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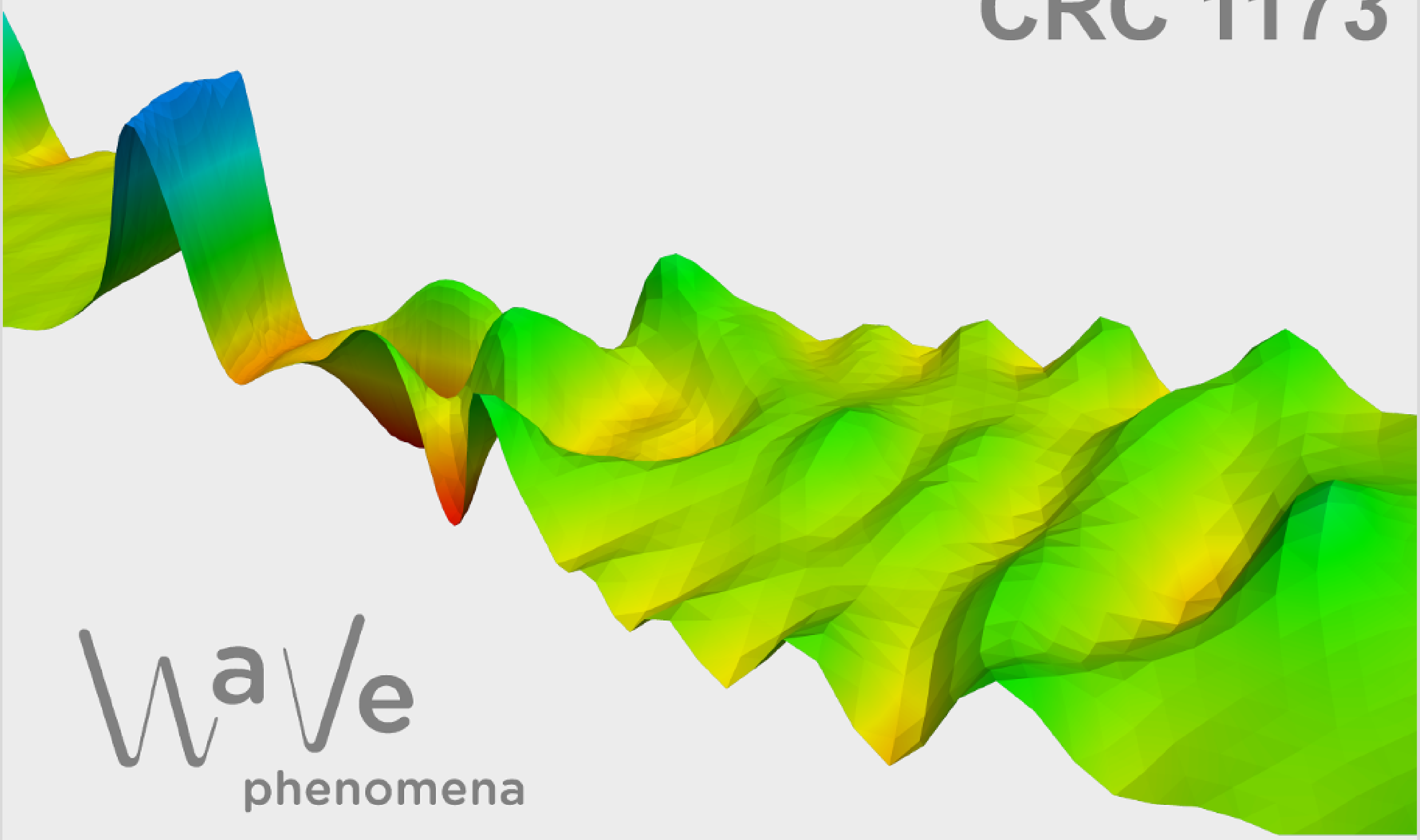

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RAYLEIGH–RITZ APPROXIMATION OF THE INF-SUP CONSTANT FOR THE DIVERGENCE

DIETMAR GALLISTL

Abstract. A numerical scheme for computing approximations to the inf-sup constant of the divergence operator in bounded Lipschitz polytopes in $\mathbb{R}^n$ is proposed. The method is based on a conforming approximation of the pressure space based on piecewise polynomials of some fixed degree $k \geq 0$. The scheme can be viewed as a Rayleigh–Ritz method and it gives monotonically decreasing approximations of the inf-sup constant under mesh refinement. The new approximation replaces the H^{-1} norm of a gradient by a discrete H^{-1} norm which behaves monotonically under mesh refinement. By discretizing the pressure space with piecewise polynomials, upper bounds to the inf-sup constant are obtained. Error estimates are presented that prove convergence rates for the approximation of the inf-sup constant provided it is an isolated eigenvalue of the corresponding non-compact eigenvalue problem; otherwise, plain convergence is achieved. Numerical computations on uniform and adaptive meshes are provided.

1. The inf-sup constant of the divergence

Let $\Omega \subseteq \mathbb{R}^n$ for $n \geq 2$ be an open, bounded, connected Lipschitz polytope, let $V := H^1_0(\Omega;\mathbb{R}^n)$ denote the L^2 vector fields over Ω with generalized first derivatives in $L^2(\Omega)$ and vanishing trace on the boundary, and let and $Q := L^2_0(\Omega)$ denote the L^2 functions with vanishing average over Ω. In fluid models, V usually refers to the space of velocity fields and Q is the pressure space. The inf-sup constant $\beta = \beta(\Omega)$ for the divergence is defined as

$$\beta := \inf_{p\in Q\setminus\{0\}} \frac{|\nabla p|_{-1}}{\|p\|} = \inf_{p\in Q\setminus\{0\}} \sup_{v\in V\setminus\{0\}} \frac{(p, \operatorname{div} v)_{L^2(\Omega)}}{\|p\|\|Dv\|} \tag{1}$$

where $|\nabla p|_{-1}$ denotes the H^{-1} norm of ∇p and $\|\cdot\|$ denotes the L^2 norm over the domain Ω. Throughout this article, the space V is equipped with the norm $\|D\cdot\|$. The inf-sup constant, sometimes also called Ladyzhenskaya–Babuška–Brezzi (LBB) constant, and its variants are in close connection to stability considerations in several applications such as fluid mechanics [21, 29], Korn's inequalities in the theory of elasticity [12], or the splitting of polyharmonic functions into second-order systems [20]. It is known [1, 8] that for a broad class of domains (including Lipschitz domains) the constant β is positive, $\beta > 0$, which implies that the divergence operator $\operatorname{div} : V \to Q$ possesses a continuous right-inverse. The numerical approximation of β is challenging because the Rayleigh quotient from (1) belongs to a non-compact

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eigenvalue problem. While there is a quite general theory for the numerical approximation of compact symmetric eigenvalue problems [3, 5], much less is known about non-compact problems like (1). The relation to non-compact operators becomes apparent when considering the related Stokes eigenvalue problem. It is well known [13] that the constant $\lambda := -\beta^2$ is equivalently described as the least-in-modulus element in the spectrum of the following Stokes eigenvalue problem: find $\lambda \in \mathbb{R}$ and $(u, p) \in V \times Q$ with $\|p\| = 1$ such that

$$\begin{bmatrix} -\Delta & \nabla \\ -\operatorname{div} & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ p \end{bmatrix}. \tag{2}$$

The first row in (2) equivalently reads

$$(Du, Dv)_{L^2(\Omega)} = (p, \operatorname{div} v)_{L^2(\Omega)} \quad \text{for all } v \in V,$$

which shows that u must be V-orthogonal on the divergence-free elements of V; here and throughout this work, V-orthogonality refers to orthogonality with respect to the scalar product $(D\cdot, D\cdot)_{L^2(\Omega)}$. By taking the gradient in the second row of (2), multiplying the first row by λ, and substituting $\lambda \nabla p$ in the first row, one finds that $-\lambda = \beta^2$ is the smallest nonzero element in the spectrum of the Cosserat operator

$$\Delta^{-1} \nabla \operatorname{div} : V \to V. \tag{3}$$

This means that

$$\beta^2 = \inf_{v \neq 0} \frac{\|\operatorname{div} v\|^2}{\|Dv\|^2} \tag{4}$$

where the infimum in (4) is taken over the V-orthogonal complement of the divergence-free functions in V. It is known [4, 18] that on non-smooth domains the Cosserat operator admits nontrivial essential parts in the spectrum.

