

ON THE DISTRIBUTION OF MONOCHROMATIC COMPLETE SUBGRAPHS AND ARITHMETIC PROGRESSIONS

Aaron Robertson¹, William Cipolli², and Maria Dascălu³
Department of Mathematics, Colgate University, Hamilton, New York

Abstract

We investigate the distributions of the number of: (1) monochromatic complete subgraphs over edgewise 2-colorings of complete graphs; and (2) monochromatic arithmetic progressions over 2-colorings of intervals, as statistical Ramsey theory questions. We present convincing evidence that both distributions are very well-approximated by the Delaporte distribution.

1. Introduction

Ramsey theory deals with finding order among chaos, two fundamental results espousing this being Ramsey's Theorem and van der Waerden's Theorem. Ramsey's Theorem, in particular, proves the existence, for any $k \in \mathbb{Z}^+$, of a minimal positive integer $R(k)$ such that every 2-coloring of the edges of a complete graph on $R(k)$ vertices contains a monochromatic complete subgraph on k vertices. Van der Waerden's Theorem, in particular, states that there exists a least positive integer $w(k)$ such that every 2-coloring of $\{1, 2, \dots, w(k)\}$ contains a monochromatic arithmetic progression of length k .

While the definitions of both Ramsey and van der Waerden numbers are simple, the computations of both, especially Ramsey numbers, are notoriously difficult. For examples, the most recent Ramsey number was determined by McKay and Radziszowski [11], who used almost 10 years of cpu time; Kouril [8] used over 200 processors and 250 days to show that $w(6) = 1132$. Given the exponential nature of these numbers, the remaining unknown numbers seem intractable at the present time. Given the difficulty of computing Ramsey numbers exactly we explore the potential value of a statistical approach with an ultimate goal of gaining some insight into $R(k)$ and $w(k)$. Starting with Ramsey numbers, let all edgewise 2-colorings of the complete graphs on n vertices be equally likely and define $X_k = X_k(n)$ as the random variable giving the total number of monochromatic subgraphs on k vertices (i.e, K_k). Our goal is to find a very good approximation for the probability mass function (pmf) of X_k .

In [4], it is shown that X_k is asymptotically Poisson as $k \rightarrow \infty$ (with certain conditions on n and k); that is, with an appropriate restriction on n , as $k \rightarrow \infty$ we have $\mathbb{P}(X_k = j) \approx \frac{\lambda^j e^{-\lambda}}{j!}$, where $\lambda = \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$.

However, since this is an asymptotic (in k) result, using this for small values of k is not appropriate. In this article, we present (hopefully very convincing) evidence of what the distribution of X_k for small k may be.

2. Sampling Algorithm for 2-Colored K_n

Given a user input of positive integers n , k , and g , our Python program `GraphCount`² generates g graphs, each on n vertices, using an adjacency list. It colors the edges between pairs of vertices randomly using

¹Principal corresponding author; arobertson@colgate.edu

²wcipolli@colgate.edu

³Undergraduate student, mdascalu@colgate.edu

²Available at <http://www.aaronrobertson.org>.

the Python random module. It then counts the total number of monochromatic complete subgraphs on k vertices of each given graph. Compiling all results will give us an empirical pmf.

Algorithm 1 is recursive with a base case of $k = 3$. For $k = 3$, the Triangle-Counting Algorithm (Algorithm 2) is used.

```

input : List of edge colorings of graph and  $k$ 
output: List  $\mathcal{F}$  of vertices of monochromatic cliques of size  $k$ 
1 Run Algorithm 1/2 with  $k - 1$ ; call the output subcliques;
2 for every vertex  $A$  in the graph, starting from  $A = k$  do
3   if  $A$  can support a clique of size  $k$ ; then
4     for each subclique  $S$  do
5       ensure  $A > i$  for every vertex  $i$  in  $S$ ;
6       if the subclique and  $A$  form a monochromatic clique of size  $k$  then
7         create a tuple of the subclique and  $A$ ;
8         add the tuple to  $\mathcal{F}$ ;
9       end
10    end
11  end
12 end
13 return  $\mathcal{F}$ ;

```

Algorithm 1: Recursive Counting Algorithm

GraphCount has a run-time of $O(n^2)$ for $k = 3$ and $O(n^k)$ for $k \geq 4$. Table 1 compiles a list of approximate run-times.

```

input : 2-colored graph
output: List  $\mathcal{F}$  of vertices of monochromatic triangles
1 for every pair of vertices  $A < B$  of the graph do
2   determine the color  $c$  connecting  $A$  and  $B$ ;
3   let  $\mathcal{N}$  be the set of neighbors common to  $A$  and  $B$ ;
4   for every  $C \in \mathcal{N}$  do
5     if color of edge between  $A$  and  $C$  is  $c$  and color of edge between  $B$  and  $C$  is  $c$  then
6       | add  $(A, B, C)$  to  $\mathcal{F}$ ;
7     end
8   end
9 end
10 return  $\mathcal{F}$ ;

```

Algorithm 2: Triangle-Counting Algorithm

Our goal is to run `GraphCount` with $g \geq 1,000,000$ in order to obtain an empirical probability mass function that is a fairly good approximation of the probability mass function. However, this will only allow us (given reasonable time constraints) to investigate $k = 6$ for a few values of n . Furthermore, as can be seen from the table of run-times (Table 1), gathering enough samples to attempt distribution fitting for $k = 7$ would require approximately 75 years on a single computer or a couple of years on the cluster of 36 computers we have available to us (with dedicated use, which we do not have). So, at this time, pursuit of $k = 7$ is not realistic with our algorithm.

Input k	Input n	Time per graph (sec)
3	6	0.0002
4	18	0.00275
5	43	0.15
5	49	0.296
6	102	22.59
6	165	407
7	205	2368

Table 1: Run-times for `GraphCount`

In Figure 1, we present the empirical probability mass functions for the number of monochromatic K_k subgraphs over 2-colorings of the edges of K_n for small k and n . You will notice a similar shape for all presented. This occurred in all histograms we obtained (for sufficiently large sample sizes).

