A Relational Gradient Descent Algorithm For Support Vector Machine Training

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Abstract

We consider gradient descent like algorithms for Support Vector Machine (SVM) training when the data is in relational form. The gradient of the SVM objective can not be efficiently computed by known techniques as it suffers from the "subtraction problem". We first show that the subtraction problem can not be surmounted by showing that computing any constant approximation of the gradient of the SVM objective function is #P-hard, even for acyclic joins. We however circumvent the subtraction problem by restricting our attention to stable instances, which intuitively are instances where a nearly optimal solution remains nearly optimal if the points are perturbed slightly. We give an efficient algorithm that computes a "pseudo-gradient" that guarantees convergence for stable instances at a rate comparable to that achieved by using the actual gradient. We believe that our results suggest that this sort of stability analysis would likely yield useful insight in context of designing algorithms on relational data for other learning problems in which the subtraction problem arises.

1 Introduction

Kaggle surveys [1] show that the majority of learning tasks faced by data scientists involve relational data. Most commonly the relational data is stored in tables in a relational database. So these data scientists want to compute something like

Data Science Query = Standard_Learning_Task(Relational Tables $T_1, \dots T_m$)

However, almost all standard algorithms for standard learning problems assume that the input consists of points in Euclidean space [18], and thus are not designed to operate directly on relational data. The current standard practice for a data scientist, confronted with a learning task on relational data, is:

1. Firstly, convert any nonnumeric categorical data to numeric data. As there are standard methods to accomplish this [18], and as we do not innovate with respect to this process, we will assume that all data is a priori numerical, so we need not consider this step.

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- 2. Secondly, issue a feature extraction query to extract the data from the relational database by joining together the tables to materialize a design matrix $J = T_1 \bowtie \cdots \bowtie T_m$ with say N rows and (d+1) columns. Each row of this design matrix is then interpreted as a point in d-dimensional Euclidean space with an associated label.
- 3. Finally this design matrix J is important into a standard learning algorithm to train the model.

Thus conceptually, standard practice transforms a data science query to a query of the following form:

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Data Science Query = Standard_Learning_Algorithm(Design Matrix J = T_1 \bowtie \cdots \bowtie T_m)
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where the joins are evaluated first, and the learning algorithm is then applied to the result. Note that if each table has n rows, the design matrix J can have as many as n^m entries. Thus, independent of the learning task, this standard practice necessarily has exponential worst-case time and space complexity as the design matrix can be exponentially larger than the underlying relational tables. Thus a natural research question is what we call the relational learning question:

The Relational Learning Question:

- A. Which standard learning algorithms can be implemented as relational algorithms, which informally are algorithms that are efficient when the input is in relational form?
- B. And for those standard algorithms that are not implementable by a relational algorithms, is there an alternative relational algorithm that has the same performance guarantee as the standard algorithm?
- C. And if we can't find an alternative relational algorithm that has the same performance guarantees to the standard algorithm, is there an alternative relational algorithm that has some reasonable performance guarantee (ideally similar to the performance guarantee for the standard algorithm)?

Note that a relational algorithm can not afford to explicitly join the relational tables.

One immediate difficulty that we run into is that if the tables have a sufficiently complicated structure, almost all natural problems/questions about the design matrix are NP-hard if the data is in relational form. For example, it is NP-hard to even determine whether or not the design matrix is empty or not (see for example [22, 33]). Thus, as we want to focus on the complexity of the learning problems, we conceptually want to abstract out the complexity of the tables. The simplest way to accomplish this is to primarily focus on instances where the structure of the tables is simple, with the most natural candidate for "simplicity" being that the join is acyclic. Acyclic joins are the norm in practice, and are a commonly considered special case in the database literature. For example, there are efficient algorithms to compute the size of the design matrix for acyclic joins.

Formally defining what an "relational" algorithm is problematic, as for each natural candidate definition there are plausible scenarios in which that candidate definition is not the "right" definition. But for the purposes of this paper it is sufficient to think of a "relational" algorithm as one whose runtime is polynomially bounded in n, m and d if the join is acyclic.

Our Research Question: In this paper we address the relational learning question within the context of gradient descent algorithms for the classic (soft-margin linear) Support Vector Machine (SVM) training problem. SVM is identified as one of the five most important learning problems

in [18], and is covered in almost all introductory machine learning textbooks. Gradient descent is probably the most commonly used computational technique for solving convex learning optimization problems [39]. So plan A is to find a relational implementation of gradient descent for the SVM objective. And if plan A fails, plan B is to find a relational descent algorithm that has the same performance guarantee as gradient descent. And finally, if both plan A fail and plan B fail, plan C is to find a relational algorithm that has some other reasonable performance guarantee.

1.1 Background

We now give the minimal background on gradient descent and SVM required to understand our results.

Gradient Descent: Gradient descent is a first-order iterative optimization method for finding an approximate minimum of a convex function $F : \mathbb{R}^d \to \mathbb{R}$, perhaps subject to a constraint the solution lies in some convex body \mathcal{K} . In the G descent algorithm, at each descent step t the current candidate solution $\beta^{(t)}$ is updated according to the following rule:

$$\beta^{(t)} \leftarrow \beta^{(t-1)} - \eta_t G(\beta^{(t-1)}) \tag{1}$$

where η_t is the step size. In projected G descent, the current candidate solution $\beta^{(t)}$ is updated according to the following rule:

$$\beta^{(t)} \leftarrow \Pi_{\mathcal{K}} \left(\beta^{(t-1)} - \eta_t G(\beta^{(t-1)}) \right)$$
 (2)

where $\Pi_{\mathcal{K}}(\alpha) = \operatorname{argmin}_{\beta \in \mathcal{K}} \|\alpha - \beta\|_2$ is the projection of the point α to the closest point to α in \mathcal{K} . In (projected) gradient descent, G is $\nabla F(\beta^{(t)})$, the gradient of F at $\beta^{(t)}$. There are lots of variations of gradient descent, including variations on the step size, and variations, like stochastic gradient descent [39], in which the gradient is only approximated.

SVM training: Conceptually the input to SVM training consists of a collection $X = \{x_1, x_2, \dots, x_N\}$ of points in \mathbb{R}^d , and a collection $Y = \{y_1, y_2, \dots, y_N\}$ of associated labels from $\{-1, 1\}$. For convenience let us rescale the points so that each point in X lies within the hypercube $[-1, 1]^d$. A feasible solution is a d-dimensional vector β , sometimes called a hypothesis. The objective is to minimize a linear combination $F(\beta, X, Y)$ of the average "hinge" loss function of the points $L(\beta, X, Y) = \frac{1}{N} \sum_{x_i \in X} \max(0, 1 - y_i \beta x_i)$ plus a regularizer $R(\beta)$. We will take the regularizer to be the 2-norm squared of β , as that is a standard choice [18], although this choice is not so important for our purposes. Thus the objective is to minimize:

$$F(\beta, X, Y) = \frac{1}{N} \sum_{x_i \in X} \max(0, 1 - y_i \beta x_i) + \lambda ||\beta||_2^2$$
 (3)

Here the loss function measures how well the hypothesis β explains the labels, and one of the regularizer's purposes is to prevent overfitting. The λ factor intuitively specifies the amount that the loss has to decrease to justify an increase in the norm of β . When either X and Y is understood, for notational convenience, we may drop them from the objective.

