Remarks about the unification types of some locally tabular normal modal logics

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Abstract

It is already known that unifiable formulas in normal modal logic $\mathbf{K}+\Box^2\bot$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1+\Box^2\bot$ are unitary. In this paper, we prove that for all $d\geq 3$, unifiable formulas in normal modal logic $\mathbf{K}+\Box^d\bot$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1+\Box^d\bot$ are unitary.

Keywords: Normal modal logics $\mathbf{K}+\Box^d\bot$ and $\mathbf{Alt}_1+\Box^d\bot$. Unification types.

1 Introduction

The unification problem in a propositional logic is to determine, given a formula φ, whether there exists a substitution σ such that $\sigma(\varphi)$ is in that logic. In that case, σ is a unifier of φ. A set of unifiers of a unifiable formula φ is complete if for all unifiers σ of φ, there exists a unifier τ of φ in that set such that τ is more general than σ[1]. Now, an important question is the following: determine whether a given unifiable formula has minimal complete sets of unifiers [3]. When such sets exist, it is well-known that they all have the same

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[1] A substitution σ is more general than a substitution τ in a propositional logic if there exists a substitution υ such that for all variables x, $\upsilon(\sigma(x))\leftrightarrow\tau(x)$ is in that logic.

cardinality. In that case, a unifiable formula is either infinitary, or finitary, or unitary, depending whether its complete sets of unifiers are either infinite, or with finite cardinality ≥ 2, or with cardinality 1, respectively. Otherwise, the formula is said to be nullary. To be nullary is considered to be the worst situation for a unifiable formula whereas to be unitary is considered to be better than to be finitary which is itself considered to be better than to be infinitary. The unification type of a propositional logic is the worst unification type of its unifiable formulas.

The importance of the unification problem lies in its connection with the admissibility problem. In a consistent propositional logic **L**, unification is reducible to non-admissibility, seeing that the unifiability in **L** of a formula φ is equivalent to the non-admissibility in **L** of the inference rule $\frac{\varphi}{\bot}$. As observed in [18, 19, 21] within the context of intermediate logics and transitive normal modal logics, when **L** has a decidable membership problem and **L** is either finitary, or unitary, algorithms for computing minimal complete sets of unifiers in **L** can be used as a key component of algorithms for solving the admissibility problem in **L**, seeing that the admissibility in **L** of an inference rule $\frac{\varphi_1,\ldots,\varphi_p}{\psi}$ is equivalent to the inclusion in **L** of the set $\{\sigma(\psi):\ \sigma\in\Sigma\}$, where Σ is an arbitrary minimal complete set of unifiers of $\varphi_1 \wedge \ldots \wedge \varphi_p$ in **L**.

About the unification type of normal modal logics, it is known that extensions of normal modal logic **K**5 such as **K**45, **KD**45 and **S**5 are unitary [7, 11, 14, 15, 20, 22], non-transitive normal modal logics like **K** and $\mathbf{Alt}_1$ are nullary [10, 23], transitive normal modal logics such as **K**4 and **S**4 are finitary [18, 19, 21] and normal modal logics characterized by transitive frames with a form of no branching to the right like **K**4**D**1 and **S**4.3 are unitary [16, 17, 24][2]. In this review, the nullary modal logics are the non-transitive ones: **K** and $\mathbf{Alt}_1$. Therefore, it is natural to ask the question of the unification type of other non-transitive normal modal logics and to see whether they also have a tendency to be nullary. Hence, one may interest for all $d\geq 2$, in the normal modal logics $\mathbf{K}+\square^d\bot$ (the least normal modal logic containing $\square^d\bot$) and $\mathbf{Alt}_1+\square^d\bot$ (the least normal modal logic containing $\mathbf{Alt}_1$ and $\square^d\bot$). The normal modal logics $\mathbf{K}+\square^2\bot$ and $\mathbf{Alt}_1+\square^2\bot$ are transitive and one may expect that unifiable formulas in these normal modal logics are either finitary, or unitary. Indeed, unifiable formulas in normal modal logic $\mathbf{K}+\square^2\bot$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1+\square^2\bot$ are unitary [8, 9]. However, when $d\geq 3$, the normal modal logics $\mathbf{K}+\square^d\bot$ and $\mathbf{Alt}_1+\square^d\bot$ are non-transitive and one may expect that they have the worst unification type (nullary). We prove in this paper that, surprisingly, for all $d\geq 3$, unifiable formulas in normal modal

[2]In this paper, we follow the same conventions as in [12, 13, 25] for talking about normal modal logics: **S**5 is the least normal modal logic containing the formulas usually denoted **T**, 4 and 5, **KD** is the least normal modal logic containing the formula usually denoted **D**, etc. In particular, $\mathbf{Alt}_1$ is the least normal modal logic containing $\lozenge x \to \square x$ and **K**4**D**1 is the least normal modal logic containing **K**4 and $\square(\square x \to y) \vee \square(\square y \to x)$. As usual, **K** is the least normal modal logic.

logic $\mathbf{K}+\Box^d\bot$ are either finitary, or unitary and unifiable formulas in normal modal logic $\mathbf{Alt}_1+\Box^d\bot$ are unitary.

2 Preliminaries

For all sets S, $\|S\|$ will denote the cardinality of S. For all nonempty sets S, for all equivalence relations $\sim$ on S and for all $T\subseteq S$, $T/\sim$ will denote the quotient set of T modulo $\sim$. For all nonempty sets S, for all equivalence relations $\sim$ on S and for all $\alpha\in S$, $[\alpha]$ will denote the equivalence class modulo $\sim$ with α as its representative. Notice that for all nonempty sets S, for all equivalence relations $\sim$ on S and for all $\alpha,\beta\in S$, $\alpha\sim\beta$ if and only if $\alpha\in[\beta]$ if and only if $[\alpha]\cap[\beta]\neq\emptyset$. Proposition 1 will be useful in Sections 6 and 7 for the proofs of Lemmas 6 and 16.

Proposition 1 *Let S,T be non-empty finite sets. Let $\sim$ be an equivalence relation on S. If $\|S/\sim\|\leq\|T\|\leq\|S\|$ then there exists a surjective function f from S to T such that for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$[3].*

Proof: Suppose $\|S/\sim\|\leq\|T\|\leq\|S\|$. Let h be a function from $S/\sim$ to S such that for all $\alpha\in S$, $h([\alpha])\in[\alpha]$. Notice that h is injective. Let S_0 be the range of h. Since h is injective, $\|S/\sim\|=\|S_0\|$. Since $\|S/\sim\|\leq\|T\|$, $\|S_0\|\leq\|T\|$. Let T_0 be a subset of T such that $\|T_0\|=\|S_0\|$. Let f_0 be a one-to-one correspondence between S_0 and T_0. Let $T_1=T\setminus T_0$. Notice that T_0 and T_1 make a partition of T. Since $\|T\|\leq\|S\|$ and $\|T_0\|=\|S_0\|$, $\|T_1\|\leq\|S\setminus S_0\|$. Let S_1 be a subset of $S\setminus S_0$ such that $\|S_1\|=\|T_1\|$. Let f_1 be a one-to-one correspondence between S_1 and T_1. Let $S_2=(S\setminus S_0)\setminus S_1$. Notice that S_0, S_1 and S_2 make a partition of S. Let f_2 be the function from S_2 to T defined by

- $f_2(\alpha)=f_0(h([\alpha]))$.

Let f be the function from S to T such that for all $\alpha\in S$,

- if $\alpha\in S_0$ then $f(\alpha)=f_0(\alpha)$,
- if $\alpha\in S_1$ then $f(\alpha)=f_1(\alpha)$,
- if $\alpha\in S_2$ then $f(\alpha)=f_2(\alpha)$.

It is a routine exercise to demonstrate that f is surjective and for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$. $\dashv$

A binary relation R on a nonempty set W is *irreflexive* if for all $s\in W$, not sRs.

[3]It is fairly easy to prove that if there exists a surjective function f from S to T such that for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$ then $\|S/\sim\|\leq\|T\|\leq\|S\|$. Indeed, let f be a surjective function from S to T such that for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$. Since f is surjective, $\|T\|\leq\|S\|$. Let h be a function from $S/\sim$ to S such that for all $\alpha\in S$, $h([\alpha])\in[\alpha]$. Obviously, for all $\alpha,\beta\in S$, if $h([\alpha])\sim h([\beta])$ then $[\alpha]=[\beta]$. Let g be a function from $S/\sim$ to T such that for all $\alpha\in S$, $g([\alpha])=f(h([\alpha]))$. Since for all $\alpha,\beta\in S$, if $f(\alpha)=f(\beta)$ then $\alpha\sim\beta$ and for all $\alpha,\beta\in S$, if $h([\alpha])\sim h([\beta])$ then $[\alpha]=[\beta]$, g is injective. Hence, $\|S/\sim\|\leq\|T\|$. Since $\|T\|\leq\|S\|$, $\|S/\sim\|\leq\|T\|\leq\|S\|$.

A binary relation R on a nonempty set W is *deterministic* if for all $s, t, u \in W$, if sRt and sRu then $t = u$. A *tree* is a structure of the form (W, R) where W is a nonempty finite set and R is a binary relation on W such that $W \subseteq \mathbb{N}$ and

- there exists $r \in W$ (called the *root* of (W, R)) such that for all $s \in W$, $rR^{\star}s$,
- R^{+} is irreflexive,
- R^{-1} is deterministic,

where $R^{\star}$ denotes the reflexive transitive closure of R, R^{+} denotes the transitive closure of R and R^{-1} denotes the converse of R. In a tree (W, R), for all $s \in W$, let $R(s) = \{t : t \in W \text{ and } sRt\}$ and for all $S \subseteq W$, let $\mathbf{l}S = \{s : s \in W \text{ and } R(s) \subseteq S\}$. In a tree (W, R), for all $e \in \mathbb{N}$, an *R-branch of length e* is a finite sequence $(s_1, \ldots, s_e)$ such that for all $k \in \{1, \ldots, e\}$, $s_k \in W$ and if $k < e$ then $s_k R s_{k+1}$. For all $e \in \mathbb{N}$, a tree (W, R) is *e-bounded* if W contains no R-branch of length $> e$. A tree (W, R) is *deterministic* if R is deterministic.

3 Normal modal logics

3.1 Formulas

Let **VAR** be a countably infinite set of *variables* (with typical members denoted x, y, etc). The set **FOR** of all *formulas* (with typical members denoted φ, ψ, etc) is inductively defined as follows:

- $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Box\varphi$,

where x ranges over **VAR**. We adopt the standard rules for omission of the parentheses. For all formulas φ, the *modal degree* of φ (denoted $\mathtt{deg}(\varphi)$) is defined as usual [12, Definition 2.28]. For all formulas φ, let $\mathtt{var}(\varphi)$ be the set of all variables occurring in φ. For all finite subsets $\bar{x}$ of **VAR**, a formula φ is an *$\bar{x}$-formula* if $\mathtt{var}(\varphi) \subseteq \bar{x}$. For all finite subsets $\bar{x}$ of **VAR**, let $\mathbf{FOR}_{\bar{x}}$ be the set of all $\bar{x}$-formulas.

3.2 Substitutions

A *substitution* is a triple $(\bar{x}, \bar{y}, \sigma)$ where $\bar{x}$ and $\bar{y}$ are finite subsets of **VAR** and σ is a homomorphism from $(\mathbf{FOR}_{\bar{x}}, \bot, \neg, \vee, \Box)$ to $(\mathbf{FOR}_{\bar{y}}, \bot, \neg, \vee, \Box)$[4]. Let **SUB** be the set of all substitutions.

3.3 Abbreviations

The Boolean connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined by the usual abbreviations. The modal connective $\Diamond$ is defined by

[4] That is, σ is a function from $\mathbf{FOR}_{\bar{x}}$ to $\mathbf{FOR}_{\bar{y}}$ such that for all $\varphi, \psi \in \mathbf{FOR}_{\bar{x}}$, $\sigma(\bot) = \bot$, $\sigma(\neg\varphi) = \neg\sigma(\varphi)$, $\sigma(\varphi \vee \psi) = \sigma(\varphi) \vee \sigma(\psi)$ and $\sigma(\Box\varphi) = \Box\sigma(\varphi)$.

- $\Diamond\varphi ::= \neg\Box\neg\varphi$.

For all $k\in\mathbb{N}$, the modal connective $\Box^k$ is inductively defined as follows:

- if k=0 then $\Box^k\varphi ::= \varphi$,
- otherwise, $\Box^k\varphi ::= \Box\Box^{k-1}\varphi$.

For all $k\in\mathbb{N}$, the modal connective $\Diamond^k$ is defined by

- $\Diamond^k\varphi ::= \neg\Box^k\neg\varphi$.

3.4 Syntactic presentation

A *normal modal logic* is a set $\mathbf{L}$ of formulas such that

- $\mathbf{L}$ contains all tautologies,
- $\mathbf{L}$ contains all formulas of the form $\Box(x \to y) \to (\Box x \to \Box y)$,
- $\mathbf{L}$ is closed under modus ponens (for all formulas φ, ψ, if $\varphi\in\mathbf{L}$ and $\varphi \to \psi\in\mathbf{L}$ then $\psi\in\mathbf{L}$),
- $\mathbf{L}$ is closed under generalization (for all formulas φ, if $\varphi\in\mathbf{L}$ then $\Box\varphi\in\mathbf{L}$),
- $\mathbf{L}$ is closed under uniform substitution (for all formulas φ, ψ, if $\varphi\in\mathbf{L}$ and ψ is obtained from φ by uniformly replacing variables in φ by arbitrary formulas then $\psi\in\mathbf{L}$).

For all normal modal logics $\mathbf{L}$ and for all $\varphi\in\mathbf{FOR}$, let $\mathbf{L}+\varphi$ be the least normal modal logic containing $\mathbf{L}$ and φ. For all normal modal logics $\mathbf{L}$, the equivalence relation $\equiv_{\mathbf{L}}$ on $\mathbf{FOR}$ is defined by

- $\varphi\equiv_{\mathbf{L}}\psi$ if and only if $\varphi \leftrightarrow \psi\in\mathbf{L}$,

where φ, ψ range over $\mathbf{FOR}$. We shall say that a normal modal logic $\mathbf{L}$ is *locally tabular* if for all finite subsets $\bar{x}$ of $\mathbf{VAR}$, $\mathbf{FOR}_{\bar{x}}/\equiv_{\mathbf{L}}$ is finite[5].

4 Unification in normal modal logics

From now on in this section, **let L be a normal modal logic.**

[5]Locally tabular normal modal logics are also said to be *locally finite*. For details about local finiteness in normal modal logics, see [13, Chapter 12] and [26, 27, 28].

