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Symmetric Heteroclinic Connections in the Michelson System: A Computer Assisted Proof*

Daniel Wilczak†

Abstract. In this paper we present a new technique of proving the existence of an infinite number of symmetric heteroclinic and homoclinic solutions. This technique combines the covering relations method introduced by Zgliczyński [*Topol. Methods Nonlinear Anal.*, 8 (1996), pp. 169–177; *Nonlinearity*, 10 (1997), pp. 243–252] with symmetry properties of a dynamical system. As an example we present a computer assisted proof of the existence of an infinite number of heteroclinic connections between equilibrium points in the Kuramoto–Sivashinsky ODE [D. Michelson, *Phys. D*, 19 (1986), pp. 89–111]. Moreover, we present the proof of the existence of an infinite number of heteroclinic connections between periodic orbits and equilibrium points.

Key words. differential equations, symmetric solutions, rigorous numerical analysis

AMS subject classifications. 34C37, 37C80, 37N30

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1. Introduction.

The aim of this paper is to present a new method for proving the existence of symmetric homoclinic or heteroclinic solutions in systems possessing the reversing symmetry property.

In [5] (and references given there) a method of proving the existence of time reversing symmetric homoclinic and heteroclinic solutions for dynamical systems is presented and it is called the *fixed set iteration* method. It applies to dynamical systems with continuous and discrete time. The basic idea of such a method is to search for the points u which are invariant under the symmetry and whose trajectories converge to an equilibrium point or a periodic orbit. This allows us to conclude that the trajectories of the points u must be homoclinic or heteroclinic.

Galias and Zgliczyński [3] presented the method for proving the existence of homoclinic and heteroclinic solutions for maps $\mathbb{R}^2 \to \mathbb{R}^2$. This result was applied to the planar circular restricted three body problem [1, 16], where the existence of an infinite number of homoclinic and heteroclinic connections between periodic orbits was shown.

In this paper we demonstrate how to combine these two methods for proving the existence of symmetric homoclinic or heteroclinic orbits in systems possessing the reversing symmetry property. Moreover, we present some generalization of the Galias–Zgliczyński method. We show how to prove the existence of heteroclinic orbits between objects possessing unequal dimensions—for example, the equilibrium points and periodic orbits.

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†Department of Numerical Methods, Institute of Computer Science, Jagiellonian University, Nawojki 11, 30 – 072 Kraków, Poland (wilczak@ii.uj.edu.pl).

As an application of our method we present a computer assisted proof of the existence of an infinite number of symmetric heteroclinic connections between equilibrium points in the Michelson system [10]

$$y''' + y' + \frac{1}{2}y^2 = c^2 \tag{1}$$

arising from the Kuramoto–Sivashinsky PDE as a traveling-wave solution. We rewrite (1) as the system

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& z, \\ \dot{z} &=& c^2 - y - \frac{1}{2}x^2. \end{cases} \tag{2}$$

The system (2) possesses two equilibrium points: $x_{c,-} = (-c\sqrt{2}, 0, 0)$ and $x_{c,+} = (c\sqrt{2}, 0, 0)$. McCord [9] showed that for every sufficiently large c there exists a unique nonstationary bounded solution—a heteroclinic orbit connecting equilibrium points. Mrozek and Żelawski [11] considered the related system with a parameter λ, i.e.,

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& z, \\ \dot{z} &=& 1 - \lambda y - \frac{1}{2}x^2. \end{cases} \tag{3}$$

They proved that for all the parameter values $\lambda \in [0, 1]$ the system (3) possesses a heteroclinic orbit connecting $(\sqrt{2}, 0, 0)$ with $(-\sqrt{2}, 0, 0)$. Observe that if $x(t)$ is a solution of (3) with $\lambda > 0$, then $\tilde{x}(t) := \lambda^{-3/2}x(\lambda^{-1/2}t)$ is a solution of (2) with $c = \lambda^{-3/2}$. Hence, this result implies that for every $c \geq 1$ there exists a heteroclinic solution of (2) connecting the equilibrium points $x_{c,+}$ and $x_{c,-}$.

Troy [12] proved that for the parameter value $c = 1$ there exist two symmetric (with respect to the y-axis) heteroclinic connections between equilibrium points. He conjectured the existence of more heteroclinic solutions.

Lau [7] described the structure of the heteroclinic bifurcations of (2). In particular, his work shows the possibility of the existence of an infinite number of symmetric heteroclinic solutions. The previously mentioned method gives the positive answer to the conjectures given by Troy and Lau. The examples of such orbits are presented in Figure 1 (see also [12, Figures 6, 8, 10]).

Troy [12] proved the existence of two odd periodic solutions of (1). These periodic orbits are presented in Figure 6 (see also [12, Figures 3, 4]). The main novelty about the Michelson system is the proof of the existence of infinitely many heteroclinic connections between periodic orbits established in [12] and the equilibrium points (see Figures 2 and 3).

In [13] the dynamics of reversible and conservative systems close to a homoclinic orbit is studied. In particular, it is shown [13, Thm. 5] that if the system is of an even dimension and it is R-reversible, where R is a linear involution, then the period blow-up appears close to the elementary symmetric homoclinic orbit and results in the existence of periodic orbits with all possible periods greater than some $\omega_0 > 0$.

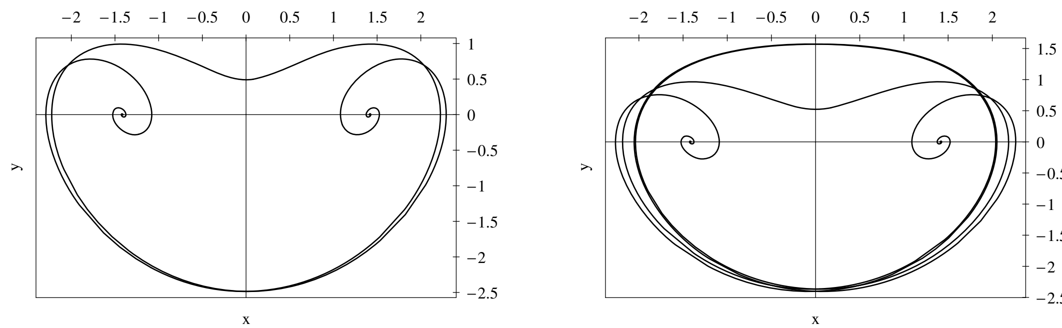

Figure 1. *Two examples of heteroclinic connection between equilibrium points (left) built on the sequence of symbols* $(2, 3, 4)$*, the initial condition* $u \approx (0, 0.4905645239087398, 0)$*, (right) built on the sequence of symbols* $(2, 3, 4, 1, 4)$*, the initial condition* $u \approx (0, 0.5229511376516929, 0)$.

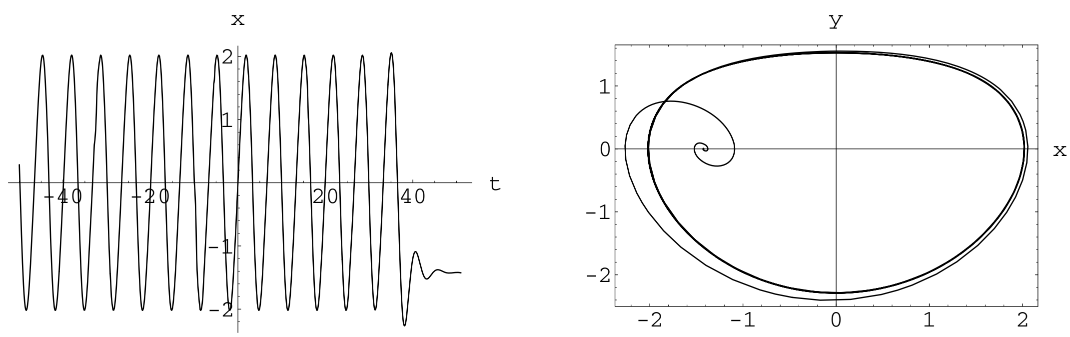

Figure 2. *An example of heteroclinic connection between the periodic orbit* S_1 *and equilibrium point* $(-\sqrt{2}, 0, 0)$.

The method presented in this paper concerns odd dimensional flows. However, it is possible to apply it in the case when the dimension of the dynamical system is even—for example, in hamiltonian systems [17]. After fixing the energy level we obtain the flow on the (usually) odd dimensional manifold where the periodic, homo-, and heteroclinic orbits can be isolated. If we can apply the method to a hamiltonian with a fixed regular energy level, then similar dynamics occurs in some range of energy values.

The interesting results about the dynamics close to the Hopf-zero bifurcation in reversible vector fields in $\mathbb{R}^3$ are presented in [6]. The application of the general result to the Michelson system leads to [6, Thm. 1.4] when the existence of extremely complicated dynamics including heteroclinic and homoclinic solutions and chaotic dynamics for the parameter values c close to zero is presented.

Another interesting question is the existence of a heteroclinic loop between equilibrium points $x_{c,\pm}$. In [4] the explicit formula for the one-dimensional heteroclinic solution connecting $x_{c,-}$ with $x_{c,+}$ was found,

$$x(t) = \alpha(-9 \tanh(\beta t) + 11 \tanh^3(\beta t)), \tag{4}$$

where $\alpha = 15\sqrt{11/19^3}$, $\beta = \frac{1}{2}\sqrt{11/19}$, and $c = c_{KT} = \alpha\sqrt{2} \approx .84952$. The method presented

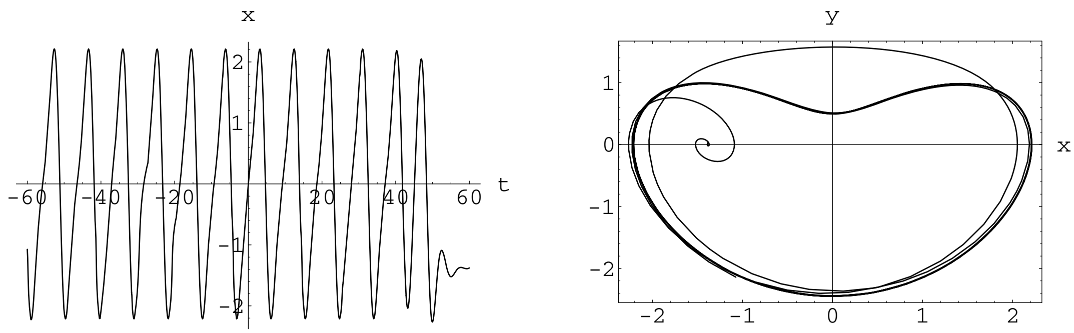

Figure 3. *An example of heteroclinic connection between the periodic orbit S_2 and equilibrium point* $(-\sqrt{2}, 0, 0)$.

in this paper may give a positive answer that for parameter value c_{KT} there exists a heteroclinic loop between equilibria.

The paper is organized as follows. In section 4 we recall the covering relations method and we prove the new topological results. In section 5 we recall and generalize the Galias–Zgliczyński method. In section 6 we give the precise statement of the main results and their proofs. In section 7 we give details of the numerical proof.

2. Basic notation and definitions.

Basic notation and definitions introduced in this subsection are used throughout the paper. By $\mathrm{int}(W)$ we denote the interior of a set W. Let $f : X \multimap Y$ be a partial function. By $\mathrm{dom}(f)$ we denote the domain of f. By $\mathrm{Fix}(f)$ we denote the set of fixed points of f

$$\mathrm{Fix}(f) := \{x \in \mathrm{dom}(f) \mid f(x) = x\}.$$

Definition 2.1. *Let I_{x_0} denote the maximal interval of the existence of the solution of*

$$\dot{x} = f(x), \; x(0) = x_0.$$

The set $\mathcal{O}(x_0) := \{x(t) \mid t \in I_{x_0}\}$ is called the trajectory of the point x_0.