Gradient Descent for SMV: In Section A we show that by a straightforward specicialization of a standard convergence analysis for projected gradient descent to SVM one obtains Theorem 1, which bounds the number of descent steps needed to reach a solution with a specified relative error.

Theorem 1. Let $F(\beta)$ be the SVM objective function. Let $\beta^* = \operatorname{argmin}_{\beta} F(\beta)$ be the optimal solution. Let $\widehat{\beta}_s = \frac{1}{s} \sum_{t=0}^{s-1} \beta^{(t)}$. Let $\eta_t = \frac{1}{8\lambda\sqrt{dt}}$. Then if $T \geq \left(\frac{4d^{3/2}}{\epsilon\lambda F(\widehat{\beta}_T)}\right)^2$ then projected gradient

descent quarantees that

$$F(\widehat{\beta}_T) \le (1 + \epsilon)F(\beta^*)$$

Thus if the algorithm returns $\widehat{\beta}$ at the first time t where $t \geq \left(\frac{4d^{3/2}}{\epsilon \lambda F(\widehat{\beta}_t)}\right)^2$, then it achieves relative error at most ϵ .

1.2 Our Results

We start by making some observations about the gradient

$$\nabla F = 2\lambda \beta - \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i \tag{4}$$

of the SVM objective function F. First note the term $2\lambda\beta$ is trivial to compute, so let us focus on the term $G = \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i$. Firstly only those points x_i that satisfy the additive constraint \mathcal{L} contribute to the gradient. Now let us focus on a particular dimension, and use x_{ik} to refer to the value of point x_i in dimension k. Let $L_k^- = \{i \mid i \in \mathcal{L} \text{ and } y_i x_{ik} < 0\}$ denote those points that satisfy \mathcal{L} and whose the gradient in the k^{th} coordinate has negative sign. Conceptually each point in L_k^- pushes the gradient in dimension k up with "force" proportional to its value in dimension k. Let $L_k^+ = \{i \mid x_i \in \mathcal{L} \text{ and } y_i x_{ik} > 0\}$ denote those points that satisfy \mathcal{L} and whose the gradient in the k^{th} coordinate has positive sign. And conceptually each point in L_k^+ pushes the gradient in dimension k down with "force" proportional to its value in dimension k.

Next we note that $G = \frac{1}{N} \sum_{i \in \mathcal{L}} y_i x_i$ is what is called a FAQ-AI(1) query in [25, 2]. [25] gives a relational approximation scheme (RAS) for certain FAQ-AI(1) queries. A RAS is a collection $\{A_{\epsilon}\}$ of relational algorithms where A_{ϵ} achieve $(1+\epsilon)$ -approximation. The results in [25] can be applied to obtain a RAS to compute a $(1+\epsilon)$ approximation \widehat{G}_k^+ to $G_k^+ = \frac{1}{N} \sum_{i \in L_k^+} y_i x_{ik}$, and a RAS to compute a $(1+\epsilon)$ approximation \widehat{G}_k^- to $G_k^- = \frac{1}{N} \sum_{i \in L_k^-} y_i x_{ik}$. However, the results in [25] can not be applied to get a RAS for computing a $(1+\epsilon)$ -approximation to $G = G_k^- + G_k^+$, as it suffers from what [25] calls the subtraction problem. Conceptually the subtraction problem is the fact that good approximations of scalars a and b are generally insufficient to deduce a good approximation of a-b. This subtraction problem commonly arises in natural problems, and several examples are given in [25]. Thus an additional reason for our interest in relational algorithms to compute the (perhaps approximate) gradient of the SVM objective function is that we want to use it as test case to see if there is some way that we can surmount/circumvent the subtraction problem, and obtain a relational algorithm with a reasonable performance guarantee, ideally using techniques that are applicable to other problems in which this subtraction problem arises.

We start with a rather discouraging negative result that shows that we can not surmount the subtraction problem in the context of computing the gradient of the SVM objective problem. In particular, we show in Section 2 that computing an O(1) approximation to the partial derivative in a specified specified dimension is #P-hard, even for acyclic joins. This kills plan A as a relational algorithm to compute the gradient would imply P=#P. This also makes it hard to imagine plan B working out since, assuming $P\neq \#P$, a relational algorithm can't even be sure that it is even approximately headed in the direction of the optimal solution, and thus its not reasonable to expect that we could find a relational algorithm to compute some sort of "pseudo-gradient" that would guarantee convergence on all instances.

Thus it seems we have no choice but to fall back to plan C. That is, we have to try to circumvent (not surmount) the subtraction problem. After some reflection, one reasonable interpretation of

our #P-hardness proof is that it shows that computing the gradient is hard on unstable instances. In this context, intuitively an instance is stable if a nearly optimal solution remains nearly optimal if the points are perturbed slightly. Intuitively one would expect real world instances, where there is a hypothesis β that explains the labels reasonably well, to be relatively stable (some discussion of the stability of SVM instances can be found in [16]). And for instances where there isn't a hypothesis that explains the labels reasonably well, it probably doesn't matter what hypothesis the algorithm returns, as it will likely be discarded by the data scientist anyways. Thus, our plan C will be to seek a gradient descent algorithm that has a similar convergence guarantee to gradient descent on stable instances.

Long story short, the main result of this paper is that this plan C works out. That is we give a relational algorithm that computes a "pseudo-gradient" that guarantees convergence for stable instances at a rate comparable to that achieved by using the actual gradient. The algorithm design can be found in Section 3, and the algorithm analysis can be found in Section 4. Postponing for the moment our formal definition of stability, we state our main result in Theorem 2. The reader should compare Theorem 2 to the analysis of gradient descent in Theorem 1.

Theorem 2. Let X be an (α, δ, γ) -stable SVM instance formed by an acyclic join. Let $\beta^* = \operatorname{argmin}_{\beta} F(\beta)$ be the optimal solution. Then there is a relational algorithm that can compute a pseudo-gradient in time $O(\frac{m}{\epsilon^2}(m^3\log^2(n))^2(d^2mn\log(n)))$, where $\epsilon = \min(\frac{\delta}{8}, \alpha)$. After $T = \left(\frac{256d^{3/2}}{\lambda \delta F(\beta_a, X_a)}\right)^2$ iterations of projected descent using this pseudo-gradient there is a relational algorithm that can compute in time $O(\frac{1}{\epsilon^2}(m^3\log^2(n))^2(d^2mn\log(n)))$ a hypothesis $\hat{\beta}$ such that:

$$F(\widehat{\beta}, X) \le (1 + \gamma)F(\beta^*, X)$$

Main Takeaway Point: In a broader context, we believe that our results suggest that this sort of stability analysis would likely yield useful insight in context of designing relational algorithms for other learning problems in which the subtraction problem arises.

1.3 Related Results

Relational algorithms are known for certain types of Sum of Sums (SumSum) and Sum of Products (SumProd) queries. In particular the Inside-Out algorithm [6] can evaluate a SumProd query in time $O(md^2n^h\log n)$, where m is the number of tables, d is the number of columns, and h is the fractional hypertree width [23] of the query. Note that h=1 for the acyclic joins, and thus Inside-Out is a polynomial time algorithm for acyclic joins. One can reduce SumSum queries to m SumProd queries [2], and thus they be solved in time $O(m^2d^2n^h\log n)$. The Inside-Out algorithm builds on several earlier papers, including [9, 21, 27, 23].

