

On Computing Optimal Temporal Branchings and Spanning Subgraphs*

Daniela Bubboloni¹, Costanza Catalano¹, Andrea Marino², and Ana Silva³

¹Department of Mathematics and Computer Sciences, University of Florence, Florence, Italy.
daniela.bubboloni@unifi.it, costanza.catalano@unifi.it

²Department of Statistics, Computer Sciences, Applications, University of Florence, Florence, Italy.
andrea.marino@unifi.it

³Departamento de Matematica, Universidade Federal Do Ceara Fortaleza, Brazil. anasilva@mat.ufc.br

December 19, 2023

Abstract

In this work we extend the concept of out/in-branchings spanning the vertices of a digraph (also called directed spanning trees) to temporal graphs, which are digraphs where arcs are available only at prescribed times. While the literature has focused on minimum weight/earliest arrival time Temporal Out-Branchings (TOB), we solve the problem for other optimization criteria. In particular, we define five different types of TOBs based on the optimization of the travel duration (FT-TOB), of the departure time (LD-TOB), of the number of transfers (MT-TOB), of the total waiting time (MW-TOB), and of the travelling time (ST-TOB). For $D \in \{LD, MT, ST\}$, we provide necessary and sufficient conditions for the existence of a spanning D-TOB; when it does not exist, we characterize the maximum vertex set that a D-TOB can span. Moreover, we provide a log linear algorithm for computing such branchings. For $D \in \{FT, MW\}$, we prove that deciding the existence of a spanning D-TOB is NP-complete; we also show that the same results hold for optimal temporal in-branchings. Finally, we investigate the related problem of computing a spanning temporal subgraph with the minimum number of arcs and optimizing a chosen criterion D . This problem turns out to be NP-hard for any D . The hardness results are quite surprising, as computing optimal paths between nodes can always be done in polynomial time.

Keywords: Temporal graph, temporal network, optimal branching, temporal branching, optimal temporal walk, temporal spanning subgraph.

1 Introduction

A temporal graph is a graph where arcs are active only at certain time instants, with a possible *travelling time* indicating the time it takes to traverse an arc. There is not a unified terminology in the literature to call these objects, as they are also known as stream graphs [23], dynamic networks [31], temporal networks [21], and time-varying graphs [22] to name a few. Important categories of temporal graphs are those of transport networks, where arcs are labeled by the times of bus/train/flight departures and arrivals [14], and communication networks as phone calls and emails networks, where each arc represents the interaction between two parties [32]. Temporal graphs find application in a vast number of fields such as neural, ecological and social networks, distributed computing and epidemiology. We refer the reader to [18] for a

*Daniela Bubboloni is partially supported by GNSAGA of INdAM (Italy). Daniela Bubboloni, Costanza Catalano and Andrea Marino are partially supported by Italian PNRR CN4 Centro Nazionale per la Mobilità Sostenibile, NextGeneration EU - CUP, B13C22001000001. Ana Silva is partially supported by: FUNCAP MLC-0191-00056.01.00/22 and PNE-0112-00061.01.00/16, CNPq 303803/2020-7 (Brazil).

Table 1: Computational time of single source shortest paths in a temporal graph with n vertices and m arcs for the different criteria.

EA	FT	LD	MT	MW	ST
$O(m)$	$O(m \log n)$	$O(m \log m)$	$O(m \log n)$	$O(m \log m)$	$O(m \log m)$
[19, 33]	[5]	[4]	[5]	[4, 5]	[4, 33, 34]

survey on temporal graphs and their applications. Walks in temporal graphs must respect the flow of time; for instance, in a public transports network a route can happen only at increasing time instants, since a person cannot catch a bus that already left. As a consequence, fundamental properties of static graphs, as the fact that concatenation of walks is a walk, do not necessarily hold in temporal graphs. This often makes temporal graphs much harder to handle: e.g. computing strongly connected components takes linear time in a static graph, but it is an NP-complete problem in a temporal graph [15]; the same happens to Eulerian walks [27]. At the same time, it is often the case that classic theorems of graph theory may or may not hold for temporal graphs depending on how some concepts are translated into the temporal framework: this applies for example to Edmonds’ result on branchings [8, 21] and Menger’s Theorem [1, 21, 28].

Figure 1 shows an example of temporal graph. Informally speaking, a temporal graph is modeled as a multidigraph¹ with no loops, such that each arc is labeled by a couple (t_s, t_a) , $t_s \leq t_a$, indicating, respectively, the starting time at which we can traverse the arc from the tail vertex and the arrival time at the head vertex. A temporal walk is a walk in the multidigraph where each arc of the walk must have an arrival time smaller than or equal to the starting time of the subsequent arc in the walk (for formal definitions see Section 2).

Shortest paths in temporal graphs. The notion of shortest path between two vertices u and v in static graphs can be generalized to temporal graphs in different ways, based on the chosen optimization criteria. For example, we may want a path from u to v that arrives the earliest possible (Earliest Arrival time, denoted by $EA(u, v)$), that minimize the overall duration of the trip (Fastest Time, denoted by $FT(u, v)$), that leaves the latest possible (Latest Departure time, denoted by $LD(u, v)$), that takes the least number of arcs (Minimum Transfers, denoted by $MT(u, v)$), that minimize the waiting time in the intermediate nodes (Minimal Waiting time, denoted by $MW(u, v)$), or that minimize the sum of the traversing times of the arcs (Shortest Travelling time, denoted by $ST(u, v)$)². Given $D \in \{EA, FT, LD, MT, MW, ST\}$, a path meeting the criteria D for a pair of vertices u and v is said to realize $D(u, v)$. For formal definitions see again Section 2; Figure 1 shows examples of such paths. Notice that the paths realizing $D(u, v)$ may not be unique. In fact $MW(1, 3)$ is realized both by the yellow and the red walk, while $MT(1, 3)$ is realized both by the blue and the green walk. Each distance is computable in polynomial-time, as reported in Table 1.

Optimal temporal branchings. In static directed graphs, spanning branchings are well-studied objects; they represent a minimal set of arcs that connect a special vertex, called the root, to any other vertex (out-branching), or any vertex to the root (in-branching). They are also called arborescences or spanning directed trees, since their underlying structure is a tree. Spanning branchings representing shortest distances are also well-studied. Their existence is guaranteed simply by the reachability of any vertex from/to the root and they can be computed in $O(m \log m)$ time by Dijkstra’s algorithm [12]. Branchings are, to cite a few, important for engineering applications as they represent the cheapest or shortest way to reach all vertices [24, 26], and in social networks in relation to information dissemination and spreading [3, 35]. We can similarly define spanning branchings in temporal graphs, here called spanning TOBs (Temporal Out-Branchings) and TIBs (Temporal In-Branchings), representing a minimal set of temporal arcs that temporally connect any vertex from/to the root. Equivalently, a TOB is a temporal graph that has a branching as underlying graph and each vertex is temporally reachable from the root (see Section 3 for formal definitions and results). This definition of TOB has already appeared in the literature [19, 20].³ In the context of urban mobility, suppose that a concert has just finished in a remote location x , and we want to guarantee that every person can

¹A directed graph where multiple arcs having the same endpoints are allowed.

²These concepts are widely used in the literature (see [4, 5, 14, 19, 33, 34]), although they may appear with different names.

³We make notice that [20] proposes it in a simplified context, while the conditions listed in the definition of [19] are not all necessary to describe the concept (see Lemma 3.1).

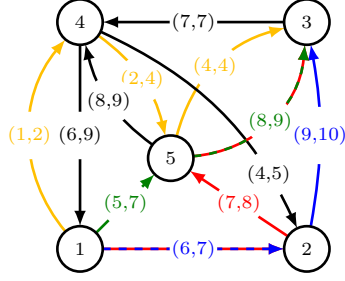


Figure 1: A temporal graph with different temporal walks from vertex 1 to vertex 3, each one represented by a color (two-tone arcs belong to two walks). **Yellow**: walk realizing EA(1, 3) and MW(1, 3). **Red**: walk realizing FT(1, 3). **Blue**: walk realizing both LD(1, 3) and ST(1, 3). **Green**: walk realizing MT(1, 3).

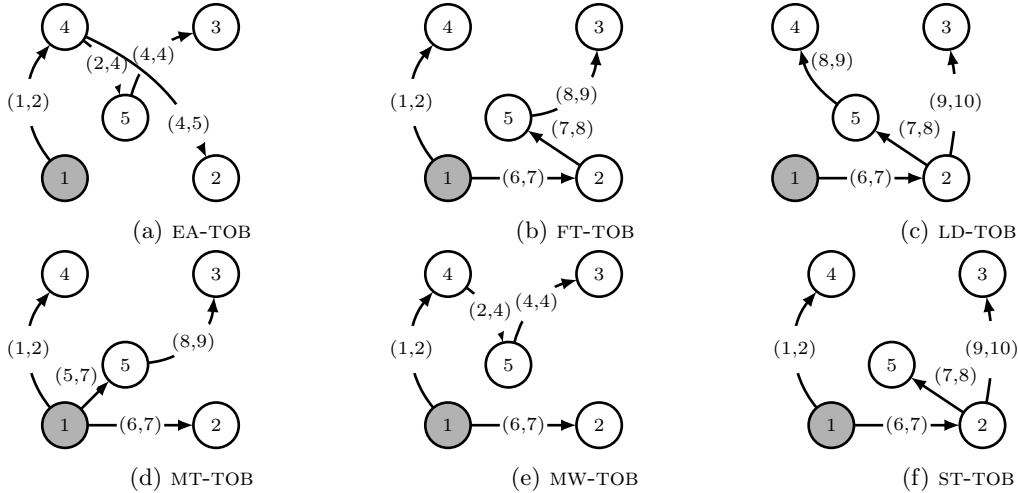


Figure 2: Example of D-TOBs of the temporal graph in Figure (1) for different distances. The grey vertex is the root of the TOB.

go back home via public transports, while optimizing the number of bus/train rides. This problem can be solved by a spanning TOB with root x . We also may ask this TOB to arrive the earliest possible in every point of interest of the city, or the trips to last the shortest possible, or to optimize any of the distances that we have introduced before. It is then natural to extend the notion of shortest distance branchings to the temporal framework. For each distance D , we call spanning D-TOB a spanning TOB such that, for every vertex v , the walk from the root to v realizes the chosen distance $D(r, v)$. Figure 2 shows, for each distance D , a spanning D-TOB with root 1 of the temporal graph in Figure 1.⁴ We define similarly spanning D-TIBs.

In [19], the authors prove that a spanning TOB, as well as an EA-TOB, exist if and only if every vertex is temporally reachable from the root. Then, they provide an algorithm to compute them in $O(m)$ time. As for all the other distances, the problem of computing optimal branchings is still open and seems to be a more difficult task. We start observing that for $D \neq EA$, the temporal reachability from the root to any vertex is no longer sufficient for the existence of a spanning D-TOB. That is showed in Figure 3 where, for each $D \neq EA$, we present a temporal graph that does not admit a spanning D-TOB even if every vertex is temporally reachable from the root. Indeed, in Figures (3b) and (3c) there is a unique temporal path P from r to y . Thus P must be included in any spanning TOB. Now, the particular structure of the temporal graphs under consideration implies that P is the only spanning TOB. However, P does not realize $D(r, x)$, which is

⁴In general D-TOBs are not unique. For instance, another spanning MT-TOB can be obtained from the one in Figure 2d by adding the arc (9, 10) from vertex 2 to vertex 3 and by deleting the arc (8, 9) from vertex 5 to vertex 3.

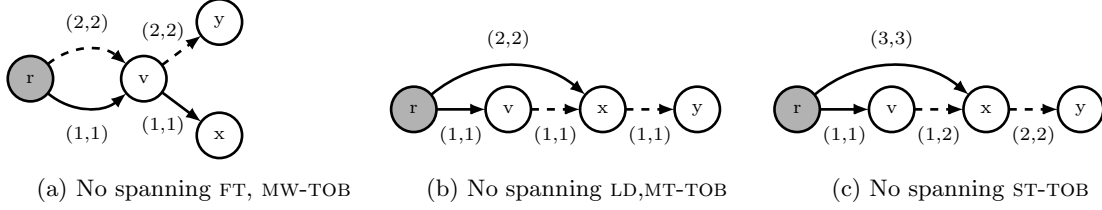


Figure 3: Examples of temporal graphs that do not admit a spanning D-TOB with root r . Solid arcs represent a maximum D-TOB.

realized by the temporal arc from r to x . Therefore, P is not a D-TOB and hence no D-TOB exists. In Figure (3a), there is a unique temporal path that realizes $D(r, x)$, namely the one with temporal arcs $(r, v, 1, 1)$ and $(v, x, 1, 1)$. Similarly, there is a unique temporal path that realizes $D(r, y)$, namely the one with temporal arcs $(r, v, 2, 2)$ and $(v, y, 2, 2)$. Thus, a possible spanning D-TOB must be equal to the graph itself, which clearly is not a branching.⁵ Notice that in all those examples, we can always find a D-TOB on the vertex set $\{r, v, x\}$; this TOB is highlighted by solid arcs in the figures.⁶ The following questions naturally arise:

1. When does a spanning D-TOB exist?
2. If it does not exist, can we identify a set of vertices of maximum size that can be spanned by a D-TOB (*maximum* D-TOB)?
3. Can we compute a maximum D-TOB in polynomial time?
4. Can we answer to all the above questions for D-TIBs?

We observe that having not all the vertices be spanned by a D-TOB might be an issue in real world situations. For example, in the public transports setting, where we might want to reach anyway all the vertices of the graph (places in a city) by optimizing some distance and while still using the least amount of connections possible (buses/trams/...). We can then approach the problem from another point of view by introducing the concept of *minimum D-Temporal Out-Spanning Subgraph* (D-TOSS) of a temporal graph, as a temporal subgraph that connects the root to any other vertex with walks realizing a given distance D and that minimizes the number of temporal arcs (see Section 6 for formal definition)⁷. Another question then arises:

5. Can we compute a minimum D-TOSS of a temporal graph?

In this paper we solve all these questions.

Our contribution. We first show some characterizations of TOBs, each of which gives different insights on these objects. Then, for each $D \in \{LD, MT, ST\}$, we provide a necessary and sufficient condition for the existence of a spanning D-TOB in a temporal graph relying on the concept of optimal substructure (question 1.). Moreover, we characterize the vertex set of maximum size that a D-TOB can span, which turns out to be uniquely identified (question 2.). This property is crucial to find efficient polynomial-time algorithms for computing a maximum D-TOB (question 3.). In particular, our algorithms compute a D-TOB whose paths from the root also arrive the earliest possible in every vertex. The characterization does not hold for $D \in \{FT, MW\}$, and in fact we show that in these cases computing a maximum D-TOB is an NP-complete problem (question 3.). We then show that the same results hold for optimal temporal in-branchings (question 4.). Finally, we

⁵Notice that in the examples, $\tau = 2$ for $D \in \{FT, LD, MT, MW\}$, which is the smallest value possible for which the temporal reachability from the root does not guarantee the existence of a spanning D-TOB, as when $\tau = 1$ the temporal graph reduces to a static graph. When $D = ST$, we have that $\tau = 3$: it can be proven that this is again the smallest value possible for which the temporal reachability from the root does not guarantee the existence of a spanning ST-TOB (Lemma 4.2).

⁶In Figure (3a) also the dashed arcs form a D-TOB on the vertex set $\{r, v, y\}$ for $D \in \{FT, MW\}$.

⁷Notice that the concepts of spanning TOB and minimum TOSS coincide, as well as the concepts of spanning EA-TOB and minimum EA-TOSS. This does not hold for all the other distances.

Table 2: Our contribution: summary results. The first row gives the time to compute a TOB/TIB/minimum TOSS. The other rows give the time to compute a D-TOB/D-TIB/minimum D-TOSS for the corresponding distance D . Results marked with * are presented in [19].

D	D-TOB	D-TIB	minimum D-TOSS
<i>none</i>	$O(m)^*$	$O(m)$	equiv. to TOB
EA	$O(m)^*$	$O(m \log m)$	equiv. to EA-TOB
FT	NP-complete	NP-complete	NP-hard
LD	$O(m \log m)$	$O(m)$	NP-hard
MT	$O(m \log n)$	$O(m \log n)$	NP-hard
MW	NP-complete	NP-complete	NP-hard
ST	$O(m \log m)$	$O(m \log m)$	NP-hard

prove that for any distance D , the problem of finding a minimum D -TOSS is again NP-complete (question 5.). A summary of our results and of the computational time of our algorithms can be found in Table 2. We stress that any algorithm computing $D(r, v)$ for all vertices v of a temporal graph cannot suffice by itself to find a D -TOB or a minimum D -TOSS. Indeed we have seen in Figure (3b) and (3c) that $D(r, y)$ is well-defined because y is temporally reachable from the root r , but no D -TOBs can span y . In other words, there are no guarantees that the union of the paths realizing D , and computed by the aforementioned algorithms, would form a D -TOB. Also applying the Dijkstra’s algorithm on the static expansion of a temporal graph would not solve the problem (see Remark 3.2). In addition, for $D \in \{FT, MW\}$ we have the extreme case where computing $D(r, v)$ is polynomial-time, but finding a maximum D -TOB or a minimum D -TOSS is NP-complete.

