Nearly Tight Low Stretch Spanning Trees

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Abstract

We prove that any graph G with n points has a distribution $\mathcal{T}$ over spanning trees such that for any edge (u,v) the expected stretch $E_{T\sim\mathcal{T}}[d_T(u,v)/d_G(u,v)]$ is bounded by $\tilde{O}(\log n)$. Our result is obtained via a new approach of building "highways" between portals and a new strong diameter probabilistic decomposition theorem.

1 Introduction

Let $G=(V,E)$ be a finite graph. For any subgraph $H=(V',E')$ of G let d_H be the induced shortest path metric with respect to H. In particular, for any edge $(u,v)\in E$ and any spanning tree T of G, $d_T(u,v)$ denotes the shortest path distance between u and v in T.

Given a distribution $\mathcal{T}$ over spanning trees of G, let $\mathrm{stretch}_{\mathcal{T}}(u,v)=\mathbb{E}_{T\sim\mathcal{T}}\left[\frac{d_T(u,v)}{d_G(u,v)}\right]$ and let $\mathrm{stretch}_{\mathcal{T}}(G)=\max_{(u,v)\in E}\mathrm{stretch}_{\mathcal{T}}(u,v)$. Let $\mathrm{stretch}(n)=\max_{G=(V,E)||V|=n}\inf_{\mathcal{T}}\{\mathrm{stretch}_{\mathcal{T}}(G)\}$.

Initial results were obtained by Alon, Karp, Peleg and West [2] showing that $\Omega(\log n)=\mathrm{stretch}(n)=\exp(O(\sqrt{\log n\log\log n}))$. The upper bound was significantly improved to $O((\log n)^2\log\log n)$ by Elkin, Emek, Spielman and Teng [10][1]. For the class of Series-Parallel graphs Emek and Peleg [11] obtained a bound of $\Theta(\log n)$. The main result of this paper is a new upper bound on $\mathrm{stretch}(n)$ that is tight up to polylogarithmic factors[2].

Theorem 1.

$$\mathrm{stretch}(n)=O\left(\log n\cdot\log\log n\cdot(\log\log\log n)^3\right)$$

Remark 1. *For ease of presentation we first show a slightly weaker bound of*

$$\mathrm{stretch}(n)=O\left(\log n\cdot(\log\log n)^2\cdot\log\log\log n\right),$$

and prove the tighter bound in Appendix B

Our result may be applied to improve the running time of the Spielman and Teng [16] solver for sparse symmetric diagonally dominant linear systems.

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[1] In fact these result apply to a similar notion, $\mathrm{avg}-\mathrm{stretch}(n)=\max_{G=(V,E)||V|=n}\inf_{T}\{\frac{1}{|E|}\sum_{(u,v)\in E}\frac{d_T(u,v)}{d_G(u,v)}\}$ which is equivalent up to a constant factor to $\mathrm{stretch}(n)$.

[2] [9] announced $\mathrm{stretch}(n)=O((\log n)^2)$, but this claim was subsequently withdrawn by the authors

1.1 Techniques

We extend the star-decomposition technique of Elkin *et. al.*[10]. A star-decomposition of a graph is a partition of the vertices into clusters that are connected into a star: a central cluster is connected to every other cluster by a single edge. As in [10] given a subgraph over a cluster X, the central cluster X_0 is formed by cutting a ball with radius r_0 around a center x_0 and the remaining clusters $X_1, X_2, \ldots$, which are called cones, are formed iteratively. Let $Y_j = X \setminus \bigcup_{0 \le k \le j} X_k$. The cone X_j is created by choosing an edge (y_j, x_j) such that $y_j \in X_0, x_j \in Y_{j-1}$ and defining X_j as the cone with radius r_j around x_j from the cluster Y_{j-1}, as all the points whose distance to x_0 going through the edge (x_j, y_j) does not increase too much relatively to the shortest path distance, formally $X_j = \{x \in Y_{j-1} \mid d_X(x_0, y_j) + d_X(y_j, x_j) + d_{Y_{j-1}}(x_j, x) - d_X(x_0, x) \le r_j\}$. Let $\mathrm{rad}_{x_0}(X) = \max_{x \in X} d(x_0, x)$, then typically the radius of the central ball is chosen so that $r_0 \approx \mathrm{rad}_{x_0}(X)/c$ for a constant c. An important parameter of a star-decomposition is the radius of the cone. We say that the star-decomposition has parameter ϵ if for any $j \ge 1$, the radius r_j of the cone X_j is at most $\epsilon \cdot \mathrm{rad}_{x_0}(X)$.

Applying star-decompositions in a recursive manner induces a spanning tree T. For a point u denote by $X^{(i)}$ the cluster that contains u in the ith recursive invocation of the hierarchical star-decomposition algorithm.

The $O(\log^2 n \log\log n)$ bound of [10] is obtained by choosing $\epsilon \approx 1/\log n$ and showing:

1. $O(1)$ *radius stretch*. For any cluster X induced by the recursive invocation of the hierarchical star-decomposition algorithm, and any $z \in X$, $d_T(x_0, z) = O(\mathrm{rad}_{x_0}(X))$.

2. $O((\log n \cdot \log\log n)/\epsilon)$ *decomposition stretch*. For any edge (u, v), $\sum_i \Pr[(u, v)$ is separated when star-decomposing $X^{(i)}] \cdot \mathrm{diam}(X^{(i)}) = O(\log n \log\log n)/\epsilon$.

Combining these two properties yields their result, noticing that if the end points of an edge (u, v) fall into different clusters in the partitioning of $X^{(i)}$ then $d_T(u, v)$ can be bounded by $d_T(u, x_0) + d_T(v, x_0) = O(\mathrm{diam}(X^{(i)}))$.

Good radius stretch is obtained by observing that in each recursive application of the star partition the radius of a cluster is stretched by at most $1 + 1/\log n$, and since there are $O(\log n)$ scales the total radius stretch is a constant. Good decomposition stretch is obtained by using a version of the decomposition of [4, 8].

Better radius stretch. In our scheme we perform a star-decomposition with a parameter $\epsilon \approx 1/\log\log n$, this significantly improves the decomposition stretch, by a factor of $\approx \log n/\log\log n$. A naive attempt to bound the radius stretch, by $1 + 1/\log\log n$ in each scale, will result in super logarithmic radius stretch over all scales.

We introduce a new approach to bound the radius stretch. We arrange all the points of X in a queue $Q = (z_1, z_2, \ldots, z_n)$, and bound the distance $d_T(x_0, z_i)$ as a function of i by building "highways" – *low stretch paths*. Roughly speaking, we obtain a bound of $d_T(x_0, z_i) = O(\log\log i) \cdot \mathrm{rad}_{x_0}(X)$. The core observation is that by choosing where to build the first cone and passing this information into the recursion, one can obtain a shortest path "highway" between x_0 and the first point z_1, such that the distance between x_0 and z_1 in the tree will be *exactly* the original distance in the graph. The challenge is to use this observation to maintain "highways" – low stretch paths – between x_0 and *all* the points. Specifically, we obtain

1. $O(\log\log n)$ *radius stretch*. For any cluster X, and any $z \in X$, $d_T(x_0, z) = O(\log\log n)\mathrm{rad}_{x_0}(X)$.

Better decomposition stretch. A relaxation of the spanning tree problem suggested by Bartal [3] is to consider a distribution of dominating tree metrics (in fact of ultrametrics) that do not necessarily span the graph. This relaxation has proven applicable for approximation algorithms, online problems and has contributed to recent solutions for the spanning tree problem (*i.e.* [10]). Initially $O(\log^2 n)$ approximation was obtained in [3] based on the truncated exponential distribution approach of [14]. This bounded was subsequently improved to $O(\log n \log\log n)$ in [4] and [8]. Finally an optimal $O(\log n)$ approximation was obtained by [12] based on the cutting scheme of [7]. Subsequently an $O(\log n)$ bound was also obtained using a truncated exponential distribution approach [5, 1].

However, all previous schemes that obtained the optimal $O(\log n)$ bound for the metric problem were insufficient for the spanning tree problem. Given a graph $G = (X, E)$, a sequence $x_1, x_2, \ldots$ of cluster centers and a sequence $r_1, r_2, \ldots$ of radiuses we can define a weak diameter decomposition by defining $W_i = B_X(x_i, r_i) \setminus \bigcup_{j<i} W_j$. We can define a strong diameter decomposition by defining $C_i = B_{X \setminus \bigcup_{j<i} C_j}(x_i, r_i)$. Observe that in a strong diameter

decomposition, for any nonempty cluster C_i, we have that $x_i \in C_i$ and C_i is a connected component of G, this may not be the case for weak diameter decompositions. Indeed the techniques of [12, 5, 1] provide a weak diameter decomposition. It was not clear how to extend these results to strong diameter decompositions that are necessary for star-decompositions. We show how to obtain a strong diameter hierarchical decomposition theorem that obtains an optimal bound in the following sense:

2. $O(\log n \log(1/\epsilon)/\epsilon)$ *decomposition stretch.* For any edge (u, v), $\sum_i \Pr[(u, v)$ is separated when star-decomposing $X^{(i)}] \cdot \mathrm{diam}(X^{(i)}) = O(\log n \log(1/\epsilon)/\epsilon)$.

As in [5, 1], our decomposition is based on the truncated exponential distribution with a parameter depending on the local growth rate of the space. The main technical difficulty arises since the space *changes* after each cluster is cut (the metric is derived from a graph, and some nodes and edges are removed at every cut). The idea is to define the local growth rate with respect to the current metric, and to show two things: that the expected sum of all growth rates (which are random variables) over all the scales telescopes to n, and that the probability to be cut is appropriately bounded in each scale. Dealing with the randomly changing graph raises some additional subtleties in the proof. Our strong diameter hierarchical decomposition theorem may be of independent interest.

1.2 Applications

One of the main applications of low stretch spanning trees is solving *sparse symmetric diagonally dominant linear systems of equations*. This approach was suggested by Boman and Hendrickson [6] and later improved by Spielman and Teng [16]. Spielman and Teng showed an algorithm that for such an n-by-n matrix A with m non-zero entries and an n-dimensional vector b, if $\epsilon > 0$ is the precision of the solution then the algorithm finds x' such that $\|x - x'\|_A \leq \epsilon$ where $Ax = b$, and the running time is $O\left(m\left(\log^{O(1)} m + \log(1/\epsilon)\right) + n \cdot \mathrm{avg} - \mathrm{stretch}(n) \cdot \log(1/\epsilon)\right)$. Improving the bound requires improvement of the second element, and we improve it by roughly an additional $O(\log \log n)$ factor over [10]. Actually, if the running time of our construction is reduced, we can obtain an $O(\log n)$ improvement. For planar graphs we obtain $O(n \cdot \log^2 n)$. See details in Corollary 6.

The *minimum communication cost spanning tree* problem introduced in [13], in which one is given a weighted graph $G = (V, E, w)$ and a matrix $A = a_{xy} \mid x, y \in V$, the objective is to find a spanning tree minimizing $c(T) = \sum_{x,y \in V} a_{xy} \cdot d_T(x, y)$. [15] showed an $O(2^{\sqrt{\log n \cdot \log \log n}})$ approximation ratio based on [2], and [10] improved to $O(\log^2 n \cdot \log \log n)$. Our results can be used to obtain $O(\log n \cdot \log \log n (\log \log \log n)^3)$ approximation ratio.

See [10] for details about more applications.