SumSum and SumProd queries with additive inequalites was fist studied in [2]. [2] gave an algorithm with worst-case time complexity $O(md^2n^{m/2}\log n)$. So this is better than the standard practice of forming the design matrix, which has worst-case time complexity $\Omega(dn^m)$. Different flavors of queries with inequalities were also studied [26, 28, 5]. [25] showed that computing even very simple types of SumSum and SumProd queries with a single inequality is NP-hard. But an RAS for special types queries is introduced in [25]. The algorithm in [25] can obtain $(1 + \epsilon)$ approximation for problems such as counting the number of rows on one side of a hyperplane in time $O(\frac{1}{\epsilon^2}(m^3\log^2(n))^2(d^2mn^h\log(n)))$.

Algorithms for linear/polynomial regression on relational data are given in [38, 3, 4, 30, 31] and an algorithm for k-means clustering on relational data is given in [19].

Stability analysis, similar in spirit to our results, has been consider before in clustering problems [37, 17, 32, 7, 13, 20, 29, 36, 10, 12, 14]. For example, the NP-hard k-means, k-medians and k-centers clustering problems are polynomially solvable for instances in which changing the distances of the points by a multiplicative factor of at most 2 does not change the optimal solution [10, 12, 14]

SVM is discuss in covered in almost every introductory machine learning textbook, for example [18]. Optimization methods for learning problems, including variations of gradient descent, are discussed in [39]. A overview of online convex optimization, which we use in our results, can be found in [15, 24].

2 Hardness of Gradient Approximation

Lemma 3. It #P hard to O(1)-approximate the partial derivative of the SVM objective function in a specified dimension.

Proof. We reduce the decision version of the counting knapsack problem to the problem of approximating the gradient of SVM. The input to the decision counting knapsack problem is a set of weights $W = \{w_1, w_2, \dots, w_m\}$, a knapsack size L, and an integer k. The output of the problem is whether there are k different combinations of the items that fit into the knapsack.

We create m+1 tables, each with two columns. The columns of the first m table are (Key, E_i) for T_i and the rows are

$$T_i = \{(1,0), (1, w_i/L), (0,0)\}.$$

The last table has two columns (Key, Value), and it has two rows (1,1), (0,-k). Note that if we take the join of these tables, there will be m+2 columns: (Key, Value, E_1, E_2, \ldots, E_m).

Let $\beta = (0, 0, 1, 1, ..., 1)$ and $\lambda = 0$, so β is 0 on the columns Key and Value and 1 everywhere else. Then we claim, if the gradient of F on the second dimension (Value) is non-negative, then the answer to the original counting knapsack is true, otherwise, it is false.

To see the reason, consider the rows in J: there are 2^m rows in the design matrix that have (1,1) in the first two dimensions and all possible combinations of the knapsack items in the other dimensions. More precisely, the concatenation of (1,1) and w_S for every $S \in [m]$ where w_S is the vector that has w_i/L in the i-th entry if item i is in S or 0 otherwise. Further, J has a single special row with values $(0, -k, 0, 0, \ldots, 0)$. Letting G_2 be the gradient of SVM on the second dimension (column Value), we have,

$$G_2 = \sum_{x \in J: 1 - \beta x \ge 0} x_2$$

For the row with Key = 1 for each $S \in [m]$, we have $1 - \beta x = 1 - \sum_{i \in S} w_i / L \ge 0$ if and only if the items in S fits into the knapsack and $x_2 = 1$. For the single row with Key = 0, we have $1 - \beta x = 1$, and its value on the second dimension is $x_2 = k$. Therefore,

$$G_2 = C_L(w_1, \dots, w_m) - k$$

where C_L is the number of subsets of items fitting into the knapsack of size L. This means if we could approximate the gradient up to any constant factor, we would be able to determine if G_2 is positive or negative, and as a result we would be able to answer the (decision version of) counting knapsack problem, which is #P-hard.

3 Algorithm Design

3.1 Review of Row Counting with a Single Additive Constraint

We now summarize algorithmic results from [25] for two different problems, that we will use as a black box.

In the first problem the input is a collection T_1,\ldots,T_m of tables, a label $\ell\in\{-1,+1\}$, and an additive inequality $\mathcal L$ of the form $\sum_{j\in[d]}g_j(x_j)\geq R$, where each function g_j can be computed in constant time. The output consists of, for each $j\in[d]$ and $e\in D(j)$, where D(j) is the domain of column/feature j, the number $C_{j,v}^\ell$ of rows in the design matrix $J=T_1\bowtie\ldots\bowtie T_m$ that satisfy constraint $\mathcal L$, that have label ℓ , and that have value v in column j. [25] gives a relational algorithm, which we will call the Row Counting Algorithm, that computes a $(1+\epsilon)$ -approximation for each such $\widehat{C}_{j,v}^\ell$ to each $C_{j,v}^\ell$, and that runs in time $O(\frac{m}{\epsilon^2}(m^3\log^2(n))^2(d^2mn^h\log(n)))$ In the second problem the input is a collection T_1,\ldots,T_m of tables, a label $\ell\in\{-1,+1\}$, and

In the second problem the input is a collection T_1, \ldots, T_m of tables, a label $\ell \in \{-1, +1\}$, and an expression in the form of $\sum_{j \in [d]} g_j(x_j)$, where the g_j functions can be computed in constant time. The output consists of, for each $k \in [0, \log_{1+\epsilon} N]$, maximum value of H_k such that the number of points in the design matrix $J = T_1 \bowtie \ldots \bowtie T_m$ with label $\ell \in \{-1, 1\}$ satisfying the additive inequality $\sum_{j \in [d]} g_j(x_j) \geq H_k$ is at least $\lfloor (1+\epsilon)^k \rfloor$. [25] gives an algorithm for this problem, which we will call the Generalized Row Counting Algorithm, that runs in time $O(\frac{1}{\epsilon^2}(m^3\log^2(n))^2(d^2mn^h\log(n)))$. Using the result of the algorithm, for any scalar distance H, it is possible to obtain a row count $\hat{N}(H)$ such that $N(H)/(1+\epsilon) \leq \hat{N}(H) \leq N(H)$, where N(H) is the number of points in the design matrix with label ℓ satisfying the inequality $\sum_{j \in [d]} g_j(x_j) \geq H_k$.