Definition 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function. A diffeomorphism $R : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called a reversing symmetry of the ODE*

$$\dot{x} = f(x) \tag{5}$$

if

$$f \circ R = -R \circ f.$$

The system (5) is called reversible if it has at least one reversing symmetry.

The existence of reversing symmetry in a given ODE has important consequences for the induced dynamical system. In particular, the R-image of a trajectory is also the trajectory of the system.

Definition 2.3. *A trajectory of a point $x \in \mathbb{R}^n$ of an R-reversible system $\dot{x} = f(x)$ is called R-symmetric if it is invariant under symmetry, i.e., $R(\mathcal{O}(x)) = \mathcal{O}(x)$.*

It is a well-known result [5, 13] that for the trajectory $\mathcal{O}(x)$ to be R-symmetric it is necessary and sufficient that $\mathcal{O}(x) \cap \mathrm{Fix}(R) \neq \emptyset$.

3. The main results. As was mentioned in the introduction the Michelson system (2) arises from the Kuramoto–Sivashinsky equation

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla u|^2 = 0$$

as a traveling-wave solution. Observe that (2) possesses the reversing symmetry

$$R(x, y, z) = (-x, y, -z), \tag{6}$$

which means that if $t \to (x(t), y(t), z(t))$ is a solution of (2), then

$$t \to R(x(-t), y(-t), z(-t)) = (-x(-t), y(-t), -z(-t))$$

is a solution, too.

Following Troy [12] we will study (1)–(2) with the parameter value $c = 1$. The system (2) possesses two equilibrium points: $x_- := (-\sqrt{2}, 0, 0)$ and $x_+ := (\sqrt{2}, 0, 0)$. These equilibria are not contained in $\mathrm{Fix}(R)$, and therefore they are R-images of themselves; i.e., $R(x_\pm) = x_\mp$.

Troy [12] gave the proof of the existence of two heteroclinic connections from x_+ to x_-. He conjectured the existence of an infinite number of heteroclinic connections between equilibrium points $x_\pm$. Here we present the answer to Troy's question.

Theorem 3.1. *There exists an infinite number of R-symmetric heteroclinic solutions of the Michelson system (2) connecting x_+ with x_-.*

Two symmetric heteroclinic solutions connecting equilibrium points $x_\pm$ are shown in Figure 1. The symmetric heteroclinic solutions are obtained in the following way. We will show that there exists an infinite number of points on the y-axis whose trajectory makes an arbitrarily large but finite number of loops close to the two symmetric periodic solutions S_1, S_2 found by Troy [12]. Finally, the trajectory is asymptotic to the equilibrium x_-. The symmetry argument shows that trajectories of these points must be heteroclinic connections between x_+ and x_-.

Theorem 3.2. *Let S_1, S_2 denote the two R-symmetric periodic solutions of (2) found by Troy [12].*

1. *There exists an infinite number of solutions of (2), such that the ω-limit set is the equilibrium point $(-\sqrt{2}, 0, 0)$ and the α-limit set is either S_1 or S_2.*
2. *There exists an infinite number of solutions of (2), such that the α-limit set is the equilibrium point $(\sqrt{2}, 0, 0)$ and the ω-limit set is either S_1 or S_2.*

Two heteroclinic connections between odd periodic orbits S_1, S_2 and the equilibrium point $(-\sqrt{2}, 0, 0)$ are shown in Figures 2 and 3.

These solutions are obtained in the following way. We will show that there are points whose forward trajectories make an arbitrarily large but finite number of loops close to the S_1 and/or S_2 periodic orbits and next are asymptotic to the equilibrium x_-. On the other hand, the whole backward trajectory is close to the fixed periodic orbit S_i. Finally, we will show that since periodic orbits S_1 and S_2 are hyperbolic in the sense of Definition 5.1, these trajectories must be backward asymptotic to the periodic solution S_i.

The assertion (2) follows from assertion (1) and the R-reversibility of the Michelson system. If $\mathcal{O}(x)$ is a heteroclinic connection between S_i and x_-, then $R(\mathcal{O}(x))$ is a heteroclinic connection between $x_+ = R(x_-)$ and $S_i = R(S_i)$.

In fact, these heteroclinic chains $\mathcal{O}(x) - R(\mathcal{O}(x))$ connecting $x_+ \to S_i \to x_-$ are the limit of symmetric heteroclinic trajectories between x_+ and x_- for which the number of loops close to the periodic orbit S_i increases to infinity.

A more precise statement of the above theorems and their connections with the symbolic dynamics proved by the author in [15] will be given in section 6 (Theorems 6.5 and 6.15). We also present there the proofs of Theorems 3.1 and 3.2.

4. Topological results.

The covering relations method was introduced by Zgliczyński [19, 20]. Here we recall the basic definitions.

4.1. t-sets.

Definition 4.1. *A t-set is a triple $N = (|N|, N^l, N^r)$ of closed subsets of $\mathbb{R}^n$ satisfying the following properties:*

1. *$|N|$ is a parallelepiped; N^l and N^r are two half-spaces.*
2. *$N^l \cap N^r = \emptyset$.*
3. *The sets $N^{lw} := N^l \cap |N|$ and $N^{rw} := N^r \cap |N|$ are two parallel walls of $|N|$.*

We call $|N|$, N^l, N^r, N^{lw}, and N^{rw} the support, the left side, the right side, the left wall, and the right wall of the t-set N, respectively.

Definition 4.1 is a generalization of [1, Def. 1]. Special cases of t-sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ are presented in Figure 4.

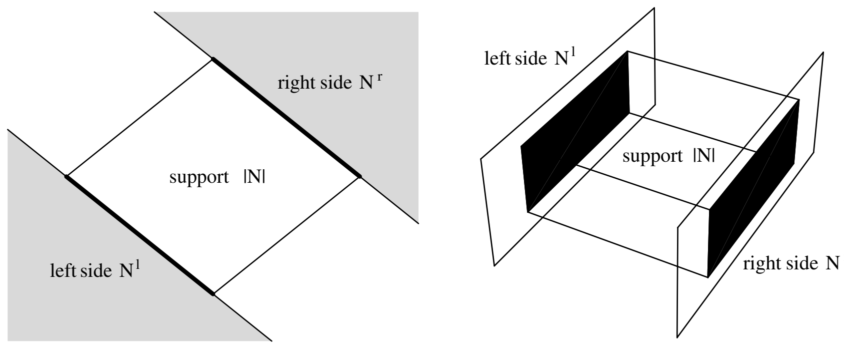

Figure 4. *An example of a t-set (left) on the plane $\mathbb{R}^2$ and (right) in $\mathbb{R}^3$.*

4.2. Representation of t-sets.

A t-set in $\mathbb{R}^n$ may be defined by specifying

$$N = t(c, u, s_1, \ldots, s_{n-1}),$$

where $c, u, s_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n-1$, are such that $u, s_1, \ldots, s_{n-1}$ are linearly independent. We set

$$\begin{aligned}
|N| &= \{x \in \mathbb{R}^n \mid \exists_{t_1,t_2,\ldots,t_n \in [-1,1]} \quad x = c + t_1 s_1 + \cdots + t_{n-1}s_{n-1} + t_n u\} \\
&= c + [-1,1] \cdot u + [-1,1] \cdot s_1 + \cdots + [-1,1] \cdot s_{n-1}, \\
N^l &= c + (-\infty, -1] \cdot u + (-\infty, \infty) \cdot s_1 + \cdots + (-\infty, \infty) \cdot s_{n-1}, \\
N^r &= c + [1, \infty) \cdot u + (-\infty, \infty) \cdot s_1 + \cdots + (-\infty, \infty) \cdot s_{n-1}.
\end{aligned}$$

In this representation c is the center point of the parallelepiped $|N|$, u is called the unstable direction, and s_i are called stable directions.

Using this representation, we have

$$N^{lw} = c - u + [-1,1] \cdot s_1 + \cdots + [-1,1] \cdot s_{n-1},$$
$$N^{rw} = c + u + [-1,1] \cdot s_1 + \cdots + [-1,1] \cdot s_{n-1}.$$

4.3. Covering relations. Let N, M be t-sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $f : \mathbb{R}^n \not\to \mathbb{R}^m$ be a continuous function such that $|N| \subset \operatorname{dom}(f)$. Below we present the generalization of [1, Def. 2] to the case of unequal dimensions of N and M.

Definition 4.2. *We say that N f-covers M if the following conditions hold:*

(a) $f(|N|) \subset \operatorname{int}(M^l \cup |M| \cup M^r)$.

(b) *Either* $f(N^{lw}) \subset \operatorname{int}(M^l)$ *and* $f(N^{rw}) \subset \operatorname{int}(M^r)$ *or* $f(N^{lw}) \subset \operatorname{int}(M^r)$ *and* $f(N^{rw}) \subset \operatorname{int}(M^l)$.

We then write $N \overset{f}{\Longrightarrow} M$.

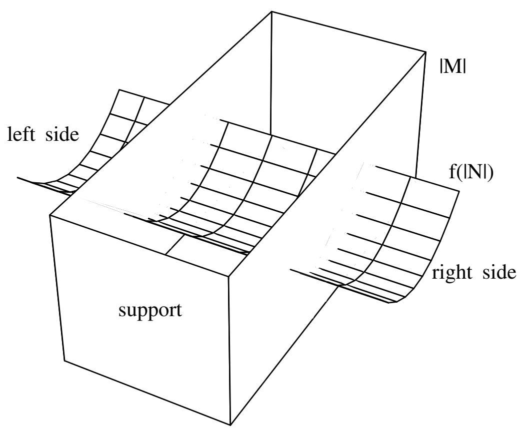

Figure 5. *An example of covering relations $N \overset{f}{\Longrightarrow} M$, where N is a t-set in $\mathbb{R}^2$ and M is a t-set in $\mathbb{R}^3$.*

The geometry of this concept is presented in Figure 5. We recall and generalize some results from [15].

Definition 4.3. *Let N be a t-set in $\mathbb{R}^n$ and let $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ be a continuous curve. We say that γ is a horizontal curve in N if the following conditions hold:*

1. $\gamma((a,b)) \subset \operatorname{int}(|N|)$.
2. *Either* $\gamma(a) \in N^{lw}$ *and* $\gamma(b) \in N^{rw}$ *or* $\gamma(a) \in N^{rw}$ *and* $\gamma(b) \in N^{lw}$.

From the geometrical point of view γ is a continuous curve connecting N^{rw} and N^{lw} inside the support of N.