4.1 Comparing substitutions

The equivalence relation $\simeq_{\mathbf{L}}$ on **SUB** is defined by

- $(\bar{x},\bar{y},\sigma)\simeq_{\mathbf{L}}(\bar{x},\bar{y},\tau)$ if and only if for all $x\in\bar{x}$, $\sigma(x)\equiv_{\mathbf{L}}\tau(x)$,

where $(\bar{x},\bar{y},\sigma),(\bar{x},\bar{y},\tau)$ range over **SUB**. That is, for all $(\bar{x},\bar{y},\sigma),(\bar{x},\bar{y},\tau)\in\mathbf{SUB}$, $(\bar{x},\bar{y},\sigma)\simeq_{\mathbf{L}}(\bar{x},\bar{y},\tau)$ if and only if for all $x\in\bar{x}$, $\sigma(x)\leftrightarrow\tau(x)\in\mathbf{L}$. The preorder $\preccurlyeq_{\mathbf{L}}$ on **SUB** is defined by

- $(\bar{x},\bar{y},\sigma)\preccurlyeq_{\mathbf{L}}(\bar{x},\bar{z},\tau)$ if and only if there exists a substitution $(\bar{y},\bar{z},\upsilon)$ such that for all $x\in\bar{x}$, $\upsilon(\sigma(x))\equiv_{\mathbf{L}}\tau(x)$,

where $(\bar{x},\bar{y},\sigma),(\bar{x},\bar{z},\tau)$ range over **SUB**. That is, for all $(\bar{x},\bar{y},\sigma),(\bar{x},\bar{z},\tau)\in\mathbf{SUB}$, $(\bar{x},\bar{y},\sigma)\preccurlyeq_{\mathbf{L}}(\bar{x},\bar{z},\tau)$ if and only if there exists a substitution $(\bar{y},\bar{z},\upsilon)$ such that for all $x\in\bar{x}$, $\upsilon(\sigma(x))\leftrightarrow\tau(x)\in\mathbf{L}$.

4.2 Unifiers

An **L**-*unifier* of $\varphi\in\mathbf{FOR}$ is a substitution $(\mathtt{var}(\varphi),\bar{y},\sigma)$ such that $\sigma(\varphi)\in\mathbf{L}$. A formula φ is **L**-*unifiable* if there exists an **L**-unifier of φ. A set Σ of **L**-unifiers of an **L**-unifiable $\varphi\in\mathbf{FOR}$ is *complete* if for all **L**-unifiers $(\mathtt{var}(\varphi),\bar{y},\sigma)$ of φ, there exists $(\mathtt{var}(\varphi),\bar{z},\tau)\in\Sigma$ such that $(\mathtt{var}(\varphi),\bar{z},\tau)\preccurlyeq_{\mathbf{L}}(\mathtt{var}(\varphi),\bar{y},\sigma)$. A complete set Σ of **L**-unifiers of an **L**-unifiable $\varphi\in\mathbf{FOR}$ is a *basis* if for all $(\mathtt{var}(\varphi),\bar{y},\sigma),(\mathtt{var}(\varphi),\bar{z},\tau)\in\Sigma$, if $(\mathtt{var}(\varphi),\bar{y},\sigma)\preccurlyeq_{\mathbf{L}}(\mathtt{var}(\varphi),\bar{z},\tau)$ then $\bar{y}=\bar{z}$ and $\sigma=\tau$[6].

Proposition 2 *Let* $\varphi\in\mathbf{FOR}$. *If* φ *is* **L**-*unifiable then for all bases* Σ,Δ *of* **L**-*unifiers of* φ, $\|\Sigma\|=\|\Delta\|$.

Proof: This is a standard result. $\dashv$

4.3 Type of L-unifiable formulas

For all **L**-unifiable $\varphi\in\mathbf{FOR}$,

- φ is **L**-*nullary* if there exists no basis of **L**-unifiers of φ,
- φ is **L**-*infinitary* if there exists an infinite basis of **L**-unifiers of φ,
- φ is **L**-*finitary* if there exists a basis of **L**-unifiers of φ with finite cardinality ≥ 2,
- φ is **L**-*unitary* if there exists a basis of **L**-unifiers of φ with cardinality 1.

For all **L**-unifiable $\varphi\in\mathbf{FOR}$,

[6] It is a routine exercise to demonstrate that for all complete sets Σ of **L**-unifiers of an **L**-unifiable $\varphi\in\mathbf{FOR}$, Σ is a basis for φ if and only if Σ is a minimal complete set of **L**-unifiers of φ.

- φ is **L**-*filtering* if for all **L**-unifiers $(\mathtt{var}(\varphi), \bar{y}, \sigma), (\mathtt{var}(\varphi), \bar{z}, \tau)$ of φ, there exists an **L**-unifier $(\mathtt{var}(\varphi), \bar{t}, \upsilon)$ of φ such that $(\mathtt{var}(\varphi), \bar{t}, \upsilon) \preccurlyeq_{\mathbf{L}} (\mathtt{var}(\varphi), \bar{y}, \sigma)$ and $(\mathtt{var}(\varphi), \bar{t}, \upsilon) \preccurlyeq_{\mathbf{L}} (\mathtt{var}(\varphi), \bar{z}, \tau)$[7],

- φ is **L**-*reasonable* if for all **L**-unifiers $(\mathtt{var}(\varphi), \bar{y}, \sigma)$ of φ, if $\|\mathtt{var}(\varphi)\| < \|y\|$ then there exists an **L**-unifier $(\mathtt{var}(\varphi), \mathtt{var}(\varphi), \tau)$ of φ such that $(\mathtt{var}(\varphi), \mathtt{var}(\varphi), \tau) \preccurlyeq_{\mathbf{L}} (\mathtt{var}(\varphi), \bar{y}, \sigma)$[8].

Proposition 3 *Let $\varphi \in$* **FOR** *be* **L**-*unifiable. If φ is* **L**-*unitary then φ is* **L**-*filtering.*

Proof: This is a standard result. $\dashv$

Proposition 4 *Let $\varphi \in$* **FOR** *be* **L**-*unifiable. If φ is* **L**-*filtering then φ is either* **L**-*nullary, or* **L**-*unitary.*

Proof: This is a standard result. $\dashv$

Proposition 5 *Let $\varphi \in$* **FOR** *be* **L**-*unifiable. If* **L** *is locally tabular and φ is* **L**-*reasonable then φ is either* **L**-*finitary, or* **L**-*unitary.*

Proof: By the fact that if **L** is locally tabular then the quotient of the set of all substitutions of the form $(\mathtt{var}(\varphi), \mathtt{var}(\varphi), \sigma)$ modulo $\simeq_{\mathbf{L}}$ is finite. $\dashv$

4.4 Type of L

We shall say that[9]

- **L** is *nullary* if there exists an **L**-nullary **L**-unifiable formula,

- **L** is *infinitary* if every **L**-unifiable formula is either **L**-infinitary, or **L**-finitary, or **L**-unitary and there exists an **L**-infinitary **L**-unifiable formula,

- **L** is *finitary* if every **L**-unifiable formula is either **L**-finitary, or **L**-unitary and there exists an **L**-finitary **L**-unifiable formula,

- **L** is *unitary* if every **L**-unifiable formula is **L**-unitary.

We shall say that

- **L** is *filtering* if every **L**-unifiable formula is **L**-filtering,

[7] Filtering unifiable formulas are also said to be *directed*. For details about directedness of unifiable formulas, see [20, 23].

[8] Reasonability of unifiable formulas has been introduced for the normal modal logics $\mathbf{K} + \square^2\bot$ and $\mathbf{Alt}_1 + \square^2\bot$ in [8, 9].

[9] Nullary (respectively, infinitary, finitary, unitary) normal modal logics are also said to be *of type* 0 (respectively, *of type* ∞, *of type* ω, *of type* 1). For details about the unification types of normal modal logics, see [15].

- **L** is *reasonable* if every **L**-unifiable formula is **L**-reasonable.

Proposition 6 *If* **L** *is unitary then* **L** *is filtering.*

Proof: By Proposition 3. ⊣

Proposition 7 *If* **L** *is filtering then* **L** *is either nullary, or unitary.*

Proof: By Proposition 4. ⊣

Proposition 8 *If* **L** *is locally tabular and reasonable then* **L** *is either finitary, or unitary.*

Proof: By Proposition 5. ⊣

5 Some locally tabular normal modal logics

From now on until the end of Section 7, **let** $d\geq 3$ **be fixed.** From now on in this section, **let** $\bar{x}$ **be a finite subset of VAR.**

5.1 Trees and models

A $\bar{x}$-*model* is a structure of the form (W,R,V) where (W,R) is a tree and V is a homomorphism from $(\mathbf{FOR}_{\bar{x}},\bot,\neg,\vee,\Box)$ to $(2^W,\emptyset,\setminus,\cup,\mathbf{l})$[10]. In a $\bar{x}$-model $\mathbf{M}=(W,R,V)$, let $V(\Phi)$ denote $\bigcap\{V(\varphi)\ :\ \varphi\in\Phi\}$ for each $\Phi\subseteq\mathbf{FOR}_{\bar{x}}$. In a $\bar{x}$-model $\mathbf{M}=(W,R,V)$, $r_{\mathbf{M}}$ will denote the root of (W,R), $W_{\mathbf{M}}$ will denote W, $R_{\mathbf{M}}$ will denote R and $V_{\mathbf{M}}$ will denote V. In a $\bar{x}$-model $\mathbf{M}$, for all $s\in W_{\mathbf{M}}$, let $\mathbf{M}_s$ be the *submodel* of $\mathbf{M}$ generated from s. For all $\bar{x}$-models $\mathbf{M}$, let $\mathbf{for}(\mathbf{M})=\{\varphi\in\mathbf{FOR}_{\bar{x}}\ :\ r_{\mathbf{M}}\in V_{\mathbf{M}}(\varphi)\}$. Notice that for all $\bar{x}$-models $\mathbf{M}$ and for all $s\in W_{\mathbf{M}}$, $s\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}_s))$. For all $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ and for all $\bar{y}$-models $\mathbf{M}$, let $\mathbf{M}^{|\sigma}$ be the $\bar{x}$-model such that $W_{\mathbf{M}^{|\sigma}}=W_{\mathbf{M}}$, $R_{\mathbf{M}^{|\sigma}}=R_{\mathbf{M}}$ and for all $x\in\bar{x}$, $V_{\mathbf{M}^{|\sigma}}(x)=V_{\mathbf{M}}(\sigma(x))$. Proposition 9 states a standard result connecting substitutions and models. In particular, see [18, Proposition 2] and [19, Proposition 1.3].

Proposition 9 *Let* $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ *and* $\mathbf{M}$ *be a* $\bar{y}$*-model. For all* $\varphi\in\mathbf{FOR}_{\bar{x}}$*,* $V_{\mathbf{M}^{|\sigma}}(\varphi)=V_{\mathbf{M}}(\sigma(\varphi))$.

Proof: By induction on φ. ⊣

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[10]That is, V is a function from $\mathbf{FOR}_{\bar{x}}$ to 2^W such that for all $\varphi,\psi\in\mathbf{FOR}_{\bar{x}}$, $V(\bot)=\emptyset$, $V(\neg\varphi)=W\setminus V(\varphi)$, $V(\varphi\vee\psi)=V(\varphi)\cup V(\psi)$ and $V(\Box\varphi)=\mathbf{l}V(\varphi)$.

5.2 Bisimulations

The $\bar{x}$-models $\mathbf{M}$ and $\mathbf{M}'$ are *bisimilar* (in symbols $\mathbf{M} \bowtie \mathbf{M}'$) if there exists $Z \subseteq W_{\mathbf{M}} \times W_{\mathbf{M}'}$ such that $r_{\mathbf{M}} Z r_{\mathbf{M}'}$ and for all $s \in W_{\mathbf{M}}$ and for all $s' \in W_{\mathbf{M}'}$, if sZs' then

- for all $x \in \bar{x}$, $s \in V_{\mathbf{M}}(x)$ if and only if $s' \in V_{\mathbf{M}'}(x)$,
- for all $t \in W_{\mathbf{M}}$, if $sR_{\mathbf{M}}t$ then there exists $t' \in W_{\mathbf{M}'}$ such that tZt' and $s'R_{\mathbf{M}'}t'$,
- for all $t' \in W_{\mathbf{M}'}$, if $s'R_{\mathbf{M}'}t'$ then there exists $t \in W_{\mathbf{M}}$ such that tZt' and $sR_{\mathbf{M}}t$.

In that case, Z is a *bisimulation between* $\mathbf{M}$ *and* $\mathbf{M}'$. As is well-known, $\bowtie$ is an equivalence relation on the set of all $\bar{x}$-models[11]. The set of all $\bar{x}$-models equivalent modulo $\bowtie$ to a $\bar{x}$-model $\mathbf{M}$ is denoted $[\mathbf{M}]$.

Proposition 10 *For all $\bar{x}$-models $\mathbf{M}, \mathbf{M}'$, for all $s \in W_{\mathbf{M}}$ and for all $s' \in W_{\mathbf{M}'}$, the following conditions are equivalent:*

1. $\mathbf{M}_s \bowtie \mathbf{M}'_{\mathbf{s}'}$,
2. $s \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'}))$.

Proof: By [12, Theorems 2.20 and 2.24]. $\dashv$

Proposition 11 *For all $(\bar{x}, \bar{y}, \sigma) \in \mathbf{SUB}$ and for all $\bar{y}$-models $\mathbf{M}, \mathbf{M}'$, if $\mathbf{M} \bowtie \mathbf{M}'$ then $\mathbf{M}^{|\sigma} \bowtie \mathbf{M}'^{|\sigma}$.*

Proof: By Proposition 10. $\dashv$

5.3 The normal modal logics $\mathbf{K} + \Box^d \bot$ and $\mathbf{Alt}_1 + \Box^d \bot$

The following normal modal logics are considered in this paper:

- $\mathbf{K}_d = \mathbf{K} + \Box^d \bot$,
- $\mathbf{A}_d = \mathbf{Alt}_1 + \Box^d \bot$,

Obviously, $\mathbf{A}_d$ contains $\mathbf{K}_d$. For all $e \in \{1, \ldots, d\}$, let $\mathcal{BT}_e^{\bar{x}}$ be the set of all $\bar{x}$-models based on e-bounded trees. For all $e \in \{1, \ldots, d\}$, let $\mathcal{DBT}_e^{\bar{x}}$ be the set of all $\bar{x}$-models based on deterministic e-bounded trees.

Proposition 12 *For all $\varphi \in \mathbf{FOR}_{\bar{x}}$,*

1. $\varphi \in \mathbf{K}_d$ *if and only if for all* $\mathbf{M} \in \mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$,

[11] Models considered in this paper are based on finite relational structures. For this reason, they constitute a set.

2. $\varphi \in \mathbf{A}_d$ *if and only if for all* $\mathbf{M} \in \mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\varphi)$.