3.2 Overview of Our Approach

Recall from the introduction that the difficulty arises when a \widehat{G}_k^+ is approximately equal to $-\widehat{G}_k^-$. In this case, it would seem that by appropriately perturbing one of L_1^- or L_1^+ by a relatively small amount one could force $G = \widehat{G}^- + \widehat{G}^+$ for this perturbed instance. In which case, if we used $2\lambda\beta^{(t)} + (\widehat{G}^- + \widehat{G}^+)$ as the pseudo-gradient, then it would be the true gradient for a slightly perturbed instance. However, this isn't quite right, as there is an additional issue. If we perturb a point x_i , then the sign of $1 - y_i\beta x_i$ may change, which means this point's contribution to the gradient may discontinuously switch between 0 and $-y_ix_i$. To address this issue, when computing the pseudo-gradient, we use a new instance X' that excludes points that are "close" to the separating hyperplane $1 - y_i\beta x_i = 0$. That is, X' excludes every point that can change sides of the hyperplane in an ϵ -perturbation of each coordinate. This will allow us to formally conclude that if we used $2\lambda\beta^{(t)} + (\widehat{G}^- + \widehat{G}^+)$, where \widehat{G}^- and \widehat{G}^+ are defined on X', as the pseudo-gradient, then it would be the true gradient for a slightly perturbed instance. After the last descent step, we choose the final hypothesis to be the ϵ -perturbation of any computed hypothesis $\beta^{(t)}$, $t \in [0,T]$ that minimizes the SVM objective.

In the analysis we interpret the sequence $\beta^{(0)}, \beta^{(1)}, \dots, \beta^{(T)}$ as solving an online convex optimization problem, and apply known techniques from this area.

3.3 Pseudo-gradient Descent Algorithm

Firstly, in linear time it is straight-forward to determine if the points in X lie in [-1,1], and if not, to rescale so that they do; This can be accomplished by, for each feature, dividing all the values of that feature in all of the input tables by maximum absolute value of that feature. The initial hypothesis $\beta^{(0)}$ is the origin. For any vector v, let u = |v| be a vector such that its entries are the absolute values of v, meaning for all j $u_j = |v_j|$.

Algorithm to Compute the Pseudo-gradient:

- 1. Run the Row Counting Algorithm to compute, for each $j \in [d]$ and $v \in D(j)$, a $(1 + \epsilon)$ approximation $\widehat{C}_{j,v}^-$ to $C_{j,v}^-$, which is the number of rows in $x \in J$ with negative label, satisfying $1 + \beta^{(t)} \cdot x \ge \epsilon \left| \beta^{(t)} \right| \cdot |x|$.
- 2. Run the Row Counting Algorithm to compute, for each $j \in [d]$ and $v \in D(j)$, a $(1 + \epsilon)$ approximation $\widehat{C}_{j,v}^+$ to $C_{j,v}^+$, which is the number of rows in $x \in J$ with positive label, satisfying $1 \beta^{(t)} \cdot x \ge \epsilon |\beta^{(t)}| \cdot |x|$.
- 3. For all $k \in [d]$, compute $\widehat{G}_{k}^{-} = \sum_{v \in D(k): v < 0} v \widehat{C}_{k,v}^{-} \sum_{v \in D(k): v > 0} v \widehat{C}_{k,v}^{+}$.
- 4. For all $k \in [d]$, compute $\widehat{G}_{k}^{+} = \sum_{v \in D(k): v \geq 0} v \ \widehat{C}_{k,v}^{-} \sum_{v \in D(k): v < 0} v \ \widehat{C}_{k,v}^{+}$
- 5. The pseudo-gradient is then

$$\widehat{G} = \frac{\widehat{G}^- + \widehat{G}^+}{N} + \lambda \beta^{(t)}$$

Algorithm for a Single Descent Step: The next hypothesis $\beta^{(t+1)}$ is

$$\beta^{(t+1)} = \Pi_{\mathcal{K}}(\beta^{(t)} - \eta_{t+1}\widehat{G})$$

Here $\eta_t = \frac{1}{\lambda \sqrt{dt}}$ and $\Pi_{\mathcal{K}}(\beta)$ is the projection of β onto a hypersphere \mathcal{K} centered at the origin with radius $\frac{\sqrt{d}}{2\lambda}$. Note that $\Pi_{\mathcal{K}}(\beta)$ is β if $\|\beta\|_2 \leq \frac{\sqrt{d}}{2\lambda}$ and $\frac{\sqrt{d}}{2\lambda \|\beta\|_2}\beta$ otherwise.

Algorithm to Compute the Final Hypothesis: After T-1 descent steps, the algorithm calls the Generalized Row Counting twice for each $t \in [0, T-1]$, with the following inputs:

- $\ell = 1$ and additive expression $1 \beta^{(t)} \cdot x_i \epsilon |\beta^{(t)}| \cdot |x_i|$
- $\ell = -1$ and additive expression $1 + \beta^{(t)} \cdot x_i \epsilon |\beta^{(t)}| \cdot |x_i|$

Note that both of these expressions are equivalent to $1 - y_i \beta^{(t)} \cdot x_i - \epsilon |\beta^{(t)}| \cdot |x_i|$. Let the array H^+ be the output for the first call and H^- be the output for the second call. Note that H^+ and H^- are monotonically decreasing by the the definition of the Generalized Row Counting algorithm. Let L^+ be the largest k such that $H_k^+ \geq 0$ and L^- be the largest k such that $H_k^- \geq 0$. The algorithm then returns as its final hypothesis $\widehat{\beta}$, the hypothesis $\beta^{(\widehat{t})}$ where \widehat{t} is defined by:

$$\widehat{t} = \underset{t \in [T]}{\operatorname{argmin}} \widehat{F}(\beta^{(t)}, X) \tag{5}$$

where

$$\widehat{F}(\beta^{(t)}, X) = \frac{1}{N} \left(\sum_{k=0}^{L^{+}-1} (1+\epsilon)^{k} (H_{k}^{+} - H_{k+1}^{+}) + (1+\epsilon)^{L^{+}} H_{L^{+}}^{+} \right) + \frac{1}{N} \left(\sum_{k=0}^{L^{-}-1} (1+\epsilon)^{k} (H_{k}^{-} - H_{k+1}^{-}) + (1+\epsilon)^{L^{-}} H_{L^{-}}^{-} \right) + \lambda \left\| \beta^{(t)} \right\|_{2}^{2}$$

$$(6)$$

Note that the values L^- , L^+ , H^+ and H^- in the definition of \widehat{F} , in equation (6), all depend upon t, which we suppressed to make the notation somewhat less ugly.

4 Algorithm Analysis

In subsection 4.1 we prove Theorem 5 which bounds the convergence of our project pseudo-gradient descent algorithm in a rather nonstandard way by applying known results on online convex optimization [15, 24]. In subsection 4.2 we introduce our definition of stability and then prove Theorem 2.

4.1 Perturbation Analysis

Before stating Theorem 5 we need some definitions.

Definition 4.

- A point p is an ϵ -perturbation of point q if every component of p is within $(1+\epsilon)$ factor of the corresponding component of q. Meaning in each dimension j we have $(1-\epsilon)q \leq p \leq (1+\epsilon)q$
- A point set X_a is an ϵ -perturbation of a point set X_b if there is a bijection between X_a and X_b such that every point in X_a is an ϵ -perturbation of its corresponding point in X_b .
- Let $\beta^* = \operatorname{argmin}_{\beta} F(\beta, X)$ to be the optimal solution at X.
- For any ϵ -perturbation X_a of X, define $\beta_a^* = \operatorname{argmin}_{\beta} F(\beta, X_a)$ to be the optimal solution at X_a .
- For a given hypothesis β , we call a point x with label y close if there is some ϵ -perturbation x' of x such that $1 y\beta x' < 0$; otherwise it is called far. In other words, a point x with label y is close if $1 y\beta \cdot x < \epsilon |\beta| \cdot |x|$

Theorem 5. Assume our projected pseudo-gradient descent algorithm ran for T-1 descent steps. Then for all hypotheses $\beta \in \mathbb{R}^d$ there exist ϵ -perturbations X_a and X_b of X such that

$$F(\widehat{\beta}, X_a) \le (1 + \epsilon)F(\beta, X_b) + \frac{32d^{3/2}}{\lambda\sqrt{T}}$$

To prove Theorem 5, our main tool is a result from the online convex optimization literature [15, 24].

