Automorphism Ensemble Decoding of Quasi-Cyclic LDPC Codes by Breaking Graph Symmetries

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Abstract—We consider automorphism ensemble decoding (AED) of quasi-cyclic (QC) low-density parity-check (LDPC) codes. Belief propagation (BP) decoding on the conventional factor graph is equivariant to the quasi-cyclic automorphisms and therefore prevents gains by AED. However, by applying small modifications to the parity-check matrix at the receiver side, we can break the symmetry without changing the code at the transmitter. This way, we can leverage a gain in error-correcting performance using an ensemble of identical BP decoders, without increasing the worst-case decoding latency. The proposed method is demonstrated using LDPC codes from the CCSDS, 802.11n and 5G standards and produces gains of 0.2 to 0.3 dB over conventional BP decoding. Compared to the similarly performing saturated BP (SBP), the proposed algorithm reduces the average decoding latency by more than eight times.

I. INTRODUCTION

Quasi-cyclic (QC) low-density parity-check (LDPC) codes are the error-correction workhorse of modern communication systems (e.g., CCSDS, Wi-Fi 802.11n and 5G [1] standards), motivated by the presence of a well-understood, lowcomplexity belief propagation (BP) decoder. Long LDPC codes constructed using classical information theoretic design tools can closely approach the Shannon limit under BP decoding [2]. However, in the short block-length regime (block-lengths of few hundreds of bits) LDPC codes perform poorly when compared to other structured algebraic coding (e.g., Bose-Chaudhuri-Hocquenghem (BCH), Reed-Muller (RM) and cyclic redundancy check (CRC)-aided polar codes), see [3] for an exhaustive comparison. The degraded error-rate performance can be attributed to the non-optimal BP decoding algorithm (when compared to the maximum likelihood (ML) decoder) and the sub-optimality of the short length LDPC code design. The problem of designing short length LDPC codes is out of the scope of this paper.

In this paper, we are interested in enhancing the decoding algorithm itself without changing the code structure. We show ways of enhancing the error-rate performance under iterative decoding with reduced latency while relaxing the complexity constraint. Remember that in the decoding problem we know the optimal solution (i.e., ML or maximum a posteriori (MAP) decoders), however, due to the infeasible complexity for practical codes, we have to rely on sub-optimal decoders with a practical decoding cost (e.g., sum-product algorithm (SPA) BP

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This work is supported by the German Federal Ministry of Education and Research (BMBF) within the project Open6GHub (grant no. 16KISK019).

decoder in the LDPC decoding context). To highlight the suboptimality of the LDPC BP decoder in the short-length regime we refer to [4, Fig. 4] and [5, Fig. 10]. For short-length LDPC codes, a huge performance gap (in E_b/N_0) exists between BP decoding and the ML bound which can be estimated via an ML-approaching ordered statistic decoding (OSD). Closing this performance gap is the main motive behind this work.¹

Ensemble decoding is a method to improve decoding performance by employing L parallel independent BP decoders each proposing a codeword estimate and then selecting the most likely candidate as the decoder output. Two instances of ensemble decoding are augmented BP [6] and saturated BP (SBP) [7] decoding, where all possible combinations of saturated log-likelihood ratio (LLR) values in the S least reliable positions of the received sequence are used as inputs to the constituent decoders. Another variant of ensemble decoding is multiple-bases belief propagation (MBBP) decoding [8] (or belief propagation list (BPL) decoding in the context of polar codes [9]), where each BP decoder uses a different decoding graph rather than a different input. When the automorphism group of the code is known, identical constituent decoders decoding permuted versions of the channel output may be used, yielding so-called automorphism ensemble decoding (AED). This has been successfully applied to high-density cyclic codes [10], RM codes [11] and polar codes [12]. Moreover, a sequential (rather than parallel) variant of automorphism-based decoding has been proposed in [13].

