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H_2 Controller Design for Multi-Agent Systems with Markovian Switching Topologies

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***Abstract*—This paper presents a H_2 controller design method for discrete-time multi-agent systems with Markovian switching topologies using linear matrix inequalities (LMIs). The consensus problem is first reformulated as an equivalent error system, allowing for consensus error dynamics of general directed graph topologies to be considered. This error dynamic system is used in controller design, and it is shown how with appropriate constraints fundamental results from Markov jump linear systems literature may be used to design a distributed controller for the multi-agent system. Applying H_2-norm control design ensures mean-square consensusability of the multi-agent system can be achieved while allowing for more intuitive controller design through selection of the performance function. Simulation results are provided to show the applicability of the proposed design method to switching general directed graphs and give an example of controller tuning with the proposed LMI problem.**

***Index Terms*—discrete-time multi-agent systems, switching topology, stochastic control**

I. Introduction

The last decade has seen substantial advances in the understanding of and interest in multi-agent systems (MASs). Driving multiple different systems to achieve a shared state value is a key capability for achieving tasks such as formation control [1], rigid body attitude synchronization [2], and synchronization of oscillators [3] to name a few examples.

Investigation of the consensus problem has primarily been done assuming the communication topology between agents is static [4]. Practically this is a restrictive assumption, as any real-life and particularly wireless communication network will have inherent restrictions on data rate as well as potential link failures, which has been thoroughly studied in the area of networked control systems [5]. This has led to intense study of MASs with switching communication topologies as well as many other constraints [6], [7].

Early results on the switching topology problem [8]–[10] focused on establishing conditions for consensusability. Intuitively the communication graph must include a spanning tree, that is, information must be able to reach all of the agents in a system for consensus to be able to be achieved. Other results on switching communication topologies were at times not intuitive, such as [11] where it was found that at times link failure at times caused more aggressive controllers to reach consensus faster than those which were optimal for the fixed topology.

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Recent results have focused significantly on the packet dropout phenomena, which is highly related to the switching topology problem. Communication networks with packet dropouts were studied in the independent and identically distributed (i.i.d.) sense [12], where necessary and sufficient conditions for mean-square consensusability for systems with identical and nonidentical i.i.d. packet dropouts were determined. This was then extended [13], where a predictor-like protocol to ensure mean-square consensusability of multi-agent systems with delays and packet dropouts was proposed. The i.i.d. packet dropout problem was generalized to obtain consensusability conditions of multi-agent systems with Markovian packet dropouts over an undirected communication topology and design an appropriate controller [14] in the case of identical and nonidentical packet dropouts. The edge Laplacian model [15] was used [12], [14] to allow for modelling of nonidentical dropouts for undirected graphs, and methods to extend analysis in [14] to directed graphs are suggested. Recently, it was shown that mean-square consensusability of multi-agent systems with packet dropouts can be achieved by designing a controller which depends on individual agent dynamics [16]. These results on the packet dropout problem in consensus [14], [16] rely on results from networked control systems [5] for control design.

While these papers give insights on the switching topology problem that are very theoretically informative, they do not explicitly address other practical aspects of the problem such as noise or intuitive (i.e. LQR-like) tuning as done in other work [3]. The consensus problem for multi-agent systems with noise, often referred to as stochastic multi-agent systems (SMASs) has shown results for mixed H_2/H_∞ control of MASs [17], but these approaches do not consider the problem of stochastic communications topologies. Optimal control for MASs with static topology has been addressed (see [4], [6], [18]), however the extension of an optimal control approach to MASs with switching topology, particularly for general directed graphs, remains less studied. In particular, to the authors' knowledge no papers consider the design of H_2 controllers for multi-agent systems from the standpoint of a switching communication topology. Research on single-agent stochastic H_2 control is very well understood [19] and has been thoroughly addressed in the study of Markov jump linear systems [20], which gives a clear point of reference that may be used in the extension to multi-agent systems.