Theorem 6. [15, 24] Let $g_1, g_2, ..., g_T : \mathbb{R}^n \to \mathbb{R}$ be G-Lipschitz functions over a convex region K, i.e., $||\nabla g_t(\beta)|| \le G$ for all $\beta \in K$ and all t. Then, starting at point $\beta^{(0)} \in \mathbb{R}^n$ and using the update rule of $\beta^{(t)} \leftarrow \Pi_K \left(\beta^{(t-1)} - \eta_t \nabla g_{t-1}(\beta^{(t-1)})\right)$, with $\eta = \frac{D}{G\sqrt{t}}$ for T-1 steps, we have

$$\frac{1}{T} \sum_{t=0}^{T-1} g_t(\beta^{(t)}) \le \frac{1}{T} \sum_{t=0}^{T-1} g_t(\beta^*) + \frac{2DG}{\sqrt{T}}$$
 (7)

for all β^* with $||\beta^{(0)} - \beta^*|| \le D$.

To apply this Theorem 6, we set $g_t = F(\beta^{(t)}, X^{(t)}, Y)$, where $X^{(t)}$ is an ϵ -perturbation of X, such that the pseudo-gradient at X is equal to the true gradient at $X^{(t)}$. We establish the existence of $X^{(t)}$ in Lemma 7. Thus our projected pseudo-gradient descent algorithm updates the hypothesis exactly the same as stated in Theorem 5 (assuming that we use the same upper bounds on D and G). Then in definition 8 we identify the ϵ -permutation Z that minimizes $F(\beta, Z)$, and then in Lemma 9 bound the relative error between $\widehat{F}(\beta, X)$ and $F(\beta, Z)$. Finally this will allow use in Lemma 10 and Lemma 11 we show the existence of X_b and X_a , respectively, that will allow us to conclude the proof of Theorem 5.

Lemma 7. In every descent step t, the computed pseudo-gradient \widehat{G} is the exact gradient of $F(\beta^{(t)}, X^{(t)})$ for some point set $X^{(t)}$ that is an ϵ -perturbation of X.

Proof. To prove the claim, we show how to find a desired $X^{(t)}$ – this is only for the sake of the proof and the algorithm doesn't need to know $X^{(t)}$. We call any point x with label y "far" if it satisfies the inequality

$$1 - y\beta^{(t)} \cdot x \ge \epsilon \left| \beta^{(t)} \right| \cdot |x| \tag{8}$$

, otherwise we call the point "close". Note that for a far point there is no ϵ -perturbation to make the derivative of the loss function 0. That is, for any point x with label y, if $1-y\beta\cdot x\geq \epsilon\sum_{j\in[d]}|\beta_j|\,|x_j|$, then we have $1-y\beta x'\geq 0$ for any x' that is ϵ -perturbation of x. To see this, note that we have $1-y\beta x'=1-\sum_{k=1}^d\beta_k x_k'\geq 1-\sum_{k=1}^d(\beta_k x_k+|\beta_k|\,|x_k|)\geq 0$ because of x' being ϵ -perturbation of x. On the other hand, for all the close points there exists a perturbation x' such that $1-y\beta^{(t)}\cdot x'<0$. We first perturb all of the close points such that they don't have any effect on the gradient.

Next, we need to show a perturbation of the far points for which the \widehat{G} is the gradient of the loss function. Let X_f^+ and X_f^- be the set of far points with positive and negative labels. Let $X_f = X_f^+ \cup X_f^-$. We show the perturbation for each dimension k separately. Based on definition of \widehat{G}_k^+ and \widehat{G}_k^- we have:

$$\begin{split} \widehat{G}_{k}^{+} + \widehat{G}_{k}^{-} &= \sum_{v \in D(k)} v \ \widehat{C}_{k,v}^{-} - \sum_{v \in D(k)} v \ \widehat{C}_{k,v}^{+} \\ &= \sum_{v \in D(k)} v \ (1 \pm \epsilon) C_{k,v}^{-} - \sum_{v \in D(k)} v \ (1 \pm \epsilon) C_{k,v}^{+} \end{split}$$

Note that $C_{k,v}^+$ is the number of points in X_f^+ with value v in dimension k. Therefore,

$$\begin{split} \widehat{G}_{k}^{+} + \widehat{G}_{k}^{-} &= \sum_{v \in D(k)} v \ (1 \pm \epsilon) C_{k,v}^{-} - \sum_{v \in D(k)} v \ (1 \pm \epsilon) C_{k,v}^{+} \\ &= \sum_{x_{i} \in X_{f}^{-}} (1 \pm \epsilon) x_{i,k} - \sum_{x_{i} \in X_{f}^{+}} (1 \pm \epsilon) x_{i,k} \\ &= - \sum_{x_{i} \in X_{f}} (1 \pm \epsilon) y_{i} x_{i,k} \end{split}$$

where the last term is $N \frac{\partial L(\beta^{(t)}, X^{(t)})}{\partial \beta_k^{(t)}}$ where $X^{(t)}$ an ϵ -perturbation of X.

Definition 8. Let $Z^{(t)}$ be an ϵ -perturbation of X such that for all $z_i \in Z^{(t)}$ and for all dimensions k

$$z_{i,k} = \begin{cases} (1 - \epsilon)x_{i,k} & y_i \beta_k^{(t)} \ge 0\\ (1 + \epsilon)x_{i,k} & y_i \beta_k^{(t)} < 0 \end{cases}$$

Note that this ϵ -perturbation minimizes $F(\beta^{(t)}, Z^{(t)})$.

Lemma 9. $\frac{1}{1+\epsilon}F(\beta^{(t)},Z^{(t)}) \leq \widehat{F}(\beta^{(t)},X) \leq F(\beta^{(t)},Z^{(t)}).$

Proof. Consider a value t and let $N^+(\tau) = \left| \left\{ x_i \mid y_i = +1 \text{ and } 1 - \beta^{(t)} \cdot x_i - \epsilon \left| \beta^{(t)} \right| \cdot |x_i| \ge \tau \right\} \right|$, and $N^-(\tau) = \left| \left\{ x_i \mid y_i = -1 \text{ and } 1 + \beta^{(t)} \cdot x_i - \epsilon \left| \beta^{(t)} \right| \cdot |x_i| \ge \tau \right\} \right|$.

Before proving the lemma we prove the following claim: $F(\beta^{(t)}, Z^{(t)}) = \frac{1}{N} \int_{\tau=0}^{\infty} N^+(\tau) d\tau + \frac{1}{N} \int_{\tau=0}^{\infty} N^-(\tau) d\tau + \lambda \left\| \beta^{(t)} \right\|^2$.

