Event-triggered output feedback stabilization via dynamic high-gain scaling

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Abstract—**This work addresses output feedback stabilization via event triggered output feedback. In the first part of the paper, linear systems are considered, whereas the second part shows that a dynamic event triggered output feedback control law can achieve feedback stabilization of the origin for a class of nonlinear systems by employing dynamic high-gain techniques.**

I. Introduction

The implementation of a control law on a process requires the use of an appropriate sampling scheme. In this regards, periodic control (with a constant sampling period) is the usual approach that is followed for practical implementation on digital platforms. Indeed, periodic control benefits from a huge literature, providing a mature theoretical background (see e.g. [11], [21], [3]) and numerous practical examples. The use of a constant sampling period makes closed-loop analysis and implementation easier, allowing solid theoretical results and a wide deployment in the industry. However, the rate of control execution being fixed by a worst case analysis (the chosen period must guarantee the stability for all possible operating conditions), this may lead to an unnecessary fast sampling rate and then to an overconsumption of available resources.

The recent growth of shared networked control systems for which communication and energy resources are often limited goes with an increasing interest in aperiodic control design. This can be observed in the comprehensive overview on event-triggered and self-triggered control presented in [15]. Event-triggered control strategies introduce a triggering condition assuming a continuous monitoring of the plant (that requires a dedicated hardware) while in self-triggered strategies, the control update time is based on predictions using previously received data. The main drawback of self-triggered control is the difficulty to guarantee an acceptable degree of robustness, especially in the case of uncertain systems.

Most of the existing results on event-triggered and self-triggered control for nonlinear systems are based on the input-to-state stability (ISS) assumption which implies the existence of a feedback control law ensuring an ISS property with respect to measurement errors ([28], [10], [2], [24]) and also [27].

In this ISS framework, an emulation approach is followed: the knowledge of an existing robust feedback law in continuous time is assumed, and some triggering conditions are proposed to preserve stability under sampling.

Another proposed approach consists in the redesign of a continuous time stabilizing control. For instance, the authors in [19] adapted the original *universal formula* introduced by Sontag for nonlinear control affine systems. The relevance of this method was experimentally shown in [30] where the regulation of an omnidirectional mobile robot was addressed.

Although aperiodic control literature has demonstrated an interesting potential, important fields still need to be further investigated to allow a wider practical deployment. In particular, literature on output feedback control for nonlinear systems is scarce ([31], [1], [18], [29]) whereas, in many control applications, the full state information is not available for measurement.

The high-gain approach is a very efficient tool to address the stabilizing control problem in the continuous time case. It has the advantage to allow uncertainties in the model and to remain simple.

Different approaches based on high-gain techniques have been followed in the literature to tackle the output feedback problem in the continuous-time case (see for instance [7], [16], [6], [9]) and more recently for the (periodic) discrete-in-time case (see [26]). In the context of observer design, [5] proposed the design of a continuous discrete time observer, revisiting high-gain techniques in order to give an adaptive sampling stepsize (see also [13], [20] for observers with constant sampling period).

In this work, we extend the results obtained in [5] to event-triggered output feedback control. In high-gain designs, the asymptotic convergence is obtained by dominating the nonlinearities with high-gain techniques. In the proposed approach, high-gain is dynamically adapted with respect to time varying nonlinearities in order to allow an efficient trade-off between the high-gain parameter and the sampling step size. Moreover, the proposed strategy is shown to ensure the existence of a minimum inter-execution time. Note that a preliminary version of this work has appeared in [22] in which only an event triggered state feedback was considered.

The paper is organized as follows. The control problem and the class of considered systems is given in Section II. In Section III, some preliminary results concerning linear system are given. The main result is stated in Section IV and its proof is given in Section V. Finally Section VI contains an illustrative example.

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II. Problem Statement

A. Class of considered systems

In this work, we consider the problem of designing an event-triggered output feedback for the class of uncertain nonlinear systems described by the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) + f(x(t)), \tag{1}$$

where the state x is in $\mathbb{R}^n$; $u : \mathbb{R} \to \mathbb{R}$ is the control signal in $\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{R})$, A is a matrix in $\mathbb{R}^{n\times n}$ and B is a vector in $\mathbb{R}^n$ in the following form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2}$$

and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field having the following triangular structure

$$f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}. \tag{3}$$

We consider the case in which the vector field f satisfies the following assumption.

Assumption 1 (Nonlinear bound): There exist a non-negative continuous function c, positive real numbers c_0, c_1 and q such that for all $x \in \mathbb{R}^n$, we have

$$|f_j(x(t))| \leq c(x_1) \left(|x_1| + |x_2| + \cdots + |x_j|\right), \tag{4}$$

with

$$c(x_1) = c_0 + c_1|x_1|^q. \tag{5}$$

Notice that Assumption 1 is more general than the incremental property introduced in [26] since the function c is not constant but depends on x_1. This bound can be also related to [25], [16] in which continuous output feedback laws were designed. Note however that in these works no bounds were imposed on the function c. Moreover, in our present context we do not consider inverse dynamics.

B. Updated sampling time controller

In the sequel, we restrict ourselves to a sample-and-hold implementation, i.e. the input is assumed to be constant between any two execution times. The control input u is defined through a sequence $(t_k, u_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+ \times \mathbb{R}$ in the following way

$$u(t) = u_k, \quad \forall\, t \in [t_k, t_{k+1}) . \tag{6}$$

It can be noticed that for u to be well defined for all positive time, we need that

$$\lim_{k\to+\infty} t_k = +\infty . \tag{7}$$

Our control objective is to design the sequence $(u_k, t_k)_{k\in\mathbb{N}}$ such that the origin of the obtained closed loop system is asymptotically stable. This sequence depends only on the output which in our considered model is simply given as

$$y(t) = Cx(t) \ , \ C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} . \tag{8}$$

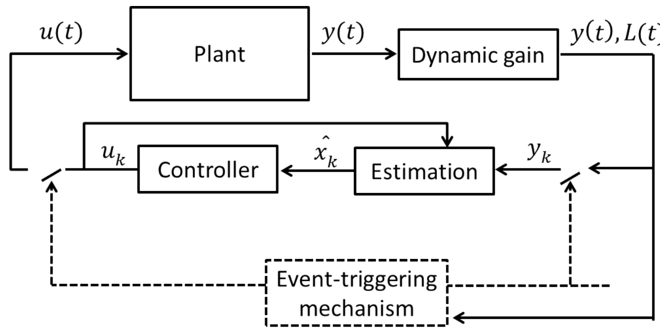

Fig. 1. Event-triggered control schematic.

Note however that in the same spirit as for the sample and hold control, we consider only a sequence of output values

$$y_k = Cx(t_k) \ , \tag{9}$$

which corresponds to the evaluation of the output $y(\cdot)$ at the same time instant t_k.

In addition to a feedback controller that computes the control input, event-triggered and self-triggered control systems need a *triggering mechanism* that determines when a new measurement occurs and when the control input has to be updated again. This rule is said to be *static* if it only involves the current state of the system, and *dynamic* if it uses an additional internal dynamic variable [14]. Our approach is summarized in Fig. 1.

C. Notation

In this paper, we denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product in $\mathbb{R}^n$ and by $|\cdot|$ the induced Euclidean norm; we use the same notation for the corresponding induced matrix norm. Also, we use the symbol $'$ to denote the transposition operation.

To simplify the presentation, we introduce the following notations: $\xi(t^-) = \lim_{\substack{\tau\to t \\ \tau<t}} \xi(\tau)$, $\xi_k = \xi(t_k)$ and $\xi_k^- = \xi(t_k^-)$.

III. Preliminary result: linear case

In high-gain design, the idea is to consider the nonlinear terms (the f_i's) as disturbances. A first step consists in synthesizing a robust control for the linear part of the system, neglecting the effects of the nonlinearities. Then, convergence and robustness are amplified through a high gain parameter to deal with the nonlinearities.

Therefore, let us first focus on a general linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{10}$$

where the state x evolves in $\mathbb{R}^n$ and the control u is in $\mathbb{R}$. The matrix A is in $\mathbb{R}^{n\times n}$ and the matrix B is in $\mathbb{R}^n$. The measured output is given as a sequence of values $(y_k)_{k\geq 0}$ in $\mathbb{R}$ as in (9) where C is a column vector in $\mathbb{R}^n$ and $(t_k)_{k\geq 0}$ is a sequence of times to be selected.

In this preliminary case, we review a well known result concerning periodic sampling approaches. Indeed, an emulation approach is adopted for the stabilization of the linear part: a feedback law is designed in continuous time and a triggering condition is chosen to preserve stability under sampling.

It is well known that if there exists a continuous time dynamical output feedback control law that asymptotically stabilizes the system, then there exists a positive inter-execution time $\delta = t_{k+1} - t_k$ such that the sampled control law renders the system asymptotically stable. This result is rephrased in the following Lemma 1 whose proof is postponed to Appendix A.

Lemma 1: Suppose that there exist a row vector K_c and a column vector K_o (both in $\mathbb{R}^n$) rendering $(A + BK_c)$ and $(A + K_oC)$ Hurwitz. Then there exists a positive real number δ^* such that for all δ in $[0; \delta^*)$ the following holds. Let the sequence $(t_k, u_k)_{k\in\mathbb{N}}$ be defined as

$$t_0 = 0 \ , \ t_{k+1} = t_k + \delta \ , \ u_k = K_c\hat{x}(t_k) \ , \ \forall\, k \in \mathbb{N} \ , \quad (11)$$

where $\hat{x}(t_0)$ is in $\mathbb{R}^n$ and for k in $\mathbb{N}^*$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_k, \quad \forall\, t \in [t_k, t_{k+1}) \, , \quad (12)$$

$$\hat{x}(t_k) = \hat{x}(t_k^-) + \delta K_o(C\hat{x}(t_k^-) - y_k). \quad (13)$$

Then $(x(t), \hat{x}(t)) = 0$ is a globally and asymptotically stable (GAS) solution for the dynamical system defined by (6), (10), (11), (12) and (13).

This result which is based on robustness is valid for general matrices A, B and C.