Theorem 4.4. *Let $M_0, M_1, \ldots, M_n$ be t-sets in $\mathbb{R}^{d_0}, \ldots, \mathbb{R}^{d_n}$, respectively, for some $d_0, \ldots, d_n \in \mathbb{N}$. Assume $f_i : M_i \longrightarrow \mathbb{R}^{d_{i+1}}$, $i = 0, \ldots, n-1$, are continuous functions such that*

$$M_0 \overset{f_0}{\Longrightarrow} M_1 \overset{f_1}{\Longrightarrow} M_2 \cdots \overset{f_{n-2}}{\Longrightarrow} M_{n-1} \overset{f_{n-1}}{\Longrightarrow} M_n.$$

If $\gamma : [a,b] \longrightarrow \mathbb{R}^{d_0}$ is a horizontal curve in M_0, then there are real numbers t_*, t^* such that the composition $(f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0) \circ (\gamma|_{[t_*,t^*]})$ is well defined. Moreover,

$$(f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0 \circ \gamma)([t_*,t^*]) \subset \operatorname{int}(|M_k|) \text{ for } k = 1,\ldots,n-1,$$
$$(f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0 \circ \gamma)|_{[t_*,t^*]} \text{ is a horizontal curve in } M_n.$$

Proof. The proof is presented in [15, Thm. 4.8] under the assumption that each of the t-sets is two-dimensional. We omit the proof, since exactly the same arguments justify the general case. ■

Theorem 4.5. *Let $(N_j)_{j\geq 0}$ be a sequence of t-sets in $\mathbb{R}^{d_j}$, respectively; $f_j : |N_j| \longrightarrow \mathbb{R}^{d_{j+1}}$, $j \geq 0$, are continuous such that*

$$N_j \overset{f_j}{\Longrightarrow} N_{j+1} \quad \text{for} \quad j \geq 0.$$

If $\gamma : [a,b] \longrightarrow \mathbb{R}^{d_0}$ is a horizontal curve in N_0, then there exists $\tau \in [a,b]$ such that

$$(f_j \circ \cdots \circ f_0)(\gamma(\tau)) \in |N_{j+1}|$$

for all $j \geq 0$.

Proof. From Theorem 4.4, for every $j = 1,2,\ldots$, there exists $t_j \in [a,b]$ such that

$$(f_k \circ \cdots \circ f_0)(\gamma(t_j)) \in |N_{i_{k+1}}| \quad \text{for} \quad k = 0,\ldots,j.$$

Since $[a,b]$ is compact, we can find a condensation point $\tau \in [a,b]$ of the set $\{t_j\}_{j>0}$. Since $|N_j|$, $j \geq 0$, are compact sets and f_j, $j \geq 0$, are continuous, we obtain that

$$(f_j \circ \cdots \circ f_0)(\gamma(\tau)) \in |N_{j+1}|$$

for all $j \geq 0$. ■

Remark 4.6. *Let $N_0,\ldots,N_j$ be t-sets in $\mathbb{R}^{d_0},\ldots,\mathbb{R}^{d_j}$, respectively. Assume $f_0,\ldots,f_{j-1},g$ are continuous such that*

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} \cdots \overset{f_{j-1}}{\Longrightarrow} N_j \overset{g}{\Longrightarrow} N_j.$$

In this special case Theorem 4.5 implies the existence of a point $u \in N_0$ such that

$$(f_k \circ \cdots \circ f_0)(u) \in |N_{k+1}|$$

for $k = 0,1,\ldots,j-1$ and

$$(g^p \circ f_{j-1} \circ \cdots \circ f_0)(u) \in |N_j|$$

for $p \geq 0$.

Since $N_j \overset{g}{\Longrightarrow} N_j$, then from [16, Thm. 3.6] there exists a fixed point x_* of g in $|N_j|$. Hence, we can ask if the fixed point x_* is unique and if the trajectory of u converges to x_*. The answer is presented in the next section.

5. Homoclinic and heteroclinic connections: $\mathcal{C}^1$-tools. The goal of this section is to describe tools for proving the existence of homo- or heteroclinic orbits for maps. We will generalize the method introduced by Galias and Zgliczyński [3] and originating from the results by Wójcik and Zgliczyński [18]. We recall some results from [3, 16].

5.1. General theorems. Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$-map. For any set X we say that an interval matrix $\boldsymbol{D}P(X) \subset \mathbb{R}^{n\times n}$ is an interval enclosure of the derivative $DP(X) := \{DP(x) \mid x \in X\}$ if

$$M \in \boldsymbol{D}P(X) \quad \Longleftrightarrow \quad \inf_{x\in X}(DP(x))_{ij} \le M_{ij} \le \sup_{x\in X}(DP(x))_{ij}, \quad i,j = 1,2,\dots,n.$$

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x) = (f_1(x), f_2(x))^T$, where $x = (x_1, x_2)^T$. We assume that $f(0) = 0$, i.e., that 0 is a fixed point of f. For a convex set U such that $0 \in U$, we define intervals $\boldsymbol{\lambda}_1(U)$, $\boldsymbol{\varepsilon}_1(U)$, $\boldsymbol{\varepsilon}_2(U)$, and $\boldsymbol{\lambda}_2(U)$ by

$$\boldsymbol{D}f(U) = \begin{pmatrix} \boldsymbol{\lambda}_1(U) & \boldsymbol{\varepsilon}_1(U) \\ \boldsymbol{\varepsilon}_2(U) & \boldsymbol{\lambda}_2(U) \end{pmatrix}. \tag{7}$$

Let

$$\begin{aligned} \varepsilon_1'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_1(U)\}, \quad & \varepsilon_2'(U) &= \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_2(U)\}, \\ \lambda_1'(U) &= \inf\{|\lambda_1| : \lambda_1 \in \boldsymbol{\lambda}_1(U)\}, \quad & \lambda_2'(U) &= \sup\{|\lambda_2| : \lambda_2 \in \boldsymbol{\lambda}_2(U)\}. \end{aligned}$$

Let us define the rectangle $N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2]$.

Definition 5.1 (see [3, Def. 1]). *Let x_* be a fixed point for the map f. We say that f is hyperbolic on $N \ni x_*$ if there exists a local coordinate system on N such that in this coordinate system*

$$\begin{aligned} x_* &= 0, \\ \varepsilon_1'(N)\varepsilon_2'(N) &< (1 - \lambda_2'(N))(\lambda_1'(N) - 1), \\ N &= N_{\alpha_1,\alpha_2}, \end{aligned}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are such that the following conditions are satisfied:

$$\frac{\varepsilon_1'(N)}{\lambda_1'(N) - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'(N)}{\epsilon_2'(N)}.$$

It is easy to see that for the map f to be hyperbolic on N it is necessary that $\lambda_1' > 1$, $\lambda_2' < 1$, and the linearization of f at x_* is hyperbolic with one stable and one unstable direction. Observe that for f to be hyperbolic on N it is not necessary that f is a diffeomorphism on $|N|$. Hence, the Jacobi matrix of f may have a determinant equal to zero at some $x \in |N|$.

Theorem 5.2 (see [3, Thm. 3]). *Assume that f is hyperbolic on N.*

1. *If $f^k(x) \in N$ for $k \ge 0$, then $\lim_{k\to\infty} f^k(x) = x_*$.*
2. *If $y_k \in N$ and $f(y_{k-1}) = y_k$ for $k \le 0$, then $\lim_{k\to-\infty} y_k = x_*$.*

Remark 5.3. *It is possible to extend Definition 5.1 and Theorem 5.2 to the case when the set N has dimension greater than two. In that case we write the matrix $\boldsymbol{D}P$ as a block matrix 2×2, where two diagonal blocks correspond to the unstable and stable parts. Next we compute $\varepsilon'_i(N),\lambda'_i(N)$, $i=1,2$, as the maximum norm of the corresponding block. We will not formulate and prove this result since we will not use it in this paper. The required convergence on some three-dimensional t-set will be proved by use of a certain energy function (see Lemma 6.9).*

Theorem 5.2 may be used to prove the existence of homoclinic and heteroclinic orbits. The following theorem is a generalization of [3, Thm. 4].

Theorem 5.4. *Let $M_0,\dots,M_n$ be t-sets in $\mathbb{R}^{d_j}$, respectively, $j=0,\dots,n$. Assume H_1,H_2 are t-sets in $\mathbb{R}^2$, $g_i:|H_i|\to\mathbb{R}^2$ are injective and hyperbolic on H_i, and $H_i \overset{g_i}{\Longrightarrow} H_i$ for $i=1,2$. Let x_i be a unique fixed point of g_i in $|H_i|$ for $i=1,2$.*

1. *If $f_0,\dots,f_n$ are such that*

$$H_1 \overset{f_0}{\Longrightarrow} M_0 \overset{f_1}{\Longrightarrow} M_1 \overset{f_2}{\Longrightarrow} \cdots \overset{f_n}{\Longrightarrow} M_n,$$

then there exists a point $x_0\in|H_1|$ satisfying
   - $g_1^{-k}(x_0)\in|H_1|$ *for all* $k\geq 0$ *and* $\lim_{k\to\infty} g_1^{-k}(x_0)=x_1$,
   - $(f_j\circ\cdots\circ f_0)(x_0)\in|M_j|$ *for* $j=0,\dots,n$.
2. *If $f_0,\dots,f_n$ are such that*

$$M_0 \overset{f_0}{\Longrightarrow} M_1 \overset{f_1}{\Longrightarrow} \cdots \overset{f_{n-1}}{\Longrightarrow} M_n \overset{f_n}{\Longrightarrow} H_2,$$

then there exists a point $x_0\in|M_0|$ satisfying
   - $(f_j\circ\cdots f_0)(x_0)\in|M_{j+1}|$ *for* $j=0,\dots,n-1$,
   - $(g_2^k\circ f_n\circ\cdots\circ f_0)(x_0)\in|H_2|$ *for* $k\geq 0$ *and* $\lim_{k\to\infty}(g_2^k\circ f_n\circ\cdots\circ f_0)(x_0)=x_2$.
3. *If $f_0,\dots,f_n,f_{n+1}$ are such that*

$$H_1 \overset{f_0}{\Longrightarrow} M_0 \overset{f_1}{\Longrightarrow} M_1 \overset{f_2}{\Longrightarrow} \cdots \overset{f_n}{\Longrightarrow} M_n \overset{f_{n+1}}{\Longrightarrow} H_2,$$

then there exists a point $x_0\in|H_1|$ satisfying
   - $g_1^{-k}(x_0)\in|H_1|$ *for all* $k\geq 0$ *and* $\lim_{k\to\infty} g_1^{-k}(x_0)=x_1$,
   - $(f_j\circ\cdots\circ f_0)(x_0)\in|M_j|$ *for* $j=0,\dots,n$ *and* $(f_{n+1}\circ\cdots\circ f_0)(x_0)\in|H_2|$,
   - $(g_2^k\circ f_{n+1}\circ\cdots\circ f_0)(x_0)\in|H_2|$ *for* $k\geq 0$ *and* $\lim_{k\to\infty}(g_2^k\circ f_{n+1}\circ\cdots\circ f_0)(x_0)=x_2$.

