

ON GAUSS-LOBATTO INTEGRATION ON THE TRIANGLE

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ABSTRACT. A recent result in [2] on the non-existence of Gauss-Lobatto cubature rules on the triangle is strengthened by establishing a lower bound for the number of nodes of such rules. A method of constructing Lobatto type cubature rules on the triangle is given and used to construct several examples.

1. INTRODUCTION

Recently in [6], motivated by hp -finite element method, Gauss-Lobatto cubature rule on the triangle

$$\triangle := \{(x, y) : 0 \leq x, y, x + y \leq 1\}$$

is studied, which requires $(n+1)(n+2)/2$ nodes, with $(n-1)(n-2)/2$ nodes in the interior, $n-1$ nodes in each side and 1 point at each vertex, and is capable of exactly integrating polynomials of degree $2n-1$. We call such a rule strict Gauss-Lobatto. The main result of [6] shows that such rules do not exist. In this note we consider cubature rules that have nodes in both interior and boundary of the triangle, and establish a lower bound for the number of nodes, from which the non-existence of the strict Gauss-Lobatto rule follows immediately. We also study the structure of rules that attain our lower bound and give a method for constructing cubature rules of degree $2n-1$ with $n-1$ nodes on each side and 1 node at each vertex. The development is based on the observation that cubature rules with nodes on the boundary can be constructed, by restricting to the class of bubble functions (functions that vanish on the boundary), from cubature rules with nodes on the interior. This leads to a bootstrapping scheme for transforming some cubature rules with interior points into higher order rules with a specific number of nodes on the boundary. Several examples are constructed to illustrate the algorithm.

2. RESULTS

For $n \in \mathbb{N}_0$, let Π_n^2 denote the space of polynomials of (total) degree n in two variables. It is known that

$$\dim \Pi_n^2 = \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

Let $W(x, y)$ be a non-negative weight function on the triangle \triangle with finite moments. A cubature rule of precision s with respect to W is a finite sum that satisfies

$$(2.1) \quad \int_{\triangle} f(x, y)W(x, y)dxdy = \sum_{k=1}^N \lambda_k f(x_k, y_k), \quad \forall f \in \Pi_s^2.$$

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We choose W to be the Jacobi weight $W_{\alpha,\beta,\gamma}(x,y) = x^\alpha y^\beta (1-x-y)^\gamma$ for $\alpha, \beta, \gamma > -1$. The case $\alpha = \beta = \gamma = 0$ corresponds to the constant weight. These weight functions are often considered together with the orthogonal polynomials, called Jacobi polynomials on the triangle, that are orthogonal with respect to them; see, for example, [5, p. 86], and [1, 2] in connection with cubature rules.

For the Jacobi weight, it is known that the number of nodes N for (2.1) satisfies

$$(2.2) \quad N \geq \begin{cases} \frac{n(n+1)}{2} + \lfloor \frac{n}{2} \rfloor & \text{if } s = 2n - 1, \\ \frac{n(n+1)}{2} & \text{if } s = 2n - 2. \end{cases}$$

This lower bound is classical for $s = 2n - 2$ ([8]) and given in [1] for $s = 2n - 1$, which agrees with Möller's lower bound for centrally symmetric weight functions [7] as well as [5]. A cubature rule that attains the lower bound is naturally minimal, meaning that it has the smallest number of nodes among all cubature rules of the same degree. The lower bound, however, is most likely not sharp; that is, a minimal cubature could require more points than what the lower bound indicates. Minimal cubature rules are sometimes called Gaussian cubature rules. Their construction is closely related to orthogonal polynomials of several variables. For discussion along this line, see [3, 9] and references therein.

