# Modal Sequents and Definability

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## ABSTRACT

We examine a certain modal consequence relation, and define the notion of validity of a modal sequent on a frame. We demonstrate that it is possible to define classes of frames, not definable by modal formulas, by modal sequents. Through the use of modal algebras and general frames, we obtain a characterization of modal sequent-definable classes of frames which are also first-order definable, and a sufficient condition for a class of frames to be definable by modal sequents.

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This thesis is dedicated to my father and to the memory of my mother.

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## Introduction

In [Go], Goldblatt developed techniques for dealing with questions regarding definability in the relational semantics for modal languages. These techniques were used to characterize modally definable classes of standard relational frames which are also first-order definable, and characterize arbitrary modally definable classes of first-order or general frames, which are a generalization of standard relational frames. In [GT], a characterization of arbitrary modally definable classes of standard relational frames was obtained.

In order to answer questions about the definability of relational frames, [Go] turns to the algebraic semantics for modal languages. Here validity of a modal formula is identified with the validity of a corresponding polynomial identity on a modal algebra. Many of the techniques used in [Go] derive from one basic result, namely that the category of descriptive frames and the category of modal algebras, with appropriate morphisms, are dual. Now it is straightforward to characterize modally definable classes of modal algebras, since they are really just equational classes. Using this along with the above-mentioned duality, it is then possible to characterize the modally definable classes of descriptive frames, and to work toward a characterization of such classes of standard frames.

A number of the techniques of [Go] are refined in [vB1] and [vB2].

In this thesis, we use these techniques to answer some questions about definability in an extended relational semantics. We introduce modal sequents, which are pairs of finite sets of modal formulas. The definition of validity of a sequent is derived from the definition of a certain modal consequence relation. This relation is fairly 'natural', insofar as it has a simple syntactic characterization, in terms of the common logics introduced in [Seg]. Having defined validity of a sequent, we can show that it is possible to define classes of frames using sequents which we cannot define using modal formulas. In order to answer questions about sequent definability, we take the algebraic approach: a class of modal algebras is is definable by modal sequents iff it is universal. Using the duality result of [Go], we then are able to characterize classes of general frames definable by modal sequents, and classes of standard frames definable by modal sequents which are also first-order definable. We are also able to provide a sufficient condition for an arbitrary class of standard frames to be definable by modal sequents.

## 1. Relational Semantics for Modal Languages

In this chapter we will introduce the standard relational semantics for modal languages. It will be shown that with respect to this interpretation, modal formulas correspond to certain kinds of second-order formulas.

We will be dealing with a number of formal languages, but our primary focus is on $L_m$, the language of propositional modal logic. We assume that the reader is already familiar with first-order and second-order logic. If not, he can refer to [Bar] and [vBD], respectively.

The language $L_m$ has three components: a countable set Var of *propositional variables*, denoted $p_0, p_1, \cdots, p, q, \cdots$, a set Con = $\{\neg, \Box, \&\}$ of *connectives*, and a set Form of *formulas*, which are strings constructed from members of Var and Con. Form is defined inductively as follows:

Form is the least set such that Var $\subseteq$ Form and

$$\alpha \in \text{Form} \Longrightarrow \neg\alpha \in \text{Form}$$
$$\alpha, \beta \in \text{Form} \Longrightarrow \&\alpha\beta \in \text{Form}$$
$$\alpha \in \text{Form} \Longrightarrow \Box\alpha \in \text{Form}$$

Formulas (in any language) are denoted by lower case Greek characters: $\alpha, \beta, \gamma, \phi, \psi$. A $L_m$–formula $\alpha$ with all propositional variables among $p_0, \cdots, p_n$ may be denoted $\alpha(p_0, \cdots, p_n)$. Sets of formulas are denoted by upper case Greek characters: $\Sigma, \Gamma, \Delta, \Theta, \Phi, \Omega$, and may be subscripted by 0 (e.g. $\Gamma_0$) if they are known to be finite. We may write $\Gamma, \Theta$ for $\Gamma \cup \Theta$ and $\Gamma, \alpha$ for $\Gamma \cup \{\alpha\}$. We introduce the following abbreviations for various $L_m$–formulas:

$$(\alpha \& \beta) \text{ for } \&\alpha\beta$$
$$(\alpha \vee \beta) \text{ for } \neg(\neg\alpha \& \neg\beta)$$
$$(\alpha \rightarrow \beta) \text{ for } (\neg\alpha \vee \beta)$$

$$(\alpha \leftrightarrow \beta) \text{ for } ((\alpha \rightarrow \beta) \& (\beta \rightarrow \alpha))$$
$$\Diamond \alpha \text{ for } \neg \Box \neg \alpha$$
$$\bot \text{ for } (\alpha \& \neg \alpha)$$
$$\top \text{ for } (\alpha \vee \neg \alpha)$$
$$\& \Gamma_0 \text{ for } \underset{\alpha \in \Gamma_0}{\&} \alpha$$

Parentheses are used freely to indicate the precedence of connectives, but may be omitted given the following implicit precedence: $\neg$, $\Box$, $\Diamond$ have the highest precedence, followed by $\&$, v and finally $\rightarrow$, $\leftrightarrow$.

We now introduce the relational semantics for $L_m$. This is considered to be the 'standard' semantics, and is based on the work of Kripke ([Kr1], [Kr2]).

**Definition 1.1** A (standard) *frame* is a structure $\mathbf{F} = \langle W,R \rangle$, where W is the *underlying set of* $\mathbf{F}$ (denoted $|\mathbf{F}|$) and $R \subseteq W \times W$. A *valuation* for a frame $\mathbf{F}$ is a mapping $V{:}\mathrm{Var} \rightarrow 2^W$.

Every valuation V for a frame $\mathbf{F} = \langle W,R \rangle$ extends uniquely to a mapping $\overline{V}{:}\mathrm{Form} \rightarrow 2^W$ via the following definitions:

$$\overline{V}(\neg\alpha) = -\overline{V}(\alpha) = \{w \in W \mid w \notin \overline{V}(\alpha)\}$$
$$\overline{V}(\alpha\&\beta) = \overline{V}(\alpha) \cap \overline{V}(\beta)$$
$$\overline{V}(\Box\alpha) = \{w \in W \mid (\forall v \in W)(wRv \Rightarrow v \in \overline{V}(\alpha))\}$$

Henceforth, we will not distinguish between V and $\overline{V}$.

**Definition 1.2** A *model* is a triple $\mathbf{M} = \langle W,R,V \rangle$, where $\langle W,R \rangle$ is a frame and V is a valuation for $\langle W,R \rangle$. $\mathbf{M}$ is a model *based on* $\langle W,R \rangle$, which is the *underlying frame* of $\mathbf{M}$. An $L_m$–formula $\alpha$ is *valid on* $\mathbf{M} = \langle W,R,V \rangle$ ($\mathbf{M} \models \alpha$) if $V(\alpha) = W$. The formula $\alpha$ is *valid on* $\mathbf{F} = \langle W,R \rangle$ ($\mathbf{F} \models \alpha$) if it is valid on every model based on $\mathbf{F}$, and is *valid* ($\models \alpha$) if it is valid on all frames. A set $\Gamma \subseteq \mathrm{Form}$ is *valid on* $\mathbf{M}$ if every member of $\Gamma$ is valid on $\mathbf{M}$.

For $w \in |\mathbf{M}|$, $\alpha$ is *true on* $\mathbf{M}$ *at* $w$ ($\langle \mathbf{M}, w \rangle \models \alpha$) if $w \in V(\alpha)$.

We will now show that with respect to the given definition of validity on a frame, every modal formula $\alpha$ defines a second-order formula $ST(\alpha)$ (This is the approach taken in [vB3]). $ST(\alpha)$ is a formula in the second-order language with one binary predicate constant $R$, and a set $\{P_i \mid i<\omega\}$ of monadic predicate variables, and is defined inductively as follows:

$$ST(p_i) = P_i(x)$$
$$ST(\neg\alpha) = \neg ST(\alpha)$$
$$ST(\alpha \& \beta) = ST(\alpha) \& ST(\beta)$$
$$ST(\Box\alpha) = (\forall y)(Rxy \rightarrow [y/x]ST(\alpha))$$

where $y$ is not free in $ST(\alpha)$ and $[y/x]\phi$ denotes the formula obtained by replacing all free occurrences of $x$ in $\phi$ by $y$.

**Theorem 1.3** Let $\mathbf{F}$ be a frame, $\alpha(p_0, \cdots, p_n) \in$ Form. Then $\mathbf{F} \models \alpha$ iff $\forall x \forall P_0 \cdots \forall P_n ST(\alpha)$ is second-order valid on $\mathbf{F}$.

**Proof** This follows directly from the definition of $ST(\alpha)$ and 1.2.

Since $\forall x \forall P_0 \cdots \forall P_n ST(\alpha)$ is second-order equivalent to $\forall P_0 \cdots \forall P_n \forall x ST(\alpha)$, $\alpha$ corresponds to a second-order sentence with a prefix of universal second order quantifiers, and no other second order quantifiers. According to the classification scheme of [CK], 4.1, such sentences are called $\Pi^1_1$ sentences.

We will now introduce modal axiom systems, and indicate their connection with the relational semantics.

**Definition 1.4** A *modal axiom system* is a pair $S = \langle Ax, Rule \rangle$ where $Ax \subseteq Form$ is the set of *axioms*, and $Rule \subseteq \{f \mid f \text{ maps } Form^i \to Form, i<\omega\}$ is the set of *rules*. For $f \in Rule$, if $f(\alpha_1, \cdots, \alpha_n) = \alpha_{n+1}$, we say that $\alpha_{n+1}$ is *inferred from* $\alpha_1, \cdots, \alpha_n$ *by* f . The formula $\alpha$ is *derivable in* S ($\vdash_S \alpha$) if there is a finite sequence $\alpha_1, \cdots, \alpha_n$ of $L_m$-formulas such that $\alpha = \alpha_n$, and for $1 \leqslant i \leqslant n$, either $\alpha_i \in Ax$ or $\alpha_i$ is inferred from some $\alpha_{i_1}, \cdots, \alpha_{i_k}$ by some $f \in Rule$, where $i_j < i$ for $1 \leqslant j \leqslant k$. For $\Gamma \subseteq Form$, $\alpha$ is *derivable from* $\Gamma$ *in* S ($\Gamma \vdash_S \alpha$) if there is some $\Gamma_0 \subseteq_{fin} \Gamma$ with $\vdash_S \& \Gamma \to \alpha$. $\Gamma$ is *S-consistent* if it is not the case that $\Gamma \vdash_S \bot$.

We now turn to the modal axiom system K . K is formed by adding to the axioms of the propositional calculus (PC) the axiom scheme $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ (that is, the set of all formulas of the given form, where $\alpha$ and $\beta$ are arbitrary formulas), and to the rules of PC the rule of necessitation: from $\alpha$ infer $\Box\alpha$. The following theorem shows the significance of this system.

**Theorem 1.5** For $\alpha \in Form$, $\models \alpha$ iff $\vdash_K \alpha$.

This is a standard result. See, e.g., [HC], 2.5. We present an outline of the 'only if' part, since some of the ideas used will be needed for later results. We construct $\mathbf{M}_K = \langle W_K, R_K, V_K \rangle$, the *canonical model* for K , as follows:

$$W_K = \{\Gamma \subseteq Form \mid \Gamma \text{ is maximally K-consistent}\}$$
$$R_K = \{\langle \Gamma, \Gamma' \rangle \mid (\forall\alpha)(\Box\alpha \in \Gamma \Rightarrow \alpha \in \Gamma')\}$$
$$V_K(p_i) = \{\Gamma \mid p_i \in \Gamma\}, i<\omega$$

The *fundamental lemma* then states that for $\alpha \in Form$ and $\Gamma \in W_K$, $\Gamma \in V_K(\alpha)$ iff $\alpha \in \Gamma$. Assuming this, suppose $\nvdash_K \alpha$. Then $\{\neg\alpha\}$ is K-consistent, and so can be extended to a maximal K-consistent set $\Gamma_\alpha$. Now $\alpha \notin \Gamma_\alpha$, so $\Gamma_\alpha \notin V_K(\alpha)$, whence $\mathbf{M}_K \not\models \alpha$. So we have that if $\nvdash_K \alpha$, there is a frame $\mathbf{F}_K = \langle W_K, R_K \rangle$ such that $\mathbf{F}_K \not\models \alpha$. The desired result is obtained by contraposition.

## 2. Common Logics and Modal Consequence Relations

Given an axiom system S, we normally identify S with the set $\{\alpha \in \mathsf{Form} \mid \vdash_S \alpha\}$ of *theorems* of S. Of course, we could also consider the set $\{\langle\Gamma,\alpha\rangle \mid \Gamma\vdash_S\alpha\}$. More generally, we can examine relations such as $\vdash_S$ in the context of arbitrary binary relations on subsets of Form. This is the approach taken in [Seg], where such relations are called *logics*, and certain conditions which characterize *common logics* are identified. In this chapter we extend $\vdash_K$ to such a logic. We also use frames to define a corresponding *consequence relation*.

**Definition 2.1** A *logic* L is a subset of $2^{\mathsf{Form}}\times 2^{\mathsf{Form}}$. Note that when dealing with an arbitrary L, we may write $\Gamma\vdash\Theta$ for $\langle\Gamma,\Theta\rangle \in$ L. A *common logic* is a logic L which meets the following closure conditions:

(Refl) $\Sigma\vdash\Sigma$ if $\Sigma \neq \varnothing$

(Mono) If $\Gamma\vdash\Theta$ then $\Gamma,\Gamma'\vdash\Theta,\Theta'$

($\mathsf{Cut}_1$) If both $\Gamma\vdash\Theta,\Omega$ and $\alpha,\Gamma'\vdash\Theta'$ for all $\alpha \in \Omega$, then $\Gamma,\Gamma'\vdash\Theta,\Theta'$

($\mathsf{Cut}_2$) If both $\Gamma\vdash\Theta,\beta$ for all $\beta \in \Omega$, and $\Omega,\Gamma'\vdash\Theta'$, then $\Gamma,\Gamma'\vdash\Theta,\Theta'$

(Susbt) If $\Gamma\vdash\Theta$ then $s\Gamma\vdash s\Theta$, where $s\Sigma$ denotes the set of substitution instances of members of $\Sigma$ for some substitution s:Var→Form.