3. Fitting the Empirical Probability Mass Function

We can view the random variable X_k as a sum of indicator random variables Y_i , where $Y_i = 1$ if the i^{th} K_k is monochromatic and $Y_i = 0$ otherwise. Since n is typically much larger than k , most pairs of

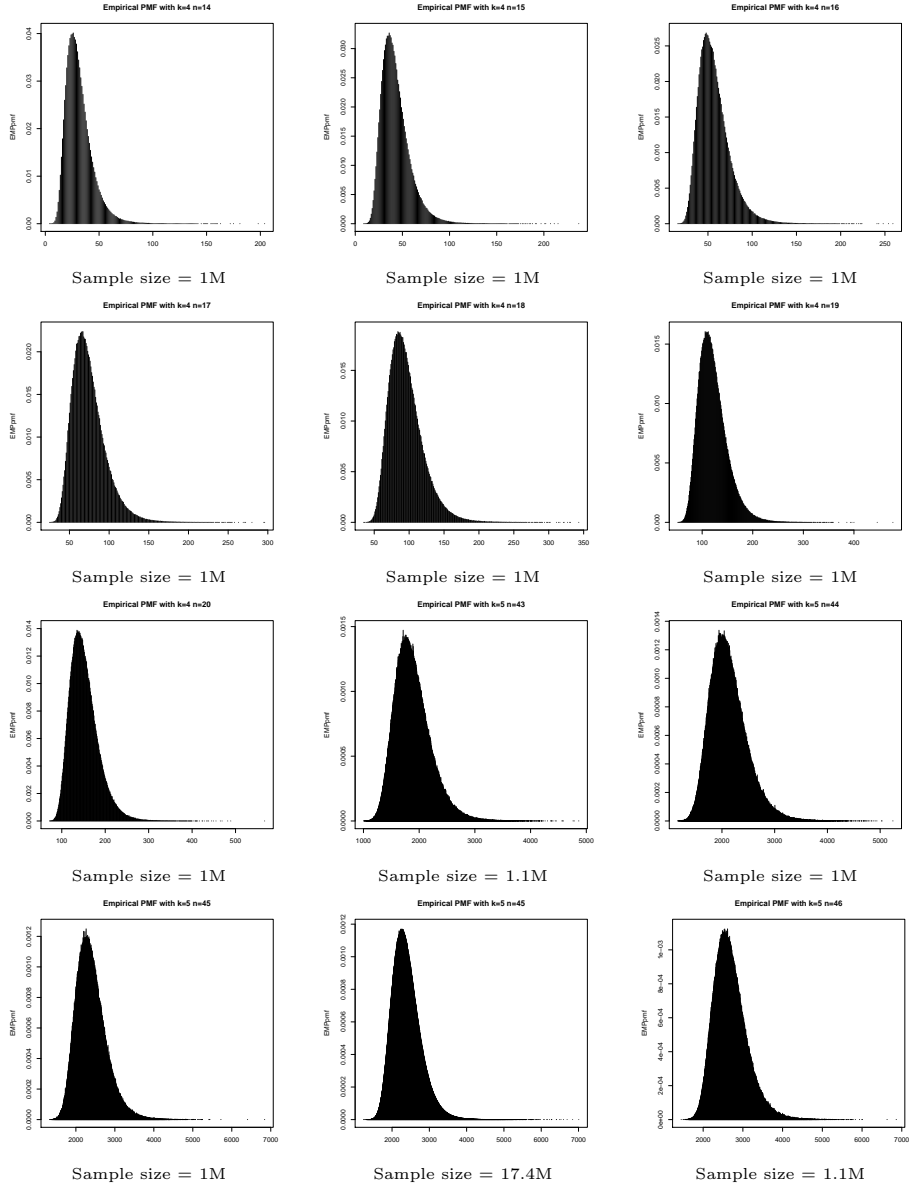


Figure 1: Empirical probability mass functions (pmfs) for various scenarios.

K_k 's are independent. Hence, we can view X_k as the sum of (somewhat) weakly dependent indicator random variables, each of which have a small probability of being 1. By weak dependence, we mean that the probability of two randomly chosen K_k 's are dependent is near 0. To see this, note that the probability that two such subgraphs are dependent requires them to share at least 2 vertices, so that this probability is

$$\frac{\binom{k}{2} \binom{n-2}{k-2}}{\binom{n}{k}} \approx \frac{k^4}{2n^2}.$$

Noting that n is typically much larger than k^2 , we see that this probability is quite low.

Now, as k tends to infinity, the proportion of dependent pairs of subgraphs goes to 0. Hence, asymptotically, we can view the set of all subgraphs as almost entirely independent. In the situation where the subgraphs are completely independent, since the probability that any given subgraph is monochromatic is small, through the Poisson process we get a Poisson distribution and the result in [4]; see [5] for a general theorem about when we can have a limiting Poisson distribution with weak overall dependence.

However, when we investigate fixed values of (small) k , as we can see from Figure 2, the Poisson distribution is not a good fit. This is to be expected since the Poisson distribution has a variance equal to its expectation, while we know that $\mathbb{E}(X_k) \neq \text{Var}(X_k)$ for $k \geq 4$ (see Lemma 1 below). For the $k = 3$ case, we do have $\mathbb{E}(X_k) \approx \text{Var}(X_k)$ for large n ; see Table 2 in the next section. We can also see from Figure 2 that the Poisson distribution appears under-dispersed (while Lemma 1 below proves this). More fundamentally, for fixed values of k , the dependence between some of the subgraphs is not accounted for with Poisson modeling.