Proof: This is a standard result. $\dashv$

Proposition 13 1. $\Box\Box^{d-1}\bot \in \mathbf{K}_d$,

2. $\neg\Diamond\Diamond^{d-1}\top \in \mathbf{K}_d$,

3. $\bigvee\{\Diamond^l\Box\bot : 0 \leq l < d\} \in \mathbf{K}_d$,

4. $\neg\bigwedge\{\Box^l\Diamond\top : 0 \leq l < d\} \in \mathbf{K}_d$,

5. $\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\} \to \Box\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\} \in \mathbf{K}_d$,

6. $\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \in \mathbf{A}_d$,

7. $\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \not\in \mathbf{K}_d$,

8. $\Box^{d-1}\bot \to \Diamond^{d-1}\top \not\in \mathbf{A}_d$.

Proof: It is a routine exercise to demonstrate that for all $\mathbf{M} \in \mathcal{BT}_d^{\emptyset}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\Box^{d-1}\bot)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg\Diamond\Diamond^{d-1}\top)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee\{\Diamond^l\Box\bot : 0 \leq l < d\})$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\neg\bigwedge\{\Box^l\Diamond\top : 0 \leq l < d\})$, for all $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\} \to \Box\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\})$ and for all $\mathbf{M} \in \mathcal{DBT}_d^{\{u\}}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\})$. Hence, by Proposition 12, $\Box\Box^{d-1}\bot \in \mathbf{K}_d$, $\neg\Diamond\Diamond^{d-1}\top \in \mathbf{K}_d$, $\bigvee\{\Diamond^l\Box\bot : 0 \leq l < d\} \in \mathbf{K}_d$, $\neg\bigwedge\{\Box^l\Diamond\top : 0 \leq l < d\} \in \mathbf{K}_d$, $\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\} \to \Box\bigwedge\{\Box^l(\neg u \vee \Diamond\top) : 0 \leq l < d\} \in \mathbf{K}_d$ and $\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \in \mathbf{A}_d$. In other respect, it is a routine exercise to demonstrate that $r_{\mathbf{M}} \not\in V_{\mathbf{M}}(\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\})$ where $\mathbf{M} \in \mathcal{BT}_d^{\{u\}}$ is such that $W_{\mathbf{M}} = \{0, 1, 2, 3\}$, $R_{\mathbf{M}} = \{(0, 1), (0, 2), (2, 3)\}$ and $V_{\mathbf{M}}(u) = \{1\}$. Thus, by Proposition 12, $\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \to \Box\bigvee\{\Diamond^l(u \wedge \Box\bot) : 0 \leq l < d\} \not\in \mathbf{K}_d$. Finally, it is a routine exercise to demonstrate that $r_{\mathbf{M}} \not\in V_{\mathbf{M}}(\Box^{d-1}\bot \to \Diamond^{d-1}\top)$ where $\mathbf{M} \in \mathcal{DBT}_d^{\emptyset}$ is such that $W_{\mathbf{M}} = \{0\}$ and $R_{\mathbf{M}} = \emptyset$. Thus, by Proposition 12, $\Box^{d-1}\bot \to \Diamond^{d-1}\top \not\in \mathbf{A}_d$. $\dashv$

Proposition 14 *For all formulas* φ, *if* $\Diamond\varphi \to \Box\varphi \in \mathbf{K}_d$ *then either* $\Box^{d-1}\bot \to \varphi \in \mathbf{K}_d$, *or* $\varphi \to \Diamond^{d-1}\top \in \mathbf{K}_d$.

Proof: Let φ be a formula. Suppose $\Diamond\varphi \to \Box\varphi \in \mathbf{K}_d$ and neither $\Box^{d-1}\bot \to \varphi \in \mathbf{K}_d$, nor $\varphi \to \Diamond^{d-1}\top \in \mathbf{K}_d$. Let $\bar{y}$ be a finite subset of **VAR** such that $\mathtt{var}(\varphi) \subseteq \bar{y}$. Since neither $\Box^{d-1}\bot \to \varphi \in \mathbf{K}_d$, nor $\varphi \to \Diamond^{d-1}\top \in \mathbf{K}_d$, by Proposition 12, let $\mathbf{M}', \mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ be such that $r_{\mathbf{M}'} \not\in V_{\mathbf{M}'}(\Box^{d-1}\bot \to \varphi)$ and $r_{\mathbf{M}''} \not\in V_{\mathbf{M}''}(\varphi \to \Diamond^{d-1}\top)$. Hence, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box^{d-1}\bot)$, $r_{\mathbf{M}'} \not\in V_{\mathbf{M}'}(\varphi)$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\varphi)$ and $r_{\mathbf{M}''} \not\in V_{\mathbf{M}''}(\Diamond^{d-1}\top)$. Thus, let $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ be such that the following conditions hold:

- for all $s \in W_{\mathbf{M}}$, if $r_{\mathbf{M}} R_{\mathbf{M}} s$, either $\mathbf{M}_s$ is isomorphic to $\mathbf{M}'$, or $\mathbf{M}_s$ is isomorphic to $\mathbf{M}''$,

- there exists $s{\in}W_{\mathbf{M}}$, such that $r_{\mathbf{M}}R_{\mathbf{M}}s$ and $\mathbf{M}_s$ is isomorphic to $\mathbf{M}'$,
- there exists $s{\in}W_{\mathbf{M}}$, such that $r_{\mathbf{M}}R_{\mathbf{M}}s$ and $\mathbf{M}_s$ is isomorphic to $\mathbf{M}''$.

Since $r_{\mathbf{M}'}{\notin}V_{\mathbf{M}'}(\varphi)$ and $r_{\mathbf{M}''}{\in}V_{\mathbf{M}''}(\varphi)$, $r_{\mathbf{M}}{\notin}V_{\mathbf{M}}(\Diamond\varphi \rightarrow \Box\varphi)$. Consequently, by Proposition 12, $\Diamond\varphi \rightarrow \Box\varphi{\notin}\mathbf{K}_d$: a contradiction. $\dashv$

Proposition 15 *1. $\mathbf{K}_d$ is locally tabular,*

2. $\mathbf{A}_d$ is locally tabular.

Proof: It is a routine exercise to demonstrate that for all finite subsets $\bar{y}$ of **VAR** and for all $\varphi{\in}\mathbf{FOR}_{\bar{y}}$, there exists $\psi{\in}\mathbf{FOR}_{\bar{y}}$ such that $\mathtt{deg}(\psi){<}d$, for all $\mathbf{M}{\in}\mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}}{\in}V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$ and for all $\mathbf{M}{\in}\mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}}{\in}V_{\mathbf{M}}(\varphi \leftrightarrow \psi)$. Hence, by [12, Proposition 2.29] and Proposition 12, $\mathbf{K}_d$ is locally tabular and $\mathbf{A}_d$ is locally tabular. $\dashv$

Proposition 16 *1. $\mathbf{K}_d$ is not filtering,*

2. $\mathbf{A}_d$ is filtering.

Proof: (**1**) Let $\varphi{=}\Diamond x \rightarrow \Box x$. We demonstrate φ is $\mathbf{K}_d$-unifiable and not $\mathbf{K}_d$-filtering. Let $(\{x\},\{x\},\sigma)$ and $(\{x\},\{x\},\tau)$ be the substitutions defined by:

- $\sigma(x){=}\Box^{d-1}\bot \vee x$,
- $\tau(x){=}\Diamond^{d-1}\top \wedge x$.

Since by Proposition 13, $\Box\Box^{d-1}\bot{\in}\mathbf{K}_d$ and $\neg\Diamond\Diamond^{d-1}\top{\in}\mathbf{K}_d$, $\sigma(\varphi){\in}\mathbf{K}_d$ and $\tau(\varphi){\in}\mathbf{K}_d$. Hence, $(\{x\},\{x\},\sigma)$ and $(\{x\},\{x\},\tau)$ are $\mathbf{K}_d$-unifiers of φ. Thus, φ is $\mathbf{K}_d$-unifiable. In order to prove that φ is not $\mathbf{K}_d$-filtering, it suffices to prove that $\{(\{x\},\{x\},\sigma),(\{x\},\{x\},\tau)\}$ is a basis of $\mathbf{K}_d$-unifiers of φ. This objective is addressed in Lemmas 1 and 2.

Lemma 1 *$\{(\{x\},\{x\},\sigma),(\{x\},\{x\},\tau)\}$ is a $\mathbf{K}_d$-complete set of $\mathbf{K}_d$-unifiers of φ.*

Proof: Let $(\{x\},\bar{y},\upsilon)$ be a $\mathbf{K}_d$-unifier of φ. Consequently, $\upsilon(\varphi){\in}\mathbf{K}_d$. Hence, by Proposition 14, either $\Box^{d-1}\bot \rightarrow \upsilon(x){\in}\mathbf{K}_d$, or $\upsilon(x) \rightarrow \Diamond^{d-1}\top{\in}\mathbf{K}_d$. In the former case, it follows immediately that $\upsilon(\sigma(x)){\equiv_{\mathbf{K}_d}}\upsilon(x)$. Thus, $(\{x\},\{x\},\sigma){\preccurlyeq_{\mathbf{K}_d}}(\{x\},\bar{y},\upsilon)$. In the latter case, it follows immediately that $\upsilon(\tau(x)){\equiv_{\mathbf{K}_d}}\upsilon(x)$. Consequently, $(\{x\},\{x\},\tau){\preccurlyeq_{\mathbf{K}_d}}(\{x\},\bar{y},\upsilon)$. Since $(\{x\},\bar{y},\upsilon)$ is an arbitrary $\mathbf{K}_d$-unifier of φ, $\{(\{x\},\{x\},\sigma),(\{x\},\{x\},\tau)\}$ is a $\mathbf{K}_d$-complete set of $\mathbf{K}_d$-unifiers of φ. $\dashv$

Lemma 2 *$\{(\{x\},\{x\},\sigma),(\{x\},\{x\},\tau)\}$ is a basis of $\mathbf{K}_d$-unifiers of φ.*

Proof: For the sake of the contradiction, suppose $\{(\{x\},\{x\},\sigma),(\{x\},\{x\},\tau)\}$ is not a basis of $\mathbf{K}_d$-unifiers of φ. Hence, either $(\{x\},\{x\},\sigma)\preccurlyeq_{\mathbf{K}_d}(\{x\},\{x\},\tau)$, or $(\{x\},\{x\},\tau)\preccurlyeq_{\mathbf{K}_d}(\{x\},\{x\},\sigma)$. In the former case, there exists a substitution $(\{x\},\bar{x},\upsilon)$ such that $\upsilon(\sigma(x))\equiv_{\mathbf{K}_d}\tau(x)$. Thus, $\Box^{d-1}\bot\vee\upsilon(x)\equiv_{\mathbf{K}_d}\Diamond^{d-1}\top\wedge x$. In the latter case, there exists a substitution $(\{x\},\bar{x},\upsilon)$ such that $\upsilon(\tau(x))\equiv_{\mathbf{K}_d}\sigma(x)$. Consequently, $\Diamond^{d-1}\top\wedge\upsilon(x)\equiv_{\mathbf{K}_d}\Box^{d-1}\bot\vee x$. In both cases, $\Box^{d-1}\bot\rightarrow\Diamond^{d-1}\top\in\mathbf{K}_d$: a contradiction with Proposition 13. $\dashv$

Consequently, $\mathbf{K}_d$ is not filtering.

(**2**) Let $\varphi\in\mathbf{FOR}$ be $\mathbf{A}_d$-unifiable. We demonstrate φ is $\mathbf{A}_d$-filtering. Let $(\mathtt{var}(\varphi),\bar{y},\sigma),(\mathtt{var}(\varphi),\bar{z},\tau)$ be $\mathbf{A}_d$-unifiers of φ. Let $\bar{t}=\bar{y}\cup\bar{z}\cup\{u\}$ where u is a new variable[12]. Let $(\mathtt{var}(\varphi),\bar{t},\mu)$ be the substitution defined by

- $\mu(x)=(\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\wedge\sigma(x))\vee(\bigwedge\{\Box^l(\neg u\vee\Diamond\top):\ 0\leq l<d\}\wedge\tau(x))$,

where x ranges over $\mathtt{var}(\varphi)$. Let $(\bar{t},\bar{y},\lambda_\top)$ and $(\bar{t},\bar{z},\lambda_\bot)$ be the substitutions defined by

- if $v\in\bar{y}$ then $\lambda_\top(v)=v$ else $\lambda_\top(v)=\top$,
- if $v\in\bar{z}$ then $\lambda_\bot(v)=v$ else $\lambda_\bot(v)=\bot$,

where v ranges over $\bar{t}$. Since by Proposition 13, $\bigvee\{\Diamond^l\Box\bot:\ 0\leq l<d\}\in\mathbf{K}_d$ and $\neg\bigwedge\{\Box^l\Diamond\top:\ 0\leq l<d\}\in\mathbf{K}_d$, for all $x\in\mathtt{var}(\varphi)$, $\lambda_\top(\mu(x))\equiv_{\mathbf{A}_d}\sigma(x)$ and $\lambda_\bot(\mu(x))\equiv_{\mathbf{A}_d}\tau(x)$. Hence, $(\mathtt{var}(\varphi),\bar{t},\mu)\preccurlyeq_{\mathbf{A}_d}(\mathtt{var}(\varphi),\bar{y},\sigma)$ and $(\mathtt{var}(\varphi),\bar{t},\mu)\preccurlyeq_{\mathbf{A}_d}(\mathtt{var}(\varphi),\bar{z},\tau)$. Moreover, by induction on $\psi\in\mathbf{FOR}_{\mathtt{var}(\varphi)}$, the reader may easily verify that $\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\rightarrow(\mu(\psi)\leftrightarrow\sigma(\psi))\in\mathbf{A}_d$ and $\bigwedge\{\Box^l(\neg u\vee\Diamond\top):\ 0\leq l<d\}\rightarrow(\mu(\psi)\leftrightarrow\tau(\psi))\in\mathbf{A}_d$[13]. Thus, $\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\rightarrow\mu(\varphi)\in\mathbf{A}_d$ and $\bigwedge\{\Box^l(\neg u\vee\Diamond\top):\ 0\leq l<d\}\rightarrow\mu(\varphi)\in\mathbf{A}_d$. Consequently, $\mu(\varphi)\in\mathbf{A}_d$. Hence, $(\mathtt{var}(\varphi),\bar{t},\mu)$ is an $\mathbf{A}_d$-unifier of φ. Since $(\mathtt{var}(\varphi),\bar{t},\mu)\preccurlyeq_{\mathbf{A}_d}(\mathtt{var}(\varphi),\bar{y},\sigma)$ and $(\mathtt{var}(\varphi),\bar{t},\mu)\preccurlyeq_{\mathbf{A}_d}(\mathtt{var}(\varphi),\bar{z},\tau)$, φ is $\mathbf{A}_d$-filtering. Since the $\mathbf{A}_d$-unifiable $\varphi\in\mathbf{FOR}$ was arbitrary, $\mathbf{A}_d$ is filtering.