Further Related Results. We have already mentioned the results of [19], where the authors also show that the problem of finding minimum weight spanning TOBs is NP-hard. Kuwata et. al. [22] are interested in the temporal reachability from the root that realizes the earliest arrival time, and they obtain it by making use of Dijkstra’s algorithm on the static expansion of the temporal graph. We already observed that that construction does not translate into a TOB in the original temporal graph. Gunturi et. al. [17] present a polynomial-time algorithm for computing what they call *minimum (weight) spanning tree* in a spatio-temporal network: the difference is that in their model, the weight of the arcs depend on a function that evolves in time but walks are not required to be time-respecting. Different versions of the problem of finding arc-disjoint TOBs in temporal graphs are investigated in [8, 20]. The concept of TOSS is closely related to the one of *spanner*. A spanner of a temporal graph $\mathcal{G} = (V, A, \tau)$ is a temporal subgraph of \mathcal{G} with vertex set V such that every vertex temporally reaches any other vertex. In contrast, in a TOSS we are only interested in the reachability of all vertices from the root. A *minimum spanner* is a spanner with the least number of temporal arcs possible. Akrida et. al. [2] proved that computing a minimum spanner is APX-hard. It is worth remarking that a very recent preprint [10] introduces new objects that are a relaxation of spanners. Still, they differ from TOBs and TOSSs for their underlying structure and their reachability properties.

Structure of the paper. Section 2 lists the notation and introduces the concepts used in the paper. In Section 3 we formally defines the Temporal Out-(In-) Branching and present some preliminary results; in particular we show the equivalence between problems on D -TIBs and on D -TOBs. Section 4 shows the theoretical results on spanning/maximum D -TOBs; the polynomial-time algorithms for computing a maximum D -TOB are then presented in Subsection 4.1 when $D = MT$ and in Subsection 4.2 for $D \in \{LD, ST\}$. Section 5 shows that the related problems for $D \in \{FT, MW\}$ are NP-complete. Finally, in Section 6 we formally define a D -Temporal Out-Spanning Subgraph and we prove that, for any D , the problem of finding a minimum D -TOSS is NP-hard.

Previous version. A preliminary version of this work has been presented at Fundamentals of Computation Theory 2023 [6]. Compared to that version, here we have added the full proofs of each result, together with

some intermediary result and explanatory remarks/examples (namely, Remark 3.2, Lemma 4.1, Figure 5, Lemma 4.2, Proposition 5.1, Remark 7.1). Also the pseudocode of Algorithm 1 has been added. Moreover, the results on the Minimum Waiting time distance are new as well as all the results of Section 6.

2 Preliminaries

We denote by \mathbb{N} the set of positive integers. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $[n] := \{x \in \mathbb{N} : x \leq n\}$ and $[n]_0 := \{x \in \mathbb{N}_0 : x \leq n\}$, for $n \in \mathbb{Z}$. Note that if $n \in \mathbb{Z}$ and $n < 0$, then $[n] = [n]_0 = \emptyset$. We instead have $[0] = \emptyset$ and $[n]_0 = \{0\}$. Given a set Q and a property \mathcal{P} , we say that Q is *minimal* for property \mathcal{P} if Q has property \mathcal{P} , and for all $R \subsetneq Q$, R does not have property \mathcal{P} . We remind that a digraph is a pair $D = (V, A)$ where V is the nonempty finite set of vertices and $A \subseteq V \times V$ is the set of arcs. Such digraph is called an *out-branching* (respectively *in-branching*) with root $r \in V$ if, for every $v \in V$, there exists a unique (r, v) -walk (respectively (v, r) -walk) in D . Note that in a branching, every walk is necessarily a path and that the underlying graph is acyclic. We will use the following well-known characterization of out-branchings.

Lemma 2.1 ([16], Theorem 4.3). *Let $D = (V, A)$ be a digraph and $r \in V$. The following facts are equivalent:*

1. D is an out-branching with root r ;
2. for every $v \in V \setminus \{r\}$, there exists a (r, v) -walk, $d_D^-(r) = 0$ and $d_D^-(v) = 1$;
3. for every $v \in V$, there exists a (r, v) -walk and $|A| = |V| - 1$.

A *multidigraph* is formalized by a quadruple $\mathcal{D} = (V, A, t, h)$, where V is the set of vertices, A the set of arcs and $t, h : A \rightarrow V$ are respectively the *head* and the *tail* function, where we require that $\forall a \in A$, $t(a) \neq h(a)$, i.e. no selfloops are allowed⁸. The *in-neighborhood* and *out-neighborhood* of a vertex v are defined as $N_{\mathcal{D}}^-(v) := \{u \in V : \exists a \in A \text{ s.t. } t(a) = u, h(a) = v\}$ and $N_{\mathcal{D}}^+(v) := \{u \in V : \exists a \in A \text{ s.t. } t(a) = v, h(a) = u\}$. The *in-degree* and *out-degree* of v are defined respectively as $d_{\mathcal{D}}^-(v) := |\{a \in A : h(a) = v\}|$, $d_{\mathcal{D}}^+(v) := |\{a \in A : t(a) = v\}|$. Let $u, v \in V$. A (u, v) -walk of length $k \in \mathbb{N}_0$ in \mathcal{D} is an alternating ordered sequence $W = (v_0 = u, a_1, v_1, \dots, v_{k-1}, a_k, v_k = v)$ of vertices $v_0, v_1, \dots, v_k \in V$ and arcs $a_1, \dots, a_k \in A$ such that $t(a_1) = u$, $h(a_k) = v$ and $h(a_i) = v_i = t(a_{i+1})$ for all $i \in [k-1]$. The set of vertices of W is defined by $V(W) := \{v_0, v_1, \dots, v_k\}$ and the set of arcs of W by $A(W) := \{a_1, \dots, a_k\}$. Note that $|V(W)| \leq k$ as well as $|A(W)| \leq k$. We use the notation $\ell(W)$ for the length k of W . Note that $\ell(W) = 0$ if and only if $u = v$; in this case W reduces to the single vertex $u = v$ and it is called a trivial walk. We say that W *traverses* a vertex v (an arc a) if $v \in V(W)$ ($a \in A(W)$). A *path* is a walk where the vertices are all distinct. If a walk X is a sub-sequence of the walk W is called a *subwalk* of W and we write $X \subseteq W$. For $h \in [k]_0$ the v_h -*prefix* of W is the subwalk of W given by $(v_0, a_1, \dots, a_h, v_h)$; the v_h -*suffix* of W is the subwalk of W given by $(v_h, a_{h+1}, \dots, a_k, v_k)$. Note that, for a fixed $z \in V(W)$, there are, in general, many z -prefixes and many z -suffixes of W ; they are unique for all $z \in V(W)$ if and only if W is a path. Given a (u, v) -walk W and a (v, s) -walk Z , we denote the walk obtained by their concatenation by $W + Z$. For $V' \subseteq V$, the *multidigraph induced* by V' in \mathcal{D} is the multidigraph $\mathcal{D}[V'] = (V', A')$, where $A' = \{a \in A : h(a), t(a) \in V'\}$.

Temporal Graphs. A temporal graph \mathcal{G} is a triple (V, A, τ) , where V is the set of vertices, $\tau \in \mathbb{N}$ is the *lifetime*, and

$$A \subseteq \{(u, v, s, t) \in V^2 \times [\tau]^2 : u \neq v \text{ and } s \leq t\}$$

is the set of *temporal arcs*. We set $m := |A|$ and $n := |V|$. Given $a \in A$, we write $a = (t(a), h(a), t_s(a), t_a(a))$, where $t(a)$ and $h(a)$ are, respectively, the *tail* and *head* vertices of the temporal arc a , and $t_s(a)$ and $t_a(a)$ are, respectively, the *starting time* and the *arrival time* of a . These functions are easily interpreted: $t_s(a)$ is the time at which it is possible to begin a trip along a from vertex $t(a)$ to vertex $h(a)$, and $t_a(a)$ is the arrival time of that trip. We also define $el(a) := t_a(a) - t_s(a)$ as the *elapsed time* of the arc $a \in A$.

The temporal graph \mathcal{G} has the multidigraph $\mathcal{D}_{\mathcal{G}} = (V, A, t, h)$ as underlying structure. When using concepts like in-neighborhood, out-neighborhood, in-degree and out-degree for a temporal graph \mathcal{G} , it is

⁸Notice that if t and h are injective, \mathcal{D} is a digraph.

intended that we are referring to its underlying multidigraph $\mathcal{D}_{\mathcal{G}}$. Given a temporal graph \mathcal{G} , we also use the notation $V(\mathcal{G})$ and $A(\mathcal{G})$ to denote, respectively, its set of vertices and temporal arcs. A temporal graph $\mathcal{G}' = (V', A', \tau')$ is a *temporal subgraph* of $\mathcal{G} = (V, A, \tau)$ if $V' \subseteq V$, $A' \subseteq A$ and $\tau' \leq \tau$. Let $(u, v) \in V^2$; we now introduce the concept of *temporal* (u, v) -walk of length $k \in \mathbb{N}_0$. If $k \in \{0, 1\}$, every (u, v) -walk of length k in the underlying multidigraph is also called a temporal (u, v) -walk of length k in \mathcal{G} . Let $k \geq 2$. A temporal (u, v) -walk of length k in \mathcal{G} is a (u, v) -walk $W = (u, a_1, v_1, \dots, v_{k-1}, a_k, v)$ in the underlying multidigraph such that $t_a(a_i) \leq t_s(a_{i+1})$ for all $i \in [k-1]$. We denote by $\mathcal{W}_{\mathcal{G}}(u, v)$ the set of temporal walks from u to v in \mathcal{G} . If $\mathcal{W}_{\mathcal{G}}(u, v) \neq \emptyset$, we say that v is *temporally reachable* from u . Observe that from every temporal walk it is possible to extract a temporal path with the same end vertices.

We now define several interesting functions from the set $\mathcal{W} := \bigcup_{(u,v) \in V^2} \mathcal{W}_{\mathcal{G}}(u, v)$ of the walks of \mathcal{G} to \mathbb{N}_0 . Let $W = (u, a_1, v_1, \dots, v_{k-1}, a_k, v) \in \mathcal{W}$. If $k \geq 1$, we define the *starting time* of W by $t_s(W) := t_s(a_1)$; the *arrival time* of W by $t_a(W) := t_a(a_k)$; the *duration* of W by $\text{dur}(W) := t_a(W) - t_s(W)$; the *waiting time* of W by $\text{wait}(W) := \sum_{i=1}^{k-1} [t_s(a_{i+1}) - t_a(a_i)]$ if $k \geq 2$ and $\text{wait}(W) := 0$ if $k = 1$; the *travelling time* of W by $\text{tt}(W) := \sum_{i=1}^k \text{el}(a_i)$. If instead $k = 0$, i.e. $W = (u)$ is a trivial walk, we set $t_s(W) = t_a(W) = \text{dur}(W) = \text{wait}(W) = \text{tt}(W) = 0$. We next define, through the functions above, a set of crucial functions from V^2 to $\mathbb{N}_0 \cup \{+\infty\}$. Let first $(u, v) \in V^2$ be such that $\mathcal{W}_{\mathcal{G}}(u, v) \neq \emptyset$. We define:

Earliest Arrival time. $\text{EA}_{\mathcal{G}}(u, v) := \min\{t_a(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$;

Fastest Time. $\text{FT}_{\mathcal{G}}(u, v) := \min\{\text{dur}(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$;

Latest Departure time. $\text{LD}_{\mathcal{G}}(u, v) := \max\{t_s(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$ if $u \neq v$, $\text{LD}_{\mathcal{G}}(u, v) := \tau + 1$ if $u = v$;

Minimum Transfers. $\text{MT}_{\mathcal{G}}(u, v) := \min\{\ell(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$;

Minimum Waiting time. $\text{MW}_{\mathcal{G}}(u, v) := \min\{\text{wait}(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$.

Shortest Travelling time. $\text{ST}_{\mathcal{G}}(u, v) := \min\{\text{tt}(W) : W \in \mathcal{W}_{\mathcal{G}}(u, v)\}$;

In this case, given $D \in \{\text{EA}, \text{FT}, \text{LD}, \text{MT}, \text{MW}, \text{ST}\}$, we say that a temporal (u, v) -walk realizes $D_{\mathcal{G}}(u, v)$ if it attains the minimum (or maximum when $D = \text{LD}$) that defines the function value $D_{\mathcal{G}}(u, v)$. If $\mathcal{W}_{\mathcal{G}}(u, v) = \emptyset$, then for every $D \in \{\text{EA}, \text{FT}, \text{LD}, \text{MT}, \text{MW}, \text{ST}\}$ we set $D_{\mathcal{G}}(u, v) = +\infty$.

We will call the functions $D_{\mathcal{G}}$ (temporal) *distances*, as it is common in the literature [7]. However, notice that they do not necessarily satisfy the classic requirements for distances, such as the triangle inequality. When the temporal graph is clear from the context, we usually omit the subscripts.

3 Temporal branching and preliminary results

3.1 Temporal out-branching

In this section, we present the formal notion of temporal out-branching, give some useful characterizations, and define the related optimization problems.

Definition 3.1. A temporal graph $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ is called a *temporal out-branching* (TOB) with root $r \in V_{\mathcal{B}}$ if A is a minimal set of temporal arcs such that, for every $v \in V_{\mathcal{B}}$, there exists a temporal (r, v) -walk in \mathcal{B} . If \mathcal{B} is a temporal subgraph of a temporal graph $\mathcal{G} = (V, A, \tau)$ we say that \mathcal{B} is a TOB of \mathcal{G} rooted in r . If in addition $V_{\mathcal{B}} = V$, we say that \mathcal{B} is a *spanning* TOB of \mathcal{G} rooted in r .

The following lemma provides different characterizations of a TOB.

Lemma 3.1. *Let $\mathcal{B} = (V, A, \tau)$ be a temporal graph and \mathcal{D} be the underlying multidigraph of \mathcal{B} . The following facts are equivalent:*

1. \mathcal{B} is a TOB with root r ;

2. For every $v \in V \setminus \{r\}$, there is a temporal (r, v) -walk in \mathcal{B} , $d_{\mathcal{B}}^-(r) = 0$ and $d_{\mathcal{B}}^-(v) = 1$;
3. For every $v \in V$, there is a temporal (r, v) -walk in \mathcal{B} , and $|A| = |V| - 1$;
4. \mathcal{D} is a digraph which is an out-branching with root r and, for every $v \in V \setminus \{r\}$, the unique (r, v) -walk in \mathcal{D} is the unique temporal (r, v) -walk in \mathcal{B} .

Proof. 1. \implies 2. Since the existence of a temporal (r, v) -walk in \mathcal{B} for all $v \in V$ is guaranteed by definition, we just need to show that $d_{\mathcal{B}}^-(r) = 0$ and that, for every $v \in V \setminus \{r\}$, $d_{\mathcal{B}}^-(v) = 1$. To that purpose, we first describe the set A of arcs of \mathcal{B} . Since from every temporal walk it is possible to extract a temporal path with the same extremes, we have that there exists a (r, v) -path in \mathcal{B} for all $v \in V$. Fix now one (r, v) -path P_v for each $v \in V$. By the minimality of A , we deduce that

$$A = \bigcup_{v \in V} A(P_v). \quad (1)$$

As an immediate consequence of (1), there exists no arc in A entering in r and hence $d_{\mathcal{B}}^-(r) = 0$. Now suppose, by contradiction, that there exists $v \in V \setminus \{r\}$ such that $d_{\mathcal{B}}^-(v) \neq 1$. If $d_{\mathcal{B}}^-(v) = 0$, then v is not reachable from r , a contradiction. Thus we must have $d_{\mathcal{B}}^-(v) \geq 2$. Let $a_1, a_2 \in A$ be two different incoming arcs of v with $t_a(a_1) \leq t_a(a_2)$. We claim that we can delete the temporal arc a_2 from A while maintaining the property that every vertex is temporally reachable from r , and thus contradicting the minimality of A . Delete a_2 . By (1), there exists $v_1 \in V$ such that $a_1 \in A(P_{v_1})$. Since in a path there cannot be two different arcs entering the same vertex, we have that $a_2 \notin A(P_{v_1})$, because $a_2 \neq a_1$. In particular, a_2 is not an arc for the v -prefix X of P_{v_1} . Let $u \in V$ and consider P_u . If $a_2 \notin A(P_u)$, surely u is temporally reachable from r after the removal of a_2 . Assume next that $a_2 \in A(P_u)$. Then, since in a path an arc appears at most once, we have that a_2 does not appear in the v -suffix S of P_u . We consider then the (r, u) -walk given by $P = X + S$. Note that $a_2 \notin A(P)$ and that P is temporal because $t_a(a_1) \leq t_a(a_2)$. Hence, again, u is temporally reachable from r after the removal of a_2 .

2. \implies 3. Since, by assumption, we have $d_{\mathcal{D}}^-(r) = 0$ and $d_{\mathcal{D}}^-(v) = 1$, the fact that $|A| = |V| - 1$ follows from 2. implies 3. in Lemma 2.1. The temporal reachability is trivially true, by assumption.

3. \implies 4. By 3. implies 1. in Lemma 2.1, we have that \mathcal{D} is an out-branching with root r . In particular, \mathcal{D} is a digraph. Let now W be a temporal (r, v) -walk in \mathcal{B} . Then W is also a (r, v) -walk in \mathcal{D} . By definition of out-branching, there is a unique (r, v) -walk in \mathcal{D} , so W is the unique temporal (r, v) -walk in \mathcal{B} .

