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Diego Gutiérrez-Oribio[1], Yury Orlov[2], Ioannis Stefanou[1] and Franck Plestan[3]

***Abstract*— Robust output tracking is addressed in this paper for a diffusion equation with Neumann boundary conditions and anti-collocated boundary input and output. The desired reference tracking is solved using the well-known flatness and Lyapunov approaches. The reference profile is obtained by solving the motion planning problem for the nominal plant. To robustify the closed-loop system in the presence of the disturbances and uncertainties, it is then augmented with PI feedback plus a discontinuous component responsible for rejecting matched disturbances with *a priori* known magnitude bounds. Such control law only requires the information of the system at the same boundary as the control input is located. The resulting dynamic controller globally exponentially stabilizes the error dynamics while also attenuating the influence of Lipschitz-in-time external disturbances and parameter uncertainties. The proposed controller relies on a discontinuous term that however passes through an integrator, thereby minimizing the chattering effect in the plant dynamics. The performance of the closed-loop system, thus designed, is illustrated in simulations under different kinds of reference trajectories in the presence of external disturbances and parameter uncertainties.**

I. INTRODUCTION

The diffusion equation is a first-order parabolic partial differential equation used to describe the diffusion and other diffusion processes such as chemical reactions, population growth, market price fluctuations, and fluids flow to name a few.

In order to perform a tracking over the diffusion equation, a reference profile of the solution should be obtained. The motion planning presented in [1] uses a flat output approach capable of parametrising the state dynamics using it as a reference. Such approach provides an open-loop control able to track the desired reference but only for the nominal system with no uncertainties and disturbances and starting at some specific initial condition.

In order to robustify such a tracking control, one may invoke a feedback control. Examples of boundary tracking control of the diffusion equation utilize output feedback [2], [3] and backstepping designs [4], [5], [6], [7], [8] as well as sliding-modes techniques [9], [10].

Clearly, more realistic models should count for uncertainties and disturbances. A classic approach to deal with constant disturbances in the finite dimension setting is the integral action [11]. Dealing with a wider class of disturbances, sliding-mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and unmodelled dynamics, even for infinite dimension systems (see, *e.g.*, [12] and a very recent monograph [13]).

The design of robust controllers in the diffusion equation has been addressed in [10] using sliding-modes and using H_∞ control in [14]. For the case of the tracking task, in [9] a control has been designed to compensate Lipschitz-in-time disturbances using sliding-modes, but using distributed control, and in [2], [3] robust boundary controls have been designed to compensate bounded disturbances and requiring to design two external systems (disturbance estimator and servo system) to fulfil the task.

More works have been presented addressing the design of boundary controls for the diffusion system under perturbations (see [15] and [16] where sliding-mode controllers were designed) and parameter uncertainties (see [16] again, and [17] where an adaptive control has been presented). Nevertheless, these last works usually require the measurement of the state throughout all the space domain and they need to solve an auxiliary PDE system in order to design the control. Therefore, the design of simpler boundary controllers capable of the boundary tracking of the diffusion equation under disturbances and parameter uncertainties calls for further investigation.

In this paper, a simple control strategy is proposed to achieve global exponential tracking of the diffusion dynamics. Using boundary state feedback only, a PI control, coupled to a discontinuous integral term, is designed. Such a controller is capable of compensating model uncertainties and Lipschitz-in-time disturbances using a continuous control signal. This strategy can be viewed as the merge of the classical integral action and the discontinuous functions commonly used in sliding-mode control applications. Using a well-known strategy for motion planning, reference profiles are obtained for the nominal diffusion equation depending on the choice of the flat output reference. Supporting simulations illustrate the closed-loop performance for different kinds of references in the presence of unbounded but Lipschitz-in-time disturbances and parameter uncertainties.

The outline of this work is as follows. The underlying diffusion model and the control objective are introduced in Section II. Using the flat approach, the trajectory generation and open-loop tracking control are described in Section III. The feedback control design for the nominal and disturbed error systems is given in Section IV. The reliability of the

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proposed control strategy is supported in the simulation study of Section V. Finally, some concluding remarks are collected in Section VI.