It is well known that the discretization of non-compact eigenvalue problems requires a careful choice of finite element spaces satisfying suitable compatibility conditions. Even if one is interested in isolated eigenvalues, which is for instance the case in models of electromagnetism where it is known that the essential part of the spectrum of the involved rot rot operator is $\{0\}$, spectral pollution may occur [5, 7]. In particular, direct conforming approximations of the Rayleigh quotient (4) will generally not assign the eigenvalue 0 to discretely divergence-free elements. As an additional difficulty, the essential spectrum of the Cosserat operator may contain intervals of finite length and there is no general theory for the numerical approximation of the bottom of the spectrum. The recent work [4] established results on the approximation of β through discrete inf-sup constants β_h of stable pairs $V_h \times Q_h \subseteq V \times Q$. Therein, it is shown that any accumulation point of the discrete inf-sup constants of such 'continuous-velocity' pairs must necessarily be less than or equal to β. Whether or not those constants converge towards β is in general unknown; there are positive and negative examples. The work [4] provides sufficient conditions for convergence of β_h to β which are in particular satisfied by schemes where the pressures and velocities are discretized on different meshes and the ratio of their mesh sizes tends to zero. On the other hand, [4] proves that, for any number between 0 and β, it is possible to construct a sequence of meshes such that the discrete inf-sup constants in the Scott–Vogelius scheme [27] converge towards that value. So far, the systematic construction of monotone sequences converging (possibly with some rate) to the inf-sup constant has been an open problem.

In contrast to finite element pairings with continuous velocities, it is not difficult to show that the discrete inf-sup constant of the nonconforming P_1 or Crouzeix–Raviart finite element method [9, 15] with piecewise constant pressure approximation always provides upper bounds of β. The reason is that the related interpolation operator is stable with constant 1. This upper bound property was also empirically observed by [28]. This work theoretically justifies this observation and shows a generalization to discretizations of the pressure of arbitrary polynomial degree. It is the interpretation of a pressure-conforming scheme with a computable discrete quantity in the numerator of (1) that provides monotonically decreasing approximations. The key tool is a suitable notion of a discrete H^{-1} norm. It is based on the following alternative description of the H^{-1} norm of a gradient. Let $H(\operatorname{div}^0, \Omega)$ denote the L^2 vector fields over Ω that are divergence-free and define the following spaces

$$\Sigma := L^2(\Omega; \mathbb{R}^{n\times n}), \qquad \mathfrak{Z} := [H(\operatorname{div}^0, \Omega)]^n, \qquad \Gamma := \mathfrak{Z}^\perp \subseteq \Sigma.$$

The space Σ is the space of $\mathbb{R}^{n\times n}$-valued L^2 fields while $\mathfrak{Z}$ is the subspace of Σ of fields whose rows are divergence-free. The symbol $\perp$ denotes the L^2-orthogonal complement so that, by the classical Helmholtz decomposition, Γ equals the space of derivatives of V, $DV = \Gamma$. Therefore, for any $q \in Q$,

$$|\nabla q|_{-1} = \sup_{v\in V\setminus\{0\}} \frac{(q, \operatorname{div} v)_{L^2(\Omega)}}{\|Dv\|} = \sup_{\gamma\in\Gamma\setminus\{0\}} \frac{(q, \operatorname{tr}\gamma)_{L^2(\Omega)}}{\|\gamma\|} \tag{5}$$

where tr denotes the trace of a matrix. This formulation gives rise to the definition of a discrete H^{-1} norm $|\cdot|_{-1,h}$ where Γ is replaced by a suitable space of "discrete gradients". The details are given in Section 2. This discrete norm is used to define a discrete inf-sup constant β_h by discretizing Q with piecewise polynomials and replacing $|\cdot|_{-1}$ by $|\cdot|_{-1,h}$ in (1). It turns out that β_h converges monotonically from above to β when the mesh is refined. The convergence is the principal result of this work and is stated in Theorem 11 below. Provided β^2 is an isolated eigenvalue of the Cosserat operator (3), β_h even converges at a rate that is determined by the smoothness of the corresponding eigenfunction. Numerical results are presented that empirically show that adaptive mesh refinement can significantly improve the computational efficiency of this low-regularity problem, which is typically related to functions of very low regularity.

The remaining parts of this article are organized as follows. The novel Rayleigh–Ritz-type approximation is based on the notion of the discrete H^{-1} norm of a gradient introduced in Section 2. The numerical method is defined in in Section 3 and its convergence is analyzed in Section 4. The numerical results in Section 5 conclude the paper.

2. The discrete H^{-1} norm of a gradient

This section presents the novel notion of the discrete H^{-1} norm of a gradient based on the observation (5). Let $\mathcal{T}$ be a regular simplicial partition of Ω and let $k \geq 0$ denote a fixed polynomial degree. The space of polynomials with respect to a domain $\omega \subseteq \mathbb{R}^n$ of degree not larger than k with values in $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ is denoted by $P_k(\omega)$, $P_k(\omega; \mathbb{R}^n)$, $P_k(\omega; \mathbb{R}^{n\times n})$, respectively. The L^2 functions over Ω that are piecewise polynomials with respect to $\mathcal{T}$ are analogously denoted by $P_k(\mathcal{T})$,

$P_k(\mathcal{T};\mathbb{R}^n)$, $P_k(\mathcal{T};\mathbb{R}^{n\times n})$. Define the following spaces

$$Q_h := P_k(\mathcal{T}) \cap Q,\ \Sigma_h := P_k(\mathcal{T};\mathbb{R}^{n\times n}),\ \mathfrak{Z}_h := RT_k(\mathcal{T})^n \cap \mathfrak{Z},\ \Gamma_h := \mathfrak{Z}_h^\perp \subseteq \Sigma_h.$$

Here, the symbol $\perp$ denotes the L^2-orthogonal complement in Σ_h and $RT_k(\mathcal{T})$ is the $H(\operatorname{div},\Omega)$ conforming Raviart–Thomas finite element space [6] defined by

$$RT_k(\mathcal{T}) := \left\{ q \in H(\operatorname{div},\Omega) \,\middle|\, \begin{array}{l} \forall T \in \mathcal{T}\ \exists (\alpha,\beta) \in P_k(T;\mathbb{R}^n) \times P_k(T), \\ \forall x \in T\ q|_T(x) = \alpha(x) + \beta(x)x \end{array} \right\}.$$

Note that generally $\mathfrak{Z}_h \subseteq \mathfrak{Z}$, but $\Gamma_h \not\subseteq \Gamma$. Let $\Pi_h : L^2(\Omega) \to P_k(\mathcal{T})$ denote the L^2 projection onto the piecewise polynomials of degree k. If applied to tensors, the action of Π_h is understood component-wise.

Lemma 1 (divergence-free RT_k fields)**.** *Any divergence-free element of $RT_k(\mathcal{T})$ is a piecewise polynomial of degree not larger than k. In other words, the following inclusion holds $\mathfrak{Z}_h \subseteq \Sigma_h$.*

Proof. The proof departs with a technical observation. Assume $\beta^{(k)} \in P_k(\mathbb{R}^n)$ is a polynomial of degree $k \geq 0$ satisfying $\beta^{(k)}(x) = \alpha^{(k-1)}(x) - n^{-1}\nabla\beta^{(k)} \cdot x$ for all $x \in \mathbb{R}^n$, where $\alpha^{(k-1)} \in P_{k-1}(\mathbb{R}^n)$ with the convention $P_{-1}(\mathbb{R}^n) = \{0\}$. Then $\beta^{(k)} \in P_{k-1}(\mathbb{R}^n)$. The proof of this claim follows from mathematical induction. For $k = 0$ it is obviously true. Let $k \geq 1$ and assume as induction hypothesis that the claim holds for $k-1$. Taking the derivative with respect to $j \in \{1,\dots,n\}$ in the representation of $\beta^{(k)}$ results in

$$\begin{aligned} \partial_j\beta^{(k)}(x) &= \partial_j\alpha^{(k-1)}(x) - n^{-1}\partial_j\beta^{(k)}(x) - n^{-1}\nabla(\partial_j\beta^{(k)})(x)\cdot x \\ &= \tilde\alpha^{(k-1)}(x) - n^{-1}\nabla(\partial_j\beta^{(k)})(x)\cdot x \end{aligned}$$

for some polynomial $\tilde\alpha^{(k-1)} \in P_{k-1}(\mathbb{R}^n)$. The induction hypothesis applied to $\partial_j\beta^{(k)} \in P_{k-1}(\mathbb{R}^n)$ shows that all partial derivatives of $\beta^{(k)}$ are polynomials of degree not larger than $(k-2)$, whence $\beta^{(k)} \in P_{k-1}(\mathbb{R}^n)$.