For QC LDPC codes, however, the decoder equivariance phenomenon [14] previously prevented successful application of AED. We show that a small variation in the decoding Tanner graph is enough to exploit AED with permutation vectors from the automorphism group of the considered OC LDPC code, i.e., quasi-cyclic shifts of the code symbols. Thus, our proposed decoding algorithm can be directly applied to standardized state-of-the-art QC LDPC codes without any special code design constraint (i.e., no changes on the encoder side, when compared to [14], [15]). Standardized codes are usually flexible in codelength by specifying different protograph lifting factors and, thus, many receiver architectures already provide parallel hardware resources used only for large block-lengths. AED may exploit these additional resources as independent parallel decoders and, thus, promises gains with minimal hardware overhead and low latency.

¹Note that all of the presented error-rate performance gains are attributed to the enhanced decoding algorithm itself (not to be confused with gains due to better code design).

II. PRELIMINARIES

A. Structure of LDPC Codes

LDPC codes were originally introduced by Gallager [16] as codes that could be conventionally represented by its corresponding $(M \times N)$ parity-check matrix $\mathbf{H} = [h_{ji}]_{M \times N}$, where N is the number of variable nodes (VNs) (i.e., also the code block-length) and M represents the number of check nodes (CNs). Therefore, the information bit block-length is $K = N - \text{rank}(\mathbf{H})$. Accordingly, the actual code rate² is designated by $R_c = K/N$. Additionally, there exists a corresponding graphical representation, namely the Tanner graph, where the bipartite sets of nodes, namely, VNs and CNs, are connected according to \mathbf{H} (i.e., a VN v_i is connected to a CN c_j if $h_{ji} = 1$, with $i \in \{0, ..., N-1\}$ and $j \in \{0, ..., M-1\}$).

B. BP Decoding

Alongside LDPC codes, Gallager introduced a suitable iterative decoding scheme [16] whose modified version is today known as the BP algorithm (also known as SPA). The algorithm passes messages, in form of extrinsic LLRs, along the edges of the Tanner graph. The result is an iterative update process at the VNs and CNs. Each VN can be interpreted as a repetition code and, thus, the update equation is

$$L_{\nu_i \to c_j} = L_{ch,i} + \sum_{j' \neq j} L_{c_{j'} \to \nu_i} \tag{1}$$

where $L_{v_i \to c_j}$ is the outgoing message from the VN v_i to the CN c_j , $L_{ch,i}$ is the *i*-th channel output LLR and $L_{c_j \to v_i}$ is the incoming message from the CN c_j to the VN v_i . In contrast, each CN can be seen as a single parity-check code which yields the update equation

$$L_{c_j \to v_i} = 2 \cdot \tanh^{-1} \left(\prod_{i' \neq i} \tanh \left(\frac{L_{v_{i'} \to c_j}}{2} \right) \right). \tag{2}$$

In the final VN calculation, all incoming messages are summed up to obtain the total LLRs. An implementation friendly variant of BP is so-called layered decoding, where CNs are processed sequentially, incorporating the output of previous CNs already within the same iteration, resulting in faster convergence. For more details about LDPC codes and BP decoding, we refer the interested reader to [17].

III. CODE SYMMETRY VS. DECODER SYMMETRY

The key aspect to enable AED for LDPC codes is the relationship between symmetries of the code and symmetries of the decoder.

A. Code Symmetry

The permutation symmetries of a code \mathscr{C} with length N are given by its *automorphism group* $\operatorname{Aut}(\mathscr{C})$. It is defined as the set of codeword symbol permutations that map every codeword onto another (not necessarily different) codeword:

$$Aut(\mathscr{C}) = \{ \pi \in \mathscr{S}_N : \pi(\mathbf{c}) \in \mathscr{C} \ \forall \mathbf{c} \in \mathscr{C} \}, \tag{3}$$

where \mathcal{S}_N denotes the symmetric group of N elements [18].

²An actual code rate could be potentially higher than the so-called design rate $r_d = (N - M)/N$.

