On SVD for Estimating Generalized Eigenvalues of Singular Matrix Pencil in Noise

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Abstract—**We review several algorithms for estimating generalized eigenvalues (GE's) of singular matrix pencils perturbed by noise. The singular value decomposition (SVD) is explored as the common structure in the three basic algorithms: direct matrix pencil algorithm, Pro-ESPRIT, and TLS-ESPRIT. We show that several SVD-based steps inherent in those algorithms are equivalent to the first-order approximation. In particular, the Pro-ESPRIT and its variant TLS-Pro-ESPRIT are shown to be equivalent, and the TLS-ESPRIT and its earlier version LS-ESPRIT are shown to be asymptotically equivalent to the first-order approximation. For the problem of estimating superimposed complex exponential signals, the state space algorithm is shown to be also equivalent to the previous matrix pencil algorithms to the first-order approximation. The second-order perturbation and the threshold phenomenon are illustrated by simulation results based on a damped sinusoidal signal. An improved state space algorithm is found to be the most robust to noise.**

I. Introduction

A MATHEMATICAL entity called matrix pencil has been utilized by many researchers [1]–[15], [22] in array processing and spectral estimation. The matrix pencil is simply a linearly combined two matrices, i.e., $Y_1 - zY_2$ where z is a scalar variable, and Y_1 and Y_2 are two (square or rectangular) N_1 by N_2 matrices. In applications [1]–[15], the matrix pencil can generally be decomposed into the following form:

$$\begin{aligned} Y_1 - zY_2 &= (X_1 + E_1) - z(X_2 + E_2) \\ &= (X_1 - zX_2) + (E_1 - zE_2) \end{aligned} \tag{1.1}$$

where E_1 and E_2 are unknown small (in norm) perturbation matrices due to some kinds of errors or noise. Y_1 and Y_2 can be two data matrices constructed directly from a data sequence (as in our simulation shown in Section V) or two covariance matrices with estimated noise covariance matrices removed (as used in [1]–[3]). In any case, we assume that Y_1 and Y_2 have been filtered by some means (e.g., low-pass or band-pass filtering, and noise cleaning at the covariance level or at higher order statistics level), and E_1 and E_2 represent small residue errors.

In (1.1), X_1 and X_2 have the same column space and the same row space, and the noiseless pencil $X_1 - zX_2$ decreases its rank by one if and only if z is one of several (say M) desired values. These desired values will be called the desired generalized eigenvalues (GE's) of $Y_1 - zY_2$, which are denoted by $z_1, z_2, \cdots, z_M$. The desired GE's contain the desired information like the directions of wave arrivals [1]–[8] and the signal poles [9]–[13]. Because of the noise (or perturbation) matrices E_1 and E_2, only noisy estimates of the desired GE's can be obtained from Y_1 and Y_2. X_1 and X_2 can be in general rectangular or/and not of full rank so that (1.1) represents a (noiseless) singular matrix pencil perturbed by noise.

This paper addresses several SVD-based techniques which exploit the singular condition of $X_1 - zX_2$ to estimate the desired GE's. These techniques are TLS-ESPRIT [3], Pro-ESPRIT [5], [6], and direct matrix pencil algorithm [9]–[11]. Our objective is to present some links and common features among those algorithms as well as others to be mentioned in the context. We need to emphasize that in the original Pro-ESPRIT and the original TLS-ESPRIT, the noise cleaning at the covariance level is incorporated with eigendecompositions. In the original Pro-ESPRIT, the Procrustes unitary approximation is applied at the data matrix level, but the eigendata of the noiseless data matrices are estimated in an asymptotically unbiased way from the eigendata of the covariance matrices. In the original TLS-ESPRIT, the eigendecomposition is directly applied to the covariance matrices. In order to compare fairly the different eigendecomposition or SVD-based steps inherent in the Pro-ESPRIT, the TLS-ESPRIT, and the direct matrix pencil algorithm, we shall not consider the effect of covariance filtering as inherent in the Pro-ESPRIT and the TLS-ESPRIT. The names Pro-ESPRIT and TLS-ESPRIT as called in the sequel only reflect the effects of the SVD's inherent in the corresponding methods unless otherwise indicated.

In Section II, we shall present different SVD approaches to estimating the desired GE's from Y_1 and Y_2. New insight will be provided into several matrix pencil algorithms. In particular, the Pro-ESPRIT and its variant TLS-Pro-ESPRIT proposed in [5], [6] will be shown to be equivalent. Since the two methods in their original versions employ the same covariance filtering approach, the equivalence in the SVD steps also means that the two methods are equivalent in their original versions.

Recently, we have studied the first-order perturbations of the Prony's method and the direct matrix pencil algorithm [9]–[11]. But we have not answered the puzzling question: Do the different matrix pencil algorithms yield the same estimates of the desired GE's to the first-order approximation? This question was prompted by many simulation results which show that various matrix pencil algorithms yield very close estimation variances at high SNR. In Section III, we shall show that all the SVD steps inherent in the pro-ESPRIT, the TLS-ESPRIT, and the direct matrix pencil algorithm have the same first-order perturbations in their estimated GE's. It implies, in particular, that the TLS-ESPRIT and its earlier version LS-ESPRIT [1], [2] both in their original forms are asymptotically equivalent to the first-order approximation.

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For the problem of estimating parameters of the superimposed exponential signals, the state space algorithm [16], [17] can be viewed as a matrix pencil algorithm which constructs a matrix pencil in a way different from the direct matrix pencil algorithm. In Section IV, we shall show that the above two matrix pencil algorithms are also equivalent to the first-order approximation.

The first-order equivalence property implies that our analysis carried out in [11] (for the direct matrix pencil algorithm) is also valid for all the above mentioned matrix pencil algorithms.

In Section V, simulation results are used to illustrate the second-order perturbations (as reflected in biases) and the threshold phenomenon of five matrix pencil algorithms. The testing data is a damped sinusoidal signal plus noise. It is observed that an improved state space algorithm is the most robust to noise.

II. SVD's in the Matrix Pencil Algorithms

We apply the basic concept of SVD to present what we think are three basic types of the matrix pencil algorithms: the Direct Matrix Pencil Algorithm, the Pro-ESPRIT and the TLS-ESPRIT. The three algorithms compress the N_1 by N_2 matrix pencil $Y_1 - zY_2$ into a smaller M-by-M matrix pencil in three different ways. The desired GE's are then estimated by the GE's of the compressed M-by-M matrix pencil.

