

LEFT-ORDERABLE COMPUTABLE GROUPS

MATTHEW HARRISON-TRAINOR

ABSTRACT. Downey and Kurtz asked whether every orderable computable group is classically isomorphic to a group with a computable ordering. By an order on a group, one might mean either a left-order or a bi-order. We answer their question for left-orderable groups by showing that there is a computable left-orderable group which is not classically isomorphic to a computable group with a computable left-order. The case of bi-orderable groups is left open.

1. INTRODUCTION

A left-ordered group is a group \mathcal{G} together with a linear order \leq such that if $a \leq b$, then $ca \leq cb$. \mathcal{G} is right-ordered if instead whenever $a \leq b$, $ac \leq bc$, and bi-ordered if \leq is both a left-order and a right-order. A group which admits a left-ordering is called left-orderable, and similarly for right- and bi-orderings. A group is left-orderable if and only if it is right-orderable. Some examples of bi-orderable groups include torsion-free abelian groups and free groups [Shi47, Vin49, Ber90]. The group $\langle x, y : x^{-1}yx = y^{-1} \rangle$ is left-orderable but not bi-orderable. For a reference on orderable groups, see [KM96].

In this paper, we will consider left-orderable computable groups. A computable group is a group with domain ω whose group operation is given by a computable function $\omega \times \omega \rightarrow \omega$. Downey and Kurtz [DK86] showed that a computable group, even a computable abelian group, which is orderable need not have a computable order. If a computable group does admit a computable order, we say that it is computably orderable. Of course, by the low basis theorem, every orderable computable group has a low ordering.

For an abelian group, any left-ordering (or right-ordering) is a bi-ordering. An abelian group is orderable if and only if it is torsion-free. Given a computable torsion-free abelian group \mathcal{G} , Dobritsa [Dob83] showed that there is another computable group \mathcal{H} , which is classically isomorphic to \mathcal{G} , which has a computable \mathbb{Z} -basis. Note that \mathcal{H} need not be computably isomorphic to \mathcal{G} . Solomon [Sol02] noted that a \mathbb{Z} -basis for a torsion-free abelian group computes an ordering of that group. Hence every orderable computable abelian group is classically isomorphic to a computably orderable group.

Downey and Kurtz asked whether this is the case even for non-abelian groups:

Question 1 (Downey and Kurtz [DR00]). Is every orderable computable group classically isomorphic to a computably orderable group?

If one takes “orderable” to mean “left-orderable” then we give a negative answer to this question. (We leave open the question for bi-orderable groups.)

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Theorem 2. *There is a computable left-orderable group which has no presentation with a computable left-ordering.*

Our strategy is to build a group

$$\mathcal{G} = \mathcal{N} \rtimes \mathcal{H}/\mathcal{R}$$

and code information into the finite orbits of certain elements of \mathcal{N} under inner automorphisms given by conjugating by elements of \mathcal{H}/\mathcal{R} . This strategy cannot work to build a bi-orderable group, as in a bi-orderable group there is no generalized torsion—i.e., no product of conjugates of a single element can be equal to the identity—and hence no inner automorphism has a non-trivial finite orbit. We leave open the case of bi-orderable groups.

2. NOTATION

We will use calligraphic letter such as \mathcal{G} , \mathcal{N} , and \mathcal{H} to denote groups. For free groups, we will use upper case latin letters such as A , B , C , U , V , and W to denote words, while using lower case letters such as a , b , and c to denote letter variables. We use ε for the empty word, 0 for the identity element of abelian groups, and 1 for the identity element of non-abelian groups (except for free groups, where we use ε).

3. THE CONSTRUCTION

Fix ψ a partial computable function which we will specify later (see Definition 8). Let p_i , q_i , and r_i be a partition of the odd primes into three lists.¹ Let \mathcal{H} be the free abelian group on α_i , β_i , and γ_i for $i \in \omega$. We write \mathcal{H} additively. Let \mathcal{R} be the set of relations

$$\mathcal{R} = \{\mathcal{R}_{i,t} : \psi_{\text{at } t}(i) \downarrow\}$$

where

$$\mathcal{R}_{i,t} = \begin{cases} p_i^t \alpha_i = q_i^t \beta_i & \text{if } \psi_{\text{at } t}(i) = 0 \\ p_i^t \alpha_i = -q_i^t \beta_i & \text{if } \psi_{\text{at } t}(i) = 1 \end{cases}.$$

By $\psi_{\text{at } t}(i) = 0$, we mean that the computation $\psi(i)$ has converged exactly at stage t (but not before) and equals zero.

The idea is that these relations force, for any ordering \leq on \mathcal{H}/\mathcal{R} , that if $\psi(i) = 0$ then $\alpha_i > 0 \iff \beta_i > 0$ (and if $\psi(i) = 1$ then $\alpha_i > 0 \iff \beta_i < 0$). The strategy is, in a very general sense, to use ψ to diagonalize against computable orderings of \mathcal{H}/\mathcal{R} . The semidirect product will add enough structure to allow us to find α_i and β_i within a computable copy of \mathcal{G} . (One cannot find α_i and β_i within a copy of \mathcal{H}/\mathcal{R} , since \mathcal{H}/\mathcal{R} is a torsion-free abelian group.) Note that

$$\mathcal{H}/\mathcal{R} = \left(\bigoplus_i \langle \alpha_i, \beta_i \rangle / \mathcal{R}_i \right) \oplus \left(\bigoplus \langle \gamma_i \rangle \right)$$

where $\mathcal{R}_i = \mathcal{R}_{i,t}$ if $\psi_{\text{at } t}(i) \downarrow$ for some t , or no relation otherwise. Define

$$\begin{aligned} \mathcal{V}_i &= \mathcal{R} \cup \{p_i \alpha_i = 0\} & \mathcal{W}_i &= \mathcal{R} \cup \{q_i \beta_i = 0\} & \mathcal{X}_i &= \mathcal{R} \cup \{r_i \gamma_i = 0\} \\ \mathcal{Y}_i &= \mathcal{R} \cup \{\alpha_i = \gamma_i\} & \mathcal{Z}_i &= \mathcal{R} \cup \{\beta_i = \gamma_i\}. \end{aligned}$$

¹We use the fact that 2 does not appear in these lists in Lemma 22.

Let \mathcal{N} be the free (non-abelian) group on the letters

$$\begin{aligned} & \{u_i : i \in \omega\} \cup \{v_{i,g} : g \in \mathcal{H}/\mathcal{V}_i, i \in \omega\} \cup \{w_{i,g} : g \in \mathcal{H}/\mathcal{W}_i, i \in \omega\} \\ & \cup \{x_{i,g} : g \in \mathcal{H}/\mathcal{X}_i, i \in \omega\} \cup \{y_{i,g} : g \in \mathcal{H}/\mathcal{Y}_i, i \in \omega\} \cup \{z_{i,g} : g \in \mathcal{H}/\mathcal{Z}_i, i \in \omega\}. \end{aligned}$$

Let $\mathcal{G} = \mathcal{N} \rtimes (\mathcal{H}/\mathcal{R})$, with $g \in \mathcal{H}/\mathcal{R}$ acting on \mathcal{N} via the automorphism φ_g as follows:

$$\begin{aligned} \varphi_g(u_i) &= u_i & \varphi_g(v_{i,h}) &= v_{i,\bar{g}+h} & \varphi_g(w_{i,h}) &= w_{i,\bar{g}+h} \\ \varphi_g(x_{i,h}) &= x_{i,\bar{g}+h} & \varphi_g(y_{i,h}) &= y_{i,\bar{g}+h} & \varphi_g(z_{i,h}) &= z_{i,\bar{g}+h}. \end{aligned}$$

Here, \bar{g} is the image of g under the quotient map $\mathcal{H}/\mathcal{R} \rightarrow \mathcal{H}/\mathcal{V}_i$ (or $\mathcal{H}/\mathcal{W}_i$, $\mathcal{H}/\mathcal{X}_i$, etc.). Recall that the semidirect product $\mathcal{G} = \mathcal{N} \rtimes (\mathcal{H}/\mathcal{R})$ is the group with underlying set $\mathcal{N} \times (\mathcal{H}/\mathcal{R})$ with group operation

$$(n, g)(m, h) = (n\varphi_g(m), g + h).$$

Note that φ_g permutes the letters of \mathcal{N} , and so given a word $A \in \mathcal{N}$, $\varphi_g(A)$ is a word of the same length as A . We write \mathcal{G} multiplicatively.

Lemma 3. *\mathcal{H}/\mathcal{R} has a computable presentation.*

Proof. It suffices to show that we can decide whether or not a relation of the form

$$\sum_{i=1}^k \ell_i \alpha_i + \sum_{i=1}^k m_i \beta_i + \sum_{i=1}^k n_i \gamma_i = 0$$

holds. This sum is equal to zero if and only if each $n_i = 0$ and for each i we have $\ell_i \alpha_i + m_i \beta_i = 0$. So it suffices to decide, for a given ℓ and m in \mathbb{Z} , whether $\ell \alpha_i = m \beta_i$.

