

# Randomness below complete theories of arithmetic\*

George Barmpalias<sup>1</sup> and Wei Wang<sup>2</sup>

<sup>1</sup>*State Key Lab of Computer Science, Inst. of Software, Chinese Acad. of Sciences, Beijing, China*

<sup>2</sup>*Inst. of Logic & Cognition and Dept. of Philosophy, Sun Yat-Sen University, Guangzhou, China*

Thursday 17<sup>th</sup> November, 2022

**Abstract.** We show that reals  $z$  which compute complete extensions of arithmetic have the random join property: for each random  $x <_T z$  there exists random  $y <_T z$  such that  $z \equiv_T x \oplus y$ . The same is true for the truth-table and the weak truth-table reducibilities.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Background and state-of-the-art</b>	<b>2</b>
<b>3</b>	<b>Random join property for PA degrees</b>	<b>6</b>
<b>4</b>	<b>Conclusion and discussion</b>	<b>12</b>

## 1 Introduction

The incompleteness phenomenon in arithmetic, discovered by Gödel (1931), implies that there is no computable binary predicate consistently evaluating the truth of each arithmetical sentence. Formal arithmetic is known as Peano arithmetic as it is generated by Peano's axioms. Theories of arithmetic can be identified with the infinite binary sequences, *reals*, under a standard fixed coding of formulas into positive integers. Let  $\text{PA}$  denote the set of reals that encode complete extensions of arithmetic; the set  $\text{PA}$  forms an effectively closed set, a  $\Pi_1^0$  class. The computational power of members of  $\text{PA}$ , in terms of their Turing degrees, has been investigated since (Scott and Tennenbaum, 1960), and its systematic study started by Jockusch and Soare (1972).

Levin (2013) pointed out that the relationships between  $\text{PA}$  and algorithmically random reals, in the sense of Martin-Löf (1966), are essential for understanding the limitations of obtaining

---

\*Supported by NSFC grant No. 11971501 and partially motivated by the participation in the virtual program of the Institute for Mathematical Sciences, National University of Singapore, in 2021.

solutions to arithmetical problems by probabilistic methods. These relationships have been under investigation since the work of Kučera (1985).

We show that the PA degrees (Turing degrees of reals in PA) have the *join property* with respect to random degrees (Turing degrees of random reals). In general, a degree  $\mathbf{a}$  is said to have the *join property* if, for every non-zero  $\mathbf{b} < \mathbf{a}$ , there exists  $\mathbf{c} < \mathbf{a}$  with  $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ .

The PA degrees do not, in general, satisfy the join property (Lewis-Pye, 2012). In contrast, we show that for every PA degree  $\mathbf{a}$ , random  $\mathbf{b} < \mathbf{a}$  there exists random  $\mathbf{c} < \mathbf{a}$  with  $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ .

**Theorem 1.1.** *If  $z$  is PA and  $x <_T z$  is random, there exists random  $y <_T z$  such that  $x \oplus y \equiv_T z$ . The same is true for truth-table and weak truth-table reducibility:  $<_{tt}, \equiv_{tt}$  and  $<_{wtt}, \equiv_{wtt}$ .*

The universality of PA reals implies that they compute some random real. Hence Theorem 1.1 generalizes an older result by Barmpalias et al. (2010), which says that for each PA real  $z$  there exist random  $x, y$  such that  $z \equiv_T x \oplus y$ . The latter was used by Moser and Stephan (2015) to derive an interesting fact about *logical depth*, invented by Bennett (1988) in order to quantify the hardness of obtaining useful information via probabilistic algorithms.

Intuitively, a real is *deep* if it cannot be produced by a randomized algorithm with non-trivial probability. The halting problem, for example, is deep. Reals that are not deep are called *shallow*, and include the computable and the random reals. Depth is not invariant with respect to Turing equivalence, but it is invariant with respect to computations with computable time-bound; equivalently, deep reals are upward closed with respect to truth-table reductions.

Moser and Stephan (2015) showed that there are shallow  $x, y$  such that  $x \oplus y$  is deep. We obtain the following extension, using Theorem 1.1. A real  $x$  is  *$\mathbf{0}$ -dominated* (also known as *hyperimmune-free*) if every  $x$ -computable function is dominated by a computable function.

**Corollary 1.2.** *Let DOM denote the class of  $\mathbf{0}$ -dominated reals.*

- (a) *For each random  $x \in \text{DOM}$ , there exists random  $y \in \text{DOM}$  such that  $x \oplus y \in \text{DOM} \cap \text{PA}$ .*
- (b) *For each random  $x$ , there exists random  $y$  such that  $x \oplus y$  is deep.*

The state-of-the-art regarding random and PA reals is given in §2, and the proofs of our results are presented in §3. We end with a brief discussion in §4, which includes questions and suggestions on the same topic.

## 2 Background and state-of-the-art

For an introductory survey on PA, random and  $\mathbf{0}$ -dominated reals and degrees, look no further than (Diamondstone et al., 2010). Let  $2^\omega$  denote the set of infinite binary sequences and let  $2^{<\omega}$  denote the set of binary strings. We also use  $2^n$  to denote the set of  $n$ -bit strings and  $2^{\geq n}$  for the set of strings of length  $\geq n$ .

We review the aspects of these topics that are directly relevant to our result.

## 2.1 Random and PA reals

A real in PA need not compute the halting set, and indeed can have low degree of unsolvability. Viewed as an infinite binary sequence, a PA real  $z$  encodes a selection with respect to every nonempty  $\Pi_1^0$  subset of  $\mathbb{N}$ :

$$\text{there exists total } f \leq_{tt} z \text{ such that if } e \text{ is an effective description of a } \Pi_1^0 \text{ set } D_e \quad (1)$$

(as the set of solutions of a  $\Pi_1^0$  predicate) either  $D_e = \emptyset$  or  $f(e) \in D_e$ .

In other words,  $z$  may not be able to tell if  $D_e$  is empty, but it can choose a number which will be in  $D_e$  unless  $D_e$  is empty. Similarly, a PA real computes a member in any given nonempty  $\Pi_1^0$  subset of  $2^\omega$ , uniformly on the index of the class.