A. Algorithm 1: Direct Matrix Pencil Algorithm [11]

In general, X_1 and X_2 are not of full rank. Hence, the traditional algorithm (e.g., the QZ algorithm [21]) for computing the GE's of $Y_1 - zY_2$ is not stable if E_1 and E_2 are small in norm. To eliminate the stability problem, we replace Y_1 and Y_2 by their truncated SVD's. The truncated SVD's of Y_1 and Y_2 are denoted by Y_{1T} and Y_{2T}, respectively, and they are defined as follows:

$$Y_1 \cong Y_{1T} = U_1 \Sigma_1 V_1^H \tag{2.1}$$

$$Y_1 \cong Y_{2T} = U_2 \Sigma_2 V_2^H \tag{2.2}$$

where $\cong$ denotes the rank-M SVD truncation. Σ_1 is the M-by-M diagonal matrix of the M principal singular values of Y_1, U_1 consists of the M principal left singular vectors of Y_1, and V_1 consists of the M principal right singular vectors of Y_1. The superscript ^{H} denotes the conjugate transposition. The notations in (2.2) are similarly defined.

Based on the above two SVD truncations, we can write

$$Y_1 - zY_2 \cong Y_{1T} - zY_{2T} = U_1 \Sigma_1 V_1^H - zU_2 \Sigma_2 V_2^H. \tag{2.3}$$

Since Y_{2T} is of rank M, the matrix pencil of (2.3) has M GE's (i.e., M rank reducing numbers). Without changing the GE's, we multiply (2.3) by U_2^H from the left and by V_2 from the right to obtain

$$U_2^H U_1 \Sigma_1 V_1^H V_2 - z\Sigma_2 \tag{2.4}$$

which is an M-by-M matrix pencil. From this pencil, the M GE's can be easily obtained without the stability problem (e.g., using the IMSL EIGZC routine). The GE's of (2.4) are the same as the eigenvalues of $\Sigma_2^{-1}(U_2^H U_1 \Sigma_1 V_1^H V_2)$ or $(U_2^H U_1 \Sigma_1 V_1^H V_2)\Sigma_2^{-1}$. The result of (2.4) was also presented in [4].

We now write (2.4) into the following form (employing (2.1) and (2.2)):

$$U_2^H U_1 \Sigma_1 V_1^H V_2 - z\Sigma_2 = \Sigma_2 V_2^H [Y_{2T}^+ Y_{1T} - zI] V_2 \tag{2.5}$$

where the superscript $^+$ denotes the pseudoinverse. It is clear from (2.5) that the GE's of (2.4) are also the M nonzero eigenvalues of the N_1-by-N_2 matrix: $Y_{2T}^+ Y_{1T}$. This matrix has been studied by the authors in [9]–[11] for estimating the parameters of the exponentially damped or/and undamped sinusoids in noise.

B. Algorithm 2: Pro-ESPRIT

Following the original pro-ESPRIT, the eigendata of the data matrices Y_1 and Y_2 (i.e., U_1, Σ_1, V_1, U_2, Σ_2, and V_2) would be expressed in terms of eigendata of the covariance matrices of $Y_1 Y_1^H$, $Y_2 Y_2^H$, and $Y_1 Y_2^H$ so that the estimated eigendata of the noiseless matrices X_1 and X_2 are asymptotically unbiased in some applications [5], [6]. In other words, a noise filtering at the covariance level and an eigendecomposition filtering are carried out at the same time in the original Pro-ESPRIT. In this paper, the effectiveness of using the covariance filtering is not considered, but rather the Procrustes unitary approximation (to be shown) used in the Pro-ESPRIT is analyzed. It is mainly the Procrustes approximation that distinguishes the Pro-ESPRIT from other matrix pencil algorithms.

In [5], [6], another method called TLS-Pro-ESPRIT was proposed as a refinement of the Pro-ESPRIT. Both of the two methods use the same covariance filtering method, and they differ only in the way (to be shown) the SVD truncation is performed. It will be shown in the following that the two (different in appearance) SVD truncations used in the Pro-ESPRIT and the TLS-Pro-ESPRIT are, in fact, equivalent.

Without noise, $Y_{1T} = X_1$ and $Y_{2T} = X_2$. Furthermore, the column space of X_1 or/and X_2 is spanned by each of U_1 and U_2 in the noiseless case, and the column space of X_1^H or/and X_2^H is spanned by each of V_1 and V_2 in the noiseless case. Therefore, the following joint rank-M SVD truncations of $[U_1, U_2]$ and $[V_1, V_2]$ are valid if the noise level is low:

$$[U_1, U_2] \cong [U_1, U_2]_T = U_U \Sigma_U V_U^H = U_U \Sigma_U [V_{U1}^H, V_{U2}^H] \tag{2.6}$$

$$[V_1, V_2] \cong [V_1, V_2]_T = U_V \Sigma_V V_V^H = U_V \Sigma_V [V_{V1}^H, V_{V2}^H] \tag{2.7}$$

where $\cong$ denotes the rank-M SVD truncation. The notations used in (2.6) and (2.7) are defined as in (2.1) and (2.2). V_{U1}, V_{U2}, V_{V1}, and V_{V2} are M-by-M matrices. Substituting (2.6) and (2.7) into (2.3) yields

$$U_1 \Sigma_1 V_1^H - zU_2 \Sigma_2 V_2^H \cong U_U \Sigma_U [V_{U1}^H \Sigma_1 V_{V1} - zV_{U2}^H \Sigma_2 V_{V2}] \Sigma_V U_V^H. \tag{2.8}$$

This expression implies that the desired GE's can be estimated by the GE's of the following M-by-M matrix pencil:

$$V_{U1}^H \Sigma_1 V_{V1} - zV_{U2}^H \Sigma_2 V_{V2}. \tag{2.9}$$

The GE's of this M-by-M matrix pencil will be shown to be the same as the estimates of the desired GE's obtained in both the Pro-ESPRIT and the TLS-Pro-ESPRIT.