Looking at \mathcal{R} , $\ell \alpha_i = m \beta_i$ if and only if either

- (1) for some t , $\psi_{\text{at } t}(i) = 0$ and there is $s \in \mathbb{Z}$ such that $\ell = sp_i^t$ and $m = sq_i^t$ or
- (2) for some t , $\psi_{\text{at } t}(i) = 1$ and there is $s \in \mathbb{Z}$ such that $\ell = sp_i^t$ and $m = -sq_i^t$.

If $t > |\ell|$ or $t > |m|$ then neither of these can hold. So we just need to check, for each $t \leq |\ell|, |m|$, whether $\psi_{\text{at } t}(i)$ converges. \square

Lemma 4. *\mathcal{G} has a computable presentation.*

Proof. We just need to check that $\mathcal{H}/\mathcal{V}_i$, $\mathcal{H}/\mathcal{W}_i$, and so on have computable presentations. We will see that the embeddings of the computable presentation (from the previous lemma) of \mathcal{H}/\mathcal{R} into these presentations are computable. Then the action φ of \mathcal{H}/\mathcal{R} on \mathcal{N} is computable. We can construct a computable presentation of \mathcal{G} as the semidirect product $\mathcal{N} \rtimes (\mathcal{H}/\mathcal{R})$ under this computable action.

We need to decide whether in $\mathcal{H}/\mathcal{V}_i$ we have a relation

$$\sum_{j=1}^k \ell_j \alpha_j + \sum_{j=1}^k m_j \beta_j + \sum_{j=1}^k n_j \gamma_j = 0.$$

It suffices to decide, for a given j , whether

$$\ell \alpha_j + m \beta_j + n \gamma_j = 0.$$

If $j \neq i$, this is just as in the previous lemma. Otherwise, this holds if and only if p_i divides ℓ , q^t divides m for some t with $\psi_{\text{at } t}(i) \downarrow$, and $n = 0$. As before, we can check this computably.

The other cases—for $\mathcal{H}/\mathcal{W}_i$, $\mathcal{H}/\mathcal{X}_i$, and so on—are similar. \square

Lemma 5. \mathcal{H}/\mathcal{R} is a torsion-free abelian group.

Proof. \mathcal{H}/\mathcal{R} is abelian as \mathcal{H} was abelian. Recall that

$$\mathcal{H}/\mathcal{R} = \left(\bigoplus_i \langle \alpha_i, \beta_i \rangle / \mathcal{R}_i \right) \oplus \left(\bigoplus_i \langle \gamma_i \rangle \right)$$

where $\mathcal{R}_i = \mathcal{R}_{i,t}$ if $\psi_{\text{at } t}(i) \downarrow$ for some t , or no relation otherwise. So it suffices to show that $\langle \alpha_i, \beta_i \rangle / \mathcal{R}_i$ is torsion-free. If \mathcal{R}_i is no relation, then this is obvious. So now suppose that $\psi_{\text{at } t}(i) = 0$ and that

$$k(m\alpha_i + n\beta_i) = \ell(p_i^t \alpha_i - q_i^t \beta_i)$$

in $\langle \alpha_i, \beta_i \rangle$. Since \mathcal{H} is torsion-free, we may assume that $\gcd(k, \ell) = 1$. Then $km = \ell p_i^t$ and $kn = -\ell q_i^t$. So we must have $k = \pm 1$, in which case $m\alpha_i + n\beta_i$ is already zero in $\langle \alpha_i, \beta_i \rangle / \mathcal{R}_i$. Thus $\langle \alpha_i, \beta_i \rangle / \mathcal{R}_i$ is torsion-free. The case where $\psi_{\text{at } t}(i) = 1$ is similar. \square

Lemma 6. \mathcal{G} is left-orderable.

Proof. Since \mathcal{H}/\mathcal{R} is a torsion-free abelian group, it is bi-orderable. \mathcal{N} is bi-orderable as it is a free group. Then by the following claim, \mathcal{G} is left-orderable (see Theorem 1.6.2 of [KM96]).

Claim 7. Let $\mathcal{A} \rtimes \mathcal{B}$ be a semi-direct product of left-orderable groups. Then $\mathcal{A} \rtimes \mathcal{B}$ is left-orderable.

Proof. Let φ be the action of \mathcal{B} on \mathcal{A} . Let $\leq_{\mathcal{A}}$ and $\leq_{\mathcal{B}}$ be left-orderings on \mathcal{A} and \mathcal{B} respectively. Define \leq on $\mathcal{A} \rtimes \mathcal{B}$ as follows: $(a, b) \leq (a', b')$ if $b <_{\mathcal{B}} b'$ or $b = b'$ and $\varphi_{b^{-1}}(a) \leq_{\mathcal{A}} \varphi_{b^{-1}}(a')$. This is clearly reflexive and symmetric. We must show that it is transitive and a left-ordering.

Suppose that $(a, b) \leq (a', b') \leq (a'', b'')$. Then $b \leq_{\mathcal{B}} b' \leq_{\mathcal{B}} b''$. If $b <_{\mathcal{B}} b''$, then $(a, b) \leq (a'', b'')$, so suppose that $b = b' = b''$. Then

$$\varphi_{b^{-1}}(a) \leq_{\mathcal{A}} \varphi_{b^{-1}}(a') = \varphi_{b'^{-1}}(a') \leq_{\mathcal{A}} \varphi_{b''^{-1}}(a'') = \varphi_{b^{-1}}(a'').$$

So $\varphi_{b^{-1}}(a) \leq_{\mathcal{A}} \varphi_{b^{-1}}(a'')$ and so $(a, b) \leq (a'', b'')$. Thus \leq is transitive.

Given $(a, b) \leq (a', b')$ we must show that $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. We have that

$$(a'', b'')(a, b) = (a'' \varphi_{b''}(a), b''b) \text{ and } (a'', b'')(a', b') = (a'' \varphi_{b''}(a'), b''b').$$

If $b <_{\mathcal{B}} b'$, then $b''b <_{\mathcal{B}} b''b'$, and so $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. Otherwise, if $b = b'$ and $\varphi_{b^{-1}}(a) \leq_{\mathcal{A}} \varphi_{b^{-1}}(a')$, then $b''b = b''b'$ and

$$\begin{aligned} \varphi_{(b''b)^{-1}}(a'' \varphi_{b''}(a)) &= \varphi_{(b''b)^{-1}}(a'') \varphi_{b^{-1}}(a) \\ &\leq_{\mathcal{A}} \varphi_{(b''b)^{-1}}(a'') \varphi_{b^{-1}}(a') \\ &= \varphi_{(b''b)^{-1}}(a'' \varphi_{b''}(a')). \end{aligned}$$

So $(a'', b'')(a, b) \leq (a'', b'')(a', b')$. \square

Note that if \leq is any left-ordering on \mathcal{G} , if $\psi_{\text{at } t}(i) = 0$ then $(\varepsilon, \alpha_i) > 1$ if and only if $(\varepsilon, \beta_i) > 1$. On the other hand, if $\psi_{\text{at } t}(i) = 1$ then $(\varepsilon, \alpha_i) > 1$ if and only if $(\varepsilon, \beta_i) < 1$. Later, in Definition 18, we will define existential formulas $\text{Same}(i)$ and $\text{Different}(i)$ (with no parameters) in the language of ordered groups. We would like to have that for any left-ordering \leq on \mathcal{G} , $(\mathcal{G}, \leq) \models \text{Same}(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$, and $(\mathcal{G}, \leq) \models \text{Different}(i)$ if and only if

$(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$. We will not quite get this for every ordering \leq , but this will be true for those against which we want to diagonalize (see Lemma 9).

Definition 8. Fix a list $(\mathcal{F}_i, \leq_i)_{i \in \omega}$ of the (partial) computable structures in the language of ordered groups. Let ψ be a partial computable function with $\psi(i) = 0$ if $(\mathcal{F}_i, \leq_i) \models \text{Different}(i)$ and $\psi(i) = 1$ if $(\mathcal{F}_i, \leq_i) \models \text{Same}(i)$. It is possible, a priori, that we have both $(\mathcal{F}_i, \leq_i) \models \text{Same}(i)$ and $(\mathcal{F}_i, \leq_i) \models \text{Different}(i)$; in this case, let $\psi(i)$ be defined according to whichever existential formula we find to be true first.

In fact, we will discover from the following lemma that we cannot have both $(\mathcal{F}_i, \leq_i) \models \text{Same}(i)$ and $(\mathcal{F}_i, \leq_i) \models \text{Different}(i)$.