This finishes the proof of Proposition 16. $\dashv$

5.4 Valuable functions

Let $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$. For all $e\in\{1,\ldots,d\}$, a function $f:\ \mathcal{BT}_e^{\bar{y}}/\bowtie\longrightarrow\mathcal{BT}_e^{\bar{x}}/\bowtie$ is *valuable* if for all $\mathbf{M}\in\mathcal{BT}_e^{\bar{y}}$ and for all $\mathbf{M}'\in\mathcal{BT}_e^{\bar{x}}$, if $f([\mathbf{M}])=[\mathbf{M}']$ then the following conditions hold:

[12] That is, neither $u\in\mathtt{var}(\varphi)$, nor $u\in\bar{y}$, nor $u\in\bar{z}$.

[13] This proof by induction uses the fact — see Proposition 13 — that $\bigwedge\{\Box^l(\neg u\vee\Diamond\top):\ 0\leq l<d\}\rightarrow\Box\bigwedge\{\Box^l(\neg u\vee\Diamond\top):\ 0\leq l<d\}\in\mathbf{K}_d$ and $\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\rightarrow\Box\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\in\mathbf{A}_d$. By the way, the fact — see Proposition 13 — that $\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\rightarrow\Box\bigvee\{\Diamond^l(u\wedge\Box\bot):\ 0\leq l<d\}\not\in\mathbf{K}_d$ explains — if it is still needed — why our argument cannot be repeated for $\mathbf{K}_d$.

- for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f([\mathbf{M}_s])=[\mathbf{M}'_{s'}]$,
- for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f([\mathbf{M}_s])=[\mathbf{M}'_{s'}]$.

Of course, in the above items, $R_{\mathbf{M}}(r_{\mathbf{M}})$ and $R_{\mathbf{M}'}(r_{\mathbf{M}'})$ are finite. For all $e \in \{1,\ldots,d\}$, let $f^{\sigma}_e : \mathcal{BT}^{\bar{y}}_e/\bowtie \longrightarrow \mathcal{BT}^{\bar{x}}_e/\bowtie$ be the function such that for all $\mathbf{M} \in \mathcal{BT}^{\bar{y}}_e$, $f^{\sigma}_e([\mathbf{M}])=[\mathbf{M}^{|\sigma}]$[14].

Proposition 17 *For all $e \in \{1,\ldots,d\}$, for all $\mathbf{M} \in \mathcal{BT}^{\bar{y}}_e$ and for all $s \in W_{\mathbf{M}}$, $f^{\sigma}_e([\mathbf{M}])_s = f^{\sigma}_e([\mathbf{M}_s])$.*

Proof: By [12, Proposition 2.6]. $\dashv$

Proposition 18 *For all $e \in \{1,\ldots,d\}$, the function f^{σ}_e is valuable.*

Proof: By Proposition 17. $\dashv$

Let $(\bar{x},\bar{y},\sigma) \in \mathbf{SUB}$. For all $e \in \{1,\ldots,d\}$, a function $f : \mathcal{DBT}^{\bar{y}}_e/\bowtie \longrightarrow \mathcal{DBT}^{\bar{x}}_e/\bowtie$ is *valuable* if for all $\mathbf{M} \in \mathcal{DBT}^{\bar{y}}_e$ and for all $\mathbf{M}' \in \mathcal{DBT}^{\bar{x}}_e$, if $f([\mathbf{M}])=[\mathbf{M}']$ then the following conditions hold:

- for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f([\mathbf{M}_s])=[\mathbf{M}'_{s'}]$,
- for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f([\mathbf{M}_s])=[\mathbf{M}'_{s'}]$.

Of course, in the above items, $R_{\mathbf{M}}(r_{\mathbf{M}})$ and $R_{\mathbf{M}'}(r_{\mathbf{M}'})$ are with cardinality ≤ 1. For all $e \in \{1,\ldots,d\}$, let $f^{\sigma}_e : \mathcal{DBT}^{\bar{y}}_e/\bowtie \longrightarrow \mathcal{DBT}^{\bar{x}}_e/\bowtie$ be the function such that for all $\mathbf{M} \in \mathcal{DBT}^{\bar{y}}_e$, $f^{\sigma}_e([\mathbf{M}])=[\mathbf{M}^{|\sigma}]$[15].

Proposition 19 *For all $e \in \{1,\ldots,d\}$, for all $\mathbf{M} \in \mathcal{DBT}^{\bar{y}}_e$ and for all $s \in W_{\mathbf{M}}$, $f^{\sigma}_e([\mathbf{M}])_s = f^{\sigma}_e([\mathbf{M}_s])$.*

Proof: By [12, Proposition 2.6]. $\dashv$

Proposition 20 *For all $e \in \{1,\ldots,d\}$, the function f^{σ}_e is valuable.*

Proof: By Proposition 19. $\dashv$

6 About the unification type of $\mathbf{A}_d$

In this section, we prove that $\mathbf{A}_d$ is unitary[16]. For all $e \in \{1,\ldots,d\}$, the surjective valuable function $f_e : \mathcal{DBT}^{\bar{y}}_e/\bowtie \longrightarrow \mathcal{DBT}^{\bar{x}}_e/\bowtie$ associated in Proposition 21 to each $(\bar{x},\bar{y},\sigma) \in \mathbf{SUB}$ such that $\|\bar{x}\| < \|\bar{y}\|$ be crucially used in the proof of Proposition 22.

[14] Notice that for all $e \in \{1,\ldots,d\}$, by Proposition 11, the function f^{σ}_e is well-defined.
[15] Notice that for all $e \in \{1,\ldots,d\}$, by Proposition 11, the function f^{σ}_e is well-defined.
[16] By Propositions 7 and 16, we know that $\mathbf{A}_d$ is either nullary, or unitary.

Proposition 21 *Let $(\bar{x}, \bar{y}, \sigma) \in$* **SUB**. *Let $e \in \{1, \ldots, d\}$. If $\|\bar{x}\| < \|\bar{y}\|$ then there exists a surjective valuable function $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \longrightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^{\sigma}([\mathbf{M}]) = f_e^{\sigma}([\mathbf{N}])$.*

Proof: Suppose $\|\bar{x}\| < \|\bar{y}\|$. By induction on e, we define a surjective valuable function $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \longrightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ such that for all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}]) = f_e([\mathbf{N}])$ then $f_e^{\sigma}([\mathbf{M}]) = f_e^{\sigma}([\mathbf{N}])$. We consider the following 2 cases.

Case e=1: Let $U = \{f_1^{\sigma}([\mathbf{M}]) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$. Notice that $U \subseteq \mathcal{DBT}_1^{\bar{x}}/\bowtie$. Let h be a function from U to $\mathcal{DBT}_1^{\bar{y}}/\bowtie$ such that for all $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$, $f_1^{\sigma}(h(f_1^{\sigma}([\mathbf{M}]))) = f_1^{\sigma}([\mathbf{M}])$. Notice that h is injective. Hence, $\|U\| = \|\{h(f_1^{\sigma}([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$. Since $\|\bar{x}\| < \|\bar{y}\|$, $\|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\| \leq \|\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^{\sigma}([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}\|$. Let S be a subset of $\mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus \{h(f_1^{\sigma}([\mathbf{M}])) : \mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}\}$ such that $\|S\| = \|\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U\|$. Let f_1^* be a one-to-one correspondence between S and $\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$. Now, we define the function f_1. Let f_1 be the function from $\mathcal{DBT}_1^{\bar{y}}/\bowtie$ to $\mathcal{DBT}_1^{\bar{x}}/\bowtie$ such that

- if $[\mathbf{M}] \in S$ then $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$ else $f_1([\mathbf{M}]) = f_1^{\sigma}([\mathbf{M}])$,

where $\mathbf{M}$ ranges over $\mathcal{DBT}_1^{\bar{y}}$. Notice that f_1 is valuable. In Lemmas 3 and 4, we show that f_1 possesses the required properties.

Lemma 3 *f_1 is surjective.*

Proof: Let $\mathbf{N} \in \mathcal{DBT}_1^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}] \in U$: Hence, let $\mathbf{M} \in \mathcal{DBT}_1^{\bar{y}}$ be such that $f_1^{\sigma}([\mathbf{M}]) = [\mathbf{N}]$. Thus, $h(f_1^{\sigma}([\mathbf{M}])) = h([\mathbf{N}])$. Consequently, $h([\mathbf{N}]) \not\in S$. Hence, $f_1(h([\mathbf{N}])) = f_1^{\sigma}(h([\mathbf{N}]))$. Since $h(f_1^{\sigma}([\mathbf{M}])) = h([\mathbf{N}])$, $f_1(h([\mathbf{N}])) = f_1^{\sigma}(h(f_1^{\sigma}([\mathbf{M}])))$. Thus, $f_1(h([\mathbf{N}])) = f_1^{\sigma}([\mathbf{M}])$. Since $f_1^{\sigma}([\mathbf{M}]) = [\mathbf{N}]$, $f_1(h([\mathbf{N}])) = [\mathbf{N}]$.

Case $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U$: Since f_1^* is one-to-one, let $[\mathbf{M}] \in S$ be such that $f_1^*([\mathbf{M}]) = [\mathbf{N}]$. Consequently, $f_1([\mathbf{M}]) = f_1^*([\mathbf{M}])$. Since $f_1^*([\mathbf{M}]) = [\mathbf{N}]$, $f_1([\mathbf{M}]) = [\mathbf{N}]$. $\dashv$

Lemma 4 *For all $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$, if $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$ then $f_1^{\sigma}([\mathbf{M}]) = f_1^{\sigma}([\mathbf{N}])$.*

Proof: Let $\mathbf{M}, \mathbf{N} \in \mathcal{DBT}_1^{\bar{y}}$. Suppose $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^*([\mathbf{N}])$. Since f_1^* is one-to-one, $[\mathbf{M}] = [\mathbf{N}]$. Hence, $f_1^{\sigma}([\mathbf{M}]) = f_1^{\sigma}([\mathbf{N}])$.

Case $[\mathbf{M}] \in S$ and $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}]) = f_1^{\sigma}([\mathbf{N}])$. Thus $(\mathcal{DBT}_1^{\bar{x}}/\bowtie \setminus U) \cap \in U \neq \emptyset$: a contradiction.

Case $[\mathbf{M}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$ and $[\mathbf{N}] \in \mathcal{DBT}_1^{\bar{y}}/\bowtie \setminus S$: Since $f_1([\mathbf{M}]) = f_1([\mathbf{N}])$,

$f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}])$. ⊣

Case $e\geq 2$**:** By induction hypothesis, let f_{e-1} be a surjective valuable function from $\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ to $\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ such that for all $\mathbf{M},\mathbf{N}\in\mathcal{DBT}_{e-1}^{\bar{y}}$, if $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$ then $f_{e-1}^\sigma([\mathbf{M}])=f_{e-1}^\sigma([\mathbf{N}])$. For all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, let

- $S([\mathbf{M}'])$ be the set of all $[\mathbf{M}]$ such that $\mathbf{M}\in\mathcal{DBT}_{e}^{\bar{y}}$ and there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M_s}])=[\mathbf{M}']$.

Notice that for all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $S([\mathbf{M}'])\subseteq\mathcal{DBT}_{e}^{\bar{y}}/\bowtie\setminus\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$. For all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, let $\sim_{\mathbf{M}'}$ be the equivalence relation on $S([\mathbf{M}'])$ defined by

- $[\mathbf{M}]\sim_{[\mathbf{M}']}[\mathbf{N}]$ if and only if $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$.

where $[\mathbf{M}],[\mathbf{N}]$ range over $S([\mathbf{M}'])$. In Lemmas 5 and 6, we compare $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\|$ and $\|S([\mathbf{M}'])\|$ with $2^{\|\bar{x}\|}$ for each $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$.

Lemma 5 *For all* $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\|\leq 2^{\|\bar{x}\|}$.

Proof: Let $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$. For the sake of the contradiction, suppose $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\|>2^{\|\bar{x}\|}$. Let $p\in\mathbb{N}$ and $[\mathbf{M_1}],\ldots,[\mathbf{M_p}]\in S([\mathbf{M}'])$ be such that $p>2^{\|\bar{x}\|}$ and for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M_q}]\not\sim_{[\mathbf{M}']}[\mathbf{M_r}]$. Hence, $\mathbf{M_1},\ldots,\mathbf{M_p}\in\mathcal{DBT}_{e}^{\bar{y}}$ and there exists $s_1\in R_{\mathbf{M_1}}(r_{\mathbf{M_1}}),\ldots,s_p\in R_{\mathbf{M_p}}(r_{\mathbf{M_p}})$ such that $f_{e-1}([\mathbf{M_{1s_1}}])=[\mathbf{M}']$, …, $f_{e-1}([\mathbf{M_{ps_p}}])=[\mathbf{M}']$. Thus, let $\mathbf{M}''\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$ be such that $f_{e-1}^\sigma([\mathbf{M_{1s_1}}])=[\mathbf{M}'']$, …, $f_{e-1}^\sigma([\mathbf{M_{ps_p}}])=[\mathbf{M}'']$. Consequently, by Proposition 20, let $\mathbf{M_1''},\ldots,\mathbf{M_p''}\in\mathcal{DBT}_{e}^{\bar{x}}$ and $s_1''\in R_{\mathbf{M_1''}}(r_{\mathbf{M_1''}}),\ldots,s_p''\in R_{\mathbf{M_p''}}(r_{\mathbf{M_p''}})$ be such that $f_e^\sigma([\mathbf{M_1}])=[\mathbf{M_1''}]$, …, $f_e^\sigma([\mathbf{M_p}])=[\mathbf{M_p''}]$ and $[\mathbf{M''_{1s_1''}}]=[\mathbf{M}'']$, …, $[\mathbf{M''_{ps_p''}}]=[\mathbf{M}'']$. Since for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M_q}]\not\sim_{[\mathbf{M}']}[\mathbf{M_r}]$, for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $f_e^\sigma([\mathbf{M_q}])\neq f_e^\sigma([\mathbf{M_r}])$. Since $f_e^\sigma([\mathbf{M_1}])=[\mathbf{M_1''}]$, …, $f_e^\sigma([\mathbf{M_p}])=[\mathbf{M_p''}]$, for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M_q''}]\neq[\mathbf{M_r''}]$. Since $[\mathbf{M''_{1s_1''}}]=[\mathbf{M}'']$, …, $[\mathbf{M''_{ps_p''}}]=[\mathbf{M}'']$, $p\leq 2^{\|\bar{x}\|}$: a contradiction. ⊣

Lemma 6 *For all* $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $2^{\|\bar{x}\|}\leq\|S([\mathbf{M}'])\|$.