Note that based on the definition of $Z^{(t)}$ it is the case that $1 - y_i \beta^{(t)} \cdot z_i = 1 - y_i \beta^{(t)} \cdot x_i - \epsilon \left| \beta^{(t)} \right| \cdot |x_i|$; therefore, $N^+(\tau) = \left| \{ y_i = +1 \in Z^{(t)} \text{ and } 1 - y_i \beta^{(t)} \cdot z_i \ge \tau \} \right|$ and $N^-(\tau) = \left| \{ y_i = -1 \in Z^{(t)} \text{ and } 1 - y_i \beta^{(t)} \cdot z_i \ge \tau \} \right|$. Hence,

$$L(\beta^{(t)}, Z^{(t)}) = \frac{1}{N} \sum_{i} \max(0, 1 - y_i \beta \cdot z_i) = \frac{1}{N} \sum_{i:1 - y_i \beta \cdot z_i \ge 0} 1 - y_i \beta \cdot z_i$$
$$= \frac{1}{N} \sum_{i:1 - y_i \beta \cdot z_i \ge 0} \int_{\tau=0}^{1 - y_i \beta \cdot z_i} d\tau = \frac{1}{N} \int_{\tau=0}^{\infty} \sum_{i:1 - y_i \beta \cdot z_i \ge \tau} d\tau$$
$$= \frac{1}{N} \int_{\tau=0}^{\infty} (N^+(\tau) + N^-(\tau)) d\tau$$

Therefore,

$$F(\beta^{(t)}, Z^{(t)}) = \frac{1}{N} \int_{\tau=0}^{\infty} N^{+}(\tau) d\tau + \frac{1}{N} \int_{\tau=0}^{\infty} N^{-}(\tau) d\tau + \lambda \left\| \beta^{(t)} \right\|^{2}$$
(9)

The number of points with label ℓ satisfying $1 - \ell \beta^{(t)} \cdot x_i - \epsilon \left| \beta^{(t)} \right| \cdot |x_i| \ge \tau$ for any $\tau \in [H_k^{\ell}, H_{k+1}^{\ell})$ is in the range $[\lfloor (1+\epsilon)^k \rfloor, \lfloor (1+\epsilon)^{(k+1)} \rfloor)$. Therefore, the claim follows by replacing $N^+(\tau)$ in Equation (9) with $(1+\epsilon)^k$ for all the values of $\tau \in [H_k^+, H_{k+1}^+)$ and replacing $N^-(\tau)$ in (9) with $(1+\epsilon)^k$ for all the values of $\tau \in [H_k^-, H_{k+1}^-)$.

Lemma 10. For all hypothesis β , there exists an ϵ -perturbation X_b of X such that

$$\min_{s} F(\beta^{(s)}, Z^{(s)}) \le F(\beta, X_b) + \frac{2DG}{\sqrt{T}}$$

Proof. By Theorem 6

$$\frac{1}{T} \sum_{t=0}^{T-1} F(\beta^{(t)}, X^{(t)}) \le \frac{1}{T} \sum_{t=0}^{T-1} F(\beta, X^{(t)}) + \frac{2DG}{\sqrt{T}}$$
(10)

Then

$$\min_{s} F(\beta^{(s)}, Z^{(s)}) \le \frac{1}{T} \sum_{t=0}^{T-1} F(\beta^{(t)}, Z^{(t)}) \le \frac{1}{T} \sum_{t=0}^{T-1} F(\beta^{(t)}, X^{(t)}). \tag{11}$$

The first inequality follows since the minimum is less than the average, and the second inequality follows from the definition of $Z^{(t)}$. Let $u = \operatorname{argmax}_t F(\beta, X^{(t)})$, and $X_b = X^{(u)}$. Then

$$\frac{1}{T} \sum_{t=0}^{T-1} F(\beta, X^{(t)}) \le \max_{t} F(\beta, X^{(t)}) = F(\beta, X_b)$$
 (12)

Thus, combining lines (10), (11) and (12) we can conclude that:

$$\min_{s} F(\beta^{(s)}, Z^{(s)}) \le F(\beta, X_b) + \frac{2DG}{\sqrt{T}}$$

$$\tag{13}$$

Lemma 11. There exists an ϵ -perturbation X_a of X such that

$$F(\widehat{\beta}, X_a) \le (1 + \epsilon) \min_{s} F(\beta^{(s)}, Z^{(s)})$$

Proof. Let $X_a = Z^{(\widehat{t})}$ where

$$F(\widehat{\beta}, X_a) \leq (1 + \epsilon)\widehat{F}(\widehat{\beta}, X)$$
 By Lemma 9
$$= (1 + \epsilon) \min_{s} \widehat{F}(\beta^{(s)}, X)$$
 By definition of $\widehat{\beta}$

$$\leq (1 + \epsilon) \min_{s} F(\beta^{(s)}, Z^{(s)})$$
 By Lemma 9

4.2 Stability Analysis

Our formal definition of stability, which we give in Definition 12 while not unnatural, is surely not the first natural formalization that one would think of. Our formal definition is more or less forced on us, which leads to the type of non-traditional approximation achieved in Theorem 5.

Definition 12. An SVM instance X is (α, δ, γ) -stable for $\delta \leq 1$ if for all X_a and X_b that are α -perturbations of X it is the case that:

- β_a^* is a $(1 + \delta)$ approximation to the optimal objective value at X_b , that is, $F(\beta_a^*, X_b) \leq (1 + \delta) \min_{\beta} F(\beta, X_b)$.
- If β_a is $(1+2\delta)$ approximation to the optimal SVM objective value at X_a then β_a is a $(1+\gamma)$ approximation to the optimal SVM objective value at X_b . That is if $F(\beta_a, X_a) \leq (1+2\delta) \min_{\beta} F(\beta, X_a)$ then $F(\beta_a, X_b) \leq (1+\gamma) \min_{\beta} F(\beta, X_b)$

Proof of Theorem 2. Let $\epsilon \leq \min(\delta/8, \alpha)$.

$$F(\widehat{\beta}, X_a) \leq (1 + \epsilon)F(\beta_a^*, X_b) + \frac{32d^{3/2}}{\lambda\sqrt{T}}$$

$$= (1 + \epsilon)(1 + \delta)F(\beta_a^*, X_a) + \frac{32d^{3/2}}{\lambda\sqrt{T}}$$
By definition of stability
$$= (1 + \epsilon)(1 + \delta)F(\beta_a^*, X_a) + \frac{\delta}{8}F(\widehat{\beta}, X_a)$$

$$\leq \frac{(1 + \delta)(1 + \epsilon)}{1 - \delta/8}F(\beta_a^*, X_a)$$
By definition of T
By algebra
$$\leq (1 + 2\delta)F(\beta_a^*, X_a)$$
by definition of ϵ

Finally since $\widehat{\beta}$ is $(1+2\delta)$ approximate solution at X_a , by the definition of stability, $\widehat{\beta}$ is a $(1+\gamma)$ approximate solution at X.

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A Analysis of Gradient Descent for SVM

Theorem 13 and Corollary 14 give bounds on the number of iterations on projected gradient descent to reach solutions with bounded absolute error and bounded relative error, respectively.