Lemma 9. Fix i . Suppose that \mathcal{F}_i is isomorphic to \mathcal{G} and \leq_i is a computable left-ordering of \mathcal{F}_i . Let \leq be an ordering on \mathcal{G} such that $(\mathcal{G}, \leq) \cong (\mathcal{F}_i, \leq_i)$. Then:

- (1) $(\mathcal{G}, \leq) \models \text{Same}(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$.
- (2) $(\mathcal{G}, \leq) \models \text{Different}(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$.

This lemma will be proved later. We will now show how to use Lemma 9 to complete proof.

Lemma 10. \mathcal{G} has no computable presentation with a computable ordering.

Proof. Let i be an index for (\mathcal{F}_i, \leq_i) a computable presentation of \mathcal{G} with a computable left-ordering. Let \leq be an ordering on \mathcal{G} such that $(\mathcal{G}, \leq) \cong (\mathcal{F}_i, \leq_i)$. Now by Lemma 9 either $(\mathcal{G}, \leq) \models \text{Same}(i)$ or $(\mathcal{G}, \leq) \models \text{Different}(i)$ (but not both). Suppose first that $(\mathcal{G}, \leq) \models \text{Same}(i)$. So $(\mathcal{F}_i, \leq_i) \models \text{Same}(i)$. By definition, $\psi(i) = 1$, say $\psi_{\text{at } t}(i) = 1$. Then, in \mathcal{H}/\mathcal{R} , $p_i^t \alpha_i = -q_i^t \beta_i$. So $(\varepsilon, \alpha_i) > 1$ if and only if $(\varepsilon, \beta_i) < 1$, contradicting Lemma 9 and the assumption that $(\mathcal{G}, \leq) \models \text{Same}(i)$. The case of $(\mathcal{G}, \leq) \models \text{Different}(i)$ is similar. Thus \mathcal{G} has no computable copy with a computable left-ordering. \square

All that remains to prove Theorem 2 is to define $\text{Same}(i)$ and $\text{Different}(i)$ and to prove Lemma 9.

4. $\text{Same}(i)$, $\text{Different}(i)$, AND THE PROOF OF LEMMA 9

To define $\text{Same}(i)$, we would like to come up with an existential formula which says that $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$. A first attempt might be to try to find an existential formula defining (ε, α_i) and an existential formula defining (ε, β_i) . This cannot be done, but it will be helpful to think about how we might try to do this.

We will consider the problem of recognizing α_i and β_i inside of \mathcal{H}/\mathcal{R} by their actions on \mathcal{N} . Note that α_i has the property that $\varphi_{\alpha_i}(v_{i,0}) = v_{i,\alpha_i} \neq 0$, but $\varphi_{p_i \alpha_i}(v_{i,0}) = v_{i,0}$. So α_i acts with order p_i on some element of \mathcal{N} . In fact, it is not hard to see that the only elements which act with order p_i on an element of \mathcal{N} are the multiples $n\alpha_i$ of α_i where $p_i \nmid n$. (Note that if α_i acts with order p_i on a word in \mathcal{N} , then it either fixes or acts with order p_i on each letter in that word, and it acts with order p_i on at least one letter.)

One difficulty we have is that \mathcal{H}/\mathcal{R} and \mathcal{N} are not existentially definable inside of \mathcal{G} . The problem is that if some element of \mathcal{G} satisfies a certain existential formula, then every conjugate of \mathcal{G} does as well. So it is only possible to define subsets of \mathcal{G} which are closed under conjugation. Given $S \subseteq \mathcal{G}$, let $S^{\mathcal{G}}$ be the set of all conjugates of S by elements of \mathcal{G} .

In this section, we will take for granted the following lemma about existential definability in \mathcal{G} . It will be proved in the following section. The lemma says that we can find \mathcal{H}/\mathcal{R} inside of \mathcal{G} , up to conjugation, by an existential formula.

Lemma 11. *$(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ is \exists -definable within \mathcal{G} without parameters.*

The different conjugates of \mathcal{H}/\mathcal{R} cannot be distinguished from each other. Instead, we will try to always work inside a single conjugate of \mathcal{H}/\mathcal{R} . The following lemma tells us when we can do this.

Lemma 12. *Suppose that $r, s \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ and $rs \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. Then there is $A \in \mathcal{N}$ and $g, h \in \mathcal{H}/\mathcal{R}$ such that*

$$r = (A, 0)(\varepsilon, g)(A^{-1}, 0)$$

and

$$s = (A, 0)(\varepsilon, h)(A^{-1}, 0).$$

Thus r and s commute.

The following remarks will be helpful not only here, but throughout the rest of the paper. They can all be checked by an easy computation.

Remark 13. If $r \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then for some $A \in \mathcal{N}$ and $f \in \mathcal{H}/\mathcal{R}$ we can write r in the form

$$r = (A, 0)(\varepsilon, f)(A^{-1}, 0).$$

Remark 14. Let $r = (A, f)$ be an element of $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. If $K \subseteq \mathcal{H}/\mathcal{R}$, then $r \in K^{\mathcal{G}}$ if and only if $f \in K$.

Remark 15. If $\varphi_g(B) = B$, then

$$(AB, 0)(\varepsilon, g)(AB, 0)^{-1} = (A, 0)(\varepsilon, g)(A, 0)^{-1}.$$

Proof of Lemma 12. Using Remark 13, let

$$\begin{aligned} r &= (A, 0)(\varepsilon, g)(A^{-1}, 0) & s &= (B, 0)(\varepsilon, h)(B^{-1}, 0) \\ rs &= (C, 0)(\varepsilon, g+h)(C^{-1}, 0). \end{aligned}$$

By conjugating r and s by some further element of \mathcal{G} (and noting that the conclusion of the lemma is invariant under conjugation), we may assume that $A^{-1}B$ is a reduced word, that is, that A and B have no common non-trivial initial segment. Using Remark 15, we may assume that $A\varphi_g(A^{-1})$, $B\varphi_h(B^{-1})$, and $C\varphi_{g+h}(C^{-1})$ are reduced words. Indeed, if, for example, $A\varphi_g(A^{-1})$ was not a reduced word, then we could write $A = A'B$ where B is a word which is fixed by φ_g , and such that $A'\varphi_g(A'^{-1})$ is a reduced word. Then, by Remark 15,

$$(A, 0)(\varepsilon, g)(A, 0)^{-1} = (A'B, 0)(\varepsilon, g)(A'B, 0)^{-1} = (A', 0)(\varepsilon, g)(A', 0)^{-1}.$$

So we may replace A by A' .

We have

$$(A, 0)(\varepsilon, g)(A^{-1}, 0)(B, 0)(\varepsilon, h)(B^{-1}, 0) = (C, 0)(\varepsilon, g+h)(C^{-1}, 0).$$

Multiplying out the first coordinates, we get

$$A\varphi_g(A^{-1})\varphi_g(B)\varphi_{g+h}(B^{-1}) = C\varphi_{g+h}(C^{-1}).$$

By the assumptions we made above, both sides are reduced words. A is an initial segment of the left hand side, so it must be an initial segment of the right hand

side, and hence an initial segment of C . On the other hand, taking inverses of both sides, we get

$$\varphi_{g+h}(B)\varphi_g(B^{-1})\varphi_g(A)A^{-1} = \varphi_{g+h}(C)C^{-1}.$$

Once again both sides are reduced words, and $\varphi_{g+h}(B)$ is an initial segment of the left hand side, and hence of $\varphi_{g+h}(C)$. But then B is an initial segment of C . So it must be that A is an initial segment of B or vice versa. This contradicts one of our initial assumptions unless A or B (or both) is the trivial word. Suppose it was A (the case of B is similar). Then

$$\varphi_g(B)\varphi_{g+h}(B^{-1}) = C\varphi_{g+h}(C^{-1})$$

and both sides are reduced words. Then we get that $C = B$ and $C = \varphi_g(B)$. So

$$r = (\varepsilon, g) = (B, 0)(\varepsilon, g)(B, 0)^{-1}$$

by Remark 15. □

Above, we noted that the set $\{n\alpha_i : p_i \nmid n\}$ is the set of elements of \mathcal{H}/\mathcal{R} which act with order p_i on an element of \mathcal{N} . Our next goal is to show that if we close under conjugation, then this set (and a few other similar sets) are definable. The key is the following remark which follows easily from Lemma 12.

Remark 16. Fix $r, s_1, s_2 \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. Suppose that $rs_1 \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ and $rs_2 \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ but s_1 and s_2 do not commute. By Lemma 12 we can write

$$\begin{aligned} r &= (A, 0)(\varepsilon, f)(A^{-1}, 0) = (B, 0)(\varepsilon, f)(B^{-1}, 0) \\ s_1 &= (A, 0)(\varepsilon, g)(A^{-1}, 0) \\ s_2 &= (B, 0)(\varepsilon, h)(B^{-1}, 0). \end{aligned}$$

Then there is some element of \mathcal{N} which is fixed by φ_f but which is not fixed by φ_g .

