

A characterization of the number of subsequences obtained via the deletion channel

Y. Liron
The Open University of Israel
yuvalal@gmail.com

M. Langberg
The Open University of Israel
mikel@openu.ac.il

Abstract—Motivated by the study of *deletion channels*, this work presents improved bounds on the number of subsequences obtained from a binary string X of length n under t deletions. It is known that the number of subsequences in this setting strongly depends on the number of *runs* in the string X ; where a run is a maximal sequence of the same character. Our improved bounds are obtained by a structural analysis of the family of r -run strings X , an analysis in which we identify the *extremal* strings with respect to the number of subsequences. Specifically, for every r , we present r -run strings with the minimum (respectively maximum) number of subsequences under any t deletions; and perform an exact analysis of the number of subsequences of these extremal strings.

I. INTRODUCTION

Let $X \in \{0,1\}^n$ be a binary string of length n , and let $t \leq n$ be a parameter. In this work, we study the size of the set $D_t(X)$ of subsequences of X that can be obtained from X via t deletions. The set $D_t(X)$ and its size play a major role in the design and analysis of communication schemes over *deletion channels*, i.e., channels in which characters of the transmitted codeword may be deleted, [3]–[7], [9].

The analysis of $D_t(X)$ is challenging as the number of subsequences of a string X obtained by deletions does not depend only on its length n and the number t of deletions, but also strongly depends on its structure. For example, $D_t(0^n)$ is of size 1 and equals the single string 0^{n-t} , while there exist strings X for which $D_t(X)$ is of size $\exp(\Omega(n-t))$. Clearly, $|D_t(X)|$ is at most 2^{n-t} (as after t deletions we remain with a binary string of length $n-t$).

In his work from 1966, Levenshtein [4] shows (as described in [5]) that the number of subsequences $|D_t(X)|$ strongly depends on the number of *runs* in the string. Here, a run is a maximal sequence of the same character, and the number of runs in a given string is denoted $r(\cdot)$. For example $r(0^n) = 1$ while $r(0101\dots 01) = n$. Specifically, Levenshtein [4] proves that

$$\binom{r(X) - t + 1}{t} \leq |D_t(X)| \leq \binom{r(X) + t - 1}{t}.$$

Bounding $|D_t(X)|$ is addressed by Calabi and Hartnett [1], which show that the maximal number of subsequences is obtained from certain strings X , denoted cyclic strings C_n , in which $r(X) = |X|$. [1] devise a recursive expression for $|D_t(C_n)|$, to obtain the bound

$$\binom{r(X) - t + 1}{t} \leq |D_t(X)| \leq |D_t(C_n)|.$$

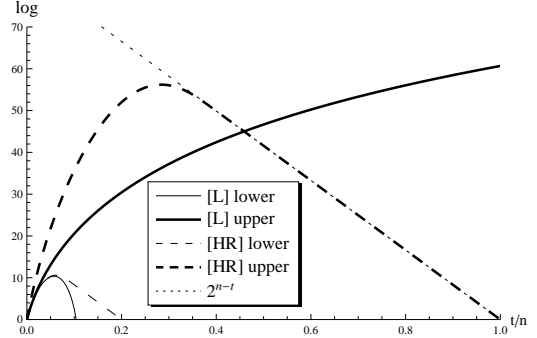


Figure 1. Previous bounds on $|D_t(X)|$. [L] marks the bounds proven by Levenshtein [4], and [HR] marks the bounds by Hirschberg et. al [2]. Also plotted is the naive bound 2^{n-t} which is the possible number of binary strings of length $n-t$. This graph shows an example for the case $n = 120$, and $r = 24$. All graphs are shown on a logarithmic scale.

Relatively recently, Hirschberg and Regnier [2] revisit the analysis of [1] and obtain an explicit upper bound together with an improved lower bound of the form

$$\sum_{i=0}^t \binom{r(X) - t}{i} \leq |D_t(X)| \leq \sum_{i=0}^t \binom{n-t}{i}.$$

Mercier et al. [8] study the setting of small values for t , and present explicit formulas for $D_t(X)$ for $t \leq 5$. However for general values of t the problem remains open. Several of the results above generalize also to arbitrary alphabets.

The bounds of [1], [2], [4] are depicted in Figure 1 for the case $n = 120$ and $r = r(X) = 24$ as a function of t . The lower bounds of both [2] and [4] depend on the number of runs $r(X)$; and it holds that the lower bound of [2] is superior (i.e., larger) to that of [4]. The upper bound of [4] depends on $r(X)$, while that of [1], [2] does not. Thus each bound is stronger (i.e., smaller) for certain settings of parameters r and t . Roughly speaking, the upper bounds of [1], [2] are stronger than those of [4] for large values of r and t ; while the opposite is true for small r and t .

A. Our results and proof techniques

In this work, we continue the study of $D_t(X)$ and present improved upper and lower bounds to those described above. Our analysis is two fold. We start by studying the family of strings X for which $r = r(X)$, and identify the *extremal* strings

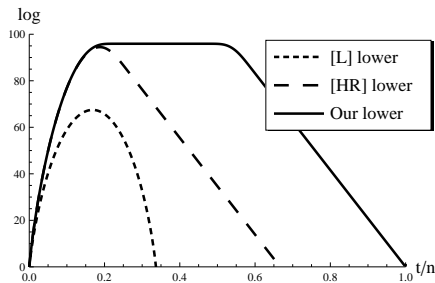


Figure 2. Comparison of lower bounds. Our lower bound based on unbalanced strings [Theorem VI.2], compared to the previous known bounds. [L] marks the lower bound proven by Levenstein [4]. [HR] marks the lower bound proven by Hirschberg et. al [2]. This graph shows an example for the case $n = 300$, and $r = 200$. The logarithmic presentation emphasizes that we obtain an *exponential* multiplicative improvement.

in this family with respect to the number of subsequences. Specifically, for every r , we identify two r -run strings, referred to as the *balanced* r -run string B_r and the *unbalanced* r -run string U_r such that for every X it holds that

$$|D_t(U_{|r(X)|})| \leq |D_t(X)| \leq |D_t(B_{|r(X)|})|. \quad (1)$$

Loosely speaking, the string $U_r = 0101 \dots 01^{n-r+1}$ is the r -run string in which each run is exactly of size 1, except the last run which is of size $n - t + 1$, and is thus referred to as ‘unbalanced’ (in the run lengths). The balanced string $B_r = 0^{n/r}1^{n/r}0^{n/r}1^{n/r} \dots 1^{n/r}0^{n/r}$ is the r -run string in which each and every run is of equal length n/r .

To obtain Equation (1), we show that any r -run string X can be transformed into the string U_r (alternatively B_r) via a series of operations that are monotonic with respect to the number of subsequences. The modifications we study include a *balancing* operation, in which given X we shorten the length of one of its runs while increasing the length of another; a *flipping* operation, in which a prefix or suffix of X is replaced by its complement; and an *insertion* operation in which characters are added to X (see Figures 4(a), 4(b) and 4(c)). A delicate combination of these (and other) operations enable us to establish Equation (1). The modifications we study and their analysis shed light on the properties of binary strings under the deletion operation and may be of independent interest. We note that for the extreme case of $r = n$, our unbalanced string U_n is exactly the cyclic string C_n ; thus we are consistent with the result of [1].