Proof. We present the proof of the first assertion. The proofs of the second and third assertions are similar. Let us fix $k\geq 0$. We have the following chain of covering relations:

$$\underbrace{H_1 \overset{g_1}{\Longrightarrow} H_1 \overset{g_1}{\Longrightarrow} \cdots \overset{g_1}{\Longrightarrow} H_1}_{k} \overset{f_0}{\Longrightarrow} M_0 \overset{f_1}{\Longrightarrow} M_1 \overset{f_2}{\Longrightarrow} \cdots \overset{f_n}{\Longrightarrow} M_n.$$

From Theorem 4.5 there exists a point $x_0^k\in|H_1|$ satisfying

$$\begin{aligned} g_1^j(x_0^k)&\in|H_1| \quad \text{for } j=1,\dots,k,\\ (f_j\circ\cdots f_0\circ g_1^k)(x_0^k)&\in|M_j| \quad \text{for } j=0,\dots,n. \end{aligned}$$

Since $|H_1|$ is compact and for every $k \geq 0$, $g_1^k(x_0^k) \in |H_1|$, we can find a condensation point $x_0 \in |H_1|$ of the set $\{g_1^k(x_0^k)\}_{k>0}$. Since g_1 is injective, we obtain $g_1^{-k}(x_0) \in |H_1|$ for all $k \geq 0$. Since g_1 is hyperbolic on H_1, Theorem 5.2 implies that $\lim_{k\to\infty} g_1^{-k}(x_0) = x_1$. The assertion

$$(f_j \circ \cdots \circ f_0)(x_0) \in |M_j| \quad \text{for } j = 0, \ldots, n$$

follows from the continuity of f_i, $i = 0, \ldots, n$, and compactness of $|M_i|$, $i = 0, \ldots, n$. ■

5.2. The fuzzy sets. To prove the existence of a heteroclinic orbit we would like to use the third assertion in Theorem 5.4. Observe that to apply this theorem directly one needs to know the exact location of the two fixed points $x_1 \in H_1$ and $x_2 \in H_2$, because the sets H_1 and H_2 are centered on x_1 and x_2, respectively. But the exact coordinates of x_1 and x_2 are usually unknown. Following [16], we overcome this obstacle in three steps.

1. *Finding very good estimates for x_1 and x_2.* In this paper we use an argument based on symmetry to obtain tight bounds for x_1 and x_2. Let F_1 and F_2 denote the estimates for x_1 and x_2, respectively.

We choose two fixed points $x_1 \in F_1$ and $x_2 \in F_2$ for further consideration.

2. *$\mathcal{C}^1$-computations, hyperbolicity.* We pick up a set U_1, $H_1 \subset U_1$, on which we rigorously compute $\boldsymbol{D}g_1(U_1)$. Then we have to choose a coordinate system, in which the matrix $\boldsymbol{D}g_1(U_1)$ will be as close as possible to the diagonal one. In this paper we take numerically obtained stable and unstable eigenvectors. Let us denote these eigenvectors by u and s, where u corresponds to unstable direction and s is pointing in the stable direction. Assume that this process gives us a coordinate frame in which

$$\varepsilon_1'(U_1)\varepsilon_2'(U_1) < (1 - \lambda_2'(U_1))(\lambda_1'(U_1) - 1). \tag{8}$$

From (8) it follows easily that there exist $\alpha_1 > 0$, $\alpha_2 > 0$ such that

$$\frac{\varepsilon_1'(U_1)}{\lambda_1'(U_1) - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'(U_1)}{\epsilon_2'(U_1)}. \tag{9}$$

Observe that the above inequality specifies only the ratio α_1/α_2; hence we can find a pair (α_1, α_2) such that condition (9) is satisfied and the following condition holds:

$$F_1 + \alpha_1 \cdot [-1, 1] \cdot u + \alpha_2 \cdot [-1, 1] \cdot s \subset U_1.$$

We now define a t-set H_1 by

$$H_1 = t(x_1, \alpha_1 u, \alpha_2 s).$$

Obviously g_1 is hyperbolic on H_1. Observe that the hyperbolicity implies uniqueness of x_1 in H_1.

We do similar construction for g_2 to obtain $H_2 = t(x_2, \beta_1 \bar{u}, \beta_2 \bar{s})$.

3. *Covering relations for fuzzy t-sets.* We have to verify the following covering relations:

$$H_1 \overset{g_1}{\Longrightarrow} H_1 \overset{f_0}{\Longrightarrow} M_0, \tag{10}$$

$$M_n \overset{f_{n+1}}{\Longrightarrow} H_2 \overset{g_2}{\Longrightarrow} H_2. \tag{11}$$

As was mentioned above we do not know the t-sets H_1, H_2 explicitly, but we know that

$$H_1 \in \widetilde{H}_1 = \{t(c, \alpha_1 u, \alpha_2 s) \mid c \in F_1\},$$
$$H_2 \in \widetilde{H}_2 = \{t(c, \beta_1 \bar{u}, \beta_2 \bar{s}) \mid c \in F_2\}.$$

The above equations define *a fuzzy t-set* as a collection of t-sets. We can now extend the definition of covering relations to fuzzy t-sets as follows.

For a given fuzzy t-set $\widetilde{N}$ we define the support of a fuzzy set as a union of supports, i.e.,

$$|\widetilde{N}| = \bigcup_{M \in \widetilde{N}} |M|.$$

Definition 5.5. *Assume $\widetilde{H}_1$, $\widetilde{H}_2$ are fuzzy t-sets on the plane and N is a t-set in $\mathbb{R}^d$.*

- *Let $f : |\widetilde{H}_1| \to \mathbb{R}^d$ be continuous. We say that $\widetilde{H}_1 \overset{f}{\Longrightarrow} N$ iff $M \overset{f}{\Longrightarrow} N$ for all $M \in \widetilde{H}_1$.*
- *Let $f : |N| \to \mathbb{R}^2$ be continuous. We say that $N \overset{f}{\Longrightarrow} \widetilde{H}_1$ iff $N \overset{f}{\Longrightarrow} M$ for all $M \in \widetilde{H}_1$.*
- *Let $f : |\widetilde{H}_1| \to \mathbb{R}^2$ be continuous. We say that $\widetilde{H}_1 \overset{f}{\Longrightarrow} \widetilde{H}_2$ iff $M_1 \overset{f}{\Longrightarrow} M_2$ for all $M_1 \in \widetilde{H}_1$ and $M_2 \in \widetilde{H}_2$.*

With the above definition it is obvious that to prove the covering relations in (10) and (11) it is enough to show that

$$\widetilde{H}_1 \overset{g_1}{\Longrightarrow} \widetilde{H}_0 \overset{f_0}{\Longrightarrow} M_0,$$
$$M_n \overset{f_{n+1}}{\Longrightarrow} \widetilde{H}_2 \overset{g_2}{\Longrightarrow} \widetilde{H}_2.$$

In practice (in rigorous numerical computations) it is convenient to think about a fuzzy t-set $\widetilde{H}$ as a parallelogram with thickened edges; hence all tools developed to verify covering relations for t-sets can be easily extended to fuzzy t-sets.

6. Proofs of the main theorems.

Let $L := \{(x, y, 0) \in \mathbb{R}^3 : y = c^2 - \frac{1}{2}x^2, x \in \mathbb{R}\}$. Observe that $\Theta := \{(x, y, 0) : x, y \in \mathbb{R}\} \setminus L$ is a local cross section for (2).

Since the third coordinate is constant and equal to 0 on Θ, we will identify points on Θ with the points on $\mathbb{R}^2$. We define five two-dimensional t-sets on Θ, $N_i := t(c_i, u_i, s_i)$, $i = 1, \ldots, 5$, where

$$\begin{aligned}
c_1 &= (0.00, 1.55), & u_1 &= (0.14, 0.06), & s_1 &= (-0.14, 0.06),\\
c_2 &= (0.00, 0.51), & u_2 &= (0.09, 0.13), & s_2 &= (-0.09, 0.13),\\
c_3 &= (1.41, 0.97), & u_3 &= (0.06, 0.05), & s_3 &= (-0.06, 0.05),\\
c_4 &= (0.00, -2.35), & u_4 &= (0.06, 0.10), & s_4 &= (-0.06, 0.10),\\
c_5 &= (-1.41, 0.97), & u_5 &= (-0.06, 0.05), & s_5 &= (0.06, 0.05).
\end{aligned}$$

The sets $N_1, \ldots, N_5$ are chosen as neighborhoods of the intersections of periodic orbits S_1 and S_2 with the section Θ. Let $N := \bigcup_{i=1}^{5} |N_i|$. In [15, Lem. 5.1] the following lemma was proved with computer assistance.

Lemma 6.1. *Let $P : \Theta \nrightarrow \Theta$ denote the Poincaré return map for the Michelson system* (2) *with the parameter value $c = 1$. Then $N \subset \operatorname{dom}(P)$ and*

$$N_4 \overset{P}{\Longrightarrow} N_5 \overset{P}{\Longrightarrow} N_2 \overset{P}{\Longrightarrow} N_3 \overset{P}{\Longrightarrow} N_4, \tag{12}$$
$$N_4 \overset{P}{\Longrightarrow} N_1 \overset{P}{\Longrightarrow} N_4. \quad \blacksquare \tag{13}$$

The numerical evidence of this fact is presented in Figure 6. Note that in (12) there are two different loops of covering relations corresponding to the two symmetric periodic solutions S_1 and S_2. The main observation is that these loops contain the common t-set N_4, which allows us to construct essentially different sequences of covering relations of an arbitrary length.

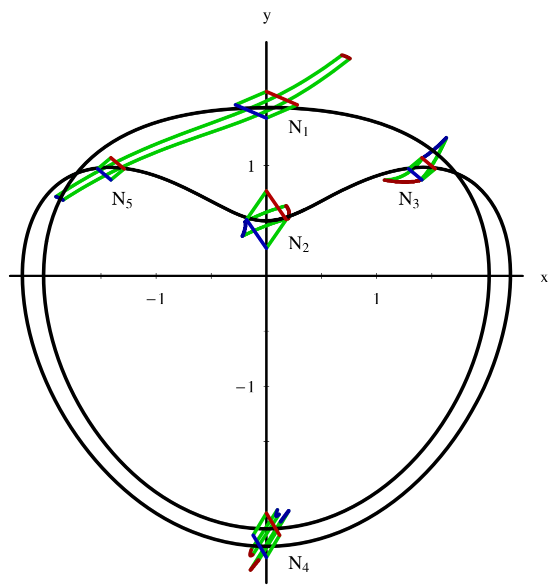

Figure 6. *Two periodic orbits established in [12], the sets $|N_1|, \ldots, |N_5|$, and their images $P(|N_1|), \ldots, P(|N_5|)$. The sets $|N_i|$, $i = 1, \ldots, 5$, are chosen as neighborhoods of the intersections of periodic orbits with the Poincaré section.*

Definition 6.2. *We say that a sequence $(i_j)_{j\in\mathbb{Z}} \in \{1, \ldots, 5\}^{\mathbb{Z}}$ is admissible with respect to the Poincaré map P if*

$$N_{i_j} \overset{P}{\Longrightarrow} N_{i_{j+1}} \quad \text{for } j \in \mathbb{Z}.$$

We formulate a similar definition for the finite sequences.

Definition 6.3. *We say that a sequence $(i_0, i_1, \ldots, i_n) \in \{1, \ldots, 5\}^{n+1}$ is admissible with respect to the Poincaré map P if*

$$N_{i_0} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_n}.$$

Recall that by R (6) we denote the reversing symmetry of (2). The existence of the reversing symmetry for (2) implies the existence of the reversing symmetry for the Poincaré map P [15, Lem. 3.3], which means that for every $x \in \operatorname{dom}(P^n)$, $n > 0$, holds $R(x) \in \operatorname{dom}(P^{-n})$ and

$$R(P^n(x)) = P^{-n}(R(x)). \tag{14}$$

The following theorem summarizes the results proved by the author in [15].