Proof: Let $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$. Since f_{e-1} is surjective, $\|S([\mathbf{M}'])\|\geq 2^{\|\bar{y}\|}$. Since $\|\bar{x}\|<\|\bar{y}\|$, $2^{\|\bar{x}\|}\leq\|S([\mathbf{M}'])\|$. ⊣

For all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, let

- $T([\mathbf{M}'])$ be the set of all $[\mathbf{N}']$ such that $\mathbf{N}'\in\mathcal{DBT}_{e}^{\bar{x}}$ and there exists $s'\in R_{\mathbf{N}'}(r_{\mathbf{N}'})$ such that $[\mathbf{N}'_{\mathbf{s}'}]=[\mathbf{M}']$.

Notice that for all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $T([\mathbf{M}'])\subseteq\mathcal{DBT}_{e}^{\bar{x}}/\bowtie\setminus\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$. Moreover, notice that for all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $\|T([\mathbf{M}'])\|=2^{\|\bar{x}\|}$. Consequently, by Lemmas 5 and 6, for all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, $\|S([\mathbf{M}'])/\sim_{[\mathbf{M}']}\|\leq\|T([\mathbf{M}'])\|\leq\|S([\mathbf{M}'])\|$. Hence, by Proposition 1, for all $\mathbf{M}'\in\mathcal{DBT}_{e-1}^{\bar{x}}\setminus\mathcal{DBT}_{e-2}^{\bar{x}}$, let $g_e^{[\mathbf{M}']}$ be a surjective function from $S([\mathbf{M}'])$ to $T([\mathbf{M}'])$ such that for all

$[\mathbf{M}],[\mathbf{N}]\in S([\mathbf{M}'])$, if $g_e^{[\mathbf{M}']}([\mathbf{M}])=g_e^{[\mathbf{M}']}([\mathbf{N}])$ then $[\mathbf{M}]\sim_{[\mathbf{M}']}[\mathbf{N}]$[17]. Now, we define the function f_e. Let $f_e : \mathcal{DBT}_e^{\bar{y}}/\bowtie \longrightarrow \mathcal{DBT}_e^{\bar{x}}/\bowtie$ be the function such that

- if $\mathbf{M}\in\mathcal{DBT}_{e-1}^{\bar{y}}$ then $f_e([\mathbf{M}])=f_{e-1}([\mathbf{M}])$ else $f_e([\mathbf{M}])=g_e^{f_{e-1}([\mathbf{M_s}])}([\mathbf{M}])$,

where $\mathbf{M}$ ranges over $\mathcal{DBT}_e^{\bar{y}}$ and $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$. Notice that f_e is valuable. In Lemmas 7 and 8, we show that f_e possesses the required properties.

Lemma 7 *f_e is surjective.*

Proof: Let $\mathbf{N}\in\mathcal{DBT}_e^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}]\in\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$: Since f_{e-1} is surjective, let $[\mathbf{M}]\in\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ be such that $f_{e-1}([\mathbf{M}])=[\mathbf{N}]$. Hence, $f_e([\mathbf{M}])=f_{e-1}([\mathbf{M}])$. Since $f_{e-1}([\mathbf{M}])=[\mathbf{N}]$, $f_e([\mathbf{M}])=[\mathbf{N}]$.

Case $[\mathbf{N}]\in\mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$: Let $s\in R_{\mathbf{N}}(r_{\mathbf{N}})$. Thus, $[\mathbf{N_s}]\in\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-2}^{\bar{x}}/\bowtie$. Moreover, $[\mathbf{N}]\in T([\mathbf{N_s}])$. Since $g_e^{[\mathbf{N_s}]}$ is surjective, let $[\mathbf{M}']\in S([\mathbf{N_s}])$ be such that $g_e^{[\mathbf{N_s}]}([\mathbf{M}'])=[\mathbf{N}]$. Let $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_{e-1}([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{N_s}]$. Since $g_e^{[\mathbf{N_s}]}([\mathbf{M}'])=[\mathbf{N}]$, $g_e^{f_{e-1}([\mathbf{M}'_{\mathbf{s}'}])}([\mathbf{M}'])=[\mathbf{N}]$. Consequently, $f_e([\mathbf{M}'])=[\mathbf{N}]$. $\dashv$

Lemma 8 *For all $\mathbf{M},\mathbf{N}\in\mathcal{DBT}_e^{\bar{y}}$, if $f_e([\mathbf{M}])=f_e([\mathbf{N}])$ then $f_e^{\sigma}([\mathbf{M}])=f_e^{\sigma}([\mathbf{N}])$.*

Proof: Let $\mathbf{M},\mathbf{N}\in\mathcal{DBT}_e^{\bar{y}}$. Suppose $f_e([\mathbf{M}])=f_e([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}]\in\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}]\in\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$. Hence, $f_{e-1}^{\sigma}([\mathbf{M}])=f_{e-1}^{\sigma}([\mathbf{N}])$ Thus, $f_e^{\sigma}([\mathbf{M}])=f_e^{\sigma}([\mathbf{N}])$

Case $[\mathbf{M}]\in\mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}]\in\mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}])=g_e^{f_{e-1}([\mathbf{N_s}])}([\mathbf{N}])$ where $s\in R_{\mathbf{N}}(r_{\mathbf{N}})$. Consequently, $\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie \cap (\mathcal{DBT}_e^{\bar{x}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie)\neq\emptyset$: a contradiction.

Case $[\mathbf{M}]\in\mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$ and $[\mathbf{N}]\in\mathcal{DBT}_e^{\bar{y}}/\bowtie \setminus \mathcal{DBT}_{e-1}^{\bar{y}}/\bowtie$: Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $g_e^{f_{e-1}([\mathbf{M_s}])}([\mathbf{M}])=g_e^{f_{e-1}([\mathbf{N_t}])}([\mathbf{N}])$ where $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ and $t\in R_{\mathbf{N}}(r_{\mathbf{N}})$. Consequently, $f_{e-1}([\mathbf{M_s}])=f_{e-1}([\mathbf{N_t}])$. Let $[\mathbf{O}]\in\mathcal{DBT}_{e-1}^{\bar{x}}/\bowtie$ be such that $f_{e-1}([\mathbf{M_s}])=[\mathbf{O}]$ and $f_{e-1}([\mathbf{N_t}])=[\mathbf{O}]$. Since $g_e^{f_{e-1}([\mathbf{M_s}])}([\mathbf{M}])=g_e^{f_{e-1}([\mathbf{N_t}])}([\mathbf{N}])$, $[\mathbf{M}]\sim_{[\mathbf{O}]}[\mathbf{N}]$. Hence, $f_e^{\sigma}([\mathbf{M}])=f_e^{\sigma}([\mathbf{N}])$. $\dashv$

This finishes the proof of Proposition 21. $\dashv$

Proposition 22 *Let $\varphi\in\mathbf{FOR}_{\bar{x}}$. If $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ is an $\mathbf{A}_d$-unifier of φ then there exists an $\mathbf{A}_d$-unifier $(\bar{x},\bar{x},\tau)\in\mathbf{SUB}$ of φ such that $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{A}_d}(\bar{x},\bar{y},\sigma)$.*

[17] Notice that this is the only place in this section where we use Proposition 1.

Proof: Suppose $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ is an $\mathbf{A}_d$-unifier of φ. Hence, for all $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$.

Case $\|\bar{x}\|\geq\|\bar{y}\|$: Let $(\bar{x},\bar{x},\tau)\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$, $\tau(x)=\lambda(\sigma(x))$ for some injective $(\bar{y},\bar{x},\lambda)\in\mathbf{SUB}$ such that $\lambda(\bar{y})\subseteq\bar{x}$. Notice that since for all $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$, for all $\mathbf{M}\in\mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\tau(\varphi))$. Thus, $(\bar{x},\bar{x},\tau)$ is an $\mathbf{A}_d$-unifier of φ. Let $(\bar{x},\bar{y},\lambda')\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$, if there exists $y\in\bar{y}$ such that $\lambda(y)=x$ then $\lambda'(x)=y$. Notice that for all $x\in\bar{x}$, $\lambda'(\tau(x))\equiv_{\mathbf{A}_d}\sigma(x)$. Consequently, $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{A}_d}(\bar{x},\bar{y},\sigma)$.

Case $\|\bar{x}\|<\|\bar{y}\|$: By Proposition 21, let $f_d : \mathcal{DBT}_d^{\bar{y}}/\bowtie \longrightarrow \mathcal{DBT}_d^{\bar{x}}/\bowtie$ be a surjective valuable function such that for all $\mathbf{M},\mathbf{N}\in\mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}])=f_d([\mathbf{N}])$ then $f_d^{\sigma}([\mathbf{M}])=f_d^{\sigma}([\mathbf{N}])$. Let $(\bar{x},\bar{x},\tau),(\bar{x},\bar{y},\nu)\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$,

- $\tau(x)=\bigvee\{\mathbf{for}(\mathbf{M}') : \mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{x}}$ are such that $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$ and $f_d([\mathbf{M}])=[\mathbf{M}']\}$,
- $\nu(x)=\bigvee\{\mathbf{for}(\mathbf{M}) : \mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{x}}$ are such that $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$ and $f_d([\mathbf{M}])=[\mathbf{M}']\}$.

Lemmas 9, 10, 11 and 12 state results connecting the substitutions $(\bar{x},\bar{x},\tau)$ and $(\bar{x},\bar{y},\nu)$ with the models in $\mathcal{DBT}_d^{\bar{x}}$ and $\mathcal{DBT}_d^{\bar{y}}$.

Lemma 9 *Let $\varphi\in\mathbf{FOR}_{\bar{x}}$. For all $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{x}}$, the following conditions are equivalent:*

1. *there exists $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}])=[\mathbf{M}']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$,*
2. *for all $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}])=[\mathbf{M}']$ then $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$,*
3. *$r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\tau(\varphi))$.*

Proof: By induction on φ.

Case $\varphi=x$: Let $\mathbf{M}\in\mathcal{DBT}_d^{\bar{x}}$.

$(\mathbf{1}\Rightarrow\mathbf{2})$ Suppose $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$. Let $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since $f_d([\mathbf{M}'])=[\mathbf{M}]$, $f_d([\mathbf{M}'])=f_d([\mathbf{M}''])$. Hence, $f_d^{\sigma}([\mathbf{M}'])=f_d^{\sigma}([\mathbf{M}''])$. Thus, $[\mathbf{M}'^{|\sigma}]=[\mathbf{M}''^{|\sigma}]$. Since $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'^{|\sigma}}(x)$. Since $[\mathbf{M}'^{|\sigma}]=[\mathbf{M}''^{|\sigma}]$, $r_{\mathbf{M}''}\in V_{\mathbf{M}''^{|\sigma}}(x)$. Consequently, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$. Since $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}''])=[\mathbf{M}]$ is arbitrary, for all $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}''])=[\mathbf{M}]$ then $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$.

$(\mathbf{2}\Rightarrow\mathbf{3})$ Suppose for all $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$. Since f_d is surjective, let $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since for all $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$. Hence, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\tau(x)$. Since by Proposition 10, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\tau(x))$.

($\mathbf{3} \Rightarrow \mathbf{1}$) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\tau(x))$. Let $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$ and $\mathbf{M}'' \in \mathcal{DBT}^{\bar{x}}_{d}$ be such that $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}'])=[\mathbf{M}'']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$. Thus, $f_d([\mathbf{M}'])=[\mathbf{M}]$. Consequently, there exists $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\sigma(x))$.

Case $\varphi = \bot$**:** Left to the reader.

Case $\varphi = \neg\psi$**:** Left to the reader.

Case $\varphi = \psi \vee \chi$**:** Left to the reader.

Case $\varphi = \Box\psi$**:**

($\mathbf{1} \Rightarrow \mathbf{2}$) Suppose $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$ is such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $\mathbf{M}'' \in \mathcal{DBT}^{\bar{y}}_{d}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$. Since f_d is valuable and $f_d([\mathbf{M}''])=[\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$. Since $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$, $f_d([\mathbf{M}'_{\mathbf{s}'}])=f_d([\mathbf{M}''_{\mathbf{s}''}])$. Hence, $f^{\sigma}_{d}([\mathbf{M}'_{\mathbf{s}'}])=f^{\sigma}_{d}([\mathbf{M}''_{\mathbf{s}''}])$. Thus, $[\mathbf{M}'_{\mathbf{s}'}{}^{|\sigma}]=[\mathbf{M}''_{\mathbf{s}''}{}^{|\sigma}]$. Since $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Consequently, $s' \in V_{\mathbf{M}'^{|\sigma}}(\psi)$. Since $[\mathbf{M}'_{\mathbf{s}'}{}^{|\sigma}]=[\mathbf{M}''_{\mathbf{s}''}{}^{|\sigma}]$, $s'' \in V_{\mathbf{M}''^{|\sigma}}(\psi)$. Hence, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ is arbitrary, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $\mathbf{M}'' \in \mathcal{DBT}^{\bar{y}}_{d}$ such that $f_d([\mathbf{M}''])=[\mathbf{M}]$ is arbitrary, for all $\mathbf{M}'' \in \mathcal{DBT}^{\bar{y}}_{d}$, if $f_d([\mathbf{M}''])=[\mathbf{M}]$ then $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$.

($\mathbf{2} \Rightarrow \mathbf{3}$) Suppose for all $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$. Since f_d is surjective, let $\mathbf{M}'' \in \mathcal{DBT}^{\bar{y}}_{d}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since f_d is valuable and $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, let $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ be such that $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$. Since for all $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $f_d([\mathbf{M}''])=[\mathbf{M}]$, $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, $s'' \in V_{\mathbf{M}''}(\sigma(\psi))$. Since $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ is arbitrary, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$.

($\mathbf{3} \Rightarrow \mathbf{1}$) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$. Since f_d is surjective, let $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$ be such that $f_d([\mathbf{M}'])=[\mathbf{M}]$. Let $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$. Since $r_{\mathbf{M}} \in V_{\mathbf{M}}(\Box\tau(\psi))$, $s \in V_{\mathbf{M}}(\tau(\psi))$. Since $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s' \in V_{\mathbf{M}'}(\sigma(\psi))$. Since $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ is arbitrary, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Consequently, there exists $\mathbf{M}' \in \mathcal{DBT}^{\bar{y}}_{d}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\Box\sigma(\psi))$. $\dashv$

Lemma 10 *For all* $\mathbf{M} \in \mathcal{DBT}^{\bar{y}}_{d}$ *and for all* $\mathbf{M}' \in \mathcal{DBT}^{\bar{x}}_{d}$, *if* $f_d([\mathbf{M}])=[\mathbf{M}']$ *then for all* $x \in \bar{x}$, *the following conditions are equivalent:*

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$,
2. $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

Proof: Let $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{x}}$. Suppose $f_d([\mathbf{M}])=[\mathbf{M}']$. Let $x\in\bar{x}$.