4. \implies 1. The temporal reachability from the root is guaranteed by hypothesis. Since \mathcal{D} is an out-branching, by Lemma 2.1, it has $|V| - 1$ arcs. Thus, if we delete any arc, then we necessarily disconnect some vertex from the root. Hence A is minimal. \square

Note that, in particular, Lemma 3.1 guarantees that in a TOB there is a unique temporal walk from the root to any vertex and such a walk is necessarily a temporal path.

We now want to specialize the concept of TOB to the various distances. The idea is that we are not only interested in temporally reaching the maximum number of vertices from the root, but we want also to minimize their distance from the root, according to the chosen distance.

Definition 3.2. Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $D \in \{\text{EA, FT, LD, MT, MW, ST}\}$. A TOB $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ of \mathcal{G} , rooted in r , is called a D -TOB of \mathcal{G} rooted in r if, for every $v \in V_{\mathcal{B}}$, we have $D_{\mathcal{B}}(r, v) = D_{\mathcal{G}}(r, v)$. If in addition \mathcal{B} is spanning, we say that \mathcal{B} is a *spanning* D -TOB of \mathcal{G} rooted in r ; if instead, in addition, $|V_{\mathcal{B}}|$ is the largest possible among all the D -TOB of \mathcal{G} rooted in r , we say that \mathcal{B} is a *maximum* D -TOB of \mathcal{G} rooted in r .

Remark 3.1. Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph and $r \in V$. Then the following facts hold:

- (i) Let $\{r\} \subseteq Z \subseteq V$. Then \mathcal{G} has a TOB \mathcal{B} rooted in r , with vertex set Z , if and only if every $v \in Z$ is temporally reachable from r in \mathcal{G} . In particular, \mathcal{G} has a spanning TOB \mathcal{B} rooted in r if and only if every $v \in V$ is temporally reachable from r in \mathcal{G} .

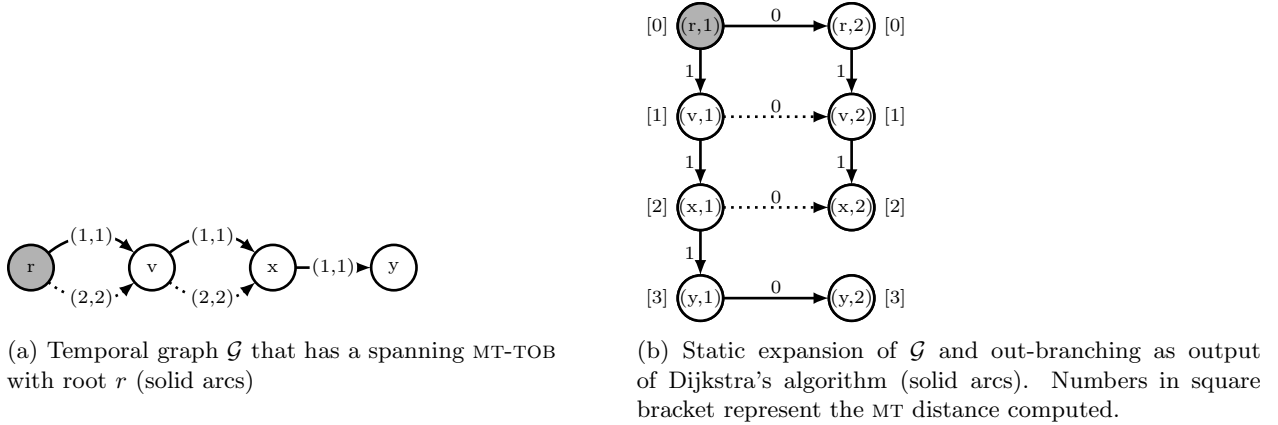


Figure 4: Explanatory figure for Remark 3.2.

(ii) Let $D \in \{EA, FT, LD, MT, MW, ST\}$. If \mathcal{B} is a D -TOB of \mathcal{G} rooted in r and with vertex set Z , then \mathcal{B} is a spanning D -TOB of $\mathcal{G}[Z]$ rooted in r .

Proof. (i) If \mathcal{G} has a TOB \mathcal{B} rooted in r and with vertex set Z , then, by Lemma 3.1, every $v \in Z$ is temporally reachable from r in \mathcal{B} and hence also in \mathcal{G} . Conversely, assume that $Z \subseteq V$ is such that every $v \in Z$ is temporally reachable from r in \mathcal{G} . Let $\mathcal{U} := \mathcal{G}[Z]$ and let $A_{\mathcal{U}}$ be its set of arcs. If $A_{\mathcal{U}}$ is minimal with respect to the reachability in \mathcal{U} from the root r then, by Lemma 3.1, \mathcal{U} is a TOB with vertex set Z . If not, we can delete a finite number of arcs until we reach a minimal set of arcs capable to guarantee the reachability from the root and hence obtain, by Lemma 3.1, a TOB \mathcal{B} with vertex set Z .

(ii) Assume that \mathcal{B} is a D -TOB of \mathcal{G} rooted in r and with vertex set Z . Then \mathcal{B} is a TOB and surely \mathcal{B} is a temporal subgraph of $\mathcal{U} := \mathcal{G}[Z]$. Now, for every $v \in Z$, we have $D_{\mathcal{U}}(r, v) \leq D_{\mathcal{B}}(r, v) = D_{\mathcal{G}}(r, v) \leq D_{\mathcal{U}}(r, v)$. As a consequence, we also have $D_{\mathcal{U}}(r, v) = D_{\mathcal{B}}(r, v)$. This completes the proof. \square

Problem 3.1 (Maximum D -TOB). Let $D \in \{EA, FT, LD, MT, MW, ST\}$ and \mathcal{G} be a temporal graph. Find a maximum D -TOB of \mathcal{G} .

Problem 3.1 has already been solved for $D = EA$ in [19]. Their result also implies that a maximum EA -TOB spans all the vertices that are temporally reachable from the root, thus the vertex set of a maximum EA -TOB is uniquely identified. One could be tempted to solve Problem 3.1 by simply applying the Dijkstra's algorithm on the static expansion of the temporal graph. This does not produce unfortunately the correct solution. We address the reasons in the following remark.

Remark 3.2. We remind that the static expansion of a temporal graph $\mathcal{G} = (V, A, \tau)$ is the digraph $SE(\mathcal{G}) = (\mathcal{V}, \mathcal{A})$ where $\mathcal{V} = \{(v, t) : v \in V, t \in [\tau]\}$ and $\mathcal{A} = \mathcal{M} \cup \mathcal{W}$ where $\mathcal{M} = \{((u, s), (v, t)) : (u, v, s, t) \in A\}$ and $\mathcal{W} = \{((v, t), (v, t + 1)) : v \in V, t \in [\tau - 1]\}$, see also [23, 29]. The static expansion can be used for computing single source distances $D_{\mathcal{G}}(r, v)$ by Dijkstra's algorithm, providing each arc in \mathcal{A} a suitable weight. Unfortunately the out-branching that the Dijkstra's algorithm returns on $SE(\mathcal{G})$ does not translate into a D -TOB of \mathcal{G} . The first problem is that by collapsing back all the vertices $\{(v, t) : t \geq 1\}$ to v , it is not guaranteed that the indegree of v will remain equal to 1. For example, consider the temporal graph of Figure (4a) and the out-branching produced by Dijkstra's algorithm for the distance $D = MT$ on its static expansion in Figure (4b): if we collapse the vertices of the out-branching, we get as a result the original temporal graph itself. Moreover, notice that the vertices $(v, 1)$ and $(v, 2)$ reach the same distance through the Dijkstra's algorithm (numbers in square brackets), and the same happens to the vertices $(x, 1)$ and $(x, 2)$; but only the choice of $(v, 1)$ and $(x, 1)$ would let us achieve a maximum MT -TOB, while the choice of $(v, 2)$ and $(x, 2)$ would not. A similar example can be produced for the other distances.

The following concepts will allow us to establish a necessary and sufficient condition for the existence of a spanning D -TOB with root r in a temporal graph in Section 4.

Definition 3.3. Let \mathcal{G} be a temporal graph and W be a temporal (u, v) -walk in \mathcal{G} . For every $D \in \{\text{EA}, \text{FT}, \text{LD}, \text{MT}, \text{MW}, \text{ST}\}$ we say that:

- W is D -*prefix-optimal* if, for every $x \in V(W)$, any x -prefix of W realizes $D_{\mathcal{G}}(u, x)$;
- W is EAD -*prefix-optimal* if it is D -*prefix-optimal* and, for every $x \in V(W)$, any x -prefix of W realizes $\text{EAD}_{\mathcal{G}}(u, x)$.

3.2 Temporal in-branching

In this section, we present definitions of temporal in-branchings and prove that the related problems are computationally equivalent to TOBs.

Definition 3.4. A temporal graph $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ is called a *temporal in-branching* (TIB) with root r if $A_{\mathcal{B}}$ is a minimal set of temporal arcs such that for all $v \in V$, there exists a temporal (v, r) -walk in \mathcal{B} . Given $\mathcal{G} = (V, A, \tau)$ a temporal graph, $r \in V$, and $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ a temporal subgraph of \mathcal{G} that is a TIB rooted in r , we say that \mathcal{B} is *spanning* if $V_{\mathcal{B}} = V$ and *maximum* if $|V_{\mathcal{B}}|$ is the largest possible. Given $D \in \{\text{EA}, \text{LD}, \text{MT}, \text{FT}, \text{ST}, \text{MW}\}$ and \mathcal{B} a TIB with root r of \mathcal{G} , we say that \mathcal{B} is a D -TIB of \mathcal{G} if for every $v \in V_{\mathcal{B}}$, $D_{\mathcal{B}}(v, r) = D_{\mathcal{G}}(v, r)$. If in addition $V_{\mathcal{B}} = V$, then \mathcal{B} is a *spanning* D -TIB, and if $|V_{\mathcal{B}}|$ is the largest possible among the D -TIB rooted in r , then \mathcal{B} is a *maximum* D -TIB.

Problem 3.2 (Maximum D -TIB). Let $D \in \{\text{EA}, \text{FT}, \text{LD}, \text{MT}, \text{MW}, \text{ST}\}$ and \mathcal{G} be a temporal graph. Find a maximum D -TIB of \mathcal{G} .

The next proposition shows that finding maximum TIBs can be reduced to finding maximum TOBs in an auxiliary temporal graph; we first need to define the transformation \circlearrowleft of a temporal graph that reverses the order of the timesteps as well as the direction of the arcs. A similar transformation has been used e.g. in [7].

Definition 3.5. Given a temporal graph $\mathcal{G} = (V, A, \tau)$, we define the *reverse* of \mathcal{G} as the temporal graph $\mathcal{G}^{\circlearrowleft} = (V, A^{\circlearrowleft}, \tau)$ where $A^{\circlearrowleft} = \{(h(a), t(a), \tau - t_a(a) + 1, \tau - t_s(a) + 1) : a \in A\} := \{a^{\circlearrowleft} : a \in A\}$.

Proposition 3.1. *Given a temporal graph \mathcal{G} , it holds that:*

1. \mathcal{B} is a maximum EA-TIB of \mathcal{G} if and only if $\mathcal{B}^{\circlearrowleft}$ is a maximum LD-TOB of $\mathcal{G}^{\circlearrowleft}$;
2. \mathcal{B} is a maximum LD-TIB of \mathcal{G} if and only if $\mathcal{B}^{\circlearrowleft}$ is a maximum EA-TOB of $\mathcal{G}^{\circlearrowleft}$;
3. For each $D \in \{\text{FT}, \text{MT}, \text{MW}, \text{ST}\}$, \mathcal{B} is a maximum D -TIB of \mathcal{G} if and only if $\mathcal{B}^{\circlearrowleft}$ is a maximum D -TOB of $\mathcal{G}^{\circlearrowleft}$.

Proof. Observe that $(\mathcal{G}^{\circlearrowleft})^{\circlearrowleft} = \mathcal{G}$ and that $W = (u, a_1, v_2, a_2, \dots, v_k, a_k, v)$ is a temporal (u, v) -walk in \mathcal{G} if and only if $W^{\circlearrowleft} = (v, a_k^{\circlearrowleft}, v_k, \dots, a_2^{\circlearrowleft}, v_2, a_1^{\circlearrowleft}, u)$ is a temporal (v, u) -walk in $\mathcal{G}^{\circlearrowleft}$. Then note that for any walk W in \mathcal{G} it holds that $(W^{\circlearrowleft})^{\circlearrowleft} = W$, $t_s(W^{\circlearrowleft}) = \tau - t_a(W) + 1$, $t_a(W^{\circlearrowleft}) = \tau - t_s(W) + 1$, $\ell(W) = \ell(W^{\circlearrowleft})$, $\text{dur}(W) = \text{dur}(W^{\circlearrowleft})$, $\text{wait}(W) = \text{wait}(W^{\circlearrowleft})$, and $\text{tt}(W) = \text{tt}(W^{\circlearrowleft})$. It is also easy to see that \mathcal{B} is a TIB with root r if and only if $\mathcal{B}^{\circlearrowleft}$ is a TOB with root r . Let now W and W' be two (v, r) -walks in \mathcal{G} . We claim that W realizes EA(v, r) in \mathcal{G} if and only if W^{\circlearrowleft} realizes LD(r, v) in $\mathcal{G}^{\circlearrowleft}$. Indeed, $t_a(W) \leq t_a(W')$ if and only if $\tau - t_s(W^{\circlearrowleft}) + 1 \leq \tau - t_s(W'^{\circlearrowleft}) + 1$ if and only if $t_s(W^{\circlearrowleft}) \geq t_s(W'^{\circlearrowleft})$. Similarly, W realizes LD(v, r) in \mathcal{G} if and only if W^{\circlearrowleft} realizes EA(r, v) in $\mathcal{G}^{\circlearrowleft}$. Indeed, $t_s(W) \geq t_s(W')$ if and only if $\tau - t_a(W^{\circlearrowleft}) + 1 \geq \tau - t_a(W'^{\circlearrowleft}) + 1$ if and only if $t_a(W^{\circlearrowleft}) \leq t_a(W'^{\circlearrowleft})$. We now prove that, for $D \in \{\text{FT}, \text{MT}, \text{MW}, \text{ST}\}$, W realizes $D(v, r)$ in \mathcal{G} if and only if W^{\circlearrowleft} realizes $D(r, v)$ in $\mathcal{G}^{\circlearrowleft}$. In fact it holds that $\text{dur}(W) \leq \text{dur}(W')$ if and only if $\text{dur}(W^{\circlearrowleft}) \leq \text{dur}(W'^{\circlearrowleft})$, $\ell(W) \leq \ell(W')$ if and only if $\ell(W^{\circlearrowleft}) \leq \ell(W'^{\circlearrowleft})$, $\text{wait}(W) \leq \text{wait}(W')$ if and only if $\text{wait}(W^{\circlearrowleft}) \leq \text{wait}(W'^{\circlearrowleft})$, and $\text{tt}(W) \leq \text{tt}(W')$ if and only if $\text{tt}(W^{\circlearrowleft}) \leq \text{tt}(W'^{\circlearrowleft})$. This concludes the proof. \square

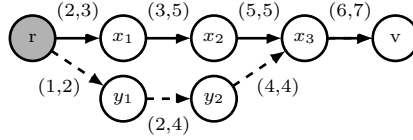


Figure 5: Example showing that Lemma 4.1 does not hold for $D \in \{\text{FT}, \text{MW}\}$.

4 Computing maximum D -temporal out-branchings for latest departure time, minimal transfers and shortest time distances

Before stating necessary and sufficient conditions for the existence of a spanning D -TOB in a temporal graph, we need the following lemma. It intuitively says that a v -prefix in a D -prefix-optimal walk can be replaced by another D -prefix-optimal walk that arrives in v earlier.