Theorem 6.4. *Let $P : \Theta \nrightarrow \Theta$ denote the Poincaré return map for the Michelson system with the parameter value $c = 1$.*

(i) *Assume* $(i_j)_{j\in\mathbb{Z}}$ *is an admissible sequence with respect to* P*. Then there exists a point* $x_0 \in |N_{i_0}|$ *such that* $P^j(x_0) \in |N_{i_j}|$ *for* $j \in \mathbb{Z}$*. Moreover, if* $(i_j)_{j\in\mathbb{Z}}$ *is a periodic sequence, then* x_0 *may be chosen as a periodic point of* P *with the same principal period.*

(ii) *Assume* $(i_0, \ldots, i_n)$*,* $n > 0$*, is an admissible sequence with respect to* P*. If* $i_0, i_n \in \{1, 2, 4\}$*, then there exists a point* $x_0 \in |N_{i_0}|$ *such that*
 (a) $R(x_0) = x_0$*,* $R(P^n(x_0)) = P^n(x_0)$*, where* R *is the reversing symmetry of* P*,*
 (b) $P^j(x_0) \in |N_{i_j}|$*,* $P^{-j}(x_0) \in R(|N_{i_j}|)$ *for* $j = 0, \ldots, n$*,*
 (c) $P^{2n}(x_0) = x_0$*.*
 In particular, the solution of (1) *with the initial condition* $(y, y', y'') = x_0$ *is a periodic odd function.*

6.1. Odd heteroclinic connections between equilibrium points. The goal of this section is to prove the following theorem.

Theorem 6.5. *Let* $(i_0, i_1, \ldots, i_k)$ *be an admissible sequence with respect to* P*. Assume* $i_0 \in \{1, 2, 4\}$*,* $i_k = 4$*. Then there exists a solution* u *of the Michelson system with parameter value* $c = 1$ *satisfying the following properties:*

1. *The solution* u *is defined for all* $t \in \mathbb{R}$*.*
2. *There is a sequence* $0 = t_0 < t_1 < \cdots < t_k$ *such that* $u(t_j) \in |N_{i_j}|$*,* $u(-t_j) \in R(|N_{i_j}|)$ *for* $j = 1, \ldots, k$*, and* $u(t_0 = 0) \in |N_{i_0}| \cap \operatorname{Fix}(R)$*.*
3. $\lim_{t\to\infty} u(t) = (-\sqrt{2}, 0, 0)$ *and* $\lim_{t\to-\infty} u(t) = (\sqrt{2}, 0, 0)$*.*

Two heteroclinic solutions resulting from Theorem 6.5 are presented in Figure 1.

Let $\phi : \mathbb{R} \times \mathbb{R}^3 \multimap \mathbb{R}^3$ denote the local dynamical system induced by (2). The proof of Theorem 6.5 is a consequence of the following steps:

1. We construct a three-dimensional t-set H centered in the equilibrium point $(-\sqrt{2}, 0, 0)$ and we find a time $T_H > 0$ such that
$$H \overset{\Phi_H}{\Longrightarrow} H \quad \text{and} \quad \phi([0, T_H], H) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\},$$
where $\Phi_H := \phi(T_H, \cdot)$.
2. We construct a two-dimensional t-set M and we find a time $T_M > 0$ such that $|M| \subset |N_4|$, $N_1 \overset{P}{\Longrightarrow} M$, $N_3 \overset{P}{\Longrightarrow} M$, and $M \overset{\Phi_M}{\Longrightarrow} H$, where $\Phi_M := \phi(T_M, \cdot)$.
3. We find a symmetric trajectory associated with the sequence of covering relations
$$N_{i_0} \overset{P}{\Longrightarrow} N_{i_1} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_{k-1}} \overset{P}{\Longrightarrow} M \overset{\Phi_M}{\Longrightarrow} H \overset{\Phi_H}{\Longrightarrow} H \overset{\Phi_H}{\Longrightarrow} \cdots$$
and we show that it must converge to the equilibrium point. Next we use the symmetry argument to show that this trajectory must be a heteroclinic connection between equilibrium points $(\pm\sqrt{2}, 0, 0)$.

Let us denote $x_- = (-\sqrt{2}, 0, 0)$. One observes that the linearized flow in x_- possesses one real eigenvalue $\lambda_1 > 0$ and a pair of complex eigenvalues λ_2, λ_3 with negative real parts. Therefore, we have a one-dimensional unstable manifold and a two-dimensional stable manifold. We define a three-dimensional t-set $H = t(x_-, 0.33 \cdot u, 0.33 \cdot s_1, 0.33 \cdot s_2)$ (see section 4.2),

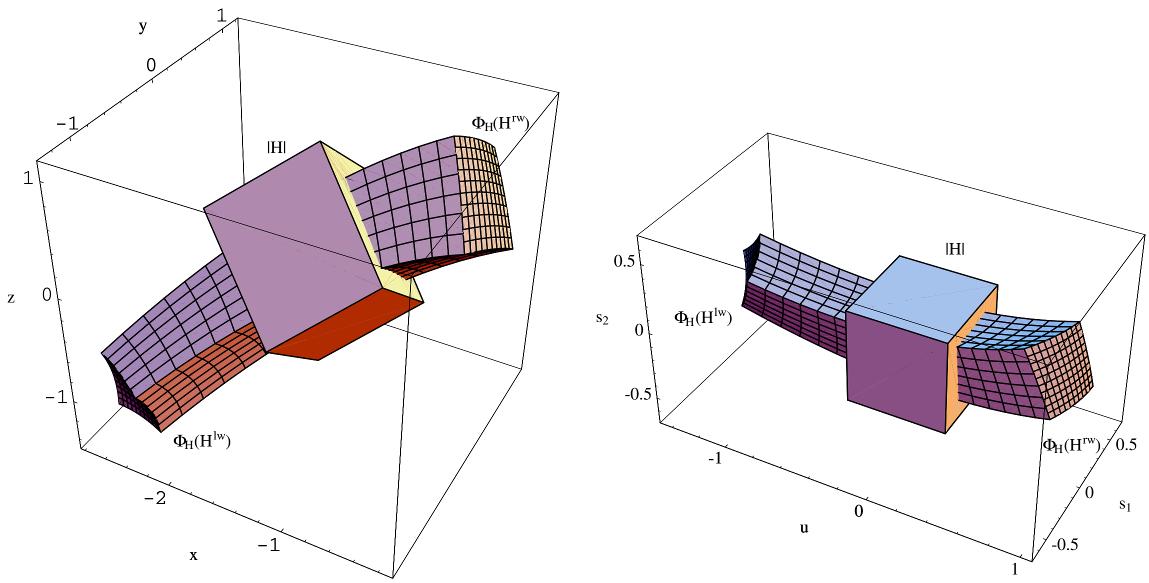

Figure 7. *The sets $|H|$ and $\Phi_H(|H|)$, (left) in (x, y, z) coordinates, (right) in (u, s_1, s_2) coordinates.*

where the vectors

$$
\begin{aligned}
u &= (1, 0.8340388297674541, 0.6956207695598643),\\
s_1 &= (0, 1.2335783628006363, -1.028852254136695),\\
s_2 &= (1, -0.4170194148837272, -1.347810384779932)
\end{aligned}
$$

are good numerical approximations of the stable and unstable eigenvectors (in fact the stable and unstable eigenvectors may be computed exactly, but this is not necessary for our method). We proved the following lemma with computer assistance.

Lemma 6.6. *Let $T_H = 1.4$ and $\Phi_H := \phi(T_H, \cdot)$. Then for each $u \in |H|$ the solution of (2) with the initial condition u is defined on the interval $[0, T_H]$. Moreover,*

$$\phi([0, T_H], |H|) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\} \quad \textit{and} \quad H \overset{\Phi_H}{\Longrightarrow} H. \qquad \blacksquare$$

The numerical evidence of this fact is presented in Figures 7 and 11.

Next we construct the set M as was required in step 2. Let us recall that the set N_4 is defined by $N_4 = t(c_4, u_4, s_4)$, where

$$c_4 = (0.00, -2.35), u_4 = (0.06, 0.10), s_4 = (-0.06, 0.10).$$

The numerical simulation shows that the intersection of the stable manifold of x_- with the Poincaré section Θ crosses the set N_4; see Figure 8. Therefore, we define the set M as a subset of N_4 containing a part of this intersection. Put $M = t(c_M, u_M, s_M)$, where

$$c_M = c_4 - 0.84 \cdot u_4, s_M = s_4, u_M = 0.1 \cdot u_4. \tag{15}$$

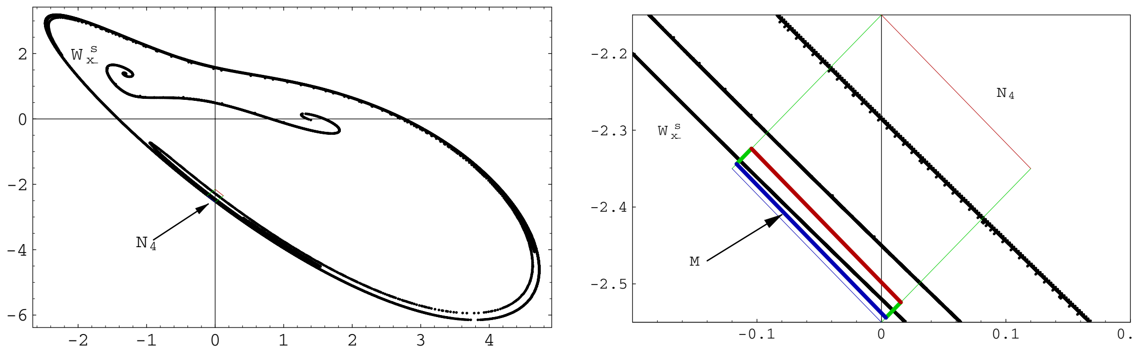

Figure 8. *A part of the intersection of the stable manifold $W^s_{x_-}$ of x_- with the Poincaré section Θ.*

Lemma 6.7. *The following covering relations hold:*

$$N_1 \overset{P}{\Longrightarrow} M, \quad N_3 \overset{P}{\Longrightarrow} M.$$

Proof. The t-set M (15) has been chosen such that the following inclusions hold:

$$\mathrm{int}(N_4^l \cup |N_4| \cup N_4^r) \subset \mathrm{int}(M^l \cup |M| \cup M^r),$$
$$\mathrm{int}(N_4^r) \subset \mathrm{int}(M^r), \quad \mathrm{int}(N_4^l) \subset \mathrm{int}(M^l)$$

(see also Figure 8, right panel). From Lemma 6.1 we know that

$$N_1 \overset{P}{\Longrightarrow} N_4, \quad N_3 \overset{P}{\Longrightarrow} N_4.$$

Therefore, all the required inclusions from the definition of covering relations are satisfied. ∎

With computer assistance we proved the following lemma.

Lemma 6.8. *Let $T_M = 6.4$ and $\Phi_M := \phi(T_M, \cdot)$. Then for each $u \in |M|$ the solution of* (2) *with the initial condition u is defined on interval $[0, T_M]$. Moreover, $M \overset{\Phi_M}{\Longrightarrow} H$.* ∎

The numerical evidence of Lemma 6.8 is presented in Figure 9.

