Variational collision avoidance problems on Riemannian manifolds

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Abstract—In this article we introduce a variational approach to collision avoidance of multiple agents evolving on a Riemannian manifold and derive necessary conditions for extremals. The problem consists of finding non-intersecting trajectories of a given number of agents, among a set of admissible curves, to reach a specified configuration, based on minimizing an energy functional that depends on the velocity, covariant acceleration and an artificial potential function used to prevent collision among the agents. The results are validated through numerical experiments on the manifolds \mathbb{R}^2 and S^2 .

I. INTRODUCTION

Path planning and collision avoidance of multiple agents have been areas of significant interest in the past few decades due to its broad applications in power networks, biological networks, social networks, mechanical networks and so on. Finding trajectories that take a set of agents from one configuration to another while avoiding collisions and minimizing some quantity like energy or time has been an important problem with applications in a variety of domains [15]. Distributed protocols were proposed for various agent networks, including general linear dynamical networks, nonlinear system networks, and mobile robotic networks in the last years [6]. Nevertheless there still exist some gaps in the literature in the bridge of knowledge between multi-agent systems and geometric mechanics.

Calculus of variations in the large, as presented in Milnor [16], has been exploited in the past for various applications. In Crouch and Silva Leite [11] the authors have used it to develop a theory of generalized cubic polynomials for dynamic interpolation problems on Riemannian manifolds. More recently, Bloch, Camarinha and Colombo [3] have used these variational methods to solve obstacle avoidance problems on Riemannian manifolds. In this article, inspired by the recent work [3], we seek to extend this method to find necessary conditions for optimal trajectories of multiple agents on a Riemannian manifold that seek to achieve a specified configuration while avoiding collisions among themselves. Specifically, the problem studied in this paper consists of finding non-intersecting trajectories of a given number of agents, among a set of admissible curves, to reach a specified configuration and minimizing an energy functional that depends on the velocity, covariant acceleration

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and an artificial potential function used to prevent collision. To solve the problem, we employ techniques from calculus of variations on Riemannian manifolds taking into account that the problem under study can be seen as a higher order variational problem [4], [8], [10], [14].

The article is organized as follows. In section 2, we introduce some concepts from Riemannian geometry relevant to the rest of the article. In the next section, we define the variational collision avoidance problem on Riemannian manifolds and derive necessary conditions for the existence of extrema. In the following section we extend our analysis to Lie groups endowed with a left-invariant metric. We show how to apply the results of this work in particular examples: two agents moving on a Euclidean space, the sphere, and the collision avoidance of multiple rigid bodies on SO(3).

II. PRELIMINARIES

A. Preliminaries on Riemannian Geometry

Let M be a smooth (C^{∞}) Riemannian manifold with Riemannian metric denoted by $\langle \cdot, \cdot \rangle : T_x M \times T_x M \to \mathbb{R}$ at each point $x \in M$, where $T_x M$ is the tangent space of M at x. The length of a tangent vector is determined by its norm, $||v_x|| = \langle v_x, v_x \rangle^{1/2}$ with $v_x \in T_x M$, for each point $x \in M$

A Riemannian affine connection ∇ on M, is a map that assigns to any two smooth vector fields X and Y on M a new vector field, $\nabla_X Y$, called the covariant derivative of Y with respect to X satisfying

$$\nabla_{fX}Y = f\nabla_{X}Y$$
, and $\nabla_{X}(fY) = X(f)Y + f\nabla_{X}Y$

for all vector fields $X,Y\in\mathfrak{X}(M)$ and $f\in\mathcal{C}^\infty(M)$, where $\mathfrak{X}(M)$ denotes the set of vector fields on M. For the properties of ∇ , we refer the reader to [5], [7], [16].

Consider a vector field W along a curve x on M. The n-th order covariant derivative of W along x is denoted by $\frac{D^nW}{dt^n}$ with $n\geq 1$. We denote by $\frac{D^{n+1}x}{dt^{n+1}}$ the n-th order covariant derivative along x of the velocity vector field of x, $n\geq 1$.

Given vector fields X, Y and Z on M, the vector field R(X,Y)Z given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad (1)$$

defines the *curvature tensor* of M, where [X,Y] denotes the *Lie bracket* of the vector fields X and Y. R is trilinear in X, Y and Z and a tensor of type (1,3). Hence for vector fields X,Y,Z,W on M the curvature tensor satisfies ([16], p. 53)

$$\langle R(X,Y)Z,W\rangle = \langle R(W,Z)Y,X\rangle.$$
 (2)

Let S be a submanifold of M and $\Omega \subset M$ be the set of all C^1 piecewise smooth curves $x:[0,T] \to M$ such that $x(0), \frac{dx}{dt}(0)$ and $x(T) \in S$ are fixed, and $\frac{dx}{dt}(T) \in T_{x(T)}S$. The set Ω is called the *admissible set*. For the class of curves in Ω we introduce the C^1 piecewise smooth *one parameter admissible variation* of a curve $x \in \Omega$ by $\alpha: (-\epsilon, \epsilon) \times [0,T] \to M; (r,t) \mapsto \alpha(r,t) = \alpha_r(t)$ that satisfy $\alpha_0 = x$ and $\alpha_r \in \Omega$, for each $r \in (-\epsilon, \epsilon)$.