We then turn to obtain analytic expressions for $|D_t(U_r)|$ and $|D_t(B_r)|$ of Equation (1). Our expressions are at least as good as previous bounds in [1], [2], [4] as they are based on specific r -run strings (U_r and B_r), and for a large range of parameters our bounds are strictly tighter. For our improved lower bound, we devise a recursive expression for $|D_t(U_r)|$ and present a closed form formula for its evaluation. We then perform an asymptotic evaluation of $|D_t(U_r)|$ (assuming large r). A comparison of our improved lower bound with that previously known is depicted in Figure 2. Specifically, we show that for values of t which are greater than $r/3$ our lower

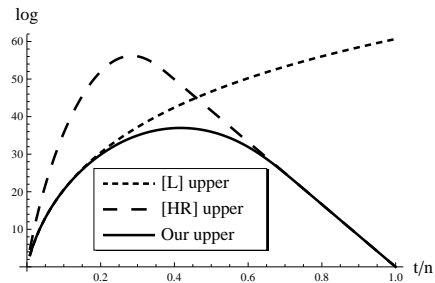


Figure 3. Comparison of upper bounds. Our upper bounds based on balanced strings [Corollary IV.1], compared to the previous best known bounds. [L] marks the upper bound proven by Levenstein [4]. [HR] marks the upper bound proven by Hirschberg et. al [2]. This graph shows an example for the case $n = 120$ and $r = 24$ as a function of t (in logarithmic scale).

bound improves on those previously known by an exponential multiplicative factor of roughly $2^{t-r/3}$.

To address our improved upper bounds, we first present a recursive formula for the computation of $|D_t(B_r)|$. We then extract a closed form solution to our recursive definition which yields an exact expression for $|D_t(B_r)|$. For example, a numerical comparison of $|D_t(B_r)|$ with the upper bounds previously known is depicted in Figure 3 for the value of $n = 120$ and $r = 24$ as a function of t . We note that the expression we obtain for $|D_t(B_r)|$ involves several summations of certain combinatorial expressions. An asymptotic analysis of our expression is left open in this work and is subject to future research.

B. Structure

The remainder of the paper is organized as follows. In section II we present the set of structural operations and tools we use for comparing and bounding the number of subsequences obtained via deletion. This section includes our balancing, flipping, and insertion modifications. In Section III, we study our first family of *balanced* strings, and show that (for any given number of runs r and deletions t) they have the largest number of subsequences under deletion. In Section IV, we analyze the number of subsequences of balanced strings and in such obtain our upper bound. In Section V, we present our second family of *unbalanced* strings, and prove that they have the least number of subsequences under any number of deletions t . We prove our lower bound by analyzing the number of subsequences of unbalanced strings in Section VI. Finally, in Section VII, we study the connection between subsequences and the closely related notion of *deletion patterns*. Using this connection, we show exponential multiplicative gaps between our improved upper bound and those previously presented.

II. TOOLS FOR ANALYZING THE NUMBER OF SUBSEQUENCES

The number of subsequences of a string obtained by deletions highly depends on the string’s structure. In order to determine the number of subsequences for a given number of deletions, it is not enough to know the length of the

string, and not even the number of the string's runs. Inspired by previous works, we looked for tools that will enable us to analyze the number of subsequences. In this section we present these tools. In subsection II-A we present a method of counting the number of subsequences by partitioning the set of subsequences into subsets characterized by their prefix, thus forming a recursive relation. In subsection II-B we present basic operations on strings that always increase (or decrease) the number of subsequences under deletion. Such basic operations allow comparison between the number of subsequences of strings, and are very useful for finding bounds on the number of subsequences.

$S(x_1, \dots, x_r)$ denotes a binary string with r runs, in which the i^{th} run is of length x_i and the first symbol is 0, E.g. $S(1, 2, 3) = 011000$. We will use the notation $n \times a$ to indicate n sequential runs of length a , E.g. $S(2, 3 \times 1, 2) = S(2, 1, 1, 1, 2) = 0010100$. $D_t(x_1, \dots, x_r)$ will be used as short form for $D_t(S(x_1, \dots, x_r))$. C_n denotes the binary cyclic string $S(n \times 1)$. We assume the following conventions: $\sum_{i=j}^k a_i = 0$ when $j > k$. $\binom{n}{i} = 0$ when $i < 0$ or $i > n$. $|D_t(X)| = 1$ for $t = |X|$ and $t = 0$, and $|D_t(X)| = 0$ for $t > |X|$.

A. Partitioning the set of subsequences

We found the following lemma (from [2]) very useful. We restate it here and derive a corollary for binary strings.

Lemma II.1. [2] For any Σ -string X :

- (i) $D_t(X) = \sum_{a \in \Sigma} D_t^{(a)}(X)$, where for a set G of strings $G^{(a)}$ denotes all members of G starting with a .
- (ii) $D_t^{(a)}(X) = aD_{t+1-f(a)}(X[f(a)+1:n])$, where $f(a)$ denotes the index of the first appearance of a in X , and $X[i:j]$ denotes the substring $x_i \dots x_j$ of X .

We derive the following lemma for binary strings.

Lemma II.2.

- (i) For any binary string X , s.t. $X = \sigma^i \epsilon^j Y$ for some $i, j > 0$ and $Y \in \{\sigma, \epsilon\}^*$, $|D_t(X)| = |D_t(\sigma^{i-1} \epsilon^j Y)| + |D_{t-i}(\epsilon^{j-1} Y)|$ for any $t < |X|$.
- (ii) Symmetrically, $|D_t(Y \epsilon^j \sigma^i)| = |D_t(Y \epsilon^j \sigma^{i-1})| + |D_{t-i}(Y \epsilon^{j-1})|$.

Proof: (i) Following the notation of Lemma II.1, $f(\sigma) = 1$ and $f(\epsilon) = i + 1$. Using Lemma II.1(ii), $D_t^{(\sigma)} = \sigma D_{t+1-1}(X[2:n])$ and $D_t^{(\epsilon)} = \epsilon D_{t+1-(i+1)}(X[i+2:n])$. Applying Lemma II.1(i) we get the result.

(ii) The proof for the symmetric case is identical. \blacksquare

Applying Lemma II.2 repeatedly, we get the following lemma.

Lemma II.3. For any binary string $S(x_1, \dots, x_r)$, s.t. $n = \sum_{i=1}^r x_i$:

- (i) $|D_t(x_1, \dots, x_r)| = |D_t(x_2, \dots, x_r)| + \sum_{i=1}^{x_1} |D_{t-i}(x_2 - 1, x_3, \dots, x_r)| + 1_{|t| > n - x_1}$.
- (ii) Symmetrically, $|D_t(x_1, \dots, x_r)| = |D_t(x_1, \dots, x_{r-1})| + \sum_{i=1}^{x_r} |D_{t-i}(x_1, \dots, x_{r-2}, x_{r-1} - 1)| + 1_{|t| > n - x_r}$.

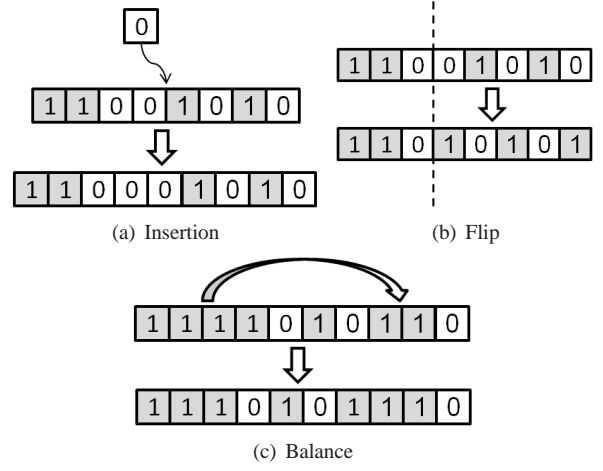


Figure 4. Basic operations on strings. In all diagrams the lower string has more subsequences under any number of deletions.