Lemma 6.9. *Assume $u \in |H|$ is such that $\Phi_H^n(u)$ is defined for all $n > 0$ and $\Phi_H^n(u) \in |H|$ for $n > 0$. Then*

$$\lim_{t\to\infty} \phi(t, u) = (-\sqrt{2}, 0, 0).$$

Proof. Let

$$V(x, y, z) = z^2/2 + y(y - 2 + x^2)/2$$

be an energy function whose derivative along the solution is given by

$$\frac{d}{dt} V(x(t), y(t), z(t)) = x(t)(y(t))^2. \tag{16}$$

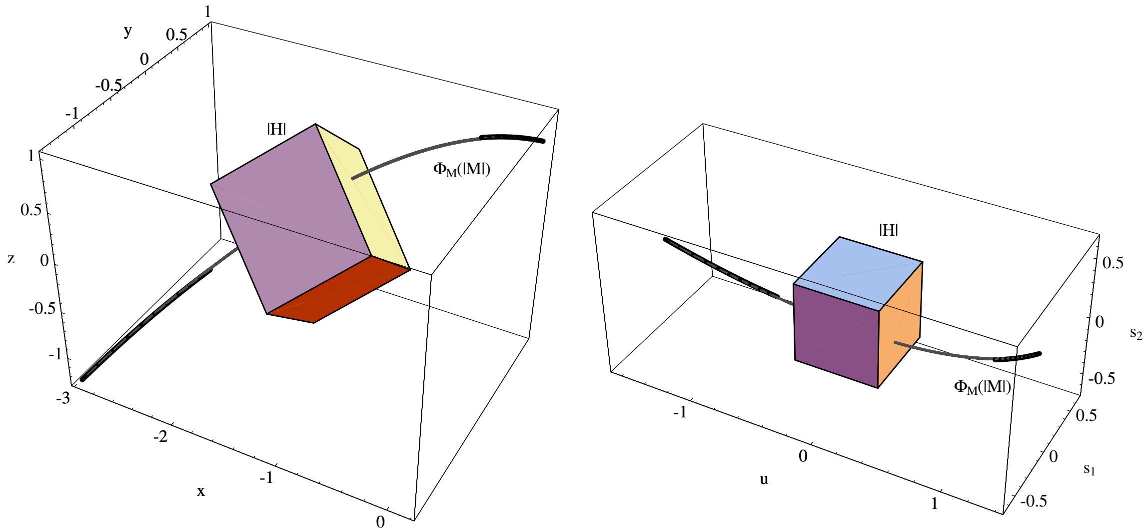

Figure 9. *The sets $|M|$ and $\Phi_M(|M|)$, (left) in (x, y, z) coordinates, (right) in (u, s_1, s_2) coordinates.*

From Lemma 6.6 we have

$$\phi([0, T_H], |H|) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}.$$

From the assumptions we know that the trajectory of u is defined on interval $[0, \infty)$ and $\phi([0, \infty), u) \subset \Phi_H([0, T_H], |H|)$. Since $\Phi_H([0, T_H], |H|)$ is a compact set, we get that the ω-limit set $\omega(u)$ is a nonempty compact set and it satisfies

$$\omega(u) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}. \tag{17}$$

From (16)–(17) we obtain that the x coordinate is nonzero on $\omega(u)$ and the y coordinate is equal to zero on $\omega(u)$. Hence $x' = y = 0$ and the x coordinate must be constant on $\omega(u)$. Therefore,

$$z' = 1 - y - x^2/2$$

is a constant function on $\omega(u)$. Since $\omega(u)$ is bounded and the Lie derivative of V is less than or equal to zero the structure of the graph of V implies that $\omega(u) = \{(-\sqrt{2}, 0, 0)\}$, the only critical value of V in $\{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}$. ∎

Now we are in the position to present the proof of Theorem 6.5.

Proof of Theorem 6.5. Let $(i_0, i_1, \ldots, i_n)$, $n > 0$, be an admissible sequence with respect to P, such that $i_0 \in \{1, 2, 4\}$ and $i_n = 4$. We have

$$N_{i_0} \overset{P}{\Longrightarrow} N_{i_1} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_n} = N_4.$$

From Lemma 6.1 either $i_{n-1} = 1$ or $i_{n-1} = 3$. From this and Lemma 6.7 we get

$$N_{i_0} \overset{P}{\Longrightarrow} N_{i_1} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_{n-1}} \overset{P}{\Longrightarrow} M.$$

Lemmas 6.8 and 6.6 imply

$$N_{i_0} \overset{P}{\Longrightarrow} N_{i_1} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_{n-1}} \overset{P}{\Longrightarrow} M \overset{\Phi_M}{\Longrightarrow} H \overset{\Phi_H}{\Longrightarrow} H. \tag{18}$$

Recall that by R (6) we denoted the reversing symmetry of (2). Observe that the set $\mathrm{Fix}(R) \cap |N_{i_0}|$, $i_0 \in \{1, 2, 4\}$, may be parameterized as a horizontal curve $\gamma : [a, b] \to \mathbb{R}^2$ in N_{i_0}. Hence, from Theorem 4.5 there exists $\tau \in [a, b]$ such that

$$P^k(\gamma(\tau)) \in |N_{i_k}|, \quad k = 0, \ldots, n$$

(recall that $|M| \subset |N_{i_n}| = |N_4|$), and

$$\left(\Phi_H^k \circ \Phi_M \circ P^n\right)(\gamma(\tau)) \in |H|, \quad k = 0, 1, \ldots.$$

From Lemma 6.9 we obtain

$$\lim_{t\to\infty} \phi(t, \Phi_M(P^n(\gamma(\tau)))) = (-\sqrt{2}, 0, 0) = x_-.$$

Put $u = \gamma(\tau) \in \mathrm{Fix}(R) \cap |N_{i_0}|$. By the definition of the Poincaré map there are numbers $0 = t_0 < t_1 < \cdots < t_n$ such that

$$\phi(t_j, u) \in |N_{i_j}|, \quad j = 0, \ldots, n.$$

Since $R(u) = u$, we get

$$\phi(-t_j, u) = \phi(-t_j, R(u)) = R(\phi(t_j, u)) \in R(|N_{i_j}|), \quad j = 0, \ldots, n.$$

A similar argument shows that

$$\lim_{t\to-\infty} \phi(t, u) = \lim_{t\to-\infty} \phi(t, R(u)) = \lim_{t\to\infty} R(\phi(t, u)) = R(x_-) = x_+ = (\sqrt{2}, 0, 0),$$

and the proof is completed. ■

Proof of Theorem 3.1. Consider the family of the admissible sequences $\{(1, 4)^n \in \mathbb{N}^{2n}, n > 0\}$. All of them satisfy the assumptions of Theorem 6.5 and each of them gives a different R-symmetric heteroclinic orbit. ■

6.2. Heteroclinic connections between periodic orbits and equilibrium points. We use the method introduced in section 5.2 in order to prove the existence of heteroclinic connections between the odd periodic orbits and equilibrium points. As was remarked in the introduction, the main difference between the method presented in [3] and that in section 5.2 is that we use t-sets which have unequal dimensions.

In [12, Thm. 1] it was proven that (1) possesses at least two R-symmetric periodic solutions (see Figure 6). The same assertion follows from Theorem 6.4 (ii) for the sequence of symbols $(1, 4)$ and $(2, 3, 4)$.

As remarked in section 5.2, we will need a very good estimation of such orbits. Unfortunately, the above mentioned theorems do not give a sufficiently precise location of such orbits.

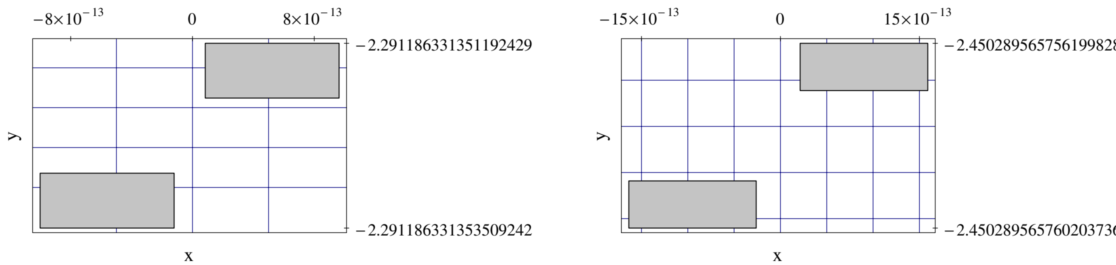

Figure 10. *(Left) the bounds of $P(q_1 \pm \eta)$ computed in the rigorous routine; (right) the bounds of $P^2(q_2 \pm \eta)$ computed in the rigorous routine.*

In this section we will find a very good approximation of such periodic orbits and prove the existence of heteroclinic connections between them and the equilibrium points.

As remarked in section 5.2, we will proceed in three steps.

1. *The existence of two R-symmetric periodic solutions.*

Lemma 6.10. *Let $I_1 = [q_1 - \eta, q_1 + \eta]$, $I_2 = [q_2 - \eta, q_2 + \eta]$, where*

$$q_1 = 1.5259617305037,$$
$$q_2 = 0.5000256485352,$$
$$\eta = 4 \cdot 10^{-13}.$$

Then there exist $q_1^* \in I_1$, $q_2^* \in I_2$ such that the solutions S_1, S_2 of (2) with the initial conditions $(0, q_j^*, 0)$, $j = 1, 2$, respectively, are R-symmetric periodic solutions.

Proof. Let π_x denote the projection onto the x coordinate. Since the interval $[0] \times I_1 \subset |N_1|$, by Lemma 6.1 the Poincaré map is defined on the set $[0] \times I_1 \subset |N_1|$. With computer assistance we verified the following conditions:

1. P^2 is well defined and continuous on $[0] \times I_2$.
2. $\pi_x(P(q_1 - \eta)) > 0, \quad \pi_x(P(q_1 + \eta)) < 0$.
3. $\pi_x(P^2(q_2 - \eta)) < 0, \quad \pi_x(P^2(q_2 + \eta)) > 0$.

The Darboux property implies the existence of a $q_1^* \in I_1$ and $q_2^* \in I_2$ such that $\pi_x(P(0, q_1^*)) = 0$ and $\pi_x(P^2(0, q_2^*)) = 0$. Hence, from (14) we obtain

$$P(0, q_1^*) = R(P(0, q_1^*)) = P^{-1}(0, R(q_1^*)) = P^{-1}(0, q_1^*),$$

and the trajectory of $(0, q_1^*, 0)$ must be an R-symmetric periodic function. Similarly,

$$P^2(0, q_2^*) = R(P^2(0, q_2^*)) = P^{-2}(0, R(q_2^*)) = P^{-2}(0, q_2^*),$$

and the proof is finished. ■

The numerical evidence of Lemma 6.10 is presented in Figure 10. Now we define the sets $F_1 = [0] \times I_1$, $F_2 = [0] \times I_2$. We define two fuzzy t-sets $\widetilde{H}_1 = t(F_1, \alpha_1 u_1, \alpha_1 s_1)$, $\widetilde{H}_2 =$

$(F_2, \alpha_2 u_2, \alpha_2 s_2)$, where

$$
\begin{aligned}
s_1 &= (-1, 0.45749078844956629796), \quad u_1 = R(s_1),\\
s_2 &= (-1, 0.45389272853821893\,70), \quad u_2 = R(s_2), \qquad (19)\\
&\qquad \alpha_1 = 3 \cdot 10^{-4}, \quad \alpha_2 = 3 \cdot 10^{-5}.
\end{aligned}
$$

These vectors appear to be good approximations for unstable (u_i) and stable eigenvectors (s_i) at $(0, q_i)$ of the Poincaré map P^2 and P^4, respectively. With computer assistance we proved the following lemma.