1.3 Structure of the Paper

In Section 2 we describe a star-decomposition framework, that for any unweighted n point graph G induces a tree such that $\mathrm{diam}(T) \leq O(\mathrm{diam}(G) \cdot \log \log n)$. In Section 3 we describe a distribution on star-partitions that follows the framework of Section 2. We analyze the expected stretch of an edge and prove the bound of $\mathrm{stretch}(n) = O(\left(\log n \cdot (\log \log n)^2 \cdot \log \log \log n\right))$. In Appendix A we discuss briefly how to extend the result for weighted graphs. In Appendix B we show the tighter result stated in Theorem 1.

2 Highways

Let $G = (V, E)$ be a finite graph. For any $X \subseteq V$ let $d_X : X^2 \to \mathbb{R}^+$ be the shortest path metric induced by the subgraph on X. Let $\mathrm{diam}(X) = \max_{y,z \in X}\{d_X(y, z)\}$. For $x \in X$ let $\mathrm{rad}_x(X) = \max_{y \in X} d_X(x, y)$, we omit the subscript when clear from context (note that $\mathrm{diam}(X)/2 \leq \mathrm{rad}(X) \leq \mathrm{diam}(X)$). For any $x \in X$ and $r \geq 0$ let $B_{X,d}(x, r) = \{y \in X \mid d_X(x, y) \leq r\}$. Let $c = 2^{16}$ be a constant. We use the uppercase letter Q to denote a *queue*, a sequence of points. Given a point x not in the queue we say that we enqueue x into Q meaning that we add x as the last element of the sequence and given a queue Q, the dequeue operation removes and returns the first element of the sequence.

Definition 1 (cone metric[3]). *Given a graph $G = (V, E)$, subsets $Y \subset X \subseteq V$, points $x \in X \setminus Y$, $y \in Y$ define the* cone-metric $\rho = \rho(X, Y, x, y) : Y^2 \to \mathbb{R}^+$ *as* $\rho(u, v) = |(d_X(x, u) - d_Y(y, u)) - (d_X(x, v) - d_Y(y, v))|$.

Note that a ball $B_{Y,\rho}(y, r)$ in the cone-metric $\rho = \rho(X, Y, x, y)$ is the set of all points $z \in Y$ such that $d_X(x, y) + d_Y(y, z) - d_X(x, z) \le r$.

Hierarchical-Star-Partition algorithm. See Figure 1 for the algorithm. Given an unweighted graph $G = (V, E)$, create a spanning tree $T = (V, E')$ by choosing some $x_0 \in V$, letting Q be an arbitrary ordering of $V \setminus \{x_0\}$ and calling: `hierarchical-star-partition`(V, x_0, Q).

> $T =$ `hierarchical-star-partition`(X, x_0, Q):
>
> 1. If $\mathrm{rad}_{x_0}(X) \le 16c$ return $\mathrm{BFS}(X)$.
> 2. $(X_0, \dots, X_m, (y_1, x_1), \dots, (y_m, x_m), Q_0, Q_1, \dots, Q_m) =$ `star-partition`(X, x_0, Q);
> 3. For each $i \in [0, \dots, m]$:
> 4. $T_i =$ `hierarchical-star-partition`(X_i, x_i, Q_i);
> 5. Let T be the tree formed by connecting T_0 with T_i using edge (y_i, x_i) for each $i \in [1, \dots, m]$;

Figure 1: `hierarchical-star-partition` algorithm

Star-Partition algorithm. See Figure 2 for our `star-partition` algorithm. We highlight the main differences of our algorithm from that of [10]. In addition to X, x_0 it receives as input an ordering of the points in X, implemented as a queue data structure and denoted by Q. In addition to returning a star decomposition $X_0, X_1, \dots, X_m$ it returns for each $0 \le j \le m$ an ordering of the points in X_j, implemented as a queue data structure and denoted by Q_j.

Since as noted above the trivial radius bound (loosing $(1 + \epsilon)$ in every scale) does not work anymore we attempt to directly bound $d_T(x_0, z)$ for all $z \in X$. The arrangement of $X \setminus \{x_0\}$ in a queue $Q = (z_1, \dots, z_{n-1})$ determines "how hard" we try to give a tight bound for the point z_i - roughly speaking the smaller value of i means the harder we try to give a better bound on $d_T(x_0, z_i)$. The star partition algorithm therefore changes to try hardest for the first point z_1, and indeed by choosing the first portal edge (y_1, x_1) on a shortest path to z_1 and keeping z_1, y_1 in the head of the recursive queues we obtain a "highway" from x_0 to z_1, *i.e.* preserving the original distance. Surprisingly, this small change is enough to give a good bound on $d_T(x_0, z_i)$ for all $i > 1$, and we obtain $d_T(x_0, z_i) = O(\log\log i)\mathrm{rad}_{x_0}(X)$. The intuition is that since every cluster contains less points, z_i advances in the recursive queues, and when it becomes the first we get a "highway" to it. For this intuition to work one must delicately define the ordering of the queues $Q_0, \dots, Q_m$ for the clusters $X_0, \dots, X_m$ created by the star partition algorithm. The main difficulty is defining Q_0, as the portals y_j play a dual part - we need to maintain their original position in Q and also make sure that the tree distance to them is small enough: as it determines the distance from x_0 to all the points in X_j.

Suppose $z_i \in Y_j$ for some $i > 1$. By Claim 2 there is an inherent loss of a $1+\epsilon$ factor due to star-partition algorithm. Hence it is not sufficient for the inductive argument to simply obtain a bound of $d_T(x_0, y_j) = O(\log\log i)\mathrm{rad}_{x_0}(X_0)$ in the ball X_0 and $d_T(x_j, z_i) = O(\log\log i)\mathrm{rad}_{x_j}(X_j)$ in the cone X_j. We must "gain" inductively either in $d_T(x_0, y_j)$ (the ball part of the path) or in $d_T(x_j, z_i)$ (the cone part of the path). This is done by choosing the queues in the following manner: Given a star decomposition $X_0, X_1, \dots, X_m$ we create the queue Q_j for $j > 0$ simply as the restriction of Q on $X_j \setminus \{x_j\}$. The queue Q_0 is the created by first adding either z_1 or the portal y_1 which is chosen on a shortest path to z_1, thus making sure the distance from x_0 to z_1 is preserved in the recursion. Then interleaving three different queues $Q_0^{\text{(ball)}}, Q_0^{\text{(fat)}}, Q_0^{\text{(reg)}}$.

- $Q_0^{\text{(ball)}}$ is the restriction of Q on X_0. This queue provides the required bound on $d_T(x_0, z_i)$ when $z_i \in X_0$.
- $Q_0^{\text{(reg)}}$ is a queue of portals y_j ordered by the minimal point of Q that their cones X_j contains. When a cone contains relatively few point we "gain" in the cone part of the path to z_i. This queue guarantees that for any $z_i \in X_j$ the "central ball" part of the path to z_i is not stretched too much.

[3] In fact, the cone-metric is a pseudo-metric.

- $Q_0^{(\text{fat})}$ is a queue of portals y_j that lead to cones that contain "many" points relative to the ordering Q of the points in X_j. When a cone is "fat" we cannot gain in the cone part, this queue guarantees that we gain in the ball part.

The exact way these three queues are created is detailed in Line 5 of Figure 2.

$(X_0, \dots, X_m, (y_1, x_1), \dots, (y_m, x_m), Q_0, Q_1, \dots, Q_m) = \texttt{star-partition}(X, x_0, Q)$:

1. Let $j = 2$; Denote the (ordered) elements of Q by $Q = (z_1, z_2, \dots, z_k)$; Let $\epsilon = \epsilon(X) \in (0, \frac{1}{170c}]$;
2. Creating the ball X_0:
   (a) Choose r_0 uniformly at random from the interval $[1/(16c), 1/(8c)]$;
   (b) Let $X_0 = B(x_0, r_0 \cdot \text{rad}_{x_0}(X))$; Let $Y_0 = X \setminus X_0$;
3. Creating the first cone X_1:
   (a) If $z_1 \in Y_0$ let $z = z_1$ otherwise let $z \in Y_0$ be an arbitrary point. Let (y_1, x_1) be an edge such that $y_1 \in X_0$, $x_1 \in Y_0$ and $d_X(x_0, z) = d_X(x_0, y_1) + d_X(y_1, x_1) + d_{Y_0}(x_1, z)$ (*i.e.* an edge on a shortest path from x_0 to z);
   (b) Let $\rho = \rho(X, Y_0, x_0, x_1)$ be the cone-metric;
   (c) Choose r_1 uniformly at random from the interval $[\epsilon/4, \epsilon/2]$;
   (d) Let $X_1 = B_{(Y_0, \rho)}(x_1, r_1 \cdot \text{rad}_{x_0}(X))$; Let $Y_1 = Y_0 \setminus X_1$;
4. Creating the remaining cones $X_2, \dots, X_m$:
   (a) While $Y_{j-1} \neq \emptyset$:
      i. Let $(x_j, y_j, r_j) = \texttt{cone-cut}(X, x_0, X_0, Y_{j-1}, \epsilon)$; (has the property that $r_j \leq \epsilon/2$)
      ii. Let $\rho = \rho(Y_{j-1} \cup X_0, Y_{j-1}, x_0, x_j)$;
      iii. Let $X_j = B_{(Y_{j-1}, \rho)}(x_j, r_j \cdot \text{rad}_{x_0}(X))$; $Y_j = Y_{j-1} \setminus X_j$;
      iv. Let $j = j + 1$;
5. Creating the queues $Q_0^{(\text{ball})}, Q_0^{(\text{fat})}, Q_0^{(\text{reg})}, Q_1, \dots, Q_m$:
   (a) For $i = 1, \dots, |X| - 1$:
      i. If $z_i \in X_0$ then enqueue z_i into $Q_0^{(\text{ball})}$;
      ii. Otherwise let $\ell \geq 1$ be such that $z_i \in X_\ell$:
         - If $z_i \neq x_\ell$ then enqueue z_i into Q_ℓ.
         - If $y_\ell \notin Q_0^{(\text{reg})}$ then enqueue y_ℓ into $Q_0^{(\text{reg})}$.
         - If $|X_\ell \cap \{z_1, \dots, z_i\}| > \sqrt{i}$ and $y_\ell \notin Q_0^{(\text{fat})}$ then enqueue y_ℓ into $Q_0^{(\text{fat})}$.
6. Creating the queue Q_0:
   (a) Denote $Q_0^{(\text{ball})} = z_1^1, \dots, z_{m_1}^1$, $Q_0^{(\text{fat})} = z_1^2, \dots, z_{m_2}^2$, $Q_0^{(\text{reg})} = z_1^3, \dots, z_{m_3}^3$.
   (b) Create Q_0 by interleaving the three queues $Q_0^{(\text{ball})}, Q_0^{(\text{fat})}, Q_0^{(\text{reg})}$ such that:
      - If $z_1 \in X_0$ then z_1 is the first element of Q_0. Otherwise y_1 is the first element of Q_0.
      - For any $x \in X$, $\ell \in \{1, 2, 3\}$, $1 \leq i \leq n$ if $x = z_i^\ell$ then x is in the first $3i$ elements of Q_0.

Figure 2: `star-partition` algorithm

2.1 Bounding the radius stretch

In this part we show that the radius stretch induced by the `hierarchical-star-partition` algorithm is at most $O(\log \log n)$.

The following two claims imply that the `star-partition` algorithm on a cluster X induces a partition on X and that radial distances are stretched by a most $1+\epsilon$. These claims are essentially proven in [10] we provide a proof for completeness.