**Proposition 2.2** Any common logic L meets the following closure conditions:

(Overl) $\Gamma\vdash\Theta$ if $\Gamma\cap\Theta \neq \varnothing$

(Trans) If $\alpha\vdash\beta$ and $\beta\vdash\gamma$ then $\alpha\vdash\gamma$.

**Proof** (Overl): Let $\Sigma = \Gamma\cap\Theta$. By (Refl), $\Sigma\vdash\Sigma$, so by (Mono), $\Gamma\vdash\Theta$.

(Trans): This is a direct result of ($\mathsf{Cut}_1$).

Let S be a modal axiom system. $L_S$ is defined as the smallest common logic such that $\langle\{\bot\},\emptyset\rangle \in L_S$, $\langle\{p\},\{\Box p\}\rangle \in L_S$ and $\langle\{\alpha_1,\cdots,\alpha_n\},\{\alpha_{n+1}\}\rangle \in L_S$ whenever $\vdash_S\alpha_1\&\cdots\&\alpha_n\rightarrow\alpha_{n+1}$. We write $\Gamma\vdash_S\Theta$ for $\langle\Gamma,\Theta\rangle \in L_S$.

**Definition 2.3** A (modal) *sequent* is a pair $\langle\Gamma_0,\Theta_0\rangle$, where $\Gamma_0,\Theta_0 \subseteq_{fin}\mathsf{Form}$. We use $\sigma$ to denote an arbitrary sequent. A logic L is *finitary* if whenever $\Gamma\vdash\Theta$ there is a sequent $\langle\Gamma_0,\Theta_0\rangle$ with $\Gamma_0 \subseteq \Gamma,\Theta_0 \subseteq \Theta$ such that $\Gamma_0\vdash\Theta_0$.

Since the logic $L_S$ is the smallest logic containing a specified set L' of sequents, we say that $L_S$ is *generated* by a set of sequents.

**Lemma 2.4** Any common logic L which is generated by a set of sequents is finitary.

**Proof** This is done by induction on members of L. The result holds trivially for the basis elements (i.e., those in the generating set). For the induction step, we consider the case where $\langle\Gamma,\Theta\rangle \in L$ is obtained via the $(\mathsf{Cut}_1)$ rule. (Other rules are handled similarly). We have $\Gamma = \Gamma'\cup\Gamma''$ and $\Theta = \Theta'\cup\Theta''$ such that $\Gamma'\vdash\Theta',\Omega$ and $\alpha,\Gamma''\vdash\Theta''$ for all $\alpha \in \Omega$, for some $\Omega \in \mathsf{Form}$. But then $\Gamma'_0\vdash\Theta'_0,\Omega_0$, where $\Gamma'_0 \subseteq_{fin}\Gamma'$, $\Theta'_0 \subseteq_{fin}\Theta'$ and $\Omega_0 \subseteq_{fin}\Omega$, and for all $\alpha \in \Omega_0$, $\alpha,(\Gamma''_0)^\alpha\vdash(\Theta''_0)^\alpha$, where $(\Gamma''_0)^\alpha$ and $(\Theta''_0)^\alpha$ are finite subsets of $\Gamma''$ and $\Theta''$ which depend on $\alpha$. Let $\Gamma''_0 = \bigcup_{\alpha\in\Omega_0}(\Gamma''_0)^\alpha$ and $\Theta''_0 = \bigcup_{\alpha\in\Omega_0}(\Theta''_0)^\alpha$. Then $\Gamma''_0 \subseteq_{fin}\Gamma''$, $\Theta''_0 \subseteq_{fin}\Theta''$, and for all $\alpha \in \Omega_0$, $\alpha,\Gamma''_0\vdash\Theta''_0$, by (Mono). So we have by $(\mathsf{Cut}_1)$ that $\Gamma'_0,\Gamma''_0\vdash\Theta'_0,\Theta''_0$. But $\Gamma'_0\cup\Gamma''_0 \subseteq_{fin}\Gamma$ and $\Theta'_0\cup\Theta''_0 \subseteq_{fin}\Theta$. Hence L is finitary.

As a result, $L_S$ is finitary. By [Seg], 2.3.6, it follows that $L_S$ can be defined using the **rule** $(\mathsf{Cut}_G)$:

$$\text{If } \Gamma_0\vdash_S\Theta_0,\alpha \text{ and } \alpha,\Gamma'_0\vdash_S\Theta'_0 \text{ then } \Gamma_0,\Gamma'_0\vdash_S\Theta_0,\Theta'_0$$

in place of the rules ($Cut_1$) and ($Cut_2$).

**Definition 2.4** Let $\mathbf{X}$ be a class of frames, $\Gamma,\Theta \subseteq$ Form. $\Theta$ is a *consequence of* $\Gamma$ *on* $\mathbf{X}$ ($\Gamma \models \Theta(\mathbf{X})$) if for all $\mathbf{F} \in \mathbf{X}$ and valuations V for $\mathbf{F}$,

$$(\forall \alpha \in \Gamma)(V(\alpha) = |\mathbf{F}|) \Longrightarrow (\exists \beta \in \Theta)(V(\beta) = |\mathbf{F}|)$$

We write $\mathbf{F} \models \langle \Gamma, \Theta \rangle$ for $\Gamma \models \Theta(\{\mathbf{F}\})$.

We want to establish now that $\Gamma \models \Theta(\mathsf{F})$ iff $\Gamma \vdash_K \Theta$, where $\mathsf{F}$ is the class of all frames. We will use the canonical model $\mathbf{M}_K$ constructed in Ch. 1.

**Definition 2.5** Let $\mathbf{F} = \langle W,R \rangle$, $\mathbf{F}' = \langle W',R' \rangle$ be frames. $\mathbf{F}'$ is a *generated subframe* of $\mathbf{F}$ if

1) $W' \subseteq W$
2) $(\forall w,w' \in W)(w \in W'$ and $wRw' \Longrightarrow w' \in W')$
3) $R' = R \cap (W' \times W')$

Suppose $\mathbf{F}'$ is a generated subframe of $\mathbf{F}$ and V',V are valuations for $\mathbf{F}'$ and $\mathbf{F}$, respectively. $\langle \mathbf{F}', V' \rangle$ is a *generated submodel* of $\langle \mathbf{F}, V \rangle$ if for all p $\in$ Var, $V'(p) = V(p) \cap W'$.

It is a standard result that if $\langle W',R',V' \rangle$ is a generated submodel of $\langle W,R,V \rangle$, then for all $\alpha \in$ Form, $V'(\alpha) = V(\alpha) \cap W'$ (see, e.g. [HC], p. 80). This result is used to show the following:

**Lemma 2.6** Suppose $\mathbf{M}' = \langle W',R',V' \rangle$ is a generated submodel of the canonical model $\mathbf{M}_K = \langle W_K,R_K,V_K \rangle$ for K. Then for all $\alpha \in$ Form and $w \in W'$, $\alpha \in w$ iff $w \in V'(\alpha)$.

**Proof** $w \in W' \subseteq W$, so

$$\alpha \in w \iff w \in V_K(\alpha), \quad \text{by the fundamental lemma}$$
$$\iff w \in V_K(\alpha) \cap W', \quad \text{since } w \in W'$$

$\iff w \in V'(\alpha)$, since $V'(\alpha) = V_K(\alpha) \cap W'$.

**Lemma 2.7** If $\alpha_i = \Box^{k_i}\alpha_i^0$, $\alpha_i^0 \in$ Form, $k_i \geqslant 0$, $1 \leqslant i \leqslant n$, then for any modal axiom system S $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\alpha_i\}$, for $1 \leqslant i \leqslant n$.

**Proof** If $k_i = 0$, then $\{\alpha_i^0\} \vdash_S \{\alpha_i\}$ by (Refl), whence $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\alpha_i\}$ by (Mono). Otherwise, since $\{p\} \vdash_S \{\Box p\}$, we have by (Susbt) that $\{\alpha_i^0\} \vdash_S \{\Box\alpha_i^0\}$, $\{\Box\alpha_i^0\} \vdash_S \{\Box\Box\alpha_i^0\}, \cdots$, $\{\Box^{k_i-1}\alpha_i^0\} \vdash_S \{\Box^{k_i}\alpha_i^0\}$. So by the repeated application of (Trans), $\{\alpha_i^0\} \vdash_S \{\Box^{k_i}\alpha_i^0\}$, i.e., $\{\alpha_i^0\} \vdash_S \{\alpha_i\}$. Then by (Mono), $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\alpha_i\}$.

**Lemma 2.8** Suppose $\Gamma, \Theta \subseteq$ Form, $\Gamma \nvdash_S \Theta$. Let $\Box\Gamma = \{\Box^i\alpha \mid \alpha \in \Gamma, i < \omega\}$.

1) If $\Theta \neq \varnothing$, then for all $\beta \in \Theta$, $\Box\Gamma \cup \{\neg\beta\}$ is S-consistent.

2) If $\Theta = \varnothing$, then $\Box\Gamma$ is S-consistent.

**Proof** 1) Assume for contradiction that there is some $\beta \in \Theta$ for which $\Box\Gamma \cup \{\neg\beta\}$ is not S-consistent. Then there are $\alpha_1, \cdots, \alpha_n \in \Box\Gamma$ such that $\vdash_S \alpha_1 \& \cdots \& \alpha_n \to \beta$, and so $\{\alpha_1, \cdots, \alpha_n\} \vdash_S \{\beta\}$. But by 2.7, $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\alpha_i\}$, where $\alpha_i^0 \in \Gamma$ for $1 \leqslant i \leqslant n$. Then by (Mono), $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\alpha_1, \cdots, \alpha_n\}$, and by $(\text{Cut}_1)$, $\{\alpha_1^0, \cdots, \alpha_n^0\} \vdash_S \{\beta\}$. But then by (Mono), $\Gamma \vdash_S \Theta$, contrary to hypothesis.

2) Assume for contradiction that $\Box\Gamma$ is not S-consistent. Then there are $\alpha_1, \cdots, \alpha_n \in \Box\Gamma$ such that $\vdash_S \alpha_1 \& \cdots \& \alpha_n \to \bot$, and hence $\{\alpha_1, \cdots, \alpha_n\} \vdash_S \{\bot\}$. Now $\{\bot\} \vdash_S \varnothing$. So by $(\text{Cut}_1)$ $\{\alpha_1, \cdots, \alpha_n\} \vdash_S \varnothing$. We can now proceed as in (1) to obtain $\Gamma \vdash_S \Theta$.

**Theorem 2.9** For $\Gamma, \Theta \subseteq$ Form, $\Gamma \models \Theta(\mathsf{F})$ iff $\Gamma \vdash_K \Theta$.

**Proof** ( $\Longrightarrow$ ) Assume $\Gamma \nvdash_K \Theta$. We want to construct a frame $\mathbf{F} = \langle W, R \rangle$ and a valuation V for $\mathbf{F}$ so that for all $\alpha \in \Gamma$, $V(\alpha) = W$, and for all $\beta \in \Theta$, $V(\beta) \neq W$. This will mean that $\Gamma \not\models \Theta\{(\langle W, R \rangle)\}$, so that $\Gamma \not\models \Theta(\mathsf{F})$. By 2.8 $\Box\Gamma$ is K-consistent. Let

$$W = \{\Sigma \supseteq \Box\Gamma \mid \Sigma \text{ is maximally K-consistent}\}$$
$$R = \{\langle\Sigma,\Sigma'\rangle \mid \forall\alpha(\Box\alpha \in \Sigma \Rightarrow \alpha \in \Sigma'\}$$
$$V(p) = \{\Sigma \mid p \in \Sigma\},\ p \in \text{Var}$$

Now

1) $W \subseteq W_K$.

2) For $\Sigma,\Sigma' \in W_K$, if $\Sigma R \Sigma'$ and $\Sigma \in W$, then for any $\alpha \in \Box\Gamma$, $\Box\alpha \in \Box\Gamma$, and so $\Box\alpha \in \Sigma$ since $\Sigma \supseteq \Box\Gamma$. Thus $\alpha \in \Sigma'$ since $\Sigma R \Sigma'$ and so $\Sigma' \supseteq \Box\Gamma$, whence $\Sigma' \in W$.

3) $R = R_K \cap (W \times W)$.

4) For $p \in \text{Var}$, $V(p) = V_K(p) \cap W$.

So $\mathbf{M} = \langle W,R,V\rangle$ is a generated submodel of the canonical model. Now by 2.6 $V(\alpha) = W$ for all $\alpha \in \Gamma$. If $\Theta = \emptyset$, then $\Gamma \not\models \Theta(\{\langle W,R\rangle\})$. Otherwise by 2.8, $\Box\Gamma \cup \{\neg\beta\}$ is K-consistent for each $\beta \in \Theta$, so for each $\beta \in \Theta$ there is a $\Gamma_\beta \in W$ such that $\neg\beta \in \Gamma_\beta$. But then since $\mathbf{M}$ is a generated submodel of $\mathbf{M}_K$, we have by 2.6 that $V(\beta) \subseteq W - \{\Gamma_\beta\}$, so $V(\beta) \neq W$ for all $\beta \in \Theta$. Hence $\Gamma \not\models \Theta(\{\langle W,R\rangle\})$ and so $\Gamma \not\models \Theta(\mathbf{F})$.

