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Local Averaging for Consensus Over Communication Links with Random Dropouts

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***Abstract*— This letter proposes a mechanism for consensusability of multi-agent systems which communicate through packet loss channels. A distinguishing feature of our approach is that we assume that each agent may act on an estimate of the state of its neighbours when packet losses occur. A system using the proposed compensation method subject to nonidentical Markovian dropouts is analysed and conditions for consensusability are proposed. Simulation results are provided which verify the proposed conditions and show that it allows for consensusability over a wider range of dropout probabilities than methods previously documented in the literature.**

***Index Terms*— Control over communications, Distributed control, Markov processes**

I. Introduction

WITH the rise of computing abilities and ubiquity of wireless communication devices, coordination of multi-agent systems (MASs) to achieve a shared objective has become an increasingly well-studied problem. Allowing multiple agents to achieve a shared state allows for complex coordinated tasks to take place, such as formation flight, coordination of groups of autonomous ground vehicles, or shared headings for satellites.

Substantial fundamental work has been done to understand stability conditions for MASs [1] and how to design optimal controllers [2]. With perfect communications these issues are generally well understood [3]. Recent results extend to large scale multi-agent systems. Mean-field game theory [4] has been used to study such systems, with recent work analysing optimal control of nonlinear agents with unstable dynamics [5]. Extension of theoretical results for consensus of MASs subject to changing network conditions (e.g. packet dropouts) is both technically substantial and practically valuable as assumptions of perfect network conditions are typically unrealistic [6], [7], [8].

Stochastically switching systems are well studied in the single-agent sense through the framework of Markov Jump Linear systems [9]. The problem of packet dropouts over networks has been studied for the problem of Kalman filtering. It has been shown that a critical packet dropout probability exists, which when exceeded results in Kalman filter covariance being unbounded [10]. Other results investigate applying zero input or holding the last input for a networked control system with packet dropouts, and in the scalar case it was found neither can be considered superior [11].

These results for single-agent systems are fundamental in informing research on multi-agent systems. Early research on the problem of switching topologies [12] provided conditions for consensusability with Markovian switching topologies. The packet dropout phenomena for multi-agent systems was studied in the i.i.d. sense [7] which was then extended to the less restrictive Markovian packet loss model [8]. Additional work provides conditions for consensusability for systems subject to communication delays as well as packet dropouts [13] and suggests a predictor-like protocol to achieve consensus [14].

Several interesting investigations of alternative control strategies to compensate for packet dropouts have been done for the first-order consensus problem. Identical i.i.d. packet dropouts were studied and control protocols with and without memory were compared. With memory, if a dropout occurred agents would act on the last sent state; without they would not apply input [15]. A more sophisticated memory-based method using Taylor series prediction of neighbors' states showed increases in convergence rate compared to previously analysed methods [16]. In fact, memory protocols [15], [16] are shown to outperform typically analysed memoryless protocols [7], [8]. The memoryless strategy, which we refer to as the standard strategy, can result in large changes in inputs that may be unrealizable or safe for practical actuators. Memory-based protocols [16] can provide benefits and result in more realizable actuation, however the proposed control strategy uses substantial memory to store multiple past states of neighbors.

In this letter, we propose a compensation mechanism to estimate an agent's neighbor's information when packet dropouts occur. Our algorithm uses memory and is easy to implement. It allows for mean square consensusability of multi-agent systems over a wider range of dropout values than previous methods. Necessary and sufficient conditions are presented for mean square consensusability of the modified system in the case of identical and nonidentical Markovian dropouts for general directed graphs.

Section II presents the consensus problem, introduces the problem of consensus with packet dropouts, and describes the network model used. Section III details the system modelling of the local averaging strategy. In Section IV, we analyse the local averaging compensation method, providing consensusability conditions for the case of nonidentical packet dropouts and the special case of identical dropouts. Simulations are presented in Section V to compare to standard practice methods. Section VI draws conclusions.

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II. Consensus with Packet Dropouts

In this section, we outline the basic consensus problem and introduce the consensus with packet dropouts problem. Then, the method for network state modelling used in later portions this work is described.