We want to point out that the proof of Lemma 1 is based on the fact that if $A + BK_c$ and $A + K_oC$ are Hurwitz, the origin of the discrete time linear system defined for all k in $\mathbb{N}$ as

$$\begin{bmatrix}\hat{x}_{k+1} \\ e_{k+1}\end{bmatrix} = \begin{bmatrix}F_c(\delta) & \delta K_oC\exp(A\delta) \\ 0 & F_o(\delta)\end{bmatrix}\begin{bmatrix}\hat{x}_k \\ e_k\end{bmatrix} \quad (14)$$

where $e = \hat{x} - x$ is the estimation error, and

$$F_c(\delta) = \exp(A\delta) + \int_0^\delta \exp(A(\delta - s))BK_c ds \quad (15)$$

$$F_o(\delta) = (I + \delta K_oC)\exp(A\delta) \quad (16)$$

is asymptotically stable for δ sufficiently small.

However, when we consider the particular case in which (A, B, C) are as in (2) and (8) (i.e. an integrator chain), it is shown in the following theorem that the inter-execution time can be selected arbitrarily large as long as the control is modified.

Theorem 1 (Chain of integrator): Suppose the matrices A, B and C have the structure stated in (2)-(8). Let K_c and K_o both in $\mathbb{R}^n$, be such that $A+BK_c$ and $A+K_oC$ are Hurwitz. Then there exists a positive real number α^* such that for all α in $[0, \alpha^*)$ the following holds.
For all $\delta > 0$, let the sequence $(t_k, u_k)_{k\in\mathbb{N}}$ be defined as

$$t_0 = 0 \ , \ t_{k+1} = t_k + \delta \ , \ u_k = K_cL^{n+1}\mathcal{L}\hat{x}(t_k) \ , \quad (17)$$

where $\hat{x}(t_0)$ is in $\mathbb{R}^n$ and for k in $\mathbb{N}^*$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_k, \quad \forall\, t \in [t_k, t_{k+1}) \, , \quad (18)$$

$$\hat{x}(t_k) = \hat{x}(t_k^-) + \delta\mathcal{L}^{-1}K_o(C\hat{x}(t_k^-) - y_k), \quad (19)$$

and

$$\mathcal{L} = \mathrm{diag}\left(\frac{1}{L}, \ldots, \frac{1}{L^n}\right) \ , \ L = \frac{\alpha}{\delta} \ . \quad (20)$$

Then $(x(t), \hat{x}(t)) = 0$ is a GAS solution for the dynamical system defined by (6), (10), (17), (18) and (19).

Remark 1: Note that the difference between equation (13) and equation (19) is the $\mathcal{L}^{-1}$ factor that appears in the latter.

Remark 2: Note that in the particular case of the chain of integrator the sampling period δ can be selected arbitrarily large. To obtain this result the two *gains* K_c and K_o have to be modified as seen in equations (17) and (19)

Proof: In order to analyze the behavior of the closed-loop system, let us mention the following algebraic properties of the matrix $\mathcal{L}$:

$$\mathcal{L}A = LA\mathcal{L} \ , \ \mathcal{L}B = \frac{B}{L^n} \ , \ C\mathcal{L}^{-1} = LC \ . \quad (21)$$

Let $e = \hat{x} - x$. Consider now the following change of coordinates

$$\hat{X} = \mathcal{L}\hat{x} \ , \ E = \mathcal{L}e \quad (22)$$

Employing (21) and (17), it yields that in the new coordinates the closed-loop dynamics are for all t in $[t_k, t_{k+1})$:

$$\dot{\hat{X}}(t) = L\left(A\hat{X}(t) + BK_c\hat{X}_k\right), \quad (23)$$

$$\dot{E}(t) = LAE(t). \quad (24)$$

By integrating the previous equality and employing $L\delta = \alpha$, it yields for all k in $\mathbb{N}$:

$$\begin{aligned}\hat{X}_{k+1}^- &= \left[\exp(AL\delta) + \int_0^\delta \exp(AL(\delta - s))LBK_c ds\right]\hat{X}_k \\ &= F_c(\alpha)\hat{X}_k, \\ E_{k+1}^- &= \exp(A\alpha)E_k,\end{aligned}$$

and with (19)

$$\begin{aligned}\hat{X}_{k+1} &= \mathcal{L}\left(\hat{x}_{k+1}^- + \delta\mathcal{L}^{-1}K_oCe_{k+1}^-\right) \\ &= \hat{X}_{k+1}^- + \alpha K_oCE_{k+1}^- \\ &= F_c(\alpha)\hat{X}_k + \alpha K_oC\exp(A\alpha)E_k \ .\end{aligned}$$

Similarly, it yields:

$$\begin{aligned}E_{k+1} &= \mathcal{L}(I + \delta\mathcal{L}^{-1}K_oC)e_{k+1}^- \\ &= (I + \alpha K_oC)E_{k+1}^- \\ &= F_o(\alpha)E_k \ .\end{aligned}$$

In other words, this is the same discrete dynamic as the one given in (14). Consequently, from Lemma 1, there exists a positive real number α^* such that $(\hat{X}, E) = 0$ (and thus $(x, \hat{x}) = 0$) is a GAS equilibrium for the system (24) provided $L\delta$ is in $[0, \alpha^*)$. ■

IV. Main result: the nonlinear case

We now consider the full nonlinear system (1) with f satisfying Assumption 1. Following the high-gain paradigm, the considered control law is the one used for the chain of integrator in (17)-(18)-(19) with (6). In the context of a linear growth condition, i.e. if the bound $c(x_1)$ defined in Assumption 1 is replaced by a constant c, the authors have shown in [26] that a (well chosen) constant parameter L can guarantee the global stability, provided that L is greater than a function of the bound. However, with a bound in the form (4) of Assumption 1, we need to adapt the high-gain parameter to follow a function of the time varying bound. Following the idea presented in [5] in the context of observer design, we define L as the evaluation at time t_k^- of the following continuous discrete dynamics:

$$\dot{L}(t) = a_2 L(t)M(t)c(x_1(t)), \quad \forall t \in [t_k, t_k+\delta_k) \tag{25}$$
$$\dot{M}(t) = a_3 M(t)c(x_1(t)), \quad \forall t \in [t_k, t_k+\delta_k) \tag{26}$$
$$L_k = L_k^-(1-a_1\alpha) + a_1\alpha \tag{27}$$
$$M_k = 1, \tag{28}$$

with initial condition $L(0) \geq 1$, $M(0) = 1$ and where a_1, a_2, a_3 are positive real numbers to be chosen. For a justification of this type of high-gain update law, the interested reader may refer to [5] where it is shown that this update law is a continuous discrete version of the high-gain parameter update law introduced in [25].

With this high-gain parameter and following what has been done in Theorem 1, the sequence of control is defined as follows.

$$u_k = K_c L_k^{n+1}\mathcal{L}_k \hat{x}(t_k)\ ,\ \forall k \in \mathbb{N}, \tag{29}$$

where $\hat{x}(0)$ is in $\mathbb{R}^n$. And, for k in $\mathbb{N}^*$

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t), \quad \forall\ t \in [t_k, t_{k+1})\,, \tag{30}$$
$$\hat{x}(t_k) = \hat{x}(t_k^-) + \delta_{k-1}(\mathcal{L}_k^-)^{-1}K_o(C\hat{x}(t_k^-) - y_k). \tag{31}$$

with $\mathcal{L}_k^- = \operatorname{diag}\left(\frac{1}{L_k^-}, \dots, \frac{1}{(L_k^-)^n}\right)$.

It remains to select the sequences δ_k and the execution times t_k . These are given by the following relations,

$$t_0 = 0, \quad t_{k+1} = t_k + \delta_k, \tag{32}$$

$$\delta_k = \min\{s \in \mathbb{R}_+ \mid sL((t_k+s)^-) = \alpha\}. \tag{33}$$

Equations (32)-(33) constitute the triggering mechanism of the self-triggered strategy. It does not directly involve the state value x but the additional dynamic variable L and so can be referred as a dynamic triggering mechanism ([14]). The relationship between L_k and δ_k comes from the right hand side equation of (20). It highlights the trade-off between high-gain value and inter-execution time (see [12], [26]).

We are now ready to state our main result whose proof is given in Section V.

Theorem 2: (Stabization via event-triggered output feedback control): Assume the functions f_i's in (1) satisfy Assumption 1. Then, there exist positive numbers a_1, a_2, a_3, two gain matrices K_c, K_o and $\alpha^* > 0$ such that for all α in $[0, \alpha^*]$, there exists a positive real number $L_{\max}$ such that the set

$$\{x = 0, \hat{x} = 0, L \leq L_{\max}\} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R},$$

is GAS along the solution of system (1) with the self-triggered feedback (29)-(33). More precisely, there exists a class $\mathcal{KL}$ function β such that by denoting $(x(\cdot), \hat{x}(\cdot), L(\cdot))$ the solution initiated from $(x(0), \hat{x}(0), L(0))$ with $L(0) \geq 1$, this solution is defined for all $t \geq 0$ and satisfies

$$|x(t)| + |\hat{x}(t)| + |\tilde{L}(t)| \leq \beta(|x(0)| + |\hat{x}(0)| + |\tilde{L}(0)|, t), \tag{34}$$

where $\tilde{L}(t) = \max\{L(t) - L_{\max}\}$. Moreover there exists a positive real number $\delta_{\min}$ such that $\delta_k > \delta_{\min}$ for all k and so ensures the existence of a minimal inter-execution time.

V. Proof of Theorem 2

Following [25], let us introduce the following scaled coordinates along a trajectory of system (1) which will be used at different places in this paper (compare with (22)).

$$\hat{X}(t) = \mathcal{S}(t)\hat{x}(t)\ ,\ E(t) = \mathcal{S}(t)e(t)\ , \tag{35}$$

where

$$\mathcal{S}(t) = L(t)^{1-b}\mathcal{L}(t)\ ,\ \mathcal{L}(t) = \operatorname{diag}\left(\frac{1}{L(t)}, \dots, \frac{1}{L^n(t)}\right),$$

$e(t) = \hat{x}(t) - x(t)$, and where $1 \geq b > 0$ is such that $bq < 1$ with q given in Assumption 1.

A. Selection of the gain matrices K_c and K_o

Let D be the diagonal matrix in $\mathbb{R}^{n\times n}$ defined by $D = \operatorname{diag}(b, 1+b, \dots, n+b-1)$. Let P and Q be two symmetric positive definite matrices and K_c, K_o two vectors in $\mathbb{R}^n$ such that (always possible, see [8])

$$P(A+BK_c) + (A+BK_c)'P \leq -I, \tag{36}$$
$$p_1 I \leq P \leq p_2 I, \tag{37}$$
$$p_3 P \leq PD + DP_c \leq p_4 P, \tag{38}$$
$$Q(A+K_oC) + (A+K_oC)'Q \leq -I, \tag{39}$$
$$q_1 I \leq Q \leq q_2 I, \tag{40}$$
$$q_3 Q \leq QD + DQ \leq q_4 Q, \tag{41}$$

with $p_1, \dots, p_4, q_1, \dots, q_4$ positive real numbers.