Indeed, since $(A, 0)(\varepsilon, f)(A^{-1}, 0) = (B, 0)(\varepsilon, f)(B^{-1}, 0)$, we see that

$$B^{-1}A = \varphi_f(B^{-1}A).$$

Suppose for the sake of contradiction that φ_g also fixes $B^{-1}A$. Then

$$s_1 = (A, 0)(A^{-1}B, 0)(\varepsilon, g)(B^{-1}A, 0)(A^{-1}, 0) = (B, 0)(\varepsilon, g)(B^{-1}, 0).$$

So s_1 and s_2 would commute. This is a contradiction. So there is some element of \mathcal{N} which is fixed by φ_f but which is not fixed by φ_g .

Lemma 17. *There are \exists -formulas which express each of the following statements about an element a in \mathcal{G} :*

- (1) $a \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$.
- (2) $a \in \{n\beta_i : q_i \nmid n\}^{\mathcal{G}}$.
- (3) $a \in \{n\gamma_i : r_i \nmid n\}^{\mathcal{G}}$.
- (4) $a \in \{n(\alpha_i - \gamma_i) : p_i, r_i \nmid n\}^{\mathcal{G}}$.
- (5) $a \in \{n(\beta_i - \gamma_i) : q_i, r_i \nmid n\}^{\mathcal{G}}$.

Proof. For (1), we claim that $a \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$ if and only if $a \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ and there is $b \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ such that $a^{p_i}b \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ but a and b do not commute. This is expressed by an \exists -formula by Lemma 11.

Suppose that a satisfies this \exists -formula, as witnessed by b . Let $a = (A, f)$ and $b = (B, g)$. Then by Remark 16 (taking $r = a^{p_i}$, $s_1 = a$, and $s_2 = b$), there is an element of \mathcal{N} which is fixed by $\varphi_{p_i f}$ but not by φ_f . Thus we see that $p_i \bar{f} = 0$ but $\bar{f} \neq 0$ in $\mathcal{H}/\mathcal{V}_i$, and $f = n\alpha_i$ for some n with $p_i \nmid n$. (It must be in $\mathcal{H}/\mathcal{V}_i$, because

this cannot happen in any of $\mathcal{H}/\mathcal{V}_j$ for $j \neq i$, or $\mathcal{H}/\mathcal{W}_j$, $\mathcal{H}/\mathcal{X}_j$, $\mathcal{H}/\mathcal{Y}_j$, or $\mathcal{H}/\mathcal{Z}_j$.) Thus by Remark 14, $a \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$.

On the other hand, suppose that $a \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$. Write

$$a = (A, 0)(\varepsilon, n\alpha_i)(A^{-1}, 0).$$

with p_i not dividing n . Then let $b = (Av_{i,0}, 0)(\varepsilon, n\alpha_i)((Av_{i,0})^{-1}, 0)$. By Remark 15, since $\varphi_{np_i\alpha_i}(v_{i,0}) = v_{i,0}$, we have

$$a^{p_i} = (A, 0)(\varepsilon, np_i\alpha_i)(A^{-1}, 0) = (Av_{i,0}, 0)(\varepsilon, np_i\alpha_i)((Av_{i,0})^{-1}, 0).$$

So $a^{p_i}b \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. On the other hand,

$$ab = (A\varphi_{n\alpha_i}(v_{i,0})\varphi_{2n\alpha_i}(v_{i,0})^{-1}\varphi_{2n\alpha_i}(A^{-1}), 2n\alpha_i)$$

and

$$ba = (Av_{i,0}\varphi_{n\alpha_i}(v_{i,0})^{-1}\varphi_{2n\alpha_i}(A^{-1}), 2n\alpha_i).$$

So a does not commute with b since $\varphi_{n\alpha_i}(v_{i,0}) = v_{i,n\alpha_i} \neq v_{i,0}$. The proofs of (2) and (3) are similar.

For (4), we claim that $a \in \{n(\alpha_i - \gamma_i) : p_i, r_i \nmid n\}^{\mathcal{G}}$ if and only if there are $b_1 \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$, $b_2 \in \{n\gamma_i : r_i \nmid n\}^{\mathcal{G}}$, and $c \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ such that $a = b_1b_2^{-1}$, $ac, ab_1 \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, and c does not commute with b_1 .

Suppose that there are such b_1 , b_2 , and c . We can write $b_1 = (B_1, m\alpha_i)$ with $p_i \nmid m$ and $b_2 = (B_2, n\gamma_i)$ with $r_i \nmid \gamma_i$. Thus we can write $a = b_1b_2^{-1} = (A, m\alpha_i - n\gamma_i)$. By Remark 16 (with $r = a$, $s_1 = b_1$, and $s_2 = c$), $\varphi_{m\alpha_i - n\gamma_i}$ fixes some element of \mathcal{N} which is not fixed by $\varphi_{m\alpha_i}$. Thus, in one of $\mathcal{H}/\mathcal{V}_j$, $\mathcal{H}/\mathcal{W}_j$, $\mathcal{H}/\mathcal{X}_j$, $\mathcal{H}/\mathcal{Y}_j$, or $\mathcal{H}/\mathcal{Z}_j$ for some j we have $m\bar{\alpha}_i - n\bar{\gamma}_i = 0$ but $m\bar{\alpha}_i \neq 0$. Since $p_i \nmid m$, it must be in $\mathcal{H}/\mathcal{Y}_i$. So $n = m$. Note that p_i and r_i do not divide n .

On the other hand, suppose that $a \in \{n(\alpha_i - \gamma_i) : p_i, r_i \nmid n\}^{\mathcal{G}}$. Then write

$$a = (A, 0)(\varepsilon, n\alpha_i - n\gamma_i)(A^{-1}, 0).$$

with p_i and r_i not dividing n . Let

$$b_1 = (A, 0)(\varepsilon, n\alpha_i)(A^{-1}, 0) \text{ and } b_2 = (A, 0)(\varepsilon, n\gamma_i)(A^{-1}, 0)$$

and let

$$c = (Ay_{i,0}, 0)(\varepsilon, n\alpha_i)((Ay_{i,0})^{-1}, 0).$$

Then $a = b_1b_2^{-1}$. Clearly $ab_1 \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. Also, since $\varphi_{n\alpha_i - n\gamma_i}(y_{i,0}) = y_{i,0}$,

$$ac = ca = (Ay_{i,0}, 0)(\varepsilon, 2n\alpha_i - n\gamma_i)((Ay_{i,0})^{-1}, 0).$$

So $ac \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ and a and c commute. On the other hand, b_1 does not commute with c since $\varphi_{\ell\alpha_i}(y_{i,0}) = y_{i,\ell\alpha_i} \neq y_{i,0}$ as p_i does not divide ℓ . \square

We will now define Same(i) and Different(i).

Definition 18. Same(i) says that there are a , b , and c such that:

- (1) a , b , c , and ab are in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$,
- (2) $a > 1 \iff b > 1$,
- (3) $a \in \{n\alpha_i : p_i \nmid n\}^{\mathcal{G}}$,
- (4) $b \in \{n\beta_i : q_i \nmid n\}^{\mathcal{G}}$,
- (5) $c \in \{n\gamma_i : r_i \nmid n\}^{\mathcal{G}}$,
- (6) $ac^{-1} \in \{n(\alpha_i - \gamma_i) : p_i, r_i \nmid n\}^{\mathcal{G}}$.
- (7) $bc^{-1} \in \{n(\beta_i - \gamma_i) : q_i, r_i \nmid n\}^{\mathcal{G}}$.

Different(i) is defined in the same way as Same(i), except that in (2) we ask that $a > 1$ if and only if $b < 1$.

Suppose, for simplicity, that a , b , and c are all in \mathcal{H}/\mathcal{R} . Then we would have that $a = (\varepsilon, \ell\alpha_i)$, $b = (\varepsilon, m\beta_i)$, and $c = (\varepsilon, n\gamma_i)$. Now $ac^{-1} = (\varepsilon, \ell\alpha_i - n\gamma_i)$ is a power of $(\varepsilon, \alpha_i - \gamma_i)$, and so $\ell = n$. Similarly, $bc^{-1} = (\varepsilon, m\beta_i - n\gamma_i)$ is a power of $(\varepsilon, \beta_i - \gamma_i)$, and so $m = n$. Thus $\ell = m$. Since $(\varepsilon, \ell\alpha_i) > 1 \iff (\varepsilon, \ell\beta_i) > 1$, $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$. Checking that this works even if a , b , and c are conjugates of \mathcal{H}/\mathcal{R} is the heart of Lemma 19.