A. Consensus

We consider homogeneous agents. Each individual agent's dynamics is assumed to be linear and noise-free such that

$$x_i(t+1) = Ax_i(t) + Bu_i(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ represent the state transition and input matrix, and $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^m$ represent the state and control input vectors of agent i respectively at time t.

The consensus problem may be trivially solved for stable systems by applying no input, which is of limited practical relevance. As such, we make the following assumption:

Assumption 1. *Eigenvalues of agent state transition matrix A are assumed to be on or outside the unit disk.*

For consensus, multiple individual agents interact aiming to achieve the same value. These interactions occur over a communication network. The network may be described by a graph (Laplacian) which describes connectivity of agents, denoted $\mathcal{L} \in \mathbb{R}^{N\times N}$, where N is the number of agents in the system. The Laplacian $\mathcal{L} \triangleq \mathcal{D} - \mathcal{A}$ is formed by an adjacency matrix $\mathcal{A} \in \mathbb{R}^{N\times N}$ where each value $a_{ij} \in \{0, 1\}$ denotes a connection between agent i and j, and a degree matrix $\mathcal{D}$, which is a diagonal matrix $\mathcal{D} \triangleq diag\{d_1, \ldots, d_N\}$ where $d_i \triangleq \sum_{j=1}^{N} a_{ij}$. Each agent implements a control law of the following form, using a given gain $K \in \mathbb{R}^{m\times n}$:

$$u_i = \sum_{j=1}^{N} K a_{ij}(x_j - x_i) \tag{2}$$

The Laplacian matrix $\mathcal{L}$ and Kronecker product $\otimes$ allow for an expression of the controlled system (2) in global form:

$$X(t+1) = (I \otimes A + \mathcal{L} \otimes BK)X(t) \tag{3}$$

where $X = [x_1 \ldots x_i \ldots x_N]^T$. The objective of this control system is to drive the difference between the states of agents to zero, thus achieving consensus:

$$\lim_{t\to\infty} ||x_j(t) - x_i(t)|| = 0 \quad \forall i, j \in 1, \ldots, N \tag{4}$$

It is convenient to introduce the consensus error of (3) as:

$$\delta_i(t) \triangleq x_i(t) - x_1(t) \tag{5}$$

noting that:

$$\lim_{t\to\infty} ||\delta_i(t)|| = 0 \quad \forall i = 1, \ldots, N \tag{6}$$

is equivalent to (4). This formulation of consensus error allows for the global form of the system (3) to be redefined in terms of consensus error, with the control objective of driving this error system to zero thus being equivalent to consensus.

B. Packet Dropouts in Consensus

As in similar work in the literature (e.g. [7], [8]), we first consider the special case of identical packet dropouts in consensus, where packet dropouts are common across the entire network. The packet dropout process $\gamma(t)$ is modelled as a two-state Markov process where state 1 means all connections are active ($\gamma(t) = 1$) and state 2 models a packet dropout ($\gamma(t) = 0$). This identical packet dropout process $\gamma(t)$ results in the following modification to global dynamics (3):

$$X(t+1) = (I \otimes A + \gamma(t)\mathcal{L} \otimes BK)X(t) \tag{7}$$

This packet dropout process has a transition probability matrix Q defined as follows:

$$Q = \begin{bmatrix} 1-q & q \\ p & 1-p \end{bmatrix} \tag{8}$$

where $0 < p < 1$ and $0 < q < 1$ are the recovery and failure rates of the communication network respectively. This allows one to model wireless communication channels, where for example connections are unlikely to fail (e.g. $q = 0.05$), but if they do they may not recover quickly (e.g. $p = 0.5$).

Now that the consensus system is impacted by the Markovian dropout process $\gamma(t)$, we are interested in consensusability in the stochastic sense. Instead of a deterministic definition of consensusability as in (4), we adopt the following definition of mean-square consensusability:

Definition 1. *A multi-agent system is said to be mean-square consensusable if the following holds:*

$$\lim_{t\to\infty} \mathbb{E}\{||x_j(t) - x_i(t)||^2\} = 0, \ \ \forall i, j \in \mathcal{V} \tag{9}$$

For multi-agent systems with dropouts, we are concerned with ensuring mean-square consensus in the sense of Definition 1 for all agents. This has been shown for systems with i.i.d dropout models [7] and extended to show consensusability conditions for Markovian dropout models [8, Thm. 10].