Lemma 1. *For any $k \geq 4$, we have $\lim_{n \rightarrow \infty} \frac{\mathbb{E}(X_k(n))}{\text{Var}(X_k(n))} = 0$.*

Proof. Lemma 3.5 in [6] gives the asymptotic order of $\text{Var}(X_k)$: Define $\Phi(X_k) = \min(\mathbb{E}(X_H))$, where the minimum is taken over all nontrivial subgraphs H of K_k . Then $\text{Var}(X_k) \asymp \frac{1}{2} \frac{\mathbb{E}^2(X_k)}{\Phi(X_k)}$. Taking $H = K_3$ (where K_2 is the degenerate case) we see that $\text{Var}(X_k) \asymp \frac{1}{2} \frac{\binom{n}{k}^2 / 2^{\binom{2k}{2}-2}}{\binom{n}{3} / 2^2} = \Omega\left(\binom{n}{k} \frac{n^{k-3}}{2^{2\binom{k}{2}-2}}\right)$, which agrees with the expressions given in Table 2 (in the next section). We know that $\mathbb{E}(X_k) = \frac{\binom{n}{k}}{2^{\binom{k}{2}-1}}$, and by comparison with the above expression we see that the lemma's statement holds. \square

To further illustrate the point, in Figure 2 we present overlays of the best-fitting (defined in the next paragraph) Poisson distributions over the empirical pmfs presented in Figure 1. As you can see, the Poisson distribution is clearly not a good fit for small values of k .

Our measure of best-fitting is via the ℓ_1 -distance between two probability mass functions $f(k)$ and $g(k)$: $\sum_{k \geq 0} |f(k) - g(k)|$.

To address the dependence and under-dispersion, we turn to mixed-Poisson processes, i.e., Poisson processes with a parameter L that is a random variable (as opposed to being fixed). The result will be a compound Poisson distribution. We need to maintain the asymptotic Poisson nature and so, heuristically, having the parameter contain a fixed portion λ and a random portion G addresses this. Considering $L = \lambda + G$ gives our mixed-Poisson process a ‘‘Poisson part’’ λ (for the independent subgraphs) and a ‘‘local dependence corrector’’ G , which captures unknown dependence. While this addresses the dependence issue, it also allows us to correct the under-dispersion of the Poisson approximation. This is to be expected since the under-dispersion is linked with the failure to account for some dependence between events.

A commonly used choice for mixing with Poisson is the Gamma distribution. However, even though there is ample empirical evidence for the use of Gamma as a mixing function, there is no real theoretic support for the choice of Gamma [13]. It seems that the Gamma distribution is used because of its flexibility and calculability when mixed with Poisson. However, we can turn to the Pólya-Eggenberger urn scheme for some motivation.

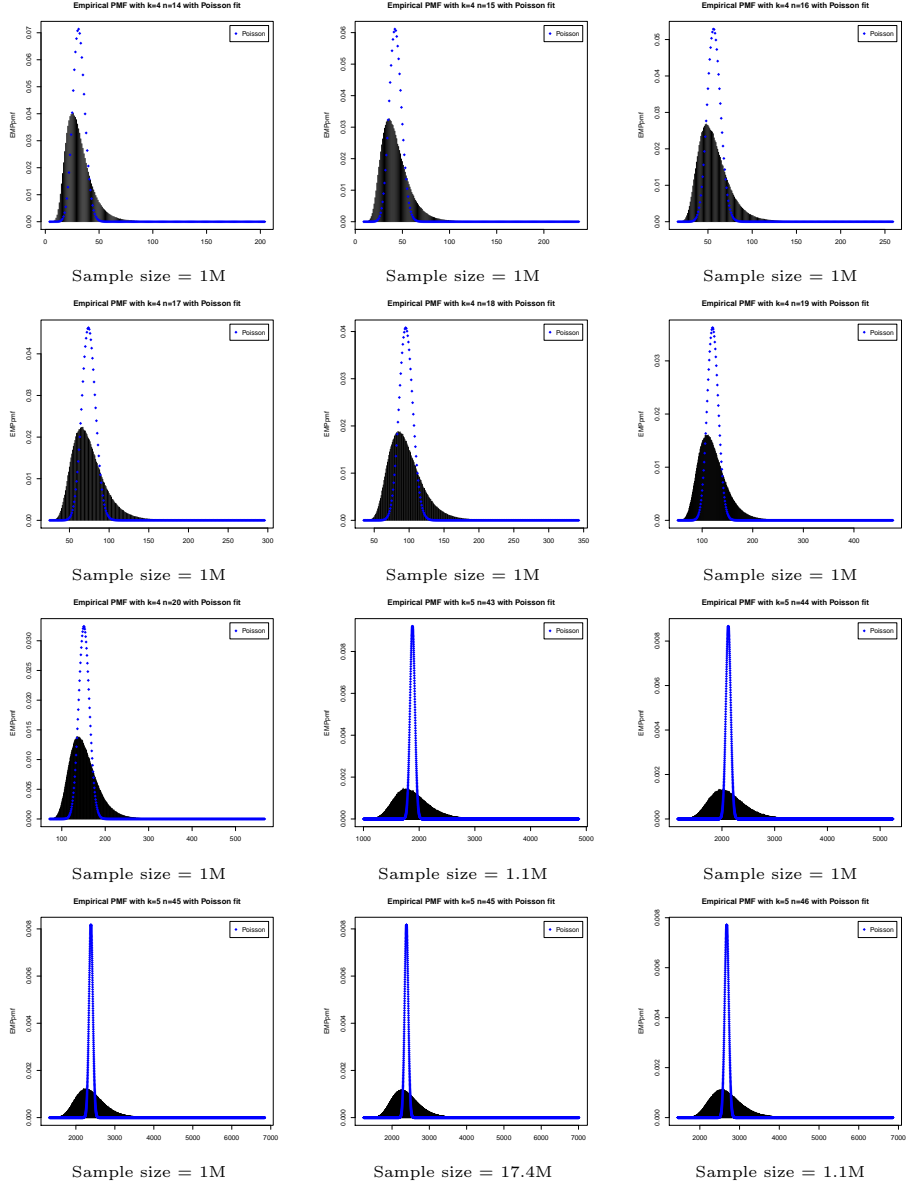


Figure 2: Empirical pmfs for various scenarios with best-fit Poisson overlay

As noted in [10], the Pólya-Eggenberger urn models have been used as contagion models. In terms of our mixed-Poisson process, a contagion model would have the property that the probabilities of future events occurring increase as events occur (see [2]). For us, this would model the change in probabilities when dealing with dependent subgraphs. Connecting this to the Pólya-Eggenberger model, we find that the negative binomial distribution is one of the limiting distributions. Since a Poisson distribution with random parameter being Gamma results in a negative binomial distribution, we have some (albeit, tangential) relationship to the Pólya-Eggenberger model as a contagion model.