( $\Leftarrow$ ) This is done by induction on members of $L_K$. For the basis, we immediately have $\{\bot\} \models \emptyset(\mathbf{F})$, $\{p\} \models \{\Box p\}(\mathbf{F})$, $p \in \text{Var}$, and $\{\alpha_1, \cdots, \alpha_n\} \models \{\alpha_{n+1}\}(\mathbf{F})$ whenever $\vdash_K \alpha_1 \& \cdots \& \alpha_n \rightarrow \alpha_{n+1}$, $\alpha_i \in \text{Form}$, $1 \leqslant i \leqslant n+1$, by the 'standard' completeness result for K (1.5). For the induction step, we consider the ($\text{Cut}_G$) rule. Here we have $\Gamma_0 \models \Theta,\gamma(\mathbf{F})$ and $\gamma,\Gamma'_0 \models \Theta'_0(\mathbf{F})$. We must show $\Gamma_0 \models \Gamma'_0(\mathbf{F})$. Now for all frames $\mathbb{F} = \langle W,R\rangle$ and valuations $V$ for $\mathbf{F}$:

$$(\forall\alpha \in \Gamma_0)(V(\alpha) = W) \Rightarrow (\exists\beta \in \Theta_0 \cup \{\gamma\})(V(\beta) = W) \tag{1}$$
$$(\forall\alpha \in \Gamma'_0 \cup \{\gamma\})(V(\alpha) = W) \Rightarrow (\exists\beta \in \Theta'_0)(V(\beta) = W) \tag{2}$$

Now choose $\mathbf{F}_0 = \langle W,R\rangle$ and $V_0$ a valuation for $\mathbf{F}_0$, and suppose $(\forall\alpha \in \Gamma_0 \cup \Gamma'_0)(V(\alpha) = W_0)$. Then by (1), $(\exists\beta \in \Theta_0 \cup \{\gamma\})(V(\beta) = W_0)$. Suppose $\beta \neq \gamma$.

Then $\beta \in \Theta_0$, so $(\exists\beta \in \Theta_0)(V(\beta) = W_0)$. Otherwise $(\forall\alpha \in \Gamma'_0 \cup \{\gamma\})(V(\alpha) = W_0)$, so by (2), $(\exists\beta \in \Theta'_0)(V(\beta) = W_0)$. In either case, $(\exists\beta \in \Theta_0 \cup \Theta'_0)(V(\beta) = W_0)$. Since $\mathbf{F}_0$ is arbitrary, $\Gamma_0, \Gamma'_0 \models \Theta_0, \Theta'_0(\mathbf{F})$.

As a corollary, we have that $\models(\mathbf{F})$ is *compact*, that is, if $\Gamma \models \Theta(\mathbf{F})$, then there are $\Gamma_0 \subsetneq_{fin} \Gamma$, $\Theta_0 \subsetneq_{fin} \Theta$ with $\Gamma_0 \models \Theta_0(\mathbf{F})$.

We now present a generalization of 2.9, which will apply to a number of well known modal systems.

**Definition 2.10** Let S be a modal system, $\mathbf{X}$ a class of frames. S is *complete with respect to* $\mathbf{X}$ if for all $\alpha \in$ Form and $\mathbf{F} \in \mathbf{X}$, $\vdash_S \alpha$ iff $\mathbf{F} \models \alpha$.

We have seen that validity of a modal formula on a frame corresponds to a certain kind of 'second-order validity. We will now show that truth of a modal formula on a model at a point $w \in |\mathbf{M}|$ corresponds to the validity of a first-order sentence on a structure derived from $\mathbf{M}$ and $w$. For $\alpha \in$ Form, $ST_1(\alpha) = [c_w/x]ST(\alpha)$, where $c_w$ is a new constant symbol and $\{P_i \mid i < \omega\}$ is a set of predicate constants, rather than variables. We then have that the validity of $\alpha$ on a model $\langle \mathbf{F}, V \rangle$ at a point $w$, is equivalent to the first-order validity of $ST_1(\alpha)$ on the structure $\langle W, R, V(p_0), V(p_1), \cdots, w \rangle$ where $P_i$, $i < \omega$, is interpreted as $V(p_i)$, and $c_w$ is interpreted as $w$.

From the preceding comments, we see that it is possible to define an ultraproduct $\mathbf{M}_U$ of modal models, using the ultraproduct construction for first-order structures, such that $\langle \mathbf{M}_U, w_U \rangle \models \sigma$ iff $\{i \mid \langle \mathbf{M}_i, w_i \rangle \models \sigma\} \in U$.

**Definition 2.11** Let $\{\mathbf{F}_i \mid i \in I\}$ be a family of frames $\mathbf{F}_i = \langle W_i, R_i \rangle$, $V_i$ a valuation for $\mathbf{F}_i$, $w_i \in W_i$, and $U$ an ultrafilter in $2^I$. $\prod_{i \in I} \mathbf{F}_i / U$, the *ultraproduct of the* $\mathbf{F}_i$'s *over* $U$ is defined in the standard way for first-order structures ([BS], 5.2.1). If there is a frame **F** such that $\mathbf{F}_i = \mathbf{F}$ for all $i \in I$, then we denote $\prod_{i \in I} \mathbf{F}_i/U$ by $\mathbf{F}^I/U$, the *ultrapower of* **F** *over* U. $\langle \mathbf{M}_U, w_U \rangle = \prod_{i \in I} \langle \mathbf{F}_i, V_i, w_i \rangle /U$ is then defined to be the ultraproduct $\prod_{i \in I} \langle W_i, R_i, V_i(p_0), V_i(p_1), \cdots, w_i \rangle /U$.

**Lemma 2.12** Suppose $\Gamma \subseteq$ Form. Let $\{\Gamma_0^i \mid i < \omega\}$ be an enumeration of the finite subsets of $\Gamma$. If for each $i < \omega$, there is an $\mathbf{M}_i = \langle W_i, R_i, V_i \rangle$ and $w_i \in W_i$ with $\langle \mathbf{M}_i, w_i \rangle \models \&(\Gamma_0^0 \cup \cdots \cup \Gamma_0^i)$, then there is an ultrafilter U in $2^\omega$ such that for $\alpha \in \Gamma$, $\langle \mathbf{M}_U, w_u \rangle \models \alpha$.

**Proof** Let F be the collection of cofinite subsets of $\omega$. Then the intersection of any finite subset of F is nonempty, so F is contained in an ultrafilter U in $2^\omega$ ([BS], 1.3.5). Now for any $\alpha \in \Gamma$, $\alpha \in \Gamma_0^i$ for some $i < \omega$. But then for all $j \geqslant i$, $\langle \mathbf{M}_j, w_j \rangle \models \alpha$, so $\{j \mid ST_1(\alpha)$ is valid on $\langle W_j, R_j, V_j(p_0), \cdots, w_j \rangle\} \supseteq \{j \mid j \geqslant i\} \in U$. Then by Los' Theorem ([BS], 5.2.1), $ST_1(\alpha)$ is valid on $\langle W_U, R_U, V_U(p_0), \cdots, w_u \rangle$, so $\langle \mathbf{M}_U, w_u \rangle \models \alpha$.

**Definition 2.13** Let $\{\mathbf{F}_i \mid i \in I\}$ be a non-empty family of frames, $\mathbf{F}_i = \langle W_i, R_i \rangle$, with $W_i \cap W_j = \varnothing$ whenever $i \neq j$. $\sum_{i \in I} \mathbf{F}_i$, the *disjoint union of the* $\mathbf{F}_i$'s is the frame $\langle \bigcup_{i \in I} W_i, \bigcup_{i \in I} R_i \rangle$. Note that by letting $W'_i = W_i \times \{i\}$, we can define $\sum_{i \in I} \mathbf{F}_i$ even if $W_i \cap W_j \neq \varnothing$ for some i,j. For a class **X** of frames, **U**(**X**) denotes the class of all disjoint unions formed from members of **X**.

**Theorem 2.14** Let **X** be a class of frames closed under the formation of ultraproducts, generated subframes and disjoint unions, and S a modal axiom system complete with respect to **X**. Then for $\Gamma, \Theta \subseteq$ Form, $\Gamma \vdash_S \Theta$ iff $\Gamma \models \Theta(\mathbf{X})$.

**Proof** ( $\Leftarrow$ ) Suppose $\Gamma \nvdash_S \Theta$. Let $\{\Gamma_0^i \mid i<\omega\}$ be an enumeration of the finite subsets of $\Box\Gamma$. Assuming $\Theta \neq \emptyset$, choose $\beta \in \Theta$. By 2.8, $\Box\Gamma \cup \{\neg\beta\}$ is S-consistent, so for $i<\omega$, $\nvdash_S \&(\Gamma_0^0, \cdots \Gamma_0^i) \rightarrow \beta$. So for some $\mathbf{M}_i = \langle W_i, R_i, V_i \rangle$ and $w_i \in W_i$, such that $\langle W_i, R_i \rangle \in \mathbf{X}$, $\langle \mathbf{M}_i, w_i \rangle \models (\Gamma_0^0 \cup \cdots \cup \Gamma_0^i)$ and $\langle \mathbf{M}_i, w_i \rangle \not\models \beta$. Let U be as in 2.13. Then for $\alpha \in \Box\Gamma$, $\langle \mathbf{M}_U, w_U \rangle \models \alpha$. However, $\{i \mid \langle \mathbf{M}_i, w_i \rangle \models \beta\} = \emptyset \notin U$, so by Los' Theorem $\langle \mathbf{M}_U, w_U \rangle \not\models \beta$. Let $\mathbf{M}_U'$ be the least generated submodel of $\mathbf{M}_U$ containing $w_u$. Then for $\alpha \in \Gamma$, $\mathbf{M}_U' \models \alpha$ and $\mathbf{M}_U' \not\models \beta$. Since $\beta$ was chosen arbitrarily, we have that for all $\beta \in \Theta$, there is some $\mathbf{F}_\beta \in \mathbf{X}$ and valuation $V_\beta$ for $\mathbf{F}_\beta$ such that for all $\alpha \in \Gamma$, $\langle \mathbf{F}_\beta, V_\beta \rangle \models \alpha$, while $\langle \mathbf{F}_\beta, V_\beta \rangle \not\models \beta$. Let $\mathbf{F} = \sum_{\beta \in \Theta} \mathbf{F}_\beta$. Define the valuation V for $\mathbf{F}$ by $V(p) = \bigcup_{\beta \in \Theta} V_\beta(p)$, for $p \in \mathbf{Var}$. A straightforward induction shows that for $\alpha \in \mathsf{Form}$, $V(\alpha) = \bigcup_{\beta \in \Theta} V_\beta(\alpha)$. So for all $\alpha \in \Gamma$ $\langle \mathbf{F}, V \rangle \models \alpha$, while for all $\beta \in \Theta$, $\langle \mathbf{F}, V \rangle \not\models \beta$. So $\Gamma \not\models \Theta(\{\mathbf{F}\})$. Since $\mathbf{F} \in \mathbf{X}$, $\Gamma \not\models \Theta(\mathbf{X})$. In case $\Theta = \emptyset$, we set $\Theta = \{\bot\}$ and proceed as above.

( $\Rightarrow$ ) As in the proof of 2.9, with $\mathbf{F}$ replaced by $\mathbf{X}$.

We may wonder if there are any modal axiom systems complete with respect to a class of frames which meets the closure conditions of 2.14. It is a standard result (cf. 3.7, 2.6) that a system S is complete with respect to $\mathbf{X}$ iff it is complete with respect to $\mathbf{U}(\mathbf{X})$ and $\mathbf{G}(\mathbf{X})$, the class of generated subframes of members of $\mathbf{X}$. Also, many well-known modal systems S (e.g. T, S4, S5) are complete with respect to a class $\mathbf{X}_S$ of frames which is first-order definable (cf 3.2) and hence closed under ultraproducts ([BS], 7.3.4). So we have for these systems that $\Gamma \vdash_S \Theta$ iff $\Gamma \models \Theta(\mathbf{UG}(\mathbf{X}_S))$.

## 3. A Survey of Modal Definability Results

In Chs. 1 and 2 we have been concerned primarily with completeness results, that is, with showing that various semantic notions such as validity and consequence can be characterized syntactically, via axiom systems and logics. We turn now to an examination of definability results based on the notion of 'validity on a frame' given in Ch. 1. In subsequent chapters we will use some ideas from Ch. 2 to extend the 'traditional' modal definability results examined in this chapter.

Traditionally ([Go], [GT], [vB1]), modal definability theory has been concerned with what can be 'said' about properties of frames using modal formulas. Some definitions are required to make this idea more precise.

**Definition 3.1** A class $\mathbf{X}$ of frames is *modal axiomatic* if there is a set $\Sigma \subseteq \mathsf{Form}$ such that $\mathbf{X} = \mathbf{Fr}(\Sigma) = \{\mathbf{F} \mid (\forall\alpha \in \Sigma)(\mathbf{F}\models\alpha)\}$.

We will use the terms class and property interchangeably. A property $\mathbf{X}$ is *modally definable* if $\mathbf{X}$ is modal axiomatic.

**Definition 3.2** ([BS], 7.1) A class $\mathbf{X}$ of frames is *Δ-elementary* if there is a set $\Sigma$ of first-order sentences such that $\mathbf{X} = \mathbf{Mod}(\Sigma) = \{\mathbf{F} \mid (\forall\phi \in \Sigma)(\phi \text{ is first-order valid on } \mathbf{F})\}$.

So a property $\mathbf{X}$ of frames is *first-order definable* if it is Δ-elementary. Some questions that arise now are the following ([vB2], p. 13):

(3.3) When is a property of frames which is modally definable first-order definable?

(3.4) When is a property of frames which is first-order definable modally definable?

(3.5) When is an arbitrary class of frames modal axiomatic?

The first two questions can be seen as comparing the 'expressive power' of modal formulas and first-order sentences. They are particularly interesting given the fact that, according to 1.3, modal formulas correspond to certain second-order sentences (when used to define properties of frames).

Our first step will be to examine various constructions on frames which preserve validity of modal formulas. Closure under these constructions is then a necessary condition for a class of frames to be modal axiomatic. Note that we will present a few results without proof, since they are well-known. Results using ideas needed in subsequent chapters will be presented in more detail.

**Lemma 3.6** If $\mathbf{F}'$ is a generated subframe of $\mathbf{F}$ and $\mathbf{F}\models\alpha$, $\alpha \in$ Form, then $\mathbf{F}'\models\alpha$.

**Proof** [HC], 5.8

**Lemma 3.7** If $\{\mathbf{F}_i \mid i \in I\}$ is a nonempty family of frames and $\mathbf{F}_i\models\alpha$, $\alpha \in$ Form, for $i \in I$, then $\sum_{i \in I} \mathbf{F}_i\models\alpha$.