Theorem: (Notations used here can be treated independently although they are consistent with notations throughout the paper.) IF each of U_1 and U_2 consists of M orthonormal vectors and they have the complete joint SVD:

$$[U_1, U_2] = [U_U, U_U'] \begin{bmatrix} \Sigma_U & \\ & \Sigma_U' \end{bmatrix} \begin{bmatrix} V_{U1} & V_{U1}' \\ V_{U2} & V_{U2}' \end{bmatrix}^H = U_U \Sigma_U [V_{U1}^H, V_{U2}^H] + U_U' \Sigma_U' [V_{U1}'^H, V_{U2}'^H] \tag{2.10}$$

where the quantities without primes consist of the M principal components and those with primes consist of the M nonprincipal components ($\Sigma'_U < \Sigma_U$ is assumed to hold in the strict sense), then,

$$\begin{bmatrix} V_{U1} & V'_{U1} \\ V_{U2} & V'_{U2} \end{bmatrix} = 1/\sqrt{2} \begin{bmatrix} G_1 H_1^H & G_1 H_2^H \\ G_2 H_1^H & -G_2 H_2^H \end{bmatrix} \quad (2.11)$$

where G_1, G_2, H_1, and H_2 are unitary matrices.

The proof is provided in the Appendix.

Applying the Theorem to (2.6) and (2.7) yields that $\sqrt{2V_{U1}}$, $\sqrt{2V_{U2}}$, $\sqrt{2V_{V1}}$, and $\sqrt{2V_{V2}}$ are all unitary matrices. Notice that those unitary matrices are used in the matrix pencil of (2.9).

1) The Pro-ESPRIT: According to [5], [6], the Pro-ESPRIT estimates the M desired GE's from the following M-by-M matrix pencil:

$$Q_U \Sigma_1 Q_V^H - z\Sigma_2 \quad (2.12)$$

where Q_U is the Procrustes ("best") unitary approximation of $U_2^H U_1$, and Q_V the Procrustes unitary approximation of $V_2^H V_1$. Specifically, Q_U is the SVD of $U_2^H U_1$ with all its singular values set to one. Q_V is similarly obtained.

Note that (2.12) is a direct modification of (2.4) based on the fact that $U_2^H U_1$ and $V_2^H V_1$ are unitary in the noiseless case.

For convenience, we write

$$Q_U = (U_2^H U_1)_{\text{unitary}} \quad (2.13)$$

$$Q_V = (V_2^H V_1)_{\text{unitary}}. \quad (2.14)$$

From (2.13) and (2.10), it follows that

$$Q_U = (V_{U2} \Sigma_U^2 V_{U1}^H + V'_{U2} \Sigma_U'^2 V_{U1}'^H)_{\text{unitary}}. \quad (2.15)$$

Invoking (2.11), (2.15) becomes

$$\begin{aligned} Q_U &= [G_2(1/2 H_1^H \Sigma_U^2 H_1 - 1/2 H_2^H \Sigma_U'^2 H_2) G_1^H]_{\text{unitary}} \\ &= G_2(1/2 H_1^H \Sigma_U^2 H_1 - 1/2 H_2^H \Sigma_U'^2 H_2)_{\text{unitary}} G_1^H. \end{aligned} \quad (2.16)$$

The validity of taking the unitary matrices G_1 and G_2 out of the operator $(\cdot)_{\text{unitary}}$ can be easily proved.

For a symmetrical positive definite matrix, the left singular vectors are the same as the right singular vectors, and hence the unitary operator on a symmetrical positive definite matrix yields an identity matrix. It is a simple matter to show that the matrix inside the unitary operator in (2.16) is symmetrical positive definite (because $\Sigma_U - \Sigma'_U$, and H_1 and H_2 are unitary). Therefore, (2.16) becomes

$$Q_U = G_2 G_1^H = V_{U2}^{-H} V_{U1}^H \quad (2.17)$$

where the superscript $^{-H}$ denotes the inverse and the conjugate transpose. Similarly, we can show

$$Q_V = V_{V2}^{-H} V_{V1}^H. \quad (2.18)$$

If we multiply (2.9) by V_{U2}^{-H} from the left and by V_{U2}^{-1} from the right, and substituting (2.17) and (2.18) into (2.12), it follows immediately that (2.9) and (2.12) are equivalent.

2) The TLS-Pro-ESPRIT: According to [5], [6], the TLS-Pro-ESPRIT estimates the desired GE's from the following M-by-M matrix pencil:

$$Q_{U1} \Sigma_1 Q_{V1}^H - z Q_{U2} \Sigma_2 Q_{V2}^H \quad (2.19)$$

where

$$Q_{U1} = (U_U^H U_1)_{\text{unitary}} \quad (2.20)$$

$$Q_{U2} = (U_U^H U_2)_{\text{unitary}} \quad (2.21)$$

$$Q_{V1} = (U_V^H V_1)_{\text{unitary}} \quad (2.22)$$

$$Q_{V2} = (U_V^H V_2)_{\text{unitary}}. \quad (2.23)$$

Substituting (2.10) into (2.20) for U_1, we can write (2.20) as

$$Q_{U1} = (\Sigma_U V_{U1}^H)_{\text{unitary}} = \sqrt{2} V_{U1}^H. \quad (2.24)$$

Similarly, $Q_{U2} = \sqrt{2} V_{U2}^H$, $Q_{V1} = \sqrt{2} V_{V1}^H$, and $Q_{V2} = \sqrt{2} V_{V2}^H$. Therefore, (2.19) and (2.9) are also equivalent.

We have now shown that the Pro-ESPRIT and the TLS-Pro-ESPRIT proposed in [5], [6] are equivalent *provided* that the eigendata of Y_1 and Y_2 are estimated in the same way and the estimated singular vectors (i.e., U_1, U_2, V_1, and V_2) are orthonormal. But it can be easily observed that the Pro-ESPRIT requires less computations than the TLS-Pro-ESPRIT.

C. Algorithm 3: TLS-ESPRIT

The noise robustness of the TLS-ESPRIT proposed in [3] is uniquely due to the SVD-based steps to be shown in the following. As we mentioned before, the noise filtering at the covariance level inherent in the original ESPRIT algorithms is not addressed in this paper.

According to one of the earliest versions of the ESPRIT algorithm [1], [2], the rank-M joint SVD truncation of $[Y_1, Y_2]$ is carried out in order to reduce the noise effect. This is based on the fact that Y_1 and Y_2 span the same column space (of dimension M) in the noiseless case. The rank-M SVD truncation of $[Y_1, Y_2]$ is defined as follows:

$$[Y_1, Y_2] \cong [Y_1, Y_2]_T = U_Y \Sigma_U V_Y^H = U_Y \Sigma_Y [V_{Y1}^H, V_{Y2}^H] \quad (2.25)$$

where the SVD notations are defined similarly as for (2.6) and (2.7). V_{Y1} and V_{Y2} are N_2-by-M matrices. The LS-ESPRIT (as called in [3]) computes the GE's of the pencil $V_{Y1} - zV_{Y2}$ to obtain the estimates of the desired GE's.