The cubature rules that we consider are of precision s and are of the form

$$(2.3) \quad \int_{\Delta} f(x,y)W_{\alpha,\beta,\gamma}(x,y)dxdy = \sum_{k=1}^{N_0} \lambda_{k,0}f(x_{k,0},y_{k,0}) + \sum_{k=1}^{N_1} \lambda_{k,1}f(x_{k,1},0) \\ + \sum_{k=1}^{N_2} \lambda_{k,2}f(0,y_{k,2}) + \sum_{k=1}^{N_3} \lambda_{k,3}f(x_{k,3},1-x_{k,3}) \\ + \mu_0f(0,0) + \mu_1f(1,0) + \mu_2f(0,1),$$

where $(x_{k,0}, y_{k,0})$ are distinct points in the interior of Δ , $(x_{k,1}, 0)$, $(0, y_{k,2})$, and $(x_{k,3}, 1 - x_{k,3})$ are distinct points on the side $y = 0$, $x = 0$, and $x + y = 1$ (but not on the corners) of Δ , respectively, $\lambda_{k,j} > 0$ and $\mu_i > 0$. Such a cubature has

$$N := N_0 + N_1 + N_2 + N_3 + 3$$

nodes. The main result in [6] states that such a cubature rule does not exist if $s = 2n - 1$ and

$$N_0 = \frac{(n-2)(n-1)}{2}, \quad N_1 = N_2 = N_3 = n-1,$$

which has a total number of nodes $N = (n+2)(n+1)/2$. This follows as an immediate corollary of the following theorem.

Theorem 2.1. *If a cubature rule of the form (2.3) exists with precision $s = 2n - 1$ or $s = 2n$, then*

$$(2.4) \quad N_0 \geq \begin{cases} \frac{n(n-1)}{2} & \text{if } s = 2n - 1, \\ \frac{n(n-1)}{2} + \lfloor \frac{n-1}{2} \rfloor & \text{if } s = 2n, \end{cases}$$

$$(2.5) \quad N_0 + N_i \geq \begin{cases} \frac{n(n-1)}{2} + \lfloor \frac{n-1}{2} \rfloor & \text{if } s = 2n - 1, \\ \frac{n(n+1)}{2} & \text{if } s = 2n, \end{cases} \quad i = 1, 2, 3.$$

Proof. The cubature (2.3) exactly integrates degree s polynomials of the form $f(x, y) = xy(1 - x - y)g(x, y)$ if

$$(2.6) \quad \int_{\Delta} g(x, y)W_{\alpha+1, \beta+1, \gamma+1}(x, y)dxdy = \sum_{k=1}^{N_0} \lambda_{k,0}^* g(x_{k,0}, y_{k,0}), \quad \forall g \in \Pi_{s-3}^2,$$

where $\lambda_{k,0}^* = \lambda_{k,0}x_{k,0}y_{k,0}(1 - x_{k,0} - y_{k,0})$, which is a cubature rule of precision $s - 3$ for the weight function $W_{\alpha+1, \beta+1, \gamma+1}$ so that, by (2.2), N_0 has to satisfy the lower bound in the inequality of (2.4). On the other hand, the cubature (2.3) exactly integrates degree s polynomials of the form $f(x, y) = x(1 - x - y)g(x, y)$ if $\forall g \in \Pi_{s-3}^2$,

$$\int_{\Delta} g(x, y)W_{\alpha+1, \beta, \gamma+1}(x, y)dxdy = \sum_{k=1}^{N_0} \tilde{\lambda}_{k,0} g(x_{k,0}, y_{k,0}) + \sum_{k=1}^{N_1} \lambda_{k,1}^* g(x_{k,1}, 0)$$

where $\tilde{\lambda}_{k,0} = \lambda_{k,0}x_{k,0}(1 - x_{k,0} - y_{k,0})$ and $\lambda_{k,1}^* = \lambda_{k,1}x_{k,1}(1 - x_{k,1})$, which is a cubature rule of precision $s - 2$ for the weight function $W_{\alpha+1, \beta, \gamma+1}(x, y)$, so that $N_0 + N_1$ satisfies the lower bound in (2.2), which gives the inequality of (2.5) for $i = 1$. Similarly, we can derive lower bound for $N_0 + N_2$ and $N_0 + N_3$. \square