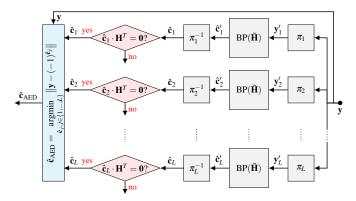


Fig. 1: Block diagram of automorphism ensemble decoding (AED) of a noisy codeword \mathbf{y} with L identical BP-based constituent decoders.

B. Automorphism Ensemble Decoding

Let $Dec(\cdot): \mathscr{Y}^N \to \mathscr{C}$ denote the decoding function, where \mathscr{Y} is the set of possible channel outputs. For instance \mathscr{Y} is the set of real numbers \mathbb{R} in case of the binary-input additive white Gaussian noise (BI-AWGN) channel. AED [11] attempts to decode multiple, differently permuted versions of the noisy codeword \mathbf{y} , using a subset $\mathscr{P} \subseteq Aut(\mathscr{C})$ of L permutations. Each permutation $\pi_i \in \mathscr{P}$ contributes one codeword candidate

$$\hat{\mathbf{c}}_j = \pi_i^{-1}(\mathrm{Dec}(\pi_j(\mathbf{y}))),\tag{4}$$

from which the final AED codeword estimate is chosen using the ML criterion

$$\hat{\mathbf{c}}_{AED} = \underset{\hat{\mathbf{c}}_{j}, j \in \{1, 2, \dots, L\}}{\operatorname{argmax}} P(\hat{\mathbf{c}}_{j} | \mathbf{y}). \tag{5}$$

Fig. 1 shows the block diagram of AED with constituent BP decoders and a selection criterion based on Euclidean distance, which is the ML criterion for the BI-AWGN channel. It is easy to see that permuted decoding with permutation π and parity-check matrix **H** is identical to decoding on the column-permuted parity-check matrix

$$\mathbf{H}' = \boldsymbol{\pi}^{-1}(\mathbf{H}). \tag{6}$$

Therefore, AED with BP decoders is a special case of MBBP [8], where the used **H**-matrices only differ by column permutations out of the automorphism group of the code. In this work, we use the notation AED-L to denote an AED with ensemble size L.

C. Decoder Symmetry

Not all permutations are useful for AED, as they result in the same codeword candidates. To analyze this, we say a decoder is *equivariant* to a permutation π , if permuting its input y is the same as permuting its output \hat{c} . In other words, the permutation operation commutes with the decoding operation:

$$\mathrm{Dec}\left(\pi(\mathbf{y})\right) = \pi\left(\mathrm{Dec}(\mathbf{y})\right) \quad \forall \mathbf{y} \in \mathscr{Y}^{N}. \tag{7}$$

We say that π is *absorbed* by $Dec(\cdot)$ [11]. Each absorbed permutation π induces sets $\{\pi \circ \sigma | \sigma \in Aut(\mathscr{C})\}$ of equivalent

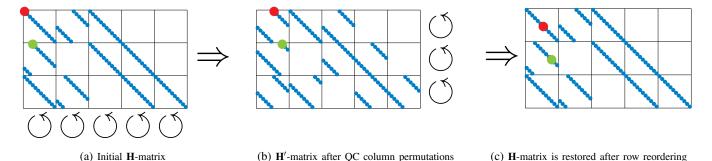


Fig. 2: Decoder equivariance: The conventional parity-check matrix of a QC LDPC code absorbs quasi-cyclic permutations of the columns. While the highlighted elements change positions, the overall **H**-matrix remains unchanged.

