Bounds on Shannon Capacity and Ramsey Numbers from Product of Graphs

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Abstract

In this paper we study Shannon capacity of channels in the context of classical Ramsey numbers. We overview some of the results on capacity of noisy channels modelled by graphs, and how some constructions may contribute to our knowledge of this capacity.

We present an improvement to the constructions by Abbott and Song and thus establish new lower bounds for a special type of multicolor Ramsey numbers. We prove that our construction implies that the supremum of the Shannon capacity over all graphs with independence number 2 cannot be achieved by any finite graph power. This can be generalized to graphs with bounded independence number.

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1 Introduction and Notation

In this article we study lower bound constructions on some multicolor Ramsey numbers and their relation to Shannon capacity of noisy channels modelled by graphs. All graphs are undirected and loopless, and all colorings are edge-colorings. The independence number of a graph G, i.e. the maximum number of mutually independent vertices in G, will be denoted by $\alpha(G)$.

For arbitrary graphs $G_1, \ldots, G_n$, where $G_i = (V_i, E_i)$, we define the *graph product* $G_1 \times \cdots \times G_n$ to be a graph G on the vertex set $V = V_1 \times \cdots \times V_n$, whose edges are all pairs of distinct vertices $\{(u_1, \ldots, u_n), (v_1, \ldots, v_n)\}$, such that for each i from 1 to n, $u_i = v_i$ or $\{u_i, v_i\} \in E_i$. This product is associative, and also commutative up to isomorphisms permuting the coordinates. G^n will denote the n-fold product of the same graph, namely $G^n = \underbrace{G \times \cdots \times G}_{n}$. The *capacity* $c(G)$ of a graph G was defined by Shannon [14] as the limit

$$c(G) = \lim_{n \to \infty} \alpha(G^n)^{1/n}, \tag{1}$$

and is now called the Shannon capacity of a noisy channel modelled by graph G (see also [6], [3]). The quantity $c(G)$ is often simply referred to as the *Shannon capacity* of G. The study of $c(G)$ within information theory was initiated by Shannon [14] and has grown to be an extensive area involving electrical engineering, communication theory, coding theory, and other fields that typically use probability theory as a tool. It may be less known that $c(G)$ attracted attention of many graph theorists trying to compute it [3, 4, 6, 10, 13]. The definitions above, the intuition below, and our work are representing this graph-theoretic perspective.

Suppose that we have a set Σ of k characters which we wish to send over a noisy channel one at a time. Let $V(G) = \Sigma$, and assume further that the edges of G indicate a possible confusion between pairs of characters when transmitted over the channel. When sending a single character, the maximum number of characters we can fix, and then choose from for transmission without danger of confusion, is clearly $\alpha(G)$. When we use the same channel repeatedly n times, we could obviously send $\alpha(G)^n$ words of length n by using an independent set in G at each coordinate. However, we might be able to do better by sending words from Σ^n corresponding to vertices of an independent set of order $\alpha(G^n)$ in graph G^n, in cases when the general inequality $\alpha(G^n) \geq \alpha(G)^n$ is strict. The Shannon capacity $c(G)$ measures the efficiency of the best possible strategy when sending long words over a noisy channel modelled by G, since the limit (1) defining it can be seen as approaching the effective alphabet size in zero-error transmissions.

A $(k_1, k_2, \ldots, k_n)$-*coloring*, for some n and $k_i \geq 1$, is an assignment of one of n colors to each edge in a complete graph, such that the coloring does not contain any monochromatic complete subgraph K_{k_i} in color i, for $1 \leq i \leq n$. Similarly, a $(k_1, k_2, \ldots, k_n; s)$-*coloring* is a $(k_1, \ldots, k_n)$-coloring of the complete graph on s vertices K_s. Let $\mathcal{R}(k_1, \ldots, k_r)$ and $\mathcal{R}(k_1, \ldots, k_n; s)$ denote the set of all $(k_1, \ldots, k_n)$- and $(k_1, \ldots, k_n; s)$-colorings, respectively. The Ramsey number $R(k_1, \ldots, k_n)$ is defined to be the least $s > 0$ such that $\mathcal{R}(k_1, \ldots, k_n; s)$ is empty. In the diagonal case $k_1 = \ldots = k_n = k$, we will use simpler notation $\mathcal{R}_n(k)$ and $\mathcal{R}_n(k; s)$ for sets of colorings and $R_n(k)$ for the Ramsey numbers. The second author maintains a regularly updated survey [12] of the most recent results on the best known bounds on various types of Ramsey numbers.