With the matrices K_c and K_o selected it remains to select the parameters a_1, a_2, a_3 and α^*.
This is done on two steps: in Proposition 1 we focus on the existence of the sequence (x_k, L_k) for all k in $\mathbb{N}$. Then, Proposition 2 shows using a Lyapunov analysis that a sequence of quadratic function of scaled coordinates is decreasing.

Based on these two propositions, the proof of Theorem 2 is given in Section V-D where it is shown that the time function L satisfies an ISS property (see Proposition 3).

B. Existence of the sequence $(t_k, \hat{x}_k, e_k, L_k)_{k\in\mathbb{N}}$

The first step of the proof is to show that the sequence $(\hat{x}_k, e_k, L_k)_{k\in\mathbb{N}} = (\hat{x}(t_k), e(t_k), L(t_k))_{k\in\mathbb{N}}$ is well defined. Note that it does not imply that $(\hat{x}(t), e(t))$ is defined for all t since for the time being it has not been shown that the sequence t_k is unbounded. This will be obtained in Section V-D when proving Theorem 2.

Proposition 1 (Existence of the sequence): Let a_1, a_3 and α be positive, and $a_2 \geq \frac{3n}{q_1}$, where q_1 was defined in (40). Then, the sequence $(t_k, \hat{x}_k, e_k, L_k)_{k\in\mathbb{N}}$ is well defined.

Proof of Proposition 1: We proceed by contradiction. Assume that $k \in \mathbb{N}$ is such that $(t_k, \hat{x}_k, e_k, L_k)$ is well defined but $(t_{k+1}, \hat{x}_{k+1}, e_{k+1}, L_{k+1})$ is not. This means that there exists a time $t^* > t_k$ such that $\hat{x}(\cdot)$, $e(\cdot)$ and $L(\cdot)$ are well defined for all t in $[t_k, t^*)$ and such that

$$\lim_{t\to t^*} \big(|\hat{x}(t)| + |e(t)| + |L(t)|\big) = +\infty. \tag{42}$$

Since $L(\cdot)$ is increasing and, in addition, for all t in $[t_k, t^*)$ we have (according to (33)) $L(t) \leq \frac{\alpha}{(t-t_k)}$, we get:

$$L^* = \lim_{t\to t^*} L(t) \leq \frac{\alpha}{(t^* - t_k)} < +\infty. \tag{43}$$

Consequently, $\lim_{t\to t^*} |\hat{x}(t)| + |e(t)| = +\infty$, which together with (35) yields

$$\lim_{t\to t^*} |\hat{X}(t)| + |E(t)| = +\infty. \tag{44}$$

On the other hand, let U and W be the two quadratic functions

$$U(\hat{X}) = \hat{X}'P\hat{X}\ ,\ W(E) = E'QE. \tag{45}$$

With a slight abuse of notation, when evaluating these functions along the solution of (1), we denote $U(t) = U(\hat{X}(t))$ and $W(t) = W(E(t))$. For all t in $[t_k, t^*)$, we have

$$\dot{U}(t) = \dot{\hat{X}}(t)'P\hat{X}(t) + \hat{X}(t)'P\dot{\hat{X}}(t), \tag{46}$$
$$\dot{W}(t) = \dot{E}(t)'QE(t) + E(t)'Q\dot{E}(t), \tag{47}$$

where

$$\begin{aligned}\dot{\hat{X}}(t) &= \dot{\mathcal{S}}(t)\hat{x}(t) + \mathcal{S}(t)\dot{\hat{x}}(t),\\ &= -\frac{\dot{L}(t)}{L(t)}D\hat{X}(t) + L(t)A\hat{X}(t) + L(t)BK\hat{X}_k,\end{aligned}$$

and

$$\begin{aligned}\dot{E}(t) &= \dot{\mathcal{S}}(t)E(t) + \mathcal{S}(t)\dot{E}(t),\\ &= -\frac{\dot{L}(t)}{L(t)}DE(t) + L(t)AE(t) - \mathcal{S}(t)f\left(x(t)\right).\end{aligned}$$

With the previous equalities, (46)-(47) become for all t in $[t_k, t^*)$

$$\begin{aligned}\dot{U}(t) &= -\frac{\dot{L}(t)}{L(t)}\hat{X}(t)'(PD + DP)\hat{X}(t)\\ &\quad + L(t)[\hat{X}(t)'(A'P + PA)\hat{X}(t) + 2\hat{X}(t)'PBK\hat{X}_k],\\ \dot{W}(t) &= -\frac{\dot{L}(t)}{L(t)}E(t)'(QD + DQ)E(t)\\ &\quad + L(t)E(t)'(A'Q + QA)E(t) + 2E(t)'Q\mathcal{S}(t)f(x(t)).\end{aligned}$$

Since $M \geq 1$, we have with (25), (38) and (41) for all t in $[t_k, t^*)$

$$\begin{aligned}-\frac{\dot{L}(t)}{L(t)}\hat{X}(t)'(PD + DP)\hat{X}(t) &\leq -p_3a_2c(x_1(t))U(t),\\ -\frac{\dot{L}(t)}{L(t)}E(t)'(QD + DQ)E(t) &\leq -q_3a_2c(x_1(t))W(t).\end{aligned}$$

Moreover, using Young's inequality, we get

$$2\hat{X}(t)'PBK\hat{X}_k \leq \hat{X}(t)'P\hat{X}(t) + \hat{X}_k'(K'B'P + PBK)\hat{X}_k.$$

Hence, taking λ_1 and λ_2 such that

$$A'P + PA + I \leq \lambda_1 P\ ,\ K'B'P + PBK \leq \lambda_2 P\ ,$$

we have, for all t in $[t_k, t^*)$

$$\dot{U}(t) \leq \left(-p_3a_2c(x_1(t)) + L(t)\lambda_1\right)U(t) + L(t)\lambda_2 U_k. \tag{48}$$

On another hand, with Assumption 1 and since $L(t) \geq 1$, it yields

$$\begin{aligned}|\mathcal{S}(t)f(x(t))|^2 &= \sum_{i=1}^n \left|\frac{f_i(x(t))}{L(t)^{i+b-1}}\right|^2,\\ &\leq \sum_{i=1}^n \left(c(x_1(t))\sum_{j=1}^i |X_j(t)|\right)^2,\\ &\leq n^2c(x_1(t))^2|\hat{X}(t) - E(t)|^2.\end{aligned} \tag{49}$$

Hence, we get

$$\begin{aligned}&2E(t)'Q\mathcal{S}(t)f(x(t))\\ &\qquad\leq 2nc(x_1(t))q_3\left(\frac{3}{2}E(t)'E(t) + \hat{X}(t)'\hat{X}(t)\right).\end{aligned}$$

Taking λ_3 such that $A'Q + QA \leq \lambda_3 Q$ and since $2nq_3 I \leq \frac{2nq_3}{p_1}P$ it yields

$$\begin{aligned}\dot{W}(t) \leq &\left(\left(\frac{3n}{q_1} - a_2\right)q_3c(x_1(t)) + L(t)\lambda_3\right)W(t)\\ &+ \frac{2nq_3}{p_1}c(x_1(t))U(t).\end{aligned} \tag{50}$$

Let us denote

$$V(t) = U(t) + \mu W(t)\ , \tag{51}$$

where μ is any positive real number that will be useful in the proof of Proposition 2. Bearing in mind that $L(t) \leq L^*$ for all t in $[t_k, t^*)$ (from (43)) and with the couple (a_2, μ) selected to satisfy $a_2 \geq \frac{3n}{q_1}$ and $a_2p_3 \geq \mu\lambda_4$, inequalities (48) and (50) yield

$$\begin{aligned}\dot{V}(t) &\leq L^*\lambda_1 U(t) + L^*\lambda_2 U_k + \mu L^*\lambda_3 W(t),\\ &\leq L^*(\lambda_1 + \lambda_3)V(t) + L^*\lambda_2 V_k.\end{aligned}$$

This with (43) give for all t in $[t_k, t^*)$

$$\begin{aligned}V(t) \leq\ &\exp\left((\lambda_1 + \lambda_3)L^*(t - t_k)\right)V_k\\ &+ \int_0^{t-t_k} \exp\big((\lambda_1 + \lambda_3)L^*(t - t_k - s)\big)\lambda_2 V_k ds\\ \leq\ &k(\alpha)V_k,\end{aligned} \tag{52}$$

where $k(\alpha) = \exp\left((\lambda_1 + \lambda_3)\alpha\right) + (\exp((\lambda_1 + \lambda_3)\alpha) - 1)\frac{\lambda_2}{\lambda_1+\lambda_3}$. Hence, $\lim_{t\to t^*} |E(t)| + |\hat{X}(t)| < +\infty$ which contradicts (44) and thus, ends the proof. □

C. Lyapunov analysis

The second step of the proof of Theorem 2 consists in a Lyapunov analysis to show that a good selection of the parameters a_1, a_2 and a_3 in the high-gain update law (25)-(28) yields the decrease of the sequences $V_k = V(t_k)$ defined from (51) with a proper selection of μ.

Remark 3: Using the results obtained in [25] on lower triangular systems, the dynamic scaling (35) includes a number b. Although the decreases of V_k can be obtained with $b = 1$, it will be required that $bq < 1$ in order to ensure the boundedness of $L(\cdot)$ (see equation (87) in Section V-D).

The aim of this subsection is to show the following intermediate result.