In the TLS-ESPRIT [3], an additional rank-M joint SVD truncation is in effect performed on $[V_{Y1}, V_{Y2}]$ before the GE's are computed. This is due to that V_{Y1} and V_{Y2} span the same column space in the noiseless case. This SVD truncation can be again written into the following familiar form:

$$\begin{aligned} [V_{Y1}, V_{Y2}] &\cong [V_{Y1}, V_{Y2}]_T = U_{YV} \Sigma_{YV} V_{YV}^H \\ &= U_{YV} \Sigma_{YV} [V_{YV1}^H, V_{YV2}^H] \end{aligned} \quad (2.26)$$

where the M principal components are kept in the truncation as before. Y_{YV1} and V_{YV2} are M-by-M matrices.

From (2.25) and (2.26), we have the approximation

$$Y_1 - zY_2 \cong U_Y \Sigma_Y [V_{YV1} - zV_{YV2}] \Sigma_{YV} U_{YV}^H. \quad (2.27)$$

This approximation suggests that the desired GE's can be estimated by the GE's of the M-by-M matrix pencil:

$$V_{YV1} - zV_{YV2}. \quad (2.28)$$

It is simple to show that (2.28) is equivalent to the following matrix pencil which is the exact pencil used in the TLS-

ESPRIT:

$$V'_{YV2} + zV'_{YV1} \tag{2.29}$$

where V'_{YV1} and V'_{YV2} consist of the M nonprincipal right singular vectors of $[V_{Y1}, V_{Y2}]$. The equivalence between (2.28) and (2.29) can be seen by observing $V'^{H}_{YV1} V_{YV1} + V'^{H}_{YV2} V_{YV2} = 0$. Equation (2.29) was derived in [3] from a different approach. But from the numerical point of view, (2.28) is preferred to (2.29) since the principal components are often easier to obtain than the nonprincipal components.

It is worth noting that the original TLS-ESPRIT differs from the original LS-ESPRIT only in the additional SVD truncation (2.26). It will be shown that the SVD truncation does not change the first-order perturbations in the estimates of the desired GE's. It implies that the original LS-ESPRIT and the original TLS-ESPRIT are asymptotically equivalent in the estimation variances to the first order approximation. (In the original ESPRIT setting, E_1 and E_2 would represent analytically the residue errors in the cleaned data matrices Y_1 and Y_2 after the estimated noise covariance matrix is subtracted from $Y_1 Y_1^H$, $Y_1 Y_2^H$, and $Y_2 Y_2^H$, and the variances of the residue errors would be inversely proportional to the large number of columns available in Y_1 and Y_2.)

III. First-Order Equivalence

Although Algorithms 1–3 presented in the previous section generally yield different estimates of the desired GE's, we now show that those estimates are equally perturbed by the noise matrices E_1 and E_2 to the first-order approximation.

We define the following equations for the noiseless case:

$$\boldsymbol{p}_i^H (X_1 - z_i X_2) = 0 \tag{3.1a}$$

$$(X_1 - z_i X_2)\, \boldsymbol{q}_i = 0 \tag{3.1b}$$

where $i = 1, 2, \cdots, M$; z_i is a GE of the pencil $X_1 - zX_2$; $\boldsymbol{p}_i$ is a left generalized eigenvector restricted within the column space of X_1 (or X_2); and $\boldsymbol{q}_i$ is a right generalized eigenvector restricted within the column space of X_1^H (or X_2^H).

A. For Algorithm 1

We can show by following our approach in [9]–[11] that the first-order perturbation in the GE z_i of $Y_{1T} - zY_{2T}$ is given by

$$\delta z_i = \frac{\boldsymbol{p}_i^H \Delta Y_{1T} \boldsymbol{q}_i - z_i \boldsymbol{p}_i^H \delta Y_{2T} \boldsymbol{q}_i}{\boldsymbol{p}_i^H X_2 \boldsymbol{q}_i} \tag{3.2}$$

where δ denotes the first-order approximation operator; δY_{1T} and δY_{2T} are perturbations, respectively, in Y_{1T} and Y_{2T} due to E_1 and E_2. In (3.2), all quantities are noiseless except those preceded by δ.

Now we need to show that

$$\boldsymbol{p}_i^H \Delta Y_{1T} \boldsymbol{q}_i = \boldsymbol{p}_i^H \delta Y_1 \boldsymbol{q}_i = \boldsymbol{p}_i^H E_1 \boldsymbol{q}_i \tag{3.3a}$$

$$\boldsymbol{p}_i^H \delta Y_{2T} \boldsymbol{q}_i = \boldsymbol{p}_i^H \delta Y_2 \boldsymbol{q}_i = \boldsymbol{p}_i^H E_2 \boldsymbol{q}_i. \tag{3.3b}$$

Write the complete SVD of Y_1 as

$$Y_1 = Y_{1T} + Y'_{1T} = U_1 \Sigma_1 V_1^H + U'_1 \Sigma'_1 V'^{H}_1 \tag{3.4}$$

where the first term consists of the M principal components of Y_1, and the second term consists of the rest nonprincipals of Y_1. Taking the first-order perturbation of Y_1 leads to

$$\begin{aligned} \delta Y_1 &= \delta Y_{1T} + \delta Y'_{1T} \\ &= \delta Y_{1T} + \delta U'_1 \Sigma'_1 V'^{H}_1 + U'_1 \delta \Sigma'_1 V'^{H}_1 + U'_1 \Sigma'_1 \delta V'^{H}_1 \end{aligned} \tag{3.5}$$

where all quantities are noiseless except those preceded by δ. Since in the noiseless case, $\boldsymbol{p}_i$ belongs to the column space of U_1, and $\boldsymbol{q}_i$ belongs to the column space of V_1, it follows that

$$\boldsymbol{p}_i^H U'_1 = 0 \quad \text{and} \quad V'^{H}_1 \boldsymbol{q}_i = 0$$

in the noiseless case. Also notice that Σ'_1 is the zero matrix in the noiseless case. Therefore, multiplying δY_1 in (3.5) by $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ leads to

$$\boldsymbol{p}_i^H \delta Y_1 \boldsymbol{q}_i = \boldsymbol{p}_i^H \delta Y_{1T} \boldsymbol{q}_i \tag{3.6}$$

which implies (3.3a), (3.3b) can be proved in a similar fashion.