Another characterization is in terms of *diagonally noncomputable DNC functions*, namely functions  $f$  with  $\forall n, f(n) \neq \varphi_e(e)$ , where  $\varphi_e$  is a computable enumeration of all partial computable functions. A Turing degree is PA iff it contains a boolean DNC function. In §2.2 we will discuss *logical-depth* which is invariant up to truth-table degrees, but not up to Turing degrees. It is therefore important to distinguish PA reals from reals of PA degree. One can think a PA real as a DNC function with binary range, or a real that effectively encodes choice-functions for  $\Pi_1^0$  sets, in a truth-table way (equivalently, the retrieval of information occurs within a fixed computable time-bound).

A subclass of DNC degrees (Turing degrees that contain a DNC function) is the class of Martin-Löf random, also known as 1-random (or simply *random*), degrees. Let  $\mu$  denote the standard Lebesgue measure on  $2^\omega$ . A uniformly c.e. sequence  $U_i \subseteq 2^{<\omega}$ , viewed as effectively open sets (namely the set  $\llbracket U_i \rrbracket$  of reals which have a prefix in  $U_i$ ), with  $\mu(U_i) \leq 2^{-i}$  is called a *Martin-Löf test*. A real  $x$  is *random* if for each Martin-Löf test  $(U_i)$  only finitely many members  $U_i$  contain a prefix of  $x$ . There exists a *universal Martin-Löf test*: a test  $(U_i)$  such that for every  $i$  and every Martin-Löf test  $(V_j)$ , we have  $\llbracket V_j \rrbracket \subseteq \llbracket U_i \rrbracket$  for all but finitely many  $j$ . Hence  $2^\omega - \llbracket U_i \rrbracket$  consists entirely of random reals, and the random reals are exactly the union of these sets with respect to  $i$ . These notions can be relativized with respect to any real  $z$ , where the test  $(U_i)$  is only required to be c.e. relative to  $z$ . Randomness relative to the halting problem  $\mathbf{0}'$  is called *2-randomness*.

The complement of a member  $U_i$  of a universal test can be viewed as a  $\Pi_1^0$  class of positive measure, consisting entirely of random reals. By the properties discussed above, every real of PA degree computes a random real.

Further results connect these classes of reals and their degrees:

- (i) every random real computes a DNC function (Kučera, 1985)
- (ii) every PA degree is the join of two random degrees (Barnališ et al., 2010)
- (iii) a random  $x$  has PA degree iff  $x \geq_T \mathbf{0}'$  (Stephan, 2006)

and in (i), (ii) the reductions are truth-table.

## 2.2 Kolmogorov complexity, PA and logical depth

The length of the shortest program for the universal Turing machine that outputs  $\sigma$  is known as the *Kolmogorov complexity* of  $\sigma$ , and is denoted by  $C(\sigma)$ . The same statement in terms of self-delimiting or prefix-free machines<sup>1</sup> defines the *prefix-free Kolmogorov complexity* of  $\sigma$ , denoted by  $K(\sigma)$ ; it is a refined information content measure compared to  $C(\sigma)$ . The Martin-Löf reals defined in §2.1 are exactly the reals  $x$  whose prefixes cannot be compressed by more than a constant:  $\exists c \forall n K(x \upharpoonright_n) \geq n - c$ .

The classes of PA, DNC and Turing-hard (namely computing  $\mathbf{0}'$ ) reals can be characterized in terms of Kolmogorov complexity (Kjos-Hanssen et al., 2011, §4):

- $x$  computes a DNC function iff  $\exists f \leq_T x \forall n : C(f(n)) \geq n$
- $x$  truth-table computes a DNC function iff  $\exists f \leq_{tt} x \forall n : C(f(n)) \geq n$
- $x$  computes a real in PA iff  $\exists f \leq_{tt} x \forall n, \sigma \in 2^n (f(n) \in 2^n \wedge C(f(n)) \geq C(\sigma))$
- $x \geq \mathbf{0}' \iff \exists f \leq_T x \forall n, \sigma \in 2^n (f(n) \in 2^n \wedge K(f(n)) \geq K(\sigma))$

Greenberg et al. (2021); Franklin et al. (2011) showed that computing a PA real is equivalent to computing a martingale that majorizes the optimal c.e. supermartingale. These examples demonstrate the ubiquity of PA degrees in computability and algorithmic information.

Many of the above results about PA and random reals indicate a qualitative difference between the information in an incomplete random real and the information in a PA real. The latter is highly structured and useful, as it allows to obtain a completion of any partial computable predicate. In contrast, although the information content in random reals is very large in terms of their Kolmogorov complexity, this information appears unusable to a large degree. This type of qualification of information was formalized by Bennett (1988) through the notion of *logical-depth* which applies to strings and reals. Several variants have been studied since, but as shown by Moser (2013), all depth notions can be interpreted in the compression framework:

a sequence is *deep* if given unlimited time, a compressor can compress the sequence  $r(n)$  more bits than if given at most  $t(n)$  time steps.

By considering different time-bounds  $t(n)$  and *depth-magnitudes*  $r(n)$ , all existing depth notions can be expressed in the compression framework. Bennett (1988) defined depth in terms of prefix-free Kolmogorov complexity, and one of his simplest formulations is obtained by considering computable  $t(n)$  and  $r(n) = O(1)$ . This was extended by Moser and Stephan (2015), and we use part of their terminology in the following definition due to Bennett (1988).

A nondecreasing unbounded  $g : \mathbb{N} \rightarrow \mathbb{N}$  is called an *order*. Let  $K^t$  denote the restriction of  $K$  with respect to time-bound  $t$ .<sup>2</sup>

<sup>1</sup>Turing machines with domain a prefix-free subset of  $2^{<\omega}$ .

<sup>2</sup>Let  $U_t$  denote the computations by the universal prefix-free machine up to step  $t(n)$  for inputs of length  $n$ , and let  $K^t$  denote the prefix-free Kolmogorov complexity relative to  $U_t$ .