One naturally asks if there is any cubature rule that attains the lower bound in the theorem. For $s = 2n - 1$, this asks if there is a cubature rule of precision $2n - 1$ with

$$(2.7) \quad N_0 = \frac{n(n-1)}{2} \quad \text{and} \quad N_i = \left\lfloor \frac{n-1}{2} \right\rfloor, \quad i = 1, 2, 3.$$

We expect that the answer is negative. A heuristic argument can be given as follows: Assume that $N_0 = \frac{n(n-1)}{2}$. Then the proof of the theorem shows that (2.6) is a cubature of degree $2n - 4$ with $n(n-1)/2$ nodes, which is known to exist only for small n . Assume that it does exist. We define a linear functional \mathcal{L}_1 , acting on polynomials of one variable, by

$$(2.8) \quad \mathcal{L}_1 g := \int_{\Delta} g(x)W_{\alpha+1, \beta, \gamma+1}(x, y)dxdy - \sum_{k=1}^{N_0} \lambda_{k,0}x_{k,0}(1 - x_{k,0} - y_{k,0})g(x_{k,0}),$$

where $x_{k,0}, y_{k,0}$ and $\lambda_{k,0}$ are as in (2.3). Applying (2.3) on polynomials of the form $f(x, y) = g(x)x(1 - x - y)$ shows that

$$(2.9) \quad \mathcal{L}_1 g = \sum_{k=1}^{N_1} \lambda_{k,1}^* g(x_{k,1}), \quad \forall g \in \Pi_{2n-3},$$

where $\lambda_{k,1}^* = \lambda_{k,1}x_{k,1}(1 - x_{k,1})$. The functional \mathcal{L}_1 defined in equation (2.8) defines a bilinear form $[p, q]_1 = \mathcal{L}_1(pq)$ that could be indefinite ($[q, q]_1$ is not necessarily positive). If the bilinear form were positive definite on Π_{2n-3} , then (2.9) could be regarded as a quadrature rule of N_1 nodes and of degree $2n - 3$ for \mathcal{L}_1 and, consequently, $N_1 \geq n - 1$ by the standard result in Gaussian quadrature rule, which is stronger than the second equation of (2.7). Thus, in order for (2.7) to hold, we would need \mathcal{L}_1 to be indefinite on Π_{2n-3} , that is, $\mathcal{L}(q^2) = 0$ for some nonzero $q \in \Pi_{n-1}$, and we would need to require that \mathcal{L}_1 has a quadrature rule of degree $2n - 3$ with $N_1 = \lfloor \frac{n-1}{2} \rfloor$ nodes. A simple count of variables (the nodes and weights of the quadrature rule) and restraints (the polynomials that need to be exactly integrated) shows that this is unlikely to happen, although it still might

as the equations are nonlinear. For a linear functional that defines an indefinite bilinear form, the theory of Gaussian quadrature rule breaks down since orthogonal polynomials may not exist and, even they do, they may not have real or simple zeros. In particular, we do not have a lower bound for the number of nodes of a quadrature rule for such a linear functional.

The above argument indicates that if we want the cubature rule (2.3) to have the smallest number of interior points, then the best that we can hope for will be a cubature rule of degree $2n - 1$ that satisfies

$$(2.10) \quad N_0 = \frac{n(n-1)}{2}, \quad \text{and} \quad N_i \geq n-1, \quad 1 \leq i \leq 3.$$

We shall call a cubature rule that attains the lower bound in (2.10) *Gauss-Lobatto cubature rule*. The proof of the lower bound indicates how such a cubature rule can be constructed. Let us also define linear functionals

$$\begin{aligned} \mathcal{L}_2 g &:= \int_{\Delta} g(y) W_{\alpha, \beta+1, \gamma+1}(x, y) dx dy - \sum_{k=1}^{N_0} \lambda_{k,0} y_{k,0} (1 - x_{k,0} - y_{k,0}) g(x_{k,0}), \\ \mathcal{L}_3 g &:= \int_{\Delta} g(x) W_{\alpha+1, \beta+1, \gamma}(x, y) dx dy - \sum_{k=1}^{N_0} \lambda_{k,0} x_{k,0} y_{k,0} g(x_{k,0}). \end{aligned}$$

Then we can summarize the method of construction as follows.