We now consider the general case of nonidentical packet dropouts, which may occur over individual communication links rather than the whole network at the same time. In this work, instead of modelling dropouts over individual links as done in [8], we model the global network state as a Markov process [17]. Thus, the dropout problem is considered as a time-varying global network topology of the form:

$$X(t+1) = (I \otimes A + \mathcal{L}(t) \otimes BK)X(t) \tag{10}$$

Due to packet dropouts, the communication topology described by $\mathcal{L}(t)$ is time-varying (unlike the models described in Section II-B). More specifically, it depends on a process $\zeta(t)$ such that $\mathcal{L}(t) = \mathcal{L}(\zeta(t))$, where $\zeta(t)$ is a Markov process:

Assumption 2. *Time-homogeneous Markov process $\zeta(t)$ with l states has a transition probability matrix $Q_{ij} = [p_{ij}]$.*

The global system (10) then becomes:

$$X(t+1) = (I \otimes A + \mathcal{L}(\zeta(t)) \otimes BK)X(t) \tag{11}$$

where we are interested in ensuring mean-square consensusabilty as in Definition 1.

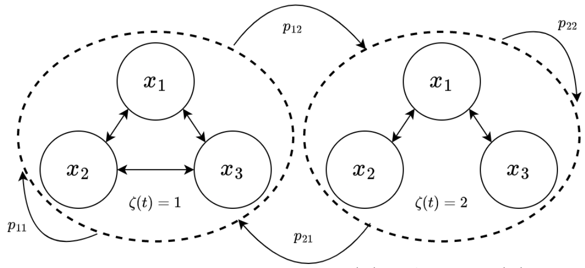

Fig. 1. Example network state graphs $\mathcal{L}(1)$ (left) and $\mathcal{L}(2)$ (right) and associated transition probabilities

Consider for example the two graph topologies shown in Figure 1. A practical interpretation of this configuration would be that the information flow between agent 1 and both others is reliable, whereas the connection between 2 and 3 is unreliable. The two possible network states result in a 2-state Markov process $\zeta(t)$. Instead of describing each edge in $\mathcal{L}(1)$ as having an individual dropout process, under Assumption 2 the Markov process $\zeta(t)$ is associated with the graph $\mathcal{L}(1)$ itself.

Remark 1. *Assumption 2 allows for any combination of packet dropouts over all edges on a communication graph to be modelled as part of the network state Markov process $\zeta(t)$. The ideal network state of a multi-agent system may not be a complete graph as depicted in $\mathcal{L}(1)$. Hence it is unlikely that the cardinality of the set describing the global network state would be as large as the possible 2^e states, where e is the number of edges in the graph.*

III. Local Averaging for Consensus with Packet Dropouts

A common assumption for multi-agent systems with dropouts in the form (7) is that no input is applied to an agent in the case that a packet dropout occurs, see Section II-B and [8]. In this work, we propose a strategy for consensus in the case of packet dropouts where instead of a zero input being applied in the case of a packet dropout, agents act on an estimate of its neighbors' states. The agent, instead of only applying an input when it is connected to its neighbors directly, acts on its locally stored information of its neighbors. This information may or may not be updated through a lossy communication network as depicted in Figure 2.