($\mathbf{1}\Rightarrow\mathbf{2}$) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$. Let $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}'''\in\mathcal{DBT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'''}\in V_{\mathbf{M}'''}(x)$, $f_d([M''])=[M''']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$. Since $f_d([\mathbf{M}])=[\mathbf{M}']$, $[M']=[M''']$. Since $r_{\mathbf{M}'''}\in V_{\mathbf{M}'''}(x)$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$.

($\mathbf{2}\Rightarrow\mathbf{1}$) Suppose $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$. Since $f_d([\mathbf{M}])=[\mathbf{M}']$, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\nu(x)$. Since by Proposition 10, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$. $\dashv$

Lemma 11 *Let $e\in\{1,\ldots,d\}$. For all $\mathbf{M}\in\mathcal{DBT}_e^{\bar{y}}$ and for all $\mathbf{M}'\in\mathcal{DBT}_e^{\bar{x}}$, $f_e([\mathbf{M}])=[\mathbf{M}']$ if and only if $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$.*

Proof: By induction on e.

Case $e=1$: Let $\mathbf{M}\in\mathcal{DBT}_1^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{DBT}_1^{\bar{x}}$. Suppose $f_1([\mathbf{M}])=[\mathbf{M}']$. Hence, by Lemma 10, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$. Since by Proposition 10, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\mathbf{for}(\mathbf{M}'))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Thus, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$. Let $\mathbf{M}''\in\mathcal{DBT}_1^{\bar{x}}$ be such that $f_1([\mathbf{M}])=[\mathbf{M}'']$. Consequently, by Lemma 10, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(x)$. Since for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$, for all $x\in\bar{x}$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(x)$. Hence, $[\mathbf{M}']=[\mathbf{M}'']$. Since $f_1([\mathbf{M}])=[\mathbf{M}'']$, $f_1([\mathbf{M}])=[\mathbf{M}']$.

Case $e\geq 2$: Let $\mathbf{M}\in\mathcal{DBT}_e^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{DBT}_e^{\bar{x}}$. Suppose $f_e([\mathbf{M}])=[\mathbf{M}']$. Thus, by Lemma 10, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$. Moreover, since f_e is valuable, for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$. Consequently, by induction hypothesis, for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$. Since for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Hence, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$. Moreover, for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$. Thus, by induction hypothesis, for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$. Let $\mathbf{M}''\in\mathcal{DBT}_e^{\bar{x}}$ be such that $f_e([\mathbf{M}])=[\mathbf{M}'']$. Consequently, by Lemma 10, for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(x)$. Moreover, since f_e is valuable, for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}''_{\mathbf{s}''}]$ and for all $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}''_{\mathbf{s}''}]$. Since for all $x\in\bar{x}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$, for all $x\in\bar{x}$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(x)$. Moreover, since for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$, for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $[\mathbf{M}'_{\mathbf{s}'}]=[\mathbf{M}''_{\mathbf{s}''}]$ and for all $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists

$s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $[\mathbf{M}'_{\mathbf{s}'}]=[\mathbf{M}''_{\mathbf{s}''}]$. Hence, $[\mathbf{M}']=[\mathbf{M}'']$. Since $f_e([\mathbf{M}])=[\mathbf{M}'']$, $f_e([\mathbf{M}])=[\mathbf{M}']$. $\dashv$

Lemma 12 *For all* $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ *and for all* $x\in\bar{x}$, *the following conditions are equivalent:*

1. $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$,
2. $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$.

Proof: Let $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$ and $x\in\bar{x}$.

($\mathbf{1}\Rightarrow\mathbf{2}$) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$. Let $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{y}}$ and $\mathbf{M}''\in\mathcal{DBT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $f_d([M'])=[M'']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'')))$. Hence, by Lemma 11, $f_d([M])=[M'']$. Since $f_d([M'])=[M'']$, $f_d([M'])=f_d([M])$. Thus, $f_d^{\sigma}([M'])=f_d^{\sigma}([M])$. Consequently, $[\mathbf{M}^{|\sigma}]=[\mathbf{M}'^{|\sigma}]$. Since $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'^{|\sigma}}(x)$. Since $[\mathbf{M}^{|\sigma}]=[\mathbf{M}'^{|\sigma}]$, $r_{\mathbf{M}}\in V_{\mathbf{M}^{|\sigma}}(x)$. Hence, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$.

($\mathbf{2}\Rightarrow\mathbf{1}$) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$. Thus, $\mathbf{for}(f_d([\mathbf{M}]))$ is a disjunct in $\tau(x)$. Since by Lemma 11, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}]))))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$. $\dashv$

Since for all $\mathbf{M}\in\mathcal{DBT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$, by Lemma 9, for all $\mathbf{M}'\in\mathcal{DBT}_d^{\bar{x}}$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\tau(\varphi))$. Hence, $(\bar{x},\bar{x},\tau)$ is an $\mathbf{A}_d$-unifier of φ. Moreover, by Lemma 12, for all $x\in\bar{x}$, $\nu(\tau(x))\leftrightarrow\sigma(x)\in\mathbf{A}_d$[18]. Thus, $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{A}_d}(\bar{x},\bar{y},\sigma)$.

This finishes the proof of Proposition 22. $\dashv$

Proposition 23 $\mathbf{A}_d$ *is reasonable.*

Proof: By Proposition 22. $\dashv$

Proposition 24 $\mathbf{A}_d$ *is unitary.*

Proof: By Propositions 15, 7, 8, 16 and 23. $\dashv$

7 About the unification type of $\mathbf{K}_d$

In this section, we prove that $\mathbf{K}_d$ is finitary[19]. For all $e\in\{1,\ldots,d\}$, the surjective valuable function $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie\longrightarrow\mathcal{BT}_e^{\bar{x}}/\bowtie$ associated in Proposition 25 to each $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ such that $\|\bar{x}\|<\|\bar{y}\|$ will be crucially used in the proof of Proposition 26.

[18]Lemmas 10 and 11 are used in the proof of Lemma 12.

[19]By Propositions 6 and 16, we know that $\mathbf{K}_d$ is not unitary.

Proposition 25 *Let* $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$. *Let* $e\in\{1,\ldots,d\}$. *If* $\|\bar{x}\|<\|\bar{y}\|$ *then there exists a surjective valuable function* $f_e:\ \mathcal{BT}_e^{\bar{y}}/\!\bowtie\ \longrightarrow \mathcal{BT}_e^{\bar{x}}/\!\bowtie$ *such that for all* $\mathbf{M},\mathbf{N}\in\mathcal{BT}_e^{\bar{y}}$, *if* $f_e([\mathbf{M}])=f_e([\mathbf{N}])$ *then* $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$.

Proof: Suppose $\|\bar{x}\|<\|\bar{y}\|$. By induction on e, we define a surjective valuable function $f_e:\ \mathcal{BT}_e^{\bar{y}}/\!\bowtie\ \longrightarrow \mathcal{BT}_e^{\bar{x}}/\!\bowtie$ such that for all $\mathbf{M},\mathbf{N}\in\mathcal{BT}_e^{\bar{y}}$, if $f_e([\mathbf{M}])=f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$. We consider the following 2 cases.

Case $e=1$**:** Let $U=\{f_1^\sigma([\mathbf{M}]):\ \mathbf{M}\in\mathcal{BT}_1^{\bar{y}}\}$. Notice that $U\subseteq\mathcal{BT}_1^{\bar{x}}/\!\bowtie$. Let h be a function from U to $\mathcal{BT}_1^{\bar{y}}/\!\bowtie$ such that for all $\mathbf{M}\in\mathcal{BT}_1^{\bar{y}}$, $f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))=f_1^\sigma([\mathbf{M}])$. Notice that h is injective. Hence, $\|U\|=\|\{h(f_1^\sigma([\mathbf{M}])):\ \mathbf{M}\in\mathcal{BT}_1^{\bar{y}}\}\|$. Since $\|\bar{x}\|<\|\bar{y}\|$, $\|\mathcal{BT}_1^{\bar{x}}/\!\bowtie\setminus U\|\leq\|\mathcal{BT}_1^{\bar{y}}/\!\bowtie\setminus\{h(f_1^\sigma([\mathbf{M}])):\ \mathbf{M}\in\mathcal{BT}_1^{\bar{y}}\}\|$. Let S be a subset of $\mathcal{BT}_1^{\bar{y}}/\!\bowtie\setminus\{h(f_1^\sigma([\mathbf{M}])):\ \mathbf{M}\in\mathcal{BT}_1^{\bar{y}}\}$ such that $\|S\|=\|\mathcal{BT}_1^{\bar{x}}/\!\bowtie\setminus U\|$. Let f_1^* be a one-to-one correspondence between S and $\mathcal{BT}_1^{\bar{x}}/\!\bowtie\setminus U$. Now, we define the function f_1. Let f_1 be the function from $\mathcal{BT}_1^{\bar{y}}/\!\bowtie$ to $\mathcal{BT}_1^{\bar{x}}/\!\bowtie$ such that

- if $[\mathbf{M}]\in S$ then $f_1([\mathbf{M}])=f_1^*([\mathbf{M}])$ else $f_1([\mathbf{M}])=f_1^\sigma([\mathbf{M}])$,

where $\mathbf{M}$ ranges over $\mathcal{BT}_1^{\bar{y}}$. Notice that f_1 is valuable. In Lemmas 13 and 14, we show that f_1 possesses the required properties.

Lemma 13 f_1 *is surjective.*

Proof: Let $\mathbf{N}\in\mathcal{BT}_1^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}]\in U$**:** Hence, let $\mathbf{M}\in\mathcal{BT}_1^{\bar{y}}$ be such that $f_1^\sigma([\mathbf{M}])=[\mathbf{N}]$. Thus, $h(f_1^\sigma([\mathbf{M}]))=h([\mathbf{N}])$. Consequently, $h([\mathbf{N}])\not\in S$. Hence, $f_1(h([\mathbf{N}]))=f_1^\sigma(h([\mathbf{N}]))$. Since $h(f_1^\sigma([\mathbf{M}]))=h([\mathbf{N}])$, $f_1(h([\mathbf{N}]))=f_1^\sigma(h(f_1^\sigma([\mathbf{M}])))$. Thus, $f_1(h([\mathbf{N}]))=f_1^\sigma([\mathbf{M}])$. Since $f_1^\sigma([\mathbf{M}])=[\mathbf{N}]$, $f_1(h([\mathbf{N}]))=[\mathbf{N}]$.

Case $[\mathbf{N}]\in\mathcal{BT}_1^{\bar{x}}/\!\bowtie\setminus U$**:** Since f_1^* is one-to-one, let $[\mathbf{M}]\in S$ be such that $f_1^*([\mathbf{M}])=[\mathbf{N}]$. Consequently, $f_1([\mathbf{M}])=f_1^*([\mathbf{M}])$. Since $f_1^*([\mathbf{M}])=[\mathbf{N}]$, $f_1([\mathbf{M}])=[\mathbf{N}]$. $\dashv$

Lemma 14 *For all* $\mathbf{M},\mathbf{N}\in\mathcal{BT}_1^{\bar{y}}$, *if* $f_1([\mathbf{M}])=f_1([\mathbf{N}])$ *then* $f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}])$.

Proof: Let $\mathbf{M},\mathbf{N}\in\mathcal{BT}_1^{\bar{y}}$. Suppose $f_1([\mathbf{M}])=f_1([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}]\in S$ **and** $[\mathbf{N}]\in S$**:** Since $f_1([\mathbf{M}])=f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}])=f_1^*([\mathbf{N}])$. Since f_1^* is one-to-one, $[\mathbf{M}]=[\mathbf{N}]$. Hence, $f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}])$.

Case $[\mathbf{M}]\in S$ **and** $[\mathbf{N}]\in\mathcal{BT}_1^{\bar{y}}/\!\bowtie\setminus S$**:** Since $f_1([\mathbf{M}])=f_1([\mathbf{N}])$, $f_1^*([\mathbf{M}])=f_1^\sigma([\mathbf{N}])$. Thus, $(\mathcal{BT}_1^{\bar{x}}/\!\bowtie\setminus U)\cap U\neq\emptyset$: a contradiction.

Case $[\mathbf{M}]\in\mathcal{BT}_1^{\bar{y}}/\!\bowtie\setminus S$ **and** $[\mathbf{N}]\in\mathcal{BT}_1^{\bar{y}}/\!\bowtie\setminus S$**:** Since $f_1([\mathbf{M}])=f_1([\mathbf{N}])$, $f_1^\sigma([\mathbf{M}])=f_1^\sigma([\mathbf{N}])$. $\dashv$

Case $e\geq2$**:** By induction hypothesis, let f_{e-1} be a surjective valuable function from $\mathcal{BT}_{e-1}^{\bar{y}}/\!\bowtie$ to $\mathcal{BT}_{e-1}^{\bar{x}}/\!\bowtie$ such that for all $\mathbf{M},\mathbf{N}\in\mathcal{BT}_{e-1}^{\bar{y}}$, if $f_{e-1}([\mathbf{M}])=$

$f_{e-1}([\mathbf{N}])$ then $f^{\sigma}_{e-1}([\mathbf{M}])=f^{\sigma}_{e-1}([\mathbf{N}])$. For all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then let

- $S(\mathcal{M}'/\bowtie)=\{[\mathbf{M}]:\ \mathbf{M}\in\mathcal{BT}^{\bar{y}}_{e}$ and $\{f_{e-1}([\mathbf{M_s}]):\ s\in R_{\mathbf{M}}(r_{\mathbf{M}})\}=\mathcal{M}'/\bowtie\}$.

Notice that for all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $S(\mathcal{M}'/\bowtie)\subseteq\mathcal{BT}^{\bar{y}}_{e}/\bowtie\setminus\mathcal{BT}^{\bar{y}}_{e-1}/\bowtie$. For all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then let $\sim_{\mathcal{M}'/\bowtie}$ be the equivalence relation on $S(\mathcal{M}'/\bowtie)$ defined by

- $[\mathbf{M}]\sim_{\mathcal{M}'/\bowtie}[\mathbf{N}]$ if and only if $f^{\sigma}_{e}([\mathbf{M}])=f^{\sigma}_{e}([\mathbf{N}])$.

where $[\mathbf{M}],[\mathbf{N}]$ range over $S(\mathcal{M}'/\bowtie)$. In Lemmas 15 and 16, we compare $\|S(\mathcal{M}'/\bowtie)/\sim_{\mathcal{M}'/\bowtie}\|$ and $\|S(\mathcal{M}'/\bowtie)\|$ with $2^{\|\bar{x}\|}$ for each $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$ such that $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$.