The variational vector field associated to an admissible variation α is a C^1 -piecewise smooth vector field X along x defined by

$$X(t) = \frac{D}{\partial r}\Big|_{r=0} \alpha(r,t) \in T_{x(t)}\Omega$$

verifying the boundary conditions

$$X(0) = 0, \quad X(T) = 0$$

$$\frac{DX}{dt}(0) = 0, \quad \frac{DX}{dt}(T) \in T_{x(T)}M$$

where the tangent space of Ω at x is the vector space $T_x\Omega$ of all C^1 piecewise smooth vector fields X along x verifying the former boundary conditions.

Lemma 2.1 ([16], p.52): The one parameter variation satisfies

$$\frac{D}{\partial r}\frac{D^2\alpha}{\partial t^2} = \frac{D^2}{\partial t^2}\frac{\partial\alpha}{\partial r} + R\Big(\frac{\partial\alpha}{\partial r},\frac{\partial\alpha}{\partial t}\Big)\frac{\partial\alpha}{\partial t}.$$
 Next, assume that M is an n -dimensional complete Rie-

Next, assume that M is an n-dimensional complete Riemannian manifold. In this context the Riemannian distance between two points in M can be defined by means of the Riemannian exponential on M, that is,

$$d(q,p) = \|\exp_q^{-1} p\|.$$

We need to guarantee that the exponential map \exp_q is a local diffeomorphism, so we assume that the point p must belong to a convex open ball around q. If we consider the geodesic from p to q given by $\gamma_{p,q}(s) = \exp_p(s\exp_p^{-1}q)$, $s \in [0,1]$, then, because $\left\|\frac{d\gamma_{p,q}}{ds}(s)\right\|$ is independent of s, we can write

$$d^{2}(p,q) = \int_{0}^{1} \left\| \frac{d\gamma_{p,q}}{ds}(s) \right\|^{2} ds.$$

The proof of the following lemmas can be found in [13]. Lemma 2.2: Let M be a complete smooth Riemannian manifold. Then $d(p,q) = \|\exp_p^{-1}q\|^2$ is well defined $\forall p,q \in M$.

Moreover, if $\alpha: (-\epsilon, \epsilon) \to M$ is a smooth curve,

$$\left.\frac{\partial}{\partial r}d(p,\alpha(r))\right|_{r=0}=-\Big\langle\frac{\partial\alpha}{\partial r}(0),\exp^{-1}_{\alpha(0)}p\Big\rangle$$

Lemma 2.3: If $X_i(t)$ is smooth vector field along $x_i(t) \in \Omega_i$, such that

$$X_i(T) = 0$$
 and $\frac{dX_i}{dt}(T) \in T_{x_i(T)}S$

then $\alpha(r,t) = \exp(rX_i(t))$ is an admissible variation of $x_i(t)$ whose variational vector field is X_i .

III. THE VARIATIONAL COLLISION AVOIDANCE PROBLEM ON RIEMANNIAN MANIFOLDS

Let M be a complete smooth Riemannian manifold. Let T, n and k be positive real numbers. Consider n agents evolving on M, and (p_0^i, v_0^i) , with i=1,2...,n, points in TM corresponding to the initial positions and velocities of the agents.

For each $i=1,\ldots,n$, consider the set $\Omega_i \subset M$ of all C^1 piecewise smooth curve on $M,\ x_i:[0,T]\to M$ verifying the boundary conditions

$$x_i(0) = p_0^i, \quad \frac{dx_i}{dt}(0) = v_0^i$$

$$x_i(T) = p_T^i \in S, \quad \frac{dx_i}{dt}(T) \in T_{x_i(T)}S$$

(the n agents reach a specified point on the submanifold S with velocity tangent to S) and define the functional J on $\dot{\Omega}=\Omega_1\times...\times\Omega_n$

$$\begin{split} J(x_1, x_2, ..., x_n) = & \frac{1}{2} \sum_{i=1}^n \int_0^T \left(\left\| \frac{D^2 x_i}{dt^2}(t) \right\|^2 + k \left\| \frac{dx_i}{dt}(t) \right\|^2 \right. \\ & + \sum_{j=1, j \neq i}^n F(\| \exp_{x_j(t)}^{-1} x_i(t) \|^2) \right) \! dt \end{split}$$

where $\exp_x: V_0 \subset T_x M \to M$ is the geodesic exponential map, which is a smooth diffeomorphism of some open set around $0 \in T_x M$ onto an open set around $x, F: \mathbb{R} \to \mathbb{R}^*$ is a smooth function from the reals to the extended reals such that $F(0) = +\infty$. The functional is constructed as the sum of a combination of the velocity and covariant acceleration of the individual trajectories regulated by a parameter k and a function that penalizes collisions between the agents.

Problem: The variational collision avoidance problem involves minimizing the functional J among $\hat{\Omega}$.

In order to minimize the functional J among the set $\hat{\Omega}$ we want to find curves $x \in \hat{\Omega}$ such that $J(x) \leq J(\tilde{x})$, for all admissible curves \tilde{x} in a C^1 neighborhood of x.

Remark 3.1: Note that the factor $\frac{1}{2}$ multiplying $\sum_{j=1,j\neq i}^n F(\|\exp_{x_j(t)}^{-1}x_i(t)\|^2)$ is to not count twice the same potential function for two agents to avoid collision between them.