Lemma 4.1. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r, u \in V$ and $D \in \{\text{MT}, \text{ST}, \text{LD}\}$. Let W be a D -prefix-optimal temporal (r, u) -walk in \mathcal{G} and $v \in V(W)$. Let S be a v -suffix of W and let W_v be a D -prefix-optimal temporal (r, v) -walk in \mathcal{G} . If $t_a(W_v) \leq t_s(S)$, then $W_v + S$ is a D -prefix-optimal temporal (r, u) -walk in \mathcal{G} .*

Proof. Since $t_a(W_v) \leq t_s(S)$ by hypothesis, then $\bar{W} := W_v + S$ is a temporal (r, u) -walk. Let $x \in V(\bar{W})$ and X be a x -prefix of \bar{W} . If $D = \text{LD}$, since W and W_v are LD -prefix-optimal, it holds that $t_s(W) = \text{LD}(r, v) = t_s(W_v)$. Since $\text{LD}(r, v) = \text{LD}(r, x)$ and $t_s(X) = t_s(\bar{W}) = t_s(W_v)$, we obtain that $t_s(X) = \text{LD}(r, x)$. Consider now $D \in \{\text{MT}, \text{ST}\}$ and let P be the v -prefix of W such that $W = P + S$. If $X \subseteq W_v$, then by hypothesis X realizes $D(r, x)$. If $X \not\subseteq W_v$, there exists $S' \subseteq S$ such that $X = W_v + S'$. Assume first $D = \text{MT}$. Then $\ell(X) = \ell(W_v) + \ell(S') = \text{MT}(r, v) + \ell(S')$ and, by hypothesis, $\text{MT}(r, x) = \ell(P + S') = \ell(P) + \ell(S') = \text{MT}(r, v) + \ell(S')$. Consequently $\ell(X) = \text{MT}(r, x)$. Assume next that $D = \text{ST}$. Then we have $\text{tt}(X) = \text{tt}(W_v) + \text{tt}(S') = \text{ST}(r, v) + \text{tt}(S')$ and, by hypothesis, $\text{ST}(r, x) = \text{tt}(P + S') = \text{tt}(P) + \text{tt}(S') = \text{ST}(r, v) + \text{tt}(S')$. This implies that $\text{tt}(X) = \text{ST}(r, x)$. \square

We emphasize that Lemma 4.1 does not hold for $D \in \{\text{FT}, \text{MW}\}$. Indeed consider the temporal graph in Figure 5. We have that $\text{FT}(r, v) = 5$ and $\text{MW}(r, v) = 1$, which is realized by the solid (r, v) -path. In particular, this path is FT -prefix-optimal and MW -prefix-optimal. Consider now the dashed (r, x_3) -path and call it W : it is both FT -prefix-optimal and MW -prefix-optimal. Let a be the temporal arc $(x_3, v, 6, 7)$. Notice that $W' = W + (x_3, a, v)$ is a temporal (r, v) -path but does not realize neither $\text{FT}(r, v)$ nor $\text{MW}(r, v)$, as $\text{dur}(W') = 6$ and $\text{wait}(W') = 2$.

Theorem 4.1. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $D \in \{\text{LD}, \text{MT}, \text{ST}\}$. Then \mathcal{G} has a spanning D -TOB with root r if and only if there exists a D -prefix-optimal temporal (r, v) -path in \mathcal{G} for all $v \in V$.*

Proof. \implies By the definition of D -TOB with root r and the uniqueness of temporal walks from the root to any other vertex in a TOB (see Lemma 3.1), it follows that all walks in a D -TOB are paths and are D -prefix-optimal.

\impliedby For every $v \in V$, let W_v be a temporal D -prefix-optimal (r, v) -path in \mathcal{G} . Let $A' := \bigcup_{v \in V} A(W_v)$. For $B \subseteq A'$, denote by $\mathcal{B}[B]$ the temporal subgraph of \mathcal{G} having vertex set V and temporal arc set B , and consider the property.

$\mathcal{P}_{D, B}$: for all $v \in V$, there exists in $\mathcal{B}[B]$ a temporal D -prefix-optimal (r, v) -walk.

Note that $\mathcal{P}_{D, A'}$ is satisfied. Thus it is possible to consider the minimal subsets B of arcs in A' satisfying $\mathcal{P}_{D, B}$. Let $A_B \subseteq A'$ be one of such minimal sets and let $\mathcal{B} := \mathcal{B}[A_B]$. We show that \mathcal{B} is a D -TOB for \mathcal{G} . Clearly, by the construction of \mathcal{B} , it is enough to show that \mathcal{B} is a TOB. Furthermore, in view of Lemma 3.1, it suffices to show that $d_B^-(r) = 0$ and that for all $v \in V \setminus \{r\}$, $d_B^-(v) = 1$, since the temporal reachability from vertex r to any other vertex is already guaranteed by property \mathcal{P}_{D, A_B} . Since A' does not contain arcs entering in r , this holds also for A_B and hence we have that $d_B^-(r) = 0$. Suppose now, by contradiction, that there exists

$v \in V \setminus \{r\}$ such that $d_{\mathcal{B}}^-(v) \neq 1$. Since $d_{\mathcal{B}}^-(v) = 0$ implies that v is not reachable from r , we necessarily have $d_{\mathcal{B}}^-(v) \geq 2$. Let $a_1, a_2 \in A_{\mathcal{B}}$ be two different incoming temporal arcs of v with $t_a(a_1) \leq t_a(a_2)$. We claim that $\mathcal{P}_{\mathcal{B}, A_{\mathcal{B}} \setminus \{a_2\}}$ is satisfied, and thus the minimality of $A_{\mathcal{B}}$ is contradicted. Indeed, by definition of $A_{\mathcal{B}}$, there exists $v_1 \in V$ and a temporal (r, v_1) -path W_{v_1} such that $a_1 \in A(W_{v_1})$. Since in a path two distinct arcs entering in the same vertex do not appear, we have that $a_2 \notin A(W_{v_1})$. In particular, a_2 is not an arc for the v -prefix X of W_{v_1} . Let $u \in V$ and consider W_u a temporal (r, u) -path in \mathcal{B} . Assume that $a_2 \in W_u$. Then, since in a path an arc appears at most once, we have that a_2 does not appear in the v -suffix S of W_u . We consider then the (r, u) -walk given by $\bar{W} = X + S$. Then $a_2 \notin A(\bar{W})$ and we have that $t_a(X) = t_a(a_1) \leq t_a(a_2) \leq t_s(S)$. As a consequence, by Lemma 4.1, \bar{W} is a D-prefix-optimal walk in $\mathcal{B}[A_{\mathcal{B}} \setminus \{a_2\}]$. \square

Theorem 4.1 does not hold for $\mathcal{D} \in \{\text{FT}, \text{MW}\}$. Indeed the temporal graph in Figure (3a) has a D-prefix-optimal path from r to any other vertex, but does not admit a spanning D-TOB as previously observed. We are now ready to characterize the vertex set of a maximum D-TOB.

Corollary 4.1. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $\mathcal{D} \in \{\text{LD}, \text{MT}, \text{ST}\}$. Then a maximum D-TOB \mathcal{B} with root r of \mathcal{G} has vertex set:*

$$V_{\mathcal{B}} = \{v \in V : \text{there exists a } \mathcal{D}\text{-prefix-optimal } (r, v)\text{-path in } \mathcal{G}\}. \quad (2)$$

Proof. Consider $\mathcal{G}[V_{\mathcal{B}}]$. Let $v \in V_{\mathcal{B}}$ and W a D-prefix-optimal (r, v) -temporal walk in \mathcal{G} . By definition of D-prefix-optimal walk, for every $u \in V(W)$, it holds that $u \in V_{\mathcal{B}}$, which implies that W is also a D-prefix-optimal (r, v) -temporal walk in $\mathcal{G}[V_{\mathcal{B}}]$. Hence, by Theorem 4.1, $\mathcal{G}[V_{\mathcal{B}}]$ has a spanning D-TOB \mathcal{B} , which is also a D-TOB of \mathcal{G} . We now show that \mathcal{B} is maximum. By Remark 3.1 it suffices to prove that if $V' \subseteq V$ is such that $V' \setminus V_{\mathcal{B}} \neq \emptyset$, then $\mathcal{G}[V']$ does not admit a spanning D-TOB with root r . Let $u \in V' \setminus V_{\mathcal{B}}$. By hypothesis there does not exist a D-prefix-optimal temporal (r, u) -walk in \mathcal{G} , hence there does not exist one in $\mathcal{G}[V']$. By Theorem 4.1, $\mathcal{G}[V']$ does not admit a spanning D-TOB. \square

The above corollary shows that for every $\mathcal{D} \in \{\text{LD}, \text{MT}, \text{ST}\}$, the vertex set of a maximum D-TOB of a temporal graph is uniquely determined. We now show that for $\mathcal{D} = \text{ST}$ and $\tau \leq 2$, a maximum ST-TOB is always spanning, as long as each vertex is temporally reachable from the root. That is no longer true for $\tau \geq 3$, as already highlighted in Figure (3c).

Lemma 4.2. *If $\tau \leq 2$, then a temporal graph $\mathcal{G} = (V, A, \tau)$ has a spanning ST-TOB with root r if and only if each vertex is temporally reachable from r .*

Proof. If $\tau = 1$ the temporal graph reduces to a static graph, so the problem reduces to find a spanning out-branching of a static graph. If $\tau = 2$, notice that every temporal label $(t_s(a), t_a(a))$ can assume only three values, namely $\{(1, 1), (1, 2), (2, 2)\}$. This implies that every temporal walk W in \mathcal{G} is such that either $\text{tt}(W) = 0$ or $\text{tt}(W) = 1$. We now want to prove that if v is temporally reachable from r , then there exists an ST-prefix-optimal (r, v) -walk in \mathcal{G} . Let W be an (r, v) -walk in \mathcal{G} . If $\text{tt}(W) = 0$, then W is necessarily ST-prefix-optimal. Suppose now that $\text{tt}(W) = 1$ and that W is not ST-prefix-optimal. Let u be the first vertex of W , starting from v , for which the u -prefix X of W does not realize $\text{ST}(r, u)$. If $\text{tt}(X) = 0$, then we also have $\text{ST}(r, u) = 0$, against the fact that X does not realize $\text{ST}(r, u)$. As a consequence, we have $\text{tt}(X) = 1$ and $\text{ST}(r, u) = 0$. This implies that $t_a(X) = 2$ and that there must exist an (r, u) -walk W_u in \mathcal{G} such that $\text{tt}(W_u) = 0$; in particular W_u is ST-prefix-optimal. Let S be a u -suffix of W such that $W = X + S$. Notice that $t_a(W_u) \leq \tau = 2 = t_a(X) \leq t_s(S) \leq 2$, so that $t_a(X) = t_s(S)$. Then all the temporal labels of the arcs in S are equal to $(2, 2)$. As a consequence $W_u + S$ is an ST-prefix-optimal (r, v) -walk in \mathcal{G} . \square

The next sections present algorithms for finding D-TOBs of a given temporal graph in polynomial time, when $\mathcal{D} \in \{\text{LD}, \text{MT}, \text{ST}\}$. In particular, we show that in such cases we can constrain ourselves to the earliest arrival paths that realize the distances. For this, we define:

Definition 4.1. Given a temporal graph $\mathcal{G} = (V, A, \tau)$ rooted in $r \in V$, for any $(u, v) \in V^2$ and $\mathcal{D} \in \{\text{EA}, \text{FT}, \text{LD}, \text{MT}, \text{MW}, \text{ST}\}$, we define $\text{EAD}_{\mathcal{G}}(u, v) := \min\{t_a(W) : W \text{ realizes } \mathcal{D}_{\mathcal{G}}(u, v)\}$. A D-TOB $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ with root r of \mathcal{G} is called an EAD-TOB if, for every $v \in V_{\mathcal{B}}$, we have that $\text{EAD}_{\mathcal{B}}(r, v) = \text{EAD}_{\mathcal{G}}(r, v)$. \mathcal{B} is called *spanning* if $V_{\mathcal{B}} = V$ and *maximum* if $|V_{\mathcal{B}}|$ is the largest possible among the EAD-TOB rooted in r .

Algorithm 1: Computing a maximum MT-TOB of a temporal graph.

Input: A temporal graph $\mathcal{G} = (V, A, \tau)$, and a vertex $r \in V$.
Output: A maximum MT-TOB $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ of \mathcal{G} with root r .

- 1 $\mathcal{EAMT}(r) \leftarrow 0; \forall v \in V \setminus \{r\}, \mathcal{EAMT}(v) \leftarrow +\infty;$
- 2 $d(r) \leftarrow 0; \forall v \in V \setminus \{r\}, d(v) \leftarrow \text{MT}_{\mathcal{G}}(r, v);$
- 3 $V_{\mathcal{B}} \leftarrow \{r\}; A_{\mathcal{B}} \leftarrow \emptyset; \tau_{\mathcal{B}} \leftarrow 0; h \leftarrow \max\{d(v) : v \in V, d(v) < +\infty\};$
- 4 **for** $i = 1, \dots, h$ **do**
- 5 **for each** $v \in V$ **such that** $d(v) = i$ **do**
- 6 $S \leftarrow \{(u', v, s', t') \in A : s' \geq \mathcal{EAMT}(u'), d(u') = i - 1\};$
- 7 **if** $S \neq \emptyset$ **then**
- 8 $a \leftarrow \text{choose } (u, v, s, t) \in \arg \min_{(u', v, s', t') \in S} t';$
- 9 $\mathcal{EAMT}(v) \leftarrow t, V_{\mathcal{B}} \leftarrow V_{\mathcal{B}} \cup \{v\}, A_{\mathcal{B}} \leftarrow A_{\mathcal{B}} \cup \{a\}, \tau_{\mathcal{B}} \leftarrow \max\{\tau_{\mathcal{B}}, t\};$
- 10 **end**
- 11 **end**
- 12 **end**

The proposed algorithms will always return an EAD-TOB. This implies that for $D \in \{\text{MT}, \text{ST}, \text{LD}\}$, the existence of a D -prefix-optimal (r, v) -path in \mathcal{G} is equivalent to the existence of an EAD-prefix-optimal (r, v) -path in \mathcal{G} . For $D = \text{FT}$ this is no longer true: indeed consider Figure (3a). The only FT -prefix-optimal (r, y) -path is $W = (r, (r, v, 2), v, (v, y, 2), y)$, but it is not EAFT-prefix-optimal: in fact, $\text{EAFT}(r, v) = 1$ since the path $(r, (r, v, 1), v)$ realizes $\text{FT}(r, v)$ and arrives in v at time 1, while W arrives in v at time 2. The same reasoning holds for $D = \text{MW}$. This difference will be crucial for showing that computing a maximum D -TOB for $D \in \{\text{FT}, \text{MW}\}$ is an NP-complete problem (Section 5).

4.1 Algorithm for minimal transfer distance

Algorithm 1 computes a maximum MT-TOB with root r of a given temporal graph. First observe that, given an MT-prefix-optimal temporal (r, v) -walk $W = (r = v_0, a_1, v_1, \dots, a_k, v_k = v)$, we have that $\text{MT}(r, v_i) = \text{MT}(r, v_{i+1}) - 1 < \text{MT}(r, v_{i+1})$ for all $i \in [k-1]$, in particular the sequence of distances in any MT-prefix-optimal walk is strictly increasing. The main idea of the algorithm is then to compute a priori the MT-distances of all vertices from the root (line 2), and then build the MT-TOB guided by these computed distances, using their strict monotonicity property. More specifically, given $h = \max\{\text{MT}(r, v) : v \in V\}$, the algorithm grows an MT-TOB starting from the root and adding, at step $i \in [h]$, all the vertices at distance i . During this process, when adding some vertex v , we choose, among its neighbors at distance $i - 1$, which one can be the parent of v . To choose the right parent, we look at the incoming temporal arcs having tail in vertices at distance $i - 1$ and we consider only the arcs $a' = (u', v, s', t')$ such that, if $W_{u'}$ is the unique temporal (r, u') -path in the MT-TOB built so far, then $s' \geq t_a(W_{u'})$, i.e. the new arc can be concatenated with $W_{u'}$ to obtain a temporal (r, v) -path. Among the arcs fulfilling these constraints, we choose a' minimizing t' , the arrival time in v ; such arc a' exists if and only if there exists an MT-prefix-optimal (r, v) -path in \mathcal{G} . We prove that such choice of a' ensures that we are actually representing in the TOB a temporal (r, v) -path that realizes the distance $\text{MT}(r, v)$ and has the earliest arrival time among the walks realizing such distance, i.e. we are computing an EAMT-TOB. The algorithm takes $O(m \log n)$ time to compute all the initial MT distances (Table 1), while the remaining part of the algorithm takes $O(m)$ time as it requires only one scan of each temporal arc.

Theorem 4.2. *Algorithm 1 returns a maximum MT-TOB of a temporal graph, for a chosen root, in $O(m \log n)$ time. Additionally, the output is an EAMT-TOB.*

Proof. Let $\mathcal{G} = (V, A, \tau)$ be the temporal graph input of the algorithm and $r \in V$, $d(v) = \text{MT}_{\mathcal{G}}(r, v)$ for all $v \in V$, $h = \max\{d(v) : v \in V, d(v) < +\infty\}$ and $V' = \{v \in V : v \text{ is temporally reachable from } r\}$. For $i \in [h]_0$ let $D_i = \{v \in V : d(v) = i\}$ and note that $\{D_i : i \in [h]_0\}$ is a partition of V' with $D_0 = \{r\}$. Since no confusion is possible, from now on we will avoid writing the subscripts \mathcal{G} . We prove the following loop invariant:

Claim 4.1. At the end of the i -th iteration of the **for** loop in lines 4-12,

$$V_{\mathcal{B}} = \{v \in V' : \exists \text{ an MT-prefix-optimal temporal } (r, v)\text{-walk in } \mathcal{G} \text{ and } d(v) \leq i\},$$

$\mathcal{EAMT}(v) = \text{EAMT}(r, v)$ for all $v \in V_{\mathcal{B}}$, and \mathcal{B} is an EAMT-TOB with root r of \mathcal{G} .