The two equations of (3.3a) and (3.3b) suggest that the SVD truncation on each of Y_1 and Y_2 does not affect the first-order approximations in the estimates of the desired GE's.

In the applications of the matrix pencil, X_1 and X_2 are often known. In that case, further analysis of (3.2) can be carried out as the authors did in [9]–[11] for the problem of estimating the complex exponential signals.

B. For Algorithm 2

We now show that the estimates of the desired GE's of Algorithm 2 are the GE's of a matrix pencil which is equivalent in the first-order approximation to the matrix pencil used in Algorithm 1.

As shown in Section II, the GE estimates of Algorithm 2 are the GE's of the matrix pencil of (2.9). Furthermore, (2.9) is equivalent to the second expression in (2.8). Hence, we can define an equivalent matrix pencil for Algorithm 2 as

$$U_{1T} \Sigma_1 V_{1T}^H - zU_{2T} \Sigma_2 V_{2T}^H \tag{3.7}$$

where U_{1T}, U_{2T}, V_{1T} and V_{2T} are defined by the following equations (refer to (2.6) and (27)):

$$\begin{aligned} [U_1, U_2] &= [U_1, U_2]_T + [U_1, U_2]'_T \\ &= [U_{1T}, U_{2T}] + [U'_{1T}, U'_{2T}] \\ &= U_U \Sigma_U [V_{U1}^H, V_{U2}^H] + U'_U \Sigma'_U [V'^{H}_{U1}, V'^{H}_{U2}] \end{aligned} \tag{3.8}$$

$$\begin{aligned} [V_1, V_2] &= [V_1, V_2]_T + [V_1, V_2]'_T \\ &= [V_{1T}, V_{2T}]\, [V'_{1T}, V'_{2T}] \\ &= U_V \Sigma_V [V_{V1}^H, V_{V2}^H] + U'_V \Sigma'_V] V'^{H}_{V1}, V'^{H}_{V2}]. \end{aligned} \tag{3.9}$$

The terms with primes consist of the nonprincipal components while those without the primes consist of the M principal components. It is known that the perturbations in singular values and singular vectors are linearly proportional to the small perturbations in the matrix to the first-order approximation [20]. Therefore, Σ'_U and Σ'_V are linearly proportional to E_1 and E_2 to the first-order approximation.

The GE's of (3.7) are not changed after it is left multiplied by U_{2T}^H and right multiplied by V_{2T}, which yields

$$(U_{2T}^H U_{1T}) \Sigma_1 (V_{1T}^H V_{2T}) - z(U_{2T}^H U_{2T}) \Sigma_2 (V_{2T}^H V_{2T}). \tag{3.10}$$

Using (3.8), we have

$$\begin{aligned} U_2^H U_1 &= U_{U2} \Sigma_U^2 V_{U1}^H + V'_{U2} \Sigma'^2_U V'^{H}_{U1} \\ &= U_{2T}^H U_{1T} + V'_{U2} \Sigma'^2_U U'^{H}_{U1}. \end{aligned} \tag{3.11}$$

Neglecting the second-order terms of E_1 and E_2, we have

$$U_{2T}^H U_{1T} = U_2^H U_1.$$

Similarly, it can be shown that to the first-order approximation

$$V_{2T}^H V_{1T} = V_2^H V_1$$

$$U_{2T}^H U_{2T} = U_2^H U_2 = I$$

$$V_{2T}^H V_{2T} = V_2^H V_2 = I.$$

Now comparing (2.4) of Algorithm 1 to (3.10) of Algorithm 2 implies that the two algorithms are equivalent to the first-order approximation. Hence, the perturbation expression of (3.2), (3.3) also applies to the Pro-ESPRIT.

3) For Algorithm 3: It suffices to show that neither the SVD truncation of (2.25) for the LS-ESPRIT nor the SVD truncation of (2.26) for the TLS-ESPRIT yields different first-order perturbations in the estimated GE's.

According to (2.25), the matrix pencil of the LS-ESPRIT can be written as

$$Y_{1T3} - zY_{2T3} \tag{3.12}$$

where $Y_{1T3} = U_Y \Sigma_Y V_{Y1}^H$ and $Y_{2T3} = U_Y \Sigma_Y V_{Y2}^H$. Note that (3.12) has the same GE's as the pencil $V_{Y1} - zV_{Y2}$.

Following the approach in [9]–[11], the first-order perturbations in the GE's of the (3.12) are given by (3.2) with ΔY_{1T} and δY_{2T} replaced by δY_{1T3} and δY_{2T3}, respectively. Following the same approach as for (3.3), it is easy to verify the following:

$$\boldsymbol{p}_i^H \delta Y_{1T3} \boldsymbol{q}_i = \boldsymbol{p}_i^H E_1 \boldsymbol{q}_i \tag{3.13a}$$

$$\boldsymbol{p}_i^H \delta Y_{2T3} \boldsymbol{q}_i = \boldsymbol{p}_i^H E_2 \boldsymbol{q}_i. \tag{3.13b}$$

These two equations implies that the joint SVD truncation as in (2.25) does not either affect the first-order perturbations in the estimated GE's.

Equation (2.26) for the TLS-ESPRIT is simply a further step of the joint SVD truncation. Hence, it can be similarly shown that the GE's of the pencil $V_{Y1} - zV_{Y2}$ are equally perturbed in the first-order approximation as the GE's of the pencil $V_{YV1} - zV_{YV2}$. In other words, the LS-ESPRIT and the TLS-ESPRIT are equivalent to the first-order approximation, and they are equivalent in the first-order approximation to Algorithms 1, 2.

We have now considered Algorithms 1–3 without knowing the detailed structure of Y_1 and Y_2. In the next section, we shall discuss the complex exponential signal problem for which the detailed structures of Y_1 and Y_2 are known.

IV. The State Space Method and the Matrix Pencil Method

For estimating the parameters of the complex exponential signals, several researchers [14]–[19] have studied the state space method. Compared to the matrix pencil algorithms, as shown in Section II, the state space algorithm [16] also exploits a matrix pencil structure but in a different way as shown in the following.