Returning to our mixed-Poisson process, we can now describe $L = \lambda + G$ as having a fixed “Poisson

part” λ (for the independent subgraphs) and a “contagion driver” G , which helps to model the weak dependence. This mixing process gives rise to the following convolution pmf called the Delaporte distribution.

Definition 2. A random variable $D = D(\lambda, \alpha, \beta)$ is called a *Delaporte random variable* if it has probability mass function

$$\mathbb{P}(D = j) = \sum_{i=0}^j \frac{\Gamma(\alpha + i)}{\Gamma(\alpha) i!} \left(\frac{\beta}{1 + \beta} \right)^i \left(\frac{1}{1 + \beta} \right)^\alpha \frac{\lambda^{j-i} e^{-\lambda}}{(j-i)!}.$$

Furthermore, $\mu = \mathbb{E}(D) = \lambda + \alpha\beta$, $\text{Var}(D) = \mathbb{E}((D - \mu)^2) = \lambda + \alpha\beta(1 + \beta)$, and $\mathbb{E}((D - \mu)^3) = \lambda + \alpha\beta(1 + 3\beta + 2\beta^2)$.

Remark. In the Delaporte pmf above, λ is the parameter for the Poisson part while α and β are parameters for the Gamma part of our $L = \lambda + G$ model for the Poisson process rate, which leads to a negative binomial distribution with parameters α and $\frac{\beta}{1+\beta}$.

In Figure 3, we present the same empirical pmfs as in Figures 1 and 2 along with an overlay of the best-fit Delaporte distribution.

In order to find the best-fitting such Delaporte distribution, we must find good estimates for the parameters λ , α , and β . The two main approaches are the maximum likelihood estimates (MLE) and the method of moments estimates (MOM).

We have found that the MLEs consistently provide better results than the MOM estimates (this is generally true because likelihood methods are more efficient). In fact, these MLEs produce near-optimal results, i.e., a total ℓ_1 -distance between the empirical pmf and the MLE-estimated Delaporte distribution very near 0. Unfortunately, closed-form formulas for the MLEs of λ , α , and β do not exist in our situation (it should also be noted that R’s calculation of the MLEs for $k = 5$ neared a day of computation time for each n). Hence, in the next section we use MOM estimators for theoretical work. Although the MOM estimators do not provide the best fit, they are still very reasonable and relatively close to the MLEs as we show in our Simulation Study section.

A remark about the support of the Delaporte distribution as an approximation for the distribution of X_k is in order. We know that X_k can only take on values in $\{0, 1, 2, \dots, \binom{n}{k}\}$ while the Delaporte distribution’s support is the nonnegative integers. The ℓ_1 -distances do include the tail of the Delaporte distribution, i.e., the value of $\sum_{j > \binom{n}{k}} \mathbb{P}(D = j)$. Hence, over the support of X_k , the total ℓ_1 -distance is smaller, although we will show that the difference is negligible.

Given $n, k \in \mathbb{Z}^+$, let $q = \mathbb{P}(D > \binom{n}{k}) = \sum_{j > \binom{n}{k}} \mathbb{P}(D = j)$. Our goal is to show that q is negligible. We will use the one-sided Chebyshev inequality: $\mathbb{P}(X - \mu \geq x) \leq \frac{\sigma^2}{\sigma^2 + x^2}$, where $\mu = \mathbb{E}(X)$ and $\sigma^2 = \text{Var}(X)$. In our situation we have $\mu = \frac{2}{2 \binom{k}{2}} \binom{n}{k}$ so we let $x = \frac{2 \binom{k}{2} - 2}{2 \binom{k}{2}} \binom{n}{k} + 1$ to bound $P(D > \binom{n}{k})$. We also know that $\text{Var}(D) = \mu + \alpha\beta^2$. In the next section, we present evidence to suggest that $\sigma^2 \sim \binom{n}{k} n^{k-3} 2^{-k^2}$ so we will use this as an assumption. Putting this all together, we find that

$$\mathbb{P}\left(D > \binom{n}{k}\right) \lesssim \frac{1}{2^k k^k n^3}.$$

In practice, we have observed – for small k and n – that this bound is quite weak. Nevertheless, this does show that the tail probabilities of our Delaporte distributions are quite negligible.