**Definition 2.1.** A real  $x$  is *O(1)-deep* if for every computable time-bound  $t$ , every  $c$  and almost all  $n$ ,  $K^t(x \upharpoonright_n) > K(x \upharpoonright_n) + c$ . A real  $x$  is *order-deep* if there exists a computable order  $g$  such that for every computable time-bound  $t$ , and almost all  $n$ ,  $K^t(x \upharpoonright_n) > K(x \upharpoonright_n) + g(n)$ .

Order-deep implies O(1)-deep. The facts we present hold for both classes, so we simply talk of *deep reals* to refer to either notion. A real that is not deep is called *shallow*. Intuitively, deep reals are compressible but any computably time-bounded process can only produce a far from optimal compression.

Bennett (1988) and Moser and Stephan (2015) observed:

- (i) computable and random reals are shallow
- (ii) the halting set and, in fact every PA real, is deep
- (iii) if  $x$  is deep then each  $y \geq_T x$  is deep
- (iv) there are shallow  $x, y$  such that  $x \oplus y$  is deep.

Item (iv) involves finding random  $x, y$  such that  $x \oplus y$  is truth-table equivalent to a PA real.

### 2.3 Join properties of PA and random degrees

Our result is a join property of PA degrees with respect to random degrees. The *join property* is one of the central algebraic properties in the study of the structure of the Turing degrees:

**Definition 2.2.** A Turing degree  $\mathbf{a}$  satisfies the

- *join property* if, for every non-zero  $\mathbf{b} < \mathbf{a}$ , there exists  $\mathbf{c} < \mathbf{a}$  with  $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$ .
- *cupping property* if, for all  $\mathbf{b} > \mathbf{a}$ , there exists  $\mathbf{c} < \mathbf{b}$  with  $\mathbf{a} \vee \mathbf{c} = \mathbf{b}$ .

Does every PA degree have the join property? That was a question raised by Kučera (1985) and answered in the negative by Lewis-Pye (2012).<sup>3</sup> Recall that  $x$  is *low* if  $x' \equiv_T \emptyset'$ , namely the halting problem relative to  $x$  is Turing-equivalent to the unrelativized halting set  $\emptyset'$ . The

- low DNC and low random degrees do not have the join property (Lewis-Pye, 2012)
- low PA degrees do not have the join property (Lewis-Pye, 2012)
- 2-random degrees have the join property (Barnali et al., 2014)
- PA degrees have the cupping property (Kučera, 1985)
- 2-random degrees do not have the cupping property (Barnali et al., 2014).

Theorem 1.1 cannot be extended with respect to other standard algebraic properties studied in the Turing degrees, such as bounding a minimal pair, being the supremum of a minimal pair, or even having the meet or complementation property.<sup>4</sup> It is known that:

<sup>3</sup>Kučera (1985) calls the degrees having the join property *cuppable*.

<sup>4</sup>An informative table of these conditions can be found in (Barnali et al., 2015).

(i) any pair of DNC degrees below  $\mathbf{0}'$  fails to be a minimal pair (Kučera, 1988)

(ii) every PA degree bounds a minimal pair (Jockusch and Soare, 1971).

Toward a strengthening of the join property with randoms shown in Theorem 1.1, one would consider the property that reals in PA bound a minimal pair of randoms (before looking at being the join of a minimal pair of randoms). However even this basic property fails due to (i) and the fact that randoms are DNC. This failure contrasts with (ii).

### 3 Random join property for PA degrees

We prove Theorem 1.1: if  $z$  is PA and  $x <_T z$  is random, there exists random  $y <_T z$  such that  $x \oplus y \equiv_T z$ . To this end, we need to code  $z$  into  $y$ , in a way that  $z$  is recoverable from  $y$  with the help of oracle  $x$ .

#### 3.1 Coding into a random

The coding we use is based on the standard coding method into randoms by Gács (1986) and Kučera (1985). The general form of this method is reviewed by Barmpalias and Lewis-Pye (2020) along with its limitations, some of which are relevant to our task. The idea is to start with a  $\Pi_1^0$  class of positive measure which contains only randoms. A  $\Pi_1^0$  class of reals is the set of paths  $[P] \subseteq 2^\omega$  of a  $\Pi_1^0$  tree  $P$ , namely a  $\Pi_1^0$  downward prefix-closed subset of  $2^{<\omega}$ , without dead-ends (every node in  $P$  has a successor in  $P$ ). Kučera (1985) showed that each  $\tau \in P$  has a computable positive lower-bound on  $\mu([P] \cap \llbracket \tau \rrbracket)$ , where  $\llbracket \tau \rrbracket := \{x \in [P] : \tau < x\}$ . This implies that there exists computable increasing  $(\ell_n)$  such that

$$\text{each } \tau \in P \cap 2^{\ell_n} \text{ has at least two extensions in } \tau \in P \cap 2^{\ell_{n+1}}. \quad (2)$$

Barmpalias and Lewis-Pye (2020) noted that if  $(\ell_n)$  is computable and increasing:

$$(2) \text{ holds for all sufficiently large } n \text{ iff } \sum_n 2^{-(\ell_{n+1} - \ell_n)} < \infty.$$

This method was used in the Kučera-Gács theorem: every real  $z$  is computable by a random  $y$ . The coding consists of choosing  $y \upharpoonright_{\ell_{n+1}}$  as the leftmost or rightmost extension of  $y \upharpoonright_{\ell_n}$  in  $P$ , according to whether  $z(n)$  is 0 or 1. The decoding of  $y$  into  $z$  is possible due to the fact that  $P$  has a computable monotone approximation.