In 1971, Erdős, McEliece and Taylor [9] were the first to discuss the connections between $\alpha(G_1 \times \ldots \times G_n)$ and Ramsey numbers. Many papers followed which studied explicitly Shannon capacity in relation to independence in product graphs and Ramsey numbers, like those by Alon et al. [6, 2, 3, 5], Bohman et al. [7, 8], and the survey papers [13, 4, 10]. Here we provide a further link between lower bounds on some multicolor Ramsey numbers and Shannon capacity. The result in Theorem 2 of Section 3 enhances our previous constructions from [17, 16] by establishing new lower bound for a special type of multicolor Ramsey numbers. This, in turn, implies that the supremum of the Shannon capacity over all graphs G with independence number $\alpha(G) = 2$ cannot be achieved by using any finite graph power. The same generalizes to graphs with bounded independence number.

2 Some Prior Results

The main results of a short but interesting paper by Erdős, McEliece and Taylor [9] are summarized in the following theorem.

Theorem 1 - Erdős, McEliece, Taylor - 1971 [9]
For arbitrary graphs $G_1, \ldots, G_n$,

$$\alpha(G_1 \times \cdots \times G_n) < R(\alpha(G_1) + 1, \ldots, \alpha(G_n) + 1), \tag{2}$$

and for all $k_1, \ldots, k_n > 0$ there exist graphs G_i with $\alpha(G_i) = k_i$, $1 \le i \le n$, such that

$$\alpha(G_1 \times \cdots \times G_n) = R(k_1 + 1, \ldots, k_n + 1) - 1. \tag{3}$$

Furthermore, for the diagonal case $k_i = k$, there exists a single graph G with $\alpha(G) = k$, such that $\alpha(G^n) = R_n(k + 1) - 1$.

This early theorem established strong links between the Shannon capacity, independence number of graph products and classical Ramsey numbers. Unfortunately, all three concepts are notoriously difficult, even for many very simple graphs. The value of the

Shannon capacity of the pentagon, $c(C_5) = \sqrt{5}$, was computed in a remarkable paper by Lovász [11] using tools from linear algebra in a surprising way. The value of $c(C_7)$ is still unknown, though significant progress has been obtained by Bohman et al. for some general cases of odd cycles [8] and their complements [7]. We only know how to compute $c(G)$ for very special graphs, like perfect graphs or self-complementary vertex-transitive graphs, and it seems plausible that even approximating $c(G)$ may be much harder than **NP**-hard [3].

If we use Theorem 1 for non-complete graphs without triangles in the complement (i.e. $\alpha(G_i) = k_i = 2$ for all i), then the Ramsey numbers in question are $R_n(3)$. It is known that $\lim_{n\to\infty} R_n(3)^{1/n}$ exists, though it may be infinite. The best established lower bound for this limit is 3.199... [16]. Clearly, $\lim_{n\to\infty}(R_n(3)-1)^{1/n} = \lim_{n\to\infty} R_n(3)^{1/n}$, and hence by (3) in Theorem 1, it is equal to the supremum of the Shannon capacity $c(G)$ over all graphs G with independence number 2.

Similarly, for any fixed integer $k \geq 3$, $\lim_{n\to\infty} R_n(k)^{1/n}$ exists, though again it may be infinite. Furthermore, this limit is equal to the supremum of the Shannon capacity $c(G)$ over all graphs G with independence number $k-1$.

3 A Ramsey Construction

This section presents a theorem which gives a new lower bound construction for some special cases of multicolor Ramsey numbers. This theorem is improving over an old result by Abbott [1] and Song [15] that $R_{n+m}(k) > (R_n(k)-1)(R_m(k)-1)$ (see also [16]). The current approach enhances our previous techniques used in [16, 17] and summarized in [12]. This result is then linked in Section 4 to the Shannon capacity of some graphs, in particular graphs with independence 2.