It is assumed that an agent i is always able to act on estimations of its neighbors' states $\hat{x}_{j|i}$, where neighbors are a set of other agents $\mathcal{N}_i$ which ideally have a direct connection to agent i, that is $a_{ij} = 1$. It is also assumed that agents have access to their own state x_i. This information is used in calculation of the control action for agent i:

$$u_i(t) = \sum_{j=1}^{N} K a_{ij}(\hat{x}_{j|i}(t) - x_i(t)) \tag{12}$$

Input (12) depends on an estimate of neighboring agents' states $\hat{x}_{j|i}$. Local averaging control can be applied to estimate this state in the case of a packet dropout:

$$\hat{x}_{j|i}(t) = \begin{cases} x_j(t) & \text{if connected} \\ A\hat{x}_{j|i}(t-1) + B\hat{u}_{j|i}(t-1) & \text{if dropout at time } t \end{cases} \tag{13}$$

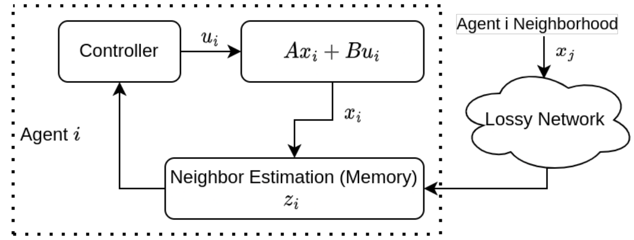

Fig. 2. Individual Agent Diagram

Further explanation of how to estimate a neighbor's input $\hat{u}_{j|i}$ is given in Section IV-B. The data available in agent i's neighbor estimation depicted in Figure 2 is a collection of its estimations of the state of its neighbors. We denote the augmented state z_i to be the information in the neighborhood estimation of agent i:

$$z_i = \begin{bmatrix} x_i & \hat{x}_{j|i} & \dots & \hat{x}_{j|i} \end{bmatrix}^T \quad \forall j \in \mathcal{N}_i \tag{14}$$

Information at agent i, encapsulated by augmented state matrix z_i, involves all of the states used in calculation of control input (12). The consensus-like feedback control proposed in (12) is not dependent on updates from the external network, and is always performed at agent i. This results in the state of agent i, and its augmented state z_i, taking the following form:

$$x_i(t+1) = \begin{bmatrix} A - d_iBK & a_{ij}BK & \dots & a_{ij}BK \end{bmatrix} z_i(t) \tag{15}$$

where d_i denotes the degree of agent i. While it is sufficient to model the evolution of agent i's state x_i using this form of difference equation that exclusively uses information locally available at agent i, it is worth noting that the information encapsulated in agent i's estimates of neighbors $\hat{x}_{j|i}$ constitutes global information coming from a network connection. Thus, it is necessary to consider a full (global) system describing the state of all agents and all estimates when describing the state evolution of an agent's estimates of its neighbors.

IV. Consensusability Analysis with Local Averaging

This section analyses the proposed local averaging mechanism to determine consensusability properties of the resulting multi-agent system. A suitable state transformation is presented which allows for analysis of a lower-order, transformed system matrix to determine consensusability properties, which are derived for the case of nonidentical Markovian dropouts.

A. System Transformation

As shown in [3, Lemma 4.2] it is convenient to analyse a multi-agent system as a problem of simultaneous stability via a state transformation. This transformation aims to express a multi-agent system with a consensus error state (5) where we may show that the mean-square stability of consensus error states $\delta_i(t)$ is equivalent to mean-square consensusability in the sense of Definition 1. Inspired by [3], [12], we introduce the following lemma:

Lemma IV.1. *Suppose that $\mathcal{W} \in \mathbb{R}^{N\times N}$ is a matrix with the constraint that each row sum is equal to some constant $c \in \mathbb{R}$.*

There exists a matrix M with the following structure:

$$M \triangleq \begin{bmatrix} \frac{1}{N} & \frac{\vec{1}}{N} \\ -\mathbf{1} & I_{N-1} \end{bmatrix} \tag{16}$$

$$M^{-1} = \begin{bmatrix} 1 & -\frac{1}{N} & \dots & -\frac{1}{N} \\ 1 & 1-\frac{1}{N} & \dots & -\frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\frac{1}{N} & \dots & 1-\frac{1}{N} \end{bmatrix} \tag{17}$$