Proof: (i) We denote $n = \sum_{i=1}^r x_i$. Using Lemma II.2 once, we get $|D_t(x_1, \dots, x_r)| = |D_t(x_1 - 1, x_2, \dots, x_r)| + |D_{t-x_1}(x_2 - 1, x_3, \dots, x_r)|$. For $x_1 > 1$ we can use Lemma II.2 again and get $|D_t(x_1, \dots, x_r)| = |D_t(x_1 - 2, x_2, \dots, x_r)| + |D_{t-x_1+1}(x_2 - 1, x_3, \dots, x_r)| + |D_{t-x_1}(x_2 - 1, x_3, \dots, x_r)|$. Likewise, for $j \leq \min(x_1, n - t)$, applying Lemma II.2 j times yields $|D_t(x_1, \dots, x_r)| = |D_t(x_1 - j, x_2, \dots, x_r)| + \sum_{i=x_1-j+1}^{x_1} |D_{t-i}(x_2 - 1, x_3, \dots, x_r)|$. When $t \leq n - x_1$, it follows that $\min(x_1, n - t) = x_1$, and so we can expand using Lemma II.2 exactly x_1 times to get $|D_t(x_1, \dots, x_r)| = |D_t(x_2, \dots, x_r)| + \sum_{i=1}^{x_1} |D_{t-i}(x_2 - 1, x_3, \dots, x_r)|$. When $t > n - x_1$ after expanding $n - t$ times, we get the expression $|D_t(x_1, \dots, x_r)| = |D_t(x_1 - (n - t), x_2, \dots, x_r)| + \sum_{i=x_1-(n-t)+1}^{x_1} |D_{t-i}(x_2 - 1, x_3, \dots, x_r)|$. As $|S(x_1 - (n - t), x_2, \dots, x_r)| = t$, and noticing that for $t > |X|$, $|D_t(X)| = 0$, we get the lemma's claim.

(ii) The proof for the symmetric case is identical. \blacksquare

B. Basic operations on strings

In the following sections we will present families of strings, for which the number of subsequences can be explicitly calculated. In order to use these families of strings to devise bounds on the number of subsequences for general strings, we use basic operations on strings, which allows us to transform one string into another, while monotonically increasing (or decreasing) the number of their subsequences. In this section we list such basic operations.

1) *Insertion operation* [Figure 4(a)]: Hirschberg et al. [2] showed that inserting a symbol anywhere in the middle of a string always increases the number of subsequences.

Lemma II.4. [Insertion increases the number of subsequences] [2] For any Σ -strings U, V and any $\sigma \in \Sigma$, $|D_t(UV)| \leq |D_t(U\sigma V)|$.

2) *Deletion chain rule:*

Lemma II.5. For any Σ -string U , and any $V \in D_t(U)$, $D_{t'}(V) \subseteq D_{t+t'}(U)$.

Proof: V was obtained from U by deleting t symbols. Any string in $D_{t'}(V)$ is obtained by deleting t' symbols from V . The same string can be created by removing the $t + t'$ symbols directly from U , and thus it belongs also to $D_{t+t'}(U)$ ■

3) *Flipping operation:* [Figure 4(b)]

Lemma II.6.[Flipping increases number of subsequences] For any binary strings U, V and for any bit σ , $|D_t(U\sigma\sigma V)| \leq |D_t(U\sigma\bar{\sigma}\bar{V})|$, where \bar{a} denotes the string a in which 0's are flipped to 1's, and vice versa.

Proof: By induction on $|U|$. When $|U| = 0$ the claim is $|D_t(\sigma\sigma V)| \leq |D_t(\sigma\bar{\sigma}\bar{V})|$. Let $V = \sigma^i \epsilon^j X$ for maximal i, j . When $j = 0$ the claim is trivial ($\sigma\sigma V$ is a constant string, with 1 possible subsequence), so we assume $j > 0$. Using Lemma II.2 we get $|D_t(\sigma\sigma V)| = |D_t(\sigma V)| + |D_{t-2-i}(\epsilon^{j-1} X)|$. We compare that to $|D_t(\sigma\epsilon\bar{V})| = |D_t(\epsilon\bar{V})| + |D_{t-1}(\bar{V})|$. $|D_t(\sigma V)| = |D_t(\epsilon\bar{V})|$ because of symmetry, and since $\epsilon^{j-1} X \subseteq D_{i+1}(V)$ we can use Lemma II.5 and get $|D_{t-2-i}(\epsilon^{j-1} X)| \leq |D_{t-1}(\bar{V})|$, and thus we prove the base of the induction.

Now for the induction step, assume the claim is true for $|U| < n$ and we look at $|U| = n$. We regard the different cases of the structure of U .

Case 1: $U = \sigma^i \epsilon^j X$ for some $i, j > 0$. We use Lemma II.2 and get $|D_t(\sigma^i \epsilon^j X \sigma \sigma V)| = |D_t(\sigma^{i-1} \epsilon^j X \sigma \sigma V)| + |D_{t-i}(\epsilon^{j-1} X \sigma \sigma V)|$. We compare that to $|D_t(\sigma^i \epsilon^j X \sigma \epsilon \bar{V})| = |D_t(\sigma^{i-1} \epsilon^j X \sigma \epsilon \bar{V})| + |D_{t-i}(\epsilon^{j-1} X \sigma \epsilon \bar{V})|$. On each of the arguments we can use our induction claim for $|U| - 1$ and $|U| - i - 1$.

Case 2: $U = \epsilon^i \sigma^j X$ for some $i, j > 0$ we use the same method.

Case 3: $U = \epsilon^i$ for some $i > 0$. $|D_t(\epsilon^i \sigma \sigma V)| = |D_t(\epsilon^{i-1} \sigma \sigma V)| + |D_{t-i}(\sigma V)|$. For the flipped string we get $|D_t(\epsilon^i \sigma \epsilon \bar{V})| = |D_t(\epsilon^{i-1} \sigma \epsilon \bar{V})| + |D_{t-i}(\epsilon \bar{V})|$. In this case, the second argument in both summations is equal due to symmetry, and we can compare the first arguments using the induction hypothesis for $|U| - 1$.

Case 4: $U = \sigma^i$ for some $i > 0$. Let $V = \sigma^j X$ for maximal j . In case $|X| = 0$ we get the trivial case of a uniform string again. For $|X| > 0$ let $X = \epsilon Y$, and then $|D_t(U\sigma\sigma V)| = |D_t(\sigma^{i+2} V)| = |D_t(\sigma^{i+1} V)| + |D_{t-i-j-2}(Y)|$. Again we compare that to $|D_t(\sigma^{i+1} \epsilon \bar{V})| = |D_t(\sigma^i \epsilon \bar{V})| + |D_{t-i-1}(\bar{V})|$, using the induction claim for the first argument, and Lemma II.5 together with symmetry for the second. ■

Corollary II.1. [Alternative proof for the maximality of C_n] Given any string X of length n , it can be transformed into the string C_n by a series of flipping operation (as defined in Lemma II.6). Each such flip can only increase the number of subsequences, and thus we get a proof for the fact that $D_t(X) \leq D_t(C_n)$.

4) *Balancing operation:* [Figure 4(c)] Informally, we refer to a string as *balanced*, if there is a low variability between the length of the string's runs. A balancing operation is one that decreases that variability, E.g. shortening a long run and increasing the length of a short run. The following lemma states terms in which balancing a string increases the number of its subsequences, and it is used later to prove maximality of string families.

Lemma II.7.[Balancing increases the number of subsequences] For $X = S(x_1, \dots, x_r)$, and for any $t > 0$, $1 \leq i < j \leq r$ s.t. $x_i - x_j > 1$, and $\{x_{i+1}, \dots, x_{j-1}\}$ is symmetric (i.e. $x_2 = x_{r-1}, x_3 = x_{r-2}, \dots$), $|D_t(x_1, \dots, x_r)| \leq |D_t(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_r)|$. In other words, decreasing the i -th run by 1, and increasing the j -th run by 1 can only increase the number of subsequences.

In order to prove Lemma II.7 we will need the following lemma that characterizes balancing operations near the edges of the string.