automorphisms. Let $\pi, \sigma_1 \in Aut(\mathscr{C})$ and π be absorbed by $Dec(\cdot)$. Then the codeword estimate from $\sigma_2 = \pi \circ \sigma_1$ is

$$\begin{split} \sigma_{2}^{-1}\left(\operatorname{Dec}\left(\sigma_{2}(\mathbf{y})\right)\right) &= \sigma_{1}^{-1}\left(\pi^{-1}\left(\operatorname{Dec}\left(\pi\left(\sigma_{1}(\mathbf{y})\right)\right)\right)\right) \\ &= \sigma_{1}^{-1}\left(\operatorname{Dec}\left(\sigma_{1}\left(\mathbf{y}\right)\right)\right), \end{split} \tag{8}$$

i.e., equivalent permutations $\sigma_1 \sim \sigma_2$ always result in the same codeword candidate under permuted decoding. It can be shown that equivalent permutations form equivalence classes which are themselves subgroups of $Aut(\mathscr{C})$ [19]. Therefore, decoder symmetries reduce the number of equivalence classes and, thus, also reduce the number of usable automorphisms for an ensemble decoder.

D. Quasi-Cyclic Codes and Decoders

A code \mathscr{C} of length N = nZ is called quasi-cyclic, if all permutations of the form

$$\pi_{d,Z}(i) = \begin{cases} i+d-Z & \text{if } i \text{ mod } Z+d \ge Z\\ i+d & \text{else} \end{cases}$$
 (9)

with $0 \le d < Z$ are automorphisms of \mathscr{C} . Therefore, $\operatorname{Aut}(\mathscr{C})$ is at least the quasi-cyclic group of size Z

$$\mathcal{Q}_Z = \{ \pi_{d,Z} : d = 0, 1, \dots, Z - 1 \}.$$
 (10)

Prominent representatives of the class of QC codes are QC LDPC codes [20]. However, in this case, the QC property mainly serves the ease of construction and implementation. A QC LDPC code is characterised by its parity-check matrix being expanded from a so-called protograph by a lifting factor Z. In the lifting process, the elements of the protograph matrix are replaced by circulant submatrices of size $Z \times Z$. Their encoding can be thus realised by a set of shift registers, with the linear complexity with respect to the total code length [21]. Moreover, various code lengths can be easily realized from a single protograph using different lifting factors Z.

The $(Zm \times Zn)$ QC LDPC code **H**-matrix can be written as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{0,0} & \mathbf{H}_{0,1} & \cdots & \mathbf{H}_{0,n-1} \\ \mathbf{H}_{1,0} & \mathbf{H}_{1,1} & \cdots & \mathbf{H}_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}_{m-1,0} & \mathbf{H}_{m-1,1} & \cdots & \mathbf{H}_{m-1,n-1} \end{bmatrix},$$

where submatrices $\mathbf{H}_{i,j}$ of size $(Z \times Z)$ are circulant.

It can be seen that both the rows and columns of the parity-check matrix fulfill the quasi-cyclic property. While quasi-cyclicity of the columns creates the automorphism group, quasi-cyclic rows result in decoder equivariance to these permutations. As shown in [14], permuted BP decoding (with the permutation $\pi_{d,Z}$) is equivalent to BP decoding on the column-permuted parity-check matrix

$$\mathbf{H}' = \pi_{d,Z}^{-1}(\mathbf{H}) = \pi_{Z-d,Z}(\mathbf{H}), \tag{11}$$

which is just a row-permuted version of **H** (as visualized in Fig. 2). For that reason, the column-permuted parity-check matrix shows exactly the same decoding behaviour in a flooding decoder as the original parity-check matrix. The same applies to layered decoding with a regular schedule, as the permutation only affects sets of independent checks. Therefore, AED using the standard **H**-matrices and QC permutations does not result in any performance gain for QC LDPC codes.