The above claim implies that the final output \mathcal{B} of the algorithm is an EAMT-TOB with root r of \mathcal{G} , which is in particular an MT-TOB. Moreover, $V_{\mathcal{B}}$ consists of all the vertices in \mathcal{G} for which there exists an MT-prefix-optimal temporal walk from the root. Thus \mathcal{B} is a maximum MT-TOB by Corollary 4.1. We are left to prove the claim. \mathcal{B} is initialized as the temporal graph made of the sole vertex r , so the loop invariant is trivially true. Suppose now that the loop invariant is true up to a certain i -th iteration. We now prove that it holds for the $(i + 1)$ -th iteration. Let $v \in V$ be such that $d(v) = i + 1$. We first prove that if there exists an MT-prefix-optimal temporal (r, v) -walk in \mathcal{G} , say W , then the set S in line 6 is non-empty. We can always choose W such that it arrives the earliest, that is $t_a(W) = \text{EAMT}(r, v)$. Let $\bar{a} = (\bar{u}, v, \bar{s}, \bar{t}) \in A$ be the last temporal arc of W . It holds that $d(\bar{u}) = i$, so by inductive hypothesis we have that $\bar{u} \in V_{\mathcal{B}}$ and $\mathcal{EAMT}(\bar{u}) = \text{EAMT}(r, \bar{u})$ at the end of the i -iteration. Since W is MT-prefix-optimal, we have that $\bar{s} \geq \text{EAMT}(r, \bar{u}) = \mathcal{EAMT}(\bar{u})$. Therefore \bar{a} fulfils the conditions to belong to S , so S is non-empty. Notice also that since $\bar{a} \in S$ and $\bar{t} = \text{EAMT}(r, v)$ then

$$\min_{(u', v, s', t') \in S} t' = \bar{t} = \text{EAMT}(r, v). \quad (3)$$

We now prove that \mathcal{B} is an EAMT-TOB with root r . We have just showed that if there exists an MT-prefix-optimal temporal (r, v) -walk in \mathcal{G} , then S in line 6 is non-empty. This implies that in line 8 we choose an arc $a = (u, v, s, t) \in S$ that minimizes the arrival time, and this arc is added to $A_{\mathcal{B}}$, while v is added to $V_{\mathcal{B}}$ and $\mathcal{EAMT}(v)$ is set to t . Since D_0, \dots, D_h is a partition of V' , no other incoming arc to v is added in the algorithm, and therefore v has in-degree equal to 1 in \mathcal{B} . Moreover $s \geq \mathcal{EAMT}(u)$ since $a \in S$, so if W_u is the unique temporal (r, u) -path in \mathcal{B} (it exists by inductive hypothesis), then $W_v = W_u + (u, a, v)$ is a temporal (r, v) -path in \mathcal{B} . Hence \mathcal{B} is a TOB with root r . It remains to show that W_v realizes $\text{EAMT}(r, v)$. By the inductive hypothesis we have that W_u is EAMT-prefix-optimal. Therefore $\ell(W_v) = \ell(W_u) + 1 = d(u) + 1 = i + 1 = d(v)$. Moreover, by equation (3) and since $a \in S$, we have that $\mathcal{EAMT}(v) = t_a(W_v) = t = \bar{t} = \text{EAMT}(r, v)$. This concludes the proof of claim.

Regarding the computational complexity of the algorithm, by Table 1 the initial computation of all distances requires $O(m \log n)$; the remaining part of the algorithm takes $O(m)$ as it requires only one scan of each temporal arc. Therefore the overall complexity is $O(m \log n)$. \square

4.2 Algorithm for latest departure time and shortest time distances

Algorithm 2 computes a maximum D-TOB \mathcal{B} with root r for a given temporal graph when $D \in \{\text{LD}, \text{ST}\}$, and it is more involved with respect to Algorithm 1. The issue is that if $W = (r = v_0, a_1, \dots, a_k, v_k = v)$ is a D-prefix-optimal walk, then it is possible to have $D(r, v_{i-1}) = D(r, v_i)$ for some $i \in [k]$. Indeed, if $D = \text{LD}$, then all the vertices in the walk share the same latest departure time, i.e. $t_s(W) = \text{LD}(r, v_i)$ for all $i \in [k]$ and $D = \text{ST}$ and $el(a_i) = 0$ for some $i \in [k]$, then $\text{ST}(r, v_{i-1}) = \text{ST}(r, v_i)$. However, in any case we have that $D(r, v_{i-1}) \leq D(r, v_i)$ for all $i \in [k]$. This implies that, by letting D_i denote the set of vertices at distance d_i from r with the distances d_i being in increasing order, to choose the parent of each vertex of D_i in \mathcal{B} , we cannot look only at vertices in $D_0 \cup \dots \cup D_{i-1}$, but also at the ones in D_i itself (in particular, only at the ones in D_i when $D = \text{LD}$). Note that this gives us an additional difficulty as we cannot simply choose an arbitrary vertex $v \in D_i$ to be the next one to be added to \mathcal{B} , as it might happen that the good parent of v (i.e. the in-neighbor of v within an EAD-prefix-optimal (r, v) -walk) has not been added to \mathcal{B} yet. To overcome this, we add vertices in D_i to \mathcal{B} in increasing order of the value of $\text{EAD}(r, v)$. Observe however that $\text{EAD}(r, v)$ is not known a priori, so to do that we use a queue that keeps the outgoing temporal arcs from vertices in \mathcal{B} in increasing order of their arrival time. These ideas are formalized below. At step i of the **for** loop at lines 5-18, Algorithm 2 adds to \mathcal{B} the vertices of D_i that are reachable by a D-prefix-optimal walk. To this aim, it uses a min priority queue Q for temporal arcs a with head vertices in D_i with weight $t_a(a)$. For $D = \text{LD}$,

Algorithm 2: Computing a maximum D-TOB, with $D \in \{\text{LD}, \text{ST}\}$.

Input: A temporal graph $\mathcal{G} = (V, A, \tau)$, a vertex $r \in V$, $D \in \{\text{LD}, \text{ST}\}$.
Output: A maximum D-TOB $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ of \mathcal{G} with root r .

```

1  $\mathcal{EAD}(r) \leftarrow 0; \forall v \in V \setminus \{r\}, \mathcal{EAD}(v) \leftarrow +\infty;$ 
2  $d(r) \leftarrow 0; \forall v \in V \setminus \{r\}, d(v) \leftarrow D_{\mathcal{G}}(r, v);$ 
3  $\langle d_1, \dots, d_h \rangle \leftarrow$  ordered list of finite  $d$  values with no repetitions;
4  $V_{\mathcal{B}} \leftarrow \{r\}; A_{\mathcal{B}} \leftarrow \emptyset; \tau_{\mathcal{B}} \leftarrow 0, D_0 \leftarrow \{r\};$ 
5 for  $i = 1, \dots, h$  do
6    $D_i \leftarrow \{v \in V \setminus \{r\} : d(v) = d_i\};$ 
7   if  $D = \text{LD}$  then enqueue all  $(r, v, s, t) \in A$  such that  $s = d_i$  in a min priority queue  $Q$  with weight  $t$ ;
8   if  $D = \text{ST}$  then enqueue all  $(u, v, s, t) \in A$  such that  $u \in D_0 \cup \dots \cup D_{i-1}$  and  $v \in D_i$  in a min priority
   queue  $Q$  with weight  $t$ ;
9   while  $Q \neq \emptyset$  do
10    dequeue  $a \leftarrow (u, v, s, t)$  from  $Q$ ;
11    while  $s < \mathcal{EAD}(u)$  or  $t \geq \mathcal{EAD}(v)$  or  $(D = \text{ST}$  and  $t - s \neq d_i - d(u))$  do
12      if  $Q = \emptyset$  then go to Line 5 with next value of  $i$ ;
13      dequeue  $a \leftarrow (u, v, s, t)$  from  $Q$ ;
14    end
15    /*  $a = (u, v, s, t)$  is s.t.  $a \in \arg \min_{(u', v', s', t') \in Q} t', s \geq \mathcal{EAD}(u), t < \mathcal{EAD}(v) = +\infty$ , and if
       $D = \text{ST}, t - s = d_i - d(u)$ . */
16     $\mathcal{EAD}(v) \leftarrow t, V_{\mathcal{B}} \leftarrow V_{\mathcal{B}} \cup \{v\}, A_{\mathcal{B}} \leftarrow A_{\mathcal{B}} \cup \{a\}, \tau_{\mathcal{B}} \leftarrow \max\{\tau_{\mathcal{B}}, t\};$ 
17    enqueue all  $(v, v', s', t') \in A$  such that  $v' \in D_i$  in  $Q$  with weight  $t'$ ;
18  end
19 end

```

Q is initialized with all the outgoing temporal arcs from r with starting time d_i , as they are the only arcs that can realize a latest departure time equal to d_i . For $D = \text{ST}$, Q is initialized with all the temporal arcs with tail in $D_0 \cup \dots \cup D_{i-1}$ and head in D_i . The vector \mathcal{EAD} in the algorithm, initialized at $+\infty$ for all the vertices but the root, keeps track of the arrival time in the vertices every time they are added to the TOB. In the **while** loop at lines 9-17, we dequeue temporal arcs from Q that cannot possibly be within an EAD-prefix-optimal walk. Formally, if such loop is not broken in line 12, then at the end we are left with an arc $a = (u, v, s, t) \in \arg \min_{(u', v', s', t') \in Q} t'$, i.e. an arc that minimizes the arrival time in the queue, satisfying:

- $s \geq \mathcal{EAD}(u)$, so that a is temporally compatible with the (r, u) -walk W_u that is already present in the TOB, i.e. $W_u + (u, a, v)$ is a temporal walk;
- $t < \mathcal{EAD}(v)$, which ensures that we add to the TOB a new vertex each time;
- $t - s = d_i - d(u)$ if $D = \text{ST}$, ensuring that $W_u + (u, a, v)$ realizes $\text{ST}(r, v)$.

We then add v and the temporal arc a to the TOB and we update the arrival time in v to $\mathcal{EAD}(v) = t$, which is equal to $\text{EAD}(r, v)$ and it will be no longer updated until the end of the algorithm. Finally, we add to Q all the outgoing arcs from v with head vertices in D_i . When at distance d_i there are no arcs satisfying these constraints, i.e. the queue Q at line 12 is empty, we go to the next distance d_{i+1} , as it means that we have already spanned all the possible vertices in D_i . The initial computation of all $D(r, v)$ requires $O(m \log m)$ by Table 1. Concerning the remaining part of the algorithm, the i -th iteration of the **for** loop considers only arcs whose head is in D_i , hence each arc is considered only in one of the iterations of the **for** loop. Moreover, each arc is dequeued from Q at most once. As the dequeue from Q costs $O(\log m)$ we obtain a running time of $O(m \log m)$.

Theorem 4.3. *For any $D \in \{\text{LD}, \text{ST}\}$, Algorithm 2 returns a maximum D-TOB of a temporal graph, for a chosen root, in $O(m \log m)$ time. Additionally, the output is an EAD-TOB.*

Proof. Let $\mathcal{G} = (V, A, \tau)$ be the temporal graph input of the algorithm, $r \in V$, $d(v) = D_{\mathcal{G}}(r, v)$ for all $v \in V$, $h = |\{d(v) : v \in V\}|$ and $\{d_0, d_1, \dots, d_h\} = \{d(v) : v \in V, d(v) < +\infty\}$, with $d_0 < d_1 < \dots < d_h$. Let $V' = \{v \in V : v \text{ is temporally reachable from } r\}$ and for all $i \in [h]_0$, let $D_i = \{v \in V : d(v) = d_i\}$. Note that $\{D_i : i \in [h]_0\}$ is a partition of V' with $D_0 = \{r\}$. Since no confusion is possible, from now on we will avoid writing the subscripts \mathcal{G} . Note that if $D = LD$, then each iteration of the **for** loop in lines 5-18 is completely independent on the others, as it deals only with vertices in D_i and temporal arcs with both tail and head in D_i . We now proceed by proving the following loop invariant:

Claim 4.2. Given $D \in \{LD, ST\}$, at the end of the i -th iteration of the **for** loop in lines 5-18, we have that $\mathcal{EAD}(v) = \text{EAD}_{\mathcal{G}}(r, v)$ for all $v \in V_{\mathcal{B}}$ and $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ is an EAD-TOB with root r of \mathcal{G} with

$$V_{\mathcal{B}} = \{v \in V' : \exists \text{ a } D\text{-prefix-optimal temp. } (r, v)\text{-walk in } \mathcal{G}, d(v) \leq d_i\}. \quad (4)$$

The claim above implies that the final output \mathcal{B} of the algorithm is an EAD-TOB with root r of \mathcal{G} , which is in particular a D -TOB. Moreover, $V_{\mathcal{B}}$ consists of all the vertices in \mathcal{G} for which there exists a D -prefix-optimal temporal walk from the root, so \mathcal{B} is a maximum D -TOB by Corollary 4.1. We are left to prove the claim. \mathcal{B} is initialized as the temporal graph made of the sole vertex r , so the loop invariant is trivially true. Suppose now that the loop invariant is true up to a certain i -th iteration. We now prove that it holds for the $(i+1)$ -th iteration. We start by proving that if $v \in D_{i+1}$ and there exists a D -prefix-optimal temporal (r, v) -walk in \mathcal{G} , say $W = (r = x_0, a_1, x_1, \dots, a_m, x_m = v)$, then $v \in V_{\mathcal{B}}$ at the end of the $(i+1)$ -th **for** loop iteration. Let x_j be the last vertex of W starting from r that is in $V_{\mathcal{B}}$ before the beginning of the $(i+1)$ -th iteration (x_j possibly equal to r). By inductive hypothesis, this implies that $d(x_j) < d_{i+1}$ and that $d(x_l) = d_{i+1}$ for all $l > j$. Then the arc a_{j+1} is added to Q in lines 7-8 at the beginning of the $(i+1)$ -th iteration. Since W is D -prefix-optimal, a_{j+1} does not fulfil the condition in line 11, unless x_{j+1} has been already added to $V_{\mathcal{B}}$. This implies that at one point of the **while** loop x_{j+1} is added to $V_{\mathcal{B}}$, which implies that a_{j+2} is put in queue Q by line 16. This iteratively proves that for every $l > j$, x_l is added to $V_{\mathcal{B}}$ at one point of the **while** loop, including $x_m = v$. This proves equation (4). To prove the rest of the claim, we will prove the following fact:

Claim 4.3. Suppose to be in the $(i+1)$ -th iteration of the **for** loop of lines 5-18. Then, at the end of each iteration of the **while** loop of lines 9-17, we have that $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ is an EAD-TOB with root r and for all $v \in V_{\mathcal{B}}$, $\mathcal{EAD}(v) = \text{EAD}_{\mathcal{G}}(r, v)$.

At the beginning of the $(i+1)$ -th **for** loop iteration, the inductive hypothesis holds, so the invariant property is true. By contradiction, consider the first iteration of the **while** loop such that the addition of the vertex v to $V_{\mathcal{B}}$ and of the arc $a = (u, v, s, t)$ to $A_{\mathcal{B}}$ makes the claim fail. Since we are at the $(i+1)$ -th **for** loop iteration, it holds that $d(v) = d_{i+1}$. Due to line 11, it must hold that $s \geq \mathcal{EAD}(u)$ and, if $D = ST$, then $t - s = d_{i+1} - d(u)$. This implies that $\mathcal{EAD}(u) < +\infty$, and since the only way for this to hold is to have $u = r$, or to have $\mathcal{EAD}(u)$ updated to a natural number (in which case u is added to $V_{\mathcal{B}}$ in line 15), we get that $u \in V_{\mathcal{B}}$. Also, u must have entered $V_{\mathcal{B}}$ before v , so by hypothesis there exists an EAD-prefix-optimal temporal (r, u) -walk W_u in \mathcal{B} ; in particular $t_a(W_u) = \text{EAD}(r, u) = \mathcal{EAD}(u)$. Since $s \geq \mathcal{EAD}(u)$, then $W_v = W_u + (u, a, v)$ is a temporal (r, v) -walk in \mathcal{B} . Moreover, if $D = LD$, then $u \in D_{i+1}$, so $t_s(W_v) = t_s(W_u) = d(u) = d_{i+1} = d(v)$. If $D = ST$, then $t - s = d_{i+1} - d(u)$, so $\text{tt}(W_v) = \text{tt}(W_u) + (t - s) = d(u) + (t - s) = d_{i+1}$. Hence in both cases W_v is D -prefix-optimal. This also implies that $t = t_a(W_v) \geq \text{EAD}(r, v)$. It remains to show that v has indegree 1 in \mathcal{B} and that $t = \mathcal{EAD}(v) = \text{EAD}(r, v)$ to derive the contradiction. Suppose first that v has indegree $\neq 1$. Then it must have indegree greater than 1, because $a \in A_{\mathcal{B}}$ is an incoming temporal arc of v . Then, at a previous step of the **while** loop, an arc $a' = (u', v, s', t')$ was added to $A_{\mathcal{B}}$, which means that v was also added to $V_{\mathcal{B}}$ and $\mathcal{EAD}(v)$ was set equal to t' . When a' was added, u' must have already been in $V_{\mathcal{B}}$. By hypothesis, there exists an EAD-prefix-optimal temporal (r, u') -walk $W_{u'}$ in \mathcal{B} , and $W' = W_{u'} + (u', a', v)$ is such that $\text{EAD}(r, v) = t_a(W') = t' = \mathcal{EAD}(v)$ by hypothesis. Since $t \geq \text{EAD}(r, v) = \mathcal{EAD}(v)$, the arc a could have never been chosen later, as it is fulfilling the condition $t \geq \mathcal{EAD}(v)$ in line 11. So v has indegree 1 in \mathcal{B} . Suppose now that $\mathcal{EAD}(v) \neq \text{EAD}(r, v)$, i.e. that $t > \text{EAD}(r, v)$. We know that v has a D -prefix-optimal (r, v) -walk in \mathcal{G} ; let W the one that arrives the earliest in v , i.e. $t_a(W) = \text{EAD}(r, v)$. Let $y \in V(W)$ be the first vertex along W such that, when v is added to $V_{\mathcal{B}}$, $y \notin V_{\mathcal{B}}$, and let $x \in V_{\mathcal{B}}$ be y 's predecessor along W