Lemma II.8. Assume $\{x_2, \dots, x_{r-1}\}$ is symmetric. It follows that:

- (i) For $X = S(x_1, \dots, x_r)$ s.t. $x_1 > x_r$, $|D_t(x_1, \dots, x_r)| \leq |D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)|$.
- (ii) For $X = S(x_1, \dots, x_r, z)$ s.t. $x_1 > x_r$ and $z > 0$, $|D_t(x_1, \dots, x_r, z)| \leq |D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1, z)|$.
- (iii) For $X = S(y, x_1, \dots, x_r)$ s.t. $x_1 - x_r > 1$ and $y > 0$, $|D_t(y, x_1, \dots, x_r)| \leq |D_t(y, x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)|$.
- (iv) For $X = S(y, x_1, \dots, x_r, z)$ s.t. $x_1 - x_r > 1$ and $y, z > 0$, $|D_t(y, x_1, \dots, x_r, z)| \leq |D_t(y, x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1, z)|$.

Proof:

- (i) When $r = 2$ the claim is reduced to $D_t(x_1, x_2) \leq D_t(x_1 - 1, x_2 + 1)$ for $x_1 > x_2$. This is easily proved because $D_t(x_1, x_2) = \min(x_1, x_2, t) + 1$. For $r > 2$ we use Lemma II.2 to get $|D_t(x_1, \dots, x_r)| = |D_t(x_1 - 1, x_2, \dots, x_r)| + |D_{t-x_1}(x_2 - 1, x_3, \dots, x_r)|$. Using Lemma II.2 and the symmetry of $\{x_2, \dots, x_{r-1}\}$ we get $|D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)| = |D_t(x_r + 1, x_2, \dots, x_{r-1}, x_1 - 1)| = |D_t(x_r, x_2, \dots, x_{r-1}, x_1 - 1)| + |D_{t-x_r-1}(x_2 - 1, x_3, \dots, x_{r-1}, x_1 - 1)|$. We compare the two expressions. Because of the symmetry $|D_t(x_1 - 1, x_2, \dots, x_r)| = |D_t(x_r, x_2, \dots, x_{r-1}, x_1 - 1)|$, and because $x_1 > x_r$ it is true that $S(x_2 - 1, x_3, \dots, x_r) \in D_{x_1-x_r-1}(x_2 - 1, x_3, \dots, x_1 - 1)$ and thus using Lemma II.5 $|D_{t-x_1}(x_2 - 1, x_3, \dots, x_r)| \leq |D_{t-x_r-1}(x_2 - 1, x_3, \dots, x_1 - 1)|$.
- (ii) Applying Lemma II.3(ii) we get $|D_t(x_1, \dots, x_r, z)| = |D_t(x_1, \dots, x_r)| + \sum_{i=1}^z |D_{t-i}(x_1 \dots x_{r-1}, x_r - 1)| + 1|_{t > n-z}$, and $|D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1, z)| = |D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)| + \sum_{i=1}^z |D_{t-i}(x_1 - 1, x_2, \dots, x_r)| + 1|_{t > n-z}$. The two expressions are comparable argument by argument using (i) above, noticing that if $x_1 > x_r$ then definitely $x_1 > x_r - 1$.

- (iii) Applying Lemma II.3 we get $|D_t(y, x_1, \dots, x_r)| = |D_t(x_1, \dots, x_r)| + \sum_{i=1}^y |D_{t-i}(x_1 - 1, x_2, \dots, x_r)| + 1|_{t > n-y}$, and $|D_t(y, x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)| = |D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1)| + \sum_{i=1}^y |D_{t-i}(x_1 - 2, x_2, \dots, x_{r-1}, x_r + 1)| + 1|_{t > n-y}$. The two expressions are comparable argument by argument using (i) above and noticing that if $x_1 - x_r > 1$ then definitely $x_1 > x_r$ and $x_1 - 1 > x_r$.
- (iv) We use Lemma II.3 to get $|D_t(y, x_1, \dots, x_r, z)| = |D_t(x_1, \dots, x_r, z)| + \sum_{i=1}^y |D_{t-i}(x_1 - 1, x_2, \dots, x_r, z)| + 1|_{t > n-y}$ and $|D_t(y, x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1, z)| = |D_t(x_1 - 1, x_2, \dots, x_{r-1}, x_r + 1, z)| + \sum_{i=1}^y |D_{t-i}(x_1 - 2, x_2, \dots, x_{r-1}, x_r + 1, z)| + 1|_{t > n-y}$. The two expressions are comparable argument by argument using (ii) above, as the condition $x_1 - x_r > 1$ guarantees that $x_1 - 1 > x_r$. ■

Now we can prove Lemma II.7 [Balancing increases the number of subsequences]:

Proof: We will prove by induction on the number of runs in X outside of the sequence $S(x_i, \dots, x_j)$, explicitly on $(i-1) + (r-j) = r+i-j-1$. We will denote these runs *outer runs*. When we have only one outer run, the lemma is reduced to Lemma II.8(ii) or II.8(iii). Now we assume that there are at least two outer runs. If the outer runs are one on each side ($i=2$ and $j=r-1$) this is the case of Lemma II.8(iv). Otherwise, at least on one of the sides there are two or more runs ($i > 2$ or $j < r-1$). We assume w.l.o.g that $i > 2$, and then we can use Lemma II.3 and the induction hypothesis on strings with the number of outer runs decreased by 1. ■

III. BALANCED STRINGS

In this section we define the family of strings named *Balanced strings*. We call a string balanced, if all the runs of symbols in the string are of equal length. Formally, we denote by $B_{r,k}$ the binary string of length rk , with r runs, each of length k . E.g. $B_{3,4} = S(4,4,4) = 000011110000$. We will prove that of all strings with length rk and r runs, the balanced string has the maximal number of subsequences, under any number of deletions.

Theorem III.1. *Let $X = S(x_1, \dots, x_r)$, $n = \sum_{i=1}^r x_i$, and $k = n/r$. If k is an integer, then $|D_t(X)| \leq |D_t(B_{r,k})|$.*

Proof: The main idea of the proof is that any such string X can be transformed into $B_{r,k}$ by repeatably applying the Balancing Lemma II.7. Each such step can only increase the number of subsequences, so if such a series of balance operations can be found, the theorem is proved. We will construct a series of strings, X_0, \dots, X_m , such that $X_0 = X$, $X_m = B_{r,k}$ and for any $0 \leq i < m$, $|D_t(X_i)| \leq |D_t(X_{i+1})|$. Given a string $X_i \neq B_{r,k}$, we denote $X_i = S(x_1^{(i)}, \dots, x_r^{(i)})$. We choose a pair (p, q) s.t $|x_p^{(i)} - x_q^{(i)}| > 1$, $p < q$ and $q-p$ is minimal. Such a pair exists, because at least one

TABLE I
EXAMPLE OF A BALANCING PROCESS AS DEFINED IN THE PROOF OF
THEOREM III.1

i	X_i	runs	$\sum x_i^2$	$D_6(X_i)$
0	000111111100100	3,7,2,1,2	67	43
1	000111111000100	3,6,3,1,2	59	56
2	000111110000100	3,5,4,1,2	55	63
3	000111110001100	3,5,3,2,2	51	85
4	000111100001100	3,4,4,2,2	49	92
5	000111100011100	3,4,3,3,2	47	102
6	000111000111000	3,3,3,3,3	45	105

run is of length different from k (w.l.o.g, bigger than k), and thus there is at least one other run with length smaller than k . Assume w.l.o.g that $x_p^{(i)} > x_q^{(i)}$, and then we can conclude that $x_p^{(i)} > x_{p+1}^{(i)} = x_{p+2}^{(i)} = \dots = x_{q-1}^{(i)} > x_q^{(i)}$, otherwise we get a contradiction to the minimality of (p, q) . We will define X_{i+1} to be the string achieved from X_i by decreasing the p^{th} run by 1, and increasing the q^{th} run by 1. Each pair of strings X_i, X_{i+1} admits to the conditions of Lemma II.7 and thus $|D_t(X_i)| \leq |D_t(X_{i+1})|$. This process is finite, because the value of $\sum_{i=0}^r x_i^2$ is a non negative integer that must decrease at every step. An example of the balancing process we use is displayed in Table I. ■

We derive the following corollary for the case where n is not divisible by r .