Our method is based on this type of coding, except that we cannot let  $y$  be locally on the boundary of  $P$ :<sup>5</sup> such reals are known to be Turing complete. Let  $\mu_\sigma([P]) := 2^{|\sigma|} \cdot \mu([P] \cap \llbracket \sigma \rrbracket)$ . Barmpalias and Lewis-Pye (2020, Corollary 2.10) observed that if

$$\sum_i q_i < \infty \wedge \forall n, \ell_{n+1} > -\log q_n + \ell_n \quad (3)$$

---

<sup>5</sup>in the sense that  $y \upharpoonright_{\ell_{n+1}}$  is the leftmost or rightmost  $\ell_{n+1}$ -bit extension of  $y \upharpoonright_{\ell_n}$  in  $P$

for positive computable rational and integer sequences  $(q_n), (\ell_n)$ , then

$$\exists k_0 \forall n \geq k_0 \forall \sigma \in 2^{\ell_n} \cap P : \mu_\sigma([P]) > q_n. \quad (4)$$

For the remaining part of our article let  $q_n := 1/(n+1)^2$ , fix  $P$  to be a nonempty  $\Pi_1^0$  tree without dead-ends, whose paths are all random, and assume that  $(\ell_n)$  is computable so by (4):

$$\text{each } \tau \in P \cap 2^{\ell_n} \text{ has at least two extensions in } \tau \in P \cap 2^{\ell_{n+1}}. \quad (5)$$

as long as  $(\ell_n)$  satisfies the latter part of (3), e.g.  $\ell_n = 2^n$ .

### 3.2 Outline of the proof

Given  $z \in \text{PA}$  and random  $x <_T z$  we will find  $y <_T z$  inside  $[P]$  such that  $x \oplus y \equiv_T z$ .

We will use property (1) of the PA reals  $z$ :

$$\text{given a nonempty } \Pi_1^0 \text{ set } Q \subseteq 2^{<\omega} \text{ we can } z\text{-effectively choose a} \quad (6)$$

$$\text{member of } Q, \text{ uniformly in any } \Pi_1^0 \text{ index of } Q.$$

Let  $\ell = (\ell_n)$  be exponentially growing (e.g.  $\ell_0 = 0, \ell_n = 2^n, n > 0$ ) and let

$$[\tau] := \{\tau' : \tau \leq \tau'\}.$$

Using (6) we can define initial segments  $y \upharpoonright_{\ell_n}$  so that  $y \leq_T z$  and  $y \in [P]$ .

A naive approach to this task is: at step  $n$ , when we choose the extension of  $y \upharpoonright_{\ell_n}$  in  $P \cap 2^{\ell_{n+1}}$  to choose one with even/odd number of 1s, in order to encode  $z(n) = 0, 1$  respectively (or, in general, use some *effective partition*  $F_0, F_1$  of  $[y \upharpoonright_{\ell_n}] \cap 2^{\ell_{n+1}}$ ).

The problem here is that at some step  $n$ ,

$$\begin{aligned} &\text{the set of extensions of } y \upharpoonright_{\ell_n} \text{ in } P \cap 2^{\ell_{n+1}} \text{ may not contain representatives} \\ &\text{of both classes in the partition: } F_0 \cap P = \emptyset \vee F_1 \cap P = \emptyset \end{aligned} \quad (7)$$

which would force us to encode wrong information about  $z$  in  $y$ . This could happen since  $P$  contains random  $y \geq_T \emptyset'$  and, by [Bienvenu et al. \(2014, Theorem 3.2\)](#),  $\inf_{\sigma < z} \mu_\sigma(P) = 0$ ; in fact, the proof of this characterization of Turing-hard random reals can be easily extended in order to show that the density  $\mu_\sigma(P)$  along  $y \geq_T \emptyset'$  drops at the expense of any effective partition  $F_0, F_1$ , making  $F_i \cap P = \emptyset$  for some  $i < 2$ .

Hence the partition of the extensions of  $y \upharpoonright_{\ell_n}$  in  $P \cap 2^{\ell_{n+1}}$  cannot be fully effective, and this is where the availability of  $x$  plays a role: it suffices to show that every random  $x$  uniformly defines a partition function, such that (7) does not happen, for sufficiently large  $n$ . We do this in §3.4: after uniformly encoding all such partition systems into reals  $x$  in §3.3, we show that the effective probability of the partition systems satisfying (7) tends to 0 as  $n$  tends to infinity. In other words, if  $x$  is random, for sufficiently large  $n$  the partition system defined by  $x$  fails (7). The coding of  $z$  into  $y$  can then be done as in the naive construction, but with the use of the partitions defined by  $x$ . By the construction,  $x \oplus y \equiv_T z$ .

### 3.3 Partition systems

Let  $\ell = (\ell_n)$  be computable and increasing with  $\ell_0 = 0$ , and  $m_n := \ell_{n+1} - \ell_n$ . An  $\ell$ -partition system is a perfect tree whose  $n$ -level nodes  $D_\sigma, \sigma \in 2^n$  are pairwise disjoint subsets of  $2^{\ell_n}$  of equal size, whose union is  $2^{\ell_n}$  and which are monotone with respect to  $\leq$ . You can think of an  $\ell$ -partition system as an interactive process which partitions the  $\ell_{n+1}$ -bit extensions of each  $\tau \in 2^{\ell_n}$  into two equal-sized sets. Given  $\tau \in 2^{\ell_n}$  there exists a unique  $\sigma \in 2^n$  such that  $\tau \in D_\sigma$  and which gives the partition of  $[\tau] \cap 2^{\ell_{n+1}}$  into  $[\tau] \cap D_{\sigma*0}, [\tau] \cap D_{\sigma*1}$ . Formally:

**Definition 3.1.** An  $\ell$ -partition system is a map  $\sigma \mapsto D_\sigma \subseteq 2^{\ell_n}$ , where  $n = |\sigma|$ , such that:

- $D_{\sigma*0} \cap D_{\sigma*1} = \emptyset$  and each string in  $D_{\sigma*0}, D_{\sigma*1}$  has a prefix in  $D_\sigma$ ,
- $\tau \in D_\sigma \Rightarrow [\tau] \cap (D_{\sigma*0} \cup D_{\sigma*1}) = [\tau] \cap 2^{\ell_{n+1}}$ ,
- $\tau \in D_\sigma \Rightarrow |[\tau] \cap D_{\sigma*0}| = |[\tau] \cap D_{\sigma*1}| = 2^{m_n-1}$ .