IV. Breaking Decoder Symmetry

To successfully apply AED to QC LDPC codes, one can either design codes whose automorphism group is larger than \mathcal{Q}_Z (such as the codes proposed in [14]), or break the symmetry group of the constituent decoders to be smaller than \mathcal{Q}_Z . We propose the latter method, as it does not require a specific code design and hence is compatible with standardized QC LDPC codes. We still apply the conventional BP decoding algorithm as introduced in Sec. II-B, however, on a different Tanner graph ($\tilde{\mathbf{H}}$) which is not quasi-cyclic. As the original Tanner graph is designed to optimize the performance under BP decoding, it serves as a natural starting point. We propose three methods to break the symmetry by modifying the original Tanner graph (i.e., three methods of finding the $\tilde{\mathbf{H}}$ -matrix):

- 1) *Row operations*: Elementary row operations on the parity-check matrix do not change the code but result in different Tanner graphs. In the case of binary codes, the only interesting row operation is adding a row onto another.
- 2) Adding Auxiliary Checks ("overcomplete"): One can add a single or multiple auxiliary checks to the parity-check matrix. The added checks should be linear combinations of the original checks, such that the resulting matrix is still a valid, overcomplete, parity-check matrix **H**.

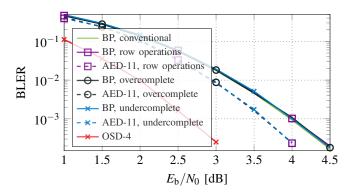


Fig. 3: Comparison of the proposed parity-check matrix modifications for the (N = 132, K = 66) 5G LDPC code. All iterative decoders use 32 iterations.

3) Removing Checks ("undercomplete"): We propose to remove some checks, resulting in an undercomplete parity-check matrix $\tilde{\mathbf{H}}$, which strictly-speaking means changing the considered code. This matrix belongs to a code that contains, besides the codewords of the original code, further invalid codewords. AED must detect when a constituent decoder converged to such an invalid codeword. Therefore, the original \mathbf{H} -matrix is used to check "code membership" and only *valid* candidates are included in the ML-in-the-list selection, as shown in Fig. 1.

Note that all proposed methods operate on the full, lifted parity-check matrix rather than the protograph.

V. RESULTS

A. Error Rate Performance

We evaluate the performance of the proposed methods on various QC LDPC codes from communications standards. Table I lists the used code parameters. All BP decoders are implemented as floating point SPA with flooding schedule and are simulated using an BI-AWGN channel with binary phase shift keying (BPSK) modulation. If available, we also plot the ML performance of the corresponding code [22] or, if computationally feasible, an approximation using OSD [23], where OSD-*t* denotes OSD with order *t*.

Table I: Parameters of the considered LDPC codes

Code	N	K	R_c	Z
802.11n	648	540	5/6	27
5G, BG 2	132	66	1/2	11
5G, BG 2	264	132	1/2	22
CCSDS	128	64	1/2	16
CCSDS	256	128	1/2	32

We first compare the three proposed methods in their errorrate performance using the (132,66) 5G LDPC code. While there exist infinite ways to combine and extend the alteration methods, we only change, add or remove a single check to demonstrate the capability of the method. The first modification adds check 0 onto check 1, i.e., changing the check 1. For the overcomplete case, we appended an additional check which is the mod-2-sum of checks 51, 53, 58 and 71 (counting from 0). This combination was chosen randomly, however,

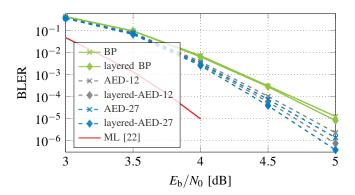


Fig. 4: Results of the Wi-Fi 802.11n code. The flooding BP decoders use 32 iterations, while the check-node layered decoders use 16 iterations. An undercomplete parity-check matrix $\tilde{\mathbf{H}}$ is used in the AED simulations.

with the constraint that the number of involved variable nodes is relatively low. In this case, the degree of the auxiliary check is 11. Lastly, we use an undercomplete $\tilde{\mathbf{H}}$ -matrix where the zeroth check has been deleted. Fig. 3 shows the block error rate (BLER) performance of the proposed methods. To fully exploit the capability of AED, we use all Z=11 available quasi-cyclic permutations, i.e., L=Z.