Lemma 19. *Let \leq be a left-ordering on \mathcal{G} . Then:*

- (1) *If $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$, then $(\mathcal{G}, \leq) \models \text{Same}(i)$.*
- (2) *If $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$, then $(\mathcal{G}, \leq) \models \text{Different}(i)$.*
- (3) *If $\psi(i) \downarrow$, then $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$ if and only if $(\mathcal{G}, \leq) \models \text{Same}(i)$.*
- (4) *If $\psi(i) \downarrow$, then $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) < 1$ if and only if $(\mathcal{G}, \leq) \models \text{Different}(i)$.*

Proof. First, for (1), suppose that $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$. Then $(\mathcal{G}, \leq) \models \text{Same}(i)$ as witnessed by $c = (\varepsilon, \alpha_i)$, $c = (\varepsilon, \beta_i)$, and $c = (\varepsilon, \gamma_i)$. (2) is similar.

Now for (3), suppose that $(\mathcal{G}, \leq) \models \text{Same}(i)$ as witnessed by a , b , and c , and that $\psi(i) \downarrow$. Let f , g , and h be the second coordinates of a , b , and c respectively. Write $f = \ell\alpha_i$ with $p_i \nmid \ell$, $g = m\beta_i$ with $q_i \nmid m$, and $h = n\gamma_i$ with $r_i \nmid h$. Then since $f - h$ is a multiple of $\alpha_i - \gamma_i$, $\ell = n$. Similarly, $m = n$, and so $\ell = m$.

Since $ab \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ and a and b commute, by Lemma 12 we can write

$$a = (B, 0)(\varepsilon, \ell\alpha_i)(B, 0)^{-1}$$

and

$$b = (B, 0)(\varepsilon, \ell\beta_i)(B, 0)^{-1}.$$

Now since $\psi(i) \downarrow$, in \mathcal{H}/\mathcal{R} either $p_i^t\alpha_i = q_i^t\beta_i$ or $p_i^t\alpha_i = -q_i^t\beta_i$ for some t . In the second case, $a^{p_i^t} = b^{-q_i^t}$ which contradicts the fact that $a > 1 \iff b > 1$. Thus $p_i^t\alpha_i = q_i^t\beta_i$, and so $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$.

(4) is proved similarly. \square

Proof of Lemma 9. We will prove (1): $(\mathcal{G}, \leq) \models \text{Same}(i)$ if and only if $(\varepsilon, \alpha_i) > 1 \iff (\varepsilon, \beta_i) > 1$. The proof of (2) is similar. The right to left direction follows immediately from (1) of Lemma 19. For the left to right direction, suppose that $(\mathcal{F}_i, \leq_i) \models \text{Same}(i)$. Then $\psi(i) \downarrow$. Then the lemma follows from (3) of Lemma 19. \square

5. AN EXISTENTIAL DEFINITION OF $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$

The goal of this section is to prove Lemma 11, which says that $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ is definable within \mathcal{G} by an existential formula. To prove this lemma, we will first have to give a detailed analysis of which elements of \mathcal{G} commute with each other.

The first lemma is the analogue of the following well-known fact about free groups: two elements a and b in a free group commute if and only if there is c such that $a = c^m$ and $b = c^n$ (see [LS01, Proposition 2.17]).

Lemma 20. *Let $r, s \in \mathcal{G}$ commute. Then there are $W, V \in \mathcal{N}$, $x, y, z \in \mathcal{H}/\mathcal{R}$, and $k, \ell \in \mathbb{Z}$ such that*

$$r = (W, 0)(V, x)^k(\varepsilon, y)(W, 0)^{-1}$$

and

$$s = (W, 0)(V, x)^\ell(\varepsilon, z)(W, 0)^{-1}.$$

If $k \neq 0$ then $\varphi_z(V) = V$, and if $\ell \neq 0$ then $\varphi_y(V) = V$.

It is easy to check that two such elements commute.

Proof. Suppose that $rs = sr$. Let $r = (A, g)$ and $s = (B, h)$. Then we find that

$$\begin{aligned} rs &= (A, g)(B, h) \\ &= (A\varphi_g(B), g + h) \\ sr &= (B, h)(A, g) \\ &= (B\varphi_h(A), g + h). \end{aligned}$$

So $A\varphi_g(B) = B\varphi_h(A)$ in \mathcal{N} . Write

$$A = a_0 \cdots a_{m-1} \text{ and } B = b_0 \cdots b_{n-1}$$

as reduced words. So

$$a_0 \cdots a_{m-1} \varphi_g(b_0) \cdots \varphi_g(b_{n-1}) = b_0 \cdots b_{n-1} \varphi_h(a_0) \cdots \varphi_h(a_{m-1}).$$

We divide into several cases.

Case 1. A is the trivial word.

We must have $B = \varphi_g(B)$. Then $r = (\varepsilon, g)$ and $s = (B, h)$. Take $W = \varepsilon$, $V = B$, $x = h$, $y = g$, $z = 0$, $k = 0$, and $\ell = 1$.

Case 2. B is the trivial word.

We must have $A = \varphi_h(A)$. Then $r = (A, g)$ and $s = (\varepsilon, h)$. Take $W = \varepsilon$, $V = A$, $x = g$, $y = 0$, $z = h$, $k = 1$, and $\ell = 0$.

Case 3. Neither A nor B is the trivial word, and both $A\varphi_g(B)$ and $B\varphi_h(A)$ are reduced words.

We have $A\varphi_g(B) = B\varphi_h(A)$ as reduced words. Assume without loss of generality that $|A| = m \geq n = |B|$. Then $n, m > 0$ and

$$a_0 \cdots a_{m-1} \varphi_g(b_0) \cdots \varphi_g(b_{n-1}) = b_0 \cdots b_{n-1} \varphi_h(a_0) \cdots \varphi_h(a_{m-1})$$

as reduced words. So

$$\begin{aligned} a_i &= b_i && \text{for } 0 \leq i < n \\ a_i &= \varphi_h(a_{i-n}) && \text{for } n \leq i < m \\ \varphi_g(b_i) &= \varphi_h(a_{m-n+i}) && \text{for } 0 \leq i < n. \end{aligned}$$

Let $d = \gcd(m, n)$. (This is where we use the fact that $m, n > 0$.) Let $n' = n/d$ and $m' = m/d$.

Given $p, q \geq 0$, write $i = qn - pm + r$ with $0 \leq r < d$ and assume that $0 \leq i < m$. Note that every i , $0 \leq i < m$, can be written in such a way. We claim that

$$a_i = \varphi_{qh-pg}(a_r).$$

We argue by induction, ordering pairs (q, p) lexicographically. For the base case $p = q = 0$ we note that $a_r = \varphi_0(a_r)$. Otherwise, if $n \leq i < m$, then we must have $q > 0$. By the induction hypothesis, $a_{i-n} = \varphi_{(q-1)h-pg}(a_r)$. So

$$a_i = \varphi_h(a_{i-n}) = \varphi_{qh-pg}(a_r).$$

If $0 \leq i < n$, and $(q, p) \neq (0, 0)$, then $q > 0$ and $p > 0$. Note that $a_{m-n+i} = \varphi_{(q-1)h-(p-1)g}(a_r)$ by the induction hypothesis and so

$$a_i = b_i = \varphi_{h-g}(a_{m-n+i}) = \varphi_{qh-pg}(a_r).$$

This completes the induction.

Write $d = qn - pm$ with $p, q \geq 0$. Let $f = qh - pg$. Then each i , $0 \leq i < m$, can be written as $i = kd + r$ with $0 \leq r < d$, and so $a_i = \varphi_{kf}(a_r)$.

Let $C = a_0 \cdots a_{d-1}$. Then

$$A = C\varphi_f(C) \cdots \varphi_{(m'-1)f}(C)$$

and so

$$r = (A, g) = (C, f)^{m'}(\varepsilon, g - m'f).$$

Since for $0 \leq i < n$, $a_i = b_i$, we have

$$s = (B, h) = (C, f)^{n'}(\varepsilon, h - n'f).$$

This is in the desired form: take $W = \varepsilon$, $V = C$, $x = f$, $y = g - m'f$, $z = h - n'f$, $k = m'$, and $\ell = n'$.

We still have to show that $\varphi_y(V) = \varphi_z(V) = V$. Noting that

$$(n'q - 1)n - (n'p)m = n'(qn - pm) - n = n'd - n = 0$$

we have, for all $0 \leq r < d$,

$$a_r = \varphi_{(n'q-1)h-n'pg}(a_r) = \varphi_{n'f-h}(a_r).$$

Similarly,

$$a_r = \varphi_{m'f-g}(a_r).$$

Hence $\varphi_{g-m'f}(C) = \varphi_{h-n'f}(C) = C$.

Case 4. Neither A nor B is the trivial word, and both $B^{-1}A$ and $\varphi_h(A)\varphi_g(B)^{-1}$ are reduced words.