Claim 1. *For any graph X, $x_0 \in X$, $j > 0$ let $Y_{j-1} \subseteq X$ be the unassigned points of X after creating j clusters $X_0, \ldots, X_{j-1}$ using the `star-partition` algorithm, then for any $z \in Y_{j-1}$ all the shortest paths from z to x_0 are fully contained in $Y_{j-1} \cup X_0$, in particular*

$$d_{Y_{j-1} \cup X_0}(x_0, z) = d_X(x_0, z).$$

Proof. Let $\Delta = \mathrm{rad}_{x_0}(X)$. Let P_{z,x_0} be a shortest path and assume by contradiction that $P_{z,x_0} \not\subseteq Y_{j-1} \cup X_0$, so let $1 \le i \le j-1$ be the minimal i such that there exists $u \in P_{z,x_0}$ and $u \in X_i$. Let x_i be the portal to the cone X_i. By Definition 1 since $u \in X_i$ it must be that in the metric $d' = d_{X_0 \cup Y_{i-1}}$

$$d'(u, x_0) + r_i \cdot \Delta \ge d'(u, x_i) + d'(x_i, x_0).$$

Since u lies on a shortest path from z to x_0, the minimality of i suggests that this shortest path is fully contained in $Y_{i-1} \cup X_0$ thus $d'(z, x_0) = d'(z, u) + d'(u, x_0)$, and conclude that

$$d'(z, x_0) + r_i \cdot \Delta = d'(z, u) + d'(u, x_0) + r_i \cdot \Delta \ge d'(z, u) + d'(u, x_i) + d'(x_i, x_0) \ge d'(z, x_i) + d'(x_i, x_0),$$

hence z should be in X_i, contradiction. ☐

Claim 2. *Let $(X_0, \ldots, X_m, (y_1, x_1), \ldots, (y_m, x_m), Q_0, Q_1, \ldots, Q_m) = \texttt{star-partition}(X, x_0, Q)$ then for any $1 \le j \le m$*

$$\mathrm{rad}_{x_0}(X_0) + d(y_j, x_j) + \mathrm{rad}_{x_j}(X_j) \le (1+\epsilon)\mathrm{rad}_{x_0}(X),$$

Proof. Let $\Delta = \mathrm{rad}_{x_0}(X)$. Let β be such that $\mathrm{rad}_{x_0}(X_0) = \beta \cdot \Delta$, let $d' = d_{X_0 \cup Y_{j-1}}$, let x_j be the portal of X_j and $\rho = \rho(X_0 \cup Y_{j-1}, Y_{j-1}, x_0, x_j)$ be the cone-metric. Take $z \in X_j$ as the farthest point from x_j (with respect to d'), take any shortest path $P_{x_j,z}$ from x_j to z and separate it into consecutive segments $x_j = u_0, v_0, u_1, v_1, \ldots, u_k, v_k = z$ such that for any $0 \le i \le k$, $\rho(u_i, v_i) = 0$, *i.e.*

$$d'(x_0, u_i) - d'(x_j, u_i) = d'(x_0, v_i) - d'(x_j, v_i)$$

and $(v_i, u_{i+1}) \in E$ (note that it could be that $u_i = v_i$). The definition of cone-metric suggests that $k \le r_i \cdot \Delta$, as otherwise $z \notin B_{Y_{j-1},\rho}(x_j, r_j \cdot \Delta) = X_j$.

Since $P_{x_j,z}$ is a shortest path we have for all $0 \le i \le k$ that $d'(x_j, u_i) + d'(u_i, v_i) = d'(x_j, v_i)$, therefore

$$\sum_{i=0}^{k} d'(x_0, v_i) = \sum_{i=0}^{k} (d'(x_0, u_i) + d'(u_i, v_i)). \tag{1}$$

Claim 1 suggests that $d_X(x_0, z) = d'(x_0, z)$, hence

$$\begin{aligned}
\Delta &\ge d_X(x_0, z) = d'(x_0, z) = d'(x_0, v_k) \\
&= \sum_{i=0}^{k-1} (d'(x_0, u_i) + d'(u_i, v_i) - d(x_0, v_i)) + d'(x_0, u_k) + d'(u_k, v_k) \\
&\ge \sum_{i=0}^{k-1} (d'(x_0, u_i) + d'(u_i, v_i) - (d'(x_0, u_{i+1}) + d'(v_i, u_{i+1}))) + d'(x_0, u_k) + d'(u_k, v_k) \\
&= d'(x_0, u_0) - d'(x_0, u_k) + \sum_{i=0}^{k-1} (d'(u_i, v_i) - 1) + d'(x_0, u_k) + d'(u_k, v_k) \\
&= (\beta\Delta + 1) - k + \sum_{i=0}^{k} d'(u_i, v_i)
\end{aligned}$$

The second line follows from (1), the third from the fact that $d'(x_0, v_i) \leq d'(x_0, u_{i+1}) + d'(u_{i+1}, v_i)$, the fourth since the sum telescopes and $d'(v_i, u_{i+1}) = 1$, and the fifth since $d'(x_0, u_0) = d'(x_0, x_j) = d'(x_0, y_j) + d'(y_j, x_j) = \mathrm{rad}_{x_0}(X_0) + 1 = \beta\Delta + 1$.

Therefore

$$\mathrm{rad}_{x_j}(X_j) = d'(x_j, z) = \sum_{i=0}^{k} d'(u_i, v_i) + \sum_{i=0}^{k-1} d'(v_i, u_{i+1}) \leq (\Delta - \beta\Delta + k - 1) + k \leq (1-\beta)\Delta + 2r_j\Delta - 1,$$

(recall that $k \leq r_j\Delta$). And now since $r_j \leq \epsilon/2$,

$$\mathrm{rad}_{x_0}(X_0) + d(y_j, x_j) + \mathrm{rad}_{x_j}(X_j) \leq \beta\Delta + 1 + (1-\beta)\Delta + \epsilon\Delta - 1 = (1+\epsilon)\Delta.$$

□

Corollary 3. *For any* $0 \leq j \leq m$, $\mathrm{rad}_{x_j}(X_j) < (1 - \frac{1}{20c})\mathrm{rad}_{x_0}(X)$

Proof. The corollary is immediate for X_0 by the construction, for $j > 0$: as $\mathrm{rad}_{x_0}(X_0) \geq \mathrm{rad}_{x_0}(X)/(16c)$ and $\epsilon \leq 1/(170c)$ using Claim 2

$$\mathrm{rad}_{x_j}(X_j) < (1+\epsilon)\mathrm{rad}_{x_0}(X) - \mathrm{rad}_{x_0}(X_0) \leq (1 - 1/(20c))\mathrm{rad}_{x_0}(X).$$

□

Lemma 4. *Let* $X \subseteq V$ *be a connected component of* $G(V, E)$*. Let* $x_0 \in X$ *and* $Q = (z_1, \ldots, z_{|X|-1})$ *be any ordering of* $X \backslash \{x_0\}$*. Let* T *be any spanning tree of* G *returned by the algorithm* `hierarchical-star-partition`(X, x_0, Q) *with parameter* $\epsilon = \epsilon(X) = \frac{1}{170c \log\log(|X|)}$*, then*

$$d_T(x_0, z_i) \leq \begin{cases} d_X(x_0, z_i) & i = 1 \\ i \cdot \mathrm{rad}_{x_0}(X) & 1 < i < c \\ c \cdot \log\log i \cdot \mathrm{rad}_{x_0}(X) & \textit{otherwise} \end{cases}$$

(where $c = 2^{16}$*)*

Proof. The proof is by induction on the radius of X. In the base case when $\mathrm{rad}_{x_0}(X) \leq 16c$ create a breadth first tree centered in x_0, and since in such a tree for every $z \in X$, $d_X(x_0, z) = d_T(x_0, z)$ the claim holds. Now we turn to the inductive step. Note that Corollary 3 guarantees that for all $j = 0, \ldots, m$ we have $0 \leq \mathrm{rad}_{x_j}(X_j) < \mathrm{rad}_{x_0}(X)$.

The main idea of the proof is to consider a single application of the star-partition algorithm, partitioning X into $X_0, X_1, \ldots, X_m$. Assuming that $z_i \in X_j$ the path between x_0 to z_i will be the path going through the edge (y_j, x_j). Then use the induction hypothesis on the sub-path x_0, y_j in X_0 and the sub path x_j, z_i in X_j. Since by Claim 2 the radius may increase by a factor of at most $1 + \epsilon$, we need to "gain" in one of the two sub paths. This "gain" will occur since our construction guarantees that either the position of z_i in the queue of X_j will improve or the position of y_j in X_0 will improve, thus the induction hypothesis will give the required bounds.

There are three main cases to consider, when $i = 1$, $i < c$ and $i \geq c$. The case $i = 1$ is simple. The case $1 < i < c$ subdivides into three more cases:

1. The first case is $z_i \in X_0$. This case is relatively straightforward.
2. The second case is that the first i points of the queue are all in X_1. Here we gain in the central ball because the portal y_1 leading to X_1 will be the first element in Q_0.
3. The remaining case is that not all of the first i points are in X_1, then there are at most $i - 1$ points in the cone X_j among $z_1, \ldots, z_i$, so by the construction of Q_j, we gain just enough in the cone (because the bound that needs to be shown is weak - linear in i) and $Q_0^{(\mathrm{reg})}$ guarantees that we do not lose too much in the central ball.

The interesting case is when $i \geq c$, this last case also subdivides into three more cases:

1. One first is that $z_i \in X_0$. Again. this case is relatively straightforward and uses the construction of $Q_0^{(\text{ball})}$.
2. The second case is that $z_i \in X_j$ and X_j is a "thin" cone - contains less than $\sqrt{i}$ of the first i points. Here we gain in the cone because the position of z_i in Q_j is at most $\sqrt{i}$ and $Q_0^{(\text{reg})}$ guarantees that we do not lose too much in the central ball.
3. The third case is that $z_i \in X_j$ and X_j is a "fat" cone - contains more than $\sqrt{i}$ of the first i points. Here we gain in the central ball, using the construction of $Q_0^{(\text{fat})}$ and Claim 5 to show that the portal y_j leading to the cone is in position $\leq i^{9/10}$ in Q_0.

We continue with the formal proof of the lemma, according to the three main cases.

Case 1: In this case $i = 1$. Note that $z_1 \in X_0 \cup X_1$. If $z_1 \in X_0$ then by the construction z_1 is going to be the first in Q_0 therefore by the induction hypothesis on X_0 it follows that $d_T(x_0, z_1) \leq d_X(x_0, z_1)$. If on the other hand $z_1 \in X_1$, then again from the construction the point y_1, which was chosen such that y_1, x_1 are on a shortest path from x_0 to z_1, will be the first in Q_0, and z_1 will be the first in X_1, so by induction $d_T(x_0, z_1) = d_T(x_0, y_1) + d_T(y_1, x_1) + d_T(x_1, z_1) \leq d_X(x_0, y_1) + d_X(y_1, x_1) + d_X(x_1, z_1) = d_X(x_0, z_1)$.

Case 2: The second case to consider is when $1 < i < c$.