Theorem 3.1: Let $x_i \in \Omega_i$. If α is an admissible variation of x_i with variational vector field X_i , then

$$\begin{split} 0 = & \frac{d}{dr} J(\alpha_r) \Big|_{r=0} = \int_0^T \left(\left\langle X_i, \frac{D^4 x_i}{dt^4} - k \frac{D^2 x_i}{dt^2} \right. \right. \\ & + R \Big(\frac{D^2 x_i}{dt^2}, \frac{d x_i}{dt} \Big) \frac{d x_i}{dt} \\ & - \sum_{j=1, j \neq i}^n F'(\| \exp_{x_i(t)}^{-1} x_j(t) \|^2) \exp_{x_i(t)}^{-1} x_j(t) \Big\rangle \right) \, dt \\ & + \sum_{j=1}^l \left[\left\langle \frac{D X_i}{dt}, \frac{D^2 x_i}{dt} \right\rangle + \left\langle X_i, k \frac{d x_i}{dt} - \frac{D^3 x_i}{dt^3} \right\rangle \right]_{t_i^{+-1}}^{t_i^{-}}. \end{split}$$

Proof: If α is an admissible variation of $x_i \in \Omega_i$ with variational vector field X_i , then

$$\frac{d}{dr}J(\alpha_r) = \int_0^T \left(\left\langle \frac{D}{dr} \frac{D^2 \alpha}{dt^2}, \frac{D^2 \alpha}{dt^2} \right\rangle + k \left\langle \frac{D^2 \alpha}{\partial r \partial t}, \frac{\partial \alpha}{\partial t} \right\rangle \right.$$
$$\left. + \sum_{j=1, j \neq i}^n F'(\|\exp_{x_j(t)}^{-1} \alpha(t)\|^2) \frac{\partial}{\partial r} \|\exp_{x_j(t)}^{-1} \alpha_r(t)\|^2 \right) dt.$$

By lemma 2.2

$$\frac{\partial}{\partial r}\| \exp_{x_j(t)}^{-1} \alpha_r(t) \|^2 = - \Big\langle \exp_{\alpha_r(t)}^{-1} x_j(t), \frac{\partial}{\partial r} \alpha_r(t) \Big\rangle$$

By lemma 2.1 and the previous equation

$$\begin{split} &\frac{d}{dr}J(\alpha_r) = \int_0^T \left(\left\langle \frac{D^2}{dt^2} \frac{\partial \alpha}{\partial r}, \frac{D^2 \alpha}{dt^2} \right\rangle \right. \\ &+ \left\langle R \left(\frac{\partial \alpha}{\partial r}, \frac{\partial \alpha}{\partial t} \right) \frac{\partial \alpha}{\partial t}, \frac{D^2 \alpha}{\partial t^2} \right\rangle + k \left\langle \frac{D^2 \alpha}{\partial r \partial t}, \frac{\partial \alpha}{\partial t} \right\rangle \\ &- \sum_{j=1, j \neq i}^n F'(\|\exp_{x_j(t)}^{-1} \alpha_r(t)\|^2) \left\langle \frac{\partial \alpha_r(t)}{\partial r}, \exp_{\alpha_r(t)}^{-1} x_j(t) \right\rangle \right) dt. \end{split}$$

Integrating the first term by parts twice and the third term once, and applying Lemma 2.1 to the second term, we obtain

$$\frac{d}{dr}J(\alpha_r) = \int_0^T \left(\left\langle \frac{\partial \alpha}{\partial r}, \frac{D^4 \alpha}{dt^4} + R\left(\frac{D^2 \alpha}{dt^2}, \frac{\partial \alpha}{\partial t}\right) \frac{\partial \alpha}{\partial t} - k \frac{D^2 \alpha}{dt^2} \right. \\ \left. - \sum_{j=1, j \neq i}^n F'(\|exp_{x_j(t)}^{-1} \alpha(t)\|^2) \left\langle \frac{\partial}{\partial r} \alpha_r(t), \exp_{\alpha_r(t)}^{-1} x_j(t) \right\rangle \right) dt \\ \left. + \sum_{i=1}^l \left[\left\langle \frac{D}{dt} \frac{\partial \alpha}{\partial r}, \frac{D^2 \alpha}{dt^2} \right\rangle + \left\langle \frac{\partial \alpha}{\partial r}, k \frac{\partial \alpha}{\partial t} - \frac{D^3 \alpha}{\partial t^3} \right\rangle \right]_{t_{i-1}^+}^{t_i^-}$$

where the interval [0,T] is partitioned as $0 = t_0 < t_1 <$ $\dots < t_l = T$ such that in each subinterval x_i is smooth. Taking r = 0 in the last equation,

$$\begin{split} \frac{d}{dr}J(\alpha_r)\Big|r &= 0 = \int_0^T \left(\left\langle X_i, \frac{D^4 x_i}{dt^4} \right. \right. \\ &+ R\left(\frac{D^2 x_i}{dt^2}, \frac{\partial x_i}{\partial t}\right) \frac{\partial x_i}{\partial t} - k \frac{D^2 x_i}{dt^2} \\ &- \left(\sum_{j=1, j \neq i}^n F'(\|\exp_{x_j(t)}^{-1} x_i(t)\|^2) . \exp_{x_i(t)}^{-1} x_j(t) \right) \right\rangle \right) dt \\ \sum_{i=1}^l \left[\left\langle \frac{DX_i}{dt}, \frac{D^2 x_i}{dt^2} \right\rangle + \left\langle X_i, k \frac{dx_i}{dt} - \frac{D^3 x_i}{dt^3} \right\rangle \right]_{t_{i-1}^+}^{t_i^-}. \end{split}$$

Theorem 3.2: If $x \in \Omega$ is a local minimizer of J, then $\forall i \in 1, 2..., n$