Note that $B^{-1}A = \varphi_h(A)\varphi_g(B)^{-1}$. We can make a transformation to reduce this to the previous case. Let

$$A' = B^{-1} \quad B' = \varphi_h(A) \quad g' = -h \quad h' = g.$$

Then $A'\varphi_{g'}(B') = B'\varphi_{h'}(A')$ and these are reduced words. Hence by the previous case there are $C \in \mathcal{N}$, $f \in \mathcal{H}/\mathcal{R}$, and $m, n \in \mathbb{Z}$ such that

$$(A', g') = (C, f)^m(\varepsilon, g' - mf)$$

and

$$(B', h') = (C, f)^n(\varepsilon, h' - nf)$$

and such that $\varphi_{g'-mf}(C) = C$ and $\varphi_{h'-nf}(C) = C$. Now

$$\begin{aligned} (A, g) &= (\varepsilon, -h)(\varphi_h(A), g)(\varepsilon, h) \\ &= (\varepsilon, -h)(B', h')(\varepsilon, h) \\ &= (\varepsilon, -h)(C, f)^n(\varepsilon, h' - nf)(\varepsilon, h) \\ &= (\varphi_{-h}(C), f)^n(\varepsilon, g - nf). \end{aligned}$$

Note that $\varphi_{g-nf}(C) = \varphi_{h'-nf}(C) = C$, and so $\varphi_{g-nf}(\varphi_{-h}(C)) = \varphi_{-h}(C)$. Similarly,

$$\begin{aligned} (B, h) &= (\varepsilon, -h)(B^{-1}, -h)^{-1}(\varepsilon, h) \\ &= (\varepsilon, -h)(A', g')^{-1}(\varepsilon, h) \\ &= (\varepsilon, -h)(\varepsilon, g' - mf)^{-1}(C, f)^{-m}(\varepsilon, h) \\ &= (\varepsilon, mf)(C, f)^{-m}(\varepsilon, h) \\ &= (\varphi_{mf}(C), f)^{-m}(\varepsilon, h + mf). \end{aligned}$$

Since $\varphi_{h+mf}(C) = \varphi_{g'-mf}(C) = C$, $\varphi_{mf}(C) = \varphi_{-h}(C)$. So

$$(B, h) = (\varphi_{-h}(C), f)^{-m}(\varepsilon, h + mf).$$

This completes this case, taking $W = \varepsilon$, $V = \varphi_{-h}(C)$, $x = f$, $y = g - nf$, $z = h + mf$, $k = n$, and $\ell = -m$.

Case 5. $|A| = 1$, B is not the trivial word, and neither $A\varphi_g(B) = B\varphi_h(A)$ nor $B^{-1}A = \varphi_h(A)\varphi_g(B^{-1})$ are reduced words.

Let $A = a$. Then $a^{-1} = \varphi_g(b_0)$ and $b_{n-1} = \varphi_h(a^{-1})$. Recall that $B = b_0 \cdots b_{n-1}$. From the non-reduced words $A\varphi_g(B) = B\varphi_h(A)$, we get, as reduced words,

$$\varphi_g(b_1)\varphi_g(b_2) \cdots \varphi_g(b_{n-1}) = b_0b_1 \cdots b_{n-2}.$$

Then, for $0 \leq i < n - 1$ we get $\varphi_g(b_{i+1}) = b_i$. Thus $a = \varphi_{ng+h}(a)$. Also, letting $C = b_0$,

$$r = (\varphi_g(C)^{-1}, g) = (C, -g)^{-1}.$$

and

$$s = (C, -g)^n(\varepsilon, h + ng)$$

Note that $\varphi_{h+ng}(C) = \varphi_{h+ng}(b_0) = b_0$ since $a = \varphi_{ng+h}(a)$ and $b_0 = \varphi_{-g}(a^{-1})$.

So in this case we take $W = \varepsilon$, $V = C$, $x = g$, $y = 0$, $z = h + ng$, $k = -1$, and $\ell = n$.

Case 6. $|B| = 1$, A is not the trivial word, and neither $A\varphi_g(B) = B\varphi_h(A)$ nor $B^{-1}A = \varphi_h(A)\varphi_g(B^{-1})$ are reduced words.

This case is similar to the previous case.

Case 7. $|A|, |B| \geq 2$ and neither $A\varphi_g(B) = B\varphi_h(A)$ nor $B^{-1}A = \varphi_h(A)\varphi_g(B^{-1})$ are reduced words.

We have $b_{n-1} = \varphi_h(a_0)^{-1}$ and $\varphi_h(a_{m-1}) = \varphi_g(b_{n-1})$ and so

$$\varphi_g(a_0) = \varphi_g(a_0^{-1})^{-1} = \varphi_{g-h}(b_{n-1})^{-1} = a_{m-1}^{-1}.$$

Letting

$$A' = a_1 \cdots a_{m-2} = a_0^{-1}A\varphi_g(a_0)$$

and

$$B' = a_0^{-1}b_0b_1 \cdots b_{n-2} = a_0^{-1}B\varphi_h(a_0)$$

we have

$$\begin{aligned} B'\varphi_h(A')\varphi_g(B')^{-1} &= B'b_{n-1}\varphi_h(a_0)\varphi_h(A')\varphi_h(a_{m-1})\varphi_g(b_{n-1})^{-1}\varphi_g(B')^{-1} \\ &= a_0^{-1}B\varphi_h(A)\varphi_g(B)^{-1}a_{m-1}^{-1} \\ &= a_0^{-1}Aa_{m-1}^{-1} \\ &= A'. \end{aligned}$$

So (A', g) and (B', h) still commute.

Note that $|A'| < |A|$ and $|B'| \leq |B|$. So we only have to repeat this finitely many times until we are in one of the other cases. Thus, for some word D we get reduced words

$$A' = DA\varphi_g(D^{-1})$$

and

$$B' = DB\varphi_h(D^{-1})$$

which fall into one of the other cases. So

$$(A', g) = (C, f)^m(\varepsilon, g - mf)$$

and

$$(B', h) = (C, f)^n(\varepsilon, h - nf).$$

Thus

$$r = (DA'\varphi_g(D^{-1}), g) = (D, 0)(A', g)(D^{-1}, 0)$$

and

$$s = (DB'\varphi_h(D^{-1}), h) = (D, 0)(B', h)(D^{-1}, 0)$$

are in the desired form. \square

The next lemma gives a criterion for knowing that an element r is in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, but it requires knowing that two particular elements s_1 and s_2 are not in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. This does not seem useful yet, but in Lemma 23 we will show that any three elements $s_1, s_2,$ and s_3 , such that r commutes with each of them but $s_1, s_2,$ and s_3 pairwise do not commute, give rise to two such elements which are not in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$.

Lemma 21. *Let $r, s_1, s_2 \in \mathcal{G}$. Suppose that r commutes with s_1 and s_2 , but s_1 and s_2 do not commute. If $s_1, s_2 \notin (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then $r \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$.*

Proof. Suppose to the contrary that $r \notin (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. Since r and s_1 commute, and r and s_2 commute, by Lemma 20 we can write

$$\begin{aligned} r &= (A, 0)(C, f_1)^{m_1}(\varepsilon, g_1)(A^{-1}, 0) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0) \\ s_1 &= (A, 0)(C, f_1)^{n_1}(\varepsilon, h_1)(A^{-1}, 0) \\ s_2 &= (B, 0)(D, f_2)^{n_2}(\varepsilon, h_2)(B^{-1}, 0) \end{aligned}$$

Since $r, s_1,$ and s_2 are not in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, C and D are non-trivial and $m_1, m_2, n_1, n_2 \neq 0$. So $\varphi_{g_1}(C) = \varphi_{h_1}(C) = C$ and $\varphi_{g_2}(D) = \varphi_{h_2}(D) = D$. Moreover, we will argue that we may assume that

$$C\varphi_{f_1}(C) \cdots \varphi_{(m_1-1)f_1}(C) \text{ and } D\varphi_{f_2}(D) \cdots \varphi_{(m_2-1)f_2}(D)$$

are reduced words. If the former is not a reduced word, then it must have length at least 2, and we can write $C = aC'\varphi_{f_1}(a^{-1})$. Then

$$C\varphi_{f_1}(C) \cdots \varphi_{(m_1-1)f_1}(C) = aC'\varphi_{f_1}(C') \cdots \varphi_{(m_1-1)f_1}(C')\varphi_{m_1f_1}(a^{-1})$$

and so, since φ_{g_1} fixes C and hence a ,

$$r = (Aa, 0)(C', f_1)^{m_1}(\varepsilon, g_1)(a^{-1}A^{-1}, 0).$$

Similarly,

$$s_1 = (Aa, 0)(C', f_1)^{n_1}(\varepsilon, h_1)(a^{-1}A^{-1}, 0).$$

So we may replace A by Aa and C by C' . We can continue to do this until $C\varphi_{f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)$ is a reduced word. The same argument works for $D\varphi_{f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)$.