Notation: The function $\lceil\cdot\rfloor^{\gamma} := |\cdot|^{\gamma}\text{sign}(\cdot)$ is determined for any $\gamma \in \mathbb{R}_{\geq 0}$. Given a differentiable function $r(t)$, notation $r^{(i)}(t)$ with $i \in \mathbb{Z}_{\geq 0}$ stands for the i-th time derivative of $r(t)$. The Sobolev space of absolutely continuous scalar functions $u(x)$ on (a,b) with square integrable derivatives $u^{(i)}(x),\ i = 1, l \ldots, l$ and the norm

$$||u(\cdot)||_{H^l(a,b)} = \sqrt{\int_a^b \sum_{i=0}^{l} [u^{(i)}(x)]^2\, dx},$$

is typically denoted by $H^l(a,b)$ with $a \leq b$ and $l = \{0, 1, 2, ...\}$. For ease of reference, the nomenclature $||u(\cdot)||_{H^0(a,b)} = ||u(\cdot)||_{L_2(a,b)} = ||u(\cdot)||$ is used throughout. The spatial derivatives are denoted by $u_x = \partial u/\partial x$ and $u_{xx} = \partial^2 u/\partial x^2$.

For later use, well-known inequalities are recalled.

Young's Inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Cauchy-Schwarz Inequality:

$$\int_a^b f(x)g(x)\, dx \leq ||f(x)||\, ||g(x)||\,.$$

Poincare's Inequality: Let $u(x) \in H^1(0,1)$. Then , the following inequality holds

$$\int_0^1 u^2(x)\, dx \leq 2u^2(i) + 2\int_0^1 u_x^2(x)\, dx, \quad i = 0, 1.$$

For $u(x) \in H^1(0,D)$ and $D \in \mathbb{R}_{>0}$, the latter inequality is specified to

$$\int_0^D u^2(x)\, dx \leq 2Du^2(i) + 2D^2\int_0^D u_x^2(x)\, dx, \quad i = 0, D.$$

II. PROBLEM STATEMENT

Consider the following diffusion equation with Neumann boundary conditions (BC)

$$\begin{aligned} u_t(x,t) &= du_{xx}(x,t), \\ u_x(0,t) &= 0, \\ u_x(D,t) &= q + \varphi(t), \end{aligned} \tag{1}$$

where $u(x,t)$ is the state vector evolving in the space $H^1(0,D)$, $x \in [0,D]$ is the space variable, $t \geq 0$ is the time variable, d is the thermal diffusivity, q is the boundary control input and the function $\varphi(t)$ is a disturbance term supposed to satisfy

$$|\dot{\varphi}| \leq L \tag{2}$$

with an *a priori* known constant L. Furthermore, the positive parameters d, D are uncertain, but bounded as

$$0 < d_m \leq d \leq d_M, \quad 0 < D_m \leq D \leq D_M, \tag{3}$$

by some known constant d_m, d_M, D_m, D_M.

It is well-known (see, *e.g.*, [13, Chapter 3]) that for arbitrary initial conditions $u(x,0)$ of class $H^1(0,D)$, there exists a mild solution of the open-loop boundary value problem (BVP) (1). Throughout the paper, only mild solutions of the corresponding BVP are in play.

The objective of this work is to design a control input q capable of driving the output

$$y(t) = u(0,t) \tag{4}$$

of the underlying BVP (1) to a desired reference $r(t)$, despite the presence of uncertainties and/or disturbances.

III. MOTION PLANNING

Following [1],[6, Chapter 12], the reference trajectory generation for the diffusion equation (1) becomes available through its flat output (4). The trajectory generation is further performed for the unperturbed system with $\varphi(t) \equiv 0$ and the perfect knowledge of the system parameters d, D.