For the proof of the lemma, let $\sigma \in RT_k(\mathcal{T})$ and let $T \in \mathcal{T}$ be arbitrary. By definition of $RT_k(\mathcal{T})$, there exist polynomials $\alpha \in P_k(T;\mathbb{R}^n)$ and $\beta \in P_k(T)$ such that

$$\sigma(x) = \alpha(x) + \beta(x)x \quad \text{for all } x \in T.$$

If $\operatorname{div}\sigma = 0$, taking the divergence of this representation leads to

$$\beta(x) = -n^{-1}\operatorname{div}\alpha(x) - n^{-1}\nabla\beta(x)\cdot x.$$

The assertion shown at the beginning of the proof reveals that $\beta \in P_{k-1}(T)$ and, thus, $\sigma|_T \in P_k(T;\mathbb{R}^n)$. ☐

The next result states a compatibility relation.

Lemma 2 (projection property)**.** *Any $v \in V$ satisfies $\Pi_h Dv \in \Gamma_h$.*

Proof. It is readily verified from the definitions of Π_h and Σ_h that $\Pi_h Dv \in \Sigma_h$ for any $v \in V$. For any $q_h \in \mathfrak{Z}_h$ the inclusion $\mathfrak{Z}_h \subseteq \Sigma_h$ from Lemma 1 implies that

$$(\Pi_h Dv, q_h)_{L^2(\Omega)} = (Dv, q_h)_{L^2(\Omega)}.$$

This equals zero because q_h is divergence-free. ☐

The foregoing lemmas have prepared the definition of the discrete H^{-1} norm as a generalization of the usual H^{-1} norm. Let $p \in Q$. The H^{-1} norm of ∇p is defined by

$$|\nabla p|_{-1} := \sup_{v \in V \setminus \{0\}} \frac{(p, \operatorname{div} v)_{L^2(\Omega)}}{\|\nabla v\|}.$$

The classical Helmholtz decomposition states that $\sigma \in L^2(\Omega; \mathbb{R}^n)$ satisfies $\sigma = \nabla v$ for some $v \in H^1_0(\Omega)$ if and only if $(\sigma, q)_{L^2(\Omega)} = 0$ for all $q \in \mathfrak{Z}$. In other words, the gradients of V form the L^2-orthogonal complement of $\mathfrak{Z}$. Thus, the H^{-1} norm can equivalently be written as

$$|\nabla p|_{-1} = \sup_{\gamma \in \Gamma \setminus \{0\}} \frac{(p, \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|\gamma\|}. \tag{6}$$

The discrete H^{-1} norm (in general a seminorm) of ∇p is a discrete analogue and is defined as

$$|\nabla p|_{-1,h} := \sup_{\gamma_h \in \Gamma_h \setminus \{0\}} \frac{(p, \operatorname{tr} \gamma_h)_{L^2(\Omega)}}{\|\gamma_h\|}.$$

The first important property is the monotonicity under mesh refinement. In particular, it shows that $|\cdot|_{h,-1}$ is a norm when restricted to discrete functions.

Lemma 3 (monotonicity of norms). *Let $\mathcal{T}_H$ be a simplicial partition of Ω (the corresponding spaces and norms are indexed by the parameter H) and let $\mathcal{T}_h$ be a regular refinement (indexed by the parameter h). Then, any $p_H \in Q_H$ satisfies*

$$|\nabla p_H|_{-1} \leq |\nabla p_H|_{-1,h} \leq |\nabla p_H|_{-1,H}.$$

Proof. Since p_H is a piecewise polynomial of degree k with respect to $\mathcal{T}_H$, the characterization (6) and the piecewise L^2 projection Π_H give

$$|\nabla p_H|_{-1} = \sup_{\gamma \in \Gamma \setminus \{0\}} \frac{(p_H, \Pi_H \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|\gamma\|} \leq \sup_{\substack{\gamma \in \Gamma \\ \Pi_H \gamma \neq 0}} \frac{(p_H, \Pi_H \operatorname{tr} \gamma)_{L^2(\Omega)}}{\|\Pi_H \gamma\|}.$$

Since by Lemma 1, $\Pi_H \Gamma \subseteq \Gamma_H$, this implies

$$|\nabla p_H|_{-1} \leq \sup_{\gamma_H \in \Gamma_H \setminus \{0\}} \frac{(p_H, \operatorname{tr} \gamma_H)_{L^2(\Omega)}}{\|\gamma_H\|} = |\nabla p_H|_{-1,H}.$$

An analogous argument for the $|\cdot|_{-1,h}$ norm concludes the proof. □

3. A Rayleigh–Ritz approximation of the inf-sup constant

The proposed Rayleigh–Ritz-type method is to replace Q by the closed subspace $Q_h \subseteq Q$ of functions that are piecewise polynomials of degree k and to determine

$$\beta_h := \inf_{p_h \in Q_h \setminus \{0\}} \frac{|\nabla p_h|_{-1,h}}{\|p_h\|} \tag{7}$$

as an approximation to (1).

Remark 4 (relation to nonconforming schemes). For $k = 0$, the value β_h from (7) coincides with the discrete inf-sup constant of the nonconforming P_1 scheme [15] for the Stokes equations because in that scheme the piecewise gradients of the (possibly discontinuous) trial functions form the orthogonal complement of the divergence–free RT_0 fields. Such discrete Helmholtz decompositions were first utilized in [2] in the numerical analysis of Reissner–Mindlin plates. Later, they formed the point of departure for a generalization of the Crouzeix–Raviart scheme to arbitrary polynomial degree [24, 26].

One observation is that these approximations form a monotonically decreasing sequence under mesh refinement.

Lemma 5 (monotonicity of inf-sup constants)**.** *Let $\mathcal{T}_H$ be a simplicial partition of Ω (indexed by the parameter H) with solution β_H to* (7) *with respect to Q_H and let $\mathcal{T}_h$ be a regular refinement (indexed by the parameter h) with solution β_h to* (7) *with respect to Q_h. Then*

$$\beta \leq \beta_h \leq \beta_H.$$

Proof. The definition (1), the inclusion $Q_h \subseteq Q$, elementary properties of the infimum, and Lemma 3 imply

$$\beta = \inf_{p\in Q\setminus\{0\}} \frac{|\nabla p|_{-1}}{\|p\|} \leq \inf_{p_h\in Q_h\setminus\{0\}} \frac{|\nabla p_h|_{-1}}{\|p_h\|} \leq \inf_{p_h\in Q_h\setminus\{0\}} \frac{|\nabla p_h|_{-1,h}}{\|p_h\|} = \beta_h.$$

Analogously, one obtains with $Q_H \subseteq Q_h$ that

$$\beta_h = \inf_{p_h\in Q_h\setminus\{0\}} \frac{|\nabla p_h|_{-1,h}}{\|p_h\|} \leq \inf_{p_H\in Q_H\setminus\{0\}} \frac{|\nabla p_H|_{-1,h}}{\|p_H\|} \leq \inf_{p_H\in Q_H\setminus\{0\}} \frac{|\nabla p_H|_{-1,H}}{\|p_H\|} = \beta_H.$$

This concludes the proof. □

The goal of the remaining parts of this section is to establish the equivalence of (7) with discrete versions of the Stokes and Cosserat eigenvalue problems. These reformulations give rise to a quantitative a priori error estimate for the difference between β_h and β.