Algorithm. We can follow the following procedure to construct a Gauss-Lobatto cubature rule of form (2.3):

Step 1. Construct a cubature rule of degree $s - 3$ for $W_{\alpha+1, \beta+1, \gamma+1}$ in the form of (2.6) with all nodes in the interior of Δ , and define

$$(2.11) \quad \lambda_{k,0} = \frac{\lambda_{k,0}^*}{(x_{k,0} y_{k,0} (1 - x_{k,0} - y_{k,0}))}.$$

Step 2. Construct Gaussian quadrature rules (2.9) and

$$\mathcal{L}_2 g = \sum_{k=1}^{N_2} \lambda_{k,2}^* g(y_{k,2}), \quad \mathcal{L}_3 g = \sum_{k=1}^{N_3} \lambda_{k,3}^* g(x_{k,3}), \quad \forall g \in \Pi_{2n-3},$$

with respect to the linear functional $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, which gives the nodes of the cubature rule (2.3) on the boundary of the triangle, and define

$$(2.12) \quad \lambda_{k,1} = \frac{\lambda_{k,1}^*}{x_{k,1}(1 - x_{k,1})}, \quad \lambda_{k,2} = \frac{\lambda_{k,2}^*}{y_{k,2}(1 - y_{k,2})}, \quad \lambda_{k,3} = \frac{\lambda_{k,3}^*}{x_{k,3}(1 - x_{k,3})}.$$

Step 3. Finally, the weight μ_0, μ_1 , and μ_2 are determined by setting $f(x, y) = 1, x, y$ in (2.3) and solve the resulted linear system of equations. \square

Proposition 2.2. *Assume, in the above algorithm, that the linear functional $\mathcal{L}_1, \mathcal{L}_2$ and \mathcal{L}_3 are positive definite on Π_{2n-3} and all $x_{k,1}, y_{k,2}, x_{k,3}$ are inside $(0, 1)$. Then the algorithm produces a cubature rule of degree $2n - 1$ in the form (2.3). In particular, if $N_0 = \frac{n(n-1)}{2}$, then the cubature rule is Gauss-Lobatto.*

Proof. We need to show that the cubature rule constructed by the algorithm holds for all $f \in \Pi_{2n-1}^2$, that is, (2.3) holds for all $f \in \Pi_s^2$ with $s = 2n - 1$. A moment of reflection shows that, with $z = 1 - x - y$, Π_s^2 can be decomposed into a direct sum.

$$(2.13) \quad \Pi_s^2 = xyz\Pi_{s-3}^2 \oplus xz\Pi_{s-2}[x] \oplus yz\Pi_{s-2}[y] \oplus xy\Pi_{s-2}[x] \oplus \Pi_1^2,$$

where $\Pi_{s-2}[x]$ and $\Pi_{s-2}[y]$ denote the space of polynomials of one variable in x -variable and y -variable, respectively. If $f \in xyz\Pi_{s-3}^2$, then (2.3) reduces to (2.6), which holds by our construction. If $f \in xz\Pi_{s-2}[x]$, then (2.3) reduces to (2.9) for \mathcal{L}_1 , which holds by our construction. The same holds for $f \in yz\Pi_{s-2}[y]$ and $f \in xy\Pi_{s-2}[x]$, whereas for $f \in \Pi_1^2$, the cubature is verified by Step 3. Thus, by (2.13), the cubature rule holds for all $f \in \Pi_{2n-1}$. \square

It should be pointed out that, as long as we have a cubature rule of degree $2n-4$ with all nodes in the interior of \triangle in the Step 1, regardless if it is a minimal one, the Step 2 and Step 3 could be carried out and Proposition 2.2 applies. Thus, the algorithm can be used to construct cubature rules of degree $2n-1$ in the form (2.3) with $N_1 = N_2 = N_3 = n-1$.