Lemma 6.11. *P^2 is smooth on $|\widetilde{H}_1|$, and P^4 is smooth on $|\widetilde{H}_2|$. Moreover,*

$$
\boldsymbol{D}P^2(|\widetilde{H}_1|) \subset \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{bmatrix}, \quad \boldsymbol{D}P^4(|\widetilde{H}_2|) \subset \begin{bmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{bmatrix},
$$

where the intervals

$$
\begin{aligned}
\boldsymbol{A}_1 &= [-16.20797124287215496, -15.5141950243296769],\\
\boldsymbol{B}_1 &= [-35.32679704133074239, -33.87386922840860848],\\
\boldsymbol{C}_1 &= [-7.301556805980839116, -7.163445906569215538],\\
\boldsymbol{D}_1 &= [-15.97740049976343535, -15.70439650055097935],\\
\boldsymbol{A}_2 &= [53.7551522870797811, 54.21349342352709755],\\
\boldsymbol{B}_2 &= [118.4351762233210224, 119.3964660170624512],\\
\boldsymbol{C}_2 &= [24.3823739267779871, 24.62135790755886333],\\
\boldsymbol{D}_2 &= [53.73353569893723148, 54.24827743305298356]. \qquad \blacksquare
\end{aligned}
$$

Let $S_1^* = (0, q_1^*)$ and $S_2^* = (0, q_2^*)$.

Lemma 6.12. *S_1^* is the unique fixed point for P^2 in $|\widetilde{H}_1|$. S_2^* is the unique fixed point for P^4 in $|\widetilde{H}_2|$.*

Proof. Easy computations show that

$$
0 \notin \det \boldsymbol{D}(P^2 - \mathrm{Id})(|\widetilde{H}_1|) = \{\det(M - \mathrm{Id}) \mid M \in \boldsymbol{D}P^2(|\widetilde{H}_1|)\}
$$

and

$$
0 \notin \det \boldsymbol{D}(P^4 - \mathrm{Id})(|\widetilde{H}_2|) = \{\det(M - \mathrm{Id}) \mid M \in \boldsymbol{D}P^4(|\widetilde{H}_2|)\}.
$$

From [16, Thm. 4.2] there exist at most one fixed point in $|\widetilde{H}_1|$ for P^2 and at most one fixed point in $|\widetilde{H}_2|$ for P^4. $\blacksquare$

2. *Hyperbolicity.* We define two t-sets $H_j = t(S_j^*, \alpha_j u_j, \alpha_j s_j)$, $j = 1, 2$, where α_j, u_j, s_j are defined as in (19).

Lemma 6.13. *P^2 is hyperbolic on H_1, and P^4 is hyperbolic on H_2.*

Proof. Observe that the transformation of $\boldsymbol{D}P^2(|\widetilde{H}_1|))$, $\boldsymbol{D}P^4(|\widetilde{H}_2|)$ to new coordinates does not depend on the exact location of S_1^*, S_2^*. In new coordinates $S_1^* = S_2^* = 0$, but we have to choose the coordinate directions in $|\widetilde{H}_1|$ and $|\widetilde{H}_2|$. It turns out that the vectors (u_i, s_i) which were used in the definition of $\widetilde{H}_i$ are good for this purpose, as they are reasonably good approximations of unstable and stable directions of the corresponding Poincaré map.

After the change of the coordinate system, $\boldsymbol{D}P^2(|\widetilde{H}_1|)$ and $\boldsymbol{D}P^4(|\widetilde{H}_2|)$ have the form

$$\boldsymbol{D}P^2(|\widetilde{H}_1|) \subset \begin{bmatrix} \boldsymbol{\lambda}_{11} & \boldsymbol{\varepsilon}_{11} \\ \boldsymbol{\varepsilon}_{21} & \boldsymbol{\lambda}_{21} \end{bmatrix}, \quad \boldsymbol{D}P^4(|\widetilde{H}_2|) \subset \begin{bmatrix} \boldsymbol{\lambda}_{12} & \boldsymbol{\varepsilon}_{12} \\ \boldsymbol{\varepsilon}_{22} & \boldsymbol{\lambda}_{22} \end{bmatrix},$$

where

$$\begin{aligned}
\boldsymbol{\lambda}_{11} &= [-32.15353216472522036, -31.1868475976410906],\\
\boldsymbol{\varepsilon}_{11} &= [-0.4631153446352325176, 0.5035692224488864666],\\
\boldsymbol{\varepsilon}_{21} &= [-0.4833845890051770011, 0.4832999780789397626],\\
\boldsymbol{\lambda}_{21} &= [-0.5151340361170263504, 0.4515505309670904133],\\
\boldsymbol{\lambda}_{12} &= [107.4819576670972197, 108.4499207461666828],\\
\boldsymbol{\varepsilon}_{12} &= [-0.4773917028128066286, 0.4905713762566483726],\\
\boldsymbol{\varepsilon}_{22} &= [-0.4839876655649711923, 0.4839754135044633254],\\
\boldsymbol{\lambda}_{22} &= [-0.4746913248681289832, 0.4932717542013055345].
\end{aligned}$$

It is clear that $\lambda_{i2} < 1 < \lambda_{i1}$ and $\varepsilon_{i1}\varepsilon_{i2} < (1-\lambda_{i2})(\lambda_{i1}-1)$. Moreover,

$$\frac{\varepsilon_{11}}{\lambda_{11}-1} < 1 < \frac{1-\lambda_{12}}{\varepsilon_{12}}, \qquad \frac{\varepsilon_{21}}{\lambda_{21}-1} < 1 < \frac{1-\lambda_{22}}{\varepsilon_{22}}.$$

Observe that by construction $|H_i| \subset |\widetilde{H}_i|$, $i = 1, 2$. This shows that P^2 is hyperbolic on H_1 and P^4 is hyperbolic on H_2. ■

3. *Covering relations for fuzzy sets.* Let $G_1 = t(q_2, \alpha_2 u_1, \alpha_2 s_1)$ and $G_2 = t(q_2, \beta_2 u_2, \beta_2 s_2)$, where $\alpha_2 = 25\alpha_1$, $\beta_2 = 90\beta_1$.

The reason for this construction is the following. The $\mathcal{C}^1$-computations are much more complicated and time-consuming than $\mathcal{C}^0$. Therefore, we want to bound $\mathcal{C}^1$-computations to the very small sets around the fixed points, namely, $\widetilde{H}_1, \widetilde{H}_2$. But the sets $\widetilde{H}_1, \widetilde{H}_2$ are small, so we cannot find the covering relations $\widetilde{H}_1 \overset{P^2}{\Longrightarrow} N_1$ and $\widetilde{H}_2 \overset{P^4}{\Longrightarrow} N_2$. Therefore, we construct new sets G_1, G_2, $|\widetilde{H}_j| \subset |G_j| \subset |N_j|$, $j = 1, 2$, with properties expressed in the following lemma.

Lemma 6.14. *The following conditions are satisfied:*

1. P^2 *is well defined and continuous on* G_1.
2. P^4 *is well defined and continuous on* G_2.
3. $H_1 \overset{P^2}{\Longrightarrow} H_1 \overset{P^2}{\Longrightarrow} G_1 \overset{P^2}{\Longrightarrow} N_1$.
4. $H_2 \overset{P^4}{\Longrightarrow} H_2 \overset{P^4}{\Longrightarrow} G_2 \overset{P^4}{\Longrightarrow} N_2$.

Proof. With computer assistance we have verified the following covering relations for the fuzzy sets:

$$\widetilde{H}_1 \overset{P^2}{\Longrightarrow} \widetilde{H}_1 \overset{P^2}{\Longrightarrow} G_1 \overset{P^2}{\Longrightarrow} N_1,$$
$$\widetilde{H}_2 \overset{P^4}{\Longrightarrow} \widetilde{H}_2 \overset{P^4}{\Longrightarrow} G_2 \overset{P^4}{\Longrightarrow} N_2.$$

Hence, the assertion 3–4 follows from the definition of covering relations for the fuzzy sets. The proof of assertions 1–2 will be presented in section 7. ■

Theorem 6.15. *Assume that $(i_0, i_1, \dots, i_n)$ is an admissible sequence with respect to P and $i_n = 4$.*

1. *If $i_0 = 1$, then there exists $x_0 \in |N_{i_0}|$ such that*
   (a) *$P^k(x_0) \in |N_{i_k}|$ for $k = 1, \dots, n$,*
   (b) *$\lim_{t\to\infty} \phi(t, x_0) = (-\sqrt{2}, 0, 0)$, and*
   (c) *$P^{-2k}(x_0)$ exists for all $k > 0$ and $\lim_{k\to\infty} P^{-2k}(x_0) = S_1^*$.*
2. *If $i_0 = 2$, then there exists $x_0 \in |N_{i_0}|$ such that*
   (a) *$P^k(x_0) \in |N_{i_k}|$ for $k = 1, \dots, n$,*
   (b) *$\lim_{t\to-\infty} \phi(t, x_0) = (-\sqrt{2}, 0, 0)$, and*
   (c) *$P^{-4k}(x_0)$ exists for all $k > 0$ and $\lim_{k\to\infty} P^{-4k}(x_0) = S_2^*$.*

Proof. Let $(i_0, i_2, \dots, i_n)$ be an admissible sequence with respect to P and $i_0 = 1$, $i_n = 4$ as in assertion (1). From Lemmas 6.14, 6.6, and 6.8 we get

$$H_1 \overset{P^2}{\Longrightarrow} H_1 \overset{P^2}{\Longrightarrow} G_1 \overset{P^2}{\Longrightarrow} N_1 = N_{i_0} \overset{P}{\Longrightarrow} \cdots \overset{P}{\Longrightarrow} N_{i_{n-1}} \overset{P}{\Longrightarrow} M \overset{\Phi_M}{\Longrightarrow} H \overset{\Phi_H}{\Longrightarrow} H.$$

Since P^2 is hyperbolic on H_1 Theorem 5.2 shows that for every $k \geq 0$ there exists a point $x_0^k \in |N_{i_0}|$ such that

1. $P^j(x_0^k) \in |N_{i_j}|$ for $j = 1, \dots, n$,
2. $(\Phi_H^j \circ \Phi_M \circ P^n)(x_0^k) \in |H|$ for $j = 0, \dots, k$, and
3. $P^{-2j}(x_0^k)$ exists for all $j > 0$ and $\lim_{j\to\infty} P^{-2j}(x_0) = S_1^*$.

Since $|N_{i_0}|$ is a compact set, we can find $x_0 \in |N_{i_0}|$ such that

1. $P^k(x_0) \in |N_{i_j}|$ for $j = 1, \dots, n$,
2. $(\Phi_H^k \circ \Phi_M \circ P^n)(x_0^k) \in |H|$ for all $k \geq 0$,
3. $P^{-2k}(x_0)$ exists for all $k > 0$ and $\lim_{k\to\infty} P^{-2k}(x_0) = S_1^*$.

Now Lemma 6.9 implies that

$$\lim_{t\to\infty} \phi(t, x_0) = \lim_{t\to\infty} \phi(t, \Phi_M(P^n(x_0))) = (-\sqrt{2}, 0, 0).$$

The second assertion can be proved in a similar way. ■

Proof of Theorem 3.2. The first assertion of Theorem 3.2 follows directly from Theorem 6.15. The second assertion is a consequence of the reversing symmetry property of the Michelson system. Namely, if $u : \mathbb{R} \to \mathbb{R}^3$ is the solution of (2) connecting the periodic orbit S_j, $j = 1, 2$, and the equilibrium point $x_- = (-\sqrt{2}, 0, 0)$, then $R(u)$ is a heteroclinic orbit connecting $R(x_-) = x_+ = (\sqrt{2}, 0, 0)$ with $R(S_j) = S_j$. ■

7. Numerical proofs.

In this section we give details of the computer assisted proofs of Lemmas 6.6, 6.8, 6.10, 6.11, and 6.14. In these lemmas we can find three types of assertions: the existence of the map, the existence of covering relations or some inclusions, and the hyperbolicity of some sets ($\mathcal{C}^1$-computations).