The discrete Stokes eigenvalue problem is to find $\lambda_h \in \mathbb{R}$ and $(\sigma_h, p_h) \in \Gamma_h \times Q_h$ with $\|p_h\| = 1$ such that

$$(\sigma_h, \tau_h)_{L^2(\Omega)} - (p_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = 0 \quad \text{for all } \tau_h \in \Gamma_h \tag{8a}$$

$$-(\operatorname{tr}\sigma_h, q_h)_{L^2(\Omega)} = \lambda_h (p_h, q_h)_{L^2(\Omega)} \quad \text{for all } q_h \in Q_h. \tag{8b}$$

This is the discrete counterpart to (2). Similarly as in the continuous case, the relation (8a) states a certain orthogonality relation. Based on this, define the space

$$X_h := \{\tau_h \in \Gamma_h : (\tau_h, \eta_h)_{L^2(\Omega)} \text{ for all } \eta_h \in \Gamma_h \text{ with } \operatorname{tr}\eta_h = 0\}.$$

The discrete Cosserat eigenvalue problem seeks $\mu_h \in \mathbb{R}$ and $\xi_h \in X_h$ with $\|\xi_h\| = 1$ such that

$$(\operatorname{tr}\xi_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = \mu_h (\xi_h, \tau_h)_{L^2(\Omega)} \quad \text{for all } \tau_h \in X_h. \tag{9}$$

The following result states the equivalence of (8) and (9) and their relation to (7).

Proposition 6 (algebraic equivalence)**.** *The eigenpairs of* (8) *and* (9) *are in one-to-one correspondence: if $(\lambda_h, p_h, \sigma_h) \in \mathbb{R} \times Q_h \times \Gamma_h$ with $\|p_h\| = 1$ solves* (8)*, then $\|\sigma\| > 0$ and the pair (μ_h, ξ_h) defined by $\mu_h := -\lambda_h$, $\xi_h := \|\sigma_h\|^{-1}\sigma_h$ belongs to $\mathbb{R} \times X_h$ and solves* (9)*. If conversely $(\mu_h, \xi_h) \in \mathbb{R} \times X_h$ is an eigenpair of* (9)*, then $\lambda_h := -\mu_h$, $p_h := \mu_h^{-1/2} \operatorname{tr}\xi_h \in Q_h$, $\sigma_h := \mu_h^{1/2}\xi_h \in \Gamma_h$ form an eigensolution of* (8)*. All eigenvalues μ_h to* (9) *are positive. The square root of the smallest eigenvalue μ_h of* (9) *coincides with β_h from* (7)*.*

Proof. To see that tr maps Γ_h to Q_h, let $\tau_h \in \Gamma_h$. Then $\operatorname{tr}\tau_h \in P_k(\mathcal{T})$ and, for the $n\times n$ unit matrix $I_{n\times n}$,

$$\int_\Omega \operatorname{tr}\tau_h\,dx = \int_\Omega \tau_h : I_{n\times n}\,dx = 0 \tag{10}$$

because $I_{n\times n} \in \mathfrak{Z}_h$. Thus, tr is a surjective map from Γ_h to Q_h. For any $q_h \in Q_h$ with $\|q_h\| = 1$, the definition (1) and Lemma 3 imply

$$\beta\|q_h\| \le |q_h|_{-1} \le |q_h|_{-1,h}. \tag{11}$$

This is a discrete inf-sup condition and implies that $\operatorname{tr} : \Gamma_h \to Q_h$ is surjective.

Condition (11) shows that the saddle-point problem (8) has full rank and all eigenvalues λ_h are nonzero. Let $(\lambda_h, p_h, \sigma_h) \in \mathbb{R}\times Q_h \times \Gamma_h$ with $\|p_h\| = 1$ solve (8). Since $\|p_h\| = 1$, the equations (8b) and (8a) imply

$$0 \ne \lambda_h = -(\operatorname{tr}\sigma_h, p_h)_{L^2(\Omega)} = -\|\sigma_h\|^2.$$

This shows that $\|\sigma_h\| > 0$. Let $\tau_h \in \Gamma_h$ be arbitrary and define $q_h := \operatorname{tr}\tau_h$. By (10), $q_h \in Q_h$. Thus, (8b) implies

$$(\operatorname{tr}\sigma_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = (\operatorname{tr}\sigma_h, q_h)_{L^2(\Omega)} = -\lambda_h(p_h, q_h)_{L^2(\Omega)} = -\lambda_h(p_h, \operatorname{tr}\tau_h)_{L^2(\Omega)}.$$

This and (8a) lead to

$$(\operatorname{tr}\sigma_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = -\lambda_h(\sigma_h, \operatorname{tr}\tau_h)_{L^2(\Omega)}.$$

Since $\|\sigma_h\| > 0$, the field σ_h can be suitably normalized to an eigenfunction of (9) with eigenvalue $\mu_h := -\lambda_h$. The relation (8a) assures that σ_h indeed belongs to X_h.

Let conversely $(\mu_h, \xi_h) \in \mathbb{R}\times X_h$ with $\|\xi_h\| = 1$ be an eigenpair of (9). The definition of X_h shows that all eigenvalues of (9) are nonzero and, thus, $\|\operatorname{tr}\xi_h\|^2 = \mu_h > 0$. Define $p_h := \mu^{-1/2}\operatorname{tr}\xi_h \in Q_h$ and $\sigma_h := \mu_h^{1/2}\xi_h \in \Gamma_h$. Then $\|p\| = 1$, and the relation (9) implies, for any $\tau_h \in \Gamma_h$,

$$\begin{aligned}(p_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} &= \mu^{-1/2}(\operatorname{tr}\xi_h, \operatorname{tr}\tau_h)_{L^2(\Omega)}\\ &= \mu_h^{1/2}(\xi_h, \tau_h)_{L^2(\Omega)} = (\sigma_h, \tau_h)_{L^2(\Omega)}.\end{aligned} \tag{12}$$

Thus, (σ_h, p_h) satisfies (8a). By the above inf-sup condition (11), for any $q_h \in Q_h$ there exists $\tau_h \in \Gamma_h$ with $q_h = \operatorname{tr}\tau_h$. The definitions of σ_h and p_h and the relations (9) and (12) therefore show

$$\begin{aligned}-(\operatorname{tr}\sigma_h, q_h)_{L^2(\Omega)} &= -\mu_h^{1/2}(\operatorname{tr}\xi_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = -\mu_h^{3/2}(\xi_h, \tau_h)_{L^2(\Omega)}\\ &= -\mu_h(\sigma_h, \tau_h)_{L^2(\Omega)} = -\mu_h(p_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} = -\mu_h(p_h, q_h)_{L^2(\Omega)}.\end{aligned}$$

Thus, (σ_h, p_h) solves (8b) with $\lambda_h := -\mu_h$. This establishes the one-to-one correspondence of the eigenvalues.

For the proof that the least-in-modulus eigenvalues determine the discrete inf-sup constant, observe the following discrete version of the usual isometry from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$ through the Laplacian or, more precisely, its restriction to the orthogonal complement of the divergence-free functions. Given $q_h \in Q_h$, let $\xi(q_h) \in X_h$ solve

$$(\xi(q_h), \tau_h)_{L^2(\Omega)} = (q_h, \operatorname{tr}\tau_h)_{L^2(\Omega)} \quad \text{for all } \tau_h \in X_h. \tag{13}$$

The structure of the right-hand side in this equation shows that (13) even holds for all test functions in Γ_h because both sides in (13) vanish when tested with $\tau_h \in \Gamma_h$

that are trace-free. Thus,

$$|\nabla q_h|_{-1,h} = \sup_{\gamma_h\in\Gamma_h\setminus\{0\}} \frac{(q_h, \operatorname{tr}\gamma_h)_{L^2(\Omega)}}{\|\gamma_h\|} = \sup_{\gamma_h\in\Gamma_h\setminus\{0\}} \frac{(\xi(q_h), \gamma_h)_{L^2(\Omega)}}{\|\gamma_h\|} = \|\xi(q_h)\|$$

and, with (13),

$$|\nabla q_h|^2_{-1,h} = (q_h, \operatorname{tr}\xi(q_h))_{L^2(\Omega)}.$$

Therefore, the definition of β_h from (7) shows

$$\beta_h^2 = \inf_{q_h\in Q_h\setminus\{0\}} \frac{|\nabla q_h|^2_{-1,h}}{\|q_h\|^2} = \inf_{q_h\in Q_h\setminus\{0\}} \frac{(q_h, \operatorname{tr}\xi(q_h))_{L^2(\Omega)}}{\|q_h\|^2}.$$

The last expression is the Rayleigh quotient corresponding to (8b) and therefore equals $-\lambda_h$ for the least-in-modulus eigenvalue λ_h, which in turn equals the smallest eigenvalue μ_h of (9). □

Let $\mathfrak{P}_h : \Sigma \to X_h$ denote the L^2-orthogonal projection onto the space X_h. It satisfies the following nonexpansivity property.