In this paper our goal is to address these gaps in the literature and provide a more intuitive formulation of the

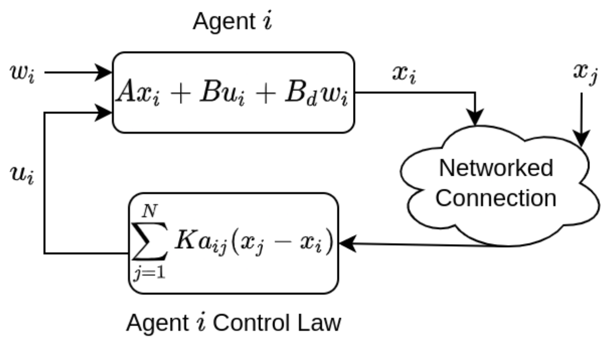

Fig. 1. Single Agent in Multi-Agent System with Stochastic Network

multi-agent consensus problem with Markovian switching topologies and noise, which allows for design trade-offs that are comparable to the traditional single-agent design problem. Our two main contributions are as follows: (1) A clear framework for transforming the consensus problem over general directed graphs to an equivalent lower-order error system and (2) LMI-based controller design to ensure mean-square consensusability of a discrete-time multi-agent system with switching communication topology and external noise.

The rest of this paper is organized as follows. Mathematical conventions and the problems considered are introduced in Section II. Section III describes the system transformation and controller design for the resulting stochastic system. Simulation results are shown to verify control design in Section IV and conclusions are given in Section V.

II. The Consensus and H_2 Problems

We consider agents with homogeneous linear dynamics. Each agent has a state transition matrix A, input matrix B, and exogenous disturbance matrix B_d as follows:

$$x_i(t+1) = Ax_i(t) + Bu_i(t) + B_d w_i(t) \tag{1}$$

where x_i, u_i, and w_i are the agent's state, input, and noise process respectively.

The consensus problem itself, as with the control problem, may be trivially solved for a stable system by assuming no input is applied. As such, we make the following assumption to avoid this trivial solution:

Assumption 1. *All eigenvalues of state tranformation matrix A are on or outside the unit disk.*

To ensure that this problem is solvable, we make the following standard assumption:

Assumption 2. *The pair (A,B) is controllable.*

Agents act via a control law of the following form:

$$u_i = \sum_{j=1}^{N} K a_{ij}(x_j - x_i) \tag{2}$$

where a_{ij} denotes the adjacency of agents i and j and N is the total number of agents. In Figure 1 an agent of the form (1) applying control law (2) is shown. Agents receive information on their own state and their neighbors' states through a network connection, which is used in the agent's control law. This input is then applied at the agent, and an additional noise signal impacts agent dynamics.

Connections between agents a_{ij} form the adjacency matrix $\mathcal{A} = [a_{ij}]$, and the number of agents which an agent is connected to is its degree $d_i = \sum_{j=1}^{N} a_{ij}$, forming a degree matrix $\mathcal{D} = diag(d_1, \ldots, d_N)$. These agents are assumed to be connected on a graph which is described by a Laplacian matrix $\mathcal{L} = \mathcal{D} - \mathcal{A}$. The state of the global system describing dynamics of all agents can be written using the graph Laplacian in the following homogeneous and non-homogeneous forms:

$$X(t+1) = (I \otimes A + \mathcal{L} \otimes BK)X(t) \tag{3}$$

$$X(t+1) = (I \otimes A + \mathcal{L} \otimes BK)X(t) + (I \otimes B_d)W(t) \tag{4}$$

where $X(t) = [x_1(t)^T, x_2(t)^T, \ldots, x_N(t)^T]^T$ is the global state vector, $W(t)$ is an appropriate size vector of disturbances, and $\otimes$ denotes the Kronecker product. We first consider the general case without noise. In a system of the form (3), the following definition of consensusability applies:

$$\lim_{t\to\infty} (x_j(t) - x_i(t)) = 0, \ \ \forall i, j \in 1, \ldots, N \tag{5}$$

For convenience, we define a new consensus error state $\delta_i(t) \triangleq x_i(t) - x_1(t), \ \forall i \in 1, \ldots, N$, which is equivalent to (5). Now we define mean-square consensuability as:

$$\lim_{t\to\infty} \mathbb{E}\{\delta_i \delta_i^T\} = 0, \ \ \forall i \in 2, \ldots, N \tag{6}$$

where clearly $\delta_1 = 0$. The goal of consensus is fundamentally to achieve a shared state between agents in the sense of (5), while mean-square consensusability is the stricter condition (6) requiring stability of the covariance of consensus error. This paper considers the mean-square consensusability problem under the assumption that the underlying topology of (3) is time-varying with external noise, that is:

$$X(t+1) = (I \otimes A + \mathcal{L}(t) \otimes BK)X(t) + (I \otimes B_d)W(t) \tag{7}$$

For convenience, we modify (6) to handle the general formulation (7) with noise and define mean-square consensusability in the following sense:

$$\lim_{t\to\infty} \mathbb{E}\{\delta_i \delta_i^T\} < \nu, \ \ \forall i \in 2, \ldots, N \tag{8}$$

where ν is some finite positive value. This definition of consensusability is a more practical one, in that it is expected that (6) has bounded covariance when subject to disturbances. It is an equivalent simultaneous version of mean-square stability as given in [20, Definition 3.8].