Proposition 2 (Decrease of scaled coordinates): There exist $a_1 > 0$ (sufficiently small), $a_2 > 0$ (sufficiently large), a continuous function N and $\alpha^* > 0$ such that for $a_3 = 2n$ and for all α in $[0, \alpha^*]$ there exists μ such that with the time function V defined in (51) the following property is satisfied:

$$V_{k+1} - V_k \leq -\alpha N(\alpha) \left(\frac{L_k}{L_{k+1}^-} \right)^{2(n-1+b)} V_k. \qquad (53)$$

Proof of Proposition 2: First of all, we assume that $a_2 \geq \frac{3n}{q_1}$. Hence, with Proposition 1, we know that the sequence (t_k, x_k, e_k, L_k) is well defined for all k in $\mathbb{N}$. Let k be in $\mathbb{N}$. The nonlinear system (1) with the control (29) gives the closed-loop dynamics

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + BK_c(L_k)^{n+1}\mathcal{L}_k\hat{x}(t_k), \\ \dot{e}(t) &= Ae(t) - f(x(t)), \qquad \forall\ t \in [t_k, t_k + \delta_k). \end{aligned}$$

Integrating the preceding equalities between t_k and t_{k+1}^- yields

$$\begin{aligned} \hat{x}_{k+1}^- =&\ \exp(A\delta_k)\hat{x}_k \\ &+ \int_0^{\delta_k} \exp(A(\delta_k - s))BK_cL_k^{n+1}\mathcal{L}_k\hat{x}_k ds, \\ e_{k+1}^- =&\ \exp(A\delta_k)e_k - \int_0^{\delta_k} \exp(A(\delta_k - s))f(x(s))ds, \end{aligned} \qquad (54)$$

and with (31), we get

$$\begin{aligned} \hat{x}_{k+1} =&\ \exp(A\delta_k)\hat{x}_k + \delta_k(\mathcal{L}_{k+1}^-)^{-1}K_oCe_{k+1}^- \\ &+ \int_0^{\delta_k} \exp(A(\delta_k - s))BK_cL_k^{n+1}\mathcal{L}_k\hat{x}_k ds \end{aligned}$$

$$\begin{aligned} e_{k+1} =&\ (I + \delta_k(\mathcal{L}_{k+1}^-)^{-1}K_oC)\Bigg(\exp(A\delta_k)e_k \\ &- \int_0^{\delta_k} \exp(A(\delta_k - s))f(x(s))ds \Bigg). \end{aligned} \qquad (55)$$

In the following, we successively consider the evolution of the e part of the dynamics and the evolution of $\hat{x}$ part.

Analysis of the term in e***:*** Employing the algebraic equality given in (21) yields that $\mathcal{L}\exp(As) = \exp(LAs)\mathcal{L}$. Hence, when left multiplying (55) by $\mathcal{S}_{k+1}^- = \left(L_{k+1}^-\right)^{1-b} \mathcal{L}_{k+1}^-$, we get the following inequality:

$$\mathcal{S}_{k+1}^- e_{k+1} = F_o(\alpha)\mathcal{S}_{k+1}^- e_k + R_o,$$

where we have used the notations

$$\begin{aligned} F_o(\alpha) &= (I + \alpha K_oC)\exp(A\alpha), \\ R_o &= -(I + \alpha K_oC)\int_0^{\delta_k} \exp(L_{k+1}^-A(\delta_k - s))\mathcal{S}_{k+1}^- f(x(s))ds. \end{aligned}$$

Let $W_k = W(E_k)$ where W and E_k are respectively defined in (45) and (35). Note that, since we have $E_{k+1} = \Psi\mathcal{S}_{k+1}^- e_{k+1}$ with $\Psi = \mathcal{S}_{k+1}(\mathcal{S}_{k+1}^-)^{-1}$, it yields from (55)

$$W_{k+1} = W(\Psi\mathcal{S}_{k+1}^- e_{k+1}) = W_k + T_{o,1} + T_{o,2},$$

with

$$\begin{aligned} T_{o,1} =& W(\Psi F_o(\alpha)\mathcal{S}_{k+1}^- e_k) - W(E_k), \\ T_{o,2} =& 2e_k'\mathcal{S}_{k+1}^- F_o(\alpha)'\Psi P\Psi R_o + R_o'\Psi P\Psi R_o. \end{aligned}$$

Let β be defined by

$$\beta = n\int_0^{\delta_k} c(x_1(t_k + s))ds.$$

The following two lemmas are devoted to upper bound the two terms $T_{o,1}$ and $T_{o,2}$. The term $T_{o,1}$ will be shown to be negative thanks to [5, Lemma 3, p109] which in our context becomes the following Lemma.

Lemma 2 ([5]): Let $a_1 \leq \frac{1}{2q_2q_4}$ and $a_3 = 2n$. There exists $\alpha_o^* > 0$ sufficiently small such that for all α in $[0, \alpha_o^*)$

$$T_{o,1} \leq -\left(\frac{a_2q_3q_1}{a_3}\left[e^{2\beta} - 1\right] + \frac{\alpha q_1}{4q_2} \right) \left|\mathcal{S}_{k+1}^- e_k\right|^2. \qquad (56)$$

For the second term, we have the following estimate.

Lemma 3: There exist two positive real valued continuous functions $N_{o,\hat{x}}$ and $N_{o,e}$ such that the following inequality holds

$$\begin{aligned} T_{o,2} \leq \left[e^{2\beta} - 1\right] \Big[& N_{o,\hat{x}}(\alpha)\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|^2 \\ & + N_{o,e}(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|^2 \Big]. \end{aligned}$$

The proof of Lemma 3 is postponed to Appendix B.

Analysis of the term in $\hat{x}$***:*** Employing the algebraic equality given in (21), we get from (55)

$$x_{k+1} = (\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k + (\mathcal{S}_{k+1}^-)^{-1}R_c,$$

where F_c is defined in (15), $\alpha_k = \delta_kL_k$ and

$$R_c = \alpha K_oCE_{k+1}^-.$$

Let $U_k = U(\hat{X}_k)$ where U is defined in (45). This yields with the former equality

$$\begin{aligned} U_{k+1} =& x_{k+1}'\mathcal{S}_{k+1}P\mathcal{S}_{k+1}x_{k+1} \\ =& U_k + T_{c,1} + T_{c,2}, \end{aligned}$$

with

$$\begin{aligned} T_{c,1} =& U(\mathcal{S}_{k+1}(\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k) - U_k, \qquad (57) \\ T_{c,2} =& 2R_c'\Psi P\mathcal{S}_{k+1}(\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k + U(\Psi R_c). \end{aligned}$$

Similarly, the following two lemmas are devoted to upper bound the two terms $T_{c,1}$ and $T_{c,2}$. The first one is [23, Lemma 5.4] which is devoted to upper bound $T_{c,1}$.

Lemma 4 ([23]): Let $a_1 \leq \frac{2}{p_4 p_2}$ and $a_3 = 2n$. Then, there exists $\alpha^* > 0$ sufficiently small such that for all α in $[0, \alpha^*)$

$$T_{c,1} \leq -\left(\frac{\alpha}{p_2}\right)^2 U(\hat{X}_k) - |\mathcal{S}_{k+1}^- \hat{x}_k|^2 (e^{2\beta} - 1)\frac{p_3 p_1 a_2}{2n}. \quad (58)$$

The proof of Lemma 4 can be found in [23].

Lemma 5: There exist three positive real valued continuous functions $N_{c,\hat{x}}$, $N_{c,e}$ and $N_{c,0}$ such that the following inequality holds

$$T_{c,2} \leq N_{c,1}(\alpha) \left|\mathcal{S}_{k+1}^- e_k\right|^2 + \frac{1}{2}\left(\frac{\alpha}{p_2}\right)^2 U(\hat{X}_k) + \left[N_{c,\hat{x}}(\alpha)\left|\mathcal{S}_{k+1}^- \hat{x}_k\right|^2 + N_{c,e}(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|^2\right][e^{2\beta} - 1]$$

The proof of Lemma 5 is postponed to Appendix C.

End of the proof of Poposition 2 *:* Let $\alpha^* = \max\{\alpha_o^*, \alpha_c^*\}$ and let $0 < \alpha < \alpha^*$, $a_1 = \min\left\{\frac{1}{2q_2 q_4}, \frac{2}{p_4 p_2}\right\}$ and $a_3 = 2n$. With Lemma 2, Lemma 3, Lemma 4 and Lemma 5, it yields

$$\begin{aligned} V_{k+1} - V_k \leq& -\frac{1}{2}\left(\frac{\alpha}{p_2}\right)^2 U_k + \left[N_{c,1}(\alpha) - \mu\frac{\alpha q_1}{4q_2}\right]\left|\mathcal{S}_{k+1}^- e_k\right|^2 \\ &+ \left[e^{2\beta} - 1\right]\left[\mu N_{o,\hat{x}}(\alpha) - \frac{p_3 p_1 a_2}{2n}\right]\left|\mathcal{S}_{k+1}^- \hat{x}_k\right|^2 \\ &+ \left[e^{2\beta} - 1\right]\left[\mu N_{o,e}(\alpha) - \mu\frac{a_2 q_3 q_1}{a_3}\right]\left|\mathcal{S}_{k+1}^- e_k\right|^2. \quad (59) \end{aligned}$$

Taking μ sufficiently large such that

$$N_{c,1}(\alpha) - \mu\frac{\alpha q_1}{4q_2} \leq -\frac{1}{2}\mu\frac{\alpha q_1}{q_2},$$

and then a_2 sufficiently large such that,

$$\mu N_{o,\hat{x}}(\alpha) - \frac{p_3 p_1 a_2}{2n} \leq 0\ ,\ \mu N_{o,e}(\alpha) - \mu\frac{a_2 q_3 q_1}{a_3} \leq 0,$$

it yields

$$V_{k+1} - V_k \leq -\alpha N_0(\alpha)\left[U_k + \left|\mathcal{S}_{k+1}^- e_k\right|^2\right].$$

where N_0 is a continuous function taking postiive values. Employing the fact that $\frac{L_k}{L_{k+1}^-} \leq 1$, it yields

$$\left|\mathcal{S}_{k+1}^- e_k\right|^2 = \left|\mathcal{S}_{k+1}^- (\mathcal{S}_k)^{-1} E_k\right|^2 \geq \left(\frac{L_k}{L_{k+1}^-}\right)^{2(n-1+b)} \frac{U_k}{p_2},$$

which gives the existence of a continuous function N such that inequality (53) holds. This ends the proof of Proposition 2.

Remark 4: Due to the jumps of the high-gain parameter L at instants t_k in equation (27), the Lyapunov function $t \mapsto V(t)$ does not decrease continuously as illustrated in Fig. 2. However, the sequence $(V_k)_{k \geq 0}$ is decreasing.

D. Boundedness of L and proof of Theorem 2

Although the construction of the updated law for the high-gain parameter (25)-(28) follows the idea developed in [5], the study of the behavior of the high-gain parameter is more involved. Indeed, in the context of observer design of [5], the nonlinear function c was assumed to be essentially bounded while in the present work, c is depending on x_1. This implies that the interconnection structure between state and high-gain dynamics must be further investigated.