We would like to note that a special product of graphs (and edge-colorings) G and H, denoted $G[H]$, which we used in a few constructions in [16, 17], is similar to but distinct from $G \times H$ usually considered in the context of Shannon capacity. The vertex set of $G[H]$ is also equal to $V(G) \times V(H)$, but for graphs G and H, $\{(u_1, v_1), (u_2, v_2)\}$ is an edge of $G[H]$ if and only if $u_1 = u_2$ and $\{v_1, v_2\} \in E(H)$, or $\{u_1, u_2\} \in E(G)$. In the case of colorings, if $u_1 = u_2$ then $\{(u_1, v_1), (u_2, v_2)\}$ in $G[H]$ has the same color as $\{v_1, v_2\}$ in H, else it has the same color as $\{u_1, u_2\}$ in G. For any edge-coloring C, let $C(u, v)$ denote the color of the edge $\{u, v\}$ in C. Thus, equivalently, for $u_1 \neq u_2$ we have $G[H]((u_1, v_1), (u_2, v_2)) = G(u_1, u_2)$, and $G[H]((u, v_1), (u, v_2)) = H(v_1, v_2)$. Observe that $G[H]$ can be seen as $|V(G)|$ disjoint copies of H interconnected by many overlapping copies of G. Specifically, there are $|V(H)|^{|V(G)|}$ of them. Note that, because of this structure, if colors used in G and H are distinct, then the orders of the largest monochromatic complete subgraphs in $G[H]$ are the same as in G or H, depending on the color. Finally, observe that in general the graphs $G[H]$ and $H[G]$ need not be isomorphic.

Theorem 2 *For integers $k, n, m, s \geq 2$, let $G \in \mathcal{R}_n(k;s)$ be a coloring containing an induced subcoloring of K_m using less than n colors. Then*

$$R_{2n}(k) \geq s^2 + m(R_n(k-1, \underbrace{k, \cdots, k}_{n-1}) - 1) + 1. \tag{4}$$

Proof. Consider coloring $G \in \mathcal{R}_n(k;s)$ with the vertex set $V(G) = \{v_1, \ldots, v_s\}$, and suppose, without loss of generality, that the set $M = \{v_1, \ldots, v_m\}, m \leq s$, does not induce any edges of color 1 in G. Let H be any critical n-coloring (on the maximum possible number of vertices) in $\mathcal{R}_n(k-1, k, \cdots, k)$ with vertices $V(H) = \{w_1, \ldots, w_t\}$, and hence $t = \mathcal{R}_n(k-1, k \ldots, k) - 1$. In order to prove the theorem, we will construct a $2n$-coloring $F \in \mathcal{R}_{2n}(k; s^2 + mt)$ with the vertex set $V(F) = (V(G) \times V(G)) \cup (M \times V(H))$.

We will use colors labeled by integers from 1 to $2n$. G and H use colors from 1 to n, and F from 1 to $2n$. The structure of coloring F induced on the set $V(G) \times V(G)$ is similar to that of the special product of $G[G]$, namely, we set the color of each edge $e = \{(v_{i_1}, v_{i_2}), (v_{j_1}, v_{j_2})\}$, for $1 \leq i_1, i_2, j_1, j_2 \leq s$, by

$$F(e) = \begin{cases} n+1 & \text{if } i_2 = j_2 \leq m \text{ and } G(v_{i_1}, v_{j_1}) = 1, \\ G(v_{i_2}, v_{j_2}) + n & \text{for } i_1 = j_1, \\ G(v_{i_1}, v_{j_1}) & \text{for other cases with } i_1 \neq j_1. \end{cases} \tag{5}$$

In addition, the coloring F contains m isomorphic copies of the coloring H on the vertex sets $U_i = \{(v_i, w_j) \mid 1 \leq j \leq t\}$ for $1 \leq i \leq m$, each of order t. The definition of the coloring of the edges connecting U_i's follows.

All the edges of the form $\{(v_{i_1}, w_{j_1}), (v_{i_2}, w_{j_2})\}$, for $1 \leq i_1 < i_2 \leq m$ and $1 \leq j_1, j_2 \leq t$, i.e. the edges between different copies of H, are assigned color $G(v_{i_1}, v_{i_2}) + n$. All the edges of the form $\{(v_{i_1}, v_{j_1}), (v_{i_2}, w_{j_2})\}$, for $1 \leq i_1, j_1 \leq s$, $1 \leq i_2 \leq m$, $j_1 \neq i_2$, and $1 \leq j_2 \leq t$, are also assigned a high index color $G(v_{j_1}, v_{i_2}) + n$. Finally, the remaining uncolored edges of the form $\{(v_{i_1}, v_q), (v_q, w_{j_2})\}$, for $1 \leq q \leq m$, $1 \leq i_1 \leq s$, and $1 \leq j_2 \leq t$, are assigned color 1.