Let the data sequence be

$$y(k) = \sum_{i=1}^{M} a_i z_i^k + n_k$$

where $k = 0, 1, \cdots, N-1$, and z_i's are called the signal poles. The state space algorithm estimates the signal poles by starting with the following data matrix:

$$Y = \begin{bmatrix} y(0) & y(1) & \cdots & y(L) \\ y(1) & y(2) & \cdots & y(L+1) \\ & & \cdots & \\ y(N-L-1) & y(N-L) & \cdots & y(N-1) \end{bmatrix} \tag{4.1}$$

where L is restricted by $M \le L \le N - M$. The SVD of this matrix is then computed:

$$Y = U \Sigma V^H + U' \Sigma' V'^H \tag{4.2}$$

where the first term consists of the M principal components, and the second term consists of the remaining nonprincipals. The signal poles are then estimated by the GE's of the full rank matrix pencil:

$$S_1 - zS_2 \tag{4.3}$$

where S_1 is V with the first row deleted, and S_2 is V with the last row deleted. In [16], the GE's of (4.3) are computed by computing the eigenvalues of the matrix $S_2^+ S_1 = (S_2^H S_2)^{-1} (S_2^H S_1)$.

After the discussions in Section II, it becomes trivial to realize that the original state space method can be improved in the following way. Since in the noiseless case, S_1 and S_2 each span the same column space, we can extract out the M principal components from S_1 and S_2 by the joint SVD:

$$[S_1, S_2] \cong [S_1, S_2]_T = U_S \Sigma_S [V_{S1}^H, V_{S2}^H]. \tag{4.4}$$

Then we estimate the signal poles from the GE's of the M-by-M matrix pencil $V_{S1} - zV_{S2}$. This algorithm will be referred to as improved state space algorithm or Algorithm 5. The original state space method will be called Algorithm 4.

In the direct matrix pencil algorithm [9]–[11], we construct a matrix pencil directly from the data as

$$Y_1 - zY_2 \tag{4.5}$$

where Y_1 is Y with the first column deleted, and Y_2 is Y with the last column deleted. Since (4.5) has all the properties stated for (1.1) in Section I, the three algorithms presented in Section II can be applied to extract the signal poles from (4.5).

To show the first-order equivalence between the state space algorithm and algorithms 1–3, we may write (4.3) into the equivalent matrix pencil:

$$Y_{1TS} - zY_{2TS} \tag{4.6}$$

where $Y_{1TS} = U \Sigma S_1$ and $Y_{2TS} = U \Sigma S_2$. Then following the approach presented in Section III, it is clear that the GE's of (4.6) have the same first-order permutations as the estimated GE's obtained by Algorithms 1–3 applied to (4.5).

It is needless to show that Algorithm 5 is equivalent to Algorithms 1–4 to the first-order approximation.

In addition to the noise sensitivity analysis of the state space algorithm carried out by researchers like Rao [19] and Kot [18], our work in [9]–[11] for the direct matrix pencil algorithm has now been shown to be valid also for the state space algorithm.

The major computations required by the five algorithms (if all applied to (4.5)) can be summarized in the following: In addition to solving for the GE's of an M-by-M matrix pencil,

Algorithm 1 requires SVD's of two "independent" $(N - L)$ by L matrices;

Algorithm 2 requires SVD's of two "independent" $(N - L)$ by L matrices and two "independent" M-by-M matrices;

Algorithm 3 requires SVD's of one $(N - L)$ by $2L$ matrix and one L-by-$2M$ matrix (or SVD's of one $2(N - L)$-by-L matrix and one $(N - L)$-by-$2M$ matrix);

Algorithm 4 requires SVD of one $(N - L)$-by-$(L + 1)$ matrix;

Algorithm 5 requires SVD's of one $(N - L)$-by-$(L + 1)$ matrix and one L-by-$2M$ matrix.

Here the "independent" means that the two matrices can be processed in parallel, which is a property stressed in [5], [6] for the Pro-ESPRIT.

It is clear that the amount of computation required by each algorithm depends on the free parameter L. It is known [9]–[11], however, that the choice of L greatly affects the noise sensitivity of the estimated signal poles. In terms of the noise sensitivity, the good choices of L are normally between $N/3$ and $2N/3$.

Finally, we like to mention the case where the underlying signals are undamped pure sinusoids. For this case, the following forward-and-backward data matrix can be used in place of (4.1)

$$Y_{FB} = \begin{bmatrix} Y \\ Y^*P \end{bmatrix}$$

where Y^* denotes the complex conjugate of Y, and P is the permutation matrix (i.e., with ones on its cross diagonal axis). It can be shown that the five algorithms are also equivalent to the first-order approximation when applied to Y_{FB}.

V. Simulations

In our Monte Carlo simulation, we considered the superimposed damped sinusoidal signal:

$$y(k) = \sum_{i=1}^{2} 2b_i r_i^k \cos(\omega_i k + \phi_i) + \sigma n(k)$$

where $k = 0, 1, \cdots, 25$, $b_1 = b_2 = 1$, $\phi_1 = 0$, $\phi_2 = 45^0$, $r_1 = 0.8$, $r_2 = 0.9$, $\omega_1 = 0.7$, and $\omega_2 = 0.5$. $n(k)$ is pseudowhite Gaussian with unit deviation, which was generated by the IMSL GGNML routine with DSEED = 123457. Two hundred independent runs were computed. The same data were applied to each of the five algorithms: 1) the direct matrix pencil algorithm; 2) the Pro-ESPRIT; 3) the TLS-ESPRIT; 4) the state space algorithm, and 5) the improved state space algorithm.

Again we mention that following the original Pro-ESPRIT or the original TLS-ESPRIT, we would have to compute the covariance matrices $Y_1 Y_1^H$, $Y_2 Y_2^H$, and $Y_1 Y_2^H$ (or the larger covariance matrix $[Y_1, Y_2][Y_1, Y_2]^H$) and remove the estimated noise covariance from the above covariance matrices. But our interest here is to illustrate the effectiveness of the SVD based truncation steps in all the five algorithms. (For the problem of only 26 data points, the covariance filtering may not do any good.)

Due to the numerical instability of the IMSL LSVDF routine, all singular values and singular vectors were computed through the IMSL EIGRS routine which computes the eigen decomposition of real symmetrical matrices. The GE's of the M-by-M matrix pencil were computed by the IMSL EIGZF. The whole simulation was run in FORTRAN-77 double precision.