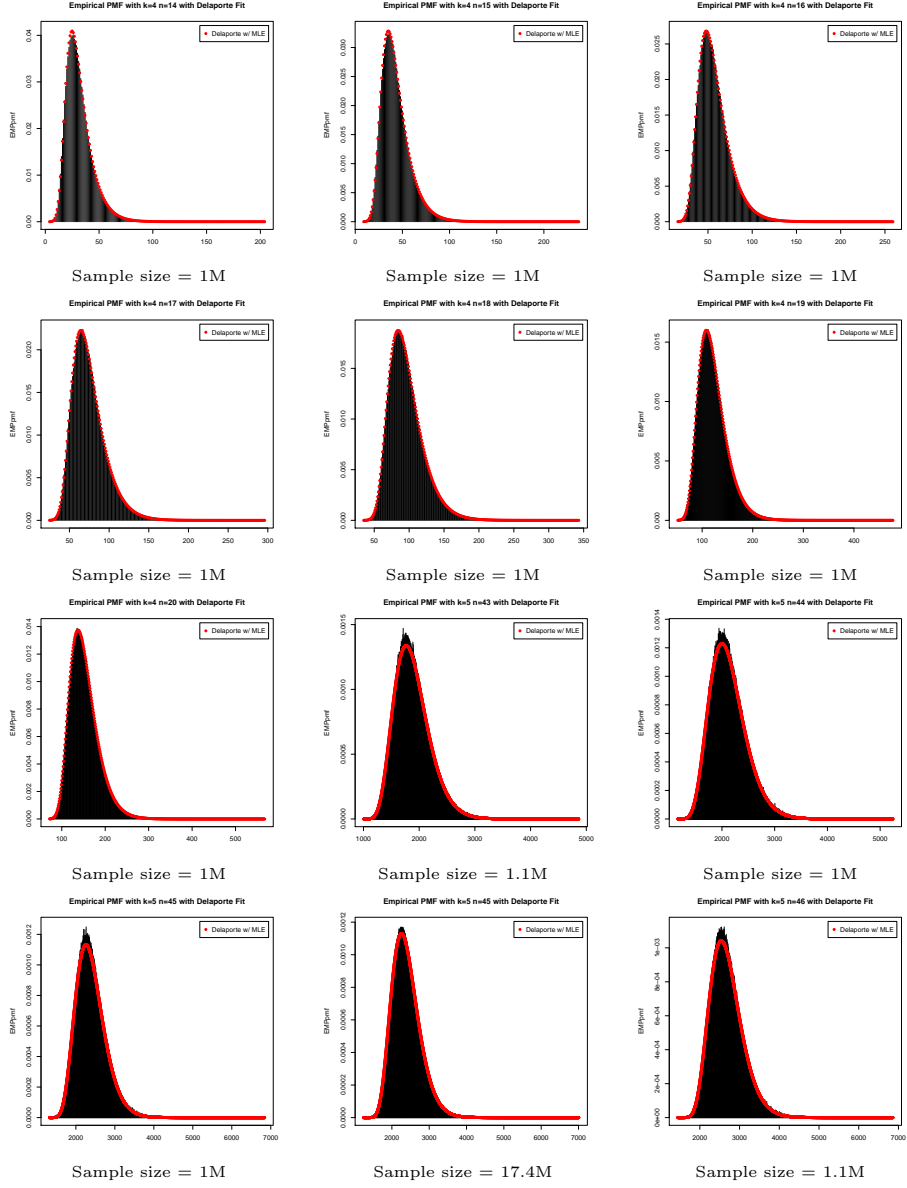


Figure 3: Empirical pmfs for various scenarios with Delaporte Overlay

4. Implications and Evidence

As stated before, it was shown in [4] that, loosely speaking, X_k is asymptotically Poisson. This was done by showing that the total ℓ_1 -distance between the distributions of X_k and a Poisson random variable with mean $\mathbb{E}(X_k)$ tends to 0 as k tends to infinity (with n bounded from above by a function of k). We will first show that the proposed Delaporte distribution is consistent with this fact. We will be using the MOM estimates and note that $\mathbb{E}(X_k) = \lambda + \alpha\beta$ under the method of moments. We use the following notation.

Notation. Let X be a random variable. We denote its moment generating function by $\text{mgf}(X)$; that is, $\text{mgf}(X) = \mathbb{E}(e^{tX})$.

Theorem 3. Let $n, k \in \mathbb{Z}^+$ with $k \geq 3$. Define $D \sim \text{Delaporte}(\lambda, \alpha, \beta)$, and $P \sim \text{Poisson}(\lambda + \alpha\beta)$. Then the $\text{mgf}(D) \rightarrow \text{mgf}(P)$ as $k \rightarrow \infty$ under the following assumptions:

$$(1) \quad \alpha \sim \frac{\binom{n}{k}}{n^{k-1}}; \quad (2) \quad \beta \sim \frac{n^{k-2}}{2^{\binom{k}{2}}}; \quad (3) \quad n = O\left(k^{1+\frac{1}{k-1}} \cdot 2^{\frac{k}{2}}\right)$$

Proof. Since D is a convolution of a Negative Binomial random variable with success probability $\frac{\beta}{1+\beta}$ and mean $\alpha\beta$ and a Poisson random variable with mean λ , using the moment generating functions of these, we easily have

$$\text{mgf}(D) = \frac{e^{\lambda(e^t-1)}}{(1 - \beta(e^t - 1))^\alpha}.$$

Isolate the denominator and use $\ln(1+x) \approx x$ for small x . Since $\beta \rightarrow \infty$ as $k, n \rightarrow \infty$, with the restriction $t \leq \ln\left(1 + \frac{1}{\beta^2}\right)$ we have $\beta(e^t - 1) \leq \frac{1}{\beta}$, so that $\beta(e^t - 1)$ is small for sufficiently large β . Hence, $\ln((1 - \beta(e^t - 1))^\alpha) \approx -\alpha\beta(e^t - 1)$. For large n and k , this gives $(1 - \beta(e^t - 1))^\alpha \approx e^{-\alpha\beta(e^t-1)}$, for $t \leq \ln\left(1 + \frac{1}{\beta^2}\right)$. Hence, we find that $\text{mgf}(D) \approx e^{(\lambda+\alpha\beta)(e^t-1)} = \text{mgf}(P)$ on a small interval including $t = 0$, which is enough to conclude the result. \square

Having Theorem 3 and knowing that there is a one-to-one correspondence between random variables and their moment generating functions, we can state that, loosely, the Delaporte random variable is asymptotically Poisson. Hence, we are not violating the ‘‘Poisson Paradigm,’’ as noted in [3].

We will now give evidence to suggest that the assumptions in Theorem 3 are satisfied. We will be using MOM estimates. We know that $\mu = \mathbb{E}(X_k) = \binom{n}{k} 2^{1-\binom{k}{2}}$. Using Zeilberger’s Maple package `SMCramsey` that accompanies [14] we find the leading terms for the second and third moments about the mean for X_k for small k :

k	$\mathbb{E}((X - \mu)^2)$	$\mathbb{E}((X - \mu)^3)$
3	$\binom{n}{3} \cdot \frac{3}{2^4}$	$\binom{n}{4} \cdot \frac{6n}{2^6}$
4	$\binom{n}{4} \cdot \frac{12n}{2^{10}}$	$\binom{n}{4} \cdot \frac{24n^3}{2^{15}}$
5	$\binom{n}{5} \cdot \frac{15n^2}{2^{18}}$	$\binom{n}{5} \cdot \frac{15n^5}{2^{27}}$
6	$\binom{n}{6} \cdot \frac{10n^3}{2^{28}}$	$\binom{n}{6} \cdot \frac{10n^7}{3 \cdot 2^{42}}$
7	$\binom{n}{7} \cdot \frac{35n^4}{2 \cdot 2^{42}}$	$\binom{n}{7} \cdot \frac{35n^9}{6 \cdot 2^{64}}$
8	$\binom{n}{8} \cdot \frac{84n^5}{15 \cdot 2^{56}}$	$\binom{n}{8} \cdot \frac{42n^{11}}{255 \cdot 2^{84}}$