The state trajectory to follow is represented in the form

$$\bar{u}(x,t) = \sum_{i=0}^{\infty} a_i(t)\frac{x^i}{i!},$$

where the time-varying coefficients $a_i(t)$ have to be determined by substituting the latter sum into (1),(4) and using the desired tracking $r(t) = \bar{u}(0,t)$. Thus, the reference state trajectory is specified to

$$\bar{u}(x,t) = \sum_{i=0}^{\infty} r^{(i)}(t)\frac{x^{2i}}{d^i(2i)!}, \tag{5}$$

and the nominal input signal from the BC at $x = D$ is defined as

$$\bar{q} = \sum_{i=1}^{\infty} r^{(i)}(t)\frac{D^{2i-1}}{d^i(2i-1)!}. \tag{6}$$

In order to guarantee that $u(x,t) \to \bar{u}(x,t)$ and $y(t) \to r(t)$, the convergence of (5) needs to be studied. The next theorem states which conditions should be imposed on the to-be-tracked reference in order to achieve this.

Theorem 1: The series (5) is absolutely convergent provided that the output reference $r(t)$ fulfils

$$\sup_{t\geq 0}\left|r^{(i+1)}(t)\right| \leq \frac{2d}{D^2}(i+1)\sup_{t\geq 0}\left|r^{(i)}(t)\right| \quad \forall \quad i \in \mathbb{Z}_{\geq 0}. \tag{7}$$

Proof: In order to check if under conditions of the theorem, (5) is a convergent series, the ratio test is performed with $S_p = r^{(p)}(t)\frac{x^{2p}}{d^p(2p)!}$. Then

$$\begin{aligned} \lim_{p\to\infty}\left|\frac{S_{p+1}}{S_p}\right| &= \lim_{p\to\infty}\left|\frac{r^{(p+1)}(t)}{r^{(p)}(t)}\right|\frac{x^{2p+2}}{x^{2p}}\frac{d^p}{d^{p+1}}\frac{(2p)!}{(2p+2)!} \\ &= \lim_{p\to\infty}\frac{x^2}{d}\left|\frac{r^{(p+1)}(t)}{r^{(p)}(t)}\right|\frac{1}{(2p+1)(2p+2)} < 1 \end{aligned}$$

for all $x \in [0,D]$ under condition (7). Theorem 1 is thus proved. ■

References $r(t)$, which are usually adopted in motion planning [1], [7], [8], are the Gevrey class defined as follows.

Definition 1: [1] A smooth function $r(t)$ is Gevrey of order α if exist $M, R > 0$ such that

$$\sup_{t\geq 0}\left|r^{(i)}(t)\right| \leq M\frac{i!^{\alpha}}{R^i},$$

for all $i \in \mathbb{Z}_{\geq 0}$.

In the works cited above, the condition to guarantee the series convergence requires $r(t)$ to be Gevrey of order $\alpha < 2$. The next lemma links references $r(t)$ satisfying condition (7) to a certain kind of the Gevrey functions.

Lemma 1: Any smooth function $r(t)$, fulfilling (7), is Gevrey of order $\alpha = 1$ such that Definition 1 holds for $r(t)$ with $R = \frac{D^2}{2d}$ and $M > 0$.

Proof: From the Definition 1, one concludes that

$$\sup_{t\geq 0}\left|r^{(i+1)}(t)\right| \leq M\frac{(i+1)!^{\alpha}}{R^{i+1}} \leq \frac{M}{R}(i+1)^{\alpha}\frac{i!^{\alpha}}{R^i},$$

thereby deriving the inequality

$$\sup_{t\geq 0}\left|r^{(i+1)}(t)\right| \leq \frac{(i+1)^{\alpha}}{R}\sup_{t\geq 0}\left|r^{(i)}(t)\right|.$$

Since (7) follows from the latter inequality with the selection of α, R, M under the lemma conditions, the proof is thus completed. ■

Although the principle reference, used in the afore-cited works, is the so-called "bump function" (see [1]), its proposed counterpart (7) allows one to exemplify more admissible references among analytical functions such as:

F.1 $\boldsymbol{r(t) = A}$ for any $A \in \Re$.
F.2 $\boldsymbol{r(t) = At}$, $t \in [0, t_f]$ for any $A \in \Re$, $t_f \geq \frac{D^2}{2d}$.
F.3 $\boldsymbol{r(t) = A\sin(\omega t)}$, $\omega \in \Re_{>0}$ for any $A \in \Re$, $\omega \leq \frac{2d}{D^2}$.
F.4 $\boldsymbol{r(t) = Ae^{\beta t}}$, $\beta \in \Re$ for any $A \in \Re$, $|\beta| \leq \frac{2d}{D^2}$.