Corollary III.1. *Let $X = S(x_1, \dots, x_r)$, $n = \sum_{i=1}^r x_i$, and $\bar{k} = n/r$. $D_t(X) \leq |D_t(B_{r, \lceil \bar{k} \rceil})|$.*

Proof: For integral \bar{k} this is the case of Theorem III.1. Otherwise, we denote $\alpha = r \lceil \bar{k} \rceil - n$, and let $Y = |D_t(x_1, \dots, x_{r-1}, x_r + \alpha)|$. Using Lemma II.4 $|D_t(X)| \leq |D_t(Y)|$, and since $|Y| = r \lceil \bar{k} \rceil$ and $r(Y) = r$, using Theorem III.1 $|D_t(Y)| \leq |D_t(B_{r, \lceil \bar{k} \rceil})|$. ■

IV. OUR UPPER BOUND

In this section we present an upper bound for the number of subsequences of a string obtained by deletions. We develop a recursive expression for the exact number of subsequences of a balanced string. We then find an explicit form for this expression, and use it to obtain a tight upper bound on the number of subsequences of a general string.

A. Recursive expression

Definition IV.1. *For all r, k , Let $B'_{r,k}$ be the string obtained from $B_{r,k}$ by removing the first symbol. E.g. $B'_{3,5} = S(4,5,5) = 0000111110000$.*

Definition IV.2. *Let $b(r, k, t) = |D_t(B_{r,k})|$ and $b'(r, k, t) = |D_t(B'_{r,k})|$.*

Lemma IV.1. *For all r, k, t , $|D_t(B_{r,k})| = |D_t(B'_{r,k})| + |D_{t-k}(B'_{r-1,k})|$*

Proof: This is derived from Lemma II.2. ■

When k is known from the context, we will use the short notations B_r and B'_r for $B_{r,k}$ and $B'_{r,k}$ respectively. Likewise $b(r, t)$ and $b'(r, t)$ denote $b(r, k, t)$ and $b'(r, k, t)$ respectively.

Lemma IV.2.[Recursive expression for b']

$$b'(r, t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t \geq kr \\ 1 + \sum_{i=1}^{k-1} b'(r-1, t-i) & \text{if } k(r-1) \leq t < kr \\ b'(r-2, t-k) + \sum_{i=0}^{k-1} b'(r-1, t-i) & \text{otherwise} \end{cases}$$

Proof: Using Lemma II.3 we get $b'(r, t) = b(r-1, t) + \sum_{i=1}^{k-1} b'(r-1, t-i) + 1_{|t > k(r-1)}$. We check the following cases:

- (i) $t < k(r-1)$: Using Lemma II.2, $b(r-1, t) = b'(r-1, t) + b'(r-2, t-k)$, and we get $b'(r, t) = b'(r-2, t-k) + \sum_{i=0}^{k-1} b'(r-1, t-i)$.
- (ii) $t = k(r-1)$: In this case $t = |B_{r-1}|$ and $b(r-1, t) = 1$. We get $b'(r, t) = 1 + \sum_{i=1}^{k-1} b'(r-1, t-i)$.
- (iii) $t > k(r-1)$: Here $t > |B_{r-1}|$ and $b(r-1, t) = 0$. We get $b'(r, t) = \sum_{i=1}^{k-1} b'(r-1, t-i) + 1$.

Rearranging the cases we get the claim of the lemma. ■

B. Solving the recursion

When calculating $b'(r, t)$ we expand the recursive expression iteratively, until all b' expressions reach their boundary condition, and get zero value. The only positive contribution in this sum is from the 1 in the second case ($1 + \sum_{i=1}^{k-1} b'(r-1, t-i)$). By counting how many times this value is added, we can get the explicit value of $b'(r, t)$. The 1 values are added exactly every time the second case is used, i.e. when expanding the value of $b'(\tilde{r}, \tilde{t})$ for \tilde{r}, \tilde{t} that fulfill the condition $k(\tilde{r}-1) \leq \tilde{t} < k\tilde{r}$. When expanding $b'(r, t)$ these are exactly the integral solutions for $\tilde{r} = \lfloor \frac{\tilde{t}}{k} \rfloor + 1, 0 \leq \tilde{t} \leq t$, which are simply the $t+1$ pairs $(r_i, t_i) = (\lfloor \frac{\tilde{t}}{k} \rfloor + 1, i)$ for $0 \leq i \leq t$. We will count the number of times that $b'(\tilde{r}, \tilde{t})$ appears in the complete expansion of $b'(r, t)$. Based on the recursion form in Lemma IV.2, the expression $b'(\tilde{r}, \tilde{t})$ can only appear in the single expansion of one of the following expressions: $b'(\tilde{r}+2, \tilde{t}+k)$, or $b'(\tilde{r}+1, \tilde{t}+i)$ when $0 \leq i \leq k-1$. Counting the number of those paths is equivalent to calculating the number of possible sets of ordered tuples $\{(r_j, t_j)\}$ selected from the set $\{(2, k), (1, 0), (1, 1), \dots, (1, k-1)\}$ s.t. $\sum r_j = r - \tilde{r}$ and $\sum t_j = t - \tilde{t}$.

Definition IV.3. We denote as S_k the set $\{(2, k), (1, 0), (1, 1), \dots, (1, k-1)\}$, and as $\#P(\Delta r, \Delta t)$ the number of possible sets of ordered tuples $\{(r_j, t_j)\}$ selected from the set S_k s.t. $\sum r_j = \Delta r$ and $\sum t_j = \Delta t$. $\#P_j(\Delta r, \Delta t)$ will denote the number of such sets using the tuple $(2, k)$ exactly j times.

Lemma IV.3.

$$\#P_0(\Delta r, \Delta t) = \sum_{i=0}^{\lfloor \frac{\Delta t}{k} \rfloor} (-1)^i \binom{\Delta r}{i} \binom{\Delta r}{\Delta t - ik}$$

Proof: In the case of $\#P_0$, the problem is reduced to finding the number of ordered partitions of t into r parts,

each of size no larger than $k-1$. The following development follows the technique used in [8] in the context of counting *deletion patterns*, similar results are calculated in [10]. This partitioning problem can be restated as counting the different solutions $\{y_i\}$ to the equations $\sum_{i=0}^r y_i = t, \forall i : y_i < k$. The number of solutions ignoring the constraints $y_i < k$ is equivalent to the number of r -partitions of t , which is $\binom{r}{t} = \binom{r+t-1}{t}$. The number of solutions that violate the constraint $y_1 < k$ is $\binom{r}{t-k}$. Subtracting this for each y_i , we get $\binom{r}{t} - r \binom{r}{t-k}$. Now we subtracted too much, because solutions that violate two constraints are subtracted twice. The number of solutions that violate the two constraints $y_1 < k$ and $y_2 < k$ is $\binom{r}{t-2k}$, and there are $\binom{r}{2}$ such pairs. Adding these cases back to the count we get $\binom{r}{t} - r \binom{r}{t-k} + \binom{r}{2} \binom{r}{t-2k}$. Now again we have to account for the solutions that violate 3 constraints, that were added too many times, and so on. Putting it all together we get $\#P_0(r, t) = \sum_{i=0}^{\lfloor \frac{t}{k} \rfloor} (-1)^i \binom{r}{i} \binom{r}{t-ik}$. ■

Lemma IV.4.

$$\#P(\Delta r, \Delta t) = \sum_{j=0}^{\lfloor \frac{\Delta t}{k} \rfloor} \binom{\Delta r - j}{j} \#P_0(\Delta r - 2j, \Delta t - jk)$$