Let us comment on how the steps in the algorithm can be realized.

For Step 1, a cubature with the specification can be constructed by solving moment equations, that is, solving the system of equations formed by setting $g(x, y) = x^i y^j$ for $i + j \leq s-3$ in (2.6) for $x_{k,0}$, $y_{k,0}$ and $\lambda_{k,0}^*$. There have been a number of papers based on this method, see e.g. [11] for the latest result and further references. Another approach is to use a characterization of the cubature rules that attain lower bound in (2.4), which is given in terms of common zeros of certain orthogonal polynomials and can be used to find cubature rules of lower order, see [7, 9]. Not all cubature rules obtained via either methods work for our purpose, since we require that all nodes are inside the domain.

For Step 2, one needs to check that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are positive linear functionals on Π_{2n-3} . Once they are, the standard procedure of constructing Gaussian quadrature rules applies. In particular, we can apply the standard algorithm to generate a sequence of orthogonal polynomials up to degree $n-1$ with respect to \mathcal{L}_i inductively; the nodes of the Gaussian quadrature rule of degree $2n-3$ for \mathcal{L}_i are the zeros of the orthogonal polynomial of degree $n-1$ with respect to \mathcal{L}_i .

In the following we give several examples of Gauss-Lobatto cubature rules of degree 5 and 7 for the unit weight function. The minimal cubature rules for the degree 5 and 7 have nodes 7 and 12, respectively ([4]). For Gauss-Lobatto rules, the number of nodes are necessarily larger, as seen in (2.7).

Example 1. There is a cubature formula of degree 5 with 12 nodes, 3 interior, 2 on each side and 1 at each corner of the triangle. To illustrate our procedure, we shall present the nodes and weights in steps.

The three interior points and weights are given in the first table. While the nodes are those of a cubature rule of degree 2 for the weight function $W_{1,1,1}(x, y) = xy(1-x-y)$, the weights are related to those of the latter cubature rule by (2.11). These nodes are common zeros of quasi-orthogonal polynomials (see [10] for definition) of degree 2 which were found by solving the nonlinear system of equations in Theorem 4.1 of [10].

$x_{k,0}$	$y_{k,0}$	$\lambda_{k,0}$
0.15881702219143	0.19201873632215	0.101342396527698
0.56219234596964	0.19201873632215	0.117181247909596
0.22100936816107	0.55798126367785	0.118066904793533

where the nodes are the common zeros of following three polynomials of degree 2:

$$\begin{aligned} & 19 - \sqrt{105} + x(-91 + \sqrt{105} + 112x) + 2(-7 + \sqrt{105})y, \\ & 49 - 3\sqrt{105} - 175x + 5\sqrt{105}x + 112x^2 - 84y + 4\sqrt{105}y + 224xy, \\ & 154 - 6\sqrt{105} - 301x + 11\sqrt{105}x + 112x^2 - 609y + 7\sqrt{105}y + 560xy + 560y^2. \end{aligned}$$

The nodes on the edges of the triangle, but not on the corners, are the nodes of Gaussian quadrature rules for $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$, respectively, where the weights are related to those of Gaussian quadrature rules by (2.12). They are given in the following three tables:

$x_{k,1}$	$\lambda_{k,1}$
0.3931870086016	0.02991955921794
0.8595419130359	0.01756588222187

where the nodes are zeros of the orthogonal polynomials p_1 of degree 2 for \mathcal{L}_1 ,

$$p_1(x) = x^2 + \frac{1}{448} \left(-469 - 9\sqrt{105} \right) x + \frac{1}{4480} \left(889 + 61\sqrt{105} \right);$$