and $a_{xy} = (x, y, s_{xy}, t_{xy})$ be the temporal arc connecting them in W (x may coincide with r). By inductive hypothesis, we have that $y \in D_{i+1}$. Notice that $t_{xy} \leq t_a(W) = \text{EAD}(r, v)$. Since $x \in V_{\mathcal{B}}$ and we chose v as the first vertex for which $\mathcal{EAD}(v) \neq \text{EAD}(r, v)$, we have that $\mathcal{EAD}(x) = \text{EAD}(r, x)$ when x was added to $V_{\mathcal{B}}$. This implies that the arc a_{xy} is enqueued in Q when x is added to $V_{\mathcal{B}}$. Indeed if $\text{D} = \text{LD}$, then $d(x) = d(y) = d(v) = d_{i+1}$ and so $a_{xy} \in \{(x, v', s', t') \in A : v' \in D_{i+1}\}$ in line 16 and if $\text{D} = \text{ST}$, let i' such that $d_{i'} = d(x) \leq d_{i+1}$. If $i' = i + 1$ we conclude as above. If $i' < i + 1$, since $y \in D_{i+1}$, then a_{xy} is enqueued in Q at the beginning of the $(i + 1)$ -th **for** loop iteration (line 8). We claim that when v was added to $V_{\mathcal{B}}$, a_{xy} was still in Q . Indeed, a_{xy} could have not been dequeued from Q and added to $A_{\mathcal{B}}$ since otherwise $y \in V_{\mathcal{B}}$ before v was added to $V_{\mathcal{B}}$, which contradicts the hypothesis. If a_{xy} was dequeued from Q without being added to $A_{\mathcal{B}}$, since W is D -prefix-optimal (and so $t_{xy} - s_{xy} = d(y) - d(x) = d_{i+1} - d(x)$ if $\text{D} = \text{ST}$) and $s_{xy} \geq \text{EAD}(r, x) = \mathcal{EAD}(x)$, then it must hold that $\mathcal{EAD}(y) < +\infty$. This implies that y was already in $V_{\mathcal{B}}$ before v was added to $V_{\mathcal{B}}$, which again contradicts the hypothesis. Therefore, when v was added to $V_{\mathcal{B}}$, it must hold that $t \leq t_{xy}$. But $t_{xy} \leq t_a(W) = \text{EAD}(r, v) \leq t$ and so $t = \text{EAD}(r, v)$. This concludes the proof.

Regarding the computational complexity, the initial computation of all $\text{D}(r, v)$, $v \in V$, requires $O(m \log m)$ by Table 1. Concerning the remaining part of the algorithm, notice that the i -th iteration of the **for** loop considers only arcs whose head is in D_i . This means that each arc is considered only in one of the iterations of the **for** loop. Moreover, each arc is dequeued from Q at most once. As the dequeue from Q costs $O(\log m)$ we obtain a total running time of $O(m \log m)$. \square

5 Finding a maximum D -temporal out-branching is NP-hard for fastest time and minimum waiting time distances

As previously observed, Theorem 4.1 does not hold for $\text{D} \in \{\text{FT}, \text{MW}\}$. Indeed in these cases the problem becomes NP-complete even in the following very constrained situations: when $el(a) = 0$ for all $a \in A$, also called *nonstrict* temporal graphs, and when $el(a) = 1$ for all $a \in A$, also called *strict* temporal graphs (see e.g. [9]). The nonstrict model is used when the time-scale of the measured phenomenon is relatively big: this is the case in a disease-spreading scenario [36, 13] where the spreading speed might be unclear, or in time-varying graphs [30], where a single snapshot corresponds e.g. to all the streets available within a day.⁹ Notice that in the following theorem we consider we consider a particular case of the decision problems, namely, the existence of a spanning D -TOB.

Theorem 5.1. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $\text{D} \in \{\text{FT}, \text{MW}\}$. Deciding whether \mathcal{G} has a spanning D -TOB with root r is NP-complete, even if $\tau = 2$ and $el(a) = 0$ for every $a \in A$, or if $\tau = 4$ and $el(a) = 1$ for every $a \in A$.*

Proof. The problem is in NP, since computing $\text{D}_{\mathcal{G}}(r, v)$ for every vertex v can be done in polynomial time (Table 1), as well as testing whether a given temporal subgraph \mathcal{B} is a D -TOB (see Lemma 3.1). To prove hardness, we make a reduction from 3-SAT, largely known to be NP-complete [11, 25]. For this, consider a formula ϕ in CNF form on variables $X = \{x_1, \dots, x_n\}$ and on clauses $C = \{c_1, \dots, c_m\}$. We first construct $\mathcal{G} = (V, A, \tau)$ for the case where every arc has elapsed time 0 (observe Figure (6a) to follow the construction). Let $V = X \cup C \cup \{r\}$. For each variable x_i , add to A the temporal arcs $(r, x_i, 1, 1)$ and $(r, x_i, 2, 2)$. Then, for each clause c_j and each variable x_i appearing in c_j , add temporal arc $(x_i, c_j, 1, 1)$ if x_i appears in c_j positively, while add the temporal arc $(x_i, c_j, 2, 2)$ if x_i appears in c_j negatively. We now prove that ϕ is satisfiable if and only if there exists a spanning D -TOB rooted in r . Suppose first that ϕ has a satisfying assignment; we show how to construct a spanning D -TOB $\mathcal{B} = (V, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ rooted in r . For each variable x_i , add to $A_{\mathcal{B}}$ the temporal arc $(r, x_i, 1, 1)$ if x_i is true, while add to $A_{\mathcal{B}}$ the temporal arc $(r, x_i, 2, 2)$ if x_i is false. Now consider a clause c_j and choose one of the variables that validates c_j , say x_{i_j} . Add to $A_{\mathcal{B}}$ the unique temporal arc with head c_j and tail x_{i_j} . Now observe that the vertices in X are connected to r in \mathcal{B} through direct arcs; hence we get that $\text{D}_{\mathcal{B}}(r, x_i) = 0$ for every $x_i \in X$. For a clause c_j , if x_{i_j} appears

⁹The literature often focused on nonstrict/strict variations to provide stronger negative results. In this paper, we have used the more general model to provide stronger positive results, while using the nonstrict/strict when providing negative ones.



(a) All temporal arcs have elapsed time 0.

(b) All temporal arcs have elapsed time 1.

Figure 6: Example of the construction in the proof of Theorem 5.1. Clause c_1 is equal to $(x_1 \vee x_2 \vee \neg x_3)$. The value on top of each arc represents the starting time.

positively in c_j , then x_{i_j} is true, and $(r, x_{i_j}, 1, 1)$ and $(x_{i_j}, c_j, 1, 1)$ are in $A_{\mathcal{B}}$; therefore $D_{\mathcal{B}}(r, c_j) = 0$. If x_{i_j} appears negatively in c_j , then x_{i_j} is false, so $(r, x_{i_j}, 2, 2)$ and $(x_{i_j}, c_j, 2, 2)$ are in $A_{\mathcal{B}}$; therefore $D_{\mathcal{B}}(r, c_j) = 0$. Finally, observe that each vertex different from the root has indegree 1. By Lemma 3.1, we get that \mathcal{B} is a spanning TOB, and since $D_{\mathcal{B}}(r, v) = 0$ for every $v \in V$, it follows that \mathcal{B} is a spanning D-TOB.

Suppose now that $\mathcal{B} = (V, A_{\mathcal{B}}, \tau_{\mathcal{B}})$ is a spanning D-TOB rooted in r . Since the only possible (r, x_i) -walk is through an arc, we get that either $(r, x_i, 1, 1) \in A_{\mathcal{B}}$ or $(r, x_i, 2, 2) \in A_{\mathcal{B}}$. If the former occurs, then set x_i to true, while if the latter occurs, then set x_i to false. We now argue that this must be a satisfying assignment to ϕ . For this, consider a clause c_j . By Lemma 3.1, we know that $d_{\mathcal{B}}^-(c_j) = 1$; so let $a = (x_{i_j}, c_j, t, t)$ be the temporal arc incident to c_j in \mathcal{B} . If x_{i_j} appears positively in c_j , then we know that $a = (x_{i_j}, c_j, 1, 1)$ by construction. And since the temporal (r, c_j) -walk must pass by x_{i_j} , we get that $(r, x_{i_j}, 1, 1) \in A_{\tau}$, in which case x_{i_j} is set to true and hence satisfies c_j . If x_{i_j} appears in c_j negatively, then $a = (x_{i_j}, c_j, 2, 2)$. Notice that $D_{\mathcal{G}}(r, c_j) = 0$; since \mathcal{B} is a D-TOB, we must also have $D_{\mathcal{B}}(r, c_j) = 0$. This implies that $(r, x_{i_j}, 2, 2) \in A_{\mathcal{B}}$ and hence x_{i_j} is set to false, satisfying c_j .

In the case where $el(a) = 1$ for every arc a , the reduction is similar to the previous one. Specifically, for each $x_i \in X$, we add arcs $(r, x_i, 1, 2)$ and $(r, x_i, 2, 3)$. For each clause c_j , if x_i appears positively in c_j we add the temporal arc $(x_i, c_j, 2, 3)$, while if x_i appears negatively in c_j we add the temporal arc $(x_i, c_j, 3, 4)$. Analogous arguments to the previous ones apply. \square

The gaps left by the above theorem are when \mathcal{G} has lifetime 1 or when \mathcal{G} has lifetime $\tau \in \{2, 3\}$ and all arcs have elapsed time at least 1. Those cases are investigated in the following proposition.

Proposition 5.1. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $D \in \{\text{FT}, \text{MW}\}$. If $\tau = 1$ or, if $\tau \in \{2, 3\}$ and $el(a) \geq 1$ for all $a \in A$, then a maximum D-TOB with root r of \mathcal{G} is computable in polynomial time.*

Proof. If $\tau = 1$, the temporal graph reduces to a static graph, so the problem is solvable in polynomial time by Dijkstra's algorithm. Suppose now that $\tau = 2$ and $el(a) \geq 1$ for all $a \in A$. Then a maximum D-TOB rooted in r contains exactly r and every $u \in V$ such that $(r, u, 1, 2)$ is an arc in \mathcal{G} . Finally, let $\tau = 3$ and $el(a) \geq 1$ for all $a \in A$. Note that each arc has a temporal label belonging to the set $\{(1, 2), (2, 3), (1, 3)\}$. This implies that a temporal walk has length of at most 2 and that every temporal walk realizing D is also D-prefix-optimal, because it is either made of just one arc, or it is made by two arcs consecutively labeled by $(1, 2)$ and $(2, 3)$. Intuitively, to build a maximum D-TOB, we first add all the vertices reachable from the root by an arc labeled by $(1, 2)$, and secondly we add all the vertices (different from the previous ones) that are reached from the root by an arc labeled by $(1, 3)$. Finally, we add all the vertices (not yet added) that are reachable from r by a temporal path of length 2. More formally, we set $A_1 = \{(r, v, 1, 2) \in A : v \in V\}$, $V_1 = \{v \in V : (r, v, 1, 2) \in A_1\}$, $A_2 = \{(r, v, 2, 3) \in A : v \in V \setminus V_1\}$, $V_2 = \{v \in V : (r, v, 2, 3) \in A_2\}$, $A_3 = \{(r, v, 1, 3) \in A : v \in V \setminus (V_1 \cup V_2)\}$, $V_3 = \{v \in V : (r, v, 1, 3) \in A_3\}$, $A_4 = \{(u, v, 2, 3) \in A : u \in V_1, v \in V \setminus (V_1 \cup V_2 \cup V_3)\}$, $V_4 = \{v \in V : u \in V_1, (u, v, 2, 3) \in A_4\}$, $V_{\mathcal{B}} = \{r\} \cup V_1 \cup V_2 \cup V_3 \cup V_4$ and $A_{\mathcal{B}} = A_1 \cup A_2 \cup A_3 \cup A_4$. Then a maximum D-TOB for \mathcal{G} rooted in r is the subgraph $\mathcal{B} = (V_{\mathcal{B}}, A_{\mathcal{B}}, 3)$. \square

6 Reaching vertices with no prefix-optimal paths: minimum temporal spanning subgraphs

We have seen that, for $D \in \{LD, MT, ST\}$, when a vertex v in a temporal graph does not have a D -prefix-optimal path from the root, then no D -TOB reaching v is possible. We can tackle the problem from another point of view, where we want to reach anyway all the vertices of the graph from the root optimizing some distance, while still using the least amount of connections possible.

Definition 6.1. Let $\mathcal{G} = (V, A, \tau)$ a temporal graph, $r \in V$ and D a distance. Then $\mathcal{G}' = (V, A', \tau')$ is a D -Temporal Out-Spanning Subgraph (D -TOSS) with root r of \mathcal{G} if for all $v \in V$, there exists a temporal (r, v) -walk in \mathcal{G}' realizing $D_{\mathcal{G}}(r, v)$; if in addition $|A'|$ is the smallest possible, then \mathcal{G}' is a *minimum* D -TOSS.

A spanning D -TOB with root r is always a minimum D -TOSS with root r . On the other hand, notice that the only D -TOSS with root r of the temporal graphs in Figure 3, for the corresponding distances, are the graph themselves. Clearly, for $D \in \{LD, MT, ST\}$, if every vertex of the temporal graph has a D -prefix-optimal path from the root, then a minimum D -TOSS with root r is a spanning D -TOB with root r . We now focus on finding a minimum D -TOSS of a given temporal graph, for a chosen root and distance D . In particular, we study the computational complexity of the following problem:

Problem 6.1. Given $\mathcal{G} = (V, A, \tau)$ a temporal graph, $r \in V$, $k \in \mathbb{N}$ and $D \in \{EA, FT, LD, MT, MW, ST\}$, decide whether \mathcal{G} has a D -TOSS with root r and with at most k temporal arcs.

Notice that for $D = EA$, the concepts of spanning EA -TOB and minimum EA -TOSS coincide¹⁰. This implies that Problem 6.1 for $D = EA$ is solvable in polynomial time: if some vertex is not temporally reachable from the root the answer is simply NO; if every vertex is temporally reachable from the root, then the answer is YES if and only if $k \geq |V| - 1$ [19]. On the other hand, for $D \in \{FT, MW\}$, Problem 6.1 becomes NP-hard as it suffices to set $k = |V| - 1$ and apply Theorem 5.1. For all the other distances, the problem turns out to be a difficult task also in very constrained situations.

Theorem 6.1. Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$, $k \in \mathbb{N}$, and suppose that there exists $a \in A$ such that $el(a) \neq 0$. Then deciding whether \mathcal{G} has an ST -TOSS with root r and with at most k temporal arcs is NP-complete even when $\tau = 3$.

Proof. Let $\mathcal{G}' = (V, A', \tau')$ be a temporal subgraph of \mathcal{G} . Computing $ST_{\mathcal{G}}(r, v)$ and $ST_{\mathcal{G}'}(r, v)$ for every vertex v can be done in polynomial time (Table 1), as well as checking if $|A'| \leq k$, so the problem is in NP.

To prove hardness we make a reduction from 3-SAT. Consider a formula ϕ in CNF form on variables $X = \{x_1, \dots, x_l\}$ and clauses $C = \{c_1, \dots, c_m\}$. We construct a temporal graph $\mathcal{G} = (V, A, 3)$ in the following way. Let $V = \{x_1^p, \dots, x_l^p\} \cup \{x_1^n, \dots, x_l^n\} \cup \{y_1, \dots, y_l\} \cup C \cup \{r\}$. Notice that $|V| = 3l + m + 1$. For each $\alpha \in \{p, n\}$ and $i \in [l]$, add to A the temporal arcs $(r, x_i^\alpha, 1, 2)$, $(r, x_i^\alpha, 3, 3)$ and $(x_i^\alpha, y_i, 2, 2)$. Then, for each clause c_j and each variable x_i appearing in c_j , add the temporal arc $(x_i^p, c_j, 2, 2)$ if x_i appears in c_j positively, while add the temporal arc $(x_i^n, c_j, 2, 2)$ if x_i appears in c_j negatively (see Figure 7). Observe that $ST_{\mathcal{G}}(r, x_i^\alpha) = 0$ for all $\alpha \in \{p, n\}$, $i \in [l]$, $ST_{\mathcal{G}}(r, y_i) = 1$ for all $i \in [l]$ and $ST_{\mathcal{G}}(r, c_j) = 1$ for all $j \in [m]$. We now prove that ϕ is satisfiable if and only if there exists an ST -TOSS $\mathcal{G}' = (V, A', 3)$ of \mathcal{G} with root r such that $|A'| \leq 4l + m$.