Remark 1: The output (4) matches the reference trajectory $r(t)$ only if the state trajectory to follow satisfies the same initial condition as that of the plant, *i.e.*, if $u(x, 0) = \bar{u}(x, 0)$. Furthermore, the nominal control (6) is not capable of compensating any non-trivial disturbance $\varphi(t) \neq 0$. Next, the tracking problem of interest is addressed for the arbitrarily initialized state trajectory to follow in the presence of matched disturbances.

IV. TRACKING CONTROL DESIGN

Setting the state deviation

$$\tilde{u}(x, t) = u(x, t) - \bar{u}(x, t), \tag{8}$$

from the reference trajectory $\bar{u}(x, t)$, given by (5), the error dynamics (8) are then governed by

$$\begin{aligned} &\tilde{u}_t(x, t) = d\tilde{u}_{xx}(x, t), \\ &\tilde{u}_x(0, t) = 0, \\ &\tilde{u}_x(D, t) = q - \bar{q} + \varphi(t) = \tilde{q} + \varphi(t), \end{aligned} \tag{9}$$

where the control input

$$q = \tilde{q} + \bar{q} \tag{10}$$

is pre-composed in terms of the reference input $\bar{q}$, determined by (6), and the virtual component $\tilde{q}$. Now the tracking objective for the nominal reference (5) is reduced to the virtual input design $\tilde{q}$ exponentially stabilizing the error dynamics (9) in the origin.

A. Disturbance-free tracking

First, the disturbance-free case $\varphi(t) \equiv 0$ is analysed. Selecting the control $\tilde{q}$ as

$$\tilde{q} = -\frac{\lambda_1}{D_n}\tilde{u}(D, t), \tag{11}$$

where λ_1 is a gain to be tuned and D_n the nominal value of D. The next result is in order.

Theorem 2: Let $\varphi(t) \equiv 0$ and let the control parameters in (11) be such that

$$\lambda_1 > \frac{D_n}{D_m}. \tag{12}$$

Then the error dynamics (9), driven by (11), are globally exponentially stable. ■

Proof: Consider the positive definite Lyapunov functional candidate

$$V = \frac{1}{2}\left|\left|\tilde{u}(x, t)\right|\right|^2.$$

Its derivative along the system (9) reads as

$$\begin{aligned} \dot{V} &= \int_0^D \tilde{u}(x, t)\tilde{u}_t(x, t)\,dx \\ &= d\int_0^D \tilde{u}(x, t)\tilde{u}_{xx}(x, t)\,dx. \end{aligned}$$

Applying the integration by parts and then employing the BC and Poincare's inequality, it follows that

$$\begin{aligned} \dot{V} &= -d\int_0^D \tilde{u}_x^2(x, t)\,dx + d\left[\tilde{u}(x, t)\tilde{u}_x(x, t)\right]_0^D \\ &\leq -\frac{d}{2D^2}\left|\left|\tilde{u}(x, t)\right|\right|^2 + \frac{d}{D}\tilde{u}^2(D, t) + d\tilde{u}(D, t)\tilde{q}. \end{aligned}$$

Now, taking into account the control law (11), coupled to (12), the Lyapunov derivative is further estimated as

$$\begin{aligned} \dot{V} &\leq -\frac{d}{2D^2}\left|\left|\tilde{u}(x, t)\right|\right|^2 - \left(\frac{\lambda_1}{D_n} - \frac{1}{D}\right)d\tilde{u}^2(D, t) \\ &\leq -\frac{d}{2D^2}\left|\left|\tilde{u}(x, t)\right|\right|^2 \\ &\leq -\frac{d}{D^2}V, \end{aligned}$$

that guarantees the exponential decay of the system dynamics (9). Since the Lyapunov functional is radially unbounded, (9) is concluded to be globally exponentially stable. ■

B. Tracking under disturbances

In the presence of the external disturbance $\varphi(t) \neq 0$, the proposed control $\tilde{q}$ in (11) is no longer capable of stabilizing the error dynamics. To robustify the control law in the disturbance-corrupted case, it is modified to

$$\begin{aligned} \tilde{q} &= -\frac{\lambda_1}{D_n}\tilde{u}(D, t) + \nu, \\ \dot{\nu} &= -\lambda_2\tilde{u}(D, t) - \lambda_3\left\lceil\tilde{u}(D, t)\right\rfloor^0, \end{aligned} \tag{13}$$

where $\lambda_1, \lambda_2, \lambda_3$ are gains to be tuned and D_n is the nominal value of D.