1)

$$\begin{split} &\frac{D^4 x_i}{dt^4} + R\Big(\frac{D^2 x_i}{dt^2}, \frac{dx_i}{dt}\Big) \frac{dx_i}{dt} - k \frac{D^2 x_i}{dt^2} \\ &= \sum_{j=1, j \neq i}^n F'(\|\exp_{x_j(t)}^{-1} x_i(t)\|^2).(\exp_{x_i(t)}^{-1} x_j(t)) \end{split}$$

2)
$$x_i$$
 is smooth on $[0,T]$
3) $\frac{D^2x_i}{dt^2}(T) \perp T_{x_i(T)}S$

Proof: Assume $x \in \Omega$ is a local minimizer of J. Consider a variation of \dot{x} , $\dot{\alpha}_{r,i}(t) := (x_1(t), ..., \alpha_{r,i}(t), ..., x_n(t)),$ where $\alpha_{r,i}(t)$ is an admissible variation of Ω_i with variational vector field X_i . Then $\frac{d}{dr}J(\alpha_{r,i})\Big|_{r=0}=0 \ \forall i\in 1,2,...n$.

$$\begin{split} & f \Big[\frac{D^4 x_i}{dt^4} + R \Big(\frac{D^2 x}{dt^2}, \frac{dx}{dt} \Big) \frac{dx}{dt} - k \frac{D^2 x}{dt^2} \\ & - \Big(\sum_{j=1, j \neq i}^n F'(\| \exp_{x_i(t)}^{-1} x_j(t) \|^2). (\exp_{x_i(t)}^{-1} x_j(t)) \Big) \Big] \end{split}$$

where f is a smooth real valued function on [0, T] such that $f(t_i) = f'(t_i) = 0$ and $f(t) > 0, t \neq t_i, i = i, ..., l$. So, we

$$0 = \frac{d}{dr}J(\alpha_r)\Big|_{r=0} = \int_0^T \left(f \| \frac{D^4 x_i}{dt^4} + R\left(\frac{D^2 x_i}{dt^2}, \frac{dx_i}{dt}\right) \frac{dx_i}{dt} - k \frac{D^2 x_i}{dt^2} - \sum_{j=1, j \neq i}^n F'(\|\exp_{x_i(t)}^{-1} x_j(t)\|^2) \exp_{x_i(t)}^{-1} x_j(t)\|^2 \right) dt$$

Since f(t) is greater then zero outside a set of measure zero,

$$\begin{split} \left\| \frac{D^4 x_i}{dt^4} + R \Big(\frac{D^2 x_i}{dt^2}, \frac{dx_i}{dt} \Big) \frac{dx_i}{dt} - k \frac{D^2 x_i}{dt^2} - \\ \Big(\sum_{j=1, j \neq i}^n F'(\| \exp_{x_i(t)}^{-1} x_j(t) \|^2) \exp_{x_i(t)}^{-1} x_j(t) \Big) \right\| = 0 \end{split}$$

from which statement 1 follows.

Now, choose $X_i \in T_{x_i}\Omega_i$ such that

$$X_{i}(t_{j}) = \frac{D^{3}x_{i}}{dt^{3}}(t_{j}^{+}) - \frac{D^{3}x_{i}}{dt^{3}}(t_{j}^{-}) \quad \forall j = 1, ..., l - 1$$

$$\frac{DX_{i}}{dt}(t_{j}) = \frac{D^{2}x_{i}}{dt^{2}}(t_{j}^{-}) - \frac{D^{2}x_{i}}{dt^{2}}(t_{j}^{+}) \quad \forall j = 1, ..., l - 1$$

$$X_{i}(T) = \frac{DX_{i}}{dt}(T) = 0$$

Therefore.

$$0 = \frac{d}{dr}J(\alpha_r)\Big|r = 0 = \sum_{i=1}^{l-1} \left\| \frac{D^2 x_i}{dt^2}(t_j^-) - \frac{D^2 x_i}{dt^2}(t_j^+) \right\|^2 + \left\| \frac{D^3 x_i}{dt^3}(t_j^+) - \frac{D^3 x_i}{dt^3}(t_j^-) \right\|^2 = 0$$

which implies that

$$\frac{D^2 x_i}{dt^2}(t_j^-) = \frac{D^2 x_i}{dt^2}(t_j^+) \qquad \frac{D^3 x_i}{dt^3}(t_j^+) = \frac{D^3 x_i}{dt^3}(t_j^-)$$

Since x_i is a C^1 curve with continuous covariant derivatives up to order 3, x_i is C^3 on [0,T]. But, we have shown that x_i is the solution of a fourth order smooth ODE, which means the fourth derivative can be expressed as a smooth function of derivatives upto order 3. The k^{th} order derivative can be expressed as a smooth function of derivatives upto order k-1, and so by induction, x_i is smooth on [0,T]. Hence, statement 2 follows.

When $X_i(T)=0,\ \frac{DX_i}{\partial t}(T)=\frac{dX_i}{dt}(T).$ Now, choose $X_i\in T_{x_i}\Omega_i$ such that

$$X_i(T)=0, \text{ and } \frac{DX_i}{dt}(T)=\Pi_{T_{x_i(T)}S}\left(\frac{D^2x_i}{\partial t^2}(T)\right)$$

where $\Pi_{T_{x_i(T)}S}V$ is the orthogonal projection onto $T_{x_i(T)}S$ of $V \in T_{x_i(T)}M$. Since $X_i(T) = 0$, $\frac{dX_i}{dt}(T) = \frac{DX_i}{\partial t}(T) \in T_{x_i(T)}S$. By lemma 2.3, X_i is the variational vector field of an admissible variation. Therefore,

$$\begin{split} \frac{d}{dr}J(\alpha_r) \Big| r &= 0 = \left\langle \Pi_{T_{x_i(T)}S} \left(\frac{D^2 x_i}{\partial t^2}(T) \right), \frac{D^2 x_i}{\partial t^2}(T) \right\rangle = 0 \\ \Longrightarrow & \Pi_{T_{x_i(T)}S} \left(\frac{D^2 x_i}{\partial t^2}(T) \right) = 0 \end{split}$$

Hence statement 3 holds.

