oriented matroid \mathcal{M} ,
- (ii) the crabbed hull $[X_1, \dots, X_k]$ of any $X_1, \dots, X_k \in \mathcal{C}^*$ is an induced subgraph of connectivity $h(X_1 \circ \dots \circ X_k) - 1$ of $G(\mathcal{C}^*)$,
- (iii) for all $X, Y \in \mathcal{C}^*$ with $X \neq \pm Y$ there is a crabbed (X, Y) -path in $G(\mathcal{C}^*)$.

Proof. (i) \implies (ii): Let $U := X_1 \circ \dots \circ X_k$ be a covector of rank $r' := h(X_1 \circ \dots \circ X_k)$ and X, Y cocircuits in $[X_1, \dots, X_k]$. Contract U^0 obtaining $\mathcal{M}' := \mathcal{M}/U^0$ of rank r' and cocircuits X', Y' . The contraction does not affect the crabbed hull we are considering, i.e. $[X_1, \dots, X_k] \cong [X'_1, \dots, X'_k]$. Now U' is a tope of \mathcal{M}' and so are $V' := X' \circ U'$ and $W' := Y' \circ U'$. By Lemma 3.1 there is a crabbed (V', U') -path $P = (V' = T_1, \dots, T_k = U')$ in $G(\mathcal{T}')$. The graphs $G(T_i)$ are all contained in $[X'_1, \dots, X'_k]$ and $(r - 1)$ -connected by Lemma 3.2. Consecutive $G(T_i)$ and $G(T_{i+1})$ intersect in at least $r' - 1$ vertices, because their intersection is a tope of a one-element-contraction minor of \mathcal{M}' . Together Menger's theorem (see e.g. [7]) yields that the graph $G(T_1) \cup \dots \cup G(T_k)$ is $(r' - 1)$ -connected. In particular there are $(r' - 1)$ internally disjoint paths connecting X' and Y' in $[X'_1, \dots, X'_k]$ and thus the analogue holds for X and Y in $[X_1, \dots, X_k]$. Hence $[X_1, \dots, X_k]$ is $(h(X_1 \circ \dots \circ X_k) - 1)$ -connected.

(ii) \implies (iii): If $X \neq \pm Y$ then $(h(X \circ Y) - 1) > 0$. Hence $[X, Y]$ is connected and there is a crabbed (X, Y) -path in $G(\mathcal{C}^*)$.

(iii) \implies (i): We have to show that (C3) holds for \mathcal{C}^* . Let $X, Y \in \mathcal{C}^*$ with $X \neq \pm Y$ and $e \in S(X, Y)$. Let P be a crabbed (X, Y) -path. Since adjacent cocircuits have empty separator, there must be $Z \in P$ with $Z(e) = 0$. Since P is crabbed Z also satisfies $Z^+ \subseteq X^+ \cup Y^+$ and $Z^- \subseteq X^- \cup Y^-$. \square

It shall be mentioned that the “(i) \implies (ii)”-part of the proof is only a slight generalization of a result in [5]. But there the characterizing quality of (ii) was not noted. Furthermore we remark that the connectivity in (ii) is best-possible, since in uniform oriented matroids X_i has exactly $h(X_1 \circ \dots \circ X_k) - 1$ neighbors in $[X_1, \dots, X_k]$.

Even if the cocircuit graph does not uniquely determine the oriented matroid, Theorem 3.3 might lead to an effective recognition algorithm for cocircuit graphs of general oriented matroids, as its uniform specialization did in [8]. In particular, if one is given $G(\mathcal{C}^*)$ with edge set \mathcal{E} it is possible to check (iii) in $\mathcal{O}(|\mathcal{C}^*||\mathcal{E}|)$, see part 5.C. of the algorithm in [8]. In contrast the naive algorithm to check (C3) takes $\mathcal{O}(|\mathcal{C}^*|^3)$. So we have an advantage for sparse cocircuit graphs, e.g., cocircuit graphs of uniform oriented matroids.

Another goal would surely be to characterize cocircuit graphs in purely graph-theoretic terms, i.e., excluding any information about signed sets at all.

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