For simplicity, given a partition-system  $\sigma \mapsto D_\sigma$  we:

- call the restriction of  $\sigma \mapsto D_\sigma$  to  $2^{\leq n}$  a *partition-system of height  $n$*
- fix  $\ell_n := 2^n$  and refer simply to *partition-systems*<sup>6</sup>

Let  $\mathcal{B}$  be the class of partition-systems and  $\mathcal{B}_n$  the class of partition-systems of height  $n$ .

For each  $\tau \in 2^{\ell_n}$  there exists a unique  $\sigma \in 2^n$  such that  $\tau \in D_\sigma$ .

Note that  $D_\sigma, \sigma \in 2^n$  is itself a partition of  $2^{\ell_n}$  into  $2^n$  parts. For  $\sigma \in 2^n, \tau \in D_\sigma$  we are interested in the partition of  $[\tau] \cap 2^{\ell_{n+1}}$  into  $D_{\sigma*0} \cap [\tau], D_{\sigma*1} \cap [\tau]$ . So  $(D_{\sigma*0}, D_{\sigma*1})$  can be viewed as the group of partitions  $(D_{\sigma*0} \cap [\tau], D_{\sigma*1} \cap [\tau]), \tau \in D_\sigma$ .<sup>7</sup>

**Definition 3.2** (Codes for partitions). Fix a computable increasing  $(u_n)$  and an effective naming  $f$  of partition-systems of finite height by strings such that:

- each  $\sigma \in 2^{u_n}$  corresponds to a unique partition-system  $f(\sigma) \in \mathcal{B}_n$
- each extension of  $\sigma$  in  $2^{u_{n+1}}$  corresponds to an extension of  $f(\sigma)$  in  $\mathcal{B}_{n+1}$ .

We get an effective naming of  $\mathcal{B}$  by reals: the computable surjection  $f : 2^\omega \rightarrow \mathcal{B}$  given by  $f(x) := \lim_n f(x \upharpoonright_{u_n})$ . Let

- $\sigma \mapsto D_\sigma^x$  denote the partition-system  $f(x)$
- $\sigma \mapsto D_\sigma^\eta$  denote the partition-system  $f(\eta)$ , when  $\eta \in 2^{u_n}$  for some  $n$ .

<sup>6</sup> $\ell_n$  is the length of the prefix of  $x$  needed for the computation of  $z$  from  $x, y$ ; later we will see that setting  $\ell_n = n \cdot \lceil \log n \rceil$  suffices for our result.

<sup>7</sup>This is why in many expressions we need to include  $[\tau]$ : to specify that we refer to the split of the  $\ell_{n+1}$ -bit extensions of  $\tau$  that the partition-system defines.



Each real computes the height- $n$  prefix of its corresponding member of  $\mathcal{B}$  with computable oracle-use  $u_n$ . Recall  $q_n := 1/(n+1)^2$  from §3.1 and that  $\ell_n := 2^n$ ,  $m_n := \ell_{n+1} - \ell_n$ . Hence:

$$2^{\ell_{n+1}} \cdot 2^{-q_n^2 \cdot 2^{m_n}} < 2^{-n} \quad \text{for sufficiently large } n \quad (8)$$

which we will use along with (5).<sup>8</sup>

*Remark.* For a later discussion on the oracle-uses of our computations it is interesting to estimate the size of  $u_n$ , which can be done by estimating  $|\mathcal{B}_n|$ :

$$\text{for } \ell_n = 2^n \text{ we get } u_n \leq 2^{2^{2^n}} \text{ and for } \ell_n = n \cdot \log n \text{ we get } u_n \leq 2^{2^{n^2}}$$

which are straightforward, and overwhelming compared to  $\ell_n$ .

### 3.4 Counting the failed partition systems

Recall that  $[\tau]$  denotes the set of extensions of  $\tau$  in  $2^{<\omega}$ . We formalize the failure discussed in §3.2, in terms of condition (7) that we try to avoid in partition systems.

**Definition 3.3** (Failure). Given  $\tau \in 2^{\ell_n}$ ,  $\eta \in 2^{u_{n+1}}$ ,  $x > \eta$  and the corresponding  $(D_\sigma^\eta)$ :

- let  $\sigma_\tau^\eta$  denote the unique  $\sigma \in 2^n$  such that  $\tau \in D_\sigma^\eta$  and  $\sigma_\tau^x := \sigma_\tau^\eta$
- we simplify the expression  $D_{\sigma_\tau^\eta}^\eta$  into  $D_{\sigma_\tau}^\eta$  and  $D_{\sigma_\tau^x}^x$  into  $D_{\sigma_\tau}^x$
- say that system  $(D_\sigma^\eta)$  fails at  $\tau$  if  $[\tau] \cap D_{\sigma_\tau^*0}^\eta \cap P = \emptyset \vee [\tau] \cap D_{\sigma_\tau^*1}^\eta \cap P = \emptyset$ .

Given  $x > \eta$  we say that  $(D_\sigma^x)$  fails at  $\tau$  if  $(D_\sigma^\eta)$  fails at  $\tau$ .

We stress that  $D_{\sigma_\tau}^\eta$  is not necessarily a subset of  $D_{\sigma_\tau}^\eta \cap [\tau]$ : distinct  $\tau, \tau'$  may have  $\sigma_\tau^\eta = \sigma_{\tau'}^\eta$ , hence  $D_{\sigma_\tau}^\eta = D_{\sigma_{\tau'}}^\eta$  and  $D_{\sigma_\tau^*i}^\eta$ ,  $i < 2$  contains the partitions of both  $[\tau] \cap 2^{\ell_{n+1}}$  and  $[\tau'] \cap 2^{\ell_{n+1}}$ .

**Definition 3.4** (Fail-sets). The set partition-systems that fail at  $\tau \in 2^{\ell_n}$  is  $G_\tau^0 \cup G_\tau^1$  where:

$$G_\tau^i := \{\eta \in 2^{u_{n+1}} : [\tau] \cap P \cap D_{\sigma_\tau^*i}^\eta = \emptyset\} \text{ so } \llbracket G_\tau^i \rrbracket = \{x : [\tau] \cap P \cap D_{\sigma_\tau^*i}^x = \emptyset\}.$$

for  $i < 2$ , and where we identify partition-systems with their codes.