$y_{k,2}$	$\lambda_{k,2}$
0.4305843026985	0.02290932968619
0.7924406473476	0.02022650113138

where the nodes are zeros of the orthogonal polynomials p_2 of degree 2 for \mathcal{L}_2 ,

$$p_2(x) = x^2 + \frac{3}{46} \left(-29 + \sqrt{105} \right) x + \frac{3}{644} \left(63 + \sqrt{105} \right);$$

$x_{k,3}$	$\lambda_{k,3}$
0.2629899118578	0.02514330117112
0.7030163143652	0.03109870484395

where the nodes are zeros of the orthogonal polynomials p_3 of degree 2 for \mathcal{L}_3 ,

$$p_3(x) = x^2 + \frac{1}{10843} \left(-10997 + 51\sqrt{105} \right) x + \frac{1}{3098} \left(665 - 9\sqrt{105} \right).$$

Finally, the weights μ_1, μ_2, μ_3 are given by

μ_1	μ_2	μ_3
0.0081170837035	0.00326155091683	0.00516753787639

□

The nodes of this Lobatto cubature rule are depicted in Figure 1.

Our next two examples are Gauss-Lobatto cubature rules that are symmetric in the sense that the nodes are invariant under the symmetric group of Δ , or the permutation of $(x, y, 1 - x - y)$. A symmetric Gauss-Lobatto rule for the constant weight takes the form

$$(2.14) \quad \int_{\Delta} f(x, y) dx dy = \sum_{k=1}^{M_0} A_k [f(u_{k,0}, v_{k,0}) + f(v_{k,0}, w_{k,0}) + f(w_{k,0}, u_{k,0})] \\ + \sum_{k=1}^{M_1} B_k [f(u_{k,1}, 0) + f(0, 1 - u_{k,1}) + f(1 - u_{k,1}, u_k)] \\ + C [f(0, 0) + f(1, 0) + f(0, 1)],$$

where $w_{k,0} = 1 - u_{k,0} - v_{k,0}$.

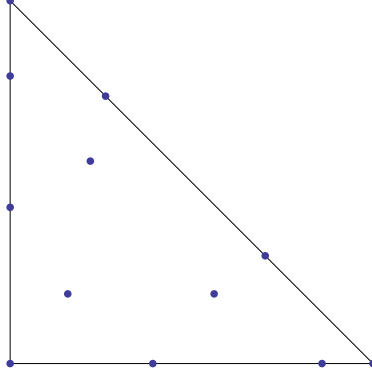


FIGURE 1. Nodes of Gauss-Lobatto cubature rules of degree 5

Both examples are constructed by following the steps in the algorithm. The symmetry makes the construction much easier, since we only need to consider polynomials that are symmetric under the symmetric group of \triangle . In particular, the cubature rule for $W_{1,1,1}(x, y)$ in Step 1 can be found by solving the reduced moment equations of symmetric polynomials. We shall skip details and only list the nodes and weights of these two cubature rules as formulated in (2.14). Their nodes are depicted in Figure 2.

Example 2. Symmetric Gauss-Lobatto cubature rules of degree 5 with 12 nodes. This formula is in the form of (2.14) with $M_0 = 1$ and $M_1 = 2$. The nodes and weights are given below:

$$\begin{aligned} u_{1,0} = v_{1,0} &= \frac{1}{21}(7 - \sqrt{7}), & A_1 &= \frac{7}{720}(14 - \sqrt{7}), \\ u_{1,1} &= \frac{1}{42}\left(21 - \sqrt{21(4\sqrt{7} - 7)}\right), & u_{1,2} &= \frac{1}{42}\left(21 + \sqrt{21(4\sqrt{7} - 7)}\right), \\ B_1 = B_2 &= \frac{1}{720}(7 + 4\sqrt{7}), & C &= \frac{1}{720}(8 - \sqrt{7}). \end{aligned}$$