1. First assume that $z_i \in X_0$. Then z_i will be at most i in the ordering of $Q_0^{(\text{ball})}$ and hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on $X_0 : d_T(x_0, z_i) \leq c \log\log(3i) \cdot \text{rad}_{x_0}(X_0) \leq i \cdot \text{rad}_{x_0}(X)$, using that $\text{rad}_{x_0}(X_0) \leq \text{rad}_{x_0}(X)/(8c)$, and that $\log\log(3i) \leq 2i$.
2. Now assume that $\{z_1, \ldots, z_i\} \subseteq X_1$. As y_1 is the first in Q_0, by the induction hypothesis on X_0 and X_1 we have that $d_T(x_0, y_1) \leq d_X(x_0, y_1) \leq \text{rad}_{x_0}(X_0)$ and $d_T(x_1, z_i) \leq i \cdot \text{rad}_{x_1}(X_1)$, so

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_1) + d_T(y_1, x_1) + d_T(x_1, z_i) \\
&\leq \text{rad}_{x_0}(X_0) + i \cdot \text{rad}_{x_1}(X_1) + d_X(y_1, x_1) \\
&\leq i(\text{rad}_{x_0}(X_0) + d_X(y_1, x_1) + \text{rad}_{x_1}(X_1)) - (i-1)\text{rad}_{x_0}(X_0) \\
&\leq i(1+\epsilon)\text{rad}_{x_0}(X) - (i-1)\text{rad}_{x_0}(X)/(16c) \\
&\leq i \cdot \text{rad}_{x_0}(X) + i \cdot \text{rad}_{x_0}(X)/(170c) - i \cdot \text{rad}_{x_0}(X)/(32c) \\
&\leq i \cdot \text{rad}_{x_0}(X).
\end{aligned}$$

   In the fourth inequality using Claim 2 and that $\text{rad}_{x_0}(X_0) \geq \text{rad}_{x_0}(X)/(16c)$ (note that by the stop condition of hierarchical-star-partition $\text{rad}_{x_0}(X) \geq 16c$, so $\text{rad}_{x_0}(X_0) \geq 1$) and in the fifth that $i - 1 \geq i/2$.
3. Now assume that $z_i \in X_j$ where not all of $z_1, \ldots, z_i$ are in X_j (note that $z_1 \in X_0 \cup X_1$, therefore there is no case for $\{z_1, \ldots, z_i\} \subseteq X_j$ where $j > 1$). First note that z_i must be at most the $i - 1$ element in Q_j. By the insert sequence to $Q_0^{(\text{reg})}$ we have that y_j is at most the $3i$ element in Q_0. Using the induction hypothesis on X_0 and X_j we get that

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\leq c \log\log(3i) \cdot \text{rad}_{x_0}(X_0) + (i-1) \cdot \text{rad}_{x_j}(X_j) + d_X(y_j, x_j) \\
&\leq (i-1)(\text{rad}_{x_0}(X_0) + d_X(y_j, x_j) + \text{rad}_{x_j}(X_j)) + 5c \cdot \text{rad}_{x_0}(X_0) \\
&\leq (i-1)(1+\epsilon)\text{rad}_{x_0}(X) + 5c \cdot \text{rad}_{x_0}(X)/(8c) \\
&\leq i \cdot \text{rad}_{x_0}(X) - \text{rad}_{x_0}(X) + (i-1) \cdot \text{rad}_{x_0}(X)/(170c) + 5\text{rad}_{x_0}(X)/8 \\
&\leq i \cdot \text{rad}_{x_0}(X).
\end{aligned}$$

   The third inequality follows since $\log\log(3i) \leq \log\log(3c) \leq 5$. The fourth using Claim 2 and that $\text{rad}_{x_0}(X_0) \leq \text{rad}_{x_0}(X)/(8c)$.

Case 3: In the third case $i \geq c$.

1. First assume that $z_i \in X_0$. Then z_i will be at most i in the ordering of $Q_0^{(\text{ball})}$, hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on X_0 we get that

$$d_T(x_0, z_i) \leq c \log\log(3i) \cdot \text{rad}_{x_0}(X_0) \leq 2c \log\log i \cdot \text{rad}_{x_0}(X_0) \leq c \log\log i \cdot \text{rad}_{x_0}(X) .$$

using that for $i \geq c$, $3i < i^2$.

2. Next assume that $z_i \in X_j$ such that $|X_j \cap \{z_1, \dots, z_i\}| \leq \sqrt{i}$, then z_i will be at most the $\sqrt{i}$ in Q_j, and y_j will be at most the i-th in $Q_0^{(\text{reg})}$ and hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on X_0 and X_j:

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\leq c \log\log(3i) \cdot \text{rad}_{x_0}(X_0) + c \log\log(\sqrt{i}) \cdot \text{rad}_{x_j}(X_j) + d_X(y_j, x_j) \\
&\leq c(\log\log i + 1) \cdot \text{rad}_{x_0}(X_0) + c(\log\log i - 1) \cdot \text{rad}_{x_j}(X_j) + d_X(y_j, x_j) \\
&\leq c(\log\log i - 1) \left(\text{rad}_{x_0}(X_0) + d_X(y_j, x_j) + \text{rad}_{x_j}(X_j)\right) + 2c \cdot \text{rad}_{x_0}(X_0) \\
&\leq c(\log\log i - 1)(1+\epsilon)\text{rad}_{x_0}(X) + \text{rad}_{x_0}(X)/4 \\
&\leq c \log\log i \cdot \text{rad}_{x_0}(X) + c \log\log i \cdot \text{rad}_{x_0}(X)/(170c \log\log i) - c \cdot \text{rad}_{x_0}(X) + \text{rad}_{x_0}(X)/4 \\
&\leq c \log\log i \cdot \text{rad}_{x_0}(X),
\end{aligned}$$

the fifth inequality using Claim 2 and that $\text{rad}_{x_0}(X_0) \leq \text{rad}_{x_0}(X)/(8c)$, the sixth that $\epsilon \leq 1/(170c \log\log i)$.

3. The last subcase is where $z_i \in X_j$ such that $|X_j \cap \{z_1, \dots, z_i\}| > \sqrt{i}$, then z_i will be at most the i in Q_j and by Claim 5 y_j will be at most the $i^{9/10}$ in Q_0. Now by the induction hypothesis, for $t \geq 2$

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_j) + d_X(y_j, x_j) + d_T(x_j, z_i) \\
&\leq c \log\log i^{9/10} \cdot \text{rad}_{x_0}(X_0) + c \log\log i \cdot \text{rad}_{x_j}(X_j) + d_X(y_j, x_j) \\
&\leq c \log\log i(\text{rad}_{x_0}(X_0) + d_X(y_j, x_j) + \text{rad}_{x_j}(X_j)) + c \log(9/10) \cdot \text{rad}_{x_0}(X_0) \\
&\leq c \log\log i \cdot \text{rad}_{x_0}(X) + \epsilon \cdot c \log\log i \cdot \text{rad}_{x_0}(X) - c \cdot \text{rad}_{x_0}(X_0)/10 \\
&\leq c \log\log i \cdot \text{rad}_{x_0}(X) + \text{rad}_{x_0}(X)/170 - \text{rad}_{x_0}(X)/160 \\
&\leq c \log\log i \cdot \text{rad}_{x_0}(X),
\end{aligned}$$

the fourth inequality using Claim 2 and the fifth that $\text{rad}_{x_0}(X_0) \geq \text{rad}_{x_0}(X)/(16c)$ and $\epsilon \leq 1/(170c \log\log i)$.

□

The following claim shows that a portal y_j leading to a point z_i that belongs to a "fat" cone will be located in an improved position in the queue of the central ball Q_0.

Claim 5. *For any $i \geq 2^{16}$, if $z_i \in X_j$ such that $|X_j \cap \{z_1, \dots, z_i\}| > \sqrt{i}$ then y_j will be at position at most $i^{9/10}$ in Q_0.*

Proof. We will show that y_j will be in the first $(3/2)i^{2/3} + 1$ elements of $Q_0^{(\text{fat})}$. Since $i \geq 2^{16}$ it follows that y_j will be in the first $3 \cdot ((3/2)i^{2/3} + 1) < i^{9/10}$ elements of Q_0.

Let $y_{i_1}, \dots, y_{i_s}$ with $i_1 < i_2 < \cdots < i_s$ be a set of s points that were inserted into $Q_0^{(\text{fat})}$ before considering the point z_i, we need to show that $s \leq (3/2)i^{2/3}$. Let $z_{i'_1}, \dots, z_{i'_s}$ be the set of points in Q such that y_{i_k} was inserted because $z_{i'_k} \in X_{i_k}$ and X_{i_k} was a "fat" cone, *i.e.* $|X_{i_k} \cap \{z_1, \dots, z_{i'_k}\}| \geq \sqrt{i'_k}$. Let $A_{i_k} = X_{i_k} \cap \{z_1, \dots, z_{i'_k}\}$ denote the set that caused y_{i_k} to enter $Q_0^{(\text{fat})}$, and note that $|A_{i_k}| \geq \sqrt{i'_k} \geq \sqrt{k}$. For any $1 \leq k < \ell \leq s$ we have that $A_{i_k} \cap A_{i_\ell} = \emptyset$, since we do not insert a point y_{i_ℓ} that already appear in $Q_0^{(\text{fat})}$, which implies $X_{i_k} \cap X_{i_\ell} = \emptyset$. Note that all the sets A_{i_k} contain points from $z_1, \ldots, z_i$, so we have that $\sum_{k=1}^{s} |A_{i_k}| \leq i$. Hence $\sum_{k=1}^{s} \sqrt{k} \leq \sum_{k=1}^{s} |A_{i_k}| \leq i$. We also bound the sum from below

$$\sum_{k=1}^{s} \sqrt{k} \geq \int_{1}^{s} \sqrt{x} dx = [(2/3)x^{3/2}]_1^s \geq (2/3)s^{3/2},$$

therefore $i \geq (2/3)s^{3/2}$ or $s \leq (3/2)i^{2/3}$. □

Corollary 6. *For any weighted graph $G = (V, E)$ denote by $|V| = n$ and $|E| = m$, invoking* `hierarchical-star-partition` *algorithm on G where in* `star partition` *algorithm we use the* `ImpConeDecompose`$(G, BS(x_0, r_0 \cdot \mathrm{rad}(X)), \mathrm{rad}(X)/\log\log n, \log\log n, m)$ *of [10], then we get a single spanning tree T such that*

$$\frac{1}{m} \sum_{(u,v)\in E} \frac{d_T(u,v)}{d_G(u,v)} \leq O(\log n \cdot (\log\log n)^3).$$

The running time is $O(m \log n)$ if G is unweighted and $O(m \log n + n \log^2 n)$ if G is weighted.

Proof. Since our algorithm works in a similar manner to the [10] algorithm, we can use their partitioning method `ImpConeDecompose`, which has a a running time of $O(m)$ if G is unweighted and $O(m+n\log n)$ if G is weighted. The only difference is that in the first iteration ($j = 1$), instead of picking an arbitrary portal x_1 we pick the node x_1 that is first on a shortest path from x_0 to the first in the queue Q. The average stretch of their cone cutting method is roughly $O(\log n \cdot \log\log n \cdot 1/\epsilon)$ (recall that $\epsilon = 1/\log\log n$), and since the radius of our spanning tree increases by $O(\log\log n)$, the corollary follows. It remains to see that our running time is no worse than [10], and indeed it is easy to see that adding the queues increase the run time only by a constant factor. □

3 Strong Diameter Probabilistic Partitions

$(x, y, r) =$ `cone cut`$(X, x_0, X_0, Y, \epsilon)$:

- Let $p \in Y$ be the point minimizing $\frac{|X|}{|B_{(Y,d_Y)}(z, \epsilon \cdot \mathrm{rad}_{x_0}(X)/16)|}$ over all $z \in Y$; Let χ denote that minimum;
- Let (y, x) be an edge such that $x \in Y$, $y \in X_0$ and $d_X(x_0, y) + d_X(y, x) + d_Y(x, p) = d_X(x_0, p)$ (*i.e.* y and x lie on some shortest path between x_0 and p);
- Choose $r \in [\epsilon/4, \epsilon/2]$ according to the following random process:
  - Divide the interval $[\epsilon/4, \epsilon/2]$ into $N = \lceil 2 \log \chi \rceil$ equal length intervals $S_1, \ldots, S_N$; Let $h = 1$;
  - LOOP: Toss a fair coin; If it turns out head and $h < N$ then let $h = h + 1$ and goto LOOP;
  - Choose r uniformly at random from the interval S_h.
- Return (x, y, r).

