

FINITE ABELIAN GROUPS VIA CONGRUENCES

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ABSTRACT. For every finite abelian group G , there are positive integers n and d such that G is isomorphic to the multiplicative group of d -th powers of reduced residues modulo n .

As every beginning student in number theory learns, when $n \in \mathbb{N}$, the integers a with $1 \leq a \leq n$ and $(a, n) = 1$ form an abelian group when equipped with the binary operation defined by multiplication modulo n . For each positive integer d , this multiplicative group U_n of reduced residues modulo n contains the subgroup of d -th powers, namely $U_n^{(d)} = \{a^d : a \in U_n\}$. The goal of this note is to show that, in fact, every finite abelian group is isomorphic to one of the latter shape.

Theorem 1. *Let G be a finite abelian group. Then there exist positive integers n and d having the property that $G \cong U_n^{(d)}$.*

Standard introductions to algebra will describe the fundamental theorem of finitely generated abelian groups (see [2, Theorem 3 of §5.2]). Thus, writing \mathbb{Z}_r for the additive group of integers modulo r , each finite abelian group G is isomorphic to a direct product of the shape $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$, where the integers m_1, \dots, m_k satisfy

$$m_i \geq 2 \quad \text{and} \quad m_{i+1} | m_i \quad (1 \leq i < k). \quad (1)$$

Given distinct odd prime numbers p_i with $p_i \equiv 1 \pmod{m_i}$, say with $p_i = 1 + m_i d_i$, one finds that \mathbb{Z}_{m_i} is isomorphic to the group $U_{p_i}^{(d_i)} = \langle g_i^{d_i} \rangle$, where g_i is any primitive root modulo p_i . Writing $n = p_1 p_2 \cdots p_k$, the Chinese Remainder Theorem delivers the familiar conclusion that G is isomorphic to a subgroup of U_n . Thus, one has

$$G \cong U_{p_1}^{(d_1)} \times U_{p_2}^{(d_2)} \times \cdots \times U_{p_k}^{(d_k)}.$$

Although it is comforting to note that every finite abelian group is isomorphic to a subgroup of the multiplicative group of residues modulo n , for some $n \in \mathbb{N}$, the need to work with a collection of exponents d_1, \dots, d_k corresponding to the prime divisors of n is somewhat inelegant. The primary motivation for establishing our theorem is to provide a self-contained description of the abelian group G with only a pair of integers.

Our goal, of identifying a simple congruence-based realization of the abelian group G of the shape $U_n^{(d)}$, would follow were one able to find distinct primes

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p_1, \dots, p_k corresponding to a common value d for the d_i . Indeed, it would then follow as a consequence of the Chinese Remainder Theorem that

$$G \cong U_{p_1}^{(d)} \times U_{p_2}^{(d)} \times \cdots \times U_{p_k}^{(d)} \cong U_n^{(d)}.$$

This approach relies on a special case of Dickson's conjecture (see [1]). When $k \geq 1$, this as yet unproven conjecture asserts that given $a_i \in \mathbb{Z}$ and $m_i \in \mathbb{N}$, there are infinitely many positive integers d for which the k -tuple $(m_1d + a_1, \dots, m_kd + a_k)$ consists of prime numbers, unless there is a congruence condition preventing such from occurring. In the case presently of interest to us, in which $a_i = 1$ for each i , congruence obstructions are absent, though the conjecture has been established only in the case $k = 1$ (a special case of Dirichlet's theorem, for which see [5, Corollary 4.10]). The challenge of obtaining an unconditional conclusion requires careful selection of arithmetic progressions in which to search for the primes p_i .