Table 2: Second and third moment orders

As noted in Definition 2, for the Delaporte random variable D , we have

$$\mathbb{E}(D) = \lambda + \alpha\beta; \quad \mathbb{E}((D - \mu)^2) = \lambda + \alpha\beta(1 + \beta); \quad \mathbb{E}((D - \mu)^3) = \lambda + \alpha\beta(1 + 3\beta + 2\beta^2).$$

By Lemma 1 and the fact that $\mathbb{E}((D - \mu)^2) = \mathbb{E}(D) + \alpha\beta^2$, we can deduce that $\alpha\beta^2 \sim \binom{n}{k} \frac{n^{k-3}}{2^{2\binom{k}{2}-2}}$. Looking at the third moments in Table 2, we have evidence to suggest that $2\alpha\beta^3 \sim \binom{n}{k} \frac{n^{2k-5}}{2^{3\binom{k}{2}-3}}$. Taking the ratio of these last two expressions yields

$$\alpha \sim \frac{\binom{n}{k}}{n^{k-1}} \quad \text{and} \quad \beta \sim \frac{n^{k-2}}{2^{\binom{k}{2}}}.$$

Remark. The interested reader can obtain more accurate results by using the following MOM formulas calculated in Mathematica:

$$\hat{\lambda} = \frac{2m^2v + 2vf - m(2f + v^2(s^2v + 7)) + 6v^3}{4m^2 - 12mv + v^2(9 - s^2v)};$$

$$\hat{\alpha} = \frac{-4(m-v)^4}{4m^2(m-4v) - f(4m-6v) + s^2v^3(m-v) + v^2(21m-9v)}; \quad \hat{\beta} = \frac{-2m^2 + f + 5mv - 3v^2}{2(m-v)^2},$$

where m is the sample mean, v is the sample variance, s is the sample skewness, and we use the notation $f = \sqrt{s^2v^3(m-v)^2}$. Note that we calculate the sample skewness using the default R command `skewness()`. A summary of sample skewness calculations can be found [7].

Remark. We are clearly extrapolating in our formulas for α and β , but it is interesting to note that the order of n required in Theorem 3 is actually slightly better (by a factor of $k^{\frac{1}{k}}$) than the best known lower bound on $R(k, k)$, while a larger order for n would void our proof of the asymptotic Poisson nature of X_k via the Delaporte distribution. Might this be evidence that $k2^{\frac{k}{2}}$ is the correct order for the Ramsey number $R(k, k)$?

5. Simulation Study: MLE vs. MOM

Though the MOM estimators can be calculated in closed-form, these estimators are quite complicated and the derivation of their expected value is impractical. MLEs are more burdensome and so far have been calculated numerically using the `optim` function in `cran` R [12].

Due to the nature of these estimators, a discussion on their accuracy and precision are difficult as closed-form expectations are impractical to calculate. To explore and compare the accuracy and precision of the MOM estimators and MLEs, consider their asymptotic behavior across three simulations motivated by estimators from the $k = 4, n = 14$; $k = 5, n = 20$; and $k = 5, n = 49$ graphs.

Tables 3 and 4 contain the results of these simulations of Delaporte data with parameter values as well as MOM estimates and MLEs across increasing sample sizes n . As n increases, the MLEs quickly approach the true values, whereas the MOM estimators appear to require higher sample sizes. These simulations suggest that both the MLEs and MOM estimates might be asymptotically unbiased where, as expected, the MLEs are more efficient and have lower variability.

		MLEs		
Parameters	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
$\alpha = 1.84$ $\beta = 7.89$ $\lambda = 16.75$	100	2.03 (1.40)	8.90 (3.41)	16.96 (3.65)
	1000	1.87 (0.39)	7.92 (0.91)	16.70 (1.22)
	10000	1.85 (0.10)	7.88 (0.27)	16.74 (0.36)
	100000	1.84 (0.03)	7.89 (0.09)	16.74 (0.11)
$\alpha = 3.74$ $\beta = 15.46$ $\lambda = 93.57$	100	4.55 (2.46)	15.48 (4.71)	90.66 (14.76)
	1000	3.68 (0.75)	15.83 (1.88)	94.46 (5.28)
	10000	3.74 (0.22)	15.46 (0.53)	93.60 (1.51)
	100000	3.74 (0.08)	15.47 (0.18)	93.58 (0.55)
$\alpha = 9.45$ $\beta = 163.51$ $\lambda = 2178.57$	100	11.39 (10.25)	183.02 (69.73)	2159.43 (560.00)
	1000	9.84 (2.20)	162.73 (19.11)	2159.16 (154.19)
	10000	9.56 (0.66)	162.66 (6.17)	2171.45 (49.41)
	100000	9.48 (0.18)	163.25 (1.67)	2176.68 (14.19)

Table 3: MLEs for Delaporte data simulated with previously calculated parameters across various sample sizes

		MOM Estimators		
Parameters	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
$\alpha = 1.84$ $\beta = 7.89$ $\lambda = 16.75$	100	3.13 (2.46)	7.96 (4.10)	14.08 (5.87)
	1000	1.98 (0.63)	7.89 (1.43)	16.41 (2.02)
	10000	1.86 (0.17)	7.88 (0.44)	16.71 (0.63)
	100000	1.85 (0.06)	7.87 (0.14)	16.72 (0.20)
$\alpha = 3.74$ $\beta = 15.46$ $\lambda = 93.57$	100	6.10 (4.20)	14.06 (4.84)	82.30 (20.90)
	1000	3.95 (1.23)	15.60 (2.71)	92.85 (8.31)
	10000	3.77 (0.37)	15.44 (0.81)	93.42 (2.69)
	100000	3.73 (0.10)	15.49 (0.24)	93.63 (0.70)
$\alpha = 9.45$ $\beta = 163.51$ $\lambda = 2178.57$	100	14.92 (12.81)	173.22 (95.54)	1950.67 (709.17)
	1000	10.49(4.11)	162.38 (29.14)	2125.06 (275.23)
	10000	9.62 (1.06)	162.60 (9.31)	2167.94 (79.22)
	100000	9.46 (0.28)	163.46 (2.55)	2178.36 (21.67)

Table 4: MOM estimators for Delaporte data simulated with previously calculated parameters across various sample sizes

The results of this simulation study lend credence to the validity of using MOMs in the last section.

