

A Tighter Relation Between Hereditary Discrepancy and Determinant Lower Bound

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Abstract

In seminal work, Lovász, Spencer, and Vesztergombi [European J. Combin., 1986] proved a lower bound for the hereditary discrepancy of a matrix $A \in \mathbb{R}^{m \times n}$ in terms of the maximum $|\det(B)|^{1/k}$ over all $k \times k$ submatrices B of A . We show algorithmically that this determinant lower bound can be off by at most a factor of $O(\sqrt{\log(m) \cdot \log(n)})$, improving over the previous bound of $O(\log(mn) \cdot \sqrt{\log(n)})$ given by Matoušek [Proc. of the AMS, 2013]. Our result immediately implies $\text{herdisc}(\mathcal{F}_1 \cup \mathcal{F}_2) \leq O(\sqrt{\log(m) \cdot \log(n)}) \cdot \max(\text{herdisc}(\mathcal{F}_1), \text{herdisc}(\mathcal{F}_2))$, for any two set systems $\mathcal{F}_1, \mathcal{F}_2$ over $[n]$ satisfying $|\mathcal{F}_1 \cup \mathcal{F}_2| = m$. Our bounds are tight up to constants when $m = O(\text{poly}(n))$ due to a construction of Pálvölgyi [Discrete Comput. Geom., 2010] or the counterexample to Beck's three permutation conjecture by Newman, Neiman and Nikolov [FOCS, 2012].

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1 Introduction

Given a matrix $A \in \mathbb{R}^{m \times n}$, the *discrepancy* of A is $\text{disc}(A) := \min_{\mathbf{x} \in \{-1, +1\}^n} \|\mathbf{A}\mathbf{x}\|_\infty$. The *hereditary discrepancy* of A is defined as $\text{herdisc}(A) := \max_{S \subseteq [n]} \text{disc}(A_S)$, where A_S denotes the restriction of the matrix A to columns in S . For a set system \mathcal{F} , $\text{disc}(\mathcal{F})$ and $\text{herdisc}(\mathcal{F})$ are defined to be $\text{disc}(A_{\mathcal{F}})$ and $\text{herdisc}(A_{\mathcal{F}})$, where $A_{\mathcal{F}}$ is the incidence matrix of \mathcal{F} .

In seminal work, Lovász, Spencer, and Vesztegombi [LSV86] introduced a powerful tool, known as the *determinant lower bound*, for bounding hereditary discrepancy:

$$\text{detLB}(A) := \max_{k \in \mathbb{N}} \max_{\substack{(S,T) \subseteq [m] \times [n] \\ |S|=|T|=k}} |\det(A_{S,T})|^{1/k},$$

where $A_{S,T}$ denotes the restriction of A to rows in S and columns in T . In particular, they showed that $\text{herdisc}(A) \geq \frac{1}{2} \text{detLB}(A)$ for any matrix A . A reverse relation was established by Matousěk [Mat13], who showed that $\text{herdisc}(A) \leq O(\log(mn) \sqrt{\log(n)}) \cdot \text{detLB}(A)$. However, Matousěk's bound does not match the largest known gap of $\Theta(\log(n))$ between $\text{herdisc}(A)$ and $\text{detLB}(A)$, given by a construction of Pálvolgyi [Pál10] or the counter-example to Beck's three permutation conjecture [NNN12].

Our main result is the following improvement over Matousěk's bound in [Mat13].

Theorem 1.1. *Given a matrix $A \in \mathbb{R}^{m \times n}$, one can efficiently find $\mathbf{x} \in \{+1, -1\}^n$ such that $\|\mathbf{A}\mathbf{x}\|_\infty \leq O(\sqrt{\log(m)} \cdot \log(n) \cdot \text{detLB}(A))$.*

Restricting to an arbitrary subset of the columns of A , one immediately obtains the following:

Corollary 1.2. *For any matrix $A \in \mathbb{R}^{m \times n}$, $\text{herdisc}(A) \leq O(\sqrt{\log(m)} \cdot \log(n) \cdot \text{detLB}(A))$.*

In light of the examples in [Pál10, NNN12] where $\text{herdisc}(A) \geq \Omega(\log n) \cdot \text{detLB}(A)$, Theorem 1.1 is tight up to constants whenever $m = \text{poly}(n)$. For the case where $m \gg \text{poly}(n)$, one cannot hope to improve the $\sqrt{\log(m)}$ dependence on m in Theorem 1.1. In particular, the set system $\mathcal{F} = 2^{[n]}$ has $\text{herdisc}(\mathcal{F}) = n$, $\text{detLB}(\mathcal{F}) = \sqrt{n}$ and therefore $\text{herdisc}(\mathcal{F}) \geq \sqrt{\log(m)} \cdot \text{detLB}(\mathcal{F})$. It remains an open problem, however, whether one can improve the $\sqrt{\log n}$ factor in the later regime.