Lemma 2. *Suppose that the integers m_1, \dots, m_k satisfy the condition (1). Then there are distinct prime numbers p_1, \dots, p_k satisfying the congruences*

$$p_i \equiv 1 + m_1 m_i \pmod{m_1^2 m_i} \quad (1 \leq i \leq k).$$

Proof. Dirichlet's theorem on prime numbers in arithmetic progressions (see [5, Corollary 4.10]) shows that for each i , there are infinitely many primes p with $p \equiv 1 + m_1 m_i \pmod{m_1^2 m_i}$, since it is evident that $(1 + m_1 m_i, m_1^2 m_i) = 1$. The desired conclusion is immediate. \square

The proof of the theorem. Let G be a finite abelian group, whence there exist integers m_1, \dots, m_k satisfying the condition (1) for which $G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$. Given distinct prime numbers p_1, \dots, p_k supplied by the lemma, define the integers u_i via the relation $p_i - 1 = m_1 m_i (1 + m_1 u_i)$, and then write $y_i = 1 + m_1 u_i$. Also, put $d = m_1 y_1 \cdots y_k$. Since $p_i - 1 = m_1 m_i y_i$, we see that for $1 \leq i \leq k$, one has $(p_i - 1, d) = m_1 y_i D_i$, where D_i is the greatest common divisor of m_i and

$$\prod_{\substack{1 \leq j \leq k \\ j \neq i}} (1 + m_1 u_j).$$

Since $m_i | m_1$, this greatest common divisor is plainly 1, and so $(p_i - 1, d) = m_1 y_i$, yielding $(p_i - 1) / (p_i - 1, d) = m_i$. Thus $U_{p_i}^{(d)} \cong \mathbb{Z}_{m_i}$ ($1 \leq i \leq k$). Consequently, if we put $n = p_1 p_2 \cdots p_k$, then it follows from the Chinese Remainder Theorem that

$$\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k} \cong U_{p_1}^{(d)} \times U_{p_2}^{(d)} \times \cdots \times U_{p_k}^{(d)} \cong U_n^{(d)},$$

whence $G \cong U_n^{(d)}$. This completes the proof of the theorem. \square

The proof of our theorem is more or less algorithmic, in the sense that it shows how to determine the integers n and d from a standard presentation of the finite abelian group G . Bounds on these integers may be given in terms of the order $|G|$ of the abelian group G of interest by using technology associated with Linnik's theorem [3, 4]. The latter shows that there is a positive number L having the property that when m and a are integers with $m \geq 2$

and $(a, m) = 1$, then the smallest prime number p satisfying $p \equiv a \pmod{m}$ satisfies $p \leq m^L$. In particular, Xylouris [6] has shown that such a prime exists with $p \leq Cm^{5.18}$, for a suitable positive constant C . By employing such results, it would be possible to show that both n and d can be taken no larger than $\exp(c(\log |G|)^2)$, for a suitable positive constant c , though less profligate bounds might well be accessible.

Plucking the example of the abelian group $G = \mathbb{Z}_3 \times \mathbb{Z}_{42}$ almost from thin air, we may illustrate the relative inefficiency of the method underlying our theorem. The smallest positive integer d satisfying the property that $3d + 1$ and $42d + 1$ are simultaneously prime is $d = 10$, so that

$$\mathbb{Z}_3 \times \mathbb{Z}_{42} \cong U_{31}^{(10)} \times U_{421}^{(10)} \cong U_{13051}^{(10)}.$$

Smaller realizations are given by

$$\mathbb{Z}_3 \times \mathbb{Z}_{42} \cong U_7^{(8)} \times U_{337}^{(8)} \cong U_{2359}^{(8)}$$

and

$$\mathbb{Z}_3 \times \mathbb{Z}_{42} \cong \mathbb{Z}_6 \times \mathbb{Z}_{21} \cong U_{13}^{(2)} \times U_{43}^{(2)} \cong U_{559}^{(2)}.$$

Meanwhile, the approach suggested by the proof of our theorem asks that we seek one prime congruent to $1 + 3 \cdot 42 = 127$ modulo $3 \cdot 42^2$, and a second congruent to $1 + 42^2$ modulo 42^3 . We find that 127 and $1 + 42^2 + 42^3 = 75853$ are the smallest such primes. We then take $n = 127 \cdot 75853 = 9633331$ and $d = 42 \cdot 1 \cdot 43 = 1806$. Thus, the realization of G suggested by the proof of our theorem is

$$\mathbb{Z}_3 \times \mathbb{Z}_{42} \cong U_{127}^{(1806)} \times U_{75853}^{(1806)} \cong U_{9633331}^{(1806)}.$$

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