7.1. The existence and continuity of the maps.

We had to prove the following assertions:

1. (In Lemma 6.6) Φ_H is well defined and continuous on $|H|$ and $\phi([0, T_H], |H|) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}$.

2. (In Lemma 6.8) Φ_M is well defined and continuous on $|M|$.
3. (In Lemma 6.14) P^2 is well defined and continuous on $|G_1|$, and P^4 is well defined and continuous on $|G_2|$. Observe that this implies the existence of P^2 on the set $[0] \times I_2 \subset G_2$ required in Lemma 6.10.

The first assertion requires us to check the existence of Φ_H on a three-dimensional set. We divided each edge of $|H|$ into three equal parts and covered the whole set $|H|$ by $3 \times 3 \times 3$ smaller parallelepipeds Q_j, $j = 1, \ldots, 27$. Next, each of them was used as an initial condition to our routine computing $\Phi_H(Q_j)$. As in [15] we used the $\mathcal{C}^0$-Lohner algorithm [8, 21] for computing the time-T flow. We used the third order Lohner method with the time step $h = 0.2$. Our routine verified that $\phi(kh + [0, h], Q_j) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\}$ for $j = 1, \ldots, 27$ and $k = 1, \ldots, 6$ (recall that $T_H = 1.4 = 7 \cdot 0.2$).

The second and third assertions were proved in a similar way. We divided the sets $|M|$, $|G_1|$, $|G_2|$ into $m \times n$ parallelograms and each of $m \times n$ sets was used as an initial condition. The parameter settings used in these computations are listed in Table 1.

Table 1

The parameter settings of the Lohner method used in the proof of the existence of the Poincaré map in Lemma 6.14 and in the existence of Φ_M and in the verification of $M \overset{\Phi_M}{\Longrightarrow} H$ in Lemma 6.8.

Set	Order	Step	Horizontal grid	Vertical grid
$\vert M\vert$	5	0.2	22	16
$\vert G_1\vert$	3	0.1	2	2
$\vert G_2\vert$	3	0.1	5	5

7.2. Verifications of covering relations. We have to prove the following assertions:

1. (In Lemma 6.6) $H \overset{\Phi_H}{\Longrightarrow} H$.
2. (In Lemma 6.8) $M \overset{\Phi_M}{\Longrightarrow} H$.
3. (In Lemma 6.14) $\widetilde{H}_1 \overset{P^2}{\Longrightarrow} \widetilde{H}_1 \overset{P^2}{\Longrightarrow} G_1 \overset{P^2}{\Longrightarrow} N_1$ and $\widetilde{H}_2 \overset{P^4}{\Longrightarrow} \widetilde{H}_2 \overset{P^4}{\Longrightarrow} G_2 \overset{P^4}{\Longrightarrow} N_2$.
4. (In Lemma 6.10) verification of the conditions 2–3.

The following lemma allows us to restrict our computations to the boundary of a t-set.

Lemma 7.1. *Let N, M be t-sets in $\mathbb{R}^m$. Assume $f : |N| \to \mathbb{R}^m$ is an injective map. Then $N \overset{f}{\Longrightarrow} M$ iff*

(a′) *$f(\mathrm{bd}|N|) \subset \mathrm{int}(M^l \cup |M| \cup M^r)$,*

(b) *either $f(N^{lw}) \subset \mathrm{int}(M^l)$ and $f(N^{rw}) \subset \mathrm{int}(M^r)$ or $f(N^{lw}) \subset \mathrm{int}(M^r)$ and $f(N^{rw}) \subset \mathrm{int}(M^l)$.*

Proof. We will show that condition (a′) implies condition (a) from Definition 4.2. Assume this is not the case, i.e., that there exists a point $x \in \mathrm{int}(|N|)$ such that

$$f(x) \notin \mathrm{int}(M^l \cup |M| \cup M^r). \tag{20}$$

From the Brouwer–Jordan theorem the set $\mathbb{R}^m \setminus f(\mathrm{bd}(|N|))$ has two connected components, one bounded and one unbounded. Moreover, $f(x)$ must be in the bounded component of $\mathbb{R}^m \setminus f(\mathrm{bd}(|N|))$. From (20) it follows that

$$f(x) \in \mathbb{R}^m \setminus \mathrm{int}(M^l \cup |M| \cup M^r) =: \overline{M}.$$

Table 2

The parameter settings of the Lohner method used in the proof of the existence of covering relation $H \overset{\Phi_H}{\Longrightarrow} H$ in Lemma 6.6.

Wall	Order	Step	Horizontal grid	Vertical grid
$x_- + u_1 + I \cdot s_1 + I \cdot s_2$	3	0.2	3	3
$x_- - u_1 + I \cdot s_1 + I \cdot s_2$	3	0.2	4	4
$x_- + I \cdot u_1 + s_1 + I \cdot s_2$	3	0.2	25	25
$x_- + I \cdot u_1 - s_1 + I \cdot s_2$	3	0.2	30	30
$x_- + I \cdot u_1 + I \cdot s_1 + s_2$	3	0.2	30	30
$x_- + I \cdot u_1 + I \cdot s_1 - s_2$	3	0.2	30	30

Moreover, $\overline{M}$ is a union of two connected, convex, unbounded sets. Therefore, one can find a half-line starting at $f(x)$ and contained in $\overline{M}$. Since $f(x)$ lies in a bounded component of $\mathbb{R}^m \setminus f(\mathrm{bd}(|N|))$, it follows that this half-line must intersect $f(\mathrm{bd}(|N|))$. Hence, we have shown that there is a point $y \in \mathrm{bd}(|N|)$ such that $f(y) \in \overline{M}$. This contradicts assumption (a′). ■

In the proof of the first assertion we covered each of six walls of $|H|$ by a finite number of smaller parallelograms. Next, each of them was used as an initial condition to our routine computing Φ_H. The parameter settings of the Lohner method are listed in Table 2 (to simplify the notation we set $I = [-1, 1]$). The bound on $\Phi_H(\mathrm{bd}|H|)$ obtained in the rigorous procedure is presented in Figure 11 (see also Figure 7).

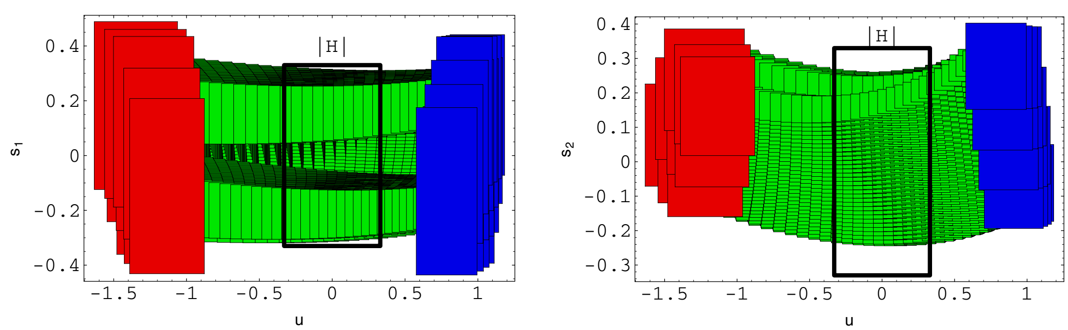

Figure 11. *The bound on $\Phi_H(\mathrm{bd}|H|)$ obtained in the rigorous procedure, (left) projected onto (u, s_1) coordinates, (right) projected onto (u, s_2) coordinates.*

In the proof of the second assertion we could not use Lemma 7.1 since the t-sets M and H have different dimensions. Therefore, the conditions from the definition of covering relations were verified together with the verification of the existence of Φ_M on $|M|$; see Table 1.

The third assertion was proved using Lemma 7.1. Let $N = t(c, u, s)$ be a t-set on the plane. To simplify notation we set $N^{te} = c + s + [-1, 1] \cdot u$ and $N^{be} = c - s + [-1, 1] \cdot u$. The parameter settings of the Lohner method are listed in Table 3.

In the proof of the fourth assertion we just inserted the points $q_j \pm \eta$, $j = 1, 2$, into our routine. The parameter settings of the Lohner method are listed in Table 4; see also Figure 10.

Table 3

The parameter settings of the Lohner method used in the proof of the existence of covering relations in Lemma 6.14.

Edge	Relation	Order	Step	Grid
$\widetilde{H}_1^{re}$ and $\widetilde{H}_1^{le}$	$\widetilde{H}_1 \overset{P^2}{\Longrightarrow} \widetilde{H}_1 \overset{P^2}{\Longrightarrow} G_1$	9	0.4	6
$\widetilde{H}_1^{te}$ and $\widetilde{H}_1^{be}$	$\widetilde{H}_1 \overset{P^2}{\Longrightarrow} \widetilde{H}_1 \overset{P^2}{\Longrightarrow} G_1$	9	0.4	45
$\widetilde{H}_2^{re}$ and $\widetilde{H}_2^{le}$	$\widetilde{H}_2 \overset{P^4}{\Longrightarrow} \widetilde{H}_2 \overset{P^4}{\Longrightarrow} G_2$	9	0.3	7
$\widetilde{H}_2^{te}$ and $\widetilde{H}_2^{be}$	$\widetilde{H}_2 \overset{P^4}{\Longrightarrow} \widetilde{H}_2 \overset{P^4}{\Longrightarrow} G_2$	9	0.3	145
G_1^{re} and G_1^{le}	$G_1 \overset{P^2}{\Longrightarrow} N_1$	9	0.4	4
G_1^{te} and G_1^{be}	$G_1 \overset{P^2}{\Longrightarrow} N_1$	9	0.4	3
G_2^{re} and G_2^{le}	$G_2 \overset{P^4}{\Longrightarrow} N_2$	9	0.3	3
G_2^{te} and G_2^{be}	$G_2 \overset{P^4}{\Longrightarrow} N_2$	9	0.3	3

Table 4

The parameter settings of the Lohner method used in the proof of the existence of odd periodic orbits S_1^*, S_2^*.

Points	Order	Step
$q_1 \pm \eta$, $q_2 \pm \eta$	10	0.1

Table 5

The parameter settings of the Lohner method used in $\mathcal{C}^1$-computations.

Set	Order	Step	Horizontal grid	Vertical grid
$\lvert\widetilde{H}_1\rvert$	9	0.3	7	7
$\lvert\widetilde{H}_2\rvert$	9	0.3	7	7

7.3. $\mathcal{C}^1$-computations. The proof of the Lemma 6.11 requires the $\mathcal{C}^1$-computations. We used the $\mathcal{C}^1$-Lohner algorithm [21]. As in the proof of the existence of the Poincaré map we covered the sets $\lvert\widetilde{H}_j\rvert$, $j = 1, 2$, by a finite numer of parallelograms, and each of them was used as an initial condition in our routine computing the derivative of the Poincaré map. Parameter settings of the Lohner method used in the proof are listed in Table 5.

7.4. Some technical data. All computations took 25 seconds on the Intel Pentium 4, 3GHz processor. The algorithms were implemented in C++ and the source codes are available from [14]. The algorithms use an implementation of interval arithmetic, vector arithmetic, and set algebra developed at Jagiellonian University by the CAPD group [2].

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