Hereditary discrepancy of union of set systems. A question of V. Sós (see [LSV86]) asks whether $\text{herdisc}(\mathcal{F}_1 \cup \mathcal{F}_2)$ can be estimated in terms of $\text{herdisc}(\mathcal{F}_1)$ and $\text{herdisc}(\mathcal{F}_2)$, for any set systems \mathcal{F}_1 and \mathcal{F}_2 over $[n]$. This is, however, not possible without any dependence on $m = |\mathcal{F}_1 \cup \mathcal{F}_2|$ or n , as first shown by an example of Hoffman (Proposition 4.11 in [Mat09]). This can also be seen from the examples in [Pál10, NNN12]. In [KMV05], it was shown that $\text{herdisc}(\mathcal{F}_1 \cup \mathcal{F}_2) \leq O(\log(n)) \cdot \text{herdisc}(\mathcal{F}_1)$ when \mathcal{F}_2 contains a single set. For more general set systems, Matousěk [Mat13] proved that $\text{herdisc}(\mathcal{F}) \leq O(\sqrt{t} \log(mn) \sqrt{\log(n)}) \cdot \max_{i \in [t]} (\text{herdisc}(\mathcal{F}_i))$, where $\mathcal{F} = \mathcal{F}_1 \cup \dots \cup \mathcal{F}_t$ and $m = |\mathcal{F}|$.

Theorem 1.1 together with Lemma 4 in [Mat13] immediately imply the following improvement of this result, whose proof is the same as in [Mat13]. For $t = 2$ and $m = \text{poly}(n)$, this bound is tight up to constants.

Theorem 1.3. *Let \mathcal{F} be a system of m sets on $[n]$ such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \dots \cup \mathcal{F}_t$. Then,*

$$\text{herdisc}(\mathcal{F}) \leq O\left(\sqrt{t \log(m) \log(n)}\right) \cdot \max_{i \in [t]}(\text{herdisc}(\mathcal{F}_i)).$$

Approximating hereditary discrepancy. It was shown in [CNN11] that $\text{disc}(A)$ cannot be approximated in polynomial time for an arbitrary matrix $A \in \{0, 1\}^{m \times n}$. The more robust notion of hereditary discrepancy, however, can be approximated within a polylog factor. The best-known result in this direction is a $O(\log(\min(m, n)) \cdot \sqrt{\log(m)})$ -approximation to hereditary discrepancy via the γ_2 -norm [MNT14]. When $m = \text{poly}(n)$, this approximation factor is $O(\log^{3/2}(n))$.

Our result in Theorem 1.1 suggests a potential approach of approximating hereditary discrepancy by approximating the determinant lower bound. There has been a recent line of work in approximating the maximum $k \times k$ subdeterminant for a given matrix A . For $k = \min(m, n)$, Nikolov [Nik15] gave a $2^{O(k)}$ -approximation; for general values of k , Anari and Vuong [AV20] showed a $k^{O(k)}$ -approximation algorithm. If these results can be strengthened to a $2^{O(k)}$ -approximation algorithm for general values of k , then together with Theorem 1.1, one would obtain the first $O(\log(n))$ -approximation algorithm for hereditary discrepancy when $m = \text{poly}(n)$.

Overview of proof of Theorem 1.1. We follow the approaches in [Ban10] and [Mat13]. The key notion to prove Theorem 1.1 is that of *hereditary partial vector discrepancy*, which is defined as follows. Given a matrix $A \in \mathbb{R}^{m \times n}$ with entries a_{ij} for $i \in [m]$ and $j \in [n]$, we consider the following SDP for a subset $S \subseteq [n]$ and a parameter $\lambda \geq 0$:

$$\begin{aligned} \left\| \sum_{j \in S} a_{ij} \mathbf{v}_j \right\|_2^2 &\leq \lambda^2 \quad \forall i \in [m], \\ \sum_{j=1}^n \|\mathbf{v}_j\|_2^2 &\geq |S|/2, \\ \|\mathbf{v}_j\|_2^2 &\leq 1 \quad \forall j \in S, \\ \|\mathbf{v}_j\|_2^2 &= 0 \quad \forall j \in [n] \setminus S. \end{aligned} \quad \text{SDP}(A, S, \lambda)$$

Define the *partial vector discrepancy* of A , denoted as $\text{pvdisc}(A)$, to be the smallest value of λ such that $\text{SDP}(A, [n], \lambda)$ is feasible, and *hereditary partial vector discrepancy* $\text{herpvdisc}(A)$ to be the smallest λ such that $\text{SDP}(A, S, \lambda)$ is feasible for any subset $S \subseteq [n]$.