Suppose first that ϕ has a satisfying assignment; we show how to construct \mathcal{G}' . For each variable x_i , if x_i is true then add to A' the temporal arcs $(r, x_i^p, 1, 2), (r, x_i^p, 3, 3), (r, x_i^n, 3, 3), (x_i^p, y_i, 2, 2)$, while if x_i is false then add to A' the temporal arcs $(r, x_i^n, 1, 2), (r, x_i^n, 3, 3), (r, x_i^p, 3, 3), (x_i^n, y_i, 2, 2)$. Now consider a clause c_j and choose one of the variables that validates c_j , say x_{i_j} . Add to A' the unique temporal arc with head c_j and tail $x_{i_j}^{\alpha_{i_j}}$, $\alpha_{i_j} \in \{p, n\}$. It holds that $|A'| = 4l + m$. Now observe that for all $v \in V$ there exists a temporal (r, v) -path in \mathcal{G}' realizing $ST_{\mathcal{G}}(r, v)$: indeed, the vertices x_i^α are directly connected to r by the arcs with time labels $(3, 3)$ realizing their shortest times. Vertices y_i are connected to r by the path $(r, x_i^p, 1, 2),$

¹⁰Since a temporal graph \mathcal{G} has a spanning EA -TOB with a given root as subgraph if and only if each vertex is reachable from the root in \mathcal{G} [19].

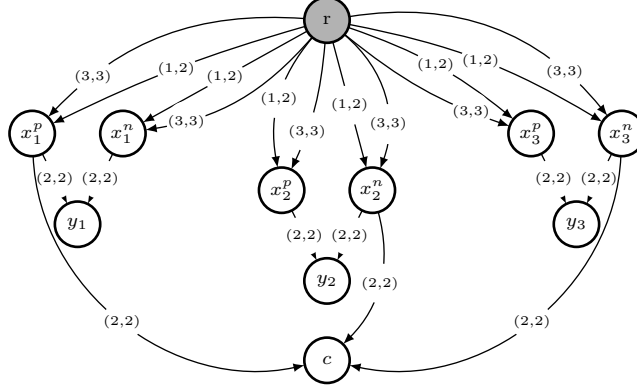


Figure 7: Example of the construction in the proof of Theorem 6.1. Clause c is equal to $(x_1 \vee \neg x_2 \vee \neg x_3)$.

$(x_i^p, y_i, 2, 2)$ if x_i is true, or by the path $(r, x_i^n, 1, 2)$, $(x_i^n, y_i, 2, 2)$ if x_i is false, which realizes $\text{ST}_{\mathcal{G}}(r, y_i)$. For a clause c_j , if x_{i_j} appears positively in c_j , then x_{i_j} is true, and so $(r, x_{i_j}^p, 1, 2)$, $(x_{i_j}^p, c_j, 2, 2)$ is a temporal path in \mathcal{G}' reaching c_j and realizing $\text{ST}_{\mathcal{G}}(r, c_j)$; if x_{i_j} appears negatively in c_j , then x_{i_j} is false, and so $(r, x_{i_j}^n, 1, 2)$, $(x_{i_j}^n, c_j, 2, 2)$ is a temporal path in \mathcal{G}' reaching c_j and realizing $\text{ST}_{\mathcal{G}}(r, c_j)$.

Suppose now that $\mathcal{G}' = (V, A', 3)$ is an ST-TOSS of \mathcal{G} with $|A'| \leq 4l + m$. Since \mathcal{G}' is spanning, each vertex different from the root must have at least one incoming arc, so $|A'| \geq 3l + m = |V| - 1$. In particular, for all $i \in [l]$, $\alpha \in \{p, n\}$, the arc $(r, x_i^\alpha, 3, 3)$ must belong to A' as it is the only one realizing $\text{ST}(r, x_i^\alpha)$. Then notice that the only way a vertex y_i can be temporally reachable from r is that at least one of the arcs $(r, x_i^p, 1, 2)$ and $(r, x_i^n, 1, 2)$ belongs to A' . Since $|A'| \leq 4l + m$, it must hold that for every $i \in [l]$, exactly one of the arcs between $(r, x_i^p, 1, 2)$ and $(r, x_i^n, 1, 2)$ belongs to A' . If the former case occurs, then set x_i to true, while if the latter case occurs, then set x_i to false. We now argue that this must be a satisfying assignment to ϕ . For this, consider a clause c_j . Each clause c_j must be reached exactly by one arc in \mathcal{G}' , as otherwise there would not be enough arcs to connect every vertex; let $(x_{i_j}^{\alpha_{i_j}}, c_j, 2, 2)$ be this arc. This also implies that $(r, x_{i_j}^{\alpha_{i_j}}, 1, 2) \in A'$, as otherwise c_j would not be temporally reachable from r . If $\alpha_{i_j} = p$, then x_{i_j} is set to true and, by construction, x_{i_j} appears positively in c_j ; so x_{i_j} satisfies c_j . If $\alpha_{i_j} = n$, then x_{i_j} is set to false and, by construction, x_{i_j} appears negatively in c_j ; so x_{i_j} satisfies c_j . \square

The gaps left by the above theorem are when \mathcal{G} has lifetime $\tau \in \{1, 2\}$ or when $el(a) = 0$ for all arcs a of \mathcal{G} . When $\tau = 1$ the temporal graph reduces to a static graph and the problem reduces to finding an out-branching of \mathcal{G} and when $\tau = 2$, by Lemma 4.2 the problem reduces to find a spanning ST-TOB, which is solvable in polynomial time. Finally, if $el(a) = 0$ for all $a \in A$, then $\text{ST}(r, v) = 0$ for all $v \in V$; hence the problem reduces to finding a TOB, which is then solvable in polynomial time (Table 2).

Theorem 6.2. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $k \in \mathbb{N}$. Then deciding whether \mathcal{G} has an LD-TOSS with root r and with at most k temporal arcs is NP-complete even when $\tau = 2$ and $el(a) = 0$ for every $a \in A$, or when $\tau = 3$ and $el(a) = 1$ for every $a \in A$.*

Proof. Let $\mathcal{G}' = (V, A', \tau')$ be a temporal subgraph of \mathcal{G} . Computing $\text{LD}_{\mathcal{G}}(r, v)$ and $\text{LD}_{\mathcal{G}'}(r, v)$ for every vertex v can be done in polynomial time (Table 1), as well as checking if $|A'| \leq k$, so the problem is in NP.

To prove hardness we make a reduction from 3-SAT, in a similar way to the proof of Theorem 6.1. Consider a formula ϕ in CNF form on variables $X = \{x_1, \dots, x_l\}$ and on clauses $C = \{c_1, \dots, c_m\}$. We construct a temporal graph $\mathcal{G} = (V, A, 2)$ in the following way. Let $V = \{x_1^p, \dots, x_l^p\} \cup \{x_1^n, \dots, x_l^n\} \cup \{y_1, \dots, y_l\} \cup C \cup \{r\}$. Notice that $|V| = 3l + m + 1$. For each $\alpha \in \{p, n\}$ and $i \in [l]$, add to A the temporal arcs $(r, x_i^\alpha, 1, 1)$, $(r, x_i^\alpha, 2, 2)$ and $(x_i^\alpha, y_i, 1, 1)$. Then, for each clause c_j and each variable x_i appearing in c_j , add the temporal arc $(x_i^p, c_j, 1, 1)$ if x_i appears in c_j positively, while add the temporal arc $(x_i^n, c_j, 1, 1)$ if x_i appears in c_j negatively (see Figure 8a). Observe that $\text{LD}_{\mathcal{G}}(r, x_i^\alpha) = 2$ for all $\alpha \in \{p, n\}$, $i \in [l]$, while $\text{LD}_{\mathcal{G}}(r, v) = 1$ for all

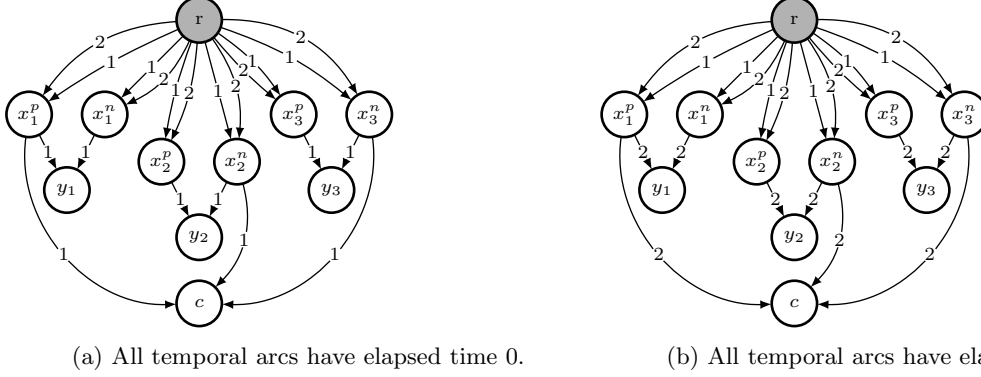


Figure 8: Example of the construction in the proof of Theorem 6.2. Clause c is equal to $(x_1 \vee \neg x_2 \vee \neg x_3)$. The value on top of each arc is the starting time.

the other vertices v . We now prove that ϕ is satisfiable if and only if there exists a LD-TOSS of \mathcal{G} with root r with at most $4l + m$ arcs. Suppose that ϕ has a satisfying assignment; we show how to construct a LD-TOSS $\mathcal{G}' = (V, A', 2)$ with root r of \mathcal{G} with $|A'| \leq 4l + m$. For each variable x_i , if x_i is true then add to A' the temporal arcs $(r, x_i^p, 1, 1), (r, x_i^p, 2, 2), (r, x_i^n, 2, 2), (x_i^p, y_i, 1, 1)$, while if x_i is false then add to A' the temporal arcs $(r, x_i^n, 1, 1), (r, x_i^n, 2, 2), (r, x_i^p, 2, 2), (x_i^n, y_i, 1, 1)$. Now consider a clause c_j and choose one of the variables that validates c_j , say x_{i_j} . Add to A' the unique temporal arc with head c_j and tail $x_{i_j}^{\alpha_{i_j}}$, $\alpha_{i_j} \in \{p, n\}$. It holds that $|A'| = 4l + m$. Observe that for all $v \in V$ there exists a temporal (r, v) -path in \mathcal{G}' realizing $\text{LD}_{\mathcal{G}}(r, v)$.

Suppose now that $\mathcal{G}' = (V, A', 2)$ is an LD-TOSS of \mathcal{G} with $|A'| \leq 4l + m$. Since \mathcal{G}' is spanning, each vertex different from the root must have at least one incoming arc, so $|A'| \geq 3l + m = |V| - 1$. In particular, for all $i \in [l]$, $\alpha \in \{p, n\}$, the arc $(r, x_i^\alpha, 2, 2)$ must belong to A' since it is the only one realizing $\text{LD}(r, x_i^\alpha)$. Then notice that the only way a vertex y_i can be temporally reachable from r is that at least one of the arcs $(r, x_i^p, 1, 1)$ and $(r, x_i^n, 1, 1)$ belongs to A' . Since $|A'| \leq 4l + m$, it must hold that for every $i \in [l]$, exactly one of the arcs between $(r, x_i^p, 1, 1)$ and $(r, x_i^n, 1, 1)$ belongs to A' . If the former case occurs, then set x_i to true, while if the latter case occurs, then set x_i to false. We now argue that this must be a satisfying assignment to ϕ . For this, consider a clause c_j . Each clause c_j must be reached exactly by one arc in \mathcal{G}' , as otherwise there would not be enough arcs to connect every vertex; let $(x_{i_j}^{\alpha_{i_j}}, c_j, 1, 1)$ be this arc. This also implies that $(r, x_{i_j}^{\alpha_{i_j}}, 1, 1) \in A'$, as otherwise c_j would not be temporally reachable from r . If $\alpha_{i_j} = p$, then x_{i_j} is set to true and, by construction, x_{i_j} appears positively in c_j ; so x_{i_j} satisfies c_j . If $\alpha_{i_j} = n$, then x_{i_j} is set to false and, by construction, x_{i_j} appears negatively in c_j ; so x_{i_j} satisfies c_j .

In the case where $\text{el}(a) = 1$ for every arc a , the reduction is similar to the previous one. Specifically, for each $i \in [l]$, $\alpha \in \{p, n\}$, we add arcs $(r, x_i^\alpha, 1, 2), (r, x_i^\alpha, 2, 3), (x_i^\alpha, y_i, 2, 3)$. For each clause c_j , if x_i appears positively in c_j we add the temporal arc $(x_i^p, c_j, 2, 3)$, while if x_i appears negatively in c_j we add the temporal arc $(x_i^n, c_j, 2, 3)$; see Figure 8b. Analogous arguments to the previous ones apply. \square

The gaps left by the above theorem are when \mathcal{G} has lifetime 1 or when \mathcal{G} has lifetime 2 and all arcs have elapsed time at least 1. In the first case, the temporal graph reduces to a static graph, so the problem is solvable in polynomial time by Dijkstra's algorithm. As for the second case, if all vertices of \mathcal{G} are temporally reachable from the root r one can see that a LD-TOSS rooted in r contains exactly all the arcs of type $(r, v, 1, 2)$, for every $v \in V$.

Theorem 6.3. *Let $\mathcal{G} = (V, A, \tau)$ be a temporal graph, $r \in V$ and $k \in \mathbb{N}$. Then deciding whether \mathcal{G} has an MT-TOSS with root r and with at most k temporal arcs is NP-complete even if $\tau = 2$ and $\text{el}(a) = 0$ for every $a \in A$, or if $\tau = 4$ and $\text{el}(a) = 1$ for every $a \in A$.*

Proof. Let $\mathcal{G}' = (V, A', \tau')$ be a temporal subgraph of \mathcal{G} . Computing $\text{MT}_{\mathcal{G}}(r, v)$ and $\text{MT}_{\mathcal{G}'}(r, v)$ for every vertex v can be done in polynomial time (Table 1), as well as checking if $|A'| \leq k$, so the problem is in NP. To prove hardness we make a reduction from 3-SAT. Consider a formula ϕ in CNF form on variables $X = \{x_1, \dots, x_l\}$ and on clauses $C = \{c_1, \dots, c_m\}$. We construct a temporal graph $\mathcal{G} = (V, A, 2)$ in the following way. Let $V = \bigcup_{i \in [l]} \{x_i^p, x_i^n, z_i^p, z_i^n, y_i\} \cup C \cup \{r\}$. Notice that $|V| = 5l + m + 1$. For each $\alpha \in \{p, n\}$ and $i \in [l]$, add to A the temporal arcs $(r, x_i^\alpha, 2, 2)$, $(r, z_i^\alpha, 1, 1)$, $(z_i^\alpha, x_i^\alpha, 1, 1)$ and $(x_i^\alpha, y_i, 1, 1)$. Then, for each clause c_j and each variable x_i appearing in c_j , add the temporal arc $(x_i^p, c_j, 1, 1)$ if x_i appears in c_j positively, while add the temporal arc $(x_i^n, c_j, 1, 1)$ if x_i appears in c_j negatively (see Figure 9a as example). Observe that $\text{MT}_{\mathcal{G}}(r, x_i^\alpha) = 1 = \text{MT}_{\mathcal{G}}(r, z_i^\alpha)$ for all $\alpha \in \{p, n\}$, $i \in [l]$, and $\text{MT}_{\mathcal{G}}(r, y_i) = 3 = \text{MT}_{\mathcal{G}}(r, c_j)$ for all $i \in [l]$ and $j \in [m]$. We now prove that ϕ is satisfiable if and only if there exists an MT-TOSS of \mathcal{G} with at most $6l + m$ temporal arcs. Suppose first that ϕ has a satisfying assignment; we show how to construct an MT-TOSS $\mathcal{G}' = (V, A', 2)$ with root r of \mathcal{G} with $|A'| \leq 6l + m$. For each variable x_i , if x_i is true then add to A' the temporal arcs

$$(r, x_i^p, 2, 2), (r, z_i^p, 1, 1), (z_i^p, x_i^p, 1, 1), (x_i^p, y_i, 1, 1), (r, x_i^n, 2, 2), (r, z_i^n, 1, 1),$$

while if x_i is false then add to A' the temporal arcs

$$(r, x_i^n, 2, 2), (r, z_i^n, 1, 1), (z_i^n, x_i^n, 1, 1), (x_i^n, y_i, 1, 1), (r, x_i^p, 2, 2), (r, z_i^p, 1, 1).$$

Now consider a clause c_j and choose one of the variables that validates c_j , say x_{i_j} . Add to A' the unique temporal arc with head c_j and tail $x_{i_j}^{\alpha_{i_j}}$, $\alpha_{i_j} \in \{p, n\}$. It holds that $|A'| = 6l + m$. Observe that for all $v \in V$ there exists a temporal (r, v) -path in \mathcal{G}' realizing $\text{MT}_{\mathcal{G}}(r, v)$.