It can be seen that while all modifications slightly degrade the performance compared to the original parity-check matrix, in all cases, the ensemble of L=Z decoders outperforms this baseline decoder by a significant margin. To our surprise, all methods show virtually identical gains. Therefore, in the following, we focus on the undercomplete $\tilde{\mathbf{H}}$ variant, as its implementation is the easiest. In fact, a conventional decoder may be used with a single check being deactivated.

In Fig. 4 we show results for the (648,540) Wi-Fi code. The AED uses an undercomplete $\tilde{\mathbf{H}}$ -matrix with the zeroth check removed. Even though the code is of moderate length and, thus, the gap to its ML performance is already less than 1 dB at a BLER of 10^{-4} , the proposed ensemble decoder (AED) produces gains of approximately 0.2 dB. We also show results for a check-node layered decoding with 16 iterations, where even larger gains are achieved by AED.

In Fig. 5 we show results for different rate-half 5G LDPC codes. Again, the AED uses an ensemble of L = Z BP decoders using an undercomplete parity-check matrix ($\tilde{\mathbf{H}}$ -matrix) with the zeroth check removed. For both block lengths, at a BLER of 10^{-3} , we see gains of 0.3 dB and 0.2 dB, respectively.

In Fig. 6, we plot results for rate-half CCSDS codes and the same AED parameters. For both block lengths, AED achieves a gain of approximately 0.2 dB at a BLER of 10^{-4} when compared to conventional BP decoding (i.e., gain due to the enhanced decoding algorithm). Moreover, we compare to SBP with the same number of constituent decoders, i.e, S = 4 for Z = 16 and S = 5 for Z = 32. We see that the performance of AED is very similar to that of SBP, while in the higher SNR regime, AED can slightly outperform SBP. Note that to make the comparison fair, we use full SPA decoders rather than the min-sum approximation as proposed in [7].

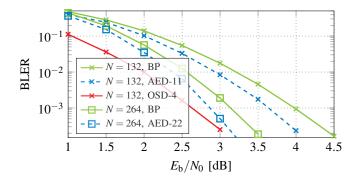


Fig. 5: Results for rate-half (N, K = N/2) 5G LDPC codes. All BP decoders use 32 iterations each, AED uses an undercomplete $\tilde{\mathbf{H}}$ matrix.

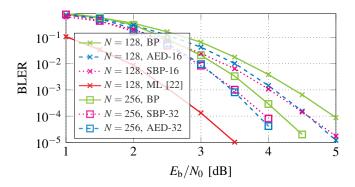


Fig. 6: Results for rate-half (N, K = N/2) CCSDS codes. All BP decoders use 32 iterations each, AED uses an undercomplete $\tilde{\mathbf{H}}$ matrix.

B. Latency

Compared to conventional BP decoding, AED has a lower worst case latency, as the number of iterations required to reach the same BLER performance is reduced. For example, for the (128,64) CCSDS code, 256 single BP iterations are required to match the AED performance with 32 iterations. Compared to SBP, a lower worst case latency is expected, as the dynamic preprocessing of finding the least reliable positions is not required. In terms of average latency, i.e., with early stopping, we find that AED with an average of 3.4 iterations significantly improves over SBP requiring 28.5 iterations at $E_b/N_0 = 4$ dB. This is because in SBP, only an average of 1.49 of the 16 constituent decoders converge at all. Note that for this analysis, in case of AED we require all constituent decoders to be done before an overall decoding result is available, while for SBP the decoding is stopped once 3 decoders have converged, as proposed in [7].

VI. CONCLUSION

In this work we demonstrated that breaking the symmetry in the parity-check matrix on the decoder side can enable AED for QC LDPC codes. Even without any optimization of how exactly the parity-check matrix is altered, consistent gains between 0.2 dB and 0.3 dB over conventional BP decoding could be achieved. Larger gains are indeed expected when further optimizations are applied. Additionally, many more ways of breaking the decoder symmetry remain to be

explored, such as non-standard schedules, which has been already successfully applied to polar codes in [24].

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