Using the above definition, we show in Lemma 2.1 of Section 2.1 that the above SDP can be rounded efficiently to obtain a coloring with discrepancy at most $O(\sqrt{\log(m) \log(n)})$.

$\text{herpvdisc}(A)$). We then prove in Lemma 2.3 of Section 2.2 that $\text{herpvdisc}(A) \leq O(\det\text{LB}(A))$, from which Theorem 1.1 immediately follows. We conjecture that $\text{herpvdisc}(A)$ is the same as $\det\text{LB}(A)$ up to constants (Conjecture 2.4).

Notations and preliminaries. Given a matrix $A \in \mathbb{R}^{m \times n}$, its rows will be denoted by $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}^n$. Define $A_{S,T}$ to be the matrix with rows restricted to some subset $S \subseteq [m]$ and columns restricted to some $T \subseteq [n]$, and $A_S := A_{[m],S}$.

Theorem 1.4 (Freedman’s Inequality, Theorem 1.6 in [Fre75]). *Consider a real-valued martingale sequence $\{X_t\}_{t \geq 0}$ such that $X_0 = 0$, and $\mathbb{E}[X_{t+1} | \mathcal{F}_t] = 0$ for all t , where $\{\mathcal{F}_t\}_{t \geq 0}$ is the filtration defined by the martingale. Assume that the sequence is uniformly bounded, i.e., $|X_t| \leq M$ almost surely for all t . Now define the predictable quadratic variation process of the martingale to be $W_t = \sum_{j=1}^t \mathbb{E}[X_j^2 | \mathcal{F}_{j-1}]$ for all $t \geq 1$. Then for all $\ell \geq 0$ and $\sigma^2 > 0$ and any stopping time τ , we have*

$$\mathbb{P} \left[\left| \sum_{j=0}^{\tau} X_j \right| \geq \ell \wedge W_{\tau} \leq \sigma^2 \text{ for some stopping time } \tau \right] \leq 2 \exp \left(- \frac{\ell^2/2}{\sigma^2 + M\ell/3} \right).$$

2 Proof of Theorem 1.1

2.1 The Algorithm

The main result of this subsection is the following lemma.

Lemma 2.1. *Given a matrix $A \in \mathbb{R}^{m \times n}$, there exists a randomized algorithm that w.h.p. constructs a coloring $\mathbf{x} \in \{+1, -1\}^n$ such that $\|\mathbf{A}\mathbf{x}\|_{\infty} \leq O(\sqrt{\log(m)\log(n)} \cdot \text{herpvdisc}(A))$. This implies that $\text{herdisc}(A) \leq O(\sqrt{\log(m)\log(n)} \cdot \text{herpvdisc}(A))$.*

The algorithm in Lemma 2.1 is given in Algorithm 1. This algorithm is a variant of the random walk in [Ban10], using the SDP for hereditary partial vector discrepancy.

Since Lemma 2.1 is invariant under rescaling of the matrix A , we may assume without loss of generality that $\max_{i,j} |a_{i,j}| = 1$. Given a coloring $\mathbf{x} \in [-1, 1]^n$, we say an element $i \in [n]$ is alive if $|x(i)| < 1 - 1/n$. The following lemma from [Ban10] states that the number of alive elements halves after $O(1/s^2)$ steps.

Lemma 2.2 (Lemma 4.1 of [Ban10]). *Let $\mathbf{y} \in [-1, +1]^n$ be an arbitrary fractional coloring with at most k alive variables. Let \mathbf{z} be the fractional coloring obtained by running algorithm 1 with $\mathbf{x}'_0 = \mathbf{y}$ for $T' = 16/s^2$ steps. Then the probability that \mathbf{z} has at least $k/2$ alive variables is at most $1/4$.*

Proof of Lemma 2.1. We first argue that after $T = 400 \log(n)/s^2$ steps, no element is alive with high probability. Divide the time horizon into epochs of size $16/s^2$. For each epoch,

Algorithm 1 HERPVDISCRROUNDING(A)