Theorem 13. [15, 24] Let K be a convex body and F be a function such that $\|\nabla F(\beta)\|_2 \leq G$ for $\beta \in K$. Let $\beta^* = \operatorname{argmin}_{\beta \in K} F(\beta)$ be the optimal solution. Let D be an upper bound on $\|\beta^{(0)} - \beta^*\|_2$, the 2-norm distance from the initial candidate solution to the optimal solution. Let $\widehat{\beta}_s = \frac{1}{s} \sum_{t=0}^{s-1} \beta^{(t)}$. Let $\eta_t = \frac{D}{G\sqrt{t}}$. Then after T-1 iterations of projected gradient descent, it must be the case that

$$F(\widehat{\beta}_T) - F(\beta^*) \le \frac{2DG}{\sqrt{T}}$$

Corollary 14. Adopting the assumptions from Theorem 13, if $T \ge \left(\frac{4DG}{\epsilon F(\widehat{\beta}_T)}\right)^2$ then

$$F(\widehat{\beta}_T) \le (1 + \epsilon)F(\beta^*)$$

That is, projected gradient descent achieves relative error ϵ .

The gradient of SVM objective F is

$$\nabla F = 2\lambda\beta - \frac{1}{N}y_i \sum_{i \in \mathcal{L}} x_i$$

where \mathcal{L} is the collection $\{i \mid \beta x_i \leq 1\}$ of indices i where x_i is currently contributing to the objective. Note that in this hinge loss function, the gradient for the points on the hyperplane $1 - \beta x = 0$ does not exist, since the gradient is not continuous at this point. In our formulation we have used the sub-gradient for the points on $1 - \beta x = 0$, meaning for a β on the hyperplane $1 - \beta x = 0$, we have used the limit of the gradient of the points that $1 - \beta' x > 0$ when β' goes to β . For all the points that $1 - \beta' x > 0$, the gradient is x; therefore, the limit is also x.

Assume $\beta^{(0)}$ is the origin and adopt the assumptions of Theorem 13. Then $\nabla F(\beta^*) = 0$ implies for any dimension j

$$\left|\beta_{j}^{*}\right| = \left|\frac{1}{2N\lambda} \sum_{i \in \mathcal{L}} x_{ij}\right| \leq \frac{1}{2\lambda}$$

where the additional subscript of j refers to dimension j. And thus

$$\|\beta^{(0)} - \beta^*\|_2 \le \|\beta^*\|_2 \le \sqrt{d} \max_{j \in [d]} |\beta_j^*| \le \frac{\sqrt{d}}{2\lambda}$$

Thus let us define our convex body \mathcal{K} to be the hypersphere with radius $\frac{\sqrt{d}}{2\lambda}$ centered at the origin.

Thus for $\beta \in \mathcal{K}$,

$$\|\nabla F(\beta)\|_{2} = \sqrt{\sum_{j \in [d]} \left(2\lambda \beta_{j} - \frac{1}{N} \sum_{i \in \mathcal{L}} x_{ij}\right)^{2}}$$

$$\leq \sqrt{\sum_{j \in [d]} 4(\lambda \beta_{j})^{2} + 2\left(\frac{1}{N} \sum_{i \in \mathcal{L}} x_{ij}\right)^{2}} \qquad \text{Since } (a - b)^{2} \leq 2a^{2} + 2b^{2}$$

$$\leq 2\lambda \sqrt{\sum_{j \in [d]} \beta_{j}^{2}} + \sqrt{2} \frac{1}{N} \sum_{j \in [d]} \sum_{i \in \mathcal{L}} |x_{ij}| \qquad \text{Since } \sqrt{\sum_{i} a_{i}^{2}} \leq \sum_{i} |a_{i}|$$

$$\leq \sqrt{d} + \sqrt{2}d$$

$$\leq 4d$$

Theorem 15. Let the convex body K be the hypersphere with radius $\frac{\sqrt{d}}{2\lambda}$ centered at the origin. Let $F(\beta)$ be the SVM objective function. Let $\beta^* = \operatorname{argmin}_{\beta} F(\beta)$ be the optimal solution. Let $\widehat{\beta}_s = \frac{1}{s} \sum_{t=0}^{s-1} \beta^{(t)}$. Let $\eta_t = \frac{1}{8\lambda\sqrt{dt}}$. Then after T-1 iterations of projected gradient descent, it must be the case that

$$F(\widehat{\beta}_T) - F(\beta^*) \le \frac{4d^{3/2}}{\lambda\sqrt{T}}$$

Theorem 1 then follows by a straightforward application of Theorem 15.

B Background

B.1 Fractional edge cover number and output size bounds

In what follows, we consider a conjunctive query Q over a relational database instance I. We use n to denote the size of the largest input relation in Q. We also use Q(I) to denote the output and |Q(I)| to denote its size. We use the query Q and its hypergraph \mathcal{H} interchangeably.

Definition 1 (Fractional edge cover number ρ^*). Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph (of some query Q). Let $B \subseteq \mathcal{V}$ be any subset of vertices. A fractional edge cover of B using edges in \mathcal{H} is a feasible solution $\vec{\lambda} = (\lambda_S)_{S \in \mathcal{E}}$ to the following linear program:

$$\begin{aligned} & \text{min} & & \sum_{S \in \mathcal{E}} \lambda_S \\ & \text{s.t.} & & \sum_{S: v \in S} \lambda_S \geq 1, & \forall v \in B \\ & & \lambda_S \geq 0, & \forall S \in \mathcal{E}. \end{aligned}$$

The optimal objective value of the above linear program is called the fractional edge cover number of B in \mathcal{H} and is denoted by $\rho_{\mathcal{H}}^*(B)$. When \mathcal{H} is clear from the context, we drop the subscript \mathcal{H} and use $\rho^*(B)$.

Given a conjunctive query Q, the fractional edge cover number of Q is $\rho_{\mathcal{H}}^*(\mathcal{V})$ where $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the hypergraph of Q.

Theorem 16 (AGM-bound [11, 23]). Given a full conjunctive query Q over a relational database instance I, the output size is bounded by

$$|Q(I)| \leq n^{\rho^*},$$

where ρ^* is the fractional edge cover number of Q.

Theorem 17 (AGM-bound is tight [11, 23]). Given a full conjunctive query Q and a non-negative number n, there exists a database instance I whose relation sizes are upper-bounded by n and satisfies

$$|Q(I)| = \Theta(n^{\rho^*}).$$

Worst-case optimal join algorithms [40, 34, 35] can be used to answer any full conjunctive query Q in time

$$O(|\mathcal{V}| \cdot |\mathcal{E}| \cdot n^{\rho^*} \cdot \log n). \tag{14}$$

B.2 Tree decompositions, acyclicity, and width parameters

Definition 2 (Tree decomposition). Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. A tree decomposition of \mathcal{H} is a pair (T, χ) where T = (V(T), E(T)) is a tree and $\chi : V(T) \to 2^{\mathcal{V}}$ assigns to each node of the tree T a subset of vertices of \mathcal{H} . The sets $\chi(t)$, $t \in V(T)$, are called the *bags* of the tree decomposition. There are two properties the bags must satisfy

- (a) For any hyperedge $F \in \mathcal{E}$, there is a bag $\chi(t)$, $t \in V(T)$, such that $F \subseteq \chi(t)$.
- (b) For any vertex $v \in \mathcal{V}$, the set $\{t \mid t \in V(T), v \in \chi(t)\}$ is not empty and forms a connected subtree of T.

