ϵ-NASH MEAN FIELD GAME THEORY FOR NONLINEAR STOCHASTIC DYNAMICAL SYSTEMS WITH MAJOR AND MINOR AGENTS[*]

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Abstract. This paper studies a large population dynamic game involving nonlinear stochastic dynamical systems with agents of the following mixed types: (i) a major agent, and (ii) a population of N minor agents where N is very large. The major and minor (MM) agents are coupled via both: (i) their individual nonlinear stochastic dynamics, and (ii) their individual finite time horizon nonlinear cost functions. This problem is approached by the so-called ϵ-Nash Mean Field Game (ϵ-NMFG) theory. A distinct feature of the mixed agent MFG problem is that even asymptotically (as the population size N approaches infinity) the noise process of the major agent causes random fluctuation of the mean field behaviour of the minor agents. To deal with this, the overall asymptotic ($N \to \infty$) mean field game problem is decomposed into: (i) two non-standard stochastic optimal control problems with random coefficient processes which yield forward adapted stochastic best response control processes determined from the solution of (backward in time) stochastic Hamilton-Jacobi-Bellman (SHJB) equations, and (ii) two stochastic coefficient McKean-Vlasov (SMV) equations which characterize the state of the major agent and the measure determining the mean field behaviour of the minor agents. This yields to a Stochastic Mean Field Game (SMFG) system which is in contrast to the deterministic mean field game system of the standard MFG problems with only minor agents. Existence and uniqueness of the solution to the SMFG system (SHJB and SMV equations) is established by a fixed point argument in the Wasserstein space of random probability measures. In the case that minor agents are coupled to the major agent only through their cost functions, the ϵ_N-Nash equilibrium property of the SMFG best responses is shown for a finite N population system where $\epsilon_N = O(1/\sqrt{N})$.

Key words. Mean field games, mixed agents, stochastic dynamic games, stochastic optimal control, decentralized control, stochastic Hamilton-Jacobi-Bellman equation, stochastic McKean-Vlasov equation, Nash equilibria

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1. Introduction. An important class of games is that of dynamic games with a very large number of minor agents in which each agent interacts with the average (or so-called mean field) effect of other agents via couplings in their individual dynamics and individual cost functions. A minor agent is an agent which, asymptotically as the population size goes to infinity, has a negligible influence on the overall system while the overall population's effect on it is significant. Stochastic dynamic games with mean field couplings arise in fields such as wireless power control [17], consensus dynamics [42], flocking [40], charging control of plug-in electric vehicles [33], synchronization of coupled nonlinear oscillators [50], crowd dynamics [8] and economics [49, 11].

For large population stochastic dynamic games with mean field couplings and no major agent, the ϵ-Nash Mean Field Game (ϵ-NMFG) (or Nash Certainty Equivalence (NCE)) theory was originally developed as a decentralized methodology in a series of papers by Huang together with Caines and Malhamé, see [17, 19] for the ϵ-NMFG linear-quadratic-Gaussian (LQG) framework, and [20, 18, 7] for a general formulation

of nonlinear McKean-Vlasov type ϵ-NMFG problems. For this class of game problems a closely related approach has been independently developed by Lasry and Lions [26, 27, 28, 11] where the term Mean Field Games (MFG) was initially used. For models of many firm industry dynamics, Weintraub et. al. proposed the notion of oblivious equilibrium by use of mean field approximations [48]. The ϵ-NMFG framework for LQG systems is extended to systems of agents with ergodic (long time average) costs in [29], while Kolokoltsov et. al. extend the ϵ-NMFG theory to general nonlinear Markov systems [23]. The extension of the ϵ-NMFG framework so as to model the collective system dynamics which include large population of leaders and followers, and an unknown (to the followers) reference trajectory for the leaders is studied in [41]. The reader is referred to the survey paper [4] for some of the research on MFG theory up to 2011.

The central idea of the ϵ-NMFG theory is to specify a certain equilibrium relationship between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent) as the population size goes to infinity [19]. Specifically, in the equilibrium: (i) the individual strategy of each agent is a best response to the infinite population mass effect in the sense of a so-called ϵ-Nash equilibrium, and (ii) the set of strategies collectively replicates the mass effect, this being a dynamical game theoretic fixed point property. The defining property of the ϵ-NMFG equilibrium with individual strategies $\{u_i^o : 1 \leq i \leq N\}$ requires that for any given $\epsilon > 0$, there exists $N(\epsilon)$ such that for any population size $N(\epsilon) \leq N$, when any agent j, $1 \leq j \leq N$, distinct from i employs u_j^o, then agent i can benefit at most ϵ by unilaterally deviating from his strategy u_i^o, and this holds for all $1 \leq i \leq N$. The estimates in [17, 20, 19] show $\epsilon = O(1/\sqrt{N})$ while distinct estimates are obtained in the framework of [23].

A stochastic maximum principle for control problems of mean field type is studied in [1] where the state process is governed by a stochastic differential equation (SDE) in which the coefficients depend on the law of the SDE. The reader is referred to [5, 6] for the analysis of forward–backward stochastic differential equations (FBSDEs) of mean field type and their related partial differential equations.

Recently, Huang [16] introduced a large population LQG dynamic game model with mean field couplings which involves not only a large number of multi-class minor agents but also a major agent with a significant influence on minor agents (see [13, 12, 34] for static cooperative games of agents with different influences or so-called mixed agents). Since all minor agents respond to the same major agent, the mean field behaviour of minor agents in each class is directly impacted by the major agent and hence is a random process [16]. This is in contrast to the situation in the standard MFG models with only minor agents. A state-space augmentation approach for the approximation of the mean field behaviour of the minor agents is taken in order to Markovianize the problem and hence to obtain ϵ-NMFG equilibrium strategies [16]. An extension of the model in [16] to the systems of agents with Markov jump parameters in their dynamics and random parameters in their cost functions is studied in [47] in a discrete-time setting. See also [21] for the extension of the model in [16] to the case of systems with egoistic and altruistic agents.

The model of [16] with finite classes of minor agents is extended in [35] to the case of minor agents parameterized by an infinite set of dynamical parameters where the state augmentation trick cannot be applied to obtain a finite dimensional Markov model. Due to the LQ structure of the problem an appropriate representation for the mean field behaviour of the minor agents as a random process is assumed which

depends linearly on the random initial state and Brownian motion of the major agent. Appropriate approximation of the model by LQG control problems with random parameters in the dynamics and costs yields non-Markovian forward adapted ϵ