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1:  $\lambda \leftarrow \text{herpvdisc}(A)$   $\triangleright$  The value of  $\lambda$  can be approximated with a binary search
2:  $\mathbf{x}_0 \leftarrow \mathbf{0} \in \mathbb{R}^n$ ,  $S_0 \leftarrow [n]$ ,  $s \leftarrow 1/m^2 n^2$ ,  $T \leftarrow 200 \log(n)/s^2$ 
3: for  $t = 1, 2, \dots, T$  do
4:    $\mathbf{v}_1, \dots, \mathbf{v}_n \leftarrow \text{SDP}(A, S_{t-1}, \lambda)$ 
5:   Sample  $\mathbf{r} \in \{-1, +1\}^n$  uniformly at random
6:   for  $i \in [n]$  do  $x_t(i) \leftarrow x_{t-1}(i) + s \cdot \langle \mathbf{r}, \mathbf{v}_i \rangle$ 
7:   end for
8:    $S_t \leftarrow S_{t-1}$ 
9:   for  $i \in [n] \setminus S_{t-1}$  do
10:    if  $|x_t(i)| \geq 1 - 1/n$  then
11:       $S_t \leftarrow S_t \setminus \{i\}$ 
12:    end if
13:  end for
14: end for
15: Round  $\mathbf{x}_T$  to a vector  $\mathbf{x} \in \{-1, +1\}^n$ 
16: Return  $\mathbf{x}$ 
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Lemma 2.2 states that regardless of the past, the number of alive elements decreases by at least half with probability at least $3/4$. It follows that no element is alive with high probability after $25 \log(n)$ epochs. Note that when no element is alive for the coloring \mathbf{x}_T , one can round it to a full coloring without changing the discrepancy of each set by more than 1.

Next we prove that with high probability, the discrepancy of each row of A is at most $O(\sqrt{\log(m) \log(n)}) \cdot \lambda$. We consider any $j \in [m]$, and denote $\text{disc}_t(j) = \langle \mathbf{a}_j, \mathbf{x}_t \rangle$ the discrepancy of row j at the end of time step $t \in [T]$. Note that $\mathbb{E}[\text{disc}_t(j) - \text{disc}_{t-1}(j) | \text{disc}_{t-1}(j)] = 0$ and $\mathbb{E}[(\text{disc}_t(j) - \text{disc}_{t-1}(j))^2 | \text{disc}_{t-1}(j)] \leq \lambda^2 s^2$. It follows from Freedman's inequality (Theorem 1.4) that

$$\mathbb{P} \left[|\text{disc}_T(j)| \geq 10 \sqrt{\log(m) \log(n)} \cdot \lambda \right] \leq 1/m^2.$$

So by the union bound, the discrepancy of the obtained coloring is at most $O(\sqrt{\log(m) \log(n)} \cdot \text{herpvdisc}(A))$ with high probability. This completes the proof of Lemma 2.1. \square

2.2 Bounding Partial Vector Discrepancy

In this subsection, we prove the following lemma which upper bounds partial vector discrepancy in terms of the determinant lower bound. The proof can be seen as a simplification of Lemma 8 in [Mat13], which gives a corresponding upper bound for *vector discrepancy* that is weaker by a factor of $\sqrt{\log n}$ due to a bucketing argument that is not needed here.

Lemma 2.3. *For any $A \in \mathbb{R}^{m \times n}$, we have $\text{herpvdisc}(A) \leq O(\text{detLB}(A))$.*

Proof. Recall that $\text{pvdisc}(A)^2$ is the optimal value of the SDP given by

$$\begin{aligned} \min \quad & t \\ \left\| \sum_{j=1}^n a_{ij} \mathbf{v}_j \right\|_2^2 & \leq t \quad \forall i \in [m] \\ \sum_{j=1}^n \|\mathbf{v}_j\|_2^2 & \geq n/2 \\ \|\mathbf{v}_j\|_2^2 & \leq 1 \quad \forall j \in [n]. \end{aligned}$$

By denoting $X_{ij} := \langle \mathbf{v}_i, \mathbf{v}_j \rangle$, we may rewrite this SDP as follows:

$$\begin{aligned} \min \quad & t \\ \langle \mathbf{a}_i \mathbf{a}_i^\top, X \rangle & \leq t \quad \forall i \in [m] \\ \langle I_n, X \rangle & \geq n/2 \\ \langle \mathbf{e}_j \mathbf{e}_j^\top, X \rangle & \leq 1 \quad \forall j \in [n] \\ X & \succeq 0, \end{aligned}$$

where \mathbf{e}_j denotes the vector with 1 on the j -th coordinate and 0 elsewhere. The dual formulation of the above SDP is given by the following:

$$\begin{aligned} \max \quad & n\gamma - \sum_{j=1}^n z_j \\ \sum_{i=1}^m w_i \mathbf{a}_i \mathbf{a}_i^\top + \sum_{j=1}^n z_j \mathbf{e}_j \mathbf{e}_j^\top & \succeq 2\gamma \cdot I_n \\ \sum_{i=1}^m w_i & = 1 \\ \mathbf{w}, \mathbf{z} & \geq 0. \end{aligned}$$