Remark 3.2: We now make some remarks on the previous result for the following scenarios:

- If M is an n-dimensional manifold and S is an m-dimensional submanifold, with m < n, then the conditions x(0) and $\frac{dx}{dt}(0)$ being fixed gives 2n independent boundary conditions, the condition x(T) being fixed gives another n independent boundary conditions, and $\frac{dx}{dt}(T) \in T_{x(T)}S$ gives yet another n-m boundary conditions. Theorem 3.2 (3) gives an additional m boundary conditions. The fourth order boundary value problem given by Theorem 3.2 (1) along with the 4n boundary conditions forms a well posed problem.
- If we were to change the endpoint boundary condition to be $\frac{dx_i}{dt}(T) = v_i$ (i.e $\frac{dx_i}{dt}(T)$ is a fixed vector), then it easily follows that conditions (1) and (2) of Theorem 3.2 still hold, but condition (3) does not hold. This still results in 4n boundary conditions.
- If we consider endpoint boundary conditions $x_i(T) \in S$ (i.e $x_i(T)$ is not fixed, but can be any element in S) and $\frac{dx_i}{dt}(T) \in T_{x_i(T)}S$, all the conditions in Theorem 3.2 still hold. But, in this case, we have 2n initial conditions, the endpoint conditions $x(T) \in S$ gives n-m conditions, and $\frac{dx}{dt}(T) \in T_{x(T)}S$ gives n-m boundary conditions. These 4n-2m conditions, along with the m conditions given by Theorem 3.2 (3) gives only 4n-m boundary conditions. We expect to find another m conditions, similar to Theorem 3.2 (3), in this case. We explore such an extension of Theorem 3.2 to this situation in a future work as we comment in Section V.

A. Example: Planar agents on an Euclidean space

We consider the case of 2 agents evolving on $M=\mathbb{R}^2$ endowed with the Euclidean Riemannian metric. At time T=0, the first agent (agent blue in Figure 1) is at $p_0^1=(0,0)$ with velocity $v_0^1=(1,0)$, and the second agent (agent green in Figure 1) is at $p_0^2=(1,0)$ with velocity $v_0^2=(0,1)$. At time T=1 the first agent is at $p_1^1=(1,1)$ with velocity $v_1^1=(0,1)$, and the second agent is at $p_1^2=(0,1)$ with velocity $v_1^2=(0,1)$. At time T=2

the first agent is at $p_2^1=(0,2)$ with velocity $v_2^1\in T_{p_2^1}S_1$, and the second agent is at $p_2^2=(1,2)$ with velocity $v_2^2\in T_{p_2^2}S_2$. Here, $S_1=\{(z^1,z^2)\in\mathbb{R}^2|\ \|(z^1,z^2)-(0.2,2)\|_2=0.2\}$, and $S_2=\{(z^1,z^2)\in\mathbb{R}^2|\ \|(z^1,z^2)-(1,1.8)\|_2=0.2\}$. Note that in this particular example, S (the submanifold in Theorem 3.2) is the disjoint union of the two circles shown in Figure 1. Here we have taken $F:(0,\infty)\to(0,\infty)$ given by $F(x)=\frac{1}{x}$, and the value of k=0. Note that at time T=1, the boundary conditions are that mentioned in Remark 3.2. In this particular case, the argument of F is the usual Euclidean distance between two points. We show in Figure 1 some simulations of the trajectories given by Theorem 3.2. Note that the trajectories exhibit the usual S-shape of cubic polynomials.

On \mathbb{R}^2 endowed with the Euclidean metric,

$$exp_{x_1}^{-1}(x_2) = x_2 - x_1.$$

We denote by $(x_i(t), y_i(t))$ the trajectory of the *i*-th agent. The conditions of Theorem 3.2 (1) translate to

$$\begin{split} x_1'''' &= \frac{x_1 - x_2}{((x_2 - x_1)^2 + ((y_2 - y_1)^2)^2)}, \\ y_1'''' &= \frac{y_1 - y_2}{((x_2 - x_1)^2 + ((y_2 - y_1)^2)^2)}, \\ x_2'''' &= \frac{x_2 - x_1}{((x_2 - x_1)^2 + ((y_2 - y_1)^2)^2)}, \\ y_2'''' &= \frac{y_2 - y_1}{((x_2 - x_1)^2 + ((y_2 - y_1)^2)^2)}. \end{split}$$

Theorem 3.2 (3) gives

$$y_1''(2) = 0, \quad x_2''(2) = 0.$$

along with the conditions

$$x_1(0) = 0,$$
 $y_1(0) = 0,$ $x_2(0) = 1,$ $y_2(0) = 0,$
 $x'_1(0) = 1,$ $y'_1(0) = 0,$ $x'_2(0) = 0,$ $y'_2(0) = 1,$
 $x_1(1) = 1,$ $y_1(1) = 1,$ $x_2(1) = 0,$ $y_2(1) = 1,$
 $x'_1(1) = 0,$ $y'_1(0) = 1,$ $x'_2(1) = 0,$ $y'_2(0) = 1,$
 $x_1(2) = 0,$ $y_1(2) = 2,$ $x_2(2) = 1,$ $y_2(2) = 2.$
 $x'_1(2) = 0,$ $y'_2(2) = 0.$