Figure 3: `cone-cut` algorithm

Consider a graph $G = (V, E)$, a connected cluster $X \subseteq V$, $x_0 \in X$ and let $\Delta = \mathrm{rad}_{x_0}(X)$. Fix some edge $(u, v) \in E$. Let $X^{(i)} = X^{(i)}(u)$ be a random variable that indicates which cluster contains u in the i-th step of the hierarchical application of the star-partition algorithm[4]. In a similar manner let $x_0^{(i)}$ be the random variable indicating the center of the cluster $X^{(i)}$, and when $X^{(i)}$ is partitioned denote the central ball as $X_0^{(i)}$ and cones as $X_1^{(i)}, \ldots X_m^{(i)}$ where m is a random variable depending on $X^{(i)}$. Let $\mathcal{E}_j(X^{(i)}, u, v)$ be the event that $u, v \in X^{(i)}$ and in the star-partition of the cluster $X^{(i)}$ with center $x_0^{(i)}$ into $X_0^{(i)}, \ldots, X_m^{(i)}$, $u \in X_j^{(i)}$, $v \notin X_j^{(i)}$. Let $\mathcal{E}(X^{(i)}, u, v)$ be the event that $\exists\, 0 \leq j \leq m$ such that $\mathcal{E}_j(X^{(i)}, u, v)$. Some notation:

[4] We abuse notation and think of $X^{(i)}$ as a function to subsets of X (instead of $\mathbb{R}$). We also refer to $X^{(i)}$ as an event.

$\mathbb{E}_{X^{(i)}}[f(X^{(i)})]$ will stand for $\sum_{X'} \Pr[X^{(i)} = X']f(X')$.

Let $\mathcal{T}$ be the support of the distribution over spanning trees induced by the hierarchical star partition algorithm. Let $\mathcal{T}^{(i)} \subseteq \mathcal{T}$ be the set of spanning trees for which event $\mathcal{E}(X^{(i)}, u, v)$ occurs.

$$\begin{aligned}
\mathbb{E}[d_T(u,v)] &\leq \sum_{i\geq 1} \sum_{T\in\mathcal{T}^{(i)}} \Pr[T] \cdot d_T(u,v) \\
&\leq \sum_{i\geq 1} \mathbb{E}_{X^{(i)}} \left[\Pr[\mathcal{E}(X^{(i)}, u, v)] \max_{T\in\mathcal{T}^{(i)}} \{d_T(u,v)\}\right] \\
&\leq O(\log\log n) \sum_{i\geq 1} \mathbb{E}_{X^{(i)}} \left[\Pr[\mathcal{E}(X^{(i)}, u, v)] \cdot \mathrm{rad}_{x_0^{(i)}}(X^{(i)})\right] .
\end{aligned}$$

The last inequality holds since for any $T \in \mathcal{T}^{(i)}$, $d_T(u,v) \leq d_T(u, x_0^{(i)}) + d_T(x_0, v) \leq 2\mathrm{rad}_{x_0^{(i)}}(T)$ and using Lemma 4 we get that $\mathrm{rad}_{x_0^{(i)}}(T) \leq O(\log\log n \cdot \mathrm{rad}_{x_0^{(i)}}(X^{(i)}))$.

In what follows we bound $\mathbb{E}_{X^{(i)}} \left[\Pr[\mathcal{E}(X^{(i)}, u, v)] \cdot \mathrm{rad}_{x_0^{(i)}}(X^{(i)})\right]$. Let $\epsilon = \frac{1}{170c\cdot\log\log|X|}$ and $k = 20c(\ln(1/\epsilon)+5)$. The main lemma to prove is the following

Lemma 7. *For any graph $G = (V,E)$, any edge $(u,v) \in E$, any connected cluster $X^{(i)} \subseteq V$ we have that*

$$\mathbb{E}_{X^{(i)}} \left[\Pr[\mathcal{E}(X^{(i)}, u, v)] \cdot \mathrm{rad}_{x_0^{(i)}}(X^{(i)})\right] \leq C \cdot d(u,v)/\epsilon \cdot \left(\mathbb{E}_{X^{(i)}}[\log|X^{(i)}|] - \mathbb{E}_{X^{(i+k)}}[\log|X^{(i+k)}|]\right) .$$

where C is a universal constant.

Once this lemma is proved, a telescopic sum argument yields that

$$\begin{aligned}
\mathbb{E}[d_T(u,v)] &\leq O(\log\log n) \sum_{i\geq 1} \mathbb{E}_{X^{(i)}} \left[\Pr[\mathcal{E}(X^{(i)}, u, v)] \cdot \mathrm{rad}_{x_0}(X^{(i)})\right] \\
&\leq O(\log\log n) \cdot d(u,v)/\epsilon \sum_{i=1}^{k} \mathbb{E}_{X^{(i)}}[\log|X^{(i)}|] \\
&\leq O(\log n \cdot \log\log n) \cdot d(u,v) \cdot \log(1/\epsilon)/\epsilon \\
&= O(\log n \cdot (\log\log n)^2 \cdot \log\log\log n) \cdot d(u,v) .
\end{aligned}$$

As we stated in the introduction, the algorithm of Figure 3 and proof of Lemma 7 are based on the truncated exponential distribution approach of [5, 1]. The main technical difficulty arises since the space *changes* after each cluster is cut. Dealing with the randomly changing graph raises some additional subtleties in the proof.

We begin with some definitions and an informal description of the algorithm and the proof idea. Fix the edge $(u,v) \in E$, a scale i and $X = X^{(i)}$. Let $Y \subseteq X$ be a random variable indicating that there exists $0 < j \leq m$ such that $Y = Y_{j-1}$ in the star partition of X. Define the local growth rate around $x \in Y$ with respect to Y as

$$\chi(X,Y,x) = \frac{|X|}{|B_{Y,d_Y}(x, \epsilon\Delta/16)|}$$

The algorithm for the partition is as follows: Choose a radius for the central ball around x_0 from a uniform distribution in a range of size $\approx \Delta/c$. The center x_1 is chosen on a shortest path to z_1, the first point in the queue, and then the radius for the cone is again sampled from a uniform distribution in a range of size $\approx \epsilon\Delta$. For $j > 1$ the jth center x_j is chosen on a shortest path to the point $p_j \in Y_{j-1}$ minimizing $\chi_j = \chi(X, Y_{j-1}, p_j)$, and then the radius of the cone is chosen from a truncated exponential distribution, with parameter χ_j.

Denote the event that $Y = Y_{j-1}$ and $u \in X_j$ as $\mathcal{Z}_j(X,Y,u)$, and let $\mathcal{Z}(X,Y,u)$ be the event that $\exists\, 0 \leq j < m$ such that $\mathcal{Z}_j(X,Y,u)$. Note that fixing Y_{j-1} determines deterministically p_j and therefore also x_j and χ_j. Similarly let $\mathcal{Z}_j(X,Y)$ be the event that $Y = Y_{j-1}$ and $\mathcal{Z}(X,Y)$ the event that $\exists\, 0 \leq j < m$ such that $\mathcal{Z}_j(X,Y)$. Let $N(j)$ be the random variable that is the number of partitions $S_1, \ldots, S_{N(j)}$ of the interval $[\epsilon/4, \epsilon/2]$ for the jth cone. Let $0 \leq h(j) \leq N(j)$ be the random variable that is the index of the interval $S_{h(j)}$ from which the radius r_j is uniformly chosen for X_j. Some more notation:

$\mathbb{E}_{Y\subseteq X}[f(Y)]$ will stand for $\sum_{Y\subseteq X}\Pr[\mathcal{Z}(X,Y)]\cdot f(Y)$ (we write $\mathbb{E}_Y$ when X is implicit).

$\mathbb{E}_{Y\subseteq X,j}[f(Y)]$ will stand for $\sum_{Y\subseteq X}\Pr[\mathcal{Z}_j(X,Y)]\cdot f(Y)$ (we write $\mathbb{E}_{Y,j}$ when X is implicit).

$\mathbb{E}_{Y\subseteq X,u}[f(Y)]$ will stand for $\sum_{Y\subseteq X}\Pr[\mathcal{Z}(X,Y,u)]\cdot f(Y)$ (we write $\mathbb{E}_{Y,u}$ when X is implicit).

We divide the event $\mathcal{E}(X,u,v)$ into three cases (by symmetry we can define all these events with respect to u).

- The first is the event that u falls into one of the first two clusters (the central ball X_0 or the first cone X_1). This event is denoted by $\mathcal{G}(X,u)$.
- The second is the event that u is contained in cluster X_j for some $j>1$, such that the cone distance between u and the center x_j is in the last interval *i.e.* that $\rho(x_j,u)/\Delta\in S_{N(j)}$. This event is denoted by $\mathcal{F}(X,u)$. We partition the event $\mathcal{F}(X,u)$ using the different values of j: For any $j>1$ let $\mathcal{F}_j(X,u)$ be the event that $\rho(x_j,u)/\Delta\in S_{N(j)}$, and note that $\mathcal{F}(X,u)$ is simply that there exists $j>1$ such that $\mathcal{F}_j(X,u)$ and also $u\in X_j$.
- The third is the completion of the first two events, that the cluster X_j containing u has $j>1$ and $\rho(x_j,u)/\Delta\notin S_{N(j)}$.

The probability of the first event can be bounded simply by the inverse of the range from which the radius is drawn, so we obtain probability at most $\approx\frac{d(u,v)}{\epsilon\Delta}$.

For the second event we note that reaching the tail of the exponential distribution requires that $N-1$ fair coin tosses turned out head, which is bounded by $\approx\frac{1}{2^N}\approx\frac{1}{\chi_j^2}$, then since we choose uniformly from the last interval, the probability that we separate u,v is $\approx\frac{\log\chi_j\cdot d(u,v)}{\epsilon\Delta\chi_j^2}\le\frac{d(u,v)}{\epsilon\Delta\chi_j}$. Since the parameter χ_j is a random variable which depends on the previous cone cuts, the proof becomes a bit more involved as we need to give a different bound for every possible $Y=Y_{j-1}$. We show that for every star-partition $\sum_{j>1}\chi_j^{-1}\le 1$, hence this also holds in expectation and the second event probability is bounded by $\approx\frac{d(u,v)}{\epsilon\Delta}$. This is shown in Claim 8

Bounding the third event relies on the memoryless property of the exponential distribution. The major technical difficulty is that the bound we show depends on the parameter χ. Hence we can only show the bound given some subspace Y from which we cut the next cone. The bound on the probability obtained here is $\approx\frac{\log\chi\cdot d(u,v)}{\epsilon\Delta}$. This is shown in Claim 9.

The last step is to sum over all scales i, and use a telescopic sum argument on the expectation of the values of the $\log\chi$ showing that they sum to $O(\log(1/\epsilon)\cdot\log n)$. This is shown in the proof of Lemma 7.