Denote $\lambda := \text{pvdisc}(A)$. By Slater's condition, there exists a feasible dual solution $(\mathbf{w}, \mathbf{z}, \gamma)$ such that $\mathbf{w}, \mathbf{z} \geq 0$ and $n\gamma - \sum_{j=1}^n z_j = \lambda^2$. Indeed, the dual has a feasible interior point (for example, $w_i = 1/m, z_j = 1$ and $\gamma = 0$) and is bounded, since we may rewrite the first constraint as

$$\sum_{i=1}^m w_i \mathbf{a}_i \mathbf{a}_i^\top \succeq \sum_{j=1}^n (2\gamma - z_j) \cdot \mathbf{e}_j \mathbf{e}_j^\top, \tag{1}$$

which implies

$$n\gamma - \sum_{j=1}^n z_j \leq n\gamma - \frac{1}{2} \sum_{j=1}^n z_j \leq \frac{1}{2} \text{tr} \left[\sum_{i=1}^m \mathbf{a}_i \mathbf{a}_i^\top \right].$$

Let \tilde{A} be the matrix obtained from A by multiplying the i -th row by $\sqrt{w_i}$ and $J \subseteq [n]$ be the set of columns for which $z_j < \frac{2}{3}\gamma$. Note that $|J| \geq \frac{1}{3}n$, for otherwise $\sum_{j=1}^n z_j > \frac{2}{3}n \cdot \frac{2}{3}\gamma = n\gamma$. Since for each $j \in J$ we have $2\gamma - z_j \geq \frac{1}{2}\gamma$, for any vector $\mathbf{x} \in \mathbb{R}^J$ it follows by (1):

$$\mathbf{x}^\top \tilde{A}_J^\top \tilde{A}_J \mathbf{x} \geq \frac{1}{2}\gamma \cdot \|\mathbf{x}\|_2^2 \geq \frac{\lambda^2}{2n} \cdot \|\mathbf{x}\|_2^2.$$

This implies that all eigenvalues of $\tilde{A}_J^\top \tilde{A}_J$ are at least $\lambda^2/2n$, so that $\det(\tilde{A}_J^\top \tilde{A}_J) \geq (\lambda^2/2n)^{|J|}$. In the other direction, the Cauchy-Binet formula also gives

$$\begin{aligned} \det(\tilde{A}_J^\top \tilde{A}_J) &= \sum_{\substack{I \subseteq [m] \\ |I|=|J|}} \det(\tilde{A}_{I,J})^2 = \sum_{\substack{I \subseteq [m] \\ |I|=|J|}} \det(A_{I,J})^2 \prod_{i \in I} w_i \\ &\leq \det\text{LB}(A)^{2|J|} \cdot \sum_{\substack{I \subseteq [m] \\ |I|=|J|}} \prod_{i \in I} w_i \leq \det\text{LB}(A)^{2|J|} \cdot \frac{1}{|J|!} \left(\sum_{i=1}^m w_i \right)^{|J|}, \end{aligned}$$

where the last inequality follows as each term $\prod_{i \in I} w_i$ appears $|J|!$ times in $\left(\sum_{i=1}^m w_i \right)^{|J|}$. Since $\sum_{i=1}^m w_i = 1$, we conclude

$$\det\text{LB}(A)^{2|J|} \cdot \frac{1}{|J|!} \geq \det(\tilde{A}_J^\top \tilde{A}_J) \geq (\lambda^2/2n)^{|J|},$$

from which $\det\text{LB}(A) \geq \Omega(\lambda \cdot \sqrt{|J|/n}) = \Omega(\lambda) = \Omega(\text{pvdisc}(A))$. Applying this result to all subsets $S \subseteq [n]$ of the columns of A proves the lemma. \square

We conjecture that the above Lemma 2.3 is tight up to constants.

Conjecture 2.4. *For any matrix $A \in \mathbb{R}^{m \times n}$, we have $\det\text{LB}(A) = \Theta(\text{herpvdisc}(A))$.*

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