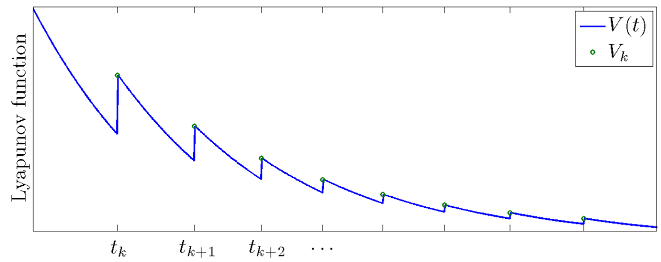

Fig. 2. Time evolution of Lyapunov function V.

Proof of Theorem 2: Assume a_1, a_2, a_3 and α^* meet the conditions of Proposition 1 and Proposition 2. Consider solutions $(\hat{x}(\cdot), e(\cdot), L(\cdot), M(\cdot))$ for system (1) with the event-triggered output feedback (29)-(33) with initial condition $\hat{x}(0)$ in $\mathbb{R}^n$, $e(0)$ in $\mathbb{R}^n$, $L(0)) \geq 1$ and $M(0) = 1$. With Proposition 1 the sequence $(t_k, \hat{x}_k, e_k, L_k)_{k \in \mathbb{N}}$ is well defined.

The existence of a strictly positive dwell time is obtained from the following proposition.

Proposition 3: There exists a positive real number $L_{\max}$ and class $\mathcal{K}$ function γ and a non decreasing function in both argument ρ such that

$$\tilde{L}_{k+1} \leq \left(1 - \frac{a_1 \alpha}{2}\right)\tilde{L}_k + \gamma(V_k), \quad \forall k \in \mathbb{N}, \quad (60)$$

where $\gamma(s) = 0$ for all s in $[0, 1]$ with

$$\tilde{L}_k = \max\{L_k - L_{\max}, 0\},$$

and for all t on the time existence of the solution, we have

$$1 \leq L(t) \leq \rho(\tilde{L}_0, V_0)\ . \quad (61)$$

The proof of this proposition is given in Appendix D.

With this proposition in hand, note that it yields for all k in $\mathbb{N}$, $\delta_k \geq \frac{\alpha}{\rho(\tilde{L}_0, V_0)} > 0$. Consequently, there is a dwell time and the solution are complete (i.e. $\sum_k \delta_k = +\infty$). Moreover, for all k in $\mathbb{N}$, $\frac{L_k}{L_{k+1}^-} \geq \frac{1}{\rho(\tilde{L}_0, V_0)}$. Consequently, inequality (53) becomes

$$V_{k+1} \leq (1 - \sigma(\tilde{L}_0, V_0))V_k,$$

where $\sigma(\tilde{L}_0, V_0) = \frac{\alpha N(\alpha)}{\rho(L_0, V_0)^{2(n-1+b)}}$ is a decreasing function of both arguments. This gives $V_k \leq (1 - \sigma(\tilde{L}_0, V_0))^k V_0$, for all k in $\mathbb{N}$. With, (60), it yields $\tilde{L}_k \leq \beta_L(\tilde{L}_0 + V_0, k)$ where

$$\beta_L(s, k) = s\left(1 - \frac{a_1 \alpha}{2}\right)^k + \sum_{j=1}^{k}\left(1 - \frac{a_1 \alpha}{2}\right)^j \gamma\left((1 - \sigma(s, s))^{k-j} s\right). \quad (62)$$

The function β_L is of class $\mathcal{K}$ in s. Moreover, since $\gamma(s) = 0$ for $s \leq 1$, this implies that there exists $k^*(s)$ such that the mapping $k \mapsto \beta_L(s, k)$ is decreasing for all $k \geq k^*(s)$. Moreover, we have $\lim_{k \to \infty} \beta_L(s, k) = 0$. On another hand, since $\delta_k \leq \alpha$, it implies that $k \leq \frac{t}{\alpha}$ for all t in $[t_k, t_{k+1})$.

$$\begin{aligned} \tilde{L}(t) &\leq \frac{\tilde{L}_{k+1}}{1 - a_1\alpha}, \\ &\leq \frac{\beta_L(\tilde{L}_0 + V_0, k+1)}{1 - a_1\alpha}. \end{aligned} \tag{63}$$

Finally, with (52), it yields

$$V(t) \leq k(\alpha)(1 - \sigma(\tilde{L}_0, V_0))^{\frac{t}{\alpha}} V_0 \ . \tag{64}$$

With the right hand side of (61) and the definition of the Lyapunov function V, we have

$$\frac{p_1 + \mu q_1}{2\rho(\tilde{L}_0, V_0)^{2(n-1+b)}} \left(|x(t)|^2 + |\hat{x}(t)|^2 \right) \leq V(t), \tag{65}$$

Moreover, we have also:

$$V_0 \leq 2(p_2 + \mu q_2) \left(|x(0)|^2 + |\hat{x}(0)|^2 \right). \tag{66}$$

From equations (63), (64), (65), (66) and the properties of the function β_L, it yields readily that there exists a class $\mathcal{KL}$ function β such that inequality (34) holds.

VI. Illustrative example

We apply our approach to the following uncertain third-order system proposed in [16]

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \theta x_1^2 x_3 + u, \end{cases} \tag{67}$$

where θ is a constant parameter which only a magnitude bound $\theta_{\max}$ is known. The stabilization of this problem is not trivial even in the case of a continuous-in-time controller. The difficulties come from the nonlinear term $x_1^2 x_3$ that makes x_3 dynamics not globally Lipschitz, and from the uncertainty on θ value, preventing the use of a feedback to cancel the nonlinearity.

However, system (67) belongs to the class of systems (1) and the Assumption 1 is satisfied with $c(x_1) = \theta_{\max} x_1^2$. Hence, by Theorem 2, an event-triggered output feedback controller (29)-(33) can be constructed. Simulation were conducted with gain matrices K_o and K_c and coefficient α selected as $K_o = \begin{bmatrix} -8 & -12 & -16 \end{bmatrix}'$, $K_c = \begin{bmatrix} -15 & -75 & -125 \end{bmatrix}$, $\alpha = 0.1$ to stabilize the linear part of the system (67).
Parameters a_1, a_2 and a_3 have then been selected through a trial and error procedure as follows:

$$a_1 = 1, \quad a_2 = .5, \quad a_3 = .5.$$

Simulation results are given in Fig. 3 and Fig. 4. The evolution of the control and state trajectories are displayed in Fig. 3 for a particular initial condition. The corresponding evolution of the Lyapunov function V and the high-gain L are shown in Fig. 3. We can see how the inter-execution times δ_k adapts to the nonlinearity. Interestingly, it allows a significant increase of δ_k when the state is close to the origin: $L(t)$ then goes to 1 and consequently δ_k increases toward α value (that was selected as $\alpha = 0.1$ in this simulation).

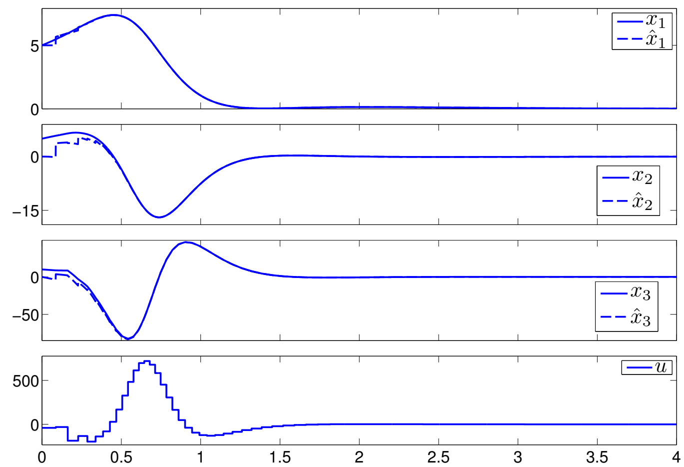

Fig. 3. Control signal and state trajectories of (67) with $(x_1, x_2, x_3) = (5, 5, 10)$ and $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = (5, 0, 0)$ as initial conditions.

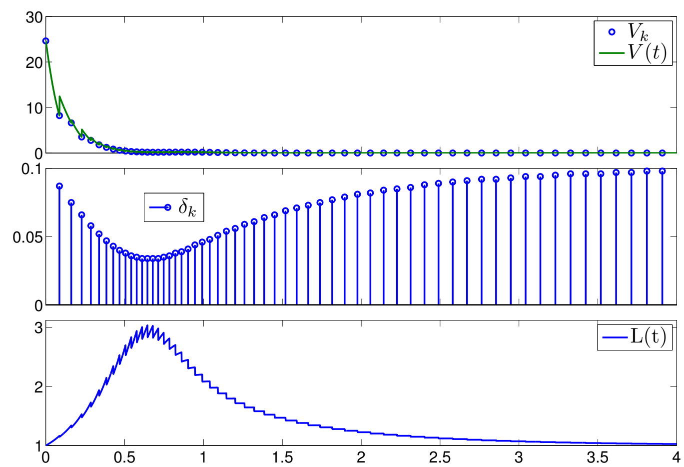

Fig. 4. Simulation results

VII. Conclusion

In conclusion, we have presented a new event triggered output feedback for a class of nonlinear systems. The triggered mechanism depends on an additional dynamics. This additional dynamics is employed to modify the output feedback following a high-gain paradigm. The stabilization of the origin of the system is demonstrated and the interest of our approach is illustrated on an example.

Appendix

A. Proof of Lemma 1

The matrix $(A + BK_c)$ being Hurwitz, let P be a symmetric positive definite matrix such that

$$\begin{aligned} P(A + BK_c) + (A + BK_c)'P \leq -I, \\ p_1 I \leq P \leq p_2 I, \end{aligned} \tag{68}$$

with p_1, p_2 positive real numbers. Likewise, let Q be a symmetric positive definite matrix such that

$$\begin{aligned} Q(A + K_oC) + (A + K_oC)'Q \leq -I, \\ q_1 I \leq Q \leq q_2 I, \end{aligned} \tag{69}$$

with q_1, q_2 positive real numbers.