The proposed feedback law (13) is composed by a PI control and a discontinuous term passing through an integrator. It generates a continuous control signal despite having a discontinuous (multi-valued) right-hand side in the manifold $\tilde{u}(D, \cdot) = 0$. The precise meaning of the solutions of the distributed parameter system (9) driven by this discontinuous controller, are viewed in the Filippov sense [18]. Extension of the Filippov concept towards the infinite-dimensional setting may be found in [13]. The present paper focuses on the tracking synthesis whereas the well-posedness analysis of the closed-loop system (9),(13) is similar to that of [9] and it remains beyond the scope of the paper. Thus, for the closed-loop system in question, it is assumed that it possesses a Filippov solution.

The closed-loop system (9), driven by (13), reads as

$$\begin{aligned} \tilde{u}_t(x,t) &= d\tilde{u}_{xx}(x,t), \\ \tilde{u}_x(0,t) &= 0, \\ \tilde{u}_x(D,t) &= -\frac{\lambda_1}{D_n}\tilde{u}(D,t) + \delta, \\ \dot{\delta} &= -\lambda_2 \tilde{u}(D,t) - \lambda_3 \lceil \tilde{u}(D,t) \rfloor^0 + \dot{\varphi}(t), \end{aligned} \tag{14}$$

with $\delta = \nu + \varphi$, being substituted into (13) for $\nu = \delta - \varphi$ for deriving δ-dynamics (14). The following result is then in force.

Theorem 3: Let the error dynamics (9) be driven by the feedback $\tilde{q}$, governed by (13) and tuned in accordance with

$$\lambda_1 > \frac{D_n}{D_m}, \quad \lambda_2 > d_M, \quad \lambda_3 > L. \tag{15}$$

Then the closed-loop error dynamics (14) are globally exponentially stable despite the presence of any boundary disturbance φ of class (2).

Proof: Consider the positive definite Lyapunov functional candidate

$$V = \frac{1}{2}||\tilde{u}(x,t)||^2 + \frac{1}{2}\delta^2, \tag{16}$$

and employing the magnitude bound (2) for $\dot{\varphi}$, compute its time derivative along the error dynamics (14), thus obtaining

$$\begin{aligned} \dot{V} \le & -\frac{d}{2D^2}||\tilde{u}(x,t)||^2 - \left(\frac{\lambda_1}{D_n} - \frac{1}{D}\right) d\tilde{u}^2(D,t) \\ & + d\tilde{u}(D,t)\delta + (-\lambda_2\tilde{u}(D,t) - \lambda_3 \lceil \tilde{u}(D,t) \rfloor^0 + \dot{\varphi})\delta \\ \le & -\frac{d}{2D^2}||\tilde{u}(x,t)||^2 - \left(\frac{\lambda_1}{D_n} - \frac{1}{D}\right) d\tilde{u}^2(D,t) \\ & + (-\lambda_2 + d - \lambda_3 |\tilde{u}(D,t)|^{-1} + L \lceil \tilde{u}(D,t) \rfloor^{-1})\tilde{u}(D,t)\delta. \end{aligned}$$