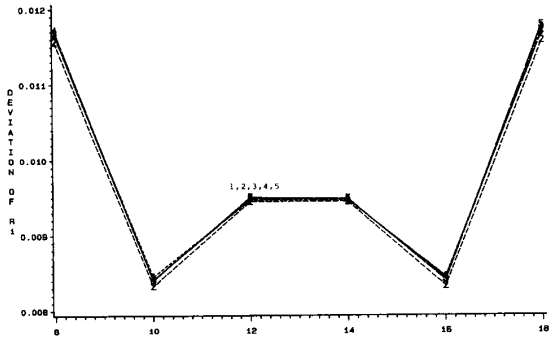

Fig. 1. Sample deviations of the estimate of r_1 versus the parameter L for Algorithms 1–5. The deviations for $L = 8, 10, 12, 14, 16, 18$ are plotted. The deviations for the five algorithms are close.

Fig. 1 shows the sample deviation of the estimated r_1 versus the free parameter L with the noise deviation $\sigma = 0.01$. r_i and ω_i were obtained from z_i according to $r_i = |z_i|$ and $\omega_i = \text{Im}\{\ln(z_i)\}$. Im $\{\cdot\}$ denotes the imaginary part. Because there are two complex conjugate pairs of the desired GE's, M was chosen to be 4. It is seen from Fig. 1 that the deviations are very close for all the five algorithms. This is because the deviations (or variations) are dominated by the first order perturbations if the noise level is not too high (i.e., not above a threshold). Fig. 2 shows the sample bias of r_1 versus L with $\sigma = 0.01$. It appears from this plot that the second-order perturbations of the five algorithms are all different since the biases are dominated by the second-order perturbations when the noise level is not too high. Figs. 1, 2 confirm our observation (made in [9]–[11] for Algorithm 1) that the good choices of L are between $N/3$ and $2N/3$. Fig. 2 also shows that Algorithm 3 has the least bias, followed by Algorithms 5 and 4.

Since r_1 was observed to be much more sensitive to the noise than the other parameters (r_2, ω_1, and ω_2), similar plots for those parameters should be of less interest. We like to stress that each of the five algorithms could be judged to be the most accurate based on some isolated cases. But our experience shows that Algorithms 3–5 tend to be more accurate than Algorithms 1, 2.

To present the threshold effect, we provide Figs. 3 and 4. Fig. 3 shows the deviation of r_1 versus σ with $L = 13$. Fig. 4 shows the deviation of ω_1 versus σ with $L = 13$. It is interesting to observe that although Algorithm 2 appears to be robust in estimating r_1 (from Fig. 3), it breaks down as early as Algorithm 1 in estimating ω_1 (from Fig. 4). In plotting Fig. 4, the deviations were limited to the value 0.1.

The large deviations of the estimated frequencies, when the noise level is above the threshold, were due to the false real valued GE's obtained from the M-by-M matrix pencil. The positive real valued GE's yield zero frequency, and the negative real valued GE's yield the frequency π. When the false real valued GE's occur, we call the corresponding run a "bad run." In Table I, the bad runs are listed for each algorithm as functions of the noise level σ. It is interesting to see from this table that the noise robustness of the five algorithms can be ranked according to their current order.

VI. Conclusion

We have reviewed the contemporary algorithms, the direct matrix pencil algorithm, the Pro-ESPRIT, and the TLS-

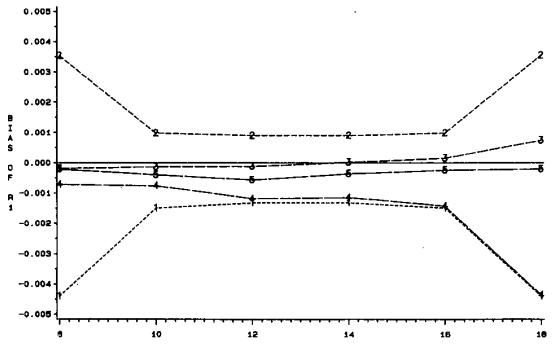

Fig. 2. Sample biases of the estimate of r_1 versus the parameter L for Algorithms 1–5. The biases for $L = 8, 10, 12, 14, 16, 18$ are plotted. The biases for the five algorithms are quite different.

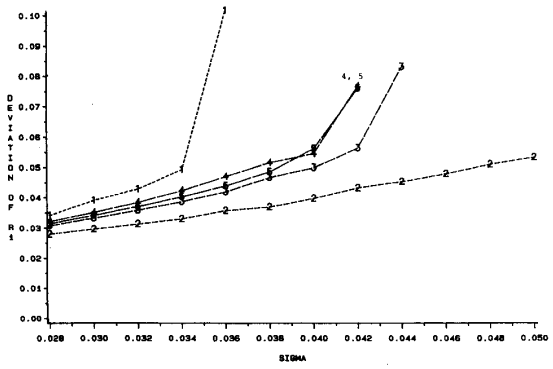

Fig. 3. Sample deviations of the estimate of r_1 versus the noise deviation σ for Algorithms 1–5. $L = 13$. Algorithm 1 appears to have the worst threshold while Algorithm 2 appears to have the best.

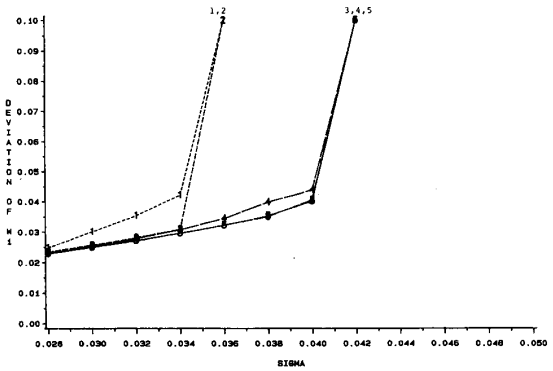

Fig. 4. Sample deviations of the estimate of ω_1 versus the noise deviation σ for Algorithms 1–5. $L = 13$. In this plot, both Algorithms 1, 2 have a close threshold which is worse than that for Algorithms 3–5.

ESPRIT, for estimating the Ge's of the singular matrix pencils perturbed by noise.