Proof: First we calculate $\#P_j(\Delta r, \Delta t)$. If we first select j times the tuple $(2, k)$, we are left with $\#P_0(\Delta r - 2j, \Delta t - jk)$ ways to select the remaining tuples. We then have $\binom{\Delta r - j}{j}$ ways to insert the $(2, k)$ tuples inside the rest, and thus $\#P_j(\Delta r, \Delta t) = \binom{\Delta r - j}{j} \#P_0(\Delta r - 2j, \Delta t - jk)$. Summing on all possible j -s, $\#P(\Delta r, \Delta t) = \sum_{j=0}^{\lfloor \frac{\Delta t}{k} \rfloor} \#P_j(\Delta r, \Delta t)$ and the lemma's claim follows. ■

Lemma IV.5.

$$b'(r, t) = \sum_{i=0}^t \#P(r - \lfloor \frac{i}{k} \rfloor - 1, t - i)$$

Proof: As mentioned in the discussion above, when expanding $b'(r, t)$, exactly $t+1$ pairs (\tilde{r}, \tilde{t}) are reached that fulfill the conditions $k(\tilde{r}-1) \leq \tilde{t} < k\tilde{r}, 0 \leq \tilde{t} \leq t$ and thus contribute to the sum. These are exactly the $t+1$ pairs $(r_i, t_i) = (\lfloor \frac{i}{k} \rfloor + 1, i)$ for $0 \leq i \leq t$, and each one of them is reached $\#P(r - r_i, t - t_i)$ times. Summing all together we get $b'(r, t) = \sum_{i=0}^t \#P(r - \lfloor \frac{i}{k} \rfloor + 1, t - i)$ which is equal to the lemma's claim. ■

Corollary IV.1. The combined results of Lemmas IV.1, IV.5, IV.4 and IV.3 give an explicit expression for $|D_t(B_{r,k})|$. We restate the results here:

$$\begin{aligned} b(r, t) &= b'(r, t) + b'(r-1, t-k) \\ b'(r, t) &= \sum_{i=0}^t \#P(r - \lfloor \frac{i}{k} \rfloor - 1, t - i) \\ \#P(\Delta r, \Delta t) &= \sum_{j=0}^{\lfloor \frac{\Delta t}{k} \rfloor} \binom{\Delta r - j}{j} \#P_0(\Delta r - 2j, \Delta t - jk) \\ \#P_0(\Delta r, \Delta t) &= \sum_{i=0}^{\lfloor \frac{\Delta t}{k} \rfloor} (-1)^i \binom{\Delta r}{i} \binom{\Delta r}{\Delta t - ik} \end{aligned}$$

Using balanced strings we have achieved upper bounds for the number of subsequences of general strings. Our bound of Corollary IV.1 (in comparison to previous bounds) is depicted in Figure 3.

V. UNBALANCED STRINGS

In the section we define a second family of strings, named *unbalanced strings*. We call a string unbalanced, if all of the runs of symbols in the string are of length 1, except for one run. Let $U_{n,r}^{(i)}$ be a binary string of length n with r runs, in which all runs are of length 1, except for the i^{th} run which is of length $n - r + 1$. We notice that due to symmetry $|D_t(U_{n,r}^{(1)})| = |D_t(U_{n,r}^{(r)})|$, and define $u(n, r, t) = |D_t(U_{n,r}^{(1)})| = |D_t(U_{n,r}^{(r)})|$. We will show that these extreme cases have the least number of subsequences among the unbalanced strings, and conclude that they have the least amount of subsequences among all strings.

Theorem V.1. [Unbalanced strings have the least subsequences] For $X = S(x_1, \dots, x_r)$, $n = \sum_{i=1}^r x_i$, and any $1 \leq t \leq n$, $|D_t(X)| \geq u(n, r, t)$.

Proof: First we will prove that there exists j s.t. $|D_t(X)| \geq |D_t(U_{n,r}^{(j)})|$, for all t . We notice that the balancing operation of Lemma II.7 can be used in the other direction, as an unbalancing operation. We will transform the string X into a string $U_{n,r}^{(j)}$ by repeatedly applying the unbalancing operation. each such step can only decrease the number of subsequences, so by constructing a series of such operations, we will prove that $D_t(X) \geq |D_t(U_{n,r}^{(j)})|$. Let j be the index of a maximal run in X . We will construct a series of strings, X_0, \dots, X_m , such that $X_0 = X$, $X_m = U_{n,r}^{(j)}$ and for any $0 \leq i < m$, $|D_t(X_i)| \geq |D_t(X_{i+1})|$. For any $i < m$, we denote $X_i = S(x_1^{(i)}, \dots, x_r^{(i)})$. We choose an index $p \neq j$ s.t. $x_p^{(i)} > 1$ and all runs between the j^{th} run and the p^{th} run are all of length 1. Such an index exists, otherwise X_i is already an unbalanced string. We define X_{i+1} to be the string obtained from X_i by increasing the j^{th} run by 1, and decreasing the p^{th} run by 1. Since x_j was the maximal run in X and each operation only made it bigger while all other runs could only shorten, we have that $x_j^{(i)} \geq x_p^{(i)}$. The runs between the j^{th} run and the p^{th} run are all of length 1, and so trivially symmetric, and so the conditions of the reverse Lemma II.7 holds, and $|D_t(X_i)| \geq |D_t(X_{i+1})|$.

To complete the proof we will prove that for any j , $|D_t(U_{n,r}^{(j)})| \geq u(n, r, t)$. For $j = 1$, $u(n, r, t) = |D_t(U_{n,r}^{(1)})|$ by definition. For $j \geq 2$ we will prove by induction on j . For $j = 2$, $|D_t(U_{n,r}^{(2)})| = |D_t(1, n - r + 1, (n - 2) \times 1)|$. using Lemma II.2 we get $|D_t(U_{n,r}^{(2)})| = |D_t(n - r + 1, (n - 2) \times 1)| + |D_{t-1}(n - r, (n - 2) \times 1)|$. We compare this to $u(n, r, t) = |D_t((n - 1) \times 1, n - r + 1)| = |D_t((n - 2) \times 1, n - r + 1)| + |D_t((n - 3) \times 1, n - r + 1)|$. Using the flipping Lemma II.6 on the second addend and symmetry on both, we get $u(n, r, t) \leq |D_t(n - r + 1, (n - 2) \times 1)| + |D_t(n - r, (n - 2) \times 1)| = |D_t(U_{n,r}^{(2)})|$. For the induction step, we assume that the claim is true for $2, \dots, j - 1$ and prove it for j . for $j > 2$, using Lemma II.2 we get $|D_t(U_{n,r}^{(j)})| = |D_t(U_{n-1,r-1}^{(j-1)})| + |D_{t-1}(U_{n-2,r-2}^{(j-2)})|$. Using the induction assumption on both

TABLE II

EXAMPLE OF A BALANCING PROCESS AS DEFINED IN THE PROOF OF THEOREM V.1

i	X_i	runs	$D_5(X_i)$
0	0011100111100	2,3,2,4,2	60
1	0011101111100	2,3,1,5,2	38
2	0011101111110	2,3,1,6,1	26
3	0011011111110	2,2,1,7,1	20
4	0010111111110	2,1,1,8,1	14
5	0101111111110	1,1,1,9,1	10
	111111110101	9,1,1,1,1	8

addends, we get that $|D_t(U_{n,r}^{(j)})| \geq u(n - 1, r - 1, t) + u(n - 2, r - 2, t)$ and using Lemma II.2 again, the last sum is equal to $u(n, r, t)$ and thus the induction step is proved. An example of the unbalancing process is displayed in Table II. ■

VI. OUR LOWER BOUND

In this section we develop a recursive expression for the number of subsequences of an unbalanced string by deletions. We will find an explicit form for this expression, and use it to obtain a lower bound on the number of subsequences of a general string. In addition, we will show the improvement that our lower bound provides.