Claim 8. *For any cluster $X\subseteq V$, edge $u,v\in X$, $(u,v)\in E$, we have*

$$\Pr[\mathcal{F}(X,u)\wedge\mathcal{E}(X,u,v)]\le 48d(u,v)/(\epsilon\Delta)\,.$$

Proof. Note that we can only bound the probability of event such as $\mathcal{E}_j(X,u,v)$ given that some $Y=Y_{j-1}$ is fixed *i.e.* that event $\mathcal{Z}_j(X,Y)$ occurred (because the parameters x_j and χ_j that govern the next cone creation are random variables depending on Y. So fix some $Y=Y_{j-1}$ and note that indeed p_j, x_j and $\chi_j=\chi(X,Y,p_j)$ are determined deterministically.

$$\begin{aligned}
&\Pr[\mathcal{F}(X,u)\wedge\mathcal{E}(X,u,v)]\\
&\quad=\ \Pr[\exists j>1,\mathcal{F}_j(X,u)\wedge\mathcal{E}_j(X,u,v)]\\
&\quad\le\ \sum_{j\ge2}\Pr[\mathcal{E}_j(X,u,v)\mid\mathcal{F}_j(X,u)]\\
&\quad=\ \sum_{j\ge2}\sum_{Y\subseteq X}\Pr[Z_j(X,Y)]\cdot\Pr[\mathcal{E}_j(X,u,v)\mid\mathcal{F}_j(X,u)\wedge\mathcal{Z}_j(X,Y)]\\
&\quad=\ \sum_{j\ge2}\mathbb{E}_{Y,j}\left[\Pr[\mathcal{E}_j(X,u,v)\mid\mathcal{F}_j(X,u)]\right]
\end{aligned}$$

The first equation holds since the probability to be cut by a cluster whose radius is "large" is the probability that some cluster X_j with large radius separates u, v. The first inequality holds by the union bound and the second equation since for every event A and pairwise disjoint events $B_1, \ldots, B_\ell$ with $\sum_{i=1}^{\ell} \Pr[B_i] = 1$ it holds that $\Pr[A] = \sum_{i=1}^{\ell} \Pr[B_i] \cdot \Pr[A \mid B_i]$. Here the events B are $\mathcal{Z}_j(X, Y)$ which are disjoint for different subgraphs Y. Note that events $\mathcal{F}_j(X, u)$ and $\mathcal{Z}_j(X, Y)$ tell us nothing of the radius of the next cone X_j, therefore the probability of $\mathcal{E}_j(X, u, v)$ given the subspace Y_{j-1} and that $\rho(x_j, u)/\Delta \in S_{N(j)}$ (where $\rho = \rho(X, Y \cup X_0, d', x_0, x_j)$ is the cone metric), is the probability that $h(j) = N(j)$ (recall that the random variable $h(j)$ is the index of the interval $S_{h(j)}$ from which the radius is uniformly chosen for X_j) and that the uniform choice in the interval $S_{N(j)}$ hits the place that separates u, v. To bound the first one

$$\Pr[h(j) = N(j)] = 2^{-(N(j)-1)} \leq 2^{-2\log \chi_j + 2} = 4/\chi^2,$$

and the probability of the second event is $\frac{d(u,v)}{\Delta |S_{N(j)}|}$. Note that $|S_{N(j)}| = \frac{\epsilon}{4\lceil 2 \log \chi_j \rceil} \geq \frac{\epsilon}{8 \log \chi_j + 4} \geq \min\{1, \frac{1}{\log \chi_j}\} \cdot \frac{\epsilon}{12}$. These two events are independent, hence

$$\begin{aligned}
\Pr[\mathcal{F}(X, u) \wedge \mathcal{E}(X, u, v)] &\leq \frac{48 d(u, v)}{\epsilon \cdot \Delta} \sum_{j \geq 1} \mathbb{E}_{Y,j} \left[\max \left\{ \frac{1}{\chi_j^2}, \frac{\log \chi_j}{\chi_j^2} \right\} \right] \\
&\leq \frac{48 d(u, v)}{\epsilon \cdot \Delta} \sum_{j \geq 1} \mathbb{E}_{Y,j}[\chi_j^{-1}]
\end{aligned}$$

For any $\bar{Y} = (\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_n) \subset X^n$ let $\mathcal{Z}(\bar{Y})$ be the event $\bigwedge_{1 \leq j \leq n} \mathcal{Z}(X, \bar{Y}_j, j)$ (where $\bar{Y}_j$ is the jth component of $\bar{Y}$). Observe that for any j and $Y \subset X$ we have $\Pr[\mathcal{Z}(X, Y, j)] = \sum_{\bar{Y} \subset X^n, \bar{Y}_j = Y} \Pr[\mathcal{Z}(\bar{Y})]$. Therefore

$$\begin{aligned}
\sum_{j > 1} \mathbb{E}_{Y,j}[\chi_j^{-1}] &= \sum_{j > 1} \sum_{Y \subseteq X} \Pr[Z(X, Y, j)] \cdot \chi_j^{-1} \\
&= \sum_{j \geq 1} \sum_{\bar{Y} \subset X^n} \Pr[\mathcal{Z}(\bar{Y})] \cdot \chi_j^{-1} \\
&= \sum_{\bar{Y} \subset X^n} \Pr[\mathcal{Z}(\bar{Y})] \sum_{j \geq 1} \chi_j^{-1}
\end{aligned}$$

Now it is enough to show that for any $X_0, X_1, \ldots, X_m$ that may occur in the start-partition algorithm (*i.e.* $\Pr[\mathcal{Z}(\bar{Y})] > 0$, given that $\bar{Y}_j = X \setminus \bigcup_{\ell < j} X_\ell$) we have $\sum_{j=1}^{m} \chi_j^{-1} \leq 1$. This holds because for any $2 \leq \ell < j \leq m$ we have that $B_{Y_\ell, d_{Y_\ell}}(p_\ell, \epsilon\Delta/16) \subseteq X_\ell$, and $Y_j \cap X_\ell = \emptyset$, *i.e.* $B_{Y_\ell, d_{Y_\ell}}(p_i, \epsilon\Delta/16) \cap B_{Y_j, d_{Y_j}}(p_j, \epsilon\Delta/16) = \emptyset$. Therefore

$$\sum_{j=1}^{m} \chi_j^{-1} \leq |X|^{-1} \sum_{j=1}^{m} B_{Y_j, d_j}(p_j, \epsilon\Delta/16) \leq 1.$$

□

Claim 9. *For any cluster $X \subseteq V$, edge $u, v \in X$, $(u, v) \in E$, subgraph $Y \subset X$ we have*

$$\Pr[\mathcal{E}(X, u, v) \wedge \neg\mathcal{F}(X, u) \mid \neg\mathcal{G}(X, u) \wedge \mathcal{Z}(X, Y, u)] \leq 12 d(u, v) \max\{1, \log \chi(X, Y, u)\}/(\epsilon \cdot \Delta)$$

Proof. If $d(u, v) \geq \epsilon \cdot \Delta/12$ the the claim is trivial, so assume it is smaller. Let $j > 1$ be such that the next cone to be cut is X_j (the value of j is not relevant, we fix it in order to simplify the notation), and recall that fixing $Y = Y_{j-1}$ determines deterministically p_j, x_j and χ_j. Let $\rho = \rho(X_0 \cup Y, Y, x_0, x_j)$ be the appropriate cone metric on Y by which the next cone is cut.

$$\begin{aligned}
\Pr[\mathcal{E}(X, u, v) \wedge \neg\mathcal{F}(X, u) \mid \mathcal{Z}(X, Y, u)] &\leq \Pr[\mathcal{E}_j(X, u, v) \wedge \neg\mathcal{F}_j(X, u) \mid \mathcal{Z}(X, Y, u) \wedge \mathcal{Z}(X, Y)] \\
&\leq \Pr[\mathcal{E}_j(X, u, v) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y, u) \wedge \mathcal{Z}(X, Y)] \\
&\leq \frac{\Pr[\mathcal{E}_j(X, u, v) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y)]}{\Pr[\mathcal{Z}(X, Y, u) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y)]}
\end{aligned}$$

The first inequality holds since event $\mathcal{Z}(X, Y, u)$ implies that $u \in X_j$ so the events $\mathcal{E}(X, u, v)$ and $\mathcal{E}_j(X, u, v)$ are equivalent (the same holds for $\neg\mathcal{F}(X, u)$), and because $\mathcal{Z}(X, Y, u) \subseteq \mathcal{Z}(X, Y)$. The second is by the definition of $\mathcal{F}(X, u)$ (given that $u \in X_j$ it cannot be that $\rho(x_j, u)/\Delta$ falls in the interval $S_{N(j)}$), and since for any events A, B, $\Pr[A \wedge B] \le \Pr[A \mid B]$. The third is by Bayes rule and since $\mathcal{E}_j(X, u, v) \wedge \mathcal{Z}(X, Y, u) = \mathcal{E}_j(X, u, v)$. Let ℓ be such that $\rho(x_j, u)/\Delta \in S_\ell$.

First we bound the denominator, noting that there is no prior information given about the distribution for the next choice of radius. Since $\ell < N(j)$ we can bound $\Pr[\mathcal{Z}(X, Y, u) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y)] \ge 2^{-\ell}$, since with this probability the radius for the cone X_j will be chosen from $S_m \cdot \Delta$ with $m > \ell$ so it will large enough to contain u. The numerator $\Pr[\mathcal{E}_j(X, u, v) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y)]$ can be bounded by $\frac{1}{2^{\ell-1}} \cdot \frac{1}{2} \cdot \frac{d(u,v)}{\Delta|S_\ell|}$, which is the probability that we reach the ℓ-th interval, not continue to the next one (note that the next interval exists because $\ell < N(j)$) and when choosing r_j uniformly from S_ℓ, it happens to be the place that separates u, v. The probability for the first event is $2^{-(\ell-1)}$, the second is $1/2$, and the third is $\frac{d(u,v)}{\Delta|S_\ell|}$. Since $|S_\ell| \ge \min\{1, \frac{1}{\log \chi_j}\} \cdot \frac{\epsilon}{12}$ it follows that $\Pr[\mathcal{E}_j(X, u, v) \mid \rho(x_j, u)/\Delta \notin S_{N(j)} \wedge \mathcal{Z}(X, Y)] \le \frac{12 d(u,v) \max\{1, \log \chi_j\}}{\epsilon \cdot \Delta \cdot 2^\ell}$. We conclude that

$$\Pr[\mathcal{E}(X, u, v) \wedge \neg\mathcal{F}(X, u) \mid \mathcal{Z}(X, Y, u)] \le \frac{12 d(u, v) \max\{1, \log \chi_j\}}{\epsilon \cdot \Delta}.$$

□

Proof of Lemma 7. Fix any $i \ge 1$ and $X^{(i)} = X^{(i)}(u)$. As described before we partition the event $\mathcal{E}(X^{(i)}, u, v)$, given a fixed cluster $X^{(i)}$ into the three cases.

$$\begin{aligned}
&\Pr[\mathcal{E}(X^{(i)}, u, v)] \\
&\quad = \Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \mathcal{F}(X^{(i)}, u)] + \Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \neg\mathcal{F}(X^{(i)}, u)] \\
&\quad = \Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \mathcal{F}(X^{(i)}, u)] + \Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \mathcal{G}(X^{(i)}, u)] + \Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \neg\mathcal{F}(X^{(i)}, u) \wedge \neg\mathcal{G}(X^{(i)}, u)]
\end{aligned}$$

The last equality holds since event $\mathcal{G}(X^{(i)}, u)$ implies that $\neg\mathcal{F}(X^{(i)}, u)$. We claim that the following hold:

$$\Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \mathcal{F}(X^{(i)}, u) \mid X^{(i)}] \le 48 d(u, v)/(\epsilon\Delta) \tag{2}$$

$$\Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \mathcal{G}(X^{(i)}, u) \mid X^{(i)}] \le 5 d(u, v)/(\epsilon\Delta) \tag{3}$$

$$\Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \neg\mathcal{F}(X^{(i)}, u) \wedge \neg\mathcal{G}(X^{(i)}, u) \mid X^{(i)}] \le 12 d(u, v)/(\epsilon\Delta) \cdot \mathbb{E}_{Y,u}[\max\{1, \log \chi(X, Y, u)\}] \tag{4}$$