**Proof** [vB2], 2.15

We are now in a position to answer 3.3. using the following:

**Lemma 3.8** Let **X** be a class of frames. If **X** is closed under isomorphism, generated subframes, disjoint unions and ultrapowers, then **X** is closed under ultraproducts.

**Proof** [Go], 16.4

**Definition 3.9** Structures $\mathbf{F}$ and $\mathbf{F}'$ for a first-order language $L$ are *elementarily equivalent*

($\mathbf{F}\equiv\mathbf{F}'$) if the same $L$-sentences are valid on both $\mathbf{F}$ and $\mathbf{F}'$. In particular, for frames $\mathbf{F}$ and $\mathbf{F}'$, $\mathbf{F}\equiv\mathbf{F}'$ if the same sentences of $L_R$, the language of one binary relation, are valid on both.

**Theorem 3.10** Let $\mathbf{X}$ be a modal axiomatic class of frames. Then $\mathbf{X}$ is first-order definable iff $\mathbf{X}$ is closed under elementary equivalence.

**Proof** ( $\Longrightarrow$ ) Say $\mathbf{X} = \mathbf{Mod}(\Sigma)$. If $\mathbf{F} \in \mathbf{X}$ and $\mathbf{F}'\equiv\mathbf{F}$, $\mathbf{F}' \in \mathbf{Mod}(\Sigma)$, so $\mathbf{F}' \in \mathbf{X}$.

( $\Longleftarrow$ ) If $\mathbf{X}$ is closed under elementary equivalence, it is closed under isomorphism and ultrapowers ([BS], 7.3.2). So by 3.6, 3.7 and 3.8 $\mathbf{X}$ is closed under ultraproducts. But then $\mathbf{X}$ is first-order definable ([BS], 7.3.4).

We now present two more constructions which preserve validity of formulas.

**Definition 3.11** Let $\mathbf{F} = \langle W,R\rangle$ and $\mathbf{F}' = \langle W',R'\rangle$. $f:W\rightarrow W'$ is a *p-morphism* if

$$(\forall w \in W)(\forall u \in W')((f(w)R'u) \Longleftrightarrow (\exists v \in W)(wRv \text{ and } f(v) = u))$$

$\mathbf{F}'$ is a *p-morphic image* of $\mathbf{F}$ if $f(W) = W'$. $\langle\mathbf{F}',V'\rangle$ is a *p-morphic image* of $\langle\mathbf{F},V\rangle$ if $\mathbf{F}'$ is a p-morphic image of $\mathbf{F}$ and for $p \in$ Var, $V(p) = f^{-1}(V'(p))$.

A straightforward inductive argument illustrates the following:

**Lemma 3.12** If $\mathbb{M}' = \langle W',R',V'\rangle$ is a p-morphic image of $\mathbb{M} = \langle W,R,V\rangle$, then for $\alpha \in$ Form $V(\alpha) = f^{-1}(V'(\alpha))$ where $f$ is a surjective p-morphism from $\langle W,R\rangle$ onto $\langle W',R'\rangle$.

**Lemma 3.13** Suppose $\mathbb{F}'$ is a p-morphic image of $\mathbb{F}$ and $\alpha \in$ Form. If $\mathbb{F}\models\alpha$ then $\mathbb{F}'\models\alpha$.

**Proof** Suppose $\mathbf{F}'\not\models\alpha$. Then there is a valuation $V'$ for $\mathbf{F}'$ with $v \notin V'(\alpha)$ for some $v \in W'$. Define a valuation $V$ for $\mathbf{F}$ by $V(p) = \{w \in W \mid f(w) \in V'(p)\}$ where $f$ is a surjective p-morphism from $\mathbf{F}$ onto $\mathbf{F}'$. Then $\langle W',R',V'\rangle$ is a p-morphic image of $\langle W,R,V\rangle$. Now since

$f(W) = W'$, $v = f(w)$ for some $w \in W$. But then $w \notin V(\alpha)$, or else by 3.12 $w \in f^{-1}(V'(\alpha))$ and $v=f(w) \in V'(\alpha)$. So $\langle \mathbf{F}, V\rangle \not\models \alpha$ and hence $\mathbf{F} \not\models \alpha$.

**Definition 3.14** ([vB2], 2.24) Let $\mathbf{F} = \langle W, R\rangle$, $\mathbf{F}' = \langle W', R'\rangle$. For $X \subseteq W$ define $l_R(X) = \{w \in W \mid (\forall v \in W)(wRv \Longrightarrow v \in X)\}$. $\mathbf{F}'$ is the *ultrafilter extension* of $\mathbf{F}$ ($\mathbf{F}' = ue(\mathbf{F})$) if

$$W' = \{u \subseteq 2^W \mid u \text{ is an ultrafilter in } 2^W\} \tag{1}$$

$$uR'u' \iff (\forall X \subseteq W)(l_R(X) \in u \Longrightarrow X \in u') \tag{2}$$

Given a valuation $V$ for $\mathbf{F}$, define the valuation $ue(V)$ for $ue(\mathbf{F})$ by $ue(V)(p) = \{u \mid V(p) \in U\}$.

Again by an inductive argument, we have the following

**Lemma 3.15** For $\mathbf{F} = \langle W, R\rangle$, $u \in |ue(\mathbf{F})|$, and $\alpha \in$ Form, $u \in ue(V)(\alpha)$ iff $V(\alpha) \in u$.

**Lemma 3.16** For $\mathbf{F} = \langle W, R\rangle$, $\alpha \in$ Form, if $ue(\mathbf{F}) \models \alpha$ then $\mathbf{F} \models \alpha$.

**Proof** Suppose $\mathbf{F} \not\models \alpha$. Let $w \in W - V(\alpha)$ where $V$ is a valuation witnessing $\mathbf{F} \not\models \alpha$. Let $u_w$ be the principle ultrafilter in $2^W$ generated by $w$. Then $V(\alpha) \notin u_w$, so $u_w \notin ue(V)(\alpha)$. Hence $\langle ue(\mathbf{F}), ue(V)\rangle \not\models \alpha$, and $ue(\mathbf{F}) \not\models \alpha$.

Considering 3.6, 3.7 and 3.13, we see that closure of **X** under disjoint unions, generated subframes and p-morphic images are necessary conditions for a class **X** to be modal axiomatic, as is the closure of −**X** under ultrafilter extensions, by 3.16. By [Go], 20.6, these conditions are also sufficient, under the assumption that **X** is closed under elementary equivalence. Since **X** is closed under elementary equivalence whenever **X** is first-order definable, this provides an answer to 3.4.

These results also point to an answer for 3.5, but not directly. A construction which 'combines' the generated subframe, p-morphic image and ultrafilter extension constructions, known as the *state-of-affairs* (SA) construction, is presented in [GT] (where its name is also explained). In general, ultrafilter extensions do not preserve validity of formulas, but SA-constructions do. The pertinent result is the following

**Lemma 3.17** ([GT], 3) A class **X** of frames is modal axiomatic iff **X** is closed under isomorphism, disjoint unions, and SA-constructions.

## 4. Definability Via Sequents

In this chapter, we introduce the notion of validity of a modal sequent on a frame. We will then demonstrate that it is possible to define properties of frames, not definable by modal formulas, by modal sequents. We will also examine some constructions that preserve validity of sequents.

**Definition 4.1** A sequent $\sigma = \langle\Gamma_0,\Theta_0\rangle$ is *valid on* a frame **F** if $\Gamma_0 \models \Theta_0(\{\mathbf{F}\})$. In this case we write $\mathbf{F}\models\sigma$. A class (property) **X** of frames is *sequent-axiomatic* (*definable by sequents*) if there is a set L of sequents such that $\mathbf{X} = \mathbf{Fr}(L) = \{\mathbf{F} \mid (\forall\sigma \in L)(\mathbf{F}\models\sigma)\}$.

**Proposition 4.2** For **X** a class of frames, let Seq(**X**) be the set of sequents for which $\sigma \in$ Seq(**X**) iff for all $\mathbf{F} \in \mathbf{X}$, $\mathbf{F}\models\sigma$. Then **X** is sequent-axiomatic iff $\mathbf{X} = \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$.

**Proof** ( $\Longrightarrow$ ) Obviously $\mathbf{X} \subseteq \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$. Since **X** is sequent-axiomatic $\mathbf{X} = \mathbf{Fr}(L)$ for some set L of sequents. Moreover, $L \subseteq \mathrm{Seq}(\mathbf{X})$. So if $\mathbf{F} \in \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$, $\mathbf{F} \in \mathbf{Fr}(L)$, and hence $\mathbf{F} \in \mathbf{X}$.

( $\Longleftarrow$ ) Clear.

Since sequents are composed of finite sets of formulas, we can establish a correspondence between sequents and $\Pi^1_1$ sentences, just as we did with modal formulas. In particular, $\sigma = \langle\Gamma_0,\Theta_0\rangle$ is valid on a frame **F** iff the sentence

$$\forall P_0 \cdots \forall P_n(\underset{\alpha\in\Gamma_0}{\&} \forall x ST(\alpha) \rightarrow \underset{\beta\in\Theta_0}{\bigvee} \forall x ST(\beta))$$

is second-order valid on **F**, where n is the largest index of any propositional variable occurring in $\Gamma_0 \cup \Theta_0$. Note that for any $L_m$–formula $\alpha$ and frame **F**, $\mathbf{F}\models\alpha$ iff $\mathbf{F}\models\langle\varnothing,\{\alpha\}\rangle$, so that the correspondence between sequents and $\Pi^1_1$ sentences is an extension of that between formulas and $\Pi^1_1$ sentences. It is our aim to show that this extension is proper, that is, to show that there are properties of frames definable by sequents but not by modal formulas. This is done by considering properties which are definable by sequents and showing that these properties are not preserved by certain formula-preserving constructions.

**Lemma 4.3** The sequent $\langle \varnothing, \{p, \neg\Box p\}\rangle$ is valid on a frame $\mathbf{F} = \langle W, R\rangle$ iff $(\forall w, v \in W)(wRv)$.

**Proof** ( $\Longrightarrow$ ) Suppose there are $w, v \in W$ with $w \not{R} v$. Then we can choose a valuation $V$ such that $\langle W, R, V\rangle \not\models \langle \varnothing, \{p, \neg\Box p\}\rangle$. Namely, let $V(p) = W - \{v\}$. Then $w \in V(\Box p)$, so $V(\neg\Box p) \neq W$

( $\Longleftarrow$ ) Suppose $(\forall x, y \in W)(xRy)$ and that for some $v \in W$ and valuation $V$ for $\mathbf{F}$, $v \notin V(p)$. Now for any $w \in W$, $wRv$ so that $w \notin V(\Box p)$, whence $w \in V(\neg\Box p)$. So $V(\neg\Box p) = W$.

Now the property $(\forall w, v \in W)(wRv)$ is not preserved by disjoint unions, although validity of formulas is. Thus we have an example of a property of frames, namely the universality of $R$, which is definable by sequents but not by formulas. Another interesting sequent-definable property of frames which is not preserved by disjoint unions is given in the following

**Lemma 4.4** The sequent $\sigma_n = \langle \varnothing, \{p_i \leftrightarrow p_j \mid 0 \leqslant i < j \leqslant 2^n\}\rangle$, $n < \omega$, is valid on $\mathbf{F}$ iff $\|\mathbf{F}\| \leqslant n$.

**Proof** ( $\Longleftarrow$ ) Suppose $\|\mathbf{F}\| = k \leqslant n$. Then for any valuation $V$ for $\mathbf{F}$, there are at most $2^k$ possible values of $V(p_i)$. Hence we must have $V(p_i) = V(p_j)$ for some $0 \leqslant i < j \leqslant 2^k$. So $\mathbf{F} \models \sigma_n$.

( $\Longrightarrow$ ) Suppose $\|\mathbf{F}\| > n$. Choose $X \subseteq |\mathbf{F}|$ with $|X| = k > n$. Let $X_0, \cdots X_{2^k-1}$ be an enumeration of the subsets of $X$. Now $2^k - 1 > 2^n$, so define $V$ with $V(p_j) = X_j$, $0 \leqslant j \leqslant 2^n$. Then $\langle \mathbf{F}, V\rangle \not\models \sigma_n$. So $\mathbf{F} \not\models \sigma_n$.

Lemmas 4.3 and 4.4 demonstrate that validity of sequents is not preserved by disjoint unions. The following result demonstrates that it is not preserved by generated subframes.

**Lemma 4.5** The sequent $\langle\{\neg p, \Box p\}, \emptyset\rangle$ is valid on a frame $\mathbf{F} = \langle W, R\rangle$ iff $(\exists w,v \in W)(wRv)$.

**Proof** ( $\Longrightarrow$ ) Suppose $(\forall w,v \in W)(w \not R v)$. Then for any valuation V for **F**, $V(\Box p) = W$. In particular, we can set $V(p) = \emptyset$ so that $V(\neg p) = W$, and also have $V(\Box p) = W$.

( $\Longleftarrow$ ) Suppose $w,v \in W$ and $wRv$. Now suppose that for some valuation V , $V(\neg p) = W$. Then $v \notin V(p)$ and $w \notin V(\Box p)$. So $\mathbf{F} \models \langle\{\neg p, \Box p\}, \emptyset\rangle$.

The preceding results demonstrate that sequents can be used to extend the 'expressive power' of the relational semantics for modal formulas. Thus the questions of Ch. 3 again become open, but now with respect to modal sequents rather than formulas. Our first step in obtaining some answers is an examination of constructions which preserve validity of sequents.

**Lemma 4.6** If $\mathbf{M}' = \langle W', R', V'\rangle$ is a p-morphic image of $\mathbf{M} = \langle W, R, V\rangle$ and for some $\alpha \in$ Form $V(\alpha) = W$, then $V'(\alpha) = W'$.

**Proof** Let f be a surjective p-morphism from **M** onto **M**'. Then $f^{-1}(V'(\alpha)) = V(\alpha) = W$. So $f(f^{-1}(V'(\alpha))) = f(W)$. Since f is onto, $f(f^{-1}(V'(\alpha))) = V'(\alpha)$ and $f(W) = W'$. So $V'(\alpha) = W'$.