By compactness, the  $G_\tau^i$  are c.e. uniformly in  $i, \tau$ .

We bound the probability of failure at  $\tau$ . Recall the  $P$ -density bound  $q_n$  from §3.2 and its property (5): for sufficiently large  $n$ , each  $\tau \in 2^{\ell_n}$  has  $\geq q_n \cdot 2^{m_n}$  successors in  $2^{\ell_{n+1}} \cap P$ .

**Lemma 3.5.** For sufficiently large  $n$ , if  $\tau \in 2^{\ell_n} \cap P$ ,  $i < 2$  then  $\mu(G_\tau^i) \leq 2^{-q_n^2 \cdot 2^{m_n}}$ .

<sup>8</sup>In the end of §3.4 we will see that setting  $\ell_n = n \cdot \lceil \log n \rceil$  suffices for our results.

**Proof.** The failure of a partition-system  $(D_\sigma^x)$  at  $\tau$  depends only on  $\eta := x \upharpoonright_{u_{n+1}}$ , and in particular on whether this partition splits the extensions of  $\tau$  in  $P \cap 2^{\ell_{n+1}}$  into two nonempty parts. This is precisely modeled by the hypergeometric distribution: randomly drawing a sample of  $k = 2^{m_{n-1}}$  balls/strings from a box containing  $N = 2^{m_n}$  balls/strings, of which  $R \geq q_n \cdot 2^{m_n}$  are black (the ones inside  $P$ , and are the *successful draws*).

Hoeffding (1963) showed that in such a sample

$$\text{the probability that no successful draws occur is } \leq e^{-2(R/N)^2 k}.$$

Since  $R/N \geq q_n$ , the proportion of strings in  $2^{u_{n+1}}$  that belong to  $G_\tau^i$  has the same bound:

$$|G_\tau^i| \leq 2^{u_{n+1}} \cdot e^{-2q_n^2 \cdot 2^{m_{n-1}}} \Rightarrow \mu(G_\tau^i) < 2^{-q_n^2 \cdot 2^{m_n}}.$$

□

An alternative route toward Lemma 3.5 is to use the explicit expression:

$$\binom{N-R}{k} / \binom{N}{k}$$

for the probability that a random sample of size  $k$  from a population of size  $N$  with  $R$  successful members, contains no successful member. A third way is to consider the probability that  $k$  independent trials with probability of success remaining above  $R/N$  all fail.

**Lemma 3.6.** *If  $x$  is random, there exists  $n_0$  such that*

$$\left( \sigma \in 2^{\geq n_0} \wedge \tau \in P \cap D_\sigma^x \right) \Rightarrow \left( [\tau] \cap P \cap D_{\sigma^*0}^x \neq \emptyset \wedge [\tau] \cap P \cap D_{\sigma^*1}^x \neq \emptyset \right).$$

**Proof.** By compactness, the  $G_\tau^i$  are c.e. uniformly in  $i, \tau$ , and by Lemma 3.5:

$$\tau \in 2^{\ell_n} \cap P \Rightarrow \mu(G_\tau^i) < 2^{-q_n^2 \cdot 2^{m_n}} \quad (9)$$

for sufficiently large  $n$ . We are almost ready to define the required Martin-Löf test for the proof of the lemma, except that the bound in (9) is conditional on  $\tau \in 2^{\ell_n} \cap P$ . Fortunately, the latter is a  $\Pi_1^0$  condition, so we may effectively restrict  $G_\tau^0, G_\tau^1$  into c.e. sets  $L_\tau, R_\tau$  which meet the bound of (9) and if  $\tau \in 2^{\ell_n} \cap P$  then they equal  $G_\tau^0, G_\tau^1$  respectively.

Considering the effective enumerations  $G_\tau^0(s), G_\tau^1(s)$ , of  $G_\tau^0, G_\tau^1$  let  $G_\tau^0(\infty) := G_\tau^0$  and let:

- $t$  be the least such that  $\mu(G_\tau^0(t+1)) \geq 2^{-q_n^2 \cdot 2^{m_n}}$ ;  $t := \infty$  if such stage does not exist
- $L_\tau := G_\tau^0(t)$ , and define  $R_\tau$  similarly, with respect to  $G_\tau^1$ .

Letting  $G_\tau := L_\tau \cup R_\tau$ , the sets  $L_\tau, R_\tau, G_\tau$  are uniformly c.e. and by (9):

$$(a) \quad \tau \in 2^{\ell_n} \cap P \Rightarrow (L_\tau = G_\tau^0 \wedge R_\tau = G_\tau^1), \text{ for sufficiently large } n$$

$$(b) \tau \in 2^{\ell_n} \Rightarrow \mu(G_\tau) \leq \mu(L_\tau) + \mu(R_\tau) < 2 \cdot 2^{-q_n^2 \cdot 2^{m_n}}.$$

Hence  $\mu(G_\tau) < 2 \cdot 2^{-q_n^2 \cdot 2^{m_n}}$  for all  $\tau \in 2^{\ell_n}$  and all  $n$ . Letting  $G_n := \cup_{\tau \in 2^{\ell_n}} G_\tau$  we get:

$$\mu(G_n) \leq \sum_{\tau \in 2^{\ell_n}} \mu(G_\tau) \leq 2^{\ell_n+1} \cdot 2^{-q_n^2 \cdot 2^{m_n}} < 2^{-n}$$

where we used (8), and the  $G_n$  are c.e. uniformly in  $n$ .

Hence  $(G_n)$  is a Martin-Löf test and if  $x$  is random,  $\exists n_0 \forall n \geq n_0 : x \notin G_n$ .