(2) holds directly from Claim 8. (3) since the radius of the central ball is chosen uniformly from interval of length $\Delta/(16c) \ge \epsilon\Delta$, and for the first cone from interval of length $\epsilon\Delta/4$. (4) holds by using Claim 9 and writing

$$\begin{aligned}
\Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \neg\mathcal{F}(X^{(i)}, u) \wedge \neg\mathcal{G}(X^{(i)}, u)] &\le \mathbb{E}_{Y,u}\left[\Pr[\mathcal{E}(X^{(i)}, u, v) \wedge \neg\mathcal{F}(X^{(i)}, u) \mid \neg\mathcal{G}(X^{(i)}, u)]\right] \\
&\le \frac{12 d(u, v)}{\epsilon \cdot \Delta} \mathbb{E}_{Y,u}[\max\{1, \log \chi(X, Y, u)\}]
\end{aligned}$$

Combining these three equation yields that for $C = 65$

$$\Pr[\mathcal{E}(X^{(i)}, u, v)] \le C \cdot d(u, v)/(\epsilon\Delta) \cdot \mathbb{E}_{Y,u}[\max\{1, \log \chi(X, Y, u)\}] .$$

Recall that $k = 20c(\ln(1/\epsilon) + 5)$, and Corollary 3 suggests that for any cluster X and any $j \ge 0$ that $\mathrm{rad}_{x_j}(X_j) \le (1 - 1/(20c))\mathrm{rad}_{x_0}(X)$, hence for any event $X^{(i+k)}$, given that $X^{(i)}$ happened

$$\mathrm{rad}(X^{(i+k)}) \le (1 - 1/(20c))^k \cdot \mathrm{rad}(X^{(i)}) \le \epsilon \cdot \mathrm{rad}(X^{(i)})/32,$$

therefore $\mathrm{diam}(X^{(i+k)}) \le \epsilon \cdot \mathrm{rad}(X^{(i)})/16$ and by definition $u \in X^{(i+k)}$, so fixing any Y such that event $\mathcal{Z}(X^{(i)}, Y, u)$

occurred then if $X^{(i+k)} \subseteq Y$ also $X^{(i+k)} \subseteq B_{Y,d_Y}(u, \epsilon \cdot \mathrm{rad}(X^{(i)})/16)$.

$$\begin{aligned}
\mathbb{E}_{Y,u}[\log \chi(X^{(i)}, Y, u)] &= \log |X^{(i)}| - \mathbb{E}_{Y,u}[\log |B_{Y,d_Y}(u, \epsilon \cdot \mathrm{rad}(X^{(i)})/16|] \\
&\leq \log |X^{(i)}| - \mathbb{E}_{Y,u}\left[\sum_{X^{(i+k)} \subseteq Y} \Pr[X^{(i+k)} \mid \mathcal{Z}(X^{(i)}, Y, u)] \cdot \log |X^{(i+k)}|\right] \\
&= \log |X^{(i)}| - \sum_{X^{(i+k)} \subseteq X^{(i)}} \Pr[X^{(i+k)} \mid X^{(i)}] \cdot \log |X^{(i+k)}|
\end{aligned}$$

We conclude that

$$\begin{aligned}
\mathbb{E}_{X^{(i)}}\left[\Pr[\mathcal{E}(X^{(i)}, u, v)]\right] &\leq \mathbb{E}_{X^{(i)}}\left[\log |X^{(i)}| - \sum_{X^{(i+k)} \subseteq X^{(i)}} \Pr[X^{(i+k)} \mid X^{(i)}] \cdot \log |X^{(i+k)}|\right] \\
&= \mathbb{E}_{X^{(i)}}[\log |X^{(i)}|] - \left[\sum_{X^{(i)}} \Pr[X^{(i)}] \sum_{X^{(i+k)} \subseteq X^{(i)}} \Pr[X^{(i+k)} \mid X^{(i)}] \cdot \log |X^{(i+k)}|\right] \\
&= \mathbb{E}_{X^{(i)}}[\log |X^{(i)}|] - \mathbb{E}_{X^{(i+k)}} \log |X^{(i+k)}|
\end{aligned}$$

□

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A Extending to Weighted Graphs

In order for our algorithm to work for general weighted graphs, we will make the following change: After choosing the points x_j, y_j in the cone-cut algorithm, create an imaginary point y'_j which lies on the edge y_j, x_j such that $d(x_0, y'_j) = \mathrm{rad}_{x_0}(X_0)$, then return the point y'_j. Note that then the inequality $\mathrm{rad}_{x_0}(X_0) + d(y'_j, x_j) + \mathrm{rad}_{x_j}(X_j) \leq (1+\epsilon)\mathrm{rad}_{x_0}(X)$ will hold, which is the only place we used the unweighted property of G. With a slight change to the algorithm the number of imaginary points added is at most the number of edges in the original graph G. This is because the point y'_j is connected only to y_j in the central ball X_0, so if in the recursion depth when cutting a cluster $\hat{X}$, the edge is cut by the central ball $\hat{X}_0$, then the cone $\hat{X}_\ell$ created will contain only one point - $y_j = x_\ell$, so in such a case it will hold that $\mathrm{rad}_{x_0}(\hat{X}_0) + d(y_j, y'_j) + \mathrm{rad}_{x_\ell}(\hat{X}_\ell) \leq (1+\epsilon)\mathrm{rad}_{x_0}(\hat{X})$, and we will not add another imaginary point.

The other change to the algorithm is contraction of small edges, following [10]. Let $G = (V, E)$ be the original graph of size n. At every recursive step of `hierarchical star partition` for a cluster X with $\Delta = \mathrm{rad}(X)$ we contract all edges shorter than $c\Delta/n$ for a constant c. Then these small edges will not be cut - it guarantees that every edge is at risk in at most $O(\log n)$ recursive steps. It remains to show that the radius does not increase - note that adding back all these edges will increase the radius by at most $c\Delta$, and also note that our inductive proof actually has a slack of $c'\Delta$, *i.e.* if we need to bound $d_T(x_0, z_i)$ by $i \cdot \Delta$ then we actually show that $d_T(x_o, z_i) \leq i \cdot \Delta - c'\Delta$. Now choosing $c < c'$ will guarantee that even after expanding back all the edges we contracted the radius bound still holds. The last issue is the choice of portals in the expanded graph. If $\hat{x_j}$ is the super node in the j-th portal (recall that y'_j is an added imaginary point), we choose $x_j \in \hat{x_j}$ which is connected to some vertex in $\hat{y_j}$ and also lies on the shortest path from x_0 to z_i.

B Improving the stretch slightly

The factor of $c \log \log i$ that was chosen as a bound on the radius increase in Lemma 4 was actually arbitrary. In fact we can replace it with almost any other monotone increasing function of i, the position in the queue. In order to optimize (asymptotically) the stretch, we take a very slowly increasing function of i, using the following definitions: Recall that $\log^{(0)} n = n$ and for any integer $1 \le t \le \log^* n$, $\log^{(t)} n = \log\left(\log^{(t-1)} n\right)$. We use $\log^* n = \min\{t \mid 1 \le \log^{(t)} n < 2\}$. For any integer $1 \le t \le \log^* n$ let $\varphi_t(n) = \prod_{k=2}^{t} \log^{(k)} n$, (when $t = 1$ let $\varphi_1(n) = 1$).

The following two technical claims are proven in Appendix C.

Claim 10. *For any $0 \le a \le 1$, $i \ge 4$ and integer $2 \le t \le \log^* i$,*

$$\log^{(t)}(i^a) \le \log^{(t)} i + (\log a)/\varphi_{t-1}(i).$$

Claim 11. *For any $a \ge 1$, $i \ge 16$ and integer $2 \le t \le \log^* i$,*

$$\log^{(t)}(ai) \le \log^{(t)} i + (2 \log a)/\log i.$$

The parameter c that was a constant can now be arbitrary number $c \ge 2^{18}$, *i.e.* it can be a function of $|X|$. We also use a different value of $\epsilon = \frac{1}{170c \cdot \varphi_t(n)}$ for the star partition. Now the lemma that gives a tighter bound on the radius is the following:

Lemma 12. *Let $1 \le t \le \log^* c$ be an integer. Let (X, d) be the metric derived from an unweighted graph $G = (V, E)$, $x_0 \in X$ and $Q = (z_1, \ldots, z_{|X|-1})$ any ordering of $X \setminus \{x_0\}$, also let T be the spanning tree of G returned by the algorithm* `hierarchical-star-partition`(X, x_0, Q) *with parameter $\epsilon = \epsilon(X, c, t) = \frac{1}{90c\varphi_t(|X|)}$, then*

$$d_T(x_0, z_i) \le \begin{cases} d(x_0, z_i) & i = 1 \\ i \cdot \mathrm{rad}_{x_0}(X) & 1 < i < c \\ c \cdot \log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) & \textit{otherwise} \end{cases}$$

Proof. The proof is by induction on the radius of X. Note that Corollary 3 guarantees that for all $j = 0, \ldots, m$ we have $1 \le \mathrm{rad}_{x_j}(X_j) < \mathrm{rad}_{x_0}(X)$.

Case 1: The case $i = 1$ is identical to Lemma 4.

Case 2: The second case to consider is when $1 < i < c$.

1. First assume that $z_i \in X_0$. Then z_i will be at most i in the ordering of $Q_0^{(\text{ball})}$ and hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on X_0 : $d_T(x_0, z_i) \le c \log^{(t)}(3i) \cdot \mathrm{rad}_{x_0}(X_0) \le i \cdot \mathrm{rad}_{x_0}(X)$, using that $\mathrm{rad}_{x_0}(X_0) \le \mathrm{rad}_{x_0}(X)/(4c)$, and that $\log^{(t)}(3i) \le 2i$.

2. Next assume that $\{z_1, \ldots, z_i\} \subseteq X_1$. As y_1 is the first in Q_0, by the induction hypothesis on X_0 and X_1 we have that $d_T(x_0, y_1) \le d_X(x_0, y_1) \le \mathrm{rad}_{x_0}(X_0)$ and $d_T(x_1, z_i) \le i \cdot \mathrm{rad}_{x_1}(X_1)$, so

$$\begin{aligned} d_T(x_0, z_i) &\le d_T(x_0, y_1) + d_T(y_1, x_1) + d_T(x_1, z_i) \\ &\le \mathrm{rad}_{x_0}(X_0) + i \cdot \mathrm{rad}_{x_1}(X_1) + d(y_1, x_1) \\ &\le i(\mathrm{rad}_{x_0}(X_0) + d(y_1, x_1) + \mathrm{rad}_{x_1}(X_1)) - (i-1)\mathrm{rad}_{x_0}(X_0) \\ &\le i(1+\epsilon)\mathrm{rad}_{x_0}(X) - (i-1)\mathrm{rad}_{x_0}(X)/(16c) \\ &\le i \cdot \mathrm{rad}_{x_0}(X) + i \cdot \mathrm{rad}_{x_0}(X)/(170c) - i \cdot \mathrm{rad}_{x_0}(X)/(32c) \\ &\le i \cdot \mathrm{rad}_{x_0}(X). \end{aligned}$$

In the fourth inequality using that $\mathrm{rad}_{x_0}(X_0) \ge \mathrm{rad}_{x_0}(X)/(16c)$, in the fifth that $i/(i-1) \le 2$.