**Theorem 4.7** If **F**' is a p-morphic image of **F**, then for any $\sigma = \langle\Gamma_0, \Theta_0\rangle$, if $\mathbf{F} \models \sigma$ then $\mathbf{F}' \models \sigma$.

**Proof** Suppose $\mathbf{F}' \not\models \sigma$. Then there is a valuation V' for **F**' such that $(\forall\alpha \in \Gamma_0)(V'(\alpha) = W')$ and $(\forall\beta \in \Theta_0)(V'(\beta) \neq W')$. Define a valuation V for **F** by $V(p) = \{w \in W \mid f(w) \in V'(p)\}$, $p \in$ Var. Then $\langle W', R', V'\rangle$ is a p-morphic image of

$\langle W,R,V\rangle$, so by 3.12 and 4.6, $(\forall\alpha \in \Gamma_0)(V(\alpha) = W)$ and $(\forall\beta \in \Theta_0)(V(\beta) \neq W)$. So $\mathbf{F}\not\models\sigma$.

**Theorem 4.8** For any sequent $\sigma = \langle\Gamma_0,\Theta_0\rangle$ and frame $\mathbf{F}$, if $ue(\mathbf{F})\models\sigma$ then $\mathbf{F}\models\sigma$.

**Proof** Suppose $\mathbf{F}\not\models\sigma$. Let the valuation $V$ for $\mathbf{F}$ witness this. Now we have that for $\alpha \in \Gamma_0$, $V(\alpha) = W$, so for every ultrafilter $u$ in $2^W$, $V(\alpha) \in u$ and hence $u \in ue(V)(\alpha)$, by 3.15. Moreover, for every $\beta \in \Theta_0$, there is some $w_\beta \in W$ with $w_\beta \notin V(\beta)$. If we let $u_\beta$ be the principle ultrafilter in $2^W$ generated by $w_\beta$ then $V(\beta) \notin u_\beta$ so $u_\beta \notin ue(V)(\beta)$. Hence $\langle ue(\mathbf{F}),ue(V)\rangle\not\models\sigma$ and so $ue(\mathbf{F})\not\models\sigma$.

We now know how the formula-preserving constructions of Ch. 3 stand with respect to preservation of sequents. We will examine one more sequent-preserving construction.

**Definition 4.9** Suppose $\langle I,\leqslant\rangle$ is a directed partial order and $\{\mathbf{F}_i \mid i \in I\}$ is a family of frames, $\mathbf{F}_i = \langle W_i,R_i\rangle$, with $\mathbf{F}_i$ a generated subframe of $\mathbf{F}_j$ whenever $i\leqslant j$. $\mathbf{F} = \langle W,R\rangle$ is the *direct union* of the $\mathbf{F}_i$'s ($\mathbf{F} = \bigcup_{i\in I}\mathbf{F}_i$) if $W = \bigcup_{i\in I} W_i$, $R = \bigcup_{i\in I} R_i$.

**Lemma 4.10** If $\mathbf{F} = \bigcup_{i\in I}\mathbf{F}_i$, then for any $X_0 \subseteq_{fin} |\mathbf{F}|$ there is some $i' \in I$ for which $X_0 \subseteq |\mathbf{F}_{i'}|$.

**Proof** Suppose $X_0 = \{w_1,\cdots,w_k\}$, $k<\omega$. Now for $1\leqslant j\leqslant k$, there is some $i_j$ with $w_j \in W_{i_j}$. Let $i'$ be an upper bound for $i_1,\cdots,i_k$. Then $X_0 \subseteq W_{i'}$.

**Lemma 4.11** If $\mathbf{F} = \bigcup_{i\in I}\mathbf{F}_i$, then for $i \in I$, $\mathbf{F}_i = \langle W_i,R_i\rangle$ is a generated subframe of $\mathbf{F} = \langle W,R\rangle$.

**Proof** Obviously $W_i \subseteq W$ and $R_i = (W_i\times W_i)\cap R$. Now suppose $w,w' \in W$, $w \in W_i$ and $wRw'$. There is some $j\geqslant i$ with $w' \in W_j$ and $wR_jw'$. Since $\mathbf{F}_i$ is a generated subframe of $\mathbf{F}_j$,

$w' \in W_i$.

**Theorem 4.12** Let $\mathbf{F} = \bigcup_{i \in I} \mathbf{F}_i$, $\sigma = \langle \Gamma_0, \Theta_0 \rangle$. If $\mathbf{F}_i \models \sigma$ for all $i \in I$, then $\mathbf{F} \models \sigma$.

**Proof** Suppose $\mathbf{F} \not\models \sigma$. Then there is a valuation $V$ for $\mathbf{F}$ with $V(\alpha) = |\mathbf{F}|$, for all $\alpha \in \Gamma_0$, while for all $\beta \in \Theta_0$, $V(\beta) \neq |\mathbf{F}|$. For each $\beta \in \Theta_0$, choose $w_\beta \notin V(\beta)$. Let $X_0 = \{w_\beta \mid \beta \in \Theta_0\}$. Let i' be as in Lemma 4.10, with $W_{i'} \supseteq X_0$. By 4.11 $\mathbf{F}_{i'}$ is a generated subframe of $\mathbf{F}$. Choose $V_{i'}$ such that $\langle \mathbf{F}_{i'}, V_{i'} \rangle$ is a generated submodel of $\langle \mathbf{F}, V \rangle$. So for $\alpha \in \Gamma_0$, $V_{i'}(\alpha) = V(\alpha) \cap W_{i'} = W_{i'}$. Moreover, for $\beta \in \Theta_0$, $w_\beta \in W - V(\beta)$ and $w_\beta \in W_{i'}$, so $X_\beta \in (W - V(\beta)) \cap W_{i'} = W_{i'} - V_{i'}(\beta)$. So $\mathbf{F}_i \not\models \sigma$.

**Corollary 4.13** The well-foundedness of $R$ is not definable by sequents.

**Proof** Let $\mathbf{F}_n = \langle \{i < \omega \mid 0 \leqslant i \leqslant n\}, \geqslant \rangle$, $n < \omega$. Then each $\mathbf{F}_n$ is well-founded. Now $\mathbf{F}_n$ is a generated subframe of $\mathbf{F}_{n+1}$, $n < \omega$, and so we may form the direct union $\bigcup_{n<\omega} \mathbf{F}_n$. But this direct union is just $\langle \omega, \geqslant \rangle$, which is not well-founded.

Interestingly, the inverse well-foundedness of $R$, that is the well-foundedness of $R^{-1} = \{\langle v, w \rangle \mid wRv\}$ is definable by sequents.

**Lemma 4.14** The sequent $\langle \{p \rightarrow \Diamond p\}, \{\neg p\} \rangle$ is valid on a frame $\mathbf{F} = \langle W, R \rangle$ iff $R^{-1}$ is well-founded on $W$.

**Proof** ( $\Longrightarrow$ ) Suppose $R^{-1}$ is not well-founded on $W$. Then there is a sequence $\{w_i\}_{i \in \omega}$ of members of $W$ with $w_i R w_{i+1}$, $i < \omega$. Letting $V(p) = \{w_i \mid i < \omega\}$, we have $V(p \rightarrow \Diamond p) = W$ and $V(\neg p) \neq W$. So $\mathbf{F} \not\models \langle \{p \rightarrow \Diamond p\}, \{\neg p\} \rangle$.

( $\Longleftarrow$ ) Suppose $R^{-1}$ is well-founded on $W$. So for any $X \subseteq W$ with $X \neq \varnothing$, there is an $w \in X$ such that for all $v \in X$, $w \not R v$. In particular, for any valuation $V$ for $\mathbf{F}$, if

$V(p) \neq \emptyset$ we have some $w \in V(p) - V(\Diamond p)$, so that $V(p \rightarrow \Diamond p) \neq W$. So if $V(p \rightarrow \Diamond p) = W$, $V(p) = \emptyset$ and $V(\neg p) = W$. So $\mathbf{F} \models \langle \{p \rightarrow \Diamond p\}, \{\neg p\} \rangle$.

It is a well-known result (see, e.g., [vB2], 2.21) that if $\alpha \in$ Form is not valid on all frames then it is invalid on some frame which is a finite irreflexive intransitive tree with no R-loops. From this it follows that the inverse well-foundedness of R is not definable by modal formulas.

We have established a number of necessary conditions for a class of frames to be sequent-axiomatic. In the next chapter we introduce algebraic semantics as a step toward determining whether these conditions are sufficient.

## 5. Algebraic Semantics

In this chapter we introduce modal algebras (MA's) and examine the notion of validity of a sequent on an MA. Having done so, we find it possible to characterize sequent-axiomatic classes of MA's using some well known results from first-order logic.

**Definition 5.1** A *modal algebra* (MA) is a structure $\mathbf{B} = \langle B, \cap, -, I \rangle$, where $\langle B, \cap, - \rangle$ is a boolean algebra, and $I$ is an operator satisfying $I(a \cap b) = Ia \cap Ib$, $a,b \in B$ and $I1 = 1$, where 1 denotes the maximum element of $\mathbf{B}$. For an MA $\mathbf{B} = \langle B, \cap, -, I \rangle$, $|\mathbf{B}|$ denotes $B$, the *underlying set* of $\mathbf{B}$.

**Definition 5.2** Let $\mathbf{B}$ be an MA, $\alpha(p_0, \cdots, p_{n-1}) \in$ Form. Then $f^{\mathbf{B}}_{\alpha}(a_0, \cdots, a_{n-1})$, the *n-ary polynomial on* $\mathbf{B}$ *induced by* $\alpha$ is defined inductively as follows:

$$f^{\mathbf{B}}_{p_i}(a_0, \cdots, a_{n-1}) = a_i,\ i<n$$
$$f^{\mathbf{B}}_{\neg\alpha}(a_0, \cdots, a_{n-1}) = -f^{\mathbf{B}}_{\alpha}(a_0, \cdots, a_{n-1})$$
$$f^{\mathbf{B}}_{\alpha\&\beta}(a_0, \cdots, a_{n-1}) = f^{\mathbf{B}}_{\alpha}(a_0, \cdots, a_{n-1}) \cap f^{\mathbf{B}}_{\beta}(a_0, \cdots, a_{n-1})$$
$$f^{\mathbf{B}}_{\Box\alpha}(a_0, \cdots, a_{n-1}) = I f^{\mathbf{B}}_{\alpha}(a_0, \cdots, a_{n-1})$$

It is easy to see that any MA polynomial $f(a_0, \cdots, a_n)$ is induced by a modal formula $\alpha_f(p_0, \cdots, p_n)$. If $\sigma = \langle \Gamma_0, \Theta_0 \rangle$ is a sequent, $\sigma$ is *valid on* $\mathbf{B}$ ($\mathbf{B} \models \sigma$) if the sentence

$$\forall \bar{x}( \mathop{\&}_{\alpha \in \Gamma_0} (f^{\mathbf{B}}_{\alpha}(\bar{x})=1) \rightarrow \bigvee_{\beta \in \Theta_0} (f^{\mathbf{B}}_{\beta}(\bar{x})=1))$$

in $L_{MA}$, the first-order language of MA's, is valid on $\mathbf{B}$. (Note that we abuse notation somewhat here, as we do not distinguish between terms $f_{\alpha}$ and functions $f^{\mathbf{B}}_{\alpha}: B \rightarrow B$). By $\bar{x}$ we mean $\langle x_0, \cdots, x_n \rangle$ where $n = \max\{i \mid i \text{ occurs in some } \alpha \in \Gamma_0 \cup \Theta_0\}$. A class $\mathbf{X}$ of MA's is *sequent-axiomatic* iff there is a set L of sequents such that $\mathbf{X} = \mathbf{Mal}(L) = \{\mathbf{B} \mid (\forall \sigma \in L)(\mathbf{B} \models \sigma)\}$. Note that as in 4.2, $\mathbf{X}$ is sequent-axiomatic iff $\mathbf{X} = \mathbf{Mal}(\mathrm{Seq}(\mathbf{X}))$. Before going on to characterizing sequent-axiomatic classes of MA's, we will demonstrate that $\mathbf{B}\models\langle\Gamma_0,\Theta_0\rangle$ for all MA's **B** iff $\Gamma_0\vdash_K\Theta_0$.

**Definition 5.3** The *Lindenbaum Algebra for* K is the MA $\mathbf{B}_K = \langle B_K, \cap, -, / \rangle$, where

$$B_K = \{ \|\alpha\| \mid \alpha \in \text{Form}\} \text{ where } \|\alpha\| = \{\beta \in \text{Form} \mid \vdash_K \alpha\leftrightarrow\beta\}$$
$$\|\alpha\| \cap \|\beta\| = \|\alpha\&\beta\|$$
$$-\|\alpha\| = \|\neg\alpha\|$$
$$/\|\alpha\| = \|\Box\alpha\|$$

It is shown in [Lem], 11, that $\mathbf{B}_K$ is a well-defined MA.

**Definition 5.4** Let $\mathbf{B} = \langle B, \cap, -, / \rangle$ and $\mathbf{B}' = \langle B', \cap', -', /' \rangle$ be MA's. Then $f{:}B\rightarrow B'$ is an *MA-homomorphism* if

$$f(a\cap b) = f(a)\cap' f(b)$$
$$f(-a) = -'f(a)$$
$$f(/a) = /'f(a)$$

If $f(B) = B'$, $\mathbf{B}'$ is a *homomorphic image* of **B**. If **X** is a class of MA's, **H(X)** denotes the class of all homomorphic images of members of **X**.

**Definition 5.5** Let $\mathbf{B} = \langle B, \cap, -, / \rangle$ be an MA. $F \subseteq B$ is a *filter* in **B** if it is a filter in the BA $\langle B, \cap, - \rangle$. For $a,b \in B$, $a\equiv_F b$ if for some $c \in F$, $a\cap c = b\cap c$. For $a \in B$, $a/F = \{b \in B \mid a\equiv_F b\}$.