In addition to MASs, in this paper we further consider stochastic systems which have the following general form:

$$\mathcal{G} : \begin{cases} x(t+1) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t) + B_{d,\theta(t)}w(t) \\ z(t+1) = C_{\theta(t)}x(t) + D_{\theta(t)}u(t) \end{cases} \tag{9}$$

where x, z, and w are the state, performance function and noise process, and $\theta(t)$ is a Markov process. Initial conditions of (9) are assumed to be constrained:

$$\mathbb{E}\{||x(0)||^2\} < \infty, \quad \theta(0) \sim \mu \tag{10}$$

where $\mu = \{\mu_1, \ldots, \mu_o\}$ is the probability distribution of the Markov process with o states at the initial time. Given bounded initial conditions and a system evolving as per (9), we are interested in the design of a linear feedback protocol K which ensures mean square stability of the closed loop system while minimizing a given cost function. We consider (9) and design a controller to minimize the H_2 norm of $\mathcal{G}$, given as follows:

$$||\mathcal{G}_0||^2 = \sum_{i=1}^{o} \mu_i ||z(0)||^2 \tag{11}$$

where z is the performance function of (9) applying control law K given the initial conditions as previously described, and o is the number of states in Markov process $\theta(t)$.

III. Augmented System and Controller Design

In this section we reformulate the consensus problem as a stability problem, then show how existing results in stochastic control may be applied with constraints to this modified problem formulation to achieve mean-square consensusability.

A. System Tranformation

We first show how a connected system of the form (3) may be transformed to a reduced-order error system. To do this, we introduce the following lemma:

Lemma III.1. *(Adapted from [10], [16]) For any graph Laplacian $\mathcal{L} \in \mathbb{R}^{N\times N}$, there exists a unitary transformation matrix $M \in \mathbb{R}^{N\times N}$ defined as follows:*

$$M \triangleq \begin{bmatrix} \frac{1}{N} & \vec{\frac{1}{N}} \\ -1 & I^{N-1} \end{bmatrix} \tag{12}$$

where $\vec{\frac{1}{N}}$ is a vector of constant value $\frac{1}{N}$ of appropriate size, and 1 denotes an appropriate size vector of value 1. For all Laplacian matrices of appropriate size, the following state transformation results in a block matrix:

$$M\mathcal{L}M^{-1} = \begin{bmatrix} 0 & \vec{\mathcal{L}} \\ 0^{N-1} & \mathcal{L}_e \end{bmatrix} \tag{13}$$

where $\mathcal{L}_e \in \mathbb{R}^{(N-1)\times(N-1)}$ denotes the error Laplacian matrix associated with $\mathcal{L}$, and $\vec{\mathcal{L}}$ is a vector of appropriate size with possibly nonzero values. Eigenvalues of $\mathcal{L}_e$ are the eigenvalues of $\mathcal{L}$, excluding a single eigenvalue equal to zero. Dynamics of $\mathcal{L}_e$ are linearly independent from dynamics associated with the zero eigenvalue of the original Laplacian.

Application of state transformation (13) and Lemma III.1 to the global state vector $X(t)$ leads to the following states:

$$(M\otimes I)X(t) = E(t) = \begin{bmatrix} \sum_{i=1}^{N} \frac{1}{N} x_i \\ x_2 - x_1 \\ \vdots \\ x_N - x_1 \end{bmatrix} \quad \tilde{E}(t) = \begin{bmatrix} x_2 - x_1 \\ \vdots \\ x_N - x_1 \end{bmatrix} \tag{14}$$

where I is an identity matrix of appropriate size. The states described by the matrix $\tilde{E}$ are equivalent to the previously established term for consensus error (5). As Lemma III.1 shows, the dynamics of $\tilde{E}(t)$ are described by the error Laplacian $\mathcal{L}_e$ and do not depend on the average state in $E(t)$, which is associated with the zero eigenvalue of the Laplacian.