A. Recursive expression

Lemma VI.1. For all $0 < r \leq n$, $0 < t < n$,

$$u(n, r, t) = \begin{cases} r & \text{if } r = 1, 2 \\ 2 & \text{if } r > 1 \text{ and } t = n - 1 \\ d(n, t) & \text{if } n = r \\ u(n - 1, r, t) + d(r - 2, t + r - n - 1) & \text{otherwise} \end{cases}$$

Where $d(r, t) = |D_t(C_r)| = \sum_{i=0}^t \binom{r-t}{i}$, as proved in [2]. We assume $d(n, 0) = 1$, and for $t < 0$, $d(n, t) = 0$.

Proof:

- When $r = 1$, $U_{n,r}$ is a constant string, and has only one possible subsequence (the constant string of length $n - t$).
- When $r = 2$, $U_{n,r}$ is of the form $\sigma \epsilon^{n-1}$, and has two possible subsequences, namely $\sigma \epsilon^{n-1-t}$ and ϵ^{n-t} .
- When $t = n - 1$ and $r > 1$ any subsequence is a single symbol. Since $r > 1$, it can be either symbol of the binary alphabet.
- When $n = r$, $U_{n,r} = C_n$, the binary cyclic string of length n . $|D_t(C_n)| = d(n, t)$ by definition.
- In the other cases ($2 < r < n$, $t < n - 1$), we regard $U_{n,r}^{(1)}$ ("tail first"). We Apply Lemma II.2 and get $|D_t(U_{n,r}^{(1)})| = |D_t(U_{n-1,r}^{(1)})| + |D_{t+r-n-1}(C_{r-2})| = u(n - 1, r, t) + d(r - 2, t + r - n - 1)$. ■

B. Solving the recursion

Theorem VI.1. [Closed form formula for $u(n, r, t)$] For all $t < n$, $2 < r \leq n$,

(i) when $r > t$:

$$u(n, r, t) = d(r, t) + \sum_{i=t+r-n-1}^{t-2} d(r-2, i).$$

(ii) when $r \leq t$:

$$u(n, r, t) = 2 + \sum_{i=t+r-n-1}^{r-3} d(r-2, i).$$

Proof: We sequentially expand $u(n, r, t)$ using Lemma VI.1, until reaching one of the boundary conditions. After one such expansion we get $u(n, r, t) = u(n-1, r, t) + d(r-2, t+r-n-1)$, after j expansions (assuming the boundary conditions weren't reached) we get $u(n, r, t) = u(n-j, r, t) + \sum_{i=t+r-n-1}^{t-2} d(r-2, i)$. We notice that $i = t+r-n+j-2$ can be negative, and in these cases $d(r-2, i)$ is defined to be zero. When $r > t$, after $n-r$ steps we get $u(n, r, t) = u(r, r, t) + \sum_{i=t+r-n-1}^{t-2} d(r-2, i)$, and as $u(r, r, t) = d(r, t)$ we get (i) above. When $r \leq t$, after $n-t-1$ steps we get $u(n, r, t) = u(t+1, r, t) + \sum_{i=t+r-n-1}^{r-3} d(r-2, i) = 2 + \sum_{i=t+r-n-1}^{r-3} d(r-2, i)$ and we get (ii) above. ■

We notice that when the number of deletions is no greater than $n-r+1$ the expression of $u(n, r, t)$ does not depend on n , as stated in the following corollary:

Corollary VI.1. For $2 < r \leq n$ and $t \leq n-r+1$:

(i) when $r > t$:

$$u(n, r, t) = d(r, t) + \sum_{i=0}^{t-2} d(r-2, i).$$

(ii) when $r \leq t$:

$$u(n, r, t) = 2 + \sum_{i=0}^{r-3} d(r-2, i) = 1 + 2^{r-2}.$$

C. Improving known lower bounds on number of subsequences

The results of Theorem V.1 together with Theorem VI.1 lead to the following:

Theorem VI.2. [lower bound on the number of subsequences] For all $t < n$, $2 < r \leq n$ and any r -run string X

$$|D_t(X)| \geq d(r, t) + \sum_{i=t+r-n-1}^{\min(t-2, r-3)} d(r-2, i).$$

We compare this result to the previous result by Hirschberg et al. $|D_t(X)| \geq d(r, t) = \sum_{i=0}^t \binom{r-t}{i}$ [2]. We limit the comparison to $t \leq r$ as for $t > r$ the previous bound gives 0.

Lemma VI.2. Let $\alpha = t/r$. for $\alpha \in [\frac{1}{3} + \frac{1}{r}, 1)$ and for $t \leq n-r+1$, $\frac{u(n, r, t)}{d(n, t)} = \Omega\left(\frac{\sqrt{1-\alpha}}{r\alpha} 2^{r(\alpha-\frac{1}{3})}\right)$.

Proof: $d(r, t) \leq (t+1) \max_{i=0}^t \binom{r-t}{i}$. The series $\binom{r-t}{i}$ reaches its maximum at $i = \lfloor (r-t)/2 \rfloor$. This value is

reached, because $t > r/3$ implies that $t > (r-t)/2$. Thus $d(r, t) \leq (t+1) \binom{r-t}{\lfloor (r-t)/2 \rfloor}$. Stirling's approximation implies that $\binom{a}{\lfloor a/2 \rfloor} = \Theta\left(\frac{2^a}{\sqrt{a}}\right)$, and thus we get $d(r, t) = O\left(\frac{t}{\sqrt{r-t}} 2^{r-t}\right)$.

On the other hand as $t-2 \geq \lfloor \frac{r-2}{3} \rfloor$, $u(r, t) \geq d(r-2, \lfloor \frac{r-2}{3} \rfloor) \geq \binom{\lfloor \frac{2}{3}(r-3) \rfloor}{\lfloor \frac{1}{3}(r-3) \rfloor} = \Theta\left(\frac{2^{\frac{2}{3}(r-3)}}{\sqrt{\frac{2}{3}(r-3)}}\right) = \Theta\left(\frac{2^{\frac{2}{3}r}}{\sqrt{r}}\right)$, thus

$$u(r, t) = \Omega\left(\frac{2^{\frac{2}{3}r}}{\sqrt{r}}\right).$$

$$\frac{u(n, r, t)}{d(n, t)} = \Omega\left(\frac{2^{\frac{2}{3}r} \sqrt{r-t}}{2^{r-t} \sqrt{r}}\right), \text{ thus } \frac{u(n, r, t)}{d(n, t)} = \Omega\left(\frac{\sqrt{1-\alpha}}{r\alpha} 2^{r(\alpha-\frac{1}{3})}\right) \quad \blacksquare$$

For large enough strings ($n > t+r$), the improvement that the bound in Theorem VI.2 gives over the result in [2] depends on the ratio between r and t . We depict our improved results in Figure 2.

VII. DELETION PATTERNS

Consider a string X . Deletion of t letters from X can be characterized by partitioning t into the number of letters deleted from each run, leading to the following definition of deletion patterns.

Definition VII.1. Let X be a string s.t. $X = S(x_1, \dots, x_r)$. A deletion pattern of size t , is a set of integers $\{y_1, \dots, y_r\}$ fulfilling $\sum_{i=1}^r y_i = t$ and for all $0 \leq i \leq r$, $y_i \in [0, x_i]$. Each y_i represents the number of letters deleted from the i -th run of X . E.g. the deletion pattern $\{2, 1, 2\}$ for the string 000110000 results in the subsequence 0100. let $\mathcal{P}_t(X)$ denote the set of deletion patterns of size t for the string X .

It is important to notice that applying different deletion patterns on a string can result in the same subsequences, E.g. For the string 11011, the deletion patterns $\{1, 1, 0\}$ and $\{0, 1, 1\}$ both result in the subsequence 111. The following lemma ties deletion patterns with the study of subsequences (and appears partially in [8]).