Lemma 7 ($\mathfrak{P}_h$ is nonexpansive with respect to $\|\operatorname{tr}\cdot\|$)**.** *Any $v \in V$ satisfies*

$$\operatorname{tr}\mathfrak{P}_h Dv = \operatorname{tr}\Pi_h Dv \qquad \text{and} \qquad \|\operatorname{tr}\mathfrak{P}_h Dv\| \le \|\operatorname{tr} Dv\|.$$

Proof. Let $v \in V$. By Lemma 1, the projection Π_h maps Γ to Γ_h. Thus, $\mathfrak{P}_h \circ \Pi_h = \mathfrak{P}_h$ and the function $\Pi_h Dv \in \Gamma_h$ can be decomposed as

$$\Pi_h Dv = \mathfrak{P}_h Dv + (1 - \mathfrak{P}_h)\Pi_h Dv.$$

By definition, $(1 - \mathfrak{P}_h)$ is the orthogonal projection onto the trace-free elements of Γ_h. Thus, taking the trace in the above relation reveals $\operatorname{tr}\mathfrak{P}_h Dv = \operatorname{tr}\Pi_h Dv$ and so

$$\|\operatorname{tr}\mathfrak{P}_h Dv\| = \|\operatorname{tr}\Pi_h Dv\| \le \|\operatorname{tr} Dv\|.$$

This concludes the proof. □

The next lemma states a first error estimate.

Lemma 8 (error estimate)**.** *Let $u \in V$ with $\|Du\| = 1$ be an arbitrary element of V that is V-orthogonal to the divergence-free subspace of V, i.e.,*

$$(Du, Dv)_{L^2(\Omega)} = 0 \quad \text{for all } v \in V \text{ with } \operatorname{div} v = 0.$$

Denote $\tilde\mu := \|\operatorname{div} u\|^2$ and let μ_h denote the smallest eigenvalue to (9). Then, the following error estimate holds

$$(1 - \|(1 - \mathfrak{P}_h)Du\|^2)(\mu_h - \tilde\mu) \le \tilde\mu\|(1 - \mathfrak{P}_h)Du\|^2.$$

Proof. The Rayleigh-Ritz principle and Lemma 7 show that

$$\mu_h\|\mathfrak{P}_h Du\|^2 \le \|\operatorname{tr}\mathfrak{P}_h Du\|^2 \le \|\operatorname{tr} Du\|^2 = \|\operatorname{div} u\|^2 = \tilde\mu.$$

The Pythagoras rule with $\|Du\|^2 = 1$ reads

$$\|\mathfrak{P}_h Du\|^2 = 1 - \|(1 - \mathfrak{P}_h)Du\|^2.$$

Thus,

$$(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\mu_h \le \tilde\mu.$$

Subtracting $(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\tilde\mu$ on both sides proves the claim. □

4. Convergence analysis

Lemma 8 established an abstract error estimate. In order to conclude convergence and quantitative approximation results, the approximation properties of $\mathfrak{P}_h$ are studied in this section. Those serve as a tool in the proof of the main result, Theorem 11.

Lemma 9 (approximation by $\mathfrak{P}_h$)**.** *Let $u \in V$ with $\|Du\| = 1$ be an element of V that is V-orthogonal to the divergence-free subspace of V. Then*

$$\|Du - \mathfrak{P}_h Du\| \to 0 \quad \text{as } h \to 0$$

where h denotes the maximum mesh size. Provided $u \in H^{1+s}(\Omega; \mathbb{R}^n)$ for some $0 < s < \infty$ and u is an eigenfunction of (2)*, then*

$$\|Du - \mathfrak{P}_h Du\| \leq C h^r \|u\|_{H^{1+s}(\Omega)}$$

for the rate $r := \min\{k+1, s\}$ and some mesh-size independent constant $C > 0$.

Remark 10 (notation)**.** In the notation in the statement of the first estimate in Lemma 9 the simplicial triangulation is tacitly identified with its maximum mesh-size. The result is true for an arbitrary choice of $\mathcal{T}$ of mesh-size h within the shape-regular triangulations.

Proof of Lemma 9. Since u is orthogonal to the divergence-free functions in V, the solution $(u_\star, p) \in V \times Q$ to the following linear Stokes system

$$\begin{aligned} (Du_\star, Dv)_{L^2(\Omega)} - (p, \operatorname{div} v)_{L^2(\Omega)} &= (Du, Dv)_{L^2(\Omega)} && \text{for all } v \in V \\ (\operatorname{div} u_\star, q) &= 0 && \text{for all } q \in Q \end{aligned}$$

satisfies $u_\star = 0$. In other words, Du has a trivial component in the subspace of trace-free derivatives, the trace-free elements of Γ. This is equivalently written with $\sigma := Du_\star \in \Gamma$ as

$$\begin{aligned} (\sigma, \gamma)_{L^2(\Omega)} - (p, \operatorname{tr} \gamma)_{L^2(\Omega)} &= (Du, \gamma)_{L^2(\Omega)} && \text{for all } \gamma \in \Gamma \\ (\operatorname{tr} \sigma, q) &= 0 && \text{for all } q \in Q. \end{aligned}$$

The property $\sigma \in \Gamma$ can be encoded with an additional multiplier such that $(\sigma, p, \mathfrak{z}) \in \Sigma \times Q \times \mathfrak{Z}$ is the unique solution to

$$\begin{aligned} (\sigma, \tau)_{L^2(\Omega)} + (\mathfrak{z}, \tau)_{L^2(\Omega)} - (p, \operatorname{tr} \tau)_{L^2(\Omega)} &= (Du, \tau)_{L^2(\Omega)} && \text{for all } \tau \in \Sigma \\ (\sigma, \alpha)_{L^2(\Omega)} &= 0 && \text{for all } \alpha \in \mathfrak{Z} \\ (\operatorname{tr} \sigma, q)_{L^2(\Omega)} &= 0 && \text{for all } q \in Q. \end{aligned} \tag{14}$$

The well-posedness of this system is readily verified with the usual criteria for saddle-point systems [6]. Indeed, one has $\mathfrak{Z} \subseteq \Sigma$ and therefore the an obvious inf-sup condition for the first saddle point. The surjectivity of $\operatorname{tr} : \Gamma \to Q$ is exactly the inf-sup condition (1). This establishes the unique existence of $(\sigma, p, \mathfrak{z})$. The discrete analogue is to seek $(\sigma_h, p_h, \mathfrak{z}_h) \in \Sigma_h \times Q_h \times \mathfrak{Z}_h$ such that

$$\begin{aligned} (\sigma_h, \tau_h)_{L^2(\Omega)} + (\mathfrak{z}_h, \tau_h)_{L^2(\Omega)} - (p_h, \operatorname{tr} \tau_h)_{L^2(\Omega)} &= (Du, \tau_h)_{L^2(\Omega)} && \text{for all } \tau_h \in \Sigma_h \\ (\sigma_h, \alpha_h)_{L^2(\Omega)} &= 0 && \text{for all } \alpha_h \in \mathfrak{Z}_h \\ (\operatorname{tr} \sigma_h, q_h)_{L^2(\Omega)} &= 0 && \text{for all } q_h \in Q_h. \end{aligned}$$