As we may apply the transformation (13) using matrix M in (12) to any graph Laplacian of size N, we may consider switching topologies using this generic state transformation. This is similar to the edge Laplacian [15] which has been used in other work to allow for reformulation of the consensus problem to a more traditional stability problem. However, instead of the requirement of the edge Laplacian to construct an incidence matrix and in turn the edge Laplacian, the key difference in transformation via Lemma III.1 is that it results in error dynamics expressed by the difference between agent 1 and all other agents as shown in (14).

Application of the state-space transformations (13) and (14) to the system described by (3) results in the following:

$$E(t+1) = M(I\otimes A + \mathcal{L}\otimes BK)M^{-1}E(t) \tag{15}$$

$$\tilde{E}(t+1) = (I\otimes A + \mathcal{L}_e\otimes BK)\tilde{E}(t) \tag{16}$$

where (16) follows from reduction of (15). Both $I\otimes A$ and $I\otimes B_d$ are diagonal and therefore are not impacted by the state transformation. For further analysis of these systems, we present the following lemma:

Lemma III.2. *The following statements are equivalent:*

- *System* (3) *is consensusable*
- *System* (4) *is consensusable*
- *System* (16) *is stable*

Proof. By Lemma III.1, the eigenvalues of reduced-order $\mathcal{L}_e$ exclude a zero eigenvalue from Laplacian $\mathcal{L}$. Then it follows from [4, Lemma 2.1] that the stability of (16) is equivalent to the consensusability of (3). Consensusability of (4) follows from the consensusability of its homogeneous counterpart (3). ■

By Lemma III.2, we may apply Lemma III.1 to a system of the form (3) to transform it to a reduced-order error system of the form (16). We may then design a control gain K which stabilizes (16) and to ensure consensusability of (3).

B. Stochastic Control Design

We now consider the problem of multi-agent systems with time-varying graph topologies. The time-varying network state is explicitly modelled as a Markov process:

Assumption 3. *Markov process $\theta(t)$ with o states has a transition probability matrix $Q \in \mathbb{R}^{o\times o}$ with $Q_{ij} = p_{ij}$ where $0 \le p_{ij} \le 1$ and $\sum_{j=1}^{o} p_{ij} = 1$ $\forall i = 1, \ldots, o$.*

The network state Markov process is associated with a set of o possible network configurations (topologies). This switching graph topology converts (7) as follows:

$$X(t+1) = (I\otimes A + \mathcal{L}(\theta(t))\otimes BK)X(t) + (I\otimes B_d)W(t) \tag{17}$$

Now we consider (17) with stochastically time-varying topology $\mathcal{L}(\theta(t))$ and convert this to an equivalent error system by applying the methodology outlined in the previous section. We define a set of error Laplacian matrices $\mathcal{L}_e(\theta(t)) \triangleq \{\mathcal{L}_e(1), \mathcal{L}_e(2), \ldots \mathcal{L}_e(o)\}$ corresponding to our set of o possible graph topologies which have a shared error state $\tilde{E}$. This set of matrices is associated with Markov process $\theta(t)$. Application of the error transformation in Lemma III.1 to (17) results in the following homogeneous and non-homogeneous consensus error systems:

$$\tilde{E}(t+1) = (I \otimes A + \mathcal{L}_e(\theta(t)) \otimes BK)\tilde{E}(t) \tag{18}$$

$$\tilde{E}(t+1) = (I \otimes A + \mathcal{L}_e(\theta(t)) \otimes BK)\tilde{E}(t) + (I \otimes B_d)W(t) \tag{19}$$

Clearly (18) is a Markov jump linear system (MJLS) with jump variable $\theta(t)$ and transition probability matrix Q. As a direct result of [20, Theorem 3.9] and Lemma III.2 we may give the following theorem:

Theorem III.3. *Define the operator $\mathcal{F}_j(K)$ as:*

$$\mathcal{F}_j(K) \triangleq I \otimes A + \mathcal{L}_e(j) \otimes BK \tag{20}$$

The following statements are equivalent:

1) *System* (17) *is mean-square consensusable*
2) *System* (18) *is mean-square stable*
3) *System* (19) *is mean-square stable*
4) *K is such that the following holds, with $j = 1, \ldots, o$:*

$$\rho((Q^T \otimes I) blkdiag(\mathcal{F}_j(K)^T \otimes \mathcal{F}_j(K))) < 1 \tag{21}$$