The following statements hold:

$$c \in \Lambda(\mathcal{W}) \tag{18}$$

$$\tilde{\mathcal{W}} \triangleq M\mathcal{W}M^{-1}, \quad \Lambda(\tilde{\mathcal{W}}) = \Lambda(\mathcal{W}) \tag{19}$$

$$\tilde{\mathcal{W}} = \begin{bmatrix} c & \vec{\mathcal{W}} \\ \mathbf{0}^{1\times(N-1)} & \mathcal{W}_e \end{bmatrix} \tag{20}$$

$$\Lambda(\mathcal{W}_e) = \Lambda(\mathcal{W}) \setminus \{c\} \tag{21}$$

where $\vec{\mathcal{W}} \in \mathbb{R}^{(N-1)\times 1}$ is a vector of possibly nonzero numbers, and Λ is the spectrum operator denoting all eigenvalues of a matrix. The eigenvalue c is isolated in the upper left block of (20)*, and dynamics $\mathcal{W}_e$ associated with other eigenvalues are linearly independent from those associated with c.*

Proof: As (19) constitutes an invertible state transformation, the equivalence of the spectrum Λ of the original and transformed matrices is evident. We may show the validity of (18) by observing that multiplication of $\mathcal{W}$ by a row vector of $\mathbf{1}^T = (1, 1, \dots, 1)$ results in the following:

$$\mathcal{W}\mathbf{1}^T = \begin{bmatrix} \sum_{j=1}^{N} \mathcal{W}_{1j} \\ \vdots \\ \sum_{j=1}^{N} \mathcal{W}_{Nj} \end{bmatrix} = c\mathbf{1}^T \tag{22}$$

Where the third equality of (22) follows noting the constraint that rows of $\mathcal{W}$ all have a row sum equal to c. Thus $\mathbf{1}^T$ is an eigenvector and c is an eigenvalue of $\mathcal{W}$.

We may prove that the first column of $\tilde{\mathcal{W}}$ is equal to a zero vector with c in the first row by examining properties of (17) and the state transformation (19). Consider $\mathcal{W}M^{-1}$ and the results of each value j of the first column $(\mathcal{W}M^{-1})_{j1}$:

$$(\mathcal{W}M^{-1})_{j1} = \sum_{j=1}^{N} \mathcal{W}_{ij} M^{-1}_{j1} = c\mathbf{1}^T \tag{23}$$

Since $\mathbf{1}^T$ is an eigenvector of matrix $\mathcal{W}$ and the first column of M^{-1}, the first column of $\mathcal{W}M^{-1}$ equals $c\mathbf{1}^T$ as in (22). Due to the choice of the first row of (16), properties of the first column of $\tilde{\mathcal{W}} \triangleq M\mathcal{W}M^{-1}$ are straightforward:

$$\tilde{\mathcal{W}}_{11} = \sum_{j=1}^{N} \frac{1}{N} c = c \qquad \tilde{\mathcal{W}}_{i1} = 0, \;\; i = 2 \dots N \tag{24}$$

(24) holds due to the specific choice of $\frac{1}{N}$ in (17), and construction of M and (23). As it is clear that the eigenvalue corresponding to the row sum of $\mathcal{W}$ is isolated in the first column of $\tilde{\mathcal{W}}$, (21) follows. ■

Suppose we have a global system of agents implementing local averaging, with augmented state z_i for each agent i as in (14). We may stack the state information of every agent to create a global state matrix Z as follows:

$$Z(t) = \begin{bmatrix} z_1 & \dots & z_i & \dots & z_N \end{bmatrix}^T \tag{25}$$

Application of the state transformation matrix M as described in Lemma IV.1 results in an augmented error state. It contains both an average of all states $\bar{x}$ and the difference between all states and the state of the first agent, following the definition of consensus error given in (6):

$$MZ(t) = \begin{bmatrix} \bar{x} \\ \hat{x}_{j|1} - x_1 \\ \vdots \\ \hat{x}_{j|N} - x_1 \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \eta_{j|1,1} \\ \vdots \\ \eta_{j|N,1} \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \tilde{E} \end{bmatrix} \tag{26}$$

where $\eta_{j|N,1}$ denotes the error between estimate $\hat{x}_{N|1}$ and the state of agent 1, and $\bar{x}$ denotes the average state of the MAS. Application of Lemma IV.1 allows for consideration of a consensus error system evaluating the difference between every estimate $\hat{x}_{j|i}$ or state of other agents x_j and the state of the first agent x_1. By forming a difference equation describing the state transition dynamics F of the global state matrix (25) of a system using local averaging, we may then apply the transformation in Lemma IV.1 to reduce the dynamics resulting in the following random process:

$$Z(t+1) = FZ(t) \tag{27}$$

$$E(t+1) = MFM^{-1} \begin{bmatrix} \bar{x} \\ \tilde{E} \end{bmatrix} = \begin{bmatrix} c & \vec{F} \\ 0 & F_e \end{bmatrix} \begin{bmatrix} \bar{x} \\ \tilde{E} \end{bmatrix} \tag{28}$$

It follows that dynamics F_e are, as shown in Lemma IV.1, independent from the value of the average state $\bar{x}$. We may then determine the mean-square consensusability of the local averaging strategy by analysing the dynamics of the augmented consensus error state $\tilde{E}$:

$$\tilde{E}(t+1) = F_e \tilde{E}(t) \tag{29}$$

B. Dropout Effects

Explicitly modelling the network state as a Markovian process as in Section II-B allows for straightforward analysis of nonidentical packet dropouts. As discussed in Section II-B, nonidentical packet dropouts result in switching graph topologies. As the state transformation in Lemma IV.1 may apply to all Laplacian-like matrices to achieve the same consensus error description as in (28), we may model switching graph topologies by applying this transform to each of the resulting system matrices of the form (27).

The single-agent control problem and consensus problem have fundamental differences. We aim to exploit these differences and propose a method to estimate the input applied at an agent's neighbors, thus providing an estimate of their state in the case that a packet dropout occurs (13). This results in a difference equation of the following form:

$$\hat{x}_{j|i}(t+1) = A\hat{x}_{j|i}(t) + B\hat{u}_{j|i}(t) \tag{30}$$

A key difficulty in (30) is to find an appropriate estimate for the input $\hat{u}$ applied at the neighbor j at time t when a packet dropout occurs. We assume all neighbors of agent i are aiming to achieve consensus within their local neighborhood. As a result, estimated input $\hat{u}_{j|i}(t)$ is given by the following:

$$\hat{u}_{j|i}(t) = \sum_{k=1}^{N} K a_{ik}(\hat{x}_{k|i} - \hat{x}_{j|i}) + K(\hat{x}_{j|i} - x_i) \tag{31}$$

Note that the above assumes that consensus-like feedback occurs between agent j and all neighbors of agent i, determined by agent i's adjacency a_{ik}. It is assumed that the neighborhood of agent i is always aiming to achieve consensus, and as such estimated inputs $\hat{u}_{j|i}$ assume that all agents in that neighborhood form a complete graph [3]. As a result, the neighbor estimates $\hat{x}_{j|i}$ evolve either directly via the dynamics at the neighbor x_j if there is not a packet dropout, or by (30) applying estimated input (31) if a dropout occurs.

This formulation of the dynamics of agents themselves, presented in Section III, and the estimation mechanism proposed above may be used to create the state transition matrix $F(\zeta_i)$ associated with the network state i of the Markov process. These matrices can be associated with the difference equation (27) for the local averaging method to form the following:

$$Z(t+1) = F(\zeta(t))Z(t) \tag{32}$$

It is clear that (32) is a Markov jump linear system. As such, we may present the following theorem:

Theorem IV.2. *Consider a multi-agent system applying local averaging subject to random packet dropouts. The system is mean-square consensusable iff the following condition holds:*

$$\bar{\Gamma} \triangleq \rho(((Q^T \otimes I)diag(\tilde{\Gamma}(\zeta_1), \tilde{\Gamma}(\zeta_2) \dots \tilde{\Gamma}(\zeta_l)) < 1 \tag{33}$$

where $\rho(\cdot)$ is the maximum eigenvalue of the matrix. Each state transition matrix $\tilde{\Gamma}(\zeta_i)$ is defined by:

$$\tilde{\Gamma}(\zeta_i) \triangleq \tilde{F}_e(\zeta_i) \otimes \tilde{F}_e(\zeta_i) \tag{34}$$

$$\tilde{F}(\zeta_i) \triangleq \begin{bmatrix} f & \vec{F} \\ 0 & \tilde{F}_e(\zeta_i) \end{bmatrix} = MF(\zeta_i)M^{-1} \tag{35}$$