Definition 3 (acyclicity). A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is acyclic iff there exists a tree decomposition (T, χ) in which every bag $\chi(t)$ is a hyperedge of \mathcal{H} .

When \mathcal{H} represents a join query, the tree T in the above definition is also called the *join tree* of the query. A query is acyclic if and only if its hypergraph is acyclic.

For non-acyclic queries, we often need a measure of how "close" a query is to being acyclic. To that end, we use *width* notions of a query.

Definition 4 (g-width of a hypergraph: a generic width notion [8]). Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph, and $g: 2^{\mathcal{V}} \to \mathbb{R}^+$ be a function that assigns a non-negative real number to each subset of \mathcal{V} . The g-width of a tree decomposition (T, χ) of \mathcal{H} is $\max_{t \in V(T)} g(\chi(t))$. The g-width of \mathcal{H} is the minimum g-width over all tree decompositions of \mathcal{H} . (Note that the g-width of a hypergraph is a Minimax function.)

Definition 5 (Treewidth and fractional hypertree width are special cases of g-width). Let s be the following function: s(B) = |B| - 1, $\forall V \subseteq \mathcal{V}$. Then the treewidth of a hypergraph \mathcal{H} , denoted by $\mathsf{tw}(\mathcal{H})$, is exactly its s-width, and the fractional hypertree width of a hypergraph \mathcal{H} , denoted by $\mathsf{fhtw}(\mathcal{H})$, is the ρ^* -width of \mathcal{H} .

From the above definitions, $\mathsf{fhtw}(\mathcal{H}) \geq 1$ for any hypergraph \mathcal{H} . Moreover, $\mathsf{fhtw}(\mathcal{H}) = 1$ if and only if \mathcal{H} is acyclic.

B.3 Algebraic Structures

In this section, we define some of the algebraic structures used in the paper. First, we discuss the definition of a monoid. A monoid is a semi-group with an identity element. Formally, it is the following.

Definition 18. Fix a set S and let \oplus be a binary operator $S \times S \to S$. The set S with \oplus is a monoid if (1) the operator satisfies associativity; that is, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all $a, b, c \in S$ and (2) there is identity element $e \in S$ such that for all $a \in S$, it is the case that $e \oplus a = a \oplus e = e$.

A commutative monoid is a moniod where the operator \oplus is commutative. That is $a \oplus b = b \oplus a$ for all $a, b \in S$.

Next, we define a semiring.

Definition 19. A semiring is a set R with two operators \oplus and \otimes . The \oplus operator is referred to as addition and the \otimes is referred to as multiplication. This is a semiring if,

- 1. it is the case that R and \oplus are a commutative monoid with 0 as the identity.
- 2. R and \otimes is a monoid with identity 1.
- 3. the multiplication distributes over addition. That is for all $a, b, c \in R$ it is the case that $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ and $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$.
- 4. the 0 element annihilates R. That is, $a \otimes 0 = 0$ and $0 \otimes a = 0$ for all $a \in R$.

A commutative semiring is a semiring where the multiplication is commutative. That is, $a \otimes b = b \otimes a$ for all $a, b \in S$.

B.4 FAQ-AI Query

The input to FAQ-AI problem consists of three components:

- A collection of relational tables $T_1, \ldots T_m$ with real-valued entries. Let $J = T_1 \bowtie T_2 \bowtie \cdots \bowtie T_m$ be the design matrix that arises from the inner join of the tables. Let n be an upper bound on the number of rows in any table T_i , let N be the number of rows in J, and let d be the number of columns in J.
- An FAQ Q(J) that is either a SumProd query or a SumSum query. We define a SumSum query to be a query of the form:

$$Q(J) = \bigoplus_{x \in J} \bigoplus_{i=1}^{d} F_i(x_i)$$

where (R, \oplus, I_0) is a commutative monoid over the arbitrary set R with identity I_0 . We define a SumProd query to be a query of the form:

$$Q(J) = \bigoplus_{x \in J} \bigotimes_{i=1}^{d} F_i(x_i)$$

where $(R, \oplus, \otimes, I_0, I_1)$ is a commutative semiring over the arbitrary set R with additive identity I_0 and multiplicative identity I_1 . In each case, x_i is the entry in column i of x, and F_i is an arbitrary function with range R.

• A collection $\mathcal{L} = \{(G_1, L_1), \dots (G_b, L_b)\}$ where G_i is a collection $\{g_{i,1}, g_{i,2}, \dots g_{i,d}\}$ of d functions that map the column domains to the reals, and each L_i is a scalar.

FAQ-AI(k) is a special case of FAQ-AI when the cardinality of \mathcal{L} is at most k.

The output for the FAQ-AI problem is the result of the query on the subset of the design matrix that satisfies the additive inequalities. That is, the output for the FAQ-AI instance with a SumSum query is:

$$Q(\mathcal{L}(J)) = \bigoplus_{x \in \mathcal{L}(J)} \bigoplus_{i=1}^{d} F_i(x_i)$$
(15)

And the output for the FAQ-AI instance with a SumProd query is:

$$Q(\mathcal{L}(J)) = \bigoplus_{x \in \mathcal{L}(J)} \bigotimes_{i=1}^{d} F_i(x_i)$$
(16)

Here $\mathcal{L}(J)$ is the set of tuples $x \in J$ that satisfy all the additive inequalities in \mathcal{L} , that is for all $i \in [1, b], \sum_{j=1}^{d} g_{i,j}(x_j) \leq L_i$, where x_j is the value of coordinate j of x.

We now illustrate how some of the SVM related problems can be reduced to FAQ-AI(1). First consider the problem of counting the number of negatively labeled points correctly classified by a linear separator. Here each row x of the design matrix J conceptually consists of a point in \mathbb{R}^{d-1} , whose coordinates are specified by the first d-1 columns in J, and a label in $\{1,-1\}$ in column d. Let the linear separator be defined by $\beta \in \mathbb{R}^{d-1}$. A negatively labeled point x is correctly classified if $\sum_{i=1}^{d-1} \beta_i x_i \leq 0$. The number of such points can be counted using SumProd query with one additive inequality as follows: \oplus is addition, \otimes is multiplication, $F_i(x_i) = 1$ for all $i \in [d-1]$, $F_d(x_d) = 1$ if $x_d = -1$, and $F_d(x_d) = 0$ otherwise, $g_{1,j}(x_j) = \beta_j x_j$ for $j \in [d-1]$, $g_{1,d}(x_d) = 0$, and $L_1 = 0$. Next, consider the problem of finding the minimum distance to the linear separator of a correctly classified negatively labeled point. This distance can be computed using a SumProd query with one additive inequality as follows: \oplus is the binary minimum operator, \otimes is addition, $F_i(x_i) = \beta_i x_i$ for all $i \in [d-1]$, $F_d(x_d) = 1$ if $x_d = -1$, and $F_d(x_d) = 0$ otherwise, $g_{1,j}(x_j) = \beta_j x_j$ for $j \in [d-1]$, $g_{1,d}(x_d) = 0$, and $L_1 = 0$.