By applying Young's inequality and the gains selection (15), it follows that

$$\begin{aligned} \dot{V} \le & -\frac{d}{2D^2}||\tilde{u}(x,t)||^2 - \left(\frac{\lambda_1}{D_n} - \frac{1}{D}\right) d\tilde{u}^2(D,t) \\ & - (\lambda_2 - d + \lambda_3 |\tilde{u}(D,t)|^{-1} - L \lceil \tilde{u}(D,t) \rfloor^{-1})\frac{1}{2}\delta^2 \\ & - (\lambda_2 - d + \lambda_3 |\tilde{u}(D,t)|^{-1} - L \lceil \tilde{u}(D,t) \rfloor^{-1})\frac{1}{2}\tilde{u}^2(D,t) \\ \le & -\frac{d}{2D^2}||\tilde{u}(x,t)||^2 \\ & - (\lambda_2 - d + \lambda_3 |\tilde{u}(D,t)|^{-1} - L \lceil \tilde{u}(D,t) \rfloor^{-1})\frac{1}{2}\delta^2 \\ \le & -\alpha V \end{aligned}$$

where $\alpha = \min\left\{\frac{d}{D^2}, \lambda_2 - d + \lambda_3 - L\right\}$. The latter inequality ensures the exponential stability of (14). Since the Lyapunov functional is radially unbounded, the result holds globally. ■

Remark 2: Due to the exponential decay of the Lyapunov functional (16) and by virtue of $\delta = \nu + \varphi$, the virtual input ν approaches the negative disturbance value, *i.e.*, $\nu(t) \to -\varphi(t)$ as $t \to \infty$.

Remark 3: The proposed feedback controls (11) and (13) require only the boundary state information $u(D,t)$. Thus, the boundary output feedback is available to perform the tracking task, minimizing the needed measurements of the system.

V. SIMULATIONS

The proposed control strategy (6),(10),(13) has been implemented in the system (1) using Matlab Simulink with Euler's integration method of fixed step and a sampling time equal to 50 [ms]. The diffusion equation was implemented using $d = 0.05$, $D = 10$ and $\varphi(t) = 0.01t + 2\sin(t)$ and the finite-differences approximation technique, discretizing the spatial domain $x \in [0, D]$ into 51 ordinary differential equations (ODE). The initial condition was set in $u(x,0) = 0$.

The tracking reference $r(t)$ and the corresponding nominal reference $\bar{u}(x,t)$ and control $\bar{q}$, are defined depending on the simulation time and expressions (5)-(6):

1) $\boldsymbol{t \in [0, 1 \times 10^4)}$ **[s]:**
   $r(t) = At$, $A = 1 \times 10^{-3}$,
   $\bar{u}(x,t) = At + A\frac{x^2}{2d}$, $\bar{q} = A\frac{D}{d}$.
2) $\boldsymbol{t \in [1 \times 10^4, 2 \times 10^4)}$ **[s]:**
   $r(t) = 20$, $\bar{u}(x,t) = 20$, $\bar{q} = 0$.
3) $\boldsymbol{t \in [2 \times 10^4, 3 \times 10^4)}$ **[s]:**
   $r(t) = Ae^{[-\beta(t-2\times10^4)]}$, $A = 20$, $\beta = 1 \times 10^{-3}$
   $\bar{u}(x,t) = Ae^{[-\beta(t-2\times10^4)]}\cos\left(x\sqrt{\frac{\beta}{d}}\right)$,
   $\bar{q} = -A\sqrt{\frac{\beta}{d}}e^{[-\beta(t-2\times10^4)]}\sin\left(D\sqrt{\frac{\beta}{d}}\right)$.
4) $\boldsymbol{t \in [3 \times 10^4, 4 \times 10^4]}$ **[s]:**
   $r(t) = A\sin(\omega t)$, $A = 10$, $\omega = 1 \times 10^{-3}$,
   $\bar{u}(x,t) = \frac{1}{2}Ae^{x\sqrt{\frac{\omega}{2d}}}\sin\left(\omega t + x\sqrt{\frac{\omega}{2d}}\right)$
   $+\frac{1}{2}Ae^{-x\sqrt{\frac{\omega}{2d}}}\sin\left(\omega t - x\sqrt{\frac{\omega}{2d}}\right)$,
   $\bar{q} = \frac{1}{2}A\sqrt{\frac{\omega}{2d}}e^{D\sqrt{\frac{\omega}{2d}}}\sin\left(\omega t + D\sqrt{\frac{\omega}{2d}}\right)$

$$
\begin{aligned}
&+\tfrac{1}{2}A\sqrt{\tfrac{\omega}{2d}}e^{D\sqrt{\frac{\omega}{2d}}}\cos\left(\omega t + D\sqrt{\tfrac{\omega}{2d}}\right)\\
&-\tfrac{1}{2}A\sqrt{\tfrac{\omega}{2d}}e^{-D\sqrt{\frac{\omega}{2d}}}\sin\left(\omega t - D\sqrt{\tfrac{\omega}{2d}}\right)\\
&-\tfrac{1}{2}A\sqrt{\tfrac{\omega}{2d}}e^{-D\sqrt{\frac{\omega}{2d}}}\cos\left(\omega t - D\sqrt{\tfrac{\omega}{2d}}\right).^{1}
\end{aligned}
$$