In order to prove that the origin of the discrete time system (14) is GAS, we consider the Lyapunov function

$$V(e,\hat{x}) = \hat{x}'P\hat{x} + \mu e'Qe, \tag{70}$$

where μ is a positive real number that will be selected later on. From (14), it comes

$$e'_{k+1}Qe_{k+1} = e'_k F_o(\delta)'QF_o(\delta)e_k. \tag{71}$$

Given v in $S^{n-1} = \{v \in \mathbb{R}^n \mid |v| = 1\}$, consider the function

$$\nu(\delta,v) = v'F_o(\delta)'QF_o(\delta)v.$$

We have

$$\begin{aligned}\nu(0,v) &= v'Qv,\\ \frac{\partial\nu}{\partial\delta}(0,v) &= v'[Q(A+K_oC)+(A+K_oC)'Q]v.\end{aligned}$$

So, using the inequalities in (69) , we get

$$\frac{\partial\nu}{\partial\delta}(0,v) \leq -\frac{1}{q_2}v'Qv. \tag{72}$$

Now, we can write

$$\nu(\delta,v) = v'Qv + \delta\frac{\partial\nu}{\partial\delta}(0,v) + \rho(\delta,v),$$

with $\lim_{\delta\to 0}\frac{\rho(\delta,v)}{\delta} = 0$. This equality together with (72) imply that

$$\nu(\delta,v) \leq (1-\frac{\delta}{q_2})v'Qv + \rho(\delta,v).$$

The vector v being in a compact set and the function ρ being continuous, there exists δ_o^* such that for all δ in $[0;\delta_o^*)$ we have $\rho(\delta,v) \leq \frac{\delta}{2q_2}v'Qv$ for all v. This gives

$$\nu(\delta,v) \leq \left(1-\frac{\delta}{2q_2}\right)v'Qv, \qquad \forall\ \delta\in[0,\delta_o^*), \forall\ v\in S^{n-1}.$$

This property being true for every v in S^{n-1}, we have

$$F_o(\delta)'QF_o(\delta) \leq \left(1-\frac{\delta}{2q_2}\right)Q,$$

and there exists δ_o^* such that for all δ in $[0;\delta_o^*)$ we have

$$e'_{k+1}Qe_{k+1} \leq \left(1-\frac{\delta}{2q_2}\right)e'_kQe_k. \tag{73}$$

Similarly, we have

$$\begin{aligned}\hat{x}'_{k+1}P\hat{x}_{k+1} = \hat{x}'_kF_c(\delta)'PF_c(\delta)\hat{x}_k + e'_kF_{oc}(\delta)'PF_{oc}(\delta)e_k\\ + 2\hat{x}'_kF_c(\delta)'PF_{oc}(\delta)e_k,\end{aligned}$$

where $F_{oc}(\delta) = \delta K_oC\exp(A\delta)$. Notice that $F_c(0) = I$ and $\frac{\partial F_c}{\partial\delta}(0) = A + BK_c$. Hence, it implies the existence of a δ_c^* such that for all δ in $[0,\delta_c^*)$, we have

$$\hat{x}'_kF_c(\delta)'PF_c(\delta)\hat{x}_k \leq \hat{x}'_kP\hat{x}_k - \frac{\delta}{2p_2}\hat{x}'_kP\hat{x}_k. \tag{74}$$

Previous inequality with (73) and (74) yields

$$\begin{aligned}V_{k+1}-V_k&\\ =&\mu e'_{k+1}Qe_{k+1} - \mu e'_kQe_k + \hat{x}'_{k+1}P\hat{x}_{k+1} - \hat{x}'_kP\hat{x}_k\\ \leq&-\mu\frac{\delta}{2q_2}e'_kQe_k - \frac{\delta}{2p_2}\hat{x}'_kP\hat{x}_k + e'_kF_{oc}(\delta)'PF_{oc}(\delta)e_k\\ &+2\hat{x}'_kF_c(\delta)'PF_{oc}(\delta)e_k\\ \leq&-\mu\frac{\delta q_1}{2q_2}|e_k|^2 - \frac{\delta p_1}{2p_2}|\hat{x}_k|^2 + |F_{oc}(\delta)|^2|P||e_k|^2\\ &+2|F_c(\delta)||F_{oc}(\delta)||P||\hat{x}_k||e_k|.\end{aligned}$$

Using Young's inequality, the preceding inequality becomes

$$V_{k+1}-V_k \leq \left(-\mu\frac{\delta q_1}{2q_2} + N(\delta)\right)|e_k|^2 - \frac{\delta p_1}{4p_2}|\hat{x}_k|^2$$

where

$$N(\delta) = |F_{ex}(\delta)|^2|P| + |F_x(\delta)|^2|F_{ex}(\delta)|^2|P|^2\frac{4p_2}{\delta p_1}.$$

Then, choosing μ as

$$\mu \geq \frac{2q_2N(\delta)}{\delta q_1},$$

ensures the decrease of V for all δ in $[0,\delta^*)$, with $\delta^* = \max\{\delta_c^*,\delta_o^*\}$.

B. *Proof of Lemma 3*

The proof of Lemma 3 uses [5, Lemma 6, p112].

Lemma 6 ([5]): The matrix Q and P satisfy the following property for all a_1 and α such that $a_1\alpha < 1$

$$\Psi Q\Psi \leq \psi_0(\alpha)Q\psi_0(\alpha)\ , \Psi P\Psi \leq \psi_0(\alpha)P\psi_0(\alpha),$$

where $\Psi = \mathcal{S}_{k+1}(\mathcal{S}_{k+1}^-)^{-1}$ and

$$\psi_0(\alpha) = \operatorname{diag}\left(\frac{1}{(1-a_1\alpha)^b},\ldots,\frac{1}{(1-a_1\alpha)^{n+b-1}}\right).$$

To prove Lemma 3, we first analyse the term R_o. Following what has been done in (49), it yields

$$\left|\mathcal{S}_{k+1}^-f(x(t_k+s))\right|^2 \leq n^2c(t_k+s)^2\left|\mathcal{S}_{k+1}^-x(t_k+s)\right|^2. \tag{75}$$

From the previous inequality, we get

$$\begin{aligned}|R_o| \leq |I+\alpha K_oC|\exp(|A|\alpha)&\\ \times\int_0^{\delta_k}nc(t_k+s)\left|\mathcal{S}_{k+1}^-x(t_k+s)\right|ds.&\end{aligned} \tag{76}$$

On the other hand, we have for all s in $[0,\delta_k)$

$$\begin{aligned}\mathcal{S}_{k+1}^-\dot{x}(t_k+s) =&\mathcal{S}_{k+1}^-\big(Ax(t_k+s) + BK_c(L_k)^{n+1}\mathcal{L}_k\hat{x}_k\\ &+f(x(t_k+s))\big)\\ =&L_{k+1}^-A\mathcal{S}_{k+1}^-x(t_k+s)\\ &+L_{k+1}^-BK_c\Omega\mathcal{S}_{k+1}^-\hat{x}_k + \mathcal{S}_{k+1}^-f(x(t_k+s)).\end{aligned}$$

where

$$\begin{aligned}\Omega =&(L_{k+1}^-)^{-n-1}(L_k)^{n+1}\mathcal{L}_k(\mathcal{L}_{k+1}^-)^{-1}\\ =&\operatorname{diag}\left\{\left(\frac{L_k}{L_{k+1}^-}\right)^n,\left(\frac{L_k}{L_{k+1}^-}\right)^{n-1},\ldots,\frac{L_k}{L_{k+1}^-}\right\}\end{aligned}$$

Note that since $L_{k+1}^- \geq L_k$, it yields $|\Omega| \leq 1$. Hence,denoting by $w(s)$ the expression $\mathcal{S}_{k+1}^- x(t_k+s)$, this gives

$$\begin{aligned}\frac{d}{ds}|w(s)| =& \frac{\langle \dot{w}(s), w(s)\rangle}{|w(s)|} \\ \leq& \left(L_{k+1}^-|A| + nc(t_k+s)\right)|w(s)| \\ &+ L_{k+1}^-|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|\end{aligned}$$

Hence, by integrating preceding inequality, it yields

$$\begin{aligned}|w(s)| \leq& \int_0^s (L_{k+1}^-|A| + nc(t_k+r))|w(r)|dr \\ &+ L_{k+1}^-|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|s + |w(0)|.\end{aligned}$$

Since $(L_{k+1}^-|A| + nc(t_k+s))$ is a continuous non-negative function and $L_{k+1}^-|BK||\mathcal{S}_{k+1}^- x_k|s + |w(0)|$ is non-decreasing, applying a variant of the Gronwall-Bellman inequality [4, Theorem 1.3.1], it comes

$$\begin{aligned}\left|\mathcal{S}_{k+1}^- x(t_k+s)\right| \leq& (L_{k+1}^-|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|s + \left|\mathcal{S}_{k+1}^- x_k\right|) \\ &\times \exp\left(L_{k+1}^-|A|s\right)\exp\left(\int_0^s (nc(t_k+r)dr\right), \quad (77)\end{aligned}$$

Consequently, according to (76), we get

$$\begin{aligned}&|R_o| \\ &\leq |I+\alpha K_o C|\exp(2|A|\alpha) \\ &\quad\times \int_0^{\delta_k} nc(t_k+s)(L_{k+1}^-|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|s + \left|\mathcal{S}_{k+1}^- x_k\right|) \\ &\quad\times \exp\left(\int_0^s nc(t_k+r)dr\right)ds \\ &\leq |I+\alpha K_o C|\exp(2|A|\alpha)\left(\alpha|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + \left|\mathcal{S}_{k+1}^- x_k\right|\right) \\ &\quad\times \int_0^{\delta_k} nc(t_k+s)\exp\left(\int_0^s nc(t_k+r)dr\right)ds \\ &= |I+\alpha K_o C|\exp(2|A|\alpha)\left(\alpha|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + \left|\mathcal{S}_{k+1}^- x_k\right|\right) \\ &\quad\times \left[\exp\left(\int_0^s nc(t_k+r)dr\right)\right]_{s=0}^{s=\delta_k} \\ &= |I+\alpha K_o C|\exp(2|A|\alpha)\left(\alpha|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + \left|\mathcal{S}_{k+1}^- x_k\right|\right) \\ &\quad\times \left[\exp\left(\int_0^{\delta_k} nc(t_k+r)dr\right) - 1\right].\end{aligned}$$

Hence, employing $e_k = \hat{x}_k - x_k$ it yields,

$$|R_o| \leq \left[M_2(\alpha)\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + M_1(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|\right] \times \left[e^\beta - 1\right].$$

where

$$\begin{aligned}M_1(\alpha) &= |I+\alpha K_o C|\exp(2|A|\alpha), \\ M_2(\alpha) &= M_1(\alpha)(\alpha|BK_c| + 1).\end{aligned}$$