6. Monochromatic Arithmetic Progressions

We follow a similar strategy for van der Waerden numbers. Let all possible 2-colorings of a given interval $[1, n]$ of positive integers be equally likely and define $Y_k = Y_k(n)$ to be the random variable giving the total number of monochromatic arithmetic progressions of length k in the interval. To approximate the distribution of Y_k , the program `APCount`³ takes a user input of positive integers n , k , and g and generates g instances of the integers between 1 and n each having one of two colors. The color of any given integer is randomly decided, again with Python’s random module. The program counts the total number of monochromatic arithmetic progressions of length k in each instance. Over all g instances, we then produce an empirical probability mass function for Y_k .

Though `APCount` is dealing with larger numbers than `GraphCount`, it works much more quickly, given that arithmetic progressions are simpler than graphs. However, like `GraphCount`, `APCount` has a

³Available at <http://www.aaronrobertson.org>.

run-time of $O(n^2)$ for $k = 3$ and $O(n^k)$ for $k \geq 4$.

Input k	Input n	Time per progression (sec)
3	9	0.00016
4	35	0.00049
5	178	0.0226
6	1132	4.9
7	3703	167
8	11495	4980

Table 5: Run-times for APCount

In Figure 4, we present the empirical ps for small k and n , with n near $w(k)$.

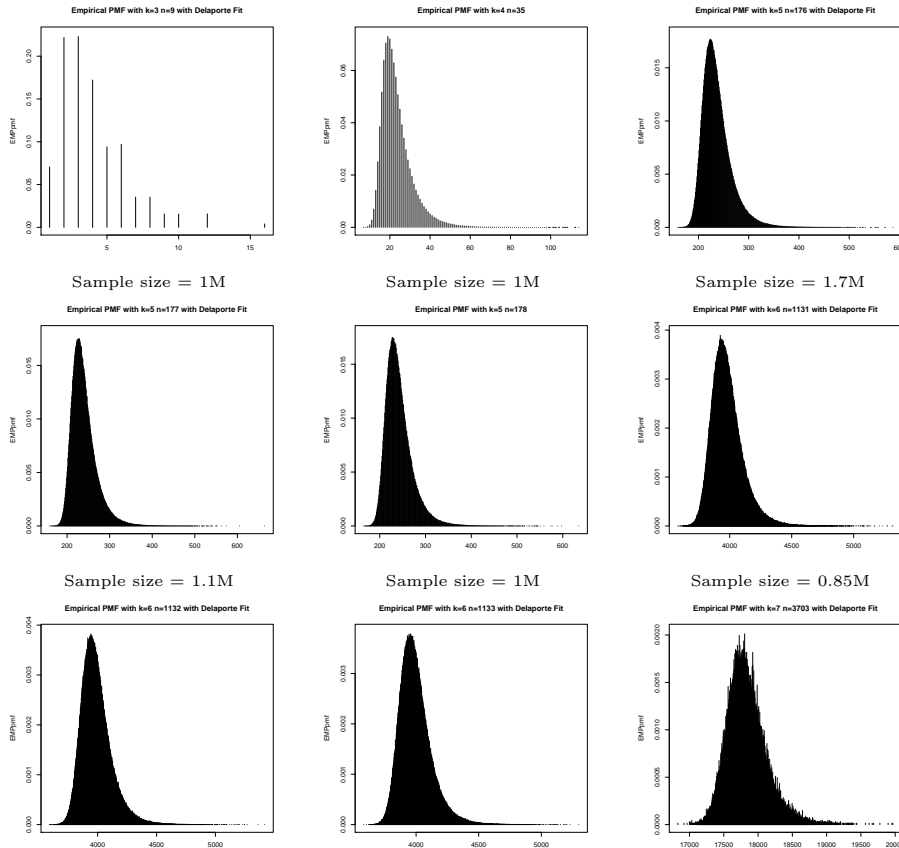


Figure 4: Empirical pmfs for the number of monochromatic arithmetic progressions

At first blush, it is quite striking that the histograms for the number of monochromatic arithmetic progressions have very similar shapes to the histograms for the number of monochromatic complete subgraphs. However, our arithmetic progressions are mostly independently colored with some dependence between some pairs of arithmetic progressions. So, heuristically, the number of monochromatic arithmetic progressions would be asymptotically Poisson by very similar reasoning to the number of monochromatic subgraphs.

We will now show that for fixed (small) k , the Poisson distribution is under-dispersed for the distribution of the number of monochromatic arithmetic progressions.

Lemma 4. *Let $k \geq 3$. We have $\lim_{n \rightarrow \infty} \frac{\mathbb{E}(Y_k(n))}{\text{Var}(Y_k(n))} < 1$.*

Proof. Let Z_i be the indicator function for whether or not the i^{th} k -term arithmetic progression is monochromatic so that $Y_k = \sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} Z_i$. By the linearity of expectation, we have $\mathbb{E}(Y_k) = \sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} \mathbb{E}(Z_i) = \frac{n^2}{2^{(k-1)}} \cdot \frac{1}{2^{k-1}} = \frac{n^2}{(k-1)2^k}$. We also have $\text{Var}(Y_k) = \text{Var}(\sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} Z_i) = \sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} \text{Var}(Z_i) + \sum_{i \neq j} \text{Cov}(Z_i, Z_j) = \sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} (\mathbb{E}(Z_i^2) - \mathbb{E}^2(Z_i)) + \sum_{i \neq j} \text{Cov}(Z_i, Z_j)$. Since $Z_i^2 = Z_i$, this simplifies to $\text{Var}(Y_k) = \sum_{i=1}^{\frac{n^2}{2^{(k-1)}}} (\frac{1}{2^{k-1}} - \frac{1}{2^{2k-2}}) + \sum_{i \neq j} \text{Cov}(Z_i, Z_j) = \frac{2^{k-1}n^2}{(k-1)2^{2k-1}} + \sum_{i \neq j} \text{Cov}(Z_i, Z_j)$.