As above, the inclusion $\mathfrak{Z}_h \subseteq \Sigma_h$ and the discrete inf-sup condition (7) (with $\beta_h > 0$ by Proposition 6) show that the discrete system is well posed. The two principal properties are the following. Firstly, $\sigma_h = (\Pi_h - \mathfrak{P}_h) Du$. This is easily verified from the fact that $(\Pi_h - \mathfrak{P}_h)Du \in \Gamma$ satisfies $\operatorname{tr}(\Pi_h - \mathfrak{P}_h)Du = 0$ by Lemma 7 as well as

$$((\Pi_h - \mathfrak{P}_h)Du, \tau_h)_{L^2(\Omega)} = (Du, \tau_h)_{L^2(\Omega)}$$

for all trace-free elements of Γ_h. Secondly, the discrete system is a conforming approximation to (14), and the well-known theory of approximation of saddle-point problems [6] shows that the following error estimate holds for some mesh-size independent constant $C > 0$

$$\|\sigma - \sigma_h\| + \|\mathfrak{z} - \mathfrak{z}_h\| + \|p - p_h\| \leq C\big(\inf_{\alpha_h \in \mathfrak{Z}_h} \|\mathfrak{z} - \alpha_h\| + \|(1 - \Pi_h)p\|\big)$$

where it has been used that $\sigma = 0$ and, thus, its best-approximation in Σ_h equals zero. The triangle inequality and $\sigma - \sigma_h = (\mathfrak{P}_h - \Pi_h)Du$ therefore show

$$\begin{aligned}\|(1 - \mathfrak{P}_h)Du\| &\leq \|(1 - \Pi_h)Du\| + \|\sigma - \sigma_h\| \\ &\leq \|(1 - \Pi_h)Du\| + C\big(\inf_{\alpha_h \in \mathfrak{Z}_h} \|\mathfrak{z} - \alpha_h\| + \|(1 - \Pi_h)p\|\big).\end{aligned}$$

The density of the spaces Σ_h in Σ, $\mathfrak{Z}_h$ in $\mathfrak{Z}$, and Q_h in Q when $\|h\|_\infty \to 0$ shows the stated convergence.

For the proof of the second asserted estimate, assume $u \in H^{1+s}(\Omega; \mathbb{R}^n)$ for some $0 < s < \infty$ is an eigenfunction of (2). Testing the first equation of (14) with elements of Γ and the fact that $\sigma = 0$ show together with (2) that $p = -\lambda^{-1} \operatorname{div} u$ for the corresponding eigenvalue λ. The first row of (14) also states that

$$\mathfrak{z} - pI_{n\times n} = Du \quad \text{almost everywhere in } \Omega.$$

Since p is a multiple of $\operatorname{div} u$, this shows that the entries of $\mathfrak{z}$ are linear combinations of partial derivatives of u, in particular

$$p \in H^s(\Omega) \quad \text{and} \quad \mathfrak{z} \in H^s(\Omega; \mathbb{R}^{n\times n}).$$

The stated convergence rate follows from standard approximation results. While those are classical for the approximation of Du and p [10], they can be inferred for $\mathfrak{z}$ with the help of the projective quasi-interpolation operator of [17], which can also be used to prove the density of the spaces $\mathfrak{Z}_h$ in $\mathfrak{Z}$. That operator is a stable projection from $H(\operatorname{div}, \Omega)$ to $RT_k(\mathcal{T})$ and maps $\mathfrak{Z}$ to $\mathfrak{Z}_h$. As a stable projection, it is quasi-optimal and shows that the best-approximation of $\mathfrak{z} \in \mathfrak{Z}$ by elements of $\mathfrak{Z}_h$ is comparable with the best-approximation in $RT_k(\mathcal{T})$, which is known [6] to give the claimed rate. □

The main result of this work the following convergence result for the discrete inf-sup constant.

Theorem 11 (convergence). *Let $\mathbb{T}$ be a shape-regular family of simplicial partitions of Ω and let $(\mathcal{T}_\ell)_{\ell\geq 0} \in \mathbb{T}^{\mathbb{N}}$ be a sequence of nested partitions such that the maximum mesh size uniformly converges to zero as $\ell \to \infty$. Let β_ℓ denote the discrete inf-sup constant (7) with respect to the mesh $\mathcal{T}_\ell$. Then the sequence $(\beta_\ell)_{\ell\geq 0}$ converges monotonically from above towards the inf-sup constant β from (1), i.e.,*

$$\beta_\ell \searrow \beta \quad \text{as } \ell \to \infty.$$

Provided the square of the inf-sup constant β^2 is an eigenvalue of the Cosserat operator (3) with eigenfunction $u \in H^{1+s}(\Omega; \mathbb{R}^n)$ for some $0 < s < \infty$, then any $\mathcal{T} \in \mathbb{T}$ with maximum mesh size h satisfies

$$(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\frac{\beta_h^2 - \beta^2}{\beta^2} \leq \|(1 - \mathfrak{P}_h)Du\|^2 \leq Ch^{2r}\|u\|^2_{H^{1+s}(\Omega)}$$

for the rate $r := \min\{k+1, s\}$ *and some mesh-size independent constant* $C > 0$.

Proof. The formula (4) implies that for any given $\varepsilon > 0$ there exists $u \in V$ with $\|Du\| = 1$ that is V-orthogonal to the divergence-free elements of V and satisfies $\beta \leq \|\operatorname{div} u\| \leq \beta + \varepsilon$. Denoting $\tilde{\mu} := \|\operatorname{div} u\|^2$, one infers

$$\tilde{\mu} \leq (\beta + \varepsilon)^2 = \beta^2 + 2\beta\varepsilon + \varepsilon^2. \tag{15}$$

The approximation properties of $\mathfrak{P}_\ell$ from Lemma 9 $(\mathcal{T}_\ell)_{\ell \geq 0}$ in Γ show that there exists $\ell_0 \in \mathbb{N}$ such that

$$\|(1 - \mathfrak{P}_\ell)Du\|^2 \leq \frac{\varepsilon}{1+\varepsilon} \quad \text{for all } \ell \geq \ell_0.$$

Thus, Lemma 8 implies for the smallest eigenvalue μ_ℓ of (9) with respect to $\mathcal{T}_\ell$ that

$$\mu_\ell - \tilde{\mu} \leq \varepsilon\tilde{\mu} \quad \text{for all } \ell \geq \ell_0.$$

Applying the relation $\mu_\ell = \beta_\ell^2$ from Proposition 6 and invoking (15) twice therefore lead to

$$\beta_\ell^2 - \beta^2 - 2\beta\varepsilon - \varepsilon^2 \leq \mu_\ell - \tilde{\mu} \leq \varepsilon\tilde{\mu} \leq \varepsilon(\beta^2 + 2\beta\varepsilon + \varepsilon^2) \quad \text{for all } \ell \geq \ell_0.$$

This and $\beta_\ell \geq \beta$ from Lemma 5 establish the convergence $\beta_\ell \searrow \beta$.

Provided $\tilde{\mu} = \beta^2$ is in addition an eigenvalue of (3) and, thus, u is an eigenfunction of (2), Lemma 8 implies

$$(1 - \|(1 - \mathfrak{P}_h)Du\|^2)\frac{\beta_h^2 - \beta^2}{\beta^2} \leq \|(1 - \mathfrak{P}_h)Du\|^2$$

The approximation results from Lemma 9 show that the right-hand side can be bounded by

$$Ch^{2r}\|u\|^2_{H^{1+s}(\Omega)}.$$

This concludes the proof. □

Remark 12. Mixed finite element approximations as generalizations of the non-conforming P_1 scheme to arbitrary polynomial degrees were introduced in [24, 25] for the computation of the velocity field of Stokes problems. Those methods are based on the characterization of the space DV as the orthogonal complement of $\operatorname{rot}^* H^1(\Omega; \mathbb{R}^2)$ $(n = 2)$ or $\operatorname{rot} H(\operatorname{rot}, \Omega)$ $(n = 3)$ in simply-connected domains. Those results can be generalized by the schemes proposed in this paper to domains in $\mathbb{R}^n$, $n \geq 2$, with arbitrary topology. While the works [24, 25] could dispense with the pressure variable through a Helmholtz decomposition for deviatoric fields, the present analysis relies on the pressure approximations p_h in (8) or the corresponding divergence-like quantity $\operatorname{tr} \xi_h$ in (9).