All of the proposed output references fulfil the condition (7), and more precisely, the conditions described in **F.1-F.4**. The respective gains of the error control $\tilde{q}$ have been selected according to (15) as $\lambda_1 = 1, \lambda_2 = 10, \lambda_3 = 2.5$ and $D_n = 9$. The results are displayed in Figs. 1-4. The solution $u(x,t)$ of the diffusion equation is performing a tracking over the space variable x. The main objective of stabilizing the output $y = u(0,t)$ over the four successive references $r(t)$ is achieved despite the presence of the unbounded but Lipschitz perturbation $\varphi(t)$ (see Fig. 2). Furthermore, the designed control strategy is able to stabilize the norm of the error despite the abrupt change between references, as seen in Fig. 3. The boundary control q shown in Fig. 4 clarifies how such disturbance compensation is performed. This is due to the presence of the discontinuous term on the control design. Nevertheless, the control signal generated is continuous throughout the tracking task.

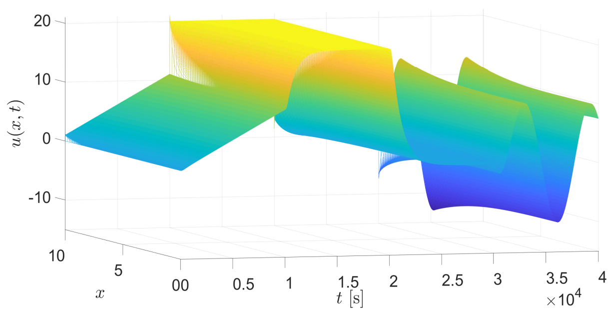

Fig. 1. Evolution of the diffusion equation state $u(t,x)$.

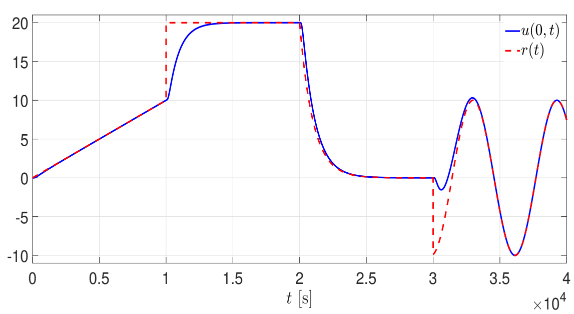

Fig. 2. Tracking of the output $y = u(0,t)$ over the successive references $r(t)$.

In order to test even more the designed control, a scenario where the motion planning was performed using the nominal values of d and D, *i.e.*, $d_n = 0.06$ and $D_n = 9$. In this case, the obtained reference $\bar{u}(x,t)$ and nominal control $\bar{q}$ introduce an error. The results are shown in Figs. 5-7. The tracking error is able again to compensate the unbounded disturbance and force the trajectories to follow this new

[1]See [6, Chapter 12] for more details on how to obtain the analytic nominal expressions for exponential and sinusoidal references.

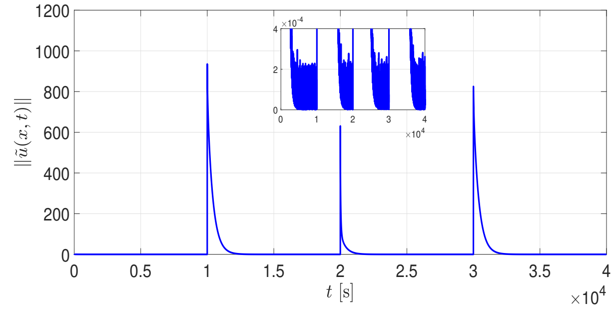

Fig. 3. Norm of the error.