Hence, employing Lemma 6 this gives

$$|R_o'\Psi P\Psi R_o| \leq \left[M_4(\alpha)\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|^2 + M_3(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|^2\right] \times [e^\beta - 1]^2$$

where

$$\begin{aligned}M_3(\alpha) &= \frac{2|Q|}{(1-a_1\alpha)^{2(n-b+1)}}M_{e,1}(\alpha)^2, \\ M_4(\alpha) &= \frac{2|Q|}{(1-a_1\alpha)^{2(n-b+1)}}M_{\hat{x},1}(\alpha)^2.\end{aligned}$$

Moreover,

$$\begin{aligned}&\left|2e_k'\mathcal{S}_{k+1}^- F_o(\alpha)'\Psi Q\Psi R_o\right| \leq \\ &\quad\left[M_6(\alpha)\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|^2 + M_5(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|^2\right] \\ &\quad\times [e^\beta - 1],\end{aligned}$$

where

$$\begin{aligned}M_5(\alpha) &= \frac{2|Q||F_o|}{(1-a_1\alpha)^{2(n-b+1)}}M_{e,1}(\alpha), \\ M_6(\alpha) &= \frac{|Q||F_o|}{(1-a_1\alpha)^{2(n-b+1)}}M_{\hat{x},1}(\alpha).\end{aligned}$$

Noticing that

$$0 \leq (e^\beta - 1)^2 \leq e^{2\beta} - 1\ ,\ 0 \leq (e^\beta - 1) \leq e^{2\beta} - 1, \quad (78)$$

the result follows with

$$N_{o,e}(\alpha) = M_3(\alpha) + M_5(\alpha)\ ,\ N_{o,\hat{x}}(\alpha) = M_4(\alpha) + M_6(\alpha).$$

C. Proof of Lemma 5

The first part of the proof is devoted to upper-bound the term $|R_c| = \alpha\left|K_o C\mathcal{S}_{k+1}^- e_{k+1}^-\right|$. From the algebraic equality given in (21) and the expression of e_{k+1}^- given in (54), it yields

$$\begin{aligned}|R_c| \leq M_7(\alpha)\Big[&\left|\mathcal{S}_{k+1}^- e_k\right| \\ &+ \int_0^{\delta_k} nc(t_k+s)\left|\mathcal{S}_{k+1}^- x(t_k+s)\right|ds\Big],\end{aligned}$$

where $M_7(\alpha) = \alpha|K_o C|\exp(|A|\alpha)$. Consequently, according to (75) and (77), we get

$$\begin{aligned}|R_c| \leq& M_7(\alpha)\Big[\left|\mathcal{S}_{k+1}^- e_k\right| + \exp(|A|\alpha)\int_0^{\delta_k} nc(t_k+s) \\ &\times (L_{k+1}^-|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right|s + \left|\mathcal{S}_{k+1}^- x_k\right|)\Big] \\ &\times \exp\left(\int_0^s (nc(t_k+r)dr\right)ds, \\ \leq& M_7(\alpha)\Big[\left|\mathcal{S}_{k+1}^- e_k\right| + \exp(|A|\alpha) \\ &\times \int_0^{\delta_k} nc(t_k+s)\exp\left(\int_0^s (nc(t_k+r)dr\right)ds \\ &\times (\alpha|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + \left|\mathcal{S}_{k+1}^- x_k\right|)\Big], \\ =& M_7(\alpha)\Big[\left|\mathcal{S}_{k+1}^- e_k\right| + (e^\beta - 1)\exp(|A|\alpha) \\ &\times (\alpha|BK_c|\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + \left|\mathcal{S}_{k+1}^- x_k\right|)\Big].\end{aligned}$$

Hence, employing $e_k = \hat{x}_k - x_k$ it yields,

$$\begin{aligned}|R_c| \leq& \left[M_8(\alpha)\left|\mathcal{S}_{k+1}^-\hat{x}_k\right| + M_9(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|\right]\left[e^\beta - 1\right] \\ &+ M_7(\alpha)\left|\mathcal{S}_{k+1}^- e_k\right|.\end{aligned}$$

where

$$M_8(\alpha) = M_7(\alpha)(\alpha\,|BK_c| + 1)\exp(|A|\,\alpha),$$
$$M_9(\alpha) = M_7(\alpha)\exp(|A|\,\alpha).$$

Hence, employing Lemma 6 and (78) this gives

$$|R_c'\Psi Q\Psi R_c| \leq \left[M_{10}(\alpha)\left|\mathcal{S}_{k+1}^{-}\hat{x}_k\right|^2 + M_{11}(\alpha)\left|\mathcal{S}_{k+1}^{-}e_k\right|^2\right] \times [e^{2\beta}-1] + M_{12}(\alpha)\left|\mathcal{S}_{k+1}^{-}e_k\right|^2 \quad (79)$$

where

$$M_{10}(\alpha) = \frac{3\,|P|\,M_8(\alpha)^2}{(1-a_1\alpha)^{2(n-b+1)}},$$
$$M_{11}(\alpha) = \frac{|P|\left[2M_9(\alpha)^2 + M_7(\alpha)^2 + 2M_9(\alpha)M_7(\alpha)\right]}{(1-a_1\alpha)^{2(n-b+1)}},$$
$$M_{12}(\alpha) = \frac{|P|\,M_7(\alpha)^2}{(1-a_1\alpha)^{2(n-b+1)}}.$$

On another hand, with the algebraic equality given in (21), we have

$$\mathcal{S}_{k+1}^{-}(\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k = \left[\exp(A\alpha) + \int_0^\alpha \exp(A(\alpha-s))ds BK_c\Lambda\right]\mathcal{S}_{k+1}^{-}\hat{x}_k, \quad (80)$$

where $\Lambda = \left(\frac{L_k}{L_{k+1}^{-}}\right)^{n+1}\mathcal{S}_k(\mathcal{S}_{k+1}^{-})^{-1}$. Note that $L_{k+1}^{-} \geq L_k$. Hence, $|\Lambda| \leq 1$ and we have

$$\left|\mathcal{S}_{k+1}^{-}(\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k\right| \leq [\exp(|A|\,\alpha)(1+|BK_c|)]\left|\mathcal{S}_{k+1}^{-}\hat{x}_k\right|.$$

Hence, employing Lemma 6, this gives

$$2R_c'\Psi P\mathcal{S}_{k+1}(\mathcal{S}_k)^{-1}F_c(\alpha_k)\hat{X}_k \leq \left[M_{13}(\alpha)\left|\mathcal{S}_{k+1}^{-}\hat{x}_k\right|^2 + M_{14}(\alpha)\left|\mathcal{S}_{k+1}^{-}e_k\right|^2\right][e^{2\beta}-1] + M_{15}(\alpha)\left|\mathcal{S}_{k+1}^{-}e_k\right|\sqrt{U_k} \quad (81)$$

where

$$M_{13}(\alpha) = \frac{|P|\,(M_8(\alpha)+\frac{1}{2})\,[\exp(|A|\,\alpha)(1+|BK_c|)]}{(1-a_1\alpha)^{2(n-b+1)}},$$
$$M_{14}(\alpha) = \frac{|P|\,M_9(\alpha)}{2(1-a_1\alpha)^{2(n-b+1)}},$$
$$M_{15}(\alpha) = \frac{|P|\,M_7(\alpha)\,[\exp(|A|\,\alpha)(1+|BK_c|)]}{\sqrt{|P|}(1-a_1\alpha)^{2(n-b+1)}}.$$

and where we have used $\left|\mathcal{S}_{k+1}^{-}\hat{x}_k\right| = \left|\mathcal{S}_{k+1}^{-}(\mathcal{S}_k)^{-1}\hat{X}_k\right| \leq \sqrt{\frac{U_k}{|P|}}$ Finally note that

$$M_{15}(\alpha)\left|\mathcal{S}_{k+1}^{-}e_k\right|\sqrt{U_k} \leq \frac{1}{2}\left(\frac{\alpha}{p_2}\right)^2 U_k + \frac{1}{2}\frac{M_{15}(\alpha)p_2^2}{\alpha^2}\left|\mathcal{S}_{k+1}^{-}e_k\right|.$$

Hence the result follows from the former inequality in combination with inequalities (79) and (81).

D. *Proof of Proposition 3*

Proof: Inequality (53) of Proposition 2 implies that $(V_k)_{k\in\mathbb{N}}$ is a nonincreasing sequence. Consequently, being nonnegative, $(V_k)_{k\in\mathbb{N}}$ is bounded. One infers, using inequality (52), that $V(t)$ is bounded. Hence, by the left parts in inequalities (37)-(40), we get that, on the time T_x $(=\sum\delta_k)$ of existence of the solution, $\hat{X}(t)$ and $E(t)$ (and consequently so are $\frac{\hat{x}_1(t)}{L(t)^b} = \hat{X}_1(t)$ and $\frac{e_1(t)}{L(t)^b} = E_1(t)$) are bounded. Then we get that $\frac{x_1(t)}{L(t)^b}$ is bounded since we have $|x_1(t)| \leq |\hat{x}_1(t)| + |e_1(t)|$.