Remark 13 (other inf-sup constants)**.** The work [20] revealed the relevance of other inf-sup conditions than that for the divergence. The newly developed tools can be adapted to the computation of the inf-sup constant of other differential operators, e.g., the rotation (curl). Let $\Omega \subseteq \mathbb{R}^3$ be a contractible Lipschitz domain and define $Q := H_0(\operatorname{div}^0, \Omega)$, the subspace of $H(\operatorname{div}^0, \Omega)$ of fields with vanishing normal trace on the boundary. It is known [23, 20] that

$$0 < \beta_{\text{rot}} = \inf_{q \in Q \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{(q, \operatorname{rot} v)_{L^2(\Omega)}}{\|Dv\| \|q\|}. \tag{16}$$

Let, for a square matrix M, its symmetric and its skew-symmetric part be denoted by $\operatorname{sym} M$ and $\operatorname{skw} M$. The space of skew-symmetric 3×3 matrices has dimension 3 and can be identified with $\mathbb{R}^3$ through the well-known map

$$\operatorname{vec} : \{A \in \mathbb{R}^{3\times 3} : \operatorname{sym} A = 0\} \to \mathbb{R}^3, \qquad \begin{pmatrix} 0 & A_{12} & A_{31} \\ & 0 & A_{23} \\ & & 0 \end{pmatrix} \mapsto \begin{pmatrix} -A_{23} \\ A_{31} \\ -A_{12} \end{pmatrix}$$

It is then easily seen that any $v \in V$ satisfies $\operatorname{rot} v = \operatorname{vec} \operatorname{skw} Dv$. Thus, in analogy to (1) and (5), β_{rot} is characterized as

$$\beta_{\mathrm{rot}} = \inf_{q \in Q\setminus\{0\}} \sup_{\gamma \in \Gamma\setminus\{0\}} \frac{(q, \operatorname{vec} \operatorname{skw} \gamma)_{L^2(\Omega)}}{\|\gamma\| \|q\|}.$$

Based on a piecewise polynomial subspace $Q_h \subseteq Q$ of degree not larger than k, the discrete approximation reads

$$\beta_{\mathrm{rot},h} = \inf_{q_h \in Q_h\setminus\{0\}} \sup_{\gamma_h \in \Gamma_h\setminus\{0\}} \frac{(q_h, \operatorname{vec} \operatorname{skw} \gamma_h)_{L^2(\Omega)}}{\|\gamma_h\| \|q_h\|}.$$

With the tools from Sections 2–3, it can be seen that these discrete inf-sup constants converge monotonically from above towards β. The details are not presented here because they are very similar to the techniques used above. Clearly, in two dimensions the constants for rot and div coincide, $\beta = \beta_{\mathrm{rot}}$. The important point is that, even in higher space dimensions, the algebraic representation of the exterior derivative as the skew-symmetric part of the Jacobian allows a Rayleigh–Ritz-type approximation of the inf-sup constant of the generalized polyharmonic Stokes problems from [20].

5. Numerical results

5.1. **Setup.** The numerical experiments of this section are devoted to the approximation of the inf-sup constant of the divergence on rectangular domains $(0, L) \times (0, 1)$ with $L \geq 1$ and aspect ratio $1/L$. It is readily verified that the inf-sup constant only depends on the aspect ratio and is independent of the actual size of the domain. It was shown in [13] that on rectangular domains the nonzero part of the essential spectrum of the Cosserat operator equals

$$\{1\} \cup \left[\frac{1}{2} - \frac{1}{\pi}, \frac{1}{2} + \frac{1}{\pi}\right]$$

and, consequently, there is a universal upper bound

$$\beta^2 \leq \frac{1}{2} - \frac{1}{\pi}$$

for all rectangular domains. Lower bounds for the inf-sup constant in polygons were proved in [14, 22] and upper bounds for rectangular domains were computed in [11, 13]. The enclosure reads [13, eq. (5.5)]

$$\sin^2\left(2^{-1} \arctan(1/L)\right) \leq \beta^2 \leq 1 - \frac{\sinh(\rho)}{\rho \cosh(\rho)} \quad \text{for } \rho = \pi/(2L).$$

But it is generally unknown for which values of L this is a sharp bound. It is known [13] that, for large values $L \geq 1.8823$, the upper bound is strictly sharper than the above mentioned value of $1/2 - 1/\pi$ and, thus, the number β^2 is isolated from the essential spectrum and in particular an eigenvalue of the Cosserat operator.

For smaller values of $L \geq 1$ it is still an unresolved question whether β^2 is an isolated eigenvalue. In particular, the conjecture [13, 28] that for the square domain $\beta^2 = 1/2 - 1/\pi$ is still unproven.

In this section, numerical computations are performed for the values $L = 2$, $L = 1.61$, and $L = 1$ (the square). The polynomial degree is chosen as $k = 3$ such that the pressure is approximated by piecewise cubics. Since the isolated eigenfunctions are known [13] to exhibit strong corner singularities when L approaches 1, an adaptive mesh-refinement algorithm is proposed. It is based on the following refinement indicator proposed in [24, 25]

$$\eta^2(T) := h_T^2 \|\operatorname{rot} \sigma_h\|^2_{L^2(T)} + h_T \sum_{F \in \mathcal{F}(T)} \|[\sigma_h]_F t_F\|^2_{L^2(F)} \quad \text{for all } T \in \mathcal{T}.$$

Here σ_h is a discrete velocity gradient corresponding to the least-in-modulus nonzero eigenvalue in (8), the set of edges of T is denoted by $\mathcal{F}(T)$, the jump across F is denoted by the bracket $[\cdot]_F$, and t_F is a tangential unit vector for the edge F. While there are proofs [24, 25] that the numbers $\eta^2(T)$ form the main part of an error estimator when applied to linear Stokes problems, in this situation it is used as a purely heuristic refinement indicator in an adaptive finite element loop. It can be shown that the sum over all $\eta^2(T)$ controls the square of the distance of σ_h to Γ. Further terms containing residuals such as, on every element, $\nabla \operatorname{tr} \sigma_h + \lambda_h \operatorname{div} \sigma_h$ could also be considered, but this possibility is disregarded here. The heuristic justification is that, by the properties of Lemma 7, the discrete equation (9) is consistent, i.e.,

$$(\operatorname{tr} \xi_h, \operatorname{tr}(Dv))_{L^2(\Omega)} = \mu_h(\xi_h, Dv)_{L^2(\Omega)}$$

for all v in the orthogonal complement of the divergence-free elements of V. Thus, it is expected that only the distance of ξ_h (and thereby σ_h) to the gradients Γ contributes to the error.

In each step of the loop, a lowest discrete eigenpair is computed and a minimal subset $\mathcal{M}$ of the current triangulation is computed such that $\theta^{-1} \sum_{T \in \mathcal{M}} \eta^2(T)$ is not smaller than the sum of the $\eta^2(T)$ over all elements in the triangulation (marking proposed by [16]). Here, $\theta = 0.3$ is chosen. Based on $\mathcal{M}$, a new triangulation of minimal cardinality is refined from the current one such that all elements in $\mathcal{M}$ are refined.

Although there is no convergence proof for this procedure, the quality of the produced mesh can be assessed *a posteriori* through the monotonicity from Theorem 11: the smaller β_h is, the better is the approximation to β.