We will prove that the coloring F constructed above does not contain any monochromatic K_k. We already noted that the part of F induced by the vertices $V(G) \times V(G)$ is similar to $G[G]$. More precisely, let's denote this part of F by F', and let G' denote the coloring obtained from G by renaming all colors from c to $c + n$. Then, if in $G[G']$ we recolor the edges specified in the first line of (5) from color 1 to color $n+1$, then we obtain exactly F'. Next, let the part of F induced by the vertices $M \times V(H)$ be denoted by F'', and the subcoloring of G' induced by vertices M be denoted by G''. Observe that F'' is isomorphic to $G''[H]$.

Since $G[G']$ and F'' are both the results of the special product with different sets of base colors, they don't contain any monochromatic K_k. Furthermore, since M doesn't induce in G any edges of color 1, then F' has no monochromatic K_k either. Thus, if there is a monochromatic K_k in F it must intersect both $V(G) \times V(G)$ and $M \times V(H)$. Next, it is not hard to see that the structure of F' and F'' and how they swap the roles of colors with labels $\leq n$ and $> n$ prevent any monochromatic K_k in all colors different from 1 and $(n+1)$. Now, note that the lastly added edges in color 1 between F' and F'' join the blocks of vertices with the same index v_q in one position, and one can conclude that no monochromatic K_k in color 1 can arise because $H \in \mathcal{R}_n(k-1, k, \cdots, k)$. Finally, no K_k is formed in color $(n+1)$ since G'' has no edges in color $(n+1)$. This completes the proof. $\diamondsuit$

We wish to comment that Theorem 2 with any lower bound better than s^2 would be sufficient for the results in Section 4. Observe that the required subcoloring with $m > 0$ exists in all nontrivial cases.

4 Shannon Capacity

It can be easily shown that $R_{2n}(3) > (R_n(3) - 1)^2$, for example by using inequalities (7) or (12) in [16]. Now, this can be improved by the construction of Theorem 2, as in the corollary below. This corollary is interesting in itself since it improves over the previous lower bound recurrence on $R_n(3)$, but first of all it will let us prove Theorem 3 on Shannon capacity of graphs with independence number 2.

Corollary *For all integers $n \geq 2$,*

$$R_{2n}(3) \geq (R_n(3) - 1)^2 + m(R_{n-1}(3) - 1) + 1, \text{ for } m = \lceil (R_n(3) - 2)/n \rceil.$$

Proof. Each vertex of any coloring in $\mathcal{R}_n(3; s)$ has at least $m = \lceil (s-1)/n \rceil$ neighbors in the same color, which must induce a coloring in $\mathcal{R}_{n-1}(3; m)$. Theorem 2 with $k = 3$ implies the claim. $\diamondsuit$

Theorem 3 *If the supremum of the Shannon capacity $c(G)$ over all graphs with independence number 2 is finite and equal to C, then $C > \alpha(G^n)^{1/n}$ for any graph G with independence number 2 and for any positive integer n.*

Proof. Suppose that C is achieved by some graph G with $\alpha(G) = 2$, and hence by (2) we have $C^n = \alpha(G^n) < R_n(3)$. By the second part of Theorem 1, we know that there exists a graph H with independence number 2 such that $\alpha(H^{2n}) = R_{2n}(3) - 1$, and by Theorem 2 we see that $\alpha(H^{2n}) > (R_n(3) - 1)^2$. This contradicts the fundamental inequality $\alpha(G_1 \times G_2) \geq \alpha(G_1)\alpha(G_2)$, Theorem 1 and the assumption that C is realized by G. $\diamondsuit$

Observe that in the case of infinite supremum, $\lim_{n\to\infty} R_n(3)^{1/n}$ must also be infinite. In other words, together with Theorem 3, this means that the supremum of the Shannon capacity over all graphs G with independence number $\alpha(G) = 2$ cannot be achieved by any finite graph power.

It is not difficult to generalize Theorem 3 for $k \geq 3$ to $\alpha(G) = k - 1$, $R_n(k)$ and the supremum of the Shannon capacity over all graphs with independence number $k - 1$, as stated in the following Theorem 4. We omit the details which are analogous to those in Corollary and Theorem 3.

Theorem 4 (*a*) *For every positive integer* n_0*,* $R_{n_0}(k)^{1/n_0} < lim_{n\to\infty} R_n(k)^{1/n}$*, and* (*b*) *the supremum of the Shannon capacity over all graphs with bounded independence number cannot be achieved by any finite graph power.*

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