Finally, by clause (a) above and the discussion in the first part of this section, for sufficiently large  $n$  the set  $G_n$  contains the partition-systems that fail at some  $\tau \in P \cap 2^{\ell_n}$ . Hence if  $x$  is random, it will stop failing from some level  $n_0$  on, which completes the proof.  $\square$

*Remark.* The choice  $\ell_n = 2^n$  that we made above (8) is far from optimal, and  $\ell_n = n \lceil \log n \rceil$  suffices for Lemma 3.6. Indeed, it suffices to ensure that  $\sum_n \mu(G_n) < \infty$ , so it suffices that  $\mu(G_n) \leq 1/n^2$ . By (9), it suffices that  $m_n := \ell_{n+1} - \ell_n > 5 \log n$ . Since

$$\sum_{i < n} 5 \log i \leq 5 \log n! < n \log n$$

for sufficiently large  $n$ , it suffices to take  $\ell_n = n \cdot \lceil \log n \rceil$ . This choice also satisfies the latter part of (3), which was a previous commitment.

### 3.5 Remaining proof for Theorem 1.1 and Corollary 1.2

We are now ready to finish the proof: let  $z$  be PA and let  $x <_T z$  be random.

We will construct random  $y <_T z$  such that  $x \oplus y \equiv_T z$ . Since  $x$  is random, there exists  $n_0$  such that for each  $\sigma \in 2^{\geq n_0}$ :

$$\tau \in P \cap D_\sigma^x \Rightarrow \left( [\tau] \cap P \cap D_{\sigma * 0}^x \neq \emptyset \wedge [\tau] \cap P \cap D_{\sigma * 1}^x \neq \emptyset \right). \quad (10)$$

We define the required  $y \leq_T z$  as follows, starting from some  $\sigma_0 \in 2^{n_0}$  and some  $\tau_0 \in P \cap D_{\sigma_0}^x$ . By (10) we may use  $z$  to choose  $\tau_1 \in [\tau_0] \cap P \cap D_{\sigma_0 * z(0)}^x$ , also letting  $\sigma_1 := \sigma_0 * z(0)$ . By the same token, we then choose  $\tau_2 \in [\tau_1] \cap P \cap D_{\sigma_1 * z(1)}^x$ , also letting  $\sigma_2 := \sigma_1 * z(1)$  and so on.

Then  $(\tau_i)$  is well-defined and  $z$ -computable since

$$\text{for } j < 2 \text{ the sets } [\tau_i] \cap P \cap D_{\sigma * j}^x \text{ are nonempty and } \Pi_1^0$$

and each can be obtained uniformly using a finite prefix of  $x$ .

Let  $y := \cup_n \tau_n$  and it remains to show that  $z \leq_T x \oplus y$ .

Starting from  $\tau_0, \sigma_0$ , we may use  $x$  to compute  $D_{\sigma_0 * j}^x, j < 2$ , and let  $z(0) = i$  for the unique  $i$  such that  $y$  has a prefix in  $D_{\sigma_0 * i}$ . In general, given  $\tau_k, \sigma_k$  we

- use  $x$  to compute  $D_{\sigma_k * j}^x, j < 2$

- let  $z(k) = i$  for the unique  $i$  such that  $y$  has a prefix in  $D_{\sigma_k * i}$ .

This shows that  $z \leq_T x \oplus y$ .

By the canonical coding of partition systems in §3.3,  $x$  truth-table computes its partition system  $D_\tau^x$ . In the above reduction  $z \leq_T x \oplus y$ , the computation of  $z$  is via a truth-table on the system  $(D_\tau^x)$  and  $y$ . Hence if  $x <_H z$ , we get  $z \equiv_H x \oplus y$ , and a similar statement holds for weak truth-table reductions.

**Proof of Corollary 1.2.** Clause (a) of Corollary 1.2 follows from Theorem 1.1 and the relativization of the  $\mathbf{0}$ -dominated basis theorem: every nonempty  $\Pi_1^0$  class contains a  $\mathbf{0}$ -dominated real. Recall that the PA reals, viewed as diagonally non-computable binary function, form a  $\Pi_1^0$  class which is universal in the sense that every infinite path can compute some path through every  $\Pi_1^0$  class, uniformly in its index (Jockusch and Soare, 1972). This class can be relativized with respect to any real  $x$ , obtaining a  $\Pi_1^0(x)$  class  $P^x$  with the property that all  $y \in P^x$  have  $y \geq_T x$  and for each nonempty  $Q \in \Pi_1^0(x)$  there exists  $w \leq_T y$ ,  $w \in Q$ .

**Lemma 3.7.** *If  $x$  is  $\mathbf{0}$ -dominated, there exists a  $\mathbf{0}$ -dominated  $z \geq_T x$  of PA degree.*

**Proof.** By the definition  $P^x$ , each  $y \in P^x$  has PA degree. By the  $\mathbf{0}$ -dominated basis theorem relativized to  $x$ , there exists  $x$ -dominated  $z \in P^x$ , so every  $f \leq_T z$  is dominated by some  $g \leq_T x$ . Hence  $z \geq_T x$  is  $\mathbf{0}$ -dominated and of PA degree.  $\square$

Any  $x \in 2^\omega$  is truth-table reducible to a PA real. Hence clause (b) of Corollary 1.2 follows from the truth-table version of Theorem 1.1 and the fact by Moser and Stephan (2015) that all reals in the truth-table degree of a PA real are deep.

*Remark.* In the computation of  $z$  from  $x, y$  that we obtained, the oracle-use of  $x$  is  $u_n$  (from the coding of partitions systems into reals that we fixed in §3.3) and the oracle-use of  $y$  is  $\ell_n$ . We estimated  $u_n$  to be more than double-exponential in  $n$  (the length the prefix of  $z$  computed) while in the end of §3.4 we explained that  $\ell_n := n \cdot \lceil \log n \rceil$  is a valid choice in our argument.

## 4 Conclusion and discussion

We showed that the complete extensions of arithmetic in the degrees of unsolvability (Turing, truth-table, weak truth-table) have the join property with respect to random reals. In §2.3 we explained that known results do not allow any obvious strengthening of this result, in terms of complementation or even a meet property with randoms.

Regarding the reductions of random reals and reals in PA, we would like to know how tight can the relevant oracle-use be: what is the

- (i) oracle-use required in a reduction of a random real to a PA real?
- (ii) oracle-use in Turing equivalence of a PA real with the join of two randoms?