We now must look at when $\text{Cov}(Z_i, Z_j) \neq 0$ so that we only need consider when Z_i and Z_j correspond to dependent arithmetic progressions, meaning that they share at least one term. Next, we note that if two arithmetic progressions A and B share only one term, then $\text{Cov}(A, B) = \mathbb{E}(AB) - \mathbb{E}(A)\mathbb{E}(B) = \frac{1}{2^{2k-2}} - \frac{1}{2^{k-1}} \cdot \frac{1}{2^{k-1}} = 0$.

We have $\text{Cov}(Z_i, Z_j) = \mathbb{E}(Z_i Z_j) - \mathbb{E}(Z_i)\mathbb{E}(Z_j) = \mathbb{E}(Z_i Z_j) - \frac{1}{2^{2k-2}}$. We will now give a lower bound for $\text{Cov}(Z_i, Z_j)$. Given i , for each of those j values that correspond to an arithmetic progression that shares (exactly) $t \geq 2$ values with the i^{th} arithmetic progression we have $\mathbb{E}(Z_i Z_j) = \frac{1}{2^{2k-t-1}} - \frac{1}{2^{2k-2}} \geq \frac{1}{2^{2k-t}}$. For each i , we use the trivial lower bound of 1 for the number of values of j for which the i^{th} and j^{th} arithmetic progressions share t terms. To prove the lemma we only need a trivial bound of 1 value of j .

Putting the above together and noting that for each i we are using $k-1$ values of j based on how many common terms each has with the i^{th} arithmetic progression, we get $\sum_{i \neq j} \text{Cov}(Z_i, Z_j) \geq \frac{n^2}{2^{(k-1)}} \sum_{t=2}^{k-1} \frac{1}{2^{2k-t}} = \frac{n^2}{2^{(k-1)}} (\frac{1}{2^{k+2}} - \frac{1}{2^{2k}}) \geq \frac{n^2}{(k-1)2^{k+4}}$. We now are done since

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(Y_k(n))}{\text{Var}(Y_k(n))} \leq \lim_{n \rightarrow \infty} \frac{\frac{n^2}{(k-1)2^k}}{\frac{n^2}{(k-1)2^k} + \frac{n^2}{(k-1)2^{k+4}}} = \frac{1}{1 + 2^{-4}} = \frac{16}{17} < 1.$$

□

Remark. We were unsuccessful in our attempts to show that the limit in the above lemma is 0 as is the case with monochromatic complete subgraphs.

Based on the under-dispersion of the Poisson distribution compared to Y_k and the weak dependence between monochromatic arithmetic progressions, we see that we are in a similar situation to the monochromatic subgraphs. Hence, relying on the same heuristics, we investigate how well the Delaporte distribution approximates these new empirical histograms (see Figure 5, below).

The Delaporte distribution is, again, an unusually good approximation, this time for the number of monochromatic arithmetic progressions. The reader may notice that for the last histogram in Figures 4 and 5 (with a sample size of 67K), there are spikes at the peak and that the Delaporte overlay misses these spikes. Based on our many simulations (only a fraction of which are shown in this article), we find that these spikes diminish as the sample size increases and that they settle near the Delaporte peak. We included this histogram to show what is expected when the sample size is relatively “small.”

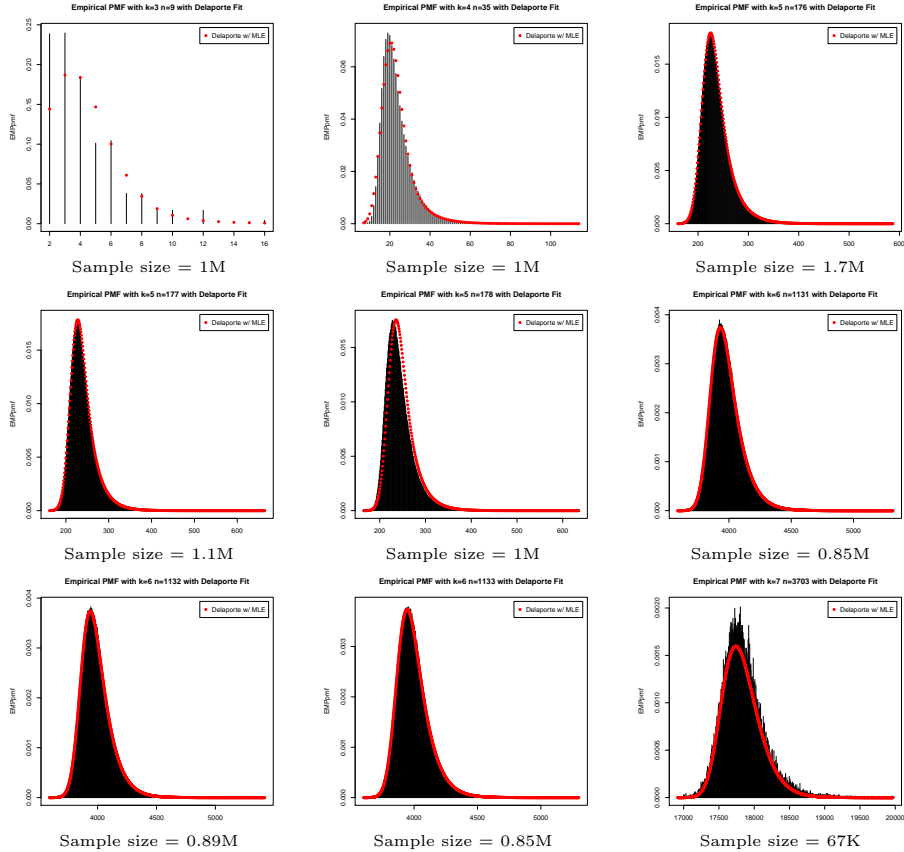


Figure 5: Empirical pmfs for various scenarios with Delaporte Overlay

Having both the number of monochromatic complete subgraphs and the number of monochromatic arithmetic progressions producing such similar empirical histograms, we end with the following question:

Is there a “Delaporte Paradigm” for Ramsey objects?

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