It is a standard result ([BS], 1.4.3) that if $\mathbf{B} = \langle B, \cap, - \rangle$ is a BA and F is a filter in **B**, then $\equiv_F$ is an equivalence relation on B, and so we can define the set $B/F = \{a/F \mid a \in B\}$. Furthermore, $f{:}B\rightarrow B/F$ defined by $f(a) = a/F$ is a well defined homomorphism of **B** onto $\mathbf{B}/F = \langle B/F, \cap', -' \rangle$, where $a/F \cap' b/F = (a\cap b)/F$ and $-'(a/F) = (-a)/F$. We can extend this result to MA's as follows:

**Lemma 5.6** If $\mathbf{B} = \langle B, \cap, -, / \rangle$ is an MA and F is a filter in **B** which is closed under $/$ (i.e., $a \in B \Longrightarrow /a \in B$), then $f: B \rightarrow B/F$ is a well defined MA-homomorphism of **B** onto $\mathbf{B}/F = \langle B/F, \cap', -', /' \rangle$, where $/'(a/F) = (/a)/F$.

**Proof** We need to show that if $a/F = b/F$ then $/'a/F = /'b/F$. Now if $a/F = b/F$, $a \cap c = b \cap c$ for some $c \in F$. Then $/(a \cap c) = /(b \cap c)$, so $/a \cap /c = /b \cap /c$ and $/c \in F$, so $(/a)/F = (/b)/F$. Thus $/'a/F = /'b/F$, so f is well defined. It is obvious that f is a homomorphism. Since $B/F = \{a/F \mid a \in B\} = \{f(a) \mid a \in B\} = f(B)$, **B**/F is a homomorphic image of **B**. So **B**/F is a well-defined MA.

We call **B**/F the *quotient MA of* **B** *modulo* F. The function f is the *canonical homomorphism* of **B** onto **B**/F.

**Lemma 5.7** Let **B** be an MA, F an $/$-closed filter in **B**, f the canonical homomorphism from **B** onto **B**/F. Then $f(a) = 1/F$ iff $a \in F$.

**Proof** ( $\Longrightarrow$ ) $f(a) = a/F = 1/F$, so $a \cap c = 1 \cap c = c$ for some $c \in F$. Thus $c \leqslant a$, so $a \in F$.

( $\Longleftarrow$ ) If $a \in F$, $a \equiv_F 1$ since $a \cap a = 1 \cap a = a$. So $f(a) = a/F = 1/F$.

**Theorem 5.8** For $\langle \Gamma_0, \Theta_0 \rangle \subseteq_{fin} \mathrm{Form}$, $\Gamma_0 \vdash_K \Theta_0$ iff for all $\mathbf{B} \in \mathbf{H}(\{\mathbf{B}_K\})$, $\mathbf{B} \models \langle \Gamma_0, \Theta_0 \rangle$.

**Proof** ( $\Longrightarrow$ ) By [Lem], 12, if $\vdash_K \alpha_1 \& \cdots \& \alpha_n \rightarrow \alpha_{n+1}$ then $\forall \bar{x}(f^{\mathbf{B}}_{\alpha_1 \& \cdots \& \alpha_n \rightarrow \alpha_{n+1}}(\bar{x}) = 1)$ is valid on **B** for any MA **B**. But then $\mathbf{B} \models \langle \{\alpha_1, \cdots, \alpha_n\}, \{\alpha_{n+1}\} \rangle$. Also for any MA **B**, if $f^{\mathbf{B}}_\alpha(\bar{x}) = 1$, then $/f^{\mathbf{B}}_\alpha(\bar{x}) = 1$, so $\mathbf{B} \models \langle \{\alpha\}, \{\Box\alpha\} \rangle$. It is routine to show that the rules sufficient for defining $L_K$, restricted to sequents, preserve algebraic validity. The result is then obtained by induction on sequents.

( $\Longleftarrow$ ) Suppose $\Gamma_0 \not\models \Theta_0$. By 2.8, $\Box\Gamma_0$ is K-consistent, and $F_0 = \{ \| \alpha \| \mid \alpha \in \Box\Gamma_0 \}$ has the finite intersection property in $\mathbf{B}_K$, and so generates a proper filter F in $\mathbf{B}_K$ ( [BS], 1.2.8 ).

Assuming that F is $I$—closed, we can conclude that $\mathbf{B}_K/F$ is a homomorphic image of $\mathbf{B}_K$ and $a/F = 1/F$ iff $a \in F$. Hence for any $\alpha \in \Gamma_0$, $f_\alpha^{\mathbf{B}_K/F}(f(\| p_0 \|), \cdots, f(\| p_n \|)) = 1/F$. So if $\Theta_0 = \emptyset$, we immediately have $\mathbf{B}_K/F \not\models \langle \Gamma_0, \Theta_0 \rangle$. Otherwise, since F is the smallest filter containing $F_0$ and for $\beta \in \Theta_0$ there is a filter $F_\beta$ extending F such that $\| \neg\beta \| \in F_\beta$ (since $\Box\Gamma_0 \cup \{\neg\beta\}$ is K—consistent), we have that for $\beta \in \Theta_0$ $\| \beta \| \notin F$ and so $f(\| \beta \|) \neq 1/F$, where f is the canonical homomorphism. So for $\beta \in \Theta_0$, $f_\beta^{\mathbf{B}_K/F}(f(\| p_0 \|), \cdots, f(\| p_n \|)) \neq 1/F$. So $\mathbf{B}_K/F \not\models \langle \Gamma_0, \Theta_0 \rangle$.

It remains to show that F is $I$—closed. By [BS], 1.2.8, $a \in F$ iff $a \geqslant a_1 \cap \cdots \cap a_n$ for some $n \geqslant 1$ and $a_1, \cdots a_n \in F_0$. Then $Ia \geqslant I(a_1 \cap \cdots \cap a_n) = Ia_1 \cap \cdots \cap Ia_n$. But for $1 \leqslant i \leqslant n$, $a_i = \| \alpha_i \|$ for some $\alpha_i \in \Gamma_0$, so $Ia_i = \| \Box\alpha_i \| \in F_0$. Hence $Ia \in F$.

By definition, a sequent is valid on an MA **B** if some corresponding universal sentence in $L_{MA}$ is valid on **B**. It is our aim now to show that any universal sentence in $L_{MA}$ holds in **B** iff some corresponding set of sequents is valid on **B**. This will enable us to characterize sequent axiomatic classes of MA's.

**Definition 5.9** A class **X** of MA's is *universal* if $\mathbf{X} = \mathbf{Mod}(\Phi)$ for some set $\Phi$ of universal $L_{MA}$-sentences. Note that as in 4.2, **X** is universal iff $\mathbf{X} = \mathbf{Mod}(\mathrm{Th}_\forall(\mathbf{X}))$, where $\mathrm{Th}_\forall(\mathbf{X})$ is the set of universal sentences valid on every member of **X**.

**Definition 5.10** ( [Gr], 7.46.1 ) A set $\Phi$ of universal sentences, written as a set of open formulas, is in *normal form* if every $\phi \in \Phi$ is of the form $\theta_1 \vee \cdots \vee \theta_n$, where $\theta_i$, $1 \leqslant i \leqslant n$, is an atomic or negated atomic formula.

**Lemma 5.11** Every set $\Phi$ of universal sentences is equivalent to a set $\Phi'$ of universal sentences in normal form (i.e., $\mathbf{Mod}(\Phi) = \mathbf{Mod}(\Phi')$ ).

**Proof** For $\phi \in \Phi$, consider $\phi$ as an open sentence $\phi_0 \& \cdots \& \phi_n$ in conjunctive normal form. Add $\phi_1, \cdots \phi_n$ to $\Phi'$.

**Lemma 5.12** Every atomic $L_{MA}$-sentence is equivalent to a sentence of the form $f(\bar{x}) = 1$ for some MA-polynomial f .

**Proof** $f(\bar{x}) = g(\bar{x})$ holds in an MA **B** iff $(-f(\bar{x}) \cup g(\bar{x})) \cap (f(\bar{x}) \cup -g(\bar{x})) = 1$ holds in **B**.

**Lemma 5.13** Let $\phi$ be a universal $L_{MA}$-sentence of the form $\forall \bar{x}(\theta_1(\bar{x}) \vee \cdots \vee \theta_n(\bar{x}))$, $\theta_i$ atomic or negated atomic, $1 \leqslant i \leqslant n$. Then there is a sequent $\sigma_\phi = \langle \Gamma_0, \Theta_0 \rangle$ such that for any MA **B**, $\phi$ is valid on **B** iff $\mathbf{B} \models \sigma_\phi$.

**Proof** By 5.12 we can assume that each $\theta_i$ is of the form $f(\bar{x}) = 1$ or $\neg(f(\bar{x}) = 1)$. Let

$$\Gamma_0 = \{\alpha_f \mid f \text{ appears in some } \theta_i \text{ of the form } \neg(f(\bar{x}) = 1),\ 1 \leqslant i \leqslant n\}$$
$$\Theta_0 = \{\alpha_f \mid f \text{ appears in some } \theta_i \text{ of the form } f(\bar{x}) = 1,\ 1 \leqslant i \leqslant n\}$$

The result then follows by definition 5.2.

**Theorem 5.14** A class **X** of MA's is sequent-axiomatic iff it is universal.

**Proof** ( $\Longrightarrow$ ) By 5.2

( $\Longleftarrow$ ) Say **X** = **Mod**($\Phi$) for a set $\Phi$ of universal sentences. By 5.11, **X** = **Mod**($\Phi'$) where $\Phi'$ is a set of universal sentences in normal form. Let $L = \{\sigma_\phi \mid \phi \in \Phi\}$. Then by 5.13, **X** = **Mal**(L). So **X** is sequent-axiomatic.

**Definition 5.15** Let **B**,**B**′ be MA's. **B**′ is *isomorphically embedded* in **B** ($\mathbf{B}' \subseteq \mathbf{B}$) if there is an injective MA-homomorphism from **B**′ into **B**. For a class **X** of MA's, **S**(**X**) denotes the class of MA's isomorphically embedded in members of **X**. For a family $\{\mathbf{B}_i \mid i \in I\}$ of MA's and ultrafilter U in $2^I$, $\prod_{i \in I} \mathbf{B}_i/U$, the *ultraproduct of the* $\mathbf{B}_i$'s *over* U is defined in the standard way for first-order structures ([BS], 5.1.3). For a class $\mathbf{X}$ of MA's, $\mathbf{P}_U(\mathbf{X})$ denotes the class of ultraproducts of members of $\mathbf{X}$.

**Theorem 5.16** A class $\mathbf{X}$ of MA's is universal iff $\mathbf{X} = \mathbf{SP}_U(\mathbf{X})$

**Proof** ( $\Longrightarrow$ ) $\mathbf{X} \subseteq \mathbf{SP}_U(\mathbf{X})$, so it suffices to show that $\mathbf{X}$ is closed under ultraproducts and isomorphic embeddings. Now by Los' Theorem ([BS], 5.2.1) ultraproducts preserve the validity of all first order sentences, and by ([CK], 5.2.4), isomorphic embeddings preserve validity of universal sentences, so these closure conditions do hold.

( $\Longleftarrow$ ) We will show that, assuming $\mathbf{X} = \mathbf{SP}_U(\mathbf{X})$, that $\mathbf{X} = \mathbf{Mod}(\mathrm{Th}_\forall(\mathbf{X}))$. Obviously, $\mathbf{X} \subseteq \mathbf{Mod}(\mathrm{Th}_\forall(\mathbf{X}))$. Suppose $\mathbf{B} \in \mathbf{Mod}(\mathrm{Th}_\forall(\mathbf{X}))$. Let $\{\phi_i \mid i<\omega\}$ be the set of existential sentences valid on $\mathbf{B}$. For each $\phi_i$, there is a $\mathbf{B}_i \in \mathbf{X}$ such that $\phi_i$ is valid on $\mathbf{B}_i$, since otherwise the universal sentence equivalent to $\neg\phi_i$ is in $\mathrm{Th}_\forall(\mathbf{X})$, which means $\phi_i$ is not valid on $\mathbf{B}$, a contradiction. Let $J_j = \{i \mid \phi_j \text{ is valid on } \mathbf{B}_i\}$. Now for any $j_1, \cdots, j_n$ we can find a common element in $J_{j_1}, \cdots, J_{j_n}$, since without loss of generality, no two of $\phi_{j_1}, \cdots, \phi_{j_n}$ have any variables in common, and so their conjunction is equivalent to an existential sentence which must be validated by some $\mathbf{B}_i \in \mathbf{X}$.. In other words, $\{J_j \mid j \in \omega\}$ has the finite intersection property, and so can be extended to an ultrafilter $U$ in $2^\omega$ ([BS], 1.3.5 ). Let $\mathbf{B}' = \prod_{i \in I} \mathbf{B}_i/U$. By Los' Theorem, every existential sentence valid on $\mathbf{B}$ is valid on $\mathbf{B}'$ (since $\{i \mid \phi_j \text{ is valid on } \mathbf{B}_i\} = J_j \in U$). Thus every universal sentence valid on $\mathbf{B}'$ is valid on $\mathbf{B}$. But then $\mathbf{B}$ can be isomorphically embedded in an ultrapower of $\mathbf{B}'$ ([BS], 9.3.8), and so $\mathbf{B} \in \mathbf{SP}_U\mathbf{P}_U(\mathbf{X})$. Then by [BS], 6.2.7, $\mathbf{B} \in \mathbf{SP}_U(\mathbf{X})$.

**Corollary 5.17** A class $\mathbf{X}$ of MA's is sequent-axiomatic iff $\mathbf{X} = \mathbf{SP}_U(\mathbf{X}) = \mathbf{Mal}(\mathrm{Seq}(\mathbf{X}))$.

## 6. General Frames

Suppose $\mathbf{F} = \langle W,R \rangle$ is a frame. Letting $/_R(X) = \{x \in W \mid (\forall y)(xRy \Longrightarrow y \in X)\}$ for $X \subseteq W$, we have $\langle 2^W, \cap, -, /_R \rangle$ is an MA, where $\langle 2^W, \cap, - \rangle$ is the power set BA of $W$. This MA, the *dual* MA of $\mathbf{F}$, is denoted $\mathbf{F}^+$. It is easy to see that there are MA's which are not of the form $\mathbf{F}^+$ for any frame $\mathbf{F}$, since not all BA's are power set BA's. In this chapter we will alter the relational semantics to obtain frames that correspond more closely to MA's. The results of Ch. 5 will then be used to characterize sequent-axiomatic classes of these frames.