5) *There exist K, $P_i > 0$, $i = 1, \ldots, o$ such that:*

$$P_i > \sum_{j=1}^{o} p_{ij} \mathcal{F}_j(K)^T P_j \mathcal{F}_j(K) \tag{22}$$

Remark 1. *As stated in [14],* (22) *is bilinear in P and K and thus computationally difficult. It is feasible but impractical to pick gain K arbitrarily and validate mean-square stability through* (21) *afterwards. The remainder of this section shows that appropriate construction of the problem allows for design of K intuitively by adapting results from MJLS literature.*

Our aim is to design a linear feedback K in (17) such that the system is mean-square consensusable in the sense of (8). As error system (18) includes gain K for state-feedback in the sense of (2), we now remove it and explicitly revert to a system with consensus error state $\tilde{E}$, system input U, and exogenous noise W. With this transformed set of graph topologies $\mathcal{L}_e(\theta(t))$, we may redefine components of (19):

$$\tilde{A} \triangleq I \otimes A, \quad \tilde{B}_{\theta(t)} \triangleq \mathcal{L}_e(\theta(t)) \otimes B, \quad \tilde{B}_d \triangleq I \otimes B_d \tag{23}$$

We additionally construct performance function variables as follows:

$$\tilde{C} \triangleq I \otimes C, \quad \tilde{D}_{\theta(t)} \triangleq \mathcal{L}_e(\theta(t)) \otimes D \tag{24}$$

We now can construct the reduced-order error dynamics (19) with time-varying interaction topology $\tilde{B}_{\theta(t)}$ as follows:

$$\tilde{E}(t+1) = \tilde{A}\tilde{E}(t) + \tilde{B}_{\theta(t)}U(t) + \tilde{B}_d W(t) \tag{25}$$

As before, it is clear that (25) is a MJLS. We now modify the general system given in (9) to the problem shown in (25):

$$\mathcal{G} : \begin{cases} \tilde{E}(t+1) = \tilde{A}\tilde{E}(t) + \tilde{B}_{\theta(t)}U(t) + \tilde{B}_d W(t) \\ z(t+1) = \tilde{C}\tilde{E}(t) + \tilde{D}_{\theta(t)}U(t) \end{cases} \tag{26}$$

where only input-related matrices $\tilde{B}_{\theta(t)}$ and $\tilde{D}_{\theta(t)}$ depend on Markov process $\theta(t)$.

Our objective is to design controller K operating in a distributed fashion as in (2) over a predefined set of communication topologies $\mathcal{L}_e(\theta(t))$. To fit with the idea of a distrubuted control law, we make the following assumption:

Assumption 4. *There is no observation available for the state of Markov process $\theta(t)$ available to system* (26).

Under this assumption, we may consider the design of a distributed control law for a multi-agent system subject to disturbances and present our main result:

Theorem III.4. *(Adapted from [19]) A multi-agent system with switching topology* (17) *is mean-square consensusable if a solution to the following LMI optimization problem exists:*

$$\min \sum_{i=1}^{o} tr(\mathcal{W}_i) \tag{27}$$

$$s.t. \begin{bmatrix} \mathcal{W}_i & \tilde{C}\tilde{G} + \tilde{D}_i\tilde{F} \\ \tilde{G}^T\tilde{C}^T + \tilde{F}^T\tilde{D}_i^T & \tilde{G} + \tilde{G}^T - \mathcal{H}_i(R) \end{bmatrix} > 0, \tag{28}$$

$$\begin{bmatrix} R_i - \mu_i \tilde{B}_d \tilde{B}_d^T & \tilde{A}\tilde{G} + \tilde{B}_i\tilde{F} \\ \tilde{G}^T\tilde{A}^T + \tilde{F}^T\tilde{B}_i^T & \tilde{G} + \tilde{G}^T - \mathcal{H}_i(R) \end{bmatrix} > 0 \tag{29}$$

$$\mathcal{H}_i(R) = \sum_{j=1}^{o} p_{ji} R_j \tag{30}$$

for all $i \in 1, \ldots, o$ where $\tilde{F} = I \otimes F$, $\tilde{G} = I \otimes G$, and μ is the probability distribution of the Markov process at the initial time. A mean-square stabilizing gain is recovered by $K = FG^{-1}$, and resulting H_2 norm calculated as:

$$inf \sum_{i=1}^{o} tr(\mathcal{W}_i) \tag{31}$$

Proof. To determine the control law $K = FG^{-1}$, we have defined $\tilde{F} = I \otimes F$ and $\tilde{G} = I \otimes G$. By the mixed-product property of the Kronecker product, the following statements are equivalent:

$$\tilde{A}\tilde{G} + \tilde{B}_i\tilde{F} \tag{32}$$

$$\tilde{A}\tilde{G} + (\mathcal{L}_e(i) \otimes B)(I \otimes F) = \tilde{A}\tilde{G} + \mathcal{L}_e(i) \otimes BF \tag{33}$$

where the same holds for $\tilde{C}\tilde{G} + \tilde{D}_i\tilde{F}$ in (28). We now define $\hat{A}_i = \tilde{A} + \tilde{B}_i\tilde{F}\tilde{G}^{-1}$ and $\hat{C}_i = \tilde{C} + \tilde{D}_i\tilde{F}\tilde{G}^{-1}$. Note that the inverse of $\tilde{G}$ as previously defined is given as $\tilde{G}^{-1} = I \otimes G^{-1}$, and rewriting (33) with $\hat{A}_i$ results in:

$$\hat{A}_i = \tilde{A} + \tilde{B}_i\tilde{F}\tilde{G}^{-1} = \tilde{A} + \mathcal{L}_e(i) \otimes BFG^{-1} \tag{34}$$

Using this formulation for both the dynamics and performance equations, we may rewrite (28) and (29) as follows:

$$\begin{bmatrix} \mathcal{W}_i & \hat{C}_i\tilde{G} \\ \tilde{G}^T\hat{C}_i^T & \tilde{G}+\tilde{G}^T-\mathcal{H}_i(R) \end{bmatrix} > 0, \tag{35}$$

$$\begin{bmatrix} R_i-\mu_i\tilde{B}_d\tilde{B}_d^T & \hat{A}_i\tilde{G} \\ \tilde{G}^T\hat{A}_i^T & \tilde{G}+\tilde{G}^T-\mathcal{H}_i(R) \end{bmatrix} > 0 \tag{36}$$

This formulation mirrors that of [19, Theorem 6] with modified construction of F and G to design for a multi-agent system, we refer to their paper for the full proof. Thus, if a feasible solution to the optimization problem (27) with constraints (28) and (29) exists, it is a mean-square stabilizing gain $K = FG^{-1}$ for system (26). The H_2 norm of the resulting error system is given as the worst-case version of [19] calculated by (31). By Theorem III.3, linear control law K which ensures mean-square stability of the reduced-order error system (19) also ensures mean-square consensusability of the multi-agent system (17) and thus the proof is complete. ■

Remark 2. *The LMI problem in Theorem III.4 corresponds to the most constrained version of the optimization problem posed in [19], with further modifications to construction of F and G. One may consider the problem without Assumption 4 and instead partial or full observation of the Markov state, with switching K_i for each observed Markov state. However, it is more practically reasonable for the multi-agent system to be unable to observe the state of the global communication topology at any given time.*

Noting that the LMI constraints in Theorem III.4 contain the error Laplacian, the complexity of the problem posed scales with the number of agents considered. For more precise bounds on complexity one may refer to [21] and references therein. Though mathematically Theorem III.4 may scale to arbitrarily large multi-agent systems, the limiting factor in controller design is more likely to be computational complexity of satisfying such high-dimension LMI constraints.

IV. Simulation Examples

We generate system matrices A, B and performance function matrices C, D as follows:

$$A = \begin{bmatrix} 1.0196 & -0.0325 \\ 0.0068 & 1.0163 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0195 \\ 0.0631 \end{bmatrix} \tag{37}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{38}$$

Noise was generated using a zero-mean normal distribution. The noise input matrix B_d is defined as:

$$B_d = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix} \tag{39}$$

Graph topologies switch between three configurations, two of which are depicted in Figure 2:

$$\mathcal{L}_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \mathcal{L}_2 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{40}$$