Lemma 15 *For all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $\|S(\mathcal{M}'/\bowtie)/\sim_{\mathcal{M}'/\bowtie}\|\leq 2^{\|\bar{x}\|}$.*

Proof: Let $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$. Suppose $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$. For the sake of the contradiction, suppose $\|S(\mathcal{M}'/\bowtie)/\sim_{\mathcal{M}'/\bowtie}\|>2^{\|\bar{x}\|}$. Let $p\in\mathbb{N}$ and $[\mathbf{M_1}],\ldots,[\mathbf{M_p}]\in S(\mathcal{M}'/\bowtie)$ be such that $p>2^{\|\bar{x}\|}$ and for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M_q}]\not\sim_{\mathcal{M}'/\bowtie}[\mathbf{M_r}]$. Hence, $\mathbf{M_1},\ldots,\mathbf{M_p}\in\mathcal{BT}^{\bar{y}}_{e}$ and $\{f_{e-1}([\mathbf{M_{1s}}]):\ s\in R_{\mathbf{M_1}}(r_{\mathbf{M_1}})\}=\mathcal{M}'/\bowtie$, …, $\{f_{e-1}([\mathbf{M_{ps}}]):\ s\in R_{\mathbf{M_p}}(r_{\mathbf{M_p}})\}=\mathcal{M}'/\bowtie$. Thus, let $\mathcal{M}''\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$ be such that $\mathcal{M}''\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ and $\{f^{\sigma}_{e-1}([\mathbf{M_{1s}}]):\ s\in R_{\mathbf{M_1}}(r_{\mathbf{M_1}})\}=\mathcal{M}''/\bowtie$, …, $\{f^{\sigma}_{e-1}([\mathbf{M_{ps}}]):\ s\in R_{\mathbf{M_p}}(r_{\mathbf{M_p}})\}=\mathcal{M}''/\bowtie$. Consequently, by Proposition 18, let $\mathbf{M''_1},\ldots,\mathbf{M''_p}\in\mathcal{BT}^{\bar{x}}_{e}$ be such that $f^{\sigma}_{e}([\mathbf{M_1}])=[\mathbf{M''_1}]$, …, $f^{\sigma}_{e}([\mathbf{M_p}])=[\mathbf{M''_p}]$ and $\{[\mathbf{M''_{1s}}]:\ s\in R_{\mathbf{M''_1}}(r_{\mathbf{M''_1}})\}=\mathcal{M}''/\bowtie$, …, $\{[\mathbf{M''_{ps}}]:\ s\in R_{\mathbf{M''_p}}(r_{\mathbf{M''_p}})\}=\mathcal{M}''/\bowtie$. Since for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M_q}]\not\sim_{\mathcal{M}'/\bowtie}[\mathbf{M_r}]$, for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $f^{\sigma}_{e}([\mathbf{M_q}])\neq f^{\sigma}_{e}([\mathbf{M_r}])$. Since $f^{\sigma}_{e}([\mathbf{M_1}])=[\mathbf{M''_1}]$, …, $f^{\sigma}_{e}([\mathbf{M_p}])=[\mathbf{M''_p}]$, for all $q,r\in\{1,\ldots,p\}$, if $q\neq r$ then $[\mathbf{M''_q}]\neq[\mathbf{M''_r}]$. Since $\{[\mathbf{M''_{1s}}]:\ s\in R_{\mathbf{M''_1}}(r_{\mathbf{M''_1}})\}=\mathcal{M}''/\bowtie$, …, $\{[\mathbf{M''_{ps}}]:\ s\in R_{\mathbf{M''_p}}(r_{\mathbf{M''_p}})\}=\mathcal{M}''/\bowtie$, $p\leq 2^{\|\bar{x}\|}$: a contradiction. $\dashv$

Lemma 16 *For all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\bowtie)\|$.*

Proof: Let $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$. Suppose $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$. Since f_{e-1} is surjective, $\|S(\mathcal{M}'/\bowtie)\|\geq 2^{\|\bar{y}\|}$. Since $\|\bar{x}\|<\|\bar{y}\|$, $2^{\|\bar{x}\|}\leq\|S(\mathcal{M}'/\bowtie)\|$. $\dashv$

For all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then let

- $T(\mathcal{M}'/\bowtie)=\{[\mathbf{N}']:\ \mathbf{N}'\in\mathcal{BT}^{\bar{x}}_{e}$ and $\{[\mathbf{N}'_{\mathbf{s}'}]:\ s'\in R_{\mathbf{N}'}(r_{\mathbf{N}'})\}=\mathcal{M}'/\bowtie\}$.

Notice that for all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $T(\mathcal{M}'/\bowtie)\subseteq\mathcal{BT}^{\bar{x}}_{e}/\bowtie\setminus\mathcal{BT}^{\bar{x}}_{e-1}/\bowtie$. Moreover, notice that for all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $\|T(\mathcal{M}'/\bowtie)\|=2^{\|\bar{x}\|}$. Consequently, by Lemmas 15 and 16, for all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then $\|S(\mathcal{M}'/\bowtie)/\sim_{\mathcal{M}'/\bowtie}\|\leq\|T(\mathcal{M}'/\bowtie)\|\leq\|S(\mathcal{M}'/\bowtie)\|$. Hence, by Proposition 1, for all $\mathcal{M}'\subseteq\mathcal{BT}^{\bar{x}}_{e-1}$, if $\mathcal{M}'\cap(\mathcal{BT}^{\bar{x}}_{e-1}\setminus\mathcal{BT}^{\bar{x}}_{e-2})\neq\emptyset$ then let $g^{\mathcal{M}'/\bowtie}_{e}$ be a surjective function from $S(\mathcal{M}'/\bowtie)$

to $T(\mathcal{M}'/\bowtie)$ such that for all $[\mathbf{M}],[\mathbf{N}]\in S(\mathcal{M}'/\bowtie)$, if $g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}])=g_e^{\mathcal{M}'/\bowtie}([\mathbf{N}])$ then $[\mathbf{M}]\sim_{\mathcal{M}'/\bowtie}[\mathbf{N}]$[20]. Now, we define the function f_e. Let $f_e : \mathcal{BT}_e^{\bar{y}}/\bowtie \longrightarrow \mathcal{BT}_e^{\bar{x}}/\bowtie$ be the function such that

- if $\mathbf{M}\in\mathcal{BT}_{e-1}^{\bar{y}}$ then $f_e([\mathbf{M}])=f_{e-1}([\mathbf{M}])$ else $f_e([\mathbf{M}])=g_e^{\mathcal{M}'/\bowtie}([\mathbf{M}])$,

where $\mathbf{M}$ ranges over $\mathcal{BT}_e^{\bar{y}}$ and $\mathcal{M}'/\bowtie=\{f_{e-1}([\mathbf{M_s}]) : s\in R_{\mathbf{M}}(r_{\mathbf{M}})\}$. Notice that f_e is valuable. In Lemmas 17 and 18, we show that f_e possesses the required properties.

Lemma 17 *f_e is surjective.*

Proof: Let $\mathbf{N}\in\mathcal{BT}_e^{\bar{x}}$. We consider the following 2 cases.

Case $[\mathbf{N}]\in\mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$**:** Since f_{e-1} is surjective, let $[\mathbf{M}]\in\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ be such that $f_{e-1}([\mathbf{M}])=[\mathbf{N}]$. Hence, $f_e([\mathbf{M}])=f_{e-1}([\mathbf{M}])$. Since $f_{e-1}([\mathbf{M}])=[\mathbf{N}]$, $f_e([\mathbf{M}])=[\mathbf{N}]$.

Case $[\mathbf{N}]\in\mathcal{BT}_e^{\bar{x}}/\bowtie \setminus \mathcal{BT}_{e-1}^{\bar{x}}/\bowtie$**:** Let $\mathcal{N}\subseteq\mathcal{BT}_{e-1}^{\bar{x}}$ be such that $\mathcal{N}/\bowtie=\{[\mathbf{N_s}] : s\in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Thus, $\mathcal{N}\cap(\mathcal{BT}_{e-1}^{\bar{x}}\setminus\mathcal{BT}_{e-2}^{\bar{x}})\neq\emptyset$. Moreover, $[\mathbf{N}]\in T(\mathcal{N}/\bowtie)$. Since $g_e^{\mathcal{N}/\bowtie}$ is surjective, let $[\mathbf{M}']\in S(\mathcal{N}/\bowtie)$ be such that $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}'])=[\mathbf{N}]$. Consequently, $\{f_{e-1}([\mathbf{M}'_\mathbf{s}]) : s\in R_{\mathbf{M}'}(r_{\mathbf{M}'})\}=\mathcal{N}/\bowtie$. Since $g_e^{\mathcal{N}/\bowtie}([\mathbf{M}'])=[\mathbf{N}]$, $f_e([\mathbf{M}'])=[\mathbf{N}]$. $\dashv$

Lemma 18 *For all $\mathbf{M},\mathbf{N}\in\mathcal{BT}_e^{\bar{y}}$, if $f_e([\mathbf{M}])=f_e([\mathbf{N}])$ then $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$.*

Proof: Let $\mathbf{M},\mathbf{N}\in\mathcal{BT}_e^{\bar{y}}$. Suppose $f_e([\mathbf{M}])=f_e([\mathbf{N}])$. We consider the following 3 cases.

Case $[\mathbf{M}]\in\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ **and** $[\mathbf{N}]\in\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$**:** Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}])=f_{e-1}([\mathbf{N}])$. Hence, $f_{e-1}^\sigma([\mathbf{M}])=f_{e-1}^\sigma([\mathbf{N}])$ Thus, $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$

Case $[\mathbf{M}]\in\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ **and** $[\mathbf{N}]\in\mathcal{BT}_e^{\bar{y}}/\bowtie\setminus\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$**:** Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $f_{e-1}([\mathbf{M}])=g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$ where $\mathcal{N}/\bowtie=\{f_{e-1}([\mathbf{N_s}]) : s\in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Consequently, $\mathcal{BT}_{e-1}^{\bar{x}}/\bowtie\cap(\mathcal{BT}_e^{\bar{x}}/\bowtie\setminus\mathcal{BT}_{e-1}^{\bar{x}}/\bowtie)\neq\emptyset$: a contradiction.

Case $[\mathbf{M}]\in\mathcal{BT}_e^{\bar{y}}/\bowtie\setminus\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$ **and** $[\mathbf{N}]\in\mathcal{BT}_e^{\bar{y}}/\bowtie\setminus\mathcal{BT}_{e-1}^{\bar{y}}/\bowtie$**:** Since $f_e([\mathbf{M}])=f_e([\mathbf{N}])$, $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}])=g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$ where $\mathcal{M}/\bowtie=\{f_{e-1}([\mathbf{M_s}]) : s\in R_{\mathbf{M}}(r_{\mathbf{M}})\}$ and $\mathcal{N}/\bowtie=\{f_{e-1}([\mathbf{N_s}]) : s\in R_{\mathbf{N}}(r_{\mathbf{N}})\}$. Consequently, $\mathcal{M}/\bowtie=\mathcal{N}/\bowtie$. Since $g_e^{\mathcal{M}/\bowtie}([\mathbf{M}])=g_e^{\mathcal{N}/\bowtie}([\mathbf{N}])$, $[\mathbf{M}]\sim_{\mathcal{M}/\bowtie}[\mathbf{N}]$ and $[\mathbf{M}]\sim_{\mathcal{N}/\bowtie}[\mathbf{N}]$. Hence, $f_e^\sigma([\mathbf{M}])=f_e^\sigma([\mathbf{N}])$. $\dashv$

This finishes the proof of Proposition 25. $\dashv$

[20]Notice that this is the only place in this section where we use Proposition 1.

Proposition 26 *Let $\varphi \in \mathbf{FOR}_{\bar{x}}$. If $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ is a $\mathbf{K}_d$-unifier of φ then there exists a $\mathbf{K}_d$-unifier $(\bar{x},\bar{x},\tau)\in\mathbf{SUB}$ of φ such that $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{K}_d}(\bar{x},\bar{y},\sigma)$.*

Proof: Suppose $(\bar{x},\bar{y},\sigma)\in\mathbf{SUB}$ is a $\mathbf{K}_d$-unifier of φ. Hence, for all $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$.

Case $\|\bar{x}\|\geq\|\bar{y}\|$: Let $(\bar{x},\bar{x},\tau)\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$, $\tau(x)=\lambda(\sigma(x))$ for some injective $(\bar{y},\bar{x},\lambda)\in\mathbf{SUB}$ such that $\lambda(\bar{y})\subseteq\bar{x}$. Notice that since for all $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$, for all $\mathbf{M}\in\mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\tau(\varphi))$. Thus, $(\bar{x},\bar{x},\tau)$ is a $\mathbf{K}_d$-unifier of φ. Let $(\bar{x},\bar{y},\lambda')\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$, if there exists $y\in\bar{y}$ such that $\lambda(y)=x$ then $\lambda'(x)=y$. Notice that for all $x\in\bar{x}$, $\lambda'(\tau(x))\equiv_{\mathbf{K}_d}\sigma(x)$. Consequently, $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{K}_d}(\bar{x},\bar{y},\sigma)$.

Case $\|\bar{x}\|<\|\bar{y}\|$: By Proposition 25, let $f_d : \mathcal{BT}_d^{\bar{y}}/\bowtie \longrightarrow \mathcal{BT}_d^{\bar{x}}/\bowtie$ be a surjective valuable function such that for all $\mathbf{M},\mathbf{N}\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}])=f_d([\mathbf{N}])$ then $f_d^{\sigma}([\mathbf{M}])=f_d^{\sigma}([\mathbf{N}])$. Let $(\bar{x},\bar{x},\tau),(\bar{x},\bar{y},\nu)\in\mathbf{SUB}$ be such that for all $x\in\bar{x}$,

- $\tau(x)=\bigvee\{\mathbf{for}(\mathbf{M}') :\ \mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{BT}_d^{\bar{x}}$ are such that $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$ and $f_d([\mathbf{M}])=[\mathbf{M}']\}$,
- $\nu(x)=\bigvee\{\mathbf{for}(\mathbf{M}) :\ \mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}'\in\mathcal{BT}_d^{\bar{x}}$ are such that $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(x)$ and $f_d([\mathbf{M}])=[\mathbf{M}']\}$.

Lemmas 19, 20, 21 and 22 state results connecting the substitutions $(\bar{x},\bar{x},\tau)$ and $(\bar{x},\bar{y},\nu)$ with the models in $\mathcal{BT}_d^{\bar{x}}$ and $\mathcal{BT}_d^{\bar{y}}$.

Lemma 19 *Let $\varphi\in\mathbf{FOR}_{\bar{x}}$. For all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{x}}$, the following conditions are equivalent:*

1. *there exists $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}])=[\mathbf{M}']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$,*
2. *for all $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}])=[\mathbf{M}']$ then $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$,*
3. *$r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\tau(\varphi))$.*

Proof: By induction on φ.

Case $\varphi=x$: Let $\mathbf{M}\in\mathcal{BT}_d^{\bar{x}}$.