5.2. **Experiment 1: $L = 2$.** As mentioned earlier, for the rectangle of length $L = 2$ it is known that β^2 is an eigenvalue of the Cosserat operator and it can be computed from the regularity theory in [13] that the corresponding pressure p of (2) with respect to $\lambda = -\beta^2$ is not smoother than $H^s(\Omega)$ for $s = 0.4760291$. Thus, the velocity u of (4) cannot be smoother than $H^{1+s}(\Omega)$ because $\operatorname{div} u = -\lambda p$. Theorem 11 predicts a convergence rate for the relative error of β_h^2 and β^2 not greater than $2s$ in terms of h when the mesh is uniformly refined. In two space dimensions, this corresponds to $\mathcal{O}(\operatorname{card}(\mathcal{T})^{-s})$. Figure 1 displays the convergence history of the relative error $(\beta_h^2 - \beta^2)/\beta^2$. The reference value 0.1499718 was provided by [4]. The observed convergence rate for uniform mesh refinement is around 0.47 as predicted. Adaptive mesh refinement leads to a rate of 4 for cubic discrete pressures, which is twice the optimal rate for the approximation of p in the

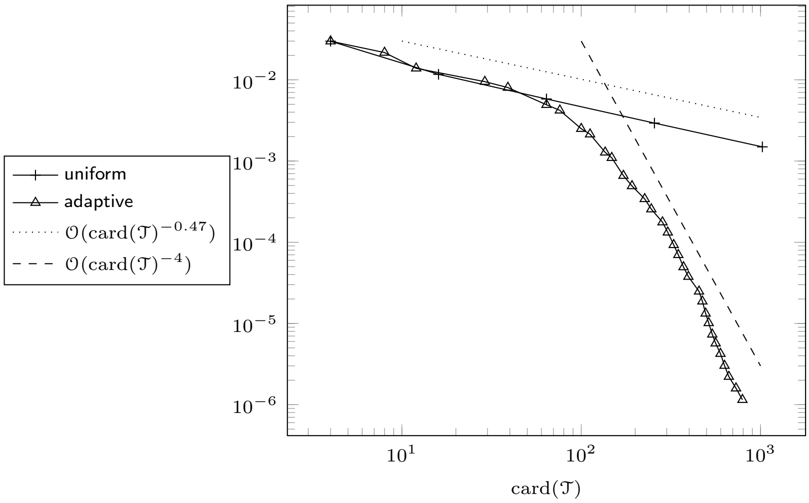

FIGURE 1. Convergence history of the relative error $(\beta_h^2 - \beta^2)/\beta^2$ for Experiment 1.

L^2 norm. An adaptive mesh is shown in Figure 4. It shows strong refinement at the corners.

5.3. **Experiment 2:** $L = 1.61$**.** The second example with $L = 1.61$ is chosen such that it is not known whether β^2 is an isolated eigenvalue. The value $L = 1.61$ is smaller than the golden ratio $(1 + \sqrt{5})/2 \geq 1.61803$. This example is of interest because it is mentioned in [13, p. 454] that the golden ratio could possibly be a transition point for β^2 to belong to the essential spectrum. The guaranteed upper bounds computed with the novel scheme offer a sufficient criterion for disproving conjectures of this type for a given domain. Indeed, the adaptive computations shown in Figure 2 result in a value $\beta_h^2 \leq 0.18159009 < 1/2 - 1/\pi = 0.1816901$ which indicates that $\beta^2 \leq \beta_h^2$ is an isolated eigenvalue.

5.4. **Experiment 3:** $L = 1$**.** This computational experiment considers the square domain with $L = 1$. As mentioned above, it is conjectured [14, 28] that $\beta^2 = 1/2 - 1/\pi$. Figure 3 displays the relative error of the numerical eigenvalues β_h^2 and the reference value $1/2 - 1/\pi = 0.1816901$. All computed values β_h^2 stay above this reference and seem to converge towards this value. The computed value on the finest adaptive mesh is 0.1826413 The empirical convergence rate under uniform mesh refinement is of the order 1/7 and can be improved by the adaptive algorithm, where the rate is 2. It is, however, less than in the first (smoother) example and on fine meshes the rate seems to become significantly slower. The slow convergence is an indication that the true value β^2 is very close to $1/2 - 1/\pi$. In the case that β^2 should be isolated, then Theorem 11 predicts a convergence rate and the fact that this rate is slow indicates a very low regularity of the corresponding eigenfunction according to the regularity theory of [13]. Should β^2 indeed belong to the essential spectrum, then no algebraic convergence rate can be expected for uniform meshes. An adaptive mesh is shown in Figure 4. The refinement towards one of the corners is extreme. Numerical tests by the author (not further reported here) have shown

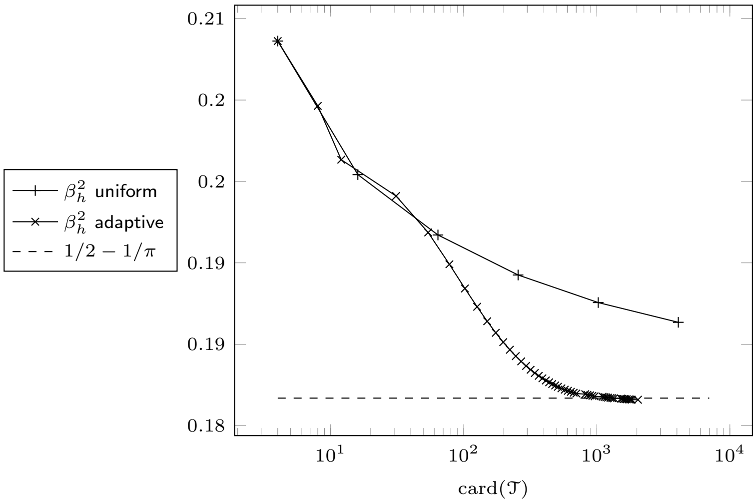

FIGURE 2. Values of β_h^2 in Experiment 2 for uniform and adaptive meshes

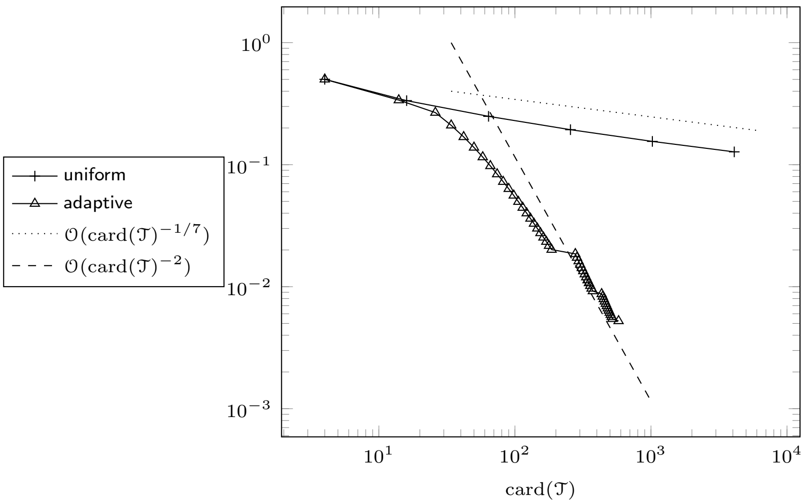

FIGURE 3. Convergence history of the relative error $(\beta_h^2 - \beta^2)/\beta^2$ for Experiment 3 with reference $\beta^2 = 1/2 - 1/\pi$.

that the smallest discrete eigenvalues form a cluster and the refinement is very much dependent on the design of the initial mesh. Depending on that, the refinement can occur at some other or at multiple corners. Such phenomena are well-understood for compact eigenvalue problems [19]. In the present case of the computation of the smallest eigenvalue, they are not expected to influence the performance of the method.

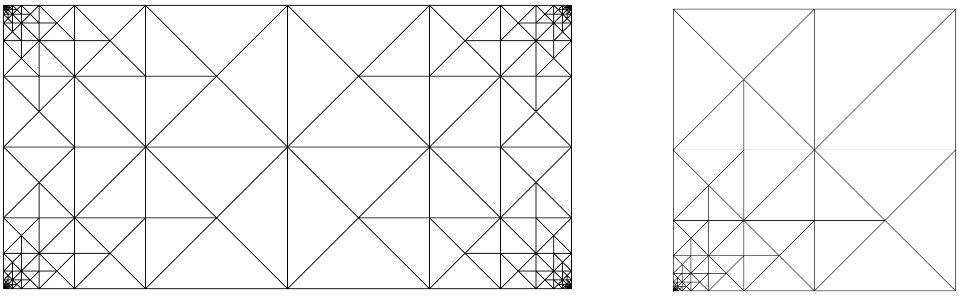

FIGURE 4. Adaptive meshes. Left: Experiment 1, 306 vertices, level 25. Right: Experiment 3, 302 vertices, level 46.

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