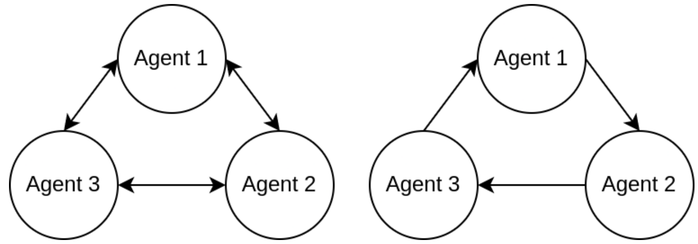

Fig. 2. Graph Topologies $\mathcal{L}_1$ (left) $\mathcal{L}_2$ (right)

Fig. 3. Sample State Trajectory of System

where a third network topology $\mathcal{L}_3$ is a fully disconnected graph. The Markovian transition probability matrix Q and stationary distribution μ for this system is given as follows:

$$Q = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.1 & 0.4 & 0.5 \\ 0.1 & 0.6 & 0.3 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0.142 & 0.462 & 0.393 \end{bmatrix} \tag{41}$$

When the LMI optimization problem in Theorem III.4 is applied to the stochastic system outlined previously assuming the initial Markov state probability is equivalent to the stationary distribution, our resulting gain K and H_2 norm are:

$$K = \begin{bmatrix} -1.322 & 2.165 \end{bmatrix}, \quad ||\mathcal{G}_0||^2 = 4.432 \tag{42}$$

We first show an example of the multi-agent system outlined above achieving consensus with one particular realization of added noise and Markovian switching. State values of each agent were randomly uniformly initialized on the interval $[-1, 1)$. The initial state of the Markov process was randomized according to the stationary distribution. Figure 3 shows that the state trajectory of the system, while subject to noise, slowly tends towards consensus. Due to the noise input, system states will not achieve consensus in the sense of (6) since the consensus error will not strictly go to zero, instead it is bounded in the sense of (8).

Though we see a system slowly moving towards consensus in Figure 3, we now show that we may increase convergence

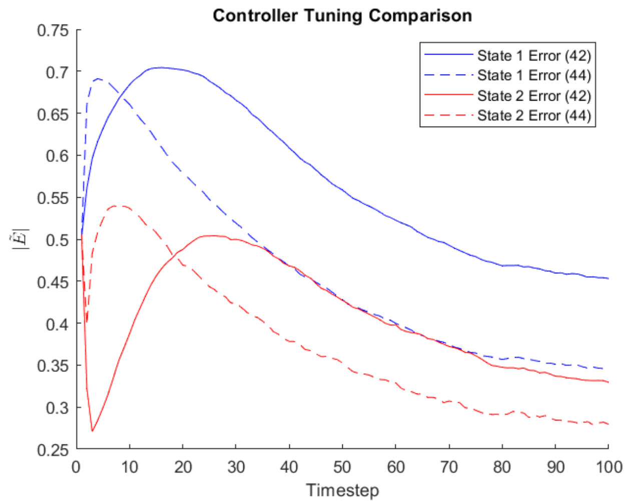

Fig. 4. Ensemble Average State Error over 1000 Trials

speed through tuning. The optimization problem in Theorem III.4 allows for more intuitive tuning of controls through the choice of the performance function z and matrices C and D by modifying D as follows:

$$D = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} \tag{43}$$

We recalculate the LMI with modified performance input matrix D and find the resulting gain and H_2 norm:

$$K = \begin{bmatrix} -4.132 & 5.569 \end{bmatrix}, \quad ||\mathcal{G}_0||^2 = 0.947 \tag{44}$$

To compare the two designed controllers, we have taken the average of the absolute value of the error of each state over 1000 separate trials. State values were initialized as done before, and all random behaviour of the system was seeded prior to simulation for each control policy to ensure accurate comparison. Figure 4 illustrates the difference between the two controllers in achieving consensus. Due to the lower weighting of control input in the performance function, the tuned controller in (44) is able to achieve consensus more rapidly than the controller designed in (42).

V. Conclusions

This paper considers the problem of H_2 control design for multi-agent systems subject to Markovian switching communication topologies. Appropriate state transformations of the multi-agent system allow for analysis of a reduced-order error system, changing the consensusability problem to the well understood stability problem. It is shown that this state transformation allows for controller design using theory developed for Markov jump linear systems. With additional constraints on design variables and exploiting properties of the Kronecker product, a distributed controller design can ensure mean-square consensusability of the multi-agent system with Markovian switching general directed topologies. Simulation results confirm the validity of the control design, provide an example of controller tuning, and show mean-square consensusability for multi-agent systems subject to noise.

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