Lemma VII.1. For any $X = S(x_1, \dots, x_r)$, let X' denote the string $S(x_1-1, \dots, x_r-1)$. Informally X' is the string obtained by deleting one letter from each run in X . It follows that $|\mathcal{P}_t(X')| \leq |D_t(X)| \leq |\mathcal{P}_t(X)|$.

Proof: Deleting letters from a given string according to a deletion pattern is a deterministic process, and so each deletion pattern yields exactly one subsequence, thus the right inequality follows. As mentioned before, several deletion patterns can yield the same subsequence, but this redundancy doesn't exist with deletion patterns that preserve the number of the runs in a string, i.e. there isn't a run in which all the symbols are deleted. In this case it is possible to reconstruct the deletion pattern from the subsequence in a unique way, and there is a one-one correspondence between the deletion patterns and the subsequences. The group of deletion patterns of X that preserve the number of runs is exactly the group of deletion patterns in which at least one symbol is not deleted from each run, and is equal to $\mathcal{P}_t(X')$. This group has a one-one correspondence to the subset of $D_t(X)$ of strings with exactly r runs, and thus the left equality holds. ■

Lemma VII.2. For any X , $|\mathcal{P}_t(X)| = |\mathcal{P}_{|X|-t}(X)|$.

Proof: Let $X = S(x_1, \dots, x_r)$ and let $\{y_1, \dots, y_r\}$ be a t -deletion pattern. It follows that $\sum_{i=1}^r y_i = t$. We define $y'_i = x_i - y_i$ for all $1 \leq i \leq r$. As $y_i \in [0, x_i]$ it follows that $y'_i \in [0, x_i]$ and $\sum_{i=1}^r y'_i = \sum_{i=1}^r (x_i - y_i) = |X| - t$, and so $\{y'_i\}$ is a $(|X| - t)$ -deletion pattern of X . Each t -deletion pattern can be mapped to a $(|X| - t)$ -deletion pattern, and this mapping is reversible, thus $|\mathcal{P}_t(X)| = |\mathcal{P}_{|X|-t}(X)|$. ■

A. The number of deletion patterns for balanced strings

We use the result obtained in Lemma IV.3 and restate it for deletion patterns to get the following result:

Lemma VII.3. $|P_t(B_{r,k})| = \sum_{i=0}^{\lfloor \frac{t}{k+1} \rfloor} (-1)^i \binom{r}{i} \binom{r}{t-i(k+1)} = \sum_{i=0}^{\lfloor \frac{t}{k+1} \rfloor} (-1)^i \binom{r}{i} \binom{r+t-i(k+1)-1}{r-1}$

We now study the multiplicative gap between $|P_t(B_{r,k})|$ and the previous bounds of [1], [2], [4] for values of t close to $n/2$ and sufficiently large r, k . This is an intriguing setting for t in the context of deletion channels [3]. It follows from basic observations (and also directly from the proof of Lemma VII.3) that

$$|P_t(B_{r,k})| \leq \min \left(\binom{r}{t}, (k+1)^r \right).$$

The first bound above is exactly that of [4], while the second bound follows from the fact that each y_i in a deletion pattern is an integer between 0 and k (notice that the former bound does not depend on the parameter k while the latter does not depend on t). In what follows, we show that the bound of $(k+1)^r$ improves on the bounds in [4] and [1], [2] for values of r and k which are sufficiently large.

For $t = n/2 = kr/2$, the bound of $\sum_{i=0}^t \binom{n-t}{i}$ from [2] is exactly $2^{n/2}$. The bound of $\binom{r+t-1}{t}$ from [4] is at least

$$\frac{1}{k} \binom{r(1+k/2)}{r} \geq \frac{1}{12k\sqrt{r}} \left(\frac{e}{1+2/k} \right)^r (1+k/2)^r$$

Here we use the fact that

$$\binom{r(1+\alpha)}{r} \geq \frac{1}{12\sqrt{r}} \left(\frac{(r(1+\alpha))^{r(1+\alpha)}}{r^r (\alpha r)^{\alpha r}} \right) = \frac{(1+\alpha)^r (1+1/\alpha)^{\alpha r}}{12\sqrt{r}}$$

derived from Stirling's formula; and the fact that for positive x , $(1+1/x)^{x+1} \geq e$. For $c = \frac{e(1+k/2)}{(k+1)(1+2/k)}$, the above implies that our bound of $(k+1)^r$ on $|P_t(B_{r,k})|$ is superior to that given in [4] (and that in [2]) by a multiplicative factor of at least

$$\frac{1}{12k\sqrt{r}} c^r.$$

Notice that for large k , $c > (1+\delta)$ for a constant $\delta > 0$. We conclude that a multiplicative gap of at least that specified above also holds between $|D_t(B_{r,k})|$ and the bounds in [1], [2], [4].

For sufficiently small $\epsilon > 0$ and $t = n(\frac{1}{2} - \epsilon)$, a similar analysis will give a gap of $\simeq c^r$ for $c = \frac{e(1+k/2-\epsilon k)}{(k+1)(1+1/(k/2-\epsilon k))}$. Here also, for small ϵ and large k ; $c > (1+\delta)$ for a constant

$\delta > 0$. All in all, we get for values t which are close to $n/2$ and for sufficiently large r and k ; that $|P_t(B_{r,k})|$, and thus our bound of $|D_t(B_{r,k})|$, improves on the bounds of [1], [2], [4] by an exponential multiplicative factor of $2^{\Omega(r)}$.

VIII. CONCLUDING REMARKS

In this work we present several operations on binary strings which are monotone with respect to the number of subsequences under deletion. We show, using the operations studied, that the balanced r -run string $B_{r,k}$ and the unbalanced one $U_{n,r}$ obtain the maximum and respectively minimum number of subsequences under deletion. By devising recursive expressions, we present a precise analysis of the number of subsequences of both $B_{r,k}$ and $U_{n,r}$ under t deletions. For our lower bound, we quantify our expressions asymptotically. For our upper bound, we analyze deletion patterns to express our asymptotic improvement over previous bounds. A direct asymptotic analysis of our expression for $|D_t(B_{r,k})|$ is left open in this work and is subject to future research.

REFERENCES

- [1] L. Calabi and W.E. Hartnett. Some general results of coding theory with applications to the study of codes for the correction of synchronization errors. *Information and Control*, 15(3):235 – 249, 1969.
- [2] D. S. Hirschberg and M. Regnier. Tight bounds on the number of string subsequences. *Journal of Discrete Algorithms*, 1(1):123–132, 2000.
- [3] I. A. Kash, M. Mitzenmacher, J. Thaler, and J. Ullman. On the zero-error capacity threshold for deletion channels. *CoRR*, abs/1102.0040, 2011.
- [4] V. I. Levenshtein. Binary codes capable of correcting deletions, insertions, and reversals. *Soviet Physics Doklady*, 10(8):707–710, 1966.
- [5] V. I. Levenshtein. Efficient reconstruction of sequences from their subsequences or supersequences. *Journal of Combinatorial Theory, Series A*, 93(2):310–332, 2001.
- [6] H. Mercier. *Communication over Channels with Symbol Synchronization Errors*. PhD thesis, The University of British Columbia, 2008.
- [7] H. Mercier, V.K. Bhargava, and V. Tarokh. A survey of error-correcting codes for channels with symbol synchronization errors. *IEEE Communications Surveys and Tutorials*, 12(1):87–96, 2010.
- [8] H. Mercier, M. Khabbazi, and V. K. Bhargava. On the number of subsequences when deleting symbols from a string. *IEEE Transactions on Information Theory*, 54(7):3279–3285, 2008.
- [9] M. Mitzenmacher. A survey of results for deletion channels and related synchronization channels. *Probability Surveys*, 6:1–33, 2009.
- [10] J. Ratsaby. Estimate of the number of restricted integer-partitions. *Appl. Anal. Discrete Math.*, 2:222–233, 2008.