In (35)*, $F(\zeta_i)$ is the state transition matrix of the system given the inputs* (12) *and* (31) *and transition equations* (15) *and* (30)*.*

Proof: Suppose a multi-agent system engaging in local averaging has states which can be described by (32). We denote the state transition matrix, which includes consensus feedback as described in Section III and neighbor estimation in the form (30), as $F(\zeta_i)$. By Lemma IV.1, transformation matrix M exists such that the following statement holds for all modelled network states associated with jump variable $\zeta(t)$. The resulting transformed system matrix $\tilde{F}(\zeta_i)$ is comprised of the following blocks:

$$\tilde{F}(\zeta_i) = MF(\zeta_i)M^{-1} = \begin{bmatrix} f & \vec{F} \\ 0 & \tilde{F}_e(\zeta_i) \end{bmatrix} \tag{36}$$

where the dynamics associated with lower right block $\tilde{F}_e(\zeta_i)$ are independent from those related to f. The state transformation applied to Z applying local averaging results in states described by (26). The state transformation M may be applied to all dynamics associated with the time-varying network state dynamics $F(\zeta_i)$. We define the set of switching consensus error dynamics associated with the consensus error state $\tilde{E}$:

$$\tilde{F}_e = \{\tilde{F}_e(\zeta_1), \tilde{F}_e(\zeta_2), \dots, \tilde{F}_e(\zeta_l)\} \tag{37}$$

This allows us to construct the switching consensus error state using the transformed consensus error dynamics (37):

$$\tilde{E}(t+1) = \tilde{F}_e(\zeta_i(t))\tilde{E}(t) \tag{38}$$

It is clear that (38) is a Markov jump linear system. Condition (33) follows from [9, Theorem 3.9]. ■

This result gives a general condition for consensusability of multi-agent systems engaged in local averaging subject to packet dropouts described with the network state model of Section II-B. We provide the condition for consensusability of local averaging in the special case of identical i.i.d. dropouts:

Corollary IV.2.1. *Consider a multi-agent system using local averaging with identical i.i.d. packet dropouts occurring with probability $0 < p < 1$. It is mean-square consensusable iff:*

$$\rho(p(\tilde{F}_e(0) \otimes \tilde{F}_e(0)) + (1-p)\tilde{F}_e(1) \otimes \tilde{F}_e(1)) < 1 \tag{39}$$

where $\tilde{F}_e(0)$ is the transition matrix associated with local averaging, and $\tilde{F}_e(1)$ is the transition matrix associated with the system operating without packet dropouts.

Remark 2. *Note that conditions* (33) *and* (39) *include the global network state for analysis purposes, in order to verify consensusability of the MAS. As a result, for systems with large numbers of agents (thus large numbers of neighbor estimates) these conditions are computationally difficult to evaluate.*

V. Simulation Results

In this section, we show that a multi-agent system using local averaging allows for consensusability of a wider class of multi-agent systems than standard methods as in [8] allow.

A. Identical Dropouts

We utilize system, input, and calculated gain matrices from [8] for identical dropouts, and switching parameters $p = 0.2$, $q = 0.7$. Construction of $F(0)$ and $F(1)$ using (13) and (15), and resulting transformation to $\tilde{F}_e(0)$ and $\tilde{F}_e(1)$ using Lemma IV.1 results in mean square consensusability with $\rho(\bar{\Gamma}) = 0.9502$.

To show the advantages of local averaging, we lower recovery rate q substantially from 0.7 to 0.05. The resulting evaluations of mean square consensusability are $\rho(\bar{\Gamma}) = 1.0139 > 1$ with the standard system, which does not apply input when dropouts occur, by [8, Thm. 10] which is not mean-square consensusable. The proposed system is evaluated by Theorem IV.2 and $\rho(\bar{\Gamma}) = 0.9956 < 1$, ensuring mean square consensusability when using local averaging.