B. Example: Agents on a 2-Sphere

In this example, we consider 2 agents evolving on the 2-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ endowed with the induced Riemannian structure from \mathbb{R}^3 . We have taken fixed position and velocity conditions

$$\begin{aligned} p_0^1 &= (0.7, 0.3, 0.648), \quad v_0^1 &= (1, 0, -1.08), \\ p_1^1 &= (-0.7, 0.3, 0.648), \quad v_1^1 &= (1, 0, 1.08), \\ p_0^2 &= (0.7, 0.6, 0.3873), \quad v_0^2 &= (0, -1, 1.55), \\ p_1^2 &= (-0.7, 0.6, 0.3873), \quad v_1^2 &= (0, 1, -1.55). \end{aligned}$$

As before, we have taken $F:(0,\infty)\to(0,\infty)$ given by $F(x)=\frac{1}{x}$, and the value of k=0. Figure 2 shows the trajectories satisfying Theorem 3.2 followed by the agents. Agent 1 and 2 follow the blue and brown trajectories

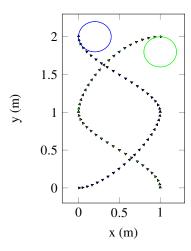


Fig. 1. Optimal Path traced by two agents evolving on \mathbb{R}^2 .

respectively in 2. For $x, y \in S^2$,

$$\begin{split} exp_x^{-1}(y) &= \cos^{-1}(\langle x,y\rangle) \frac{y - \langle x,y\rangle x}{\sqrt{1 - \langle x,y\rangle^2}}, \\ \|exp_x^{-1}(y)\| &= \cos^{-1}(\langle x,y\rangle). \end{split}$$

For the purpose of computation, we parametrize S^2 with the (θ,ϕ) coordinates as

$$x = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$$

If $(\theta(t),\phi(t))$ is the coordinate representation of the curve x(t), it can be shown that

$$\begin{split} \frac{D^4x}{dt^4} &= \left(\theta'''' + (5\sin 2\theta)\theta'^2\phi'^2 + (1 - 7\cos^2\theta)\theta''\phi'^2 \right. \\ &\quad + (5 - 17\cos^2\theta)\theta'\phi'\phi'' - (3\sin\theta\cos\theta)\phi''^2 \\ &\quad - (2\sin 2\theta)\phi'\phi''' + (\sin\theta\cos^3\theta)\phi'^4\right)\frac{\partial}{\partial\theta} + \\ &\quad \left(\phi'''' - 7\theta'\theta''\phi' - 5\theta'^2\phi'' + (4\cot\theta)\theta'''\phi' \right. \\ &\quad + (6\cot\theta)\theta''\phi'' + (4\cot\theta)\phi'''\theta' \\ &\quad + (\sin 2\theta - \cot\theta(5\cos^2\theta - 1))\theta'\phi'^3 \\ &\quad - (6\cos^2\theta)\phi'^2\phi'' - (2\cot\theta)\phi'\theta'^3\right)\frac{\partial}{\partial\phi}, \\ R\left(\frac{D^2x}{dt^2}, \frac{dx}{dt}\right)\frac{dx}{dt} = 0. \end{split}$$

If $(\theta_i(t), \phi_i(t))$ denote the coordinate representation of the trajectory of the *i*-th agent, then 3.2 in local coordinates gives a 2 point boundary value problem in the variables (θ_i, ϕ_i) , which can be solved to obtain the optimal trajectories.

IV. VARIATIONAL COLLISION AVOIDANCE PROBLEM ON LIE GROUPS

On a Lie group G, we can give an explicit notion of geodesic exponential.

Let e be the identity element on G, and \mathfrak{g} denotes the Lie algebra associated with G, that is, $\mathfrak{g} = T_e G$. A vector field $X \in \mathfrak{X}(G)$ is called *left-invariant* if $T_h L_g(X(h)) =$

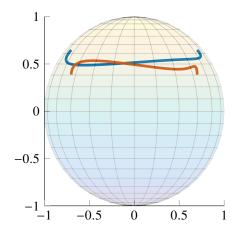


Fig. 2. Optimal Path traced by two agents evolving on S^2 .

 $X(L_g(h))=X(gh)$ for all $g,h\in G$. In particular for h=e this means that a vector field X is left-invariant if $\dot{g}=X(g)=T_eL_g\zeta$ for $\zeta=X(e)\in \mathfrak{g}$. Note that if X is a left invariant vector field, then $\zeta=X(e)=T_gL_{g^{-1}}\dot{g}$.

Given $\zeta \in \mathfrak{g}$, we denote by X_{ζ} the left-invariant vector field defined as $X_{\zeta}(g) = T_e L_g(\zeta), \forall g \in G$. Let ϕ_t^{ζ} denote the flow of X_{ζ} . Let U be some neighborhood of $0 \in \mathfrak{g}$, the *exponential function* $\exp : U \subset \mathfrak{g} \to G$ is defined as $\exp(\zeta) = \phi_1^{\zeta}(e)$.

If G is equipped with a Riemannian structure, it is not in general true that the 2 notions of an exponential coincide. Geodesics on a Lie group are not generally the flow of a left-invariant vector field on G. A connection on G for which geodesics are flows of left-invariant vector fields is called a Cartan connection.