($\mathbf{1}\Rightarrow\mathbf{2}$) Suppose $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$. Let $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since $f_d([\mathbf{M}'])=[\mathbf{M}]$, $f_d([\mathbf{M}'])=f_d([\mathbf{M}''])$. Hence, $f_d^{\sigma}([\mathbf{M}'])=f_d^{\sigma}([\mathbf{M}''])$. Thus, $[\mathbf{M}'^{|\sigma}]=[\mathbf{M}''^{|\sigma}]$. Since $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'^{|\sigma}}(x)$. Since $[\mathbf{M}'^{|\sigma}]=[\mathbf{M}''^{|\sigma}]$, $r_{\mathbf{M}''}\in V_{\mathbf{M}''^{|\sigma}}(x)$. Consequently, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$. Since $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}''])=[\mathbf{M}]$ is arbitrary, for all $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}''])=[\mathbf{M}]$ then $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$.

($\mathbf{2}\Rightarrow\mathbf{3}$) Suppose for all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$. Since f_d is surjective, let $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since for all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\sigma(x))$. Hence, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\tau(x)$. Since by Proposition 10, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\tau(x))$.

(**3** $\Rightarrow$ **1**) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\tau(x))$. Let $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}''\in\mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $f_d([\mathbf{M}'])=[\mathbf{M}'']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$. Thus, $f_d([\mathbf{M}'])=[\mathbf{M}]$. Consequently, there exists $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$.

Case $\varphi=\bot$**:** Left to the reader.

Case $\varphi=\neg\psi$**:** Left to the reader.

Case $\varphi=\psi\vee\chi$**:** Left to the reader.

Case $\varphi=\Box\psi$**:**

(**1** $\Rightarrow$ **2**) Suppose $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ is such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Let $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$. Since f_d is valuable and $f_d([\mathbf{M}''])=[\mathbf{M}]$, let $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ be such that $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$. Since $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$, $f_d([\mathbf{M}'_{\mathbf{s}'}])=f_d([\mathbf{M}''_{\mathbf{s}''}])$. Hence, $f_d^{\sigma}([\mathbf{M}'_{\mathbf{s}'}])=f_d^{\sigma}([\mathbf{M}''_{\mathbf{s}''}])$. Thus, $[{\mathbf{M}'_{\mathbf{s}'}}^{|\sigma}]=[{\mathbf{M}''_{\mathbf{s}''}}^{|\sigma}]$. Since $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, $s'\in V_{\mathbf{M}'}(\sigma(\psi))$. Consequently, $s'\in V_{\mathbf{M}'^{|\sigma}}(\psi)$. Since $[{\mathbf{M}'_{\mathbf{s}'}}^{|\sigma}]=[{\mathbf{M}''_{\mathbf{s}''}}^{|\sigma}]$, $s''\in V_{\mathbf{M}''^{|\sigma}}(\psi)$. Hence, $s''\in V_{\mathbf{M}''}(\sigma(\psi))$. Since $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ is arbitrary, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}''])=[\mathbf{M}]$ is arbitrary, for all $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}''])=[\mathbf{M}]$ then $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\Box\sigma(\psi))$.

(**2** $\Rightarrow$ **3**) Suppose for all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Let $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$. Since f_d is surjective, let $\mathbf{M}''\in\mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}''])=[\mathbf{M}]$. Since f_d is valuable and $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, let $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ be such that $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$. Since for all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$, if $f_d([\mathbf{M}'])=[\mathbf{M}]$ then $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$ and $f_d([\mathbf{M}''])=[\mathbf{M}]$, $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(\Box\sigma(\psi))$. Since $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, $s''\in V_{\mathbf{M}''}(\sigma(\psi))$. Since $f_d([\mathbf{M}''_{\mathbf{s}''}])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s\in V_{\mathbf{M}}(\tau(\psi))$. Since $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ is arbitrary, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\Box\tau(\psi))$.

(**3** $\Rightarrow$ **1**) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\Box\tau(\psi))$. Since f_d is surjective, let $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ be such that $f_d([\mathbf{M}'])=[\mathbf{M}]$. Let $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$. Since f_d is valuable and $f_d([\mathbf{M}'])=[\mathbf{M}]$, let $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ be such that $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$. Since $r_{\mathbf{M}}\in V_{\mathbf{M}}(\Box\tau(\psi))$, $s\in V_{\mathbf{M}}(\tau(\psi))$. Since $f_d([\mathbf{M}'_{\mathbf{s}'}])=[\mathbf{M}_{\mathbf{s}}]$, by induction hypothesis, $s'\in V_{\mathbf{M}'}(\sigma(\psi))$. Since $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ is arbitrary, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$. Consequently, there exists $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ such that $f_d([\mathbf{M}'])=[\mathbf{M}]$ and $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\Box\sigma(\psi))$. $\dashv$

Lemma 20 *For all* $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ *and for all* $\mathbf{M}'\in\mathcal{BT}_d^{\bar{x}}$*, if* $f_d([\mathbf{M}])=[\mathbf{M}']$ *then for all* $x\in\bar{x}$*, the following conditions are equivalent:*

1. $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$,

2. $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

Proof: Let $\mathbf{M} \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_d^{\bar{x}}$. Suppose $f_d([\mathbf{M}])=[\mathbf{M}']$. Let $x \in \bar{x}$.

($\mathbf{1} \Rightarrow \mathbf{2}$) Suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. Let $\mathbf{M}'' \in \mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}''' \in \mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'''} \in V_{\mathbf{M}'''}(x)$, $f_d([M''])=[M''']$ and $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}''))$. Since $f_d([\mathbf{M}])=[\mathbf{M}']$, $[M']=[M''']$. Since $r_{\mathbf{M}'''} \in V_{\mathbf{M}'''}(x)$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$.

($\mathbf{2} \Rightarrow \mathbf{1}$) Suppose $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since $f_d([\mathbf{M}])=[\mathbf{M}']$, $\mathbf{for}(\mathbf{M})$ is a disjunct in $\nu(x)$. Since by Proposition 10, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\mathbf{for}(\mathbf{M}))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$. $\dashv$

Lemma 21 *Let $e \in \{1, \ldots, d\}$. For all $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and for all $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$, $f_e([\mathbf{M}])=[\mathbf{M}']$ if and only if $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$.*

Proof: By induction on e.

Case e=1**:** Let $\mathbf{M} \in \mathcal{BT}_1^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_1^{\bar{x}}$. Suppose $f_1([\mathbf{M}])=[\mathbf{M}']$. Hence, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Since by Proposition 10, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(\mathbf{for}(\mathbf{M}'))$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Thus, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Let $\mathbf{M}'' \in \mathcal{BT}_1^{\bar{x}}$ be such that $f_1([\mathbf{M}])=[\mathbf{M}'']$. Consequently, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Hence, $[\mathbf{M}']=[\mathbf{M}'']$. Since $f_1([\mathbf{M}])=[\mathbf{M}'']$, $f_1([\mathbf{M}])=[\mathbf{M}']$.

Case $e \geq 2$**:** Let $\mathbf{M} \in \mathcal{BT}_e^{\bar{y}}$ and $\mathbf{M}' \in \mathcal{BT}_e^{\bar{x}}$. Suppose $f_e([\mathbf{M}])=[\mathbf{M}']$. Thus, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$. Consequently, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Reciprocally, suppose $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}')))$. Hence, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$. Moreover, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $s \in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'_{\mathbf{s}'})))$. Thus, by induction hypothesis, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s' \in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$. Let $\mathbf{M}'' \in \mathcal{BT}_e^{\bar{x}}$ be such that $f_e([\mathbf{M}])=[\mathbf{M}'']$. Consequently, by Lemma 20, for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}''} \in V_{\mathbf{M}''}(x)$. Moreover, since f_e is valuable, for all $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}''_{\mathbf{s}''}]$ and for all $s'' \in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s \in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}''_{\mathbf{s}''}]$. Since for all $x \in \bar{x}$, $r_{\mathbf{M}} \in V_{\mathbf{M}}(\nu(x))$ if and only if $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$, for all $x \in \bar{x}$, $r_{\mathbf{M}'} \in V_{\mathbf{M}'}(x)$

if and only if $r_{\mathbf{M}''}\in V_{\mathbf{M}''}(x)$. Moreover, since for all $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$ and for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s\in R_{\mathbf{M}}(r_{\mathbf{M}})$ such that $f_{e-1}([\mathbf{M}_s])=[\mathbf{M}'_{\mathbf{s}'}]$, for all $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$, there exists $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$ such that $[\mathbf{M}'_{\mathbf{s}'}]=[\mathbf{M}''_{\mathbf{s}''}]$ and for all $s''\in R_{\mathbf{M}''}(r_{\mathbf{M}''})$, there exists $s'\in R_{\mathbf{M}'}(r_{\mathbf{M}'})$ such that $[\mathbf{M}'_{\mathbf{s}'}]=[\mathbf{M}''_{\mathbf{s}''}]$. Hence, $[\mathbf{M}']=[\mathbf{M}'']$. Since $f_e([\mathbf{M}])=[\mathbf{M}'']$, $f_e([\mathbf{M}])=[\mathbf{M}']$. $\dashv$

Lemma 22 *For all $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ and for all $x\in\bar{x}$, the following conditions are equivalent:*

1. $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$,
2. $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$.

Proof: Let $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$ and $x\in\bar{x}$.

($\mathbf{1}\Rightarrow\mathbf{2}$) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$. Let $\mathbf{M}'\in\mathcal{BT}_d^{\bar{y}}$ and $\mathbf{M}''\in\mathcal{BT}_d^{\bar{x}}$ be such that $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $f_d([M'])=[M'']$ and $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(\mathbf{M}'')))$. Hence, by Lemma 21, $f_d([M])=[M'']$. Since $f_d([M'])=[M'']$, $f_d([M'])=f_d([M])$. Thus, $f_d^{\sigma}([M'])=f_d^{\sigma}([M])$. Consequently, $[\mathbf{M}^{|\sigma}]=[\mathbf{M}'^{|\sigma}]$. Since $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\sigma(x))$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'^{|\sigma}}(x)$. Since $[\mathbf{M}^{|\sigma}]=[\mathbf{M}'^{|\sigma}]$, $r_{\mathbf{M}}\in V_{\mathbf{M}^{|\sigma}}(x)$. Hence, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$.

($\mathbf{2}\Rightarrow\mathbf{1}$) Suppose $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(x))$. Thus, $\mathbf{for}(f_d([\mathbf{M}]))$ is a disjunct in $\tau(x)$. Since by Lemma 21, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\mathbf{for}(f_d([\mathbf{M}]))))$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\nu(\tau(x)))$. $\dashv$

Since for all $\mathbf{M}\in\mathcal{BT}_d^{\bar{y}}$, $r_{\mathbf{M}}\in V_{\mathbf{M}}(\sigma(\varphi))$, by Lemma 19, for all $\mathbf{M}'\in\mathcal{BT}_d^{\bar{x}}$, $r_{\mathbf{M}'}\in V_{\mathbf{M}'}(\tau(\varphi))$. Hence, $(\bar{x},\bar{x},\tau)$ is an $\mathbf{K}_d$-unifier of φ. Moreover, by Lemma 22, for all $x\in\bar{x}$, $\nu(\tau(x))\leftrightarrow\sigma(x)\in\mathbf{K}_d$[21]. Thus, $(\bar{x},\bar{x},\tau)\preccurlyeq_{\mathbf{K}_d}(\bar{x},\bar{y},\sigma)$.

This finishes the proof of Proposition 26. $\dashv$

Proposition 27 $\mathbf{K}_d$ *is reasonable.*

Proof: By Proposition 26. $\dashv$

Proposition 28 $\mathbf{K}_d$ *is finitary.*

Proof: By Propositions 15, 6, 8, 16 and 27. $\dashv$

8 Last words

Recently, the question of the unification type has been considered within the context of a semantic restriction of description logic $\mathcal{FL}_0$. The formulas of

[21]Lemmas 20 and 21 are used in the proof of Lemma 22.

$\mathcal{FL}_0$ are constructed by means of the connectives $\top$, $\wedge$ and $\Box_a$ — where a ranges over a countably infinite set **ACT** [4, 6]. The unification problem in $\mathcal{FL}_0$ is to determine, given a couple (φ, ψ) of formulas, whether there exists a substitution σ such that $\sigma(\varphi)$ and $\sigma(\psi)$ are logically equivalent in the class of all **ACT**-frames, i.e. Kripke frames of the form (W, R) where W is a nonempty set and R_a is a binary relation on W for each $a \in \mathbf{ACT}$. As is well-known, the unification type of $\mathcal{FL}_0$ is nullary [5]. Restricting the discussion to the class of all 2-bounded **ACT**-frames, Baader *et al.* [2] have proved that the unification type of $\mathcal{FL}_0$ is unitary. This leads us to the following open questions:

- for all $d \geq 2$, interpreting the formulas constructed by means of the connectives $\bot$, $\neg$, $\vee$ and $\Box_a$ — where a ranges over **ACT** — in the class of all d-bounded **ACT**-frames (W, R) such that $\bigcup\{R_a : a \in \mathbf{ACT}\}$ is deterministic, determine the unification type of unifiable formulas,
- for all $d \geq 2$, interpreting the formulas constructed by means of the connectives $\bot$, $\neg$, $\vee$ and $\Box_a$ — where a ranges over **ACT** — in the class of all d-bounded **ACT**-frames, determine the unification type of unifiable formulas,
- determine the unification type of other locally tabular normal modal logics like the ones studied in [26, 27, 28].

We conjecture that the normal modal logics mentioned in these open questions are either finitary, or unitary.

On the side of computability and complexity, it is known that unification is in **PSPACE** for $\mathbf{Alt}_1$ [10]. As for **KD** and $\mathbf{DAlt}_1$, the membership in **NP** of their unification problem is a direct consequence of the fact that in these normal modal logics, every variable-free formula is equivalent either to $\bot$, or to $\top$. This leads us to the following open questions[22]:

- determine for all $d \geq 2$, the complexity of unification for the locally tabular normal modal logics $\mathbf{K} + \Box^d \bot$ and $\mathbf{Alt}_1 + \Box^d \bot$,
- determine the complexity of unification for other locally tabular normal modal logics like the ones studied in [26, 27, 28].

Following the line of reasoning developed in [1], we conjecture that for all $d \geq 2$, unification for $\mathbf{Alt}_1 + \Box^d \bot$ is **NP**-complete.

Funding

The preparation of this paper has been supported by the Project of *Bulgarian Science Fund DN02/15/19.12.2016* — 'Space, time and modality: relational, algebraic and topological models' — and the Programme *Professeurs invités 2018* of *Université Toulouse III - Paul Sabatier*.

[22] The local tabularity of the normal modal logics mentioned in these open questions implies the decidability of their unification problem.

Acknowledgement

We are indebted to Silvio Ghilardi for his suggestion to consider the question of the unification type of $\mathbf{K} + \square^2\bot$. Special acknowledgement is heartily granted to the colleagues of the *Toulouse Institute of Computer Science Research* for their valuable remarks. Finally, we also make a point of thanking the referees for their feedback: their useful comments have been essential for improving the correctness and the readability of a preliminary version of this paper.

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