Summing up, there exists a class $\mathcal{K}$ function $\mathfrak{d}_1$ such that

$$\frac{|x_1(t)|}{L(t)^b} \leq \mathfrak{d}_1(V_k) \leq \mathfrak{d}_1(V_0), \quad \forall (t,k)\in[t_k, T^*). \quad (82)$$

With this result in hand, let us analyze the high-gain dynamics. According to equations (25) and (26), we have, for all k and all t in $[t_k, t_{k+1})$, $\dot{L}(t) = \frac{a_2}{a_3}L(t)\dot{M}(t)$, which implies that for all t in $[t_k, t_{k+1})$

$$\begin{aligned} L(t) &= \exp\left(\frac{a_2}{a_3}\int_{t_k}^t \dot{M}(s)ds\right)L_k, \\ &= \exp\left(\frac{a_2}{a_3}M(t) - \frac{a_2}{a_3}\right)L_k. \end{aligned} \quad (83)$$

Consequently, from (27) and (33)

$$L_{k+1} = \exp\left(\frac{a_2}{a_3}(M_{k+1}^{-}-1)\right)L_k(1-a_1\alpha) + a_1\alpha, \quad (84)$$

and δ_k satisfies

$$\exp\left(\frac{a_2}{a_3}(M_{k+1}^{-}-1)\right)\delta_k L_k = \alpha.$$

Since $M_{k+1}^{-} \geq 1$, $a_2 \geq 0$ and $a_3 \geq 0$ the previous equality implies

$$\delta_k L_k \leq \alpha. \quad (85)$$

Moreover, we have

$$\begin{aligned} \dot{M}(t) &= a_3 M(t)c(x_1(t)) \\ &= a_3 M(t)(c_0 + c_1|x_1|^q) \\ &\leq a_3 M(t)(c_0 + c_1\mathfrak{d}_1(V_k)^q L(t)^{bq}) && \text{(by (82))} \\ &\leq a_3(c_0 + c_1\mathfrak{d}_1(V_k)^q)M(t)L(t)^{bq} && \text{(since } L(t)\geq 1) \\ &\leq \mathfrak{d}_2(V_k)M(t)\exp\left(\frac{a_2}{a_3}bq(M(t)-1)\right)L_k^{bq}, && \text{(by (83))} \end{aligned}$$

where $\mathfrak{d}_2(V_k) = a_3(c_0 + c_1\mathfrak{d}_1(V_k)^q)$. Let $\psi(t)$ be the solution to the scalar dynamical system

$$\dot{\psi}(t) = \psi(t)\exp\left(\frac{a_2}{a_3}bq(\psi(t)-1)\right), \quad \psi(0) = 1.$$

$\psi(\cdot)$ is defined on $[0, T_\psi)$ where T_ψ is a positive real number possibly equal to $+\infty$. Note that we have (see e.g. [17, Theorem 1.10.1]) that for all t such that $0 \leq \mathfrak{d}_2(V_k)(t-t_k)L_k^{bq} < T_\psi$

$$M(t) \leq \psi\left(\mathfrak{d}_2(V_k)(t-t_k)L_k^{bq}\right).$$

Consequently, for all k such that $\mathfrak{d}_2(V_k)\delta_k L_k^{bq} < T_\psi$

$$M_{k+1}^{-} = M(t_k + \delta_k^{-}) \leq \psi\left(\mathfrak{d}_2(V_k)\delta_k L_k^{bq}\right).$$

From this, we get employing (85) that, for all k such that $\mathfrak{d}_2(V_k)\alpha L_k^{bq-1} < T_\psi$

$$1 \leq M_{k+1}^- \leq \psi\left(\mathfrak{d}_2(V_k)\alpha L_k^{bq-1}\right), \tag{86}$$

and employing (84) that, for all k such that $\mathfrak{d}_2(V_k)\alpha L_k^{bq-1} < T_\psi$

$$L_{k+1} \leq F(L_k), \tag{87}$$

where

$$F(L_k) = \exp\left(\psi\left(\mathfrak{d}_2(V_k)\alpha L_k^{bq-1}\right) - 1\right) L_k(1 - a_1\alpha) + a_1\alpha.$$

Note that, since $bq < 1$,

$$\lim_{L \to +\infty} L^{bq-1} = 0$$

and since moreover, $\psi(0) = 1$, we also get

$$\lim_{L \to +\infty} \frac{F(L)}{L} = 1 - a_1\alpha < 1.$$

Consequently, there exists an increasing function $\bar{L}_1$ such that for all $L > \bar{L}_1(V_k)$

$$\mathfrak{d}_2(V_k)\alpha L^{bq-1} < T_\psi, \quad F(L) < \left(1 - \frac{a_1\alpha}{2}\right) L. \tag{88}$$

On the other hand, consider the following nonlinear system with input χ

$$\left\{\begin{array}{l} \dot{L}(t) = a_2 L(t) M(t)\left(c_0 + c_1\chi(t)^q L(t)^{bq}\right) \\ \dot{M}(t) = a_3 M(t)\left(c_0 + c_1\chi(t)^q L(t)^{bq}\right), \end{array}\right. \tag{89}$$

We assume that the norm of the input signal satisfies the bound

$$|\chi(\cdot)| \leq \mathfrak{d}_1(v), \tag{90}$$

where v is a given positive real number. Notice that the couple (L, M) which satisfies equations (25) and (26) between $[t_k, t_{k+1})$ is also a solution of the previous nonlinear system with input $\chi(t) = \frac{x_1(t)}{L(t)^b}$ which satisfies (90) with $v = V_k$. Let $\phi_{s,t}$ denotes the flow of (89) issued from s, i.e., $\phi_{s,t}(a, b)$ is the solution of (89) that takes value (a, b) at $t = s$. Let C_1, C_2, be the two compact subsets of $\mathbb{R}^2$ defined by:

$$C_1 = \{1 \leq L \leq \bar{L}_1(v), M = 1\},$$
$$C_2 = \{1 \leq L \leq 2\bar{L}_1(v), 0 \leq M \leq 2\}.$$

The set C_1 is included in the interior of C_2, and we have the following Lemma.

Lemma 7: There exists a non increasing function t_1 such that for all input function χ which satisfies the bound (90) the following holds.

$$\forall k \in \mathbb{N}, \quad \forall t \leq t_1(v), \quad \phi_{t_k, t_k+t}(C_1) \subset C_2. \tag{91}$$

The proof of Lemma 7 is given in Appendix E. Let

$$\bar{L}_2(v) := \max\left\{2\bar{L}_1(v), \frac{\alpha}{t_1(v)}\right\}.$$

Note that L_k satisfies the following property:

1) If $L_k > \bar{L}_1(V_k)$ then $L_{k+1} \leq \left(1 - \frac{a_1\alpha}{2}\right) L_k$;
2) If $L_k \leq \bar{L}_1(V_k)$ then $L_{k+1} \leq \bar{L}_2(V_k)$.

Indeed, we have

1) If $L_k > \bar{L}_1(V_k)$. With (87) and (88), we get
$$L_{k+1} \leq \left(1 - \frac{a_1\alpha}{2}\right) L_k.$$
2) If $L_k \leq \bar{L}_1(V_k)$
   a) If $\delta_k \leq t_1(V_k)$. Because $L_{k+1}^- \geq 1$ and $a_1\alpha < 1$, (27) implies that $L_{k+1} \leq L_{k+1}^-$. It follows, using (91) with $v = V_k$ (note that $(L_k, M_k) \in C_1$), that
$$L_{k+1} \leq L_{k+1}^- = L\left((t_k + \delta_k)^-\right) \leq 2\bar{L}_1(V_k) \leq \bar{L}_2(V_k).$$
   b) If $\delta_k > t_1(V_k)$. $L_{k+1} \leq L_{k+1}^-$, and since, by (33), $\delta_k L_{k+1}^- = \alpha$, it follows that
$$L_{k+1} \leq \frac{\alpha}{\delta_k} \leq \frac{\alpha}{t_1(V_k)} \leq \bar{L}_2(V_k).$$

Note that the previous properties, implies that for all k

$$L_{k+1} \leq \left(1 - \frac{a_1\alpha}{2}\right) L_k + \bar{L}_2(V_k)$$

and the first part of the result (i.e. inequality (60)) holds with $L_{\max} = \bar{L}_2(1)$ and $\gamma(V_k) = \max\{\bar{L}_2(V_k) - \bar{L}_2(1), 0\}$.

Note that the previous properties 1) and 2) in combination with the fact that the sequence (V_k) is decreasing imply also for all k

$$L_k \leq \max\{L_0, \bar{L}_2(V_0)\}$$

Moreover, since for all k in $\mathbb{N}$ and all t in $[t_k, t_{k+1})$

$$\begin{aligned} L(t) &\leq L_{k+1}^- && \text{(since } \dot{L}(t) \geq 0) \\ &= \frac{L_{k+1} - a_1\alpha}{1 - a_1\alpha} && \text{(by (27))} \\ &\leq \frac{L_{k+1}}{1 - a_1\alpha} \\ &\leq \frac{\left(\frac{a_1\alpha}{2}\right)^{k+1} \max\{L_0, \bar{L}_2(V_0)\}}{1 - a_1\alpha}, && (92) \\ &\leq \frac{\max\{L_0, \bar{L}_2(V_0)\}}{1 - a_1\alpha}, && (93) \end{aligned}$$

and the result holds with $\rho(L_0, V_0) = \frac{\max\{\bar{L}_0 + \bar{L}_2(1), \bar{L}_2(V_0)\}}{1 - a_1\alpha}$. ■

E. Proof of Lemma 7

Let $dL_{\max}$ and $dM_{\max}$ be the increasing functions

$$dL_{\max}(v) = 4a_2\bar{L}_1(v)(c_0 + c_1\mathfrak{d}_1(v)^q(2\bar{L}_1(v))^{bq}),$$
$$dM_{\max}(v) = 2a_3(c_0 + c_1\mathfrak{d}_1(v)^q(2\bar{L}_1(v))^{bq}).$$

Note that if $(L(t), M(t))$ is in C_2 and $\chi(t)$ satisfies the bound (90), we have

$$\dot{L}(t) \leq dL_{\max}(v), \quad \dot{M}(t) \leq dM_{\max}(v). \tag{94}$$

Let t_1 be the function defined by

$$t_1(v) = \min\left\{\frac{1}{dL_{\max}(v)}, \frac{1}{dM_{\max}(v)}\right\}.$$

We show that this function satisfies the properties of Lemma 7. Assume this is not the case. Hence, there exists $M(t_k), L(t_k)$ in C_1, χ which satisfies the bound (90) and $t^* \leq t_1(v)$ such that $(L(t_k + t^*), M(t_k + t^*)) \notin C_2$. Let s^* be the time instant at which the solution leaves C_2. More precisely, let $s^* = \inf\{s, t_k \leq s \leq t_k + t^*, (L(s), M(s)) \notin C_2\}$. Note that $(L(s^*), M(s^*))$ is at the border of C_2 and $t_k < s^* < t_k + t_1(v)$. Moreover, with (94), it yields:

$$M(s^*) \leq 1 + (s^* - t_k) dM_{\max}(v) < 1 + t_1(v) dM_{\max}(v) \leq 2.$$

Similarly, we have

$$L(s^*) < L(t_k) + t_1(v) dL_{\max} \leq L(t_k) + 1 \leq 2\bar{L}_1(v),$$

where the last inequality is obtained since $\bar{L}_1(v) \geq 1$. This implies that $(L(s^*), M(s^*))$ is not at the border of C_2 which contradicts the existence of t^*.

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