The SVD truncations have been presented as the basic principle inherent in all the algorithms. An in-depth look into the Pro-ESPRIT and the TLS-Pro-ESPRIT has been made possible with the theorem shown in Section II, which shows that the two algorithms are equivalent. We have also shown that the SVD truncations employed in all the algorithms do not the change the first-order perturbations in the estimated GE's. It implies in particular that the LS-ESPRIT and the TLS-ESPRIT are asymptotically equivalent to the first-order approximation. For the special problem of estimating the parameters of the complex exponential signals, the state space algorithm has also been reviewed. It has been shown that the state space algorithm is equivalent in the first-order approximation to all the other matrix pencil algorithms. We have also presented an improved state space algorithm which turns out to be the most robust algorithm among the five matrix pencil algorithms compared.

TABLE I
NUMBER OF BAD RUNS OUT OF 200

σ =	Algrm. 1	Algrm. 2	Algrm. 3	Algrm. 4	Algrm. 5
0.032	0	0	0	0	0
0.034	1	0	0	0	0
0.036	5	3	0	0	0
0.038	7	4	0	0	0
0.040	9	5	1	1	0
0.042	11	7	2	1	1
0.044	13	7	3	3	2
0.046	13	7	4	3	3
0.048	17	8	5	3	3

APPENDIX
PROOF OF THEOREM 1

First, we let the M-by-M matrix V_{U1} have the SVD:

$$V_{U1} = G_1 \Sigma H_1^H \tag{A.1}$$

where G_1 and H_1 are unitary left and right, respectively, singular vector matrices, and Σ is the diagonal matrix of the singular values of V_{U1}. It is easy to show that each singular value of V_{U1} is larger than zero and less than one. Due to the orthonormality of the singular vectors

$$V_{U1}^H V_{U1} + V_{U2}^H V_{U2} = I. \tag{A.2}$$

So

$$V_{U2}^H V_{U2} = I - H_1 \Sigma^2 H_1^H = H_1 [I - \Sigma^2] H_1^H. \tag{A.3}$$

Hence, we can write the SVD of V_{U2} as

$$V_{U2} = G_2 (I - \Sigma^2)^{1/2} H_1^H \tag{A.4}$$

where G_2 is the unitary left singular vector matrix of V_{U2}. In the similar manner, (A.1) together with

$$V_{U1} V_{U1}^H + V_{U1} V_{U1}^H = I \tag{A.5}$$

enables us to write the SVD of V'_{U1} as

$$V'_{U1} = G_1 (I - \Sigma^2)^{1/2} H_2^H \tag{A.6}$$

where H_2 is another unitary matrix. Furthermore, (A.4) and (A.6) together with

$$V_{U2} V_{U2}^H + V'_{U2} V'^H_{U2} = I \tag{A.7}$$

and

$$V'^H_{U1} V'_{U1} + V'^H_{U2} V'_{U2} = I \tag{A.8}$$

lead to

$$V'_{U2} V'^H_{U2} = G_2 \Sigma^2 G_2^H \tag{A.9}$$

$$V'^H_{U2} V'_{U2} = H_2 \Sigma^2 H_2^H. \tag{A.10}$$

Equations (A.9) and (A.10) force V'_{U2} to have the SVD

$$V'_{U2} = G_2 K \Sigma H_2^H \tag{A.11}$$

where K is a unitary diagonal matrix (due to the fact that each singular vector is unique up to a complex scalar of the magnitude one, and the left singular vector is absolutely unique if the corresponding right singular vector is given, and vice versa). But we know that

$$V_{U1} V_{U2}^H + V'_{U1} V'^H_{U2} = 0. \tag{A.12}$$

Substituting (A.1), (A.4), (A.6), and (A.11) into (A.12) leads to

$$G_1 \Sigma (I - \Sigma^2)^{1/2} G_2 + G_1 \Sigma (I - \Sigma^2)^{1/2} K G_2 = 0 \tag{A.13}$$

which implies $K = -I$.

Now, we need to show that

$$\Sigma = 1/\sqrt{2} I. \tag{A.14}$$

Since each of U_1 and U_2 consists of M orthonormal vectors, we can write (using (2.10)):

$$U_1^H U_1 = V_{U1} \Sigma_U^2 V_{U1}^H + V'_{U1} \Sigma'^2_U V'^H_{U1} = I \tag{A.15}$$

and

$$U_2^H U_2 = V_{U2} \Sigma_U^2 V_{U2}^H + V'_{U2} \Sigma'^2_U V'^H_{U2} = I. \tag{A.16}$$

We denote by $\boldsymbol{g}_{1i}$, $\boldsymbol{g}_{2i}$, $\boldsymbol{h}_{1i}$, and $\boldsymbol{h}_{2i}$ the ith column of G_1, G_2, H_1, and H_2, respectively, and by σ_i the ith diagonal element of Σ. Then, it is easy to verify that (using (A.1), (A.6) and (A.15)):

$$\boldsymbol{g}_{1i}^H U_1^H U_1 \boldsymbol{g}_{1i} = \sigma_i^2 \boldsymbol{h}_{1i}^H \Sigma_U^2 \boldsymbol{h}_{1i} + (1 - \sigma_i^2) \boldsymbol{h}_{2i}^H \Sigma'^2_U \boldsymbol{h}_{2i} = 1 \tag{A.17}$$

and (using (A.4), (A.11) and (A.16))

$$\boldsymbol{g}_{2i}^H U_2^H U_2 \boldsymbol{g}_{2i} = (1 - \sigma_i^2) \boldsymbol{h}_{1i}^H \Sigma_U^2 \boldsymbol{h}_{1i} + \sigma_i^2 \boldsymbol{h}_{2i}^H \Sigma'^2_U \boldsymbol{h}_{2i} = 1. \tag{A.18}$$

Since

$$\boldsymbol{h}_{1i}^H \Sigma_U^2 \boldsymbol{h}_{1i} > \boldsymbol{h}_{2i}^H \Sigma'^2_U \boldsymbol{h}_{2i} \tag{A.19}$$

it follows from (A.17) and (A.18) that $\sigma_i^2 = 1 - \sigma_i^2$, i.e., $\sigma_i = 1/\sqrt{2}$.

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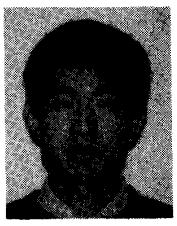

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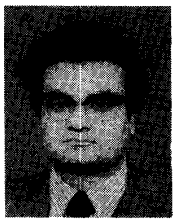

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