As discussed in §2.1, these are truth-table reductions, but we care about the length of the prefix of the oracle needed for the  $n$ -bit prefix of the output. Regarding (i), we know it cannot be  $n + O(1)$ : Barmpalias and Lewis (2007) showed that if  $x$  is random and  $x \leq_T y$  with oracle-use  $n + O(1)$  then  $y \leq_T x$ ; this cannot happen when  $x$  is incomplete since, as discussed in §2.1, incomplete randoms do not compute PA reals.

With regard to (ii), the optimal oracle-use in computations of reals by randoms was characterized by Barmpalias and Lewis-Pye (2016). Some oracle-uses in our reductions between PA reals and joins of randoms are double-exponential (recall the explicit bounds in §3.3, §3.4).<sup>9</sup> In the weaker result by Barmpalias et al. (2010) discussed in §2.1, the oracle-use is super-exponential. These sizes may be unavoidable, and a worthwhile task would be to establish such a characterization.

Another question stemming from Stephan (2006) who showed that incomplete randoms are computationally far from PA reals, is

how well can an incomplete random trace a PA real?

A common notion of tracing is *c.e.-tracing*: a function  $f$  is *x-c.e. traceable* if there exist uniformly *x-c.e.*  $(D_i)$ ,  $|D_i| \leq n$  such that  $\forall n, f(n) \in D_n$ . Some facts about traceability of PA and DNC reals were established by Kjos-Hanssen et al. (2011).

A dual aspect of this question concerns the fact that incomplete randoms compute DNC functions but not DNC functions with fixed finite range. The exact condition for the growth of the range of a DNC function which is computable by an incomplete random was given by Miller (see Bienvenu et al. (2014, §1.2)): given a computable  $f$ , an incomplete random computes an  $f$ -bounded DNC function iff  $\sum_n 1/f(n) < \infty$ .

## References

- G. Barmpalias and A. E. M. Lewis. Randomness and the linear degrees of computability. *Ann. Pure Appl. Logic*, 145(3):252–257, 2007.
- G. Barmpalias and A. Lewis-Pye. The information content of typical reals. In *Turing’s revolution*, pages 207–224. Birkhäuser/Springer, Cham, 2015.
- G. Barmpalias and A. Lewis-Pye. Optimal redundancy in computations from random oracles. *J. Comput. System Sci.*, 82:1283–1299, 2016.
- G. Barmpalias and A. Lewis-Pye. Limits of the Kučera-Gács coding method. In *Structure and Randomness in Computability and Set Theory (edited with Douglas Cenzer, Chris Porter and Jindrich Zapletal)*, pages 87–109. World Scientific Press, 2020.
- G. Barmpalias, A. E. M. Lewis, and K. M. Ng. The importance of  $\Pi_1^0$  classes in effective randomness. *J. Symb. Log.*, 75(1):387–400, 2010.

---

<sup>9</sup>In our argument, the large oracle-use is due to the coding of partition-systems by reals.

- G. Barmpalias, A. R. Day, and A. E. M. Lewis-Pye. The typical Turing degree. *Proc. Lond. Math. Soc.* (3), 109(1):1–39, 2014.
- C. H. Bennett. Logical depth and physical complexity. In R. Herken, editor, *The universal Turing machine, a half century survey*, pages 227–257. Oxford U.P., 1988.
- L. Bienvenu, R. Hölzl, J. S. Miller, and A. Nies. Denjoy, Demuth, and density. *J. Math. Logic*, 14, 2014.
- D. E. Diamondstone, D. D. Dzhafarov, and R. I. Soare.  $\Pi_1^0$  Classes, Peano Arithmetic, Randomness, and Computable Domination. *Notre Dame J. Formal Logic*, 51(1):127–159, 2010.
- J. N. Y. Franklin, F. Stephan, and L. Yu. Relativizations of randomness and genericity notions. *Bull. Lond. Math. Soc.*, 43:721–733, 2011.
- P. Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70(2-3):186–192, 1986.
- K. Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monash. Math. Phys.*, 38(1):173–198, 1931.
- N. Greenberg, J. S. Miller, and A. Nies. Highness properties close to PA completeness. *Isr. J. Math.*, 244:419–465, 2021. Arxiv:1912.03016.
- W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Am. Stat. Assoc.*, 58(301):13–30, 1963.
- C. G. Jockusch and R. I. Soare. A minimal pair of  $\Pi_1^0$  classes. *J. Symb. Log.*, 36(1):66–78, 1971.
- C. G. Jockusch, Jr. and R. I. Soare.  $\Pi_1^0$  classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.
- B. Kjos-Hanssen, W. Merkle, and F. Stephan. Kolmogorov complexity and the recursion theorem. *Trans. Amer. Math. Soc.*, 363, 2011.
- A. Kučera. Measure,  $\Pi_1^0$ -classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.
- A. Kučera. On the role of  $0'$  in recursion theory. In *Logic Colloquium '86 (Hull, 1986)*, volume 124 of *Stud. Logic Found. Math.*, pages 133–141. North-Holland, Amsterdam, 1988.
- L. A. Levin. Forbidden information. *J. ACM*, 60(2):9:1–9:9, 2013.
- A. Lewis-Pye. A note on the join property. *Proc. Amer. Math. Soc.*, 140(2):707–714, 2012.

- P. Martin-Löf. The definition of random sequences. *Inform. and Control*, 9:602–619, 1966.
- P. Moser. On the polynomial depth of various sets of random strings. *Theoret. Comput. Sci.*, 477:96–108, 2013.
- P. Moser and F. Stephan. Depth, highness and DNR degrees. In A. Kosowski and I. Walukiewicz, editors, *Fundamentals of Computation Theory*, pages 81–94, Cham, 2015. Springer International Publishing.
- D. S. Scott and S. Tennenbaum. On the degrees of complete extensions of arithmetic. *Notices Am. Math. Soc.*, 7:242–243, 1960.
- F. Stephan. Martin-Löf random and PA-complete sets. In *Logic Colloquium '02*, volume 27 of *Lect. Notes Log.*, pages 342–348. Assoc. Symbol. Logic, La Jolla, CA, 2006.