Mean-square consensusability of the averaging method subject to the identical packet dropouts as described above can be empirically investigated by the sum of the squared consensus error $\tilde{E}(t)^T\tilde{E}(t)$. Figure 3 shows the system using local averaging is able to achieve mean square consensus with the modified recovery rate while the standard system,

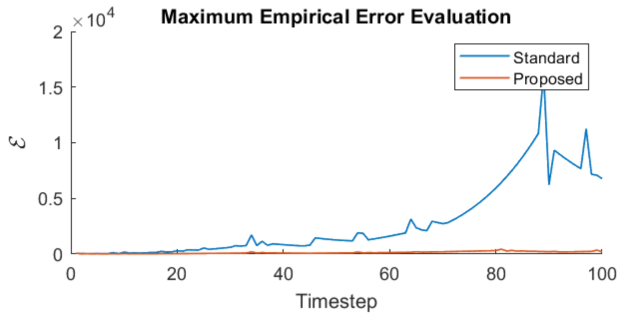

Fig. 3. Maximum empirical evaluation of sum of squared consensus error over $\mathbf{1000}$ trials with standard (—) and proposed (—) methods

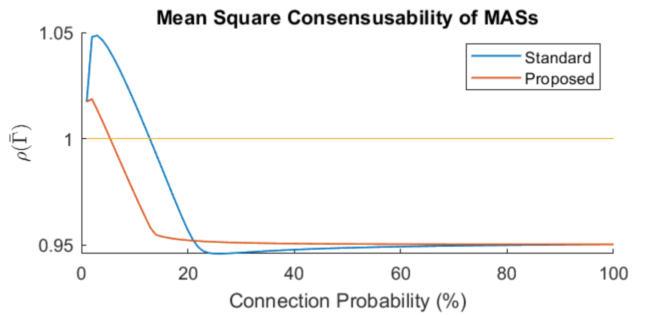

Fig. 4. Consensusability condition using standard (—) and proposed (—) methods subject to identical i.i.d. dropouts

which applies no input when a packet dropout occurs, shows unbounded error.

For ease of comparison the standard and proposed systems were tested for mean-square consensusability subject to identical i.i.d. dropouts by evaluating [8, Thm. 10] and (39) respectively. It is shown in Figure 4 that the proposed local averaging strategy is able to achieve consensus in the mean square sense for networks with a dropout probability of less than 94%, while the standard strategy is only able to do so until a dropout rate of 87%. Note that the Laplacian in [8] has many undirected links between agents. For systems which are more sparsely connected, local averaging would not be expected to provide such substantial benefit.

B. Nonidentical Dropouts

To show the accuracy of Theorem IV.2 at handling general directed graphs, we randomly generate new system matrices, choose three graph topologies to switch between, and choose a probability transition matrix Q as follows:

$$\mathcal{L}(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathcal{L}(2) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$
$$\mathcal{L}(3) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} r & \frac{1-r}{2} & \frac{1-r}{2} \\ 0.4 & 0.5 & 0.1 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \tag{40}$$

An arbitrary control gain which satisfies [3, Thm. 4.1] for $\mathcal{L}(2)$ was selected. We use the variable r to illustrate how the system behaves when it is more likely to stay in a failure state, which increases as r approaches 1. Evaluation of (33) of Theorem IV.2 for $0 < r < 1$ was used to compare consensusability of MASs using the standard approach and local averaging. It was found that using standard methods, mean-square consensusability was ensured until $r = 0.74$, while using local averaging allows for mean-square consensusability until $r = 0.83$.

VI. Conclusion

This letter proposed a compensation scheme to achieve consensusability of multi-agent systems over communication channels subject to packet dropouts. The consensus problem is first transformed to a lower-order system. Neccessary and sufficient conditions are given for mean-square consensusability of multi-agent systems using local averaging compensation in both the case of identical Markovian dropouts as well as nonidentical dropouts. Simulations show the improved performance of the proposed method when compared to standard alternatives and verify the accuracy of the proposed theorems. Controller design methods and analysis using mean-field methods (e.g. [18]) to improve scalability are left for future work.

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