3. Now assume that $z_i \in X_j$ where not all of $z_1, \dots, z_i$ are in X_j. This case further subdivides to two main cases, the second one divides to two subcases (this complication arises since c is not a constant anymore).

   (a) If $i \le c/4$: First note that z_i must be at most the $i-1$ element in Q_j. By the insert sequence to $Q_0^{(\text{reg})}$ we have that y_j is at most the $3i < c$ element in Q_0. Using the induction hypothesis on X_0 and X_j we get that

$$\begin{aligned}
d_T(x_0, z_i) &\le d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\le i \cdot \text{rad}_{x_0}(X_0) + (i-1) \cdot \text{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&\le (i-1)(\text{rad}_{x_0}(X_0) + d(y_j, x_j) + \text{rad}_{x_j}(X_j)) + \text{rad}_{x_0}(X_0) \\
&\le (i-1)(1+\epsilon)\text{rad}_{x_0}(X) + \text{rad}_{x_0}(X)/(8c) \\
&\le i \cdot \text{rad}_{x_0}(X) - \text{rad}_{x_0}(X) + (i-1) \cdot \text{rad}_{x_0}(X)/(170c) + \text{rad}_{x_0}(X)/(8c) \\
&\le i \cdot \text{rad}_{x_0}(X).
\end{aligned}$$

   (b) Otherwise $i > c/4$, then there are two cases:

   - If $|X_j \cap \{z_1, \dots, z_i\}| \le \sqrt{i}$ then z_i will be at most the $\sqrt{i}$ in Q_j and y_j will be at most the $3i$ in Q_0. Note that for $c > 100$, $\sqrt{i} < i/2$, and also $\log^{(t)}(3i) < i$ for all $t \ge 1$, hence

$$\begin{aligned}
d_T(x_0, z_i) &\le d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\le c \log^{(t)}(3i) \cdot \text{rad}_{x_0}(X_0) + (i/2) \cdot \text{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&\le c \cdot i \cdot \text{rad}_{x_0}(X_0) + (i/2) \cdot \text{rad}_{x_0}(X) \\
&\le i \cdot \text{rad}_{x_0}(X)/8 + (i/2)\text{rad}_{x_0}(X) \\
&\le i \cdot \text{rad}_{x_0}(X),
\end{aligned}$$

     using that $\text{rad}_{x_0}(X_0) \le \text{rad}_{x_0}(X)/(8c)$.
   - If $|X_j \cap \{z_1, \dots, z_i\}| > \sqrt{i}$ then z_i will be at most the i-th in Q_j and by Claim 5 y_j will be at most the $i^{9/10}$ in Q_0. Note that $i^{9/10} < i/2$, then by the induction hypothesis

$$\begin{aligned}
d_T(x_0, z_i) &\le d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\le (i/2) \cdot \text{rad}_{x_0}(X_0) + i \cdot \text{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&\le i \cdot (\text{rad}_{x_0}(X_0) + d(y_j, x_j) + \cdot\text{rad}_{x_j}(X_j)) - (i/2) \cdot \text{rad}_{x_0}(X_0) \\
&\le i \cdot \text{rad}_{x_0}(X) + \epsilon \cdot i \cdot \text{rad}_{x_0}(X) - i \cdot \text{rad}_{x_0}(X)/(32c) \\
&\le i \cdot \text{rad}_{x_0}(X),
\end{aligned}$$

     using that $\text{rad}_{x_0}(X_0) \ge \text{rad}_{x_0}(X)/(16c)$.

Case 3: The third case when $i \ge c$:

1. If $z_i \in X_0$ then it will be at most i in the ordering of $Q_0^{(\text{ball})}$ hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on X_0 we get that

$$d_T(x_0, z_i) \le c \log^{(t)}(3i) \cdot \text{rad}_{x_0}(X_0) \le 2c \log^{(t)}(i) \cdot \text{rad}_{x_0}(X_0) \le c \log^{(t)} i \cdot \text{rad}_{x_0}(X),$$

   using that for $i \ge c$, $3i < i^2$ hence $\log^{(t)}(3i) \le 2 \log^{(t)} i$ for all t.

2. The second case is when $z_i \in X_j$ such that $|X_j \cap \{z_1, \dots, z_i\}| \le \sqrt{i}$, then z_i will be at most the $\sqrt{i}$ in Q_j, and y_j will be at most the i-th in $Q_0^{(\text{reg})}$ and hence at most $3i$ in the ordering of Q_0. By the induction hypothesis on

X_0 and X_j, for $t \geq 2$:

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_j) + d_T(y_j, x_j) + d_T(x_j, z_i) \\
&\leq c\log^{(t)}(3i) \cdot \mathrm{rad}_{x_0}(X_0) + c\log^{(t)}(\sqrt{i}) \cdot \mathrm{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&\leq c(\log^{(t)} i + 4/\log i) \cdot \mathrm{rad}_{x_0}(X_0) + c(\log^{(t)} i - 1/\varphi_{t-1}(i)) \cdot \mathrm{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&= c(\log^{(t)} i - 1/\varphi_{t-1}(i)) \left(\mathrm{rad}_{x_0}(X_0) + d(y_j, x_j) + \mathrm{rad}_{x_j}(X_j)\right) + c(4/\log i + 1/\varphi_{t-1}(i)) \cdot \mathrm{rad}_{x_0}(X_0) \\
&\leq c(\log^{(t)} i - 1/\varphi_{t-1}(i))(1+\epsilon)\mathrm{rad}_{x_0}(X) + (1/(2\log i) + 1/(8\varphi_{t-1}(i)))\mathrm{rad}_{x_0}(X) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) + \mathrm{rad}_{x_0}(X) \cdot \log^{(t)} i/(170\varphi_t(i)) - \mathrm{rad}_{x_0}(X)/\varphi_{t-1}(i) + 3\mathrm{rad}_{x_0}(X)/(4\varphi_{t-1}(i)) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) + \mathrm{rad}_{x_0}(X)/(170\varphi_{t-1}(i)) - \mathrm{rad}_{x_0}(X)/(4\varphi_{t-1}(i)) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X),
\end{aligned}$$

the third inequality using Claim 10 and Claim 11. The fifth inequality holds since for every $s \geq 1$, $\log i \geq \varphi_s(i)$, and so $\mathrm{rad}_{x_0}(X)/\log i \leq \mathrm{rad}_{x_0}(X)/\varphi_{t-1}(i)$, and the sixth because $\log^{(t)} i/\varphi_t(i) = 1/\varphi_{t-1}(i)$.

In a similar manner, it can shown that the same holds for $t = 1$.

3. The last case is where $z_i \in X_j$ such that $|X_j \cap \{z_1, \ldots, z_i\}| > \sqrt{i}$, then z_i will be at most the i in Q_j and by Claim 5 y_j will be at most the $i^{9/10}$ in Q_0. Now by the induction hypothesis, for $t \geq 2$

$$\begin{aligned}
d_T(x_0, z_i) &\leq d_T(x_0, y_j) + d(y_j, x_j) + d_T(x_j, z_i) \\
&\leq c\log^{(t)} i^{9/10} \cdot \mathrm{rad}_{x_0}(X_0) + c\log^{(t)} i \cdot \mathrm{rad}_{x_j}(X_j) + d(y_j, x_j) \\
&\leq c\log^{(t)} i(\mathrm{rad}_{x_0}(X_0) + d(y_j, x_j) + \mathrm{rad}_{x_j}(X_j)) + c\log(9/10)/\varphi_{t-1}(i) \cdot \mathrm{rad}_{x_0}(X_0) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) + \epsilon \cdot c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) - c \cdot \mathrm{rad}_{x_0}(X_0)/(10\varphi_{t-1}(i)) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X) + \mathrm{rad}_{x_0}(X) \cdot \log^{(t)} i/(170\varphi_t(i)) - \mathrm{rad}_{x_0}(X)/(160\varphi_{t-1}(i)) \\
&\leq c\log^{(t)} i \cdot \mathrm{rad}_{x_0}(X),
\end{aligned}$$

the third inequality using Claim 10. In a similar manner, it can shown that the same holds for $t = 1$.

□

Proof of Theorem 1. Take $c = 2^{18}\log^{(t)} n$ (recall that $t = (\log^* n)/2$ and indeed $t \leq \log^* c$). Note that $1/\epsilon = O(c\varphi_t(n))$ and the parameter $k = O(c\log(1/\epsilon)) = O(c\log\log\log n)$. The increase in radius is $\mathrm{rad}(T) \leq O(c^2\mathrm{rad}(X))$, and plugging in these parameters to Lemma 7 implies that the expected stretch for any edge $(u, v) \in E$ is bounded by

$$\begin{aligned}
\mathbb{E}_{T\sim\mathcal{T}}[d_T(u, v)] &\leq O\left(c^2 \cdot \log n \cdot k/\epsilon\right) \\
&= O(c^4 \log n \cdot \log\log\log n \cdot \varphi_t(n)) \\
&= O\left(\log^{(1)} n \cdot \log^{(2)} n \cdot \log^{(3)} n \cdots \log^{((\log^* n)/2)} n \cdot \left(\log^{((\log^* n)/2)} n\right)^4 \cdot \log^{(3)} n\right)
\end{aligned}$$

□

C Proof of some claims

Proof of Claim 10. We prove by induction on t. The base case where $t = 2$ holds since $\log\log(i^a) = \log(a \log i) = \log\log i + \log a$. Assume the claim holds for t and we prove for $t+1$

$$\begin{aligned}
\log^{(t+1)}(i^a) &= \log\left(\log^{(t)}(i^a)\right) \\
&\leq \log\left(\log^{(t)} i + (\log a)/\varphi_{t-1}(i)\right) \\
&= \log\left(\log^{(t)} i \cdot \left(1 + (\log a)/(\varphi_{t-1}(i) \cdot \log^{(t)} i)\right)\right) \\
&\leq \log\left(\log^{(t)} i \cdot \left(2^{\log a/\varphi_t(i)}\right)\right) \\
&= \log^{(t+1)} i + (\log a)/\varphi_t(i).
\end{aligned}$$

The first inequality uses the induction hypothesis, the last inequality holds because $\log a \leq 0$, and $1 + x \leq 2^x$ for $x \leq 0$. ☐

Claim 13. *For any $c \geq 0$, $b \geq 2$ and $0 \leq t \leq \log^* b$*

$$\log^{(t)}(b + c) \leq (\log^{(t)} b) + c.$$

Proof. By induction on t, for $t = 0$ it holds since by definition $\log^{(0)}(b+c) = b+c$. Assume for $t-1$ and prove for t

$$\begin{aligned}
\log^{(t)}(b + c) &= \log^{(t-1)}(\log(b \cdot (1 + c/b))) \\
&\leq \log^{(t-1)}(\log b + (c \log e)/b) \\
&\leq \log^{(t)} b + (c \log e)/b \\
&\leq \log^{(t)} b + c.
\end{aligned}$$

We used the induction hypothesis in the second inequality. ☐

Proof of Claim 11.

$$\begin{aligned}
\log^{(t)}(ai) &= \log^{(t-1)}(\log i + \log a) \\
&= \log^{(t-1)}(\log i \cdot (1 + (\log a)/\log i)) \\
&\leq \log^{(t-1)}\left(\log i \cdot e^{(\log a)/\log i}\right) \\
&= \log^{(t-2)}(\log\log i + (\log a \cdot \log e)/\log i) \\
&\leq \log^{(t)} i + (2\log a)/\log i.
\end{aligned}$$

The last inequality we use Claim 13 with $b = \log\log i > \log e$. ☐