**Definition 6.1** A *general frame* is a structure $\mathbf{F} = \langle W,R,P \rangle$, where $\langle W,R \rangle$ is a frame and $P \subseteq 2^W$ is closed under $\cap$, $-$, and $/_R$. If $\sigma = \langle \Gamma_0, \Theta_0 \rangle$ is a sequent and $\mathbf{F} = \langle W,R,P \rangle$ is a general frame, then $\sigma$ is *valid on* $\mathbf{F}$, $(\mathbf{F} \models \sigma)$ if for all valuations $V{:}\mathrm{Form} \rightarrow P$,

$$(\forall \alpha \in \Gamma_0)(V(\alpha) = W) \Longrightarrow (\exists \beta \in \Theta_0)(V(\beta) = W)$$

(By a valuation $V$ for a general frame $\langle W,R,P \rangle$ we mean a valuation $V$ for $\langle W,R \rangle$ with $\mathrm{Im}(V) \subseteq P$).

If $\mathbf{F} = \langle W,R,P \rangle$ is a general frame, $\mathbf{F}_0$ denotes the standard frame $\langle W,R \rangle$. Now for any sequent $\sigma = \langle \Gamma_0, \Theta_0 \rangle$, $\langle W,R \rangle \models \sigma$ iff the general frame $\langle W,R,2^W \rangle \models \sigma$. Moreover for any general frame $\mathbf{F}$, if $\mathbf{F}_0 \models \sigma$ then $\mathbf{F} \models \sigma$. So $\sigma$ is valid on all general frames iff $\sigma$ is valid on all standard frames, which is the case iff $\Gamma_0 \vdash \Theta_0$.

**Definition 6.2** Let $\mathbf{F} = \langle W,R,P \rangle$ be a general frame. Define $\mathbf{F}^+$, the *dual MA of* $\mathbf{F}$, as follows: $\mathbf{F}^+ = \langle P, \cap, -, /_R \rangle$. For a standard frame $\langle W,R \rangle$, $\langle W,R \rangle^+ = \langle W,R,2^W \rangle^+$.

By a straightforward inductive argument, we can show that for $\alpha(p_0, \cdots, p_n) \in \mathrm{Form}$, $\mathbf{F}$ a general frame, and $V$ a valuation for $\mathbf{F}$, $f_\alpha^{\mathbf{F}^+}(V(p_0), \cdots, V(p_n)) = V(\alpha)$, and so we have the following

**Lemma 6.3** For a general frame $\mathbf{F}$ and sequent $\sigma$, $\mathbf{F}\models\sigma$ iff $\mathbf{F}^{+}\models\sigma$.

**Proof** By 5.2, 6.1 and the preceding remarks.

We now introduce a way to obtain general frames from MA's.

**Definition 6.4** ( [Go], 10.1 ) Let $\mathbf{B} = \langle B,\cap,-,/\rangle$ be an MA. The *dual frame* of $\mathbf{B}$ is the general frame $\mathbf{B}_{+} = \langle W_{\mathbf{B}},R_{\mathbf{B}},P^{\mathbf{B}}\rangle$, where

$$W_{\mathbf{B}} = \{w \mid w \text{ is an ultrafilter in } \mathbf{B}\}$$
$$wR_{\mathbf{B}}v \text{ iff } \{a \mid /a \in w\} \subseteq v$$
$$P^{\mathbf{B}} = \{\,|a|^{\mathbf{B}} \mid a \in B\} \text{ where } |a|^{\mathbf{B}} = \{w \in W_{\mathbf{B}} \mid a \in w\}$$

By [Go], 10.2, we have $|a|^{\mathbf{B}} = |b|^{\mathbf{B}}$ iff $a = b$, $W_{\mathbf{B}}-|a|^{\mathbf{B}} = |-a|^{\mathbf{B}}$, $|a|^{\mathbf{B}}\cap|b|^{\mathbf{B}}= |a\cap b|^{\mathbf{B}}$, and $/_{R_{\mathbf{B}}}(|a|^{\mathbf{B}}) = |/a|^{\mathbf{B}}$, so that $\mathbf{B}_{+}$ is indeed a general frame. This also means that the map $f:B\rightarrow P^{\mathbf{B}}$ defined by $f(a) = |a|^{\mathbf{B}}$ is an MA-isomorphism, so we have

**Lemma 6.5** $\mathbf{B}\simeq(\mathbf{B}_{+})^{+}$, for any MA $\mathbf{B}$.

**Corollary 6.6** For any sequent $\sigma$ and MA $\mathbf{B}$, $\mathbf{B}\models\sigma$ iff $\mathbf{B}_{+}\models\sigma$.

**Proof** $\mathbf{B}_{+}\models\sigma$ iff $(\mathbf{B}_{+})^{+}\models\sigma$, by 6.3, iff $\mathbf{B}\models\sigma$ by 6.5.

We will now examine some sequent preserving constructions of general frames.

**Definition 6.7** Let $\mathbf{F} = \langle W,R,P\rangle$, $\mathbf{F}' = \langle W',R',P'\rangle$ be general frames. A function $f:W\rightarrow W'$ is a *p-morphism* if f is a p-morphism from $\mathbf{F}_0$ to $\mathbf{F}_0'$ and for $X \in P'$, $f^{-1}(X) \in P$. If $f(W) = W'$, $\mathbf{F}'$ is a *p-morphic image* of $\mathbf{F}$.

**Lemma 6.8** If $\mathbf{F}' = \langle W',R',P'\rangle$ is a p-morphic image of $\mathbf{F} = \langle W,R,P\rangle$, then $\mathbf{F}'^{+}$ is isomorphically embedded in $\mathbf{F}^+$.

**Proof** ([Go], 5.3) Suppose f is a surjective p-morphism of **F** onto **F**'. We will show that $f^+: P' \to P$ defined by $f^+(X) = f^{-1}(X)$ is an injective MA-homomorphism. Suppose $X, Y \subseteq W'$. Obviously $f^+(-X) = f^{-1}(-X) = -f^{-1}(X) = -f^+(X)$, and likewise $f^+(X \cap Y) = f^+(X) \cap f^+(Y)$. So to show that f is a MA-homomorphism, we need $f^+(/_{R'}(X)) = /_R(f^+(X))$. Suppose $w \notin f^+(/_{R'}(X))$. Then we have $u \in W'$ with $f(w)R'u$ and $u \notin X$. Now by 6.1, we have $v \in W$ with wRv, $f(v) = u \notin X$. So wRv and $v \notin f^+(X)$, whence $w \notin /_R(f^+(X))$. Now suppose $w \notin /_R(f^+(X))$. Now by 6.1 $f(w)R'f(w)$, so $f(w) \notin /_R'(X)$ and thus $w \notin f^+(/_R'(X))$. To see that $f^+$ is injective suppose $f^+(X) = f^+(Y)$. Then $f(f^+(X)) = f(f^+(Y))$, and since f is surjective, $X = Y$.

**Lemma 6.9** Let **F**, **F**' be general frames, $\sigma$ a sequent. If **F**' is a p-morphic image of **F** and $\mathbf{F} \models \sigma$ then $\mathbf{F}' \models \sigma$.

**Proof** If $\mathbf{F} \models \sigma$, $\mathbf{F}^+ \models \sigma$ (6.3). But by 6.8, $\mathbf{F}'^+ \subseteq \mathbf{F}^+$, so $\mathbf{F}'^+ \models \sigma$ by 5.16. Then by 6.3, $\mathbf{F}' \models \sigma$.

In [Go]. Ch. 7, the ultraproduct construction is extended to general frames. The difficulty in doing this is ensuring that for an ultraproduct $\mathbf{F}_U = \langle W_U, R_U, P_U \rangle$, we in fact have that $P_U \subseteq 2^{W_U}$.

**Definition 6.10** Suppose $\{\mathbf{F}_i \mid i \in I\}$ is a family of general frames, ($\mathbf{F}_i = \langle W_i, R_i, P_i \rangle$), and U is an ultrafilter in $2^I$. For $f \in \prod_{i \in I} W_i$, $\tau \in \prod_{i \in I} P_i$, let $[f, \tau] = \{i \mid f(i) \in \tau(i)\}$. For $\tau/U \in \prod_{i \in I} P_i/U$, let $X_{\tau/U} = \{f/U \mid [f, \tau] \in U\}$. Then $X_{\tau/U} \subseteq W_U$. The *ultraproduct of the* $\mathbf{F}_i$'s *over* U is the structure $\mathbf{F}_U = \prod_{i \in I} \mathbf{F}_i/U = \langle W_U, R_U, P_U \rangle$, where $\langle W_U, R_U \rangle = \prod_{i \in I} (\mathbf{F}_i)_0/U$ and $P_U = \{X_{\tau/U} \mid \tau \in \prod_{i \in I} P_i\}$

It is shown in [Go], 7.5 and 7.7, that $X_{\tau/U}$ is well-defined and that $\mathbf{F}_U$ is indeed a general frame (i.e., $P_U$ meets the necessary closure conditions). This also provides the following lemma.

**Lemma 6.11** If $\{\mathbf{F}_i \mid i \in I\}$ is a family of general frames and U an ultrafilter in $2^I$ then $\prod_{i\in I}\mathbf{F}_i^+/U \simeq (\prod_{i\in I}\mathbf{F}_i/U)^+$.

Note that 6.11 does not hold in general for standard frames, as is shown in [Go], Ch. 17.

We can now show that ultraproducts of general frame preserve validity of sequents.

**Lemma 6.12** Suppose $\{\mathbf{F}_i \mid i \in I\}$ is a family of general frames, U an ultrafilter in $2^I$ and $\sigma$ a sequent. If for all $i \in I$, $\mathbf{F}_i \models \sigma$, then $\prod_{i\in I}\mathbf{F}_i/U \models \sigma$.

**Proof** By 6.3 $(\mathbf{F}_i)^+ \models \sigma$, $i \in I$. So $\{i \mid (\mathbf{F}_i)^+ \models \sigma\} = I \in U$. But then $\prod_{i\in I}(\mathbf{F}_i)^+/U \models \sigma$ by Los' Theorem. So by 6.11 $(\prod_{i\in I}\mathbf{F}_i/U)^+ \models \sigma$ and thus by 6.3, $\prod_{i\in I}\mathbf{F}_i/U \models \sigma$.

With the following lemma, we will be able to characterize sequent-axiomatic classes of general frames.

**Lemma 6.13** Let **B**, **B**' be MA's. If $\mathbf{B}' \subseteq \mathbf{B}$, then $\mathbf{B}'_+$ is a p-morphic image of $\mathbf{B}_+$.

**Proof** ([Go], 10.9) Let f be a injective MA-homomorphism from **B**' into **B**. Then $f_+ : \mathbf{B}_+ \to \mathbf{B}'_+$ defined by $f_+(w) = \{b \in B' \mid f(b) \in w\}$ is a surjective p-morphism.

**Definition 6.14** For a general frame **F**, the *bidual* of **F** is the frame $(\mathbf{F}^+)_+$.

**Definition 6.15** For a class **X** of general frames, $\mathbf{X}^+ = \{\mathbf{B} \mid (\exists \mathbf{F} \in \mathbf{X})(\mathbf{B} \simeq \mathbf{F}^+)\}$.

**Theorem 6.16** A class $\mathbf{X}$ of general frames is sequent-axiomatic iff $\mathbf{X}$ is closed under p-morphic images, ultraproducts and biduals, while $-\mathbf{X}$ is closed under biduals.

**Proof** ( $\Longrightarrow$ ) Closure under p-morphic images and ultraproducts follows by 6.9 and 6.12, respectively. For closure of $\mathbf{X}$ and $-\mathbf{X}$ under biduals, note that by 6.3 and 6.6, for any sequent $\sigma$, $\mathbf{F}\models\sigma$ iff $(\mathbf{F}^+)_+\models\sigma$.

( $\Longleftarrow$ ) We will show that $\mathbf{X} = \mathbf{GFr}(\mathrm{Seq}(\mathbf{X}))$, where $\mathbf{GFr}(L)$ is the class of general frames validating every sequent in L. Obviously, $\mathbf{X} \subseteq \mathbf{GFr}(\mathrm{Seq}(\mathbf{X}))$. Suppose $\mathbf{F} \in \mathbf{GFr}(\mathrm{Seq}(\mathbf{X}))$. Now by 6.3 $\mathrm{Seq}(\mathbf{X}) = \mathrm{Seq}(\mathbf{X}^+)$. Then, again by 6.3, $\mathbf{F}^+ \in \mathbf{Mal}(\mathrm{Seq}(\mathbf{X}^+))$. So by 5.17, $\mathbf{F}^+ \in \mathbf{SP}_U(\mathbf{X}^+)$, that is, $\mathbf{F}^+$ is isomorphically embedded in some ultraproduct $\prod_{i\in I}\mathbf{B}_i/U$, $\mathbf{B}_i \in \mathbf{X}^+$, $i \in I$. But then $\mathbf{F}^+$ is isomorphically embedded in the ultraproduct $\prod_{i\in I}(\mathbf{F}_i)^+/U$, $\mathbf{F}_i \in X$, $\mathbf{F}_i^+\simeq\mathbf{B}_i$, $i \in I$. So by 6.13, $(\mathbf{F}^+)_+$ is a p-morphic image of $(\prod_{i\in I}\mathbf{F}_i^+/U)_+$. Now by 6.11, $(\prod_{i\in I}\mathbf{F}_i^+/U)_+\simeq((\prod_{i\in I}\mathbf{F}_i/U)^+)_+$, so $(\mathbf{F}^+)_+$ is a p-morphic image of a bidual of an ultraproduct of members of $\mathbf{X}$, so by the closure conditions on $\mathbf{X}$, $(\mathbf{F}^+)_+ \in \mathbf{X}$. Then by the closure conditions on $-\mathbf{X}$, $\mathbf{F} \in \mathbf{X}$.