Any inner product on g induces a left invariant Riemannian metric on G (see [7] p. 271). The restriction of such a Riemannian metric to \mathfrak{g} will be denoted by $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. A Riemannian metric which is both left and right invariant is called bi-invariant. Unfortunately, the Riemannian metric being left-invariant does not guarantee the Levi-Civita connection it induces will be a Cartan connection. It can be shown that the Levi-Civita connection induced by a biinvariant Riemannian metric on G is also a Cartan connection (see [19] p. 156). In fact, on a connected Lie group, the converse of the statement is also true, i.e. if the Levi-Civita connection induced by a left-invariant Riemannian metric is also a Cartan connection, then the Riemannian metric is bi-invariant. In the following discussion, we exclusively deal with the case where G is endowed with a left-invariant Riemannian metric such that the Levi-civita connection it induces is also a Cartan connection.

Therefore, assume that G endowed with a left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$, with $\mathbb{I}: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the corresponding inner product on the Lie algebra \mathfrak{g} , a positive-definite symmetric bilinear form in \mathfrak{g} . The inner product \mathbb{I} defines the metric $\langle \cdot, \cdot \rangle$ completely via left translation (see for instance [7] pp. 273).

Let $x:I\subset\mathbb{R}\to G$ be a smooth curve on G. The body velocity of x is the curve $v:I\subset\mathbb{R}\to\mathfrak{g}$ defined

by
$$v(t) = T_{x(t)} L_{x(t)^{-1}} \left(\frac{dx}{dt}(t) \right)$$
.

Let $\{e_1,\ldots,e_n\}$ be a basis of \mathfrak{g} . We denote by u_L the left-invariant vector field associated with $u\in\mathfrak{g}$. The body velocity of x on the given basis is described by $v=\sum_{i=1}^n v_ie_i$, where v_1,\ldots,v_n are the so-called pseudo-velocities of the curve x with respect to the given basis. The velocity vector can be written in terms of the pseudo-velocities as follows

$$\frac{dx}{dt}(t) = T_e L_{x(t)} v(t) = \sum_{i=1}^{n} v_i(t)(e_i)_L(x(t)).$$
 (3)

When the body velocity is interpreted as a control on the Lie algebra, equations (3) give rise to the so called left-invariant control systems discussed in [9]. Therefore our analysis also includes this class of kinematic control systems.

To write the equations determining necessary conditions for existence of extrema in the variational collision avoidance problem, we must use the following formulas (see [1], Section 7 for more details).

$$\begin{split} \widetilde{\nabla}_{v}v &= \sum_{i,j=1}^{n} v_{i}v_{j}\widetilde{\nabla}_{e_{j}}e_{i}, \quad \exp_{x}^{-1}y = T_{e}L_{x}(exp_{e}^{-1}(x^{-1}y)), \\ \frac{D^{2}x}{dt^{2}} &= T_{e}L_{x}\Big(v' + \widetilde{\nabla}_{v}v\Big), \\ \frac{D^{3}x}{dt^{3}} &= T_{e}L_{x}\Big(v'' + \widetilde{\nabla}_{v'}v + 2\widetilde{\nabla}_{v}v' + \widetilde{\nabla}_{v}\widetilde{\nabla}_{v}v\Big), \\ \frac{D^{4}x}{dt^{4}} &= T_{e}L_{x}\Big(v''' + \widetilde{\nabla}_{v''}v + 3\widetilde{\nabla}_{v'}v' + 3\widetilde{\nabla}_{v}v'' + \widetilde{\nabla}_{v'}\widetilde{\nabla}_{v}v \\ &\quad + 2\widetilde{\nabla}_{v}\widetilde{\nabla}_{v'}v + 3\widetilde{\nabla}_{v}v' + \widetilde{\nabla}_{v}v\Big), \\ R\Big(\frac{D^{2}x}{dt^{2}}, \frac{dx}{dt}\Big)\frac{dx}{dt} &= T_{e}L_{x}\Big(\widetilde{R}(v', v)v + \widetilde{R}(\widetilde{\nabla}_{v}v, v)v\Big), \end{split}$$

where \widetilde{R} denotes the curvature tensor associated with $\widetilde{\nabla}$. Thus, as a consequence of Theorem 3.2 (1) we have the following result

Corollary 4.1: The equations giving necessary conditions for the existence of minimizers in the variational collision avoidance problem where agents are defined on a Lie group G are

$$\begin{split} 0 = & v_i''' + \widetilde{\nabla}_{v_i''} v_i + 3\widetilde{\nabla}_{v_i'} v_i' + 3\widetilde{\nabla}_{v_i} v_i'' + \widetilde{\nabla}_{v_i'} \widetilde{\nabla}_{v_i} v_i \\ & + 2\widetilde{\nabla}_{v_i} \widetilde{\nabla}_{v_i'} v_i + 3\widetilde{\nabla}_{v_i} v_i' + \widetilde{\nabla}_{v_i} v_i + \widetilde{R}(v_i', v_i) v_i \\ & + \widetilde{R}(\widetilde{\nabla}_{v_i} v_i, v_i) v_i - k \Big(v_i' + \widetilde{\nabla}_{v_i} v_i \Big) \\ & - \Big(\sum_{j=1, i \neq j}^n F' (\| \exp_e^{-1}(x_j^{-1} x_i) \|^2) \exp_e^{-1}(x_i^{-1} x_j) \Big). \end{split}$$

A. Example: Collision avoidance of rigid bodies on SO(3)

We clarify the notion of collision avoidance on SO(3) as follows: The idea is to ensure that n agents, each evolving on the manifold SO(3), do not attain the same orientation at any given instant of time. In this particular example, we consider a collision avoidance problem for the motion of three rigid bodies in the space where the configuration space of each agent is the Lie group G = SO(3).