Rearranging the two expressions for r , we get

$$(B^{-1}A, 0)(C, f_1)^{m_1}(\varphi_{g_1}(A^{-1}B), g_1) = (D, f_2)^{m_2}(\varepsilon, g_2).$$

Looking at the first coordinate,

$$\begin{aligned} & B^{-1}AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1+g_1}(A^{-1}B) \\ &= D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D). \end{aligned}$$

We claim that we can write $B^{-1}A = E_2^{-1}E_1$ where $\varphi_{g_1}(E_1) = \varphi_{h_1}(E_1) = E_1$ and $\varphi_{g_2}(E_2) = \varphi_{h_2}(E_2) = E_2$. Recall that

$$C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)$$

is a non-trivial reduced word. Taking a high enough power ℓ , the length of

$$(C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^\ell$$

as a reduced word is more than twice the length of $B^{-1}A$. Then

$$\begin{aligned} & B^{-1}A(C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^\ell\varphi_{m_1f_1+g_1}(A^{-1}B) \\ &= (D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D))^\ell. \end{aligned}$$

We can write $B^{-1}A = E_2^{-1}E_1$ as a reduced word where E_2^{-1} appears at the start of the right hand side when it is written as a reduced word, and E_1 cancels with the beginning of $(C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^\ell$. Thus E_1 is fixed by φ_{g_1} and φ_{h_1} since they fix each letter appearing in the word $(C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C))^\ell$, and E_2 is fixed by φ_{g_2} and φ_{h_2} since they fix each letter appearing in the right hand side.

Since $E_2B^{-1} = E_1A^{-1}$,

$$\begin{aligned} E_2B^{-1}rBE_2^{-1} &= (E_1, 0)(C, f_1)^{m_1}(\varepsilon, g_1)(E_1^{-1}, 0) \\ &= (E_2, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(E_2^{-1}, 0) \\ E_2B^{-1}s_1BE_2^{-1} &= (E_1, 0)(C, f_1)^{n_1}(\varepsilon, h_1)(E_1^{-1}, 0) \\ E_2B^{-1}s_2BE_2^{-1} &= (E_2, 0)(D, f_2)^{n_2}(\varepsilon, h_2)(E_2^{-1}, 0). \end{aligned}$$

So, applying the automorphism of \mathcal{G} given by conjugating by E_2B^{-1} (and noting that this automorphism fixes $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$) we may assume from the beginning that $\varphi_{g_1}(A) = \varphi_{h_1}(A) = A$ and $\varphi_{g_2}(B) = \varphi_{h_2}(B) = B$. Thus

$$\begin{aligned} r &= (A, 0)(C, f_1)^{m_1}(A^{-1}, 0)(\varepsilon, g_1) = (B, 0)(D, f_2)^{m_2}(B^{-1}, 0)(\varepsilon, g_2) \\ s_1 &= (A, 0)(C, f_1)^{n_1}(A^{-1}, 0)(\varepsilon, h_1) \\ s_2 &= (B, 0)(D, f_2)^{n_2}(B^{-1}, 0)(\varepsilon, h_2). \end{aligned}$$

Now looking at the first coordinate, we have

$$\begin{aligned} & AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1}(A)^{-1} \\ &= BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)\varphi_{m_2f_2}(B)^{-1}. \end{aligned}$$

Our next step is to argue that we may assume that these are reduced words. Suppose that there was some cancellation, say $A = A'a$ and $C = a^{-1}C'$. Let $C^* = C'\varphi_{f_1}(a^{-1})$. Then

$$\begin{aligned} & AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1}(A)^{-1} \\ &= A'C^*\varphi_{f_1}(C^*)\varphi_{2f_1}(C^*)\cdots\varphi_{(m_2-1)f_1}(C^*)\varphi_{m_1f_1}(A')^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} r &= (A', 0)(C^*, f_1)^{m_1}(\varepsilon, g_1)(A', 0)^{-1} \\ s_1 &= (A', 0)(C^*, f_1)^{n_1}(\varepsilon, h_1)(A', 0)^{-1}. \end{aligned}$$

Note that

$$(C^*, f_1)^{m_1} = C^*\varphi_{f_1}(C^*)\varphi_{2f_1}(C^*)\cdots\varphi_{(m_1-1)f_1}(C^*)$$

is still a reduced word. If it was not a reduced word, then we would have $m_1 > 0$, $|C^*| > 1$, and $\varphi_{f_1}(a^{-1}) = \varphi_{f_1}(a')^{-1}$, where a' is the first letter of C^* . Thus $a' = a$ is the second letter of C , which together with the fact that the first letter of C is a^{-1} contradicts our assumption that C is a reduced word. We have reduced the size of A , so after finitely many reductions of this form, we get

$$\begin{aligned} & AC\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)\varphi_{m_1f_1}(A)^{-1} \\ &= BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)\varphi_{m_2f_2}(B)^{-1} \end{aligned}$$

and that both sides are reduced words.

Now either $|A| \leq |B|$ or $|B| \leq |A|$. Without loss of generality, assume that we are in the first case. Then A is an initial segment of B (i.e., $B = AB'$ as a reduced word). Then by replacing r , s_1 , and s_2 with $A^{-1}rA$, $A^{-1}s_1A$, and $A^{-1}s_2A$, we may assume that A is trivial. To summarize the reductions we have made so far, we have

$$\begin{aligned} r &= (C, f_1)^{m_1}(\varepsilon, g_1) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0) \\ s_1 &= (C, f_1)^{n_1}(\varepsilon, h_1) \\ s_2 &= (B, 0)(D, f_2)^{n_2}(\varepsilon, h_2)(B^{-1}, 0). \end{aligned}$$

The automorphisms φ_{g_1} and φ_{h_1} fix C , and the automorphisms φ_{g_2} and φ_{h_2} fix D and B . Both sides of

$$\begin{aligned} & C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) \\ &= BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)\varphi_{m_2f_2}(B)^{-1} \end{aligned}$$

are reduced words.

Now we will show that either $m_1 = 1$ or B is trivial. Suppose that B was non-trivial, say $B = bB'$. First note that the length of C is greater than one, as otherwise $C = b$ and $\varphi_{(m_1-1)f_1}(C) = \varphi_{m_2f_2}(b^{-1})$; but there is no $e \in \mathcal{H}/\mathcal{R}$ such that $\varphi_e(b) = b^{-1}$. Then we must have $C = bC'\varphi_{m_2f_2-(m_1-1)f_1}(b^{-1})$ for some C' . We have $m_1f_1 + g_1 = m_2f_2 + g_2$. Since b appears both in C and in B , it is fixed by both φ_{g_1} and φ_{g_2} . Thus $C = bC'\varphi_{f_1}(b^{-1})$. But then if $m_1 > 1$,

$$C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C)$$

is not a reduced word. So we conclude that either $m_1 = 1$ or B is trivial.

Case 1. Suppose that $m_1 = 1$.

We have

$$r = (C, f_1)(\varepsilon, g_1) = (B, 0)(D, f_2)^{m_2}(\varepsilon, g_2)(B^{-1}, 0).$$

Also, as reduced words,

$$C = BD\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D)\varphi_{m_2f_2}(B)^{-1}.$$

Since the right hand side is a reduced word, φ_{g_1} and φ_{h_1} fix B and D since each letter in B and D appears in C . Thus

$$s_1 = (C, f_1)^{n_1}(\varepsilon, h_1) = [(B, 0)(D, f_2)^{m_2}(B^{-1}, 0)(\varepsilon, f_1 - m_2f_2)]^{n_1}(\varepsilon, h_1).$$

Now $f_1 + g_1 = m_2f_2 + g_2$. Since φ_{g_1} and φ_{g_2} fix B and D , $\varphi_{f_1 - m_2f_2}$ also fixes B and D . Thus

$$s_1 = (B, 0)(D, f_2)^{m_2n_1}(\varepsilon, h_1 + n_1(f_1 - m_2f_2))(B^{-1}, 0)$$

and $h_1 + n_1(f_1 - m_2f_2)$ fixes D . Thus s_1 and s_2 commute. This is a contradiction.

Case 2. B is trivial.

Let $|C| = k$ and $|D| = \ell$. Suppose without loss of generality that $k \geq \ell$. Let d_0, d_1, d_2, \dots be the reduced word

$$C\varphi_{f_1}(C)\varphi_{2f_1}(C)\cdots\varphi_{(m_1-1)f_1}(C) = D\varphi_{f_2}(D)\varphi_{2f_2}(D)\cdots\varphi_{(m_2-1)f_2}(D).$$

Then we have

$$\begin{aligned} d_i &= \varphi_{f_2}(d_{i-\ell}) && \text{for } i \geq \ell \\ \varphi_{(m_1-1)f_1}(d_{k-\ell+i}) &= \varphi_{(m_2-1)f_2}(d_i) && \text{for } 0 \leq i < \ell \end{aligned}$$

Let $e = \gcd(k, \ell)$.