□

Example 3. Symmetric Gauss-Lobatto cubature rules of degree 7 with 18 nodes. This formula is in the form of (2.14) with $M_0 = 2$ and $M_1 = 3$. The nodes and weights are given below:

$$\begin{aligned} u_{1,0} = v_{1,0} &= \frac{1}{18}(5 - \sqrt{7}), & u_{2,0} = v_{2,0} &= \frac{1}{18}(5 + \sqrt{7}), \\ A_1 &= \frac{1}{17640}(1141 - 94\sqrt{7}), & A_2 &= \frac{1}{17640}(1141 + 94\sqrt{7}), \\ u_{1,1} &= \frac{1}{6}(3 - \sqrt{3}), & u_{2,1} &= \frac{1}{2}, & u_{3,1} &= \frac{1}{6}(3 + \sqrt{3}), \\ B_1 &= \frac{3}{280}, & B_2 &= \frac{4}{315}, & B_3 &= \frac{3}{280}, & C &= \frac{1}{315}. \end{aligned}$$

□

These cubature rules appear to be new (see the list in [4]). Their numbers of nodes are more than the minimal given in (2.2). The existence of higher order Gauss-Lobatto rules depends on the existence of minimal cubature rules of degree $2n - 4$ for the weight $W_{1,1,1}$. The latter cubature rules most likely do not exist for $n \geq 6$, but the algorithm can still be applied to produce cubature formulas with $n - 1$ points in each side of the triangle.

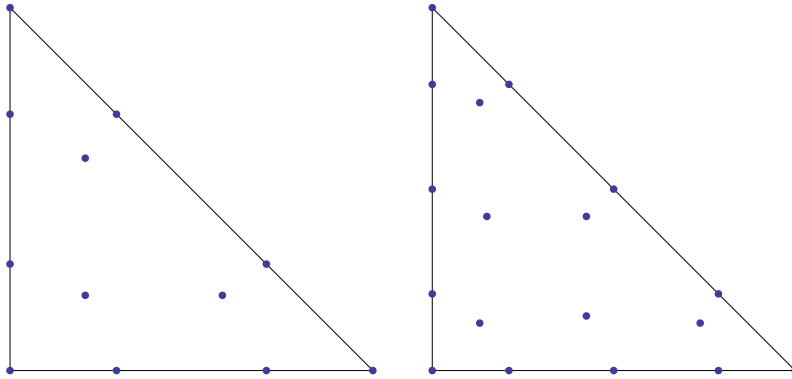


FIGURE 2. Nodes of symmetric Gauss-Lobatto cubature rules of degree 5 and 7

It should be mentioned that there are cubature rules of degree $2n - 1$ in the form of (2.3) that have fewer than $n - 1$ points on each side, naturally with more interior points according to (2.5), such rules cannot be constructed directly by our algorithm. It is possible to modify the algorithm, however, since \mathcal{L}_i for such rules cannot be positive definite on Π_{n-1} but it must be positive definite on a subspace of Π_{n-1} .

Finally, let us mention that our algorithm can also be modified for constructing cubature rules of degree $2n$. This means a cubature rule of degree $2n - 3$ for $W_{\alpha+1, \beta+1, \gamma+1}$ in Step 1, and a quadrature for \mathcal{L}_i of degree $2n - 2$ in Step 2. The quadrature of degree $2n - 2$ is generated by a quasi-orthogonal polynomial of the form $q_n := p_n + \alpha p_{n-1}$, where α is a free parameter which can be fixed by requiring, say, $q_n(1/2) = 0$, which means fixing the middle point on the corresponding side of the triangle as a node of the cubature. We have tried this construction for cubature rules of degree 6 with 4 interior points, 3 on each sides and 1 at each vertex, starting with a cubature rule for $W_{1,1,1}$ of degree 3. The Lobatto type cubature rule of degree 6 that we obtained, however, has three negative weights.

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