Suppose now that $\mathcal{G}' = (V, A', 2)$ is an MT-TOSS of \mathcal{G} with $|A'| \leq 6l + m$. Since \mathcal{G}' is spanning, each vertex different from the root must have at least one incoming arc, so $|A'| \geq 5l + m = |V| - 1$. In particular, for all $i \in [l]$, $\alpha \in \{p, n\}$, the arcs $(r, x_i^\alpha, 2, 2)$ and $(r, z_i^\alpha, 1, 1)$ must belong to A' as they are the only ones realizing the distance for such vertices. Then notice that the only way a vertex y_i can be temporally reachable from r is that at least one of the arcs $(z_i^p, x_i^p, 1, 1)$ and $(z_i^n, x_i^n, 1, 1)$ belong to A' . Since $|A'| \leq 6l + m$, it must hold that for every $i \in [l]$, exactly one of the arcs between $(z_i^p, x_i^p, 1, 1)$ and $(z_i^n, x_i^n, 1, 1)$ belongs to A' . If the former case occurs, then set x_i to true, while if the latter case occurs, then set x_i to false. We now argue that this must be a satisfying assignment to ϕ . For this, consider a clause c_j . Each clause c_j must be reached exactly by one arc in \mathcal{G}' , as otherwise there would not be enough arcs to connect every vertex; let $(x_{i_j}^{\alpha_{i_j}}, c_j, 1, 1)$ be this arc. This also implies that $(z_{i_j}^{\alpha_{i_j}}, x_{i_j}^{\alpha_{i_j}}, 1, 1) \in A'$, as otherwise c_j would not be temporally reachable from r . If $\alpha_{i_j} = p$, then x_{i_j} is set to true and, by construction, x_{i_j} appears positively in c_j ; so x_{i_j} satisfies c_j . If $\alpha_{i_j} = n$, then x_{i_j} is set to false and, by construction, x_{i_j} appears negatively in c_j ; so x_{i_j} satisfies c_j .

In the case where $\text{el}(a) = 1$ for every arc a , the reduction is similar to the previous one. Specifically, for each $i \in [l]$, $\alpha \in \{p, n\}$, the temporal graph is made of the arcs $(r, x_i^\alpha, 3, 4)$, $(r, z_i^\alpha, 1, 2)$, $(z_i^\alpha, x_i^\alpha, 2, 3)$ and $(x_i^\alpha, y_i, 3, 4)$. Then for each clause c_j , if x_i appears positively in c_j we add the temporal arc $(x_i^p, c_j, 3, 4)$, while if x_i appears negatively in c_j we add the temporal arc $(x_i^n, c_j, 3, 4)$. See Figure 9b as example. Analogous arguments to the previous ones apply. \square

The gaps left by the above theorem are when $\tau = 1$ or when $\tau \in \{2, 3\}$ and all arcs have elapsed time at least 1. In the first case, the temporal graph reduces to a static graph, so the problem is solvable in polynomial time by Dijkstra's algorithm. When $\tau = 2$ and all arcs have elapsed time at least 1, the minimum MT-TOSS rooted in r contains exactly all the arcs of type $(r, v, 1, 2)$, for every $v \in V$, if they exist; otherwise there is no MT-TOSS of \mathcal{G} . When $\tau = 3$ and all arcs have elapsed time at least 1, if v is temporally reachable from r , then either $\text{MT}(r, v) = 1$ or $\text{MT}(r, v) = 2$. Notice that every vertex such that $\text{MT}(r, v) = 2$ has a MT-prefix-optimal path from the root; hence every vertex temporally reachable from the root has a MT-prefix-optimal path from r . Then, by Theorem 4.2, a maximum MT-TOB of \mathcal{G} is computable in polynomial time. If it is also spanning, then it is in particular a minimum MT-TOSS; if it is not spanning, then there is no MT-TOSS of \mathcal{G} .

Theorems 6.1, 6.2, and 6.3, together with Theorem 5.1 prove that finding a minimum D-TOSS of a temporal graph is an NP-hard problem for each $D \neq \text{EA}$.

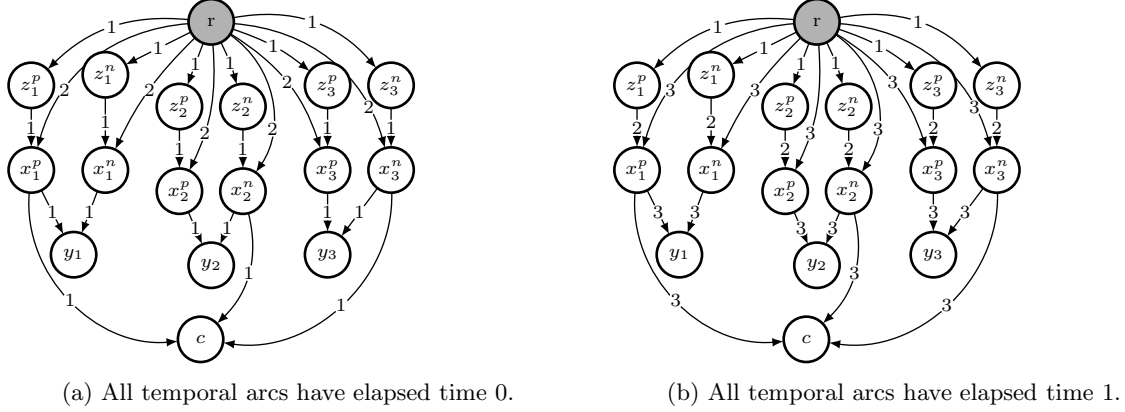


Figure 9: Example of the construction in the proof of Theorem 6.3. Clause c is equal to $(x_1 \vee \neg x_2 \vee \neg x_3)$. The value on top of each arc is the starting time.

Remark 6.1. Similarly to what done for TOBS, we could define a D-Temporal In-Spanning Subgraph (D-TISS) with root r of a temporal graph $\mathcal{G} = (V, A, \tau)$ as the subgraph where, for all $v \in V$, there exists a temporal (v, r) -walk realizing $D_{\mathcal{G}}(r, v)$. If in addition it has the least number possible of temporal arcs, then we call it a *minimum* D-TISS. We then may ask what is the complexity of finding a minimum D-TISS of a temporal graph. Thanks to the transformation \circlearrowleft presented in Definition 3.5 and by following the lines of the proof of Proposition 3.1, it is easy to show that, given a subgraph \mathcal{G}' of \mathcal{G} :

- \mathcal{G}' is a minimum EA-TISS of \mathcal{G} if and only if $\mathcal{G}'^{\circlearrowleft}$ is a minimum LD-TOSS of $\mathcal{G}^{\circlearrowleft}$;
- \mathcal{G}' is a minimum LD-TISS of \mathcal{G} if and only if $\mathcal{G}'^{\circlearrowleft}$ is a minimum EA-TOSS of $\mathcal{G}^{\circlearrowleft}$;
- For each $D \in \{\text{FT}, \text{MT}, \text{MW}, \text{ST}\}$, \mathcal{G}' is a minimum D-TISS of \mathcal{G} if and only if $\mathcal{G}'^{\circlearrowleft}$ is a minimum D-TOSS of $\mathcal{G}^{\circlearrowleft}$.

This, together with Theorems 5.1, 6.1, 6.2, and 6.3, implies that for all $D \neq \text{LD}$, finding a minimum D-TISS of a temporal graph is an NP-hard problem, while a minimum LD-TISS is computable in polynomial time (Table 2).

7 Final remarks and conclusions

We have showed that for $D \in \{\text{LD}, \text{MT}, \text{ST}\}$, a spanning D-TOB does not always exist, but computing a D-TOB that spans the maximum number of vertices can be done in polynomial-time. Moreover, the overall complexity of the algorithms mainly depends on the complexity of computing the single source distances from the root to all the other vertices. In contrast, when $D \in \{\text{FT}, \text{MW}\}$, finding a spanning D-TOB becomes NP-complete. We then introduced a D-TOSS of a temporal graph as being a temporal subgraph containing a walk that realizes $D(r, v)$ for each vertex v , and we investigated the complexity of deciding the existence of a D-TOSS with at most k arcs, for a given k . We showed that, for any distance $D \neq \text{EA}$, finding such subgraph is an NP-complete problem. Same results can be proven for D-TISS. We highlight that all the hardness results of this paper (Theorems 5.1, 6.1, 6.2 and 6.3) can be modified to meet the condition that the temporal graph in the input must have as underlying structure a digraph¹¹ (instead of a multi-digraph), i.e. for every pair of vertices u and v we require at most one temporal arc having tail u and head v . These modifications are summarized in the following remark.

¹¹Sometimes these temporal graphs are called *simple*.

Remark 7.1. List of the modifications to be implemented in the reductions of the corresponding theorems to meet the condition that the temporal graph in the input must have as underlying structure a digraph.

Theorem 5.1: In the case $el(a) = 0$ for all $a \in A$, set $V = X \cup \{y_1^1, \dots, y_n^1\} \cup \{y_1^2, \dots, y_n^2\} \cup C \cup \{r\}$, and for each variable x_i add the temporal arcs $(r, y_i^1, 1, 1)$, $(y_i^1, x_i, 1, 1)$, $(r, y_i^2, 2, 2)$ and $(y_i^2, x_i, 2, 2)$. The temporal arcs connecting the variables to the clauses remain the same. In the case $el(a) = 1$ for all $a \in A$, consider the same vertex set V and add the temporal arcs $(r, y_i^1, 1, 2)$, $(y_i^1, x_i, 2, 3)$, $(r, y_i^2, 2, 3)$ and $(y_i^2, x_i, 3, 4)$, while augmenting by 1 the starting and arrival times of the temporal arcs connecting the variables to the clauses.

Theorem 6.3: no modifications needed, the temporal graph built in the reduction is already simple.

Theorem 6.1: subdivide the arcs connecting the root to a vertex. Specifically, replace each arc $(r, x_i^\alpha, 3, 3)$ by the arcs $(r, y_i^\alpha, 3, 3)$ and $(y_i^\alpha, x_i^\alpha, 3, 3)$, while the rest remains the same.

Theorem 6.2: In the case $el(a) = 0$ for all $a \in A$, replace each arc $(r, x_i^\alpha, 2, 2)$ by the arcs $(r, y_i^\alpha, 2, 2)$ and $(y_i^\alpha, x_i^\alpha, 2, 2)$. In the case $el(a) = 1$ for all $a \in A$, replace each arc $(r, x_i^\alpha, 2, 3)$ by the arcs $(r, y_i^\alpha, 2, 3)$ and $(y_i^\alpha, x_i^\alpha, 3, 4)$ and each arc $(r, x_i^\alpha, 1, 2)$ by the arcs $(r, z_i^\alpha, 1, 2)$ and $(y_i^\alpha, z_i^\alpha, 2, 3)$. All the other arcs have both their starting and arrival times increased by one.

Notice that in the cases where the elapsed time are all equal to 1, the above transformations make the temporal graph have lifetime $\tau = 5$. We leave open the question whether the same complexity holds for simple temporal graphs with $\tau = 4$.

The hardness results presented in this paper are clearly an issue when it comes to implement solutions for real life networks. In this direction, it would be of interest to study whether some of these networks, like public transport networks, present particular structures in their topology in order to restrict the analysis to a specific class of temporal graphs, for which the results might become tractable.

References

- [1] E. C. Akrida, J. Czyzowicz, L. Gasieniec, L. Kuszner, and P. G. Spirakis. Temporal flows in temporal networks. *J. of Computer and System Sciences*, 103:46–60, 2019.
- [2] E. C. Akrida, L. Gasieniec, G. B. Mertzios, and P. G. Spirakis. The complexity of optimal design of temporally connected graphs. *Theory Comput Syst*, 61:907–944, 2017.
- [3] M. Amoroso, D. Anello, V. Auletta, and D. Ferraioli. Contrasting the spread of misinformation in online social networks. *J. of Artificial Intelligence Research*, 69:847–879, 2020.
- [4] M. Bentert, A.S. Himmel, A. Nichterlein, and R. Niedermeier. Efficient computation of optimal temporal walks under waiting-time constraints. *App. Net. Sci.*, 5, 2020.
- [5] F. Brunelli and L. Viennot. Minimum-cost temporal walks under waiting-time constraints in linear time. *arXiv:2211.12136*, 2023.
- [6] D. Bubboloni, C. Catalano, A. Marino, and A. Silva. On computing optimal temporal branchings. In *Fundamentals of Computation Theory*, pages 103–117, 2023.
- [7] M. Calamai, P. Crescenzi, and A. Marino. On computing the diameter of (weighted) link streams. *ACM J. Exp. Algorithmics*, 27:4.3:1–4.3:28, 2022.
- [8] V. A. Campos, R. Lopes, A. Marino, and A. Silva. Edge-disjoint branchings in temporal graphs. *Electronic J. of Combinatorics*, 28, 2020.
- [9] A. Casteigts. Finding structure in dynamic networks. *arXiv:1807.07801*, 2018.
- [10] A. Casteigts and T. Corsini. In search of the lost tree: Hardness and relaxation of spanning trees in temporal graphs. *arXiv:2312.06260*, 2023.
- [11] S.A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the third annual ACM symposium on Theory of computing*, pages 151–158, 1971.

- [12] T.H. Cormen, C.E. Leiserson, and R.L. Rivest. *Introduction to Algorithms*. McGraw–Hill. MIT Press, third ed. edition, 2001.
- [13] A. Deligkas and I. Potapov. Optimizing reachability sets in temporal graphs by delaying. *Information and Computation*, 285:104890, 2022.
- [14] J. Dibbelt, T. Pajor, B. Strasser, and D. Wagner. Connection scan algorithm. *ACM J. Exp. Algorithmics*, 23, 2018.
- [15] H. N. Gabow, Z. Galil, T. Spencer, and R.E. Tarjan. Efficient algorithms for finding minimum spanning trees in undirected and directed graphs. *Combinatorica*, 6:109–122, 1986.
- [16] J. Gallier. *Discrete Mathematics*. Universitext. Springer, 2011.
- [17] V. Gunturi, S. Shekhar, and A. Bhattacharya. Minimum spanning tree on spatio-temporal networks. In *Database and Expert Systems Applications*, pages 149–158, Berlin, Heidelberg, 2010. Springer Berlin Heidelberg.
- [18] P. Holme and J. Saramäki. Temporal networks. *Phys. Reports*, 519(3):97–125, 2012.
- [19] S. Huang, A. W.-C. Fu, and R. Liu. Minimum spanning trees in temporal graphs. In *ACM SIGMOD Intern. Conference on Management of Data*, pages 419–430, 2015.
- [20] N. Kamiyama and Y. Kawase. On packing arborescences in temporal networks. *Information Processing Letters*, 115(2):321–325, 2015.
- [21] D. Kempe, J. Kleinberg, and A. Kumar. Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences*, 64(4):820–842, 2002.
- [22] Y. Kuwata, L. Blackmore, M.T. Wolf, N. Fathpour, C.E. Newman, and A. Elfes. Decomposition algorithm for global reachability analysis on a time-varying graph with an application to planetary exploration. *Intelligent Robot and Syst.*, 2009. 3955-3960.
- [23] M. Latapy, T. Viard, and C. Magnien. Stream Graphs and Link Streams for the Modeling of Interactions over Time. *Social Network Analysis*, 8(1):61:1–61:29, 2018.
- [24] G. Leitao, Z. Guangshe, L. Guoqi, L. Yuming, H. Jiangshuai, and D. Lei. Containment control of directed networks with time-varying nonlinear multi-agents using minimum number of leaders. *Physica A: Statistical Mechanics and its Applications*, 526:120859, 2019.
- [25] L.A. Levin. Universal sequential search problems. *Problemy peredachi informatsii*, 9(3):115–116, 1973.
- [26] N. Li and J.C. Hou. Topology control in heterogeneous wireless networks: problems and solutions. In *IEEE INFOCOM 2004*, volume 1, page 243, 2004.
- [27] A. Marino and A. Silva. Eulerian walks in temporal graphs. *Algorithmica*, 85:805–830, 2023.
- [28] G.B. Mertzios, O. Michail, and P.G. Spirakis. Temporal network optimization subject to connectivity constraints. *Algorithmica*, 81:1416–1449, 2019.
- [29] O. Michail. An introduction to temporal graphs: An algorithmic perspective. *Internet Mathematics*, 12(4), 7 2016.
- [30] V. Nicosia, J. Tang, M. Musolesi, G. Russo, C. Mascolo, and V. Latora. Components in time-varying graphs. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2), 2012.
- [31] S. Ranshous, S. Shen, D. Koutra, S. Harenberg, C. Faloutsos, and N.F. Samatova. Anomaly detection in dynamic networks: a survey. *WIREs Computational Statistics*, 7(3):223–247, 2015.

- [32] J. K. Tang, C. Mascolo, M. Musolesi, and V. Latora. Exploiting temporal complex network metrics in mobile malware containment. *2011 IEEE International Symposium on a World of Wireless, Mobile and Multimedia Networks*, pages 1–9, 2010.
- [33] H. Wu, J. Cheng, S. Huang, Y. Ke, Y. Lu, and Y. Xu. Path problems in temporal graphs. *Proc. VLDB Endow.*, 7(9):721–732, 2014.
- [34] H. Wu, J. Cheng, Y. Ke, S. Huang, Y. Huang, and H. Wu. Efficient algorithms for temporal path computation. *Knowledge and Data Eng.*, 28(11):2927–2942, 2016.
- [35] P. Yue, Q. Cai, W. Yan, and W.-X. Zhou. Information flow networks of chinese stock market sectors. *IEEE Access*, 8:13066–13077, 2020.
- [36] P. Zschoche, T. Fluschnik, H. Molter, and R. Niedermeier. The complexity of finding small separators in temporal graphs. *J. of Comp. and Syst. Sci.*, 107:72–92, 2020.