Given $p, q \geq 0$, write $i = q\ell - pk + r$ with $0 \leq r < e$ and assume that $0 \leq i < m_1k = m_2\ell$. Note that every i , $0 \leq i < m_1k = m_2\ell$, can be written in such a way.

We claim that

$$d_i = \varphi_{qf_2+p[(m_1-1)f_1-m_1f_2]}(d_r).$$

We argue by induction, ordering pairs (q, p) lexicographically. For the base case $p = q = 0$ we note that $d_r = \varphi_0(d_r)$. If $\ell \leq i$, then we must have $q > 0$. By the induction hypothesis, $d_{i-\ell} = \varphi_{(q-1)f_2+p[(m_1-1)f_1-m_2f_2]}(d_r)$. So

$$d_i = \varphi_{f_2}(d_{i-\ell}) = \varphi_{qf_2+p[(m_1-1)f_1-m_2f_2]}(d_r).$$

If $0 \leq i < \ell$, and $(q, p) \neq (\varepsilon, 0)$, then $q > 0$ and $p > 0$. Note that

$$d_{k-\ell+i} = \varphi_{(q-1)f_2+(p-1)[(m_1-1)f_1-m_2f_2]}(d_r) = \varphi_{qf_2+p[(m_1-1)f_1-m_2f_2]-[(m_1-1)f_1-(m_2-1)f_2]}(d_r)$$

by the induction hypothesis and so

$$d_i = \varphi_{(m_1-1)f_1-(m_2-1)f_2}(d_{i+k-\ell}) = \varphi_{qf_2+p[(m_1-1)f_1-m_2f_2]}(d_r).$$

This completes the induction.

Write $e = q\ell - pk$ with $p, q \geq 0$. Let $f = qf_2 + p[(m_1-1)f_1 - m_2f_2]$. Then each i , $0 \leq i < km_1$, can be written as $i = se + r$ with $0 \leq r < e$, and so

$$d_i = \varphi_{sf}(d_r).$$

Let $E = d_1 \cdots d_e$. Then

$$C = E\varphi_f(E)\cdots\varphi_{(\frac{k}{e}-1)f}(E).$$

Similarly,

$$D = E\varphi_f(E) \cdots \varphi_{(\frac{\ell}{e}-1)f}(E).$$

Also,

$$\varphi_{f_1}(E) = d_k \cdots d_{k+e-1} = \varphi_{\frac{k}{e}f}(d_0, \dots, d_{e-1}) = \varphi_{\frac{k}{e}f}(E)$$

and

$$\varphi_{f_2}(E) = d_\ell \cdots d_{\ell+e-1} = \varphi_{\frac{\ell}{e}f}(d_0, \dots, d_{e-1}) = \varphi_{\frac{\ell}{e}f}(E).$$

So $\varphi_{f_1}(C) = \varphi_{\frac{k}{e}f}(C)$ and $\varphi_{f_2}(D) = \varphi_{\frac{\ell}{e}f}(D)$. Hence

$$s_1 = (C, f_1)^{m_1}(\varepsilon, h_1) = (E, f)^{\frac{m_1 k}{e}}(\varepsilon, h_1 + m_1 f_1 - \frac{m_1 k}{e}f)$$

and

$$s_2 = (D, f_2)^{m_2}(\varepsilon, h_2) = (E, f)^{\frac{m_2 \ell}{e}}(\varepsilon, h_2 + m_2 f_2 - \frac{m_2 \ell}{e}f)$$

Note that φ_{h_1} and φ_{h_2} both fix E , since they fix C and D respectively. Also, since $\varphi_{f_1}(E) = \varphi_{\frac{k}{e}f}(E)$, $\varphi_{m_1 f_1 - \frac{m_1 k}{e}f}$ fixes E . Similarly, $\varphi_{m_2 f_2 - \frac{m_2 \ell}{e}f}$ fixes E . So s_1 and s_2 commute. This is a contradiction. \square

Lemma 22. *Fix $r \in \mathcal{G}$. If $r^2 \in \mathcal{H}/\mathcal{R}$, then $r \in \mathcal{H}/\mathcal{R}$.*

Proof. Write $r = (A, f)$. We will show that if $r \notin \mathcal{H}/\mathcal{R}$, i.e. if $A \neq \varepsilon$, then $r^2 \notin \mathcal{H}/\mathcal{R}$. Since

$$r^2 = (A\varphi_f(A), 2f)$$

we must show that $A\varphi_f(A)$ is non-trivial. Suppose that it was trivial; then the length of A as a reduced word must be even. (If the length of A was odd, say $A = A_1 a A_2$ with A_1 and A_2 of equal lengths, then

$$A\varphi_f(A) = A_1 a A_2 \varphi_f(A_1) \varphi_f(a) \varphi_f(A_2) = \varepsilon.$$

So it must be that $\varphi_f(a) = a^{-1}$, which cannot happen for any letter a .) Write $A = BC$, where B and C are each half the length of A . Then since $A\varphi_f(A)$ is the trivial word, $C\varphi_f(B)$ is the trivial word; thus $C = \varphi_f(B^{-1})$. So $A = B\varphi_f(B^{-1})$, and

$$A\varphi_f(A) = B\varphi_f(B^{-1})\varphi_f(B)\varphi_{2f}(B^{-1}) = B\varphi_{2f}(B^{-1}).$$

Since $A\varphi_f(A)$ is the trivial word, $\varphi_{2f}(B) = B$. Since A is not the trivial word, $B \neq \varphi_f(B)$. But this is impossible, as p_i , q_i , and r_i were all chosen to be odd primes. \square

The next lemma is the heart of the existential definition of $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. The proof is to show that under the hypotheses of the lemma, elements not in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ such as in Lemma 21 must exist.

Lemma 23. *Let $r, s_1, s_2, s_3 \in \mathcal{G}$. Suppose that r commutes with s_1, s_2 , and s_3 , but that no two of s_1, s_2 , and s_3 commute. Then $r \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$.*

Proof. If at least two of s_1, s_2 , and s_3 are not in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then this follows immediately by Lemma 21. Otherwise, without loss of generality suppose that s_1 and s_2 are in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. By Lemma 12, $s_1 s_2 \notin (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$.

Note that r commutes with $s_1 s_2$ and with $s_1 (s_2)^2$. Also, $s_1 s_2$ does not commute with $s_1 (s_2)^2$, since if it did, then

$$s_1 s_2 s_1 s_2 s_2 = s_1 s_2 s_2 s_1 s_2 \Rightarrow s_1 s_2 = s_2 s_1.$$

We claim that $s_1(s_2)^2 \notin (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. If $s_1(s_2)^2$ was in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then by Lemma 12, we could write

$$s_1 = (A, 0)(\varepsilon, g)(A^{-1}, 0) \text{ and } (s_2)^2 = (A, 0)(\varepsilon, h)(A^{-1}, 0).$$

Then let $s'_2 = (A^{-1}, 0)s_2(A, 0) = (C, f)$. Then $(s'_2)^2 = (\varepsilon, h)$, and so by Lemma 22, $s'_2 = (\varepsilon, f)$. Thus $s_2 = (A, 0)(\varepsilon, f)(A^{-1}, 0)$. So s_1 and s_2 would commute; since we know that s_1 and s_2 do not commute, $s_1(s_2)^2 \notin (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$.

By Lemma 21, with r , s_1s_2 , and $s_1s_2^2$, we see that r is in $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$. \square

The existential definition of $(\mathcal{H}/\mathcal{R})^{\mathcal{G}}$ comes from the previous lemma. It remains only to show that if $r \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then the hypothesis of the previous lemma is satisfied.

Proof of Lemma 11. By the previous lemma, it suffices to show that if $r \in (\mathcal{H}/\mathcal{R})^{\mathcal{G}}$, then there are s_1 , s_2 , and s_3 such that r commutes with s_1 , s_2 , and s_3 , but no two of these commute with each other. If $r = (A, 0)(\varepsilon, g)(A^{-1}, 0)$, let $s_1 = (A, 0)(u_0, 0)(A^{-1}, 0)$, $s_2 = (A, 0)(u_1, 0)(A^{-1}, 0)$, and $s_3 = (A, 0)(u_2, 0)(A^{-1}, 0)$. Then r commutes with s_1 , s_2 , and s_3 since g fixes u_0 , u_1 , and u_2 , but no two of s_1 , s_2 , and s_3 commute with each other as u_0 , u_1 , and u_2 do not commute with each other. \square

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GROUP IN LOGIC AND THE METHODOLOGY OF SCIENCE, UNIVERSITY OF CALIFORNIA, BERKELEY, USA

E-mail address: matthew.h-t@berkeley.edu

URL: www.math.berkeley.edu/~mathtt