Denote by $t \to R_i(t) \in SO(3)$ a curve for the i^{th} agent, i=1,2,3 and let I be the (3×3) -identity matrix. The columns of the matrix $R_i(t)$ represent the directions of the principal axis of the i^{th} body at time t with respect to some reference system. Let $\mathfrak{so}(3) = T_I SO(3)$ be the Lie algebra of the Lie group SO(3), that is, the set of 3×3 skew-symmetric matrices,

$$\mathfrak{so}(3) = \{\dot{R}(0) | R(t) \in SO(3), R(0) = I\}$$
$$= \{\hat{\Omega} \in \mathbb{R}^{3 \times 3} | \hat{\Omega} \text{ is skew-symmetric} \}.$$

It is well know that (see [2] for instance) $\mathfrak{so}(3) \simeq \mathbb{R}^3$ using the isomorphism

$$\hat{\Omega}_i(t) = \left(\begin{array}{ccc} 0 & -\Omega_i^3(t) & \Omega_i^2(t) \\ \Omega_i^3(t) & 0 & -\Omega_i^1(t) \\ -\Omega_i^2(t) & \Omega_i^1(t) & 0 \end{array} \right) \simeq (\Omega_i^1, \Omega_i^2, \Omega_i^3)$$

where $\Omega_i = (\Omega_i^1, \Omega_i^2, \Omega_i^3) \in \mathbb{R}^3$.

We consider the basis of $\mathfrak{so}(3)$ represented by the canonical basis of \mathbb{R}^3 denoted by $\{e_1, e_2, e_3\}$ and endow SO(3) with the left-invariant metric defined by the inner product

$$\mathbb{I} = \sum_{k=1}^{3} J_k^i e^k \otimes e^k \text{ where } J_k^i \text{ are the elements of the diagonal}$$
matrix defining the kinematics structure of the rigid body,
the inertia moments with $k=1,2,3$ for each $i=1,2,3$

the inertia moments, with k = 1, 2, 3 for each i = 1, 2, 3 and $\{e^1, e^2, e^3\}$ the dual basis of $\{e_1, e_2, e_3\}$.

The Levi-Civita connection ∇ induced by $\langle \cdot, \cdot \rangle$ determined by the inner product is completely determined by its restriction to $\mathfrak{so}(3)$ and given by (see for instance [7] p. 281)

$$\widetilde{\nabla}_v z = \frac{1}{2} v \times z + \frac{1}{2} \left(\mathbb{I}^{-1} (v \times (\mathbb{I}z) + w \times (\mathbb{I}v) \right)$$

where $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $z = (z_1, z_2, z_3) \in \mathbb{R}^3$.

For simplicity in the exposition we consider the case of a symmetric rigid body with $J_k^i=1$ with $i,k=\{1,2,3\}$. Then the formula above for the Levi-Civita connection reduces to $\widetilde{\nabla}_v z = \frac{1}{2}(v \times z)$. By using equation (2), the restriction of the curvature tensor to $\mathfrak{so}(3)$ is defined by $\widetilde{R}(v,z)w = -\frac{1}{4}(v \times z) \times w$ where $v,z,w \in \mathbb{R}^3$. For the collision avoidance problem we consider F:

For the collision avoidance problem we consider $F: \mathbb{R}^+ \to \mathbb{R}^+$ as $F(x) = \frac{1}{x}$ where the argument of F is the usual Euclidean distance between two elements on $\mathfrak{so}(3) \simeq \mathbb{R}^3$, that is, $F(\|\exp_Q^{-1}R\|^2) = \frac{1}{\|\exp_Q^{-1}R\|^2}$ with $R,Q \in SO(3)$ and where $\exp: \mathfrak{so}(3) \to SO(3)$ representing the matrix exponential on SO(3).

By Corollary 4.1 the necessary conditions for the normal extremals for the variational collision avoidance problem are determined by the solutions of the equation

$$\begin{split} v_i''' = & v_i'' \times v_i + k v_i' + \frac{3}{4} (v_i' \times v_i) \times v_i + \frac{3}{2} v_i' \times v_i \\ & + \sum_{j=1, i \neq j}^3 \frac{\exp^{-1}(R_j^{-1} R_i)}{\|\exp^{-1}(R_i^{-1} R_j)\|^4}. \end{split}$$

together with the equation $\dot{R}_i = R_i v_i$, and the boundary conditions $R_0 = R_0^i$, $R_i(T) = R_T^i$, $v_i(0) = v_0^i$ and $v_i(T) = v_T^i$ for i = 1, 2, 3.

V. CONCLUSIONS AND FUTURE WORK

We discussed the problem of collision avoidance of multiagent systems on a complete Riemannian manifold and derived, from the point of view of calculus of variations, necessary conditions for the existence of extrema in the problem. We have shown how the main result can be applied for the particular case of a compact Lie group, a Euclidean space, and a non Lie group example, i.e., the 2-sphere, S^2 .

The study of these necessary conditions on symmetric spaces and reduction theories for variational problems has attracted considerable interest and has been carried out systematically by several authors. In future work we intend to extend the main results presented in this paper to this setting and explore numerical results for three dimensional agents as well as explore extensions of Theorem 3.2 as we commented in Remark 3.2.

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