Yet another proof of the strong law of large numbers

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Abstract

We give a short proof of the strong law of large numbers based on duality for random walk.

Let $X_1, X_2, \dots$ be i.i.d. random variables with finite expectation and let $S_n = X_1 + \cdots + X_n$ for $n \geqslant 0$ be the corresponding random walk. Kolmogorov's strong law of large numbers says that $n^{-1} S_n \to \mathbb{E}[X]$ almost surely as $n \to \infty$. Clearly, it is a consequence of the lemma:

Lemma 1. *Let $X_1, X_2, \dots$ be i.i.d. r.v. with $\mathbb{E}[X] > 0$. Then $\inf_{n \geqslant 0}(X_1 + \cdots + X_n)$ is finite a.s.*

Proof. STEP 1. BOUNDING THE INCREMENTS FROM ABOVE. Choose $C > 0$ large enough so that by dominated convergence $\mathbb{E}[X\mathbf{1}_{X<C}] > 0$. We will show that the random walk $\tilde{S}_n = X_1\mathbf{1}_{X_1<C} + \cdots + X_n\mathbf{1}_{X_n<C}$ is a.s. bounded from below which is sufficient to prove the lemma.
STEP 2. DUALITY. For every $n \geqslant 0$ we have the equality in law $(0 = \tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_n) = (\tilde{S}_n - \tilde{S}_n, \tilde{S}_n - \tilde{S}_{n-1}, \dots, \tilde{S}_n - \tilde{S}_1, \tilde{S}_n - \tilde{S}_0)$. Let $T = \inf\{i \geqslant 0 : \tilde{S}_i > 0\}$ be the first hitting time of the positive axis by the walk and recall that a time $n \geqslant 0$ is a weak descending record time if and only if $\tilde{S}_n = \min_{0 \leqslant k \leqslant n} \tilde{S}_k$. By applying the above equality in law we deduce (see Figure 1) that

$$\text{for all } n \geqslant 0, \quad \mathbb{P}(T > n) = \mathbb{P}(n \text{ is a weak descending record time}).$$

Summing over $n \geqslant 0$, we get that $\mathbb{E}[T] = \mathbb{E}[\#\text{ weak descending record times}]$ and the proof is complete if we prove that $\mathbb{E}[T] < \infty$ since this implies that almost surely there is a finite number of weak descending records for $\tilde{S}$, hence the walk is bounded from below a.s.

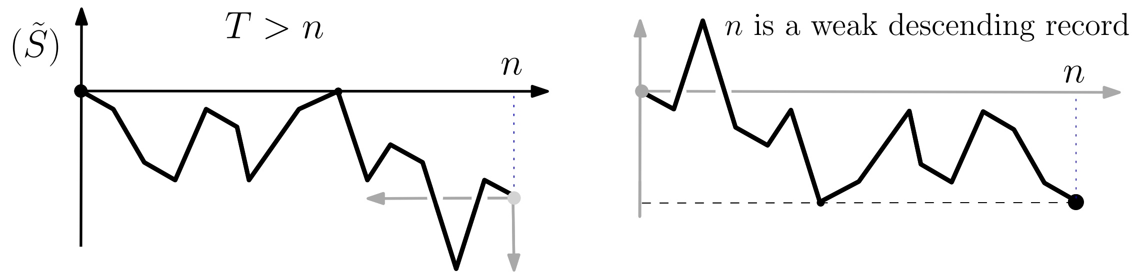

Figure 1: Time and space reversal shows that $\mathbb{P}(T > n) = \mathbb{P}(n \text{ is a descending record time})$.

STEP 3. OPTIONAL SAMPLING THEOREM. To prove $\mathbb{E}[T] < \infty$, consider the standard martingale

$$M_n = \tilde{S}_n - \mathbb{E}[X\mathbf{1}_{X<C}]n, \quad \text{for } n \geqslant 0$$

(for the filtration generated by the X_i's) and apply the optional sampling theorem to the stopping time $n \wedge T$ to deduce that

$$0 = \mathbb{E}[M_{n\wedge T}] \quad \text{or in other words} \quad \mathbb{E}[X\mathbf{1}_{X<C}]\mathbb{E}[n \wedge T] = \mathbb{E}[\tilde{S}_{n\wedge T}].$$

Since the increments of $\tilde{S}$ are bounded above by C, the right-hand side of the last display is bounded by C as well. Letting $n \to \infty$, by monotone convergence we deduce that the expectation of T is finite. Et voilà. □

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