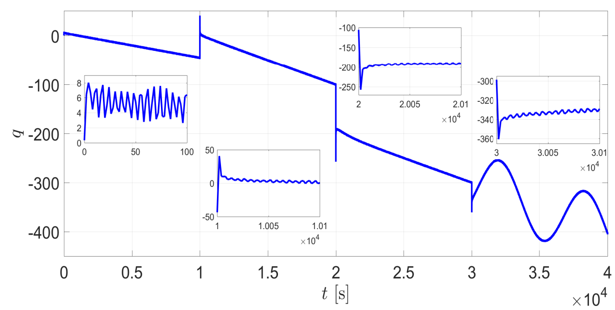

Fig. 4. Control signal q.

wrong reference, that is why the norm of the error is not zero. The magnitude of the error depends on how close the nominal values are from the real system parameters and the kind of reference to be followed. This shows how the proposed control strategy is able to get a bounded error tracking despite the wrong reference from the motion planning.

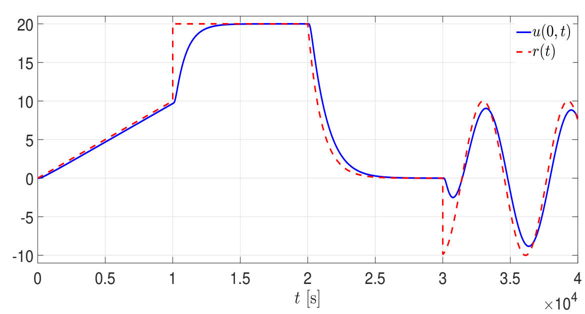

Fig. 5. Tracking of the output $y = u(0,t)$ with uncertainties in the trajectory generation.

VI. CONCLUSIONS

The diffusion equation with boundary control is analysed, and robust output tracking is developed. The proposed boundary control requires the state at the boundary only and it is composed of a PI control and an extra discontinuous term, passing through an integrator. Such a controller is typically used the sliding mode control theory and it is derived from a Lyapunov approach. The controller, thus composed, compensates Lipschitz-in-time disturbances and uncertainties in the system. It generates sliding modes in

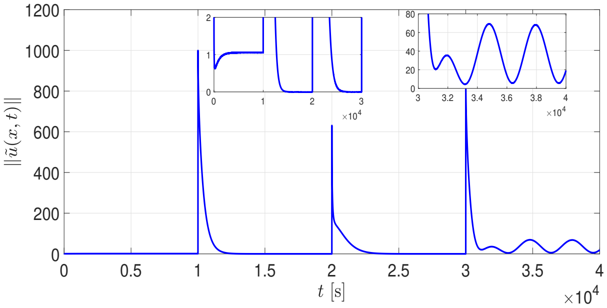

Fig. 6. Norm of the error with uncertainties in the trajectory generation.

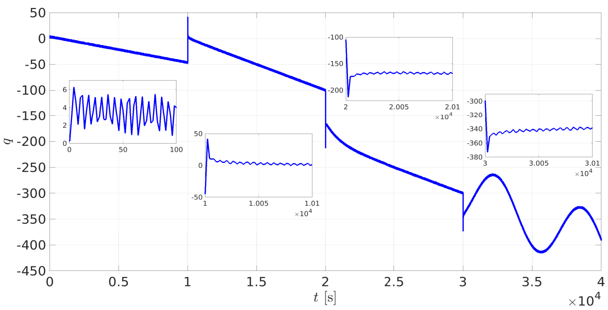

Fig. 7. Control signal q with uncertainties in the trajectory generation.

the actuator dynamics so that after passing through the integrator, a continuous control signal is applied to the underlying system, thereby diminishing the chattering effect. Capabilities of tracking diffusion dynamics along different kinds of boundary references and good robustness properties of the developed design are illustrated in the simulation study. The reference profiles are obtained for the nominal diffusion model using the flatness approach. In the case of the presence of uncertainties in the model parameters, the references obtained from the motion planning introduce a new error to the tracking. Nevertheless, simulations of this case show the boundedness of the tracking error norm. A call for further investigation to explain these results remains as future work.

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