## 7. Some Results on Sequent-Axiomatic Classes of Standard Frames

In this chapter, we will exploit the methods developed in Chs. 5 and 6 to obtain a characterization of sequent-axiomatic classes of frames, under the assumption that these classes are $\Delta$-elementary, and a relatively simple sufficient condition for an arbitrary class of frames to be sequent-axiomatic. Thus we will have obtained an answer to the sequent analogue of 3.4 and a partial answer to the sequent analogue of 3.5.

**Lemma 7.1** Let **X** be a class of frames closed under elementary equivalence and p-morphic images. Then **X** is closed under ultrafilter extensions.

We require some additional model-theoretic machinery in order to prove 7.1

**Definition 7.2** Let **F** be a structure for the first-order language $L$. By a *simple expansion* of **F** we mean a structure $\mathbf{F}_X = \langle \mathbf{F}, \langle w \rangle_{w \in X} \rangle$ for some $X \subseteq |\mathbf{F}|$. For such an expansion, $L(\mathbf{F}_X)$ denotes the language $L \cup \{c_w \mid w \in X\}$, where the constant $c_w$ is interpreted as $w$. We say that **F** is *ω-saturated* if for every $X \subseteq_{\text{fin}} |\mathbf{F}|$, every set $\Sigma(x)$ of $L(\mathbf{F}_X)$-formulas with free variable $x$ which is finitely satisfiable in $\mathbf{F}_X$ is realized in $\mathbf{F}_X$, i.e., there is some $w \in |\mathbf{F}|$ such that for $\phi(x) \in \Sigma(x)$, $\phi(w)$ is valid in $\mathbf{F}_X$.

The important fact we will use about $\omega$-saturated structures is the following:

**Lemma 7.3** Let **F** be a structure for a first order language $L$. Then there is a structure $\mathbf{F}'$ for $L$ such that $\mathbf{F}' \equiv \mathbf{F}$ and $\mathbf{F}'$ is $\omega$-saturated.

**Proof** [CK] 5.1.1(i), 5.1.2(i) and 5.1.4.

**Proof of 7.1** ([vB2], 8.9) For $\mathbf{F} = \langle W, R \rangle \in \mathbf{X}$, we will construct $\mathbf{F}'' = \langle W', R', \langle X' \rangle_{X \subseteq W} \rangle$ with $\langle W', R' \rangle \equiv \mathbf{F}$ and $ue(\mathbf{F}) = \langle W^*, R^* \rangle$ a p-morphic image of $\langle W', R' \rangle$. Let

$L_R' = L_R \cup \{P_X \mid X \subseteq W\}$, where each $P_X$ is a unary predicate constant. Expand **F** to a structure $\mathbf{F}' = \langle W,R,\langle X\rangle_{X \subseteq W}\rangle$ for $L_R'$. Now by 7.3, we have an $\omega$-saturated structure $\mathbf{F}'' = \langle W',R',\langle X'\rangle_{X \subseteq W}\rangle$, with $\mathbf{F}''\equiv\mathbf{F}'$. Define the function f by $f(w) = \{X \subseteq W \mid w \in X'\} = \{X \subseteq W \mid P_X w \text{ is valid on } \mathbf{F}''\}$, for $w \in W'$.

We want to show that f is a surjective p-morphism from $\langle W',R'\rangle$ onto ue(**F**). We must first verify that for $w \in W'$, $f(w) \in W^*$. Now $\forall y(\neg P_X y \leftrightarrow P_{W-X} y)$ and $\forall y(P_X y \& P_Y y \leftrightarrow P_{X\cap Y} y)$ are valid in **F'** and hence **F''**, so for $w \in W'$, we do have that f(w) is an ultrafilter in $2^W$.

Next we need to show that f is a p-morphism. Suppose $w,v \in W'$, $wR'v$ and $w \in (l_R(X))'$. Since $\forall y \forall z(P_{l_R(X)} y \& Ryz \leftrightarrow P_X z)$ is valid in **F'** (by 3.14), and hence on **F''**, $v \in X'$. So for $X \subseteq W$, if $l_R(X) \in f(w)$, $w \in (l_R(X))'$, whence $v \in X'$ and $X \in f(v)$. So $f(w)R^*f(v)$. Now suppose $w \in W'$, $u \in W^*$ and $f(w)R^*u$. Let $X = \{w\}$ and $\Sigma(y)$ be the set $\{P_X y \mid X \in u\} \cup \{Rc_w y\}$ of $L_R'(\mathbf{F}''_X)$-formulas with free variable y. We claim $\Sigma(y)$ is finitely satisfiable in $\mathbf{F}''_X$. Let $X_1, \cdots, X_k \in u$, $k<\omega$ and $X = X_1 \cap \cdots \cap X_k \in u$. If $\{P_{X_1}, \cdots, P_{X_k}, Rc_w y\}$ is not satisfiable in $\mathbf{F}''_X$, then $\forall y(Rc_w y \rightarrow \neg P_X y)$ is valid in $\mathbf{F}''_X$, as is $\forall y(Rc_w y \rightarrow P_{W-X} y)$. Now $\forall y \forall z((Ryz \rightarrow P_{W-X} z) \rightarrow P_{l_R(W-X)} y)$ is valid in **F'** (by 3.14) and hence in $\mathbf{F}''_X$. So we must have $P_{l_R(W-X)} c_w$ is valid in $\mathbf{F}''_X$, whence $l_R(W-X) \in f(w)$. But then since $f(w)R^*u$, $W-X \in u$, a contradiction since $X \in u$ and u is an ultrafilter in $2^W$.

Finally, we must show that f is onto. Suppose $u \in W^*$. Then $\Sigma(y) = \{P_X y \mid X \in u\}$ is finitely satisfiable in **F'** and hence in **F''**. So $\Sigma(y)$ is realized in **F''** Thus there is some $w \in W'$ such that for $X \in u$, $w \in X'$ and so $f(w) = u$.

**Theorem 7.4** Let **X** be a $\Delta$–elementary class of frames. Then **X** is sequent-axiomatic iff **X** is closed under p-morphic images and **−X** is closed under ultrafilter extensions.

**Proof** ( $\Longrightarrow$ ) By 4.7 and 4.8.

( $\Longleftarrow$ ) Since $\mathbf{X}$ is $\Delta$–elementary, $\mathbf{X}$ is closed under elementary equivalence ( [BS], 7.3.4 ), and so by 7.1 $\mathbf{X}$ is closed under ultrafilter extensions. We want to show that $\mathbf{X} = \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$. Obviously, $\mathbf{X} \subseteq \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$. Suppose $\mathbf{F} \in \mathbf{Fr}(\mathrm{Seq}(\mathbf{X}))$. Then $\langle \mathbf{F}, 2^{|\mathbf{F}|} \rangle \in \mathbf{GFr}(\mathrm{Seq}(\mathbf{X}))$, so $\mathbf{F}^+ = \langle \mathbf{F}, 2^{|\mathbf{F}|} \rangle^+ \in \mathbf{Mal}(\mathrm{Seq}(\mathbf{X}^+))$, as in the proof of 6.16. So by 5.17, $\mathbf{F}^+ \subseteq \prod_{i \in I} \mathbf{B}_i/U$, where for $i \in I$, $\mathbf{B}_i \simeq \mathbf{F}_i^+$, $\mathbf{F}_i \in X$, and $U$ is an ultrafilter in $2^I$. So $\mathbf{F}^+ \subseteq \prod_{i \in I} \mathbf{F}_i^+/U$. Say $\prod_{i \in I} \mathbf{F}_i/U = \langle W_U, R_U \rangle$. Then $\prod_{i \in I} \mathbf{F}_i^+/U = \prod_{i \in I} \langle \mathbf{F}_i, 2^{|\mathbf{F}_i|} \rangle^+/U$, which is isomorphic to $(\prod_{i \in I} \langle \mathbf{F}_i, 2^{|\mathbf{F}_i|} \rangle/U)^+ = \langle W_U, R_U, P_U \rangle^+$ where $P_U \subseteq 2^{W_U}$. Thus $\prod_{i \in I} \mathbf{F}_i^+/U \subseteq (\prod_{i \in I} \mathbf{F}_i/U)^+$ and so $\mathbf{F}^+ \subseteq (\prod_{i \in I} \mathbf{F}_i/U)^+$. Then by 6.13, $(\mathbf{F}^+)_+$ is a p-morphic image of $((\prod_{i \in I} \mathbf{F}_i/U)^+)_+$. By the definition of p-morphisms of general frames (6.7), $((\mathbf{F}^+)_+)_0$ is a p-morphic image of $(((\prod_{i \in I} \mathbf{F}_i/U)^+)_+)_0$. But for a standard frame $\mathbf{F}$, $((\mathbf{F}^+)_+)_0 = \mathrm{ue}(\mathbf{F})$. Now since $\mathbf{X}$ is $\Delta$–elementary, $\prod_{i \in I} \mathbf{F}_i/U \in \mathbf{X}$ ([BS], 7.3.4). But then $\mathrm{ue}(\mathbf{F}) \in \mathbf{X}$ since $\mathbf{X}$ is closed under p-morphic images and ultrafilter extensions. Then since $-\mathbf{X}$ is closed under ultrafilter extensions, $\mathbf{F} \in X$.

It is clear that validity of sequents is not preserved by ultraproducts. Consider the structure $\overline{\mathbb{N}} = \langle \omega, > \rangle$, which is inversely well-founded. By a well known result ([BS], 6.4.3), the ultrapower $\overline{\mathbb{N}}^\omega/U$, where $U$ is a nonprincipal ultrafilter in $2^\omega$, is not inversely well-founded. Thus $\mathbf{X}_{iw} = \mathbf{Fr}(\{\langle \{p \to \Diamond p\}, \{\neg p\} \rangle\})$ is not closed under ultraproducts, since by 4.14 $\mathbf{F} \in \mathbf{X}_{iw}$ iff $\mathbf{F}$ is inversely well-founded. Also, it is not clear that the sequent analogue of 3.10 holds, since sequent-axiomatic classes are not closed under disjoint unions or generated subframes. This means that we have not obtained an answer to the sequent analogue of question 3.3, and so we have not been able to determine whether the conclusion of 7.4 holds under any assumption weaker than **X** being $\Delta$—elementary. We also note that by [vB2], Ch. 2, sequent-axiomatic classes are not closed under ultrafilter extensions.

For an arbitrary class **X** of frames, the conditions of 7.4, along with closure under ultrafilter extensions and ultraproducts, are sufficient for **X** to be sequent-axiomatic. We now present a sufficient condition that does not require the ultraproduct construction.

**Lemma 7.5** Let **X** be a class of MA's, **B** an MA. Suppose for some finite MA $\mathbf{B}_0 \subsetneq \mathbf{B}$, $\mathbf{B}_0 \notin \mathbf{S}(\mathbf{X})$. Then there is a universal $L_{MA}$-sentence $\phi$ valid in every $\mathbf{B}' \in \mathbf{X}$ which is not valid in **B**.

**Proof** Suppose $|\mathbf{B}_0| = \{a_1, \cdots, a_n\}$. Let

$$\begin{aligned}\phi = \forall x_1 \cdots \forall x_n(&\bigvee\{x_i = x_j \mid 0<i<j\leqslant n\} \vee \\ &\bigvee\{x_i \cap x_j \neq x_k \mid a_i \cap a_j = a_k,\ i,j,k \leqslant n\} \vee \\ &\bigvee\{-x_i \neq x_j \mid -a_i = a_j,\ i,j \leqslant n\} \vee \\ &\bigvee\{/x_i \neq x_j \mid /a_i = a_j,\ i,j \leqslant n\}).\end{aligned}$$

Now $a_1, \cdots, a_n$ witness that $\phi$ is not valid in $\mathbf{B}_0$. Hence $\phi$ is not valid in **B** (5.16). Moreover, if $\phi$ is not valid in $\mathbf{B}' \in \mathbf{X}$, then $\mathbf{B}'$ has a subalgebra $\mathbf{B}_0'$ isomorphic to $\mathbf{B}_0$. Namely, $\mathbf{B}_0'$ is the subalgebra of $\mathbf{B}'$ generated by $a_1', \cdots, a_n'$, which witness that $\phi$ is not valid in $\mathbf{B}'$.

**Corollary 7.6** Let **X** be a class of MA's, **B** an MA. If $\mathbf{B} \in \mathbf{Mod}(\mathrm{Th}_\forall(\mathbf{X}))$, then every finite subalgebra of **B** is isomorphically embedded in some member of **X**.

**Theorem 7.7** Let **X** be a class of frames. For **X** to be sequent axiomatic, it is sufficient that for any frame **F**, if for every finite p-morphic image $\mathbf{F}^0$ of **F**, $\mathbf{F}^0$ is a p-morphic image of ue($\mathbf{F}'$), for some $\mathbf{F}' \in \mathbf{X}$, then $\mathbf{F} \in$ X.

**Proof** We need to show that **Fr**(Seq(**X**)) $\subseteq$ **X**. Let **F** $\in$ **Fr**(Seq(**X**)). Then as in 7.4, $\mathbf{F}^+ \in$ **Mal**(Seq($\mathbf{X}^+$)). Since every sequent corresponds to a universal $L_{MA}$-sentence, $\mathbf{F}^+ \in$ **Mod**($\mathrm{Th}_{\forall}(\mathbf{X}^+)$). Now suppose $\mathbf{F}^0$ is a finite p-morphic image of **F**. Then by 6.8 $\mathbf{F}^{0+} \subseteq \mathbf{F}^+$. So by 7.6, $\mathbf{F}^{0+} \subseteq \mathbf{B}$, $\mathbf{B} \simeq \mathbf{F}'^+$, $\mathbf{F}' \in \mathbf{X}$, whence $\mathbf{F}^{0+} \subseteq \mathbf{F}'^+$. So $(\mathbf{F}^{0+})_+$ is a p-morphic image of of $(\mathbf{F}'^+)_+$. As in 7.4, we then have that ue($\mathbf{F}^0$) is a p-morphic image of ue($\mathbf{F}'$). Since for finite **F**, $\mathbf{F} \simeq$ ue(**F**), we have by assumption that $\mathbf{F} \in \mathbf{X}$.

As there are frames with no finite p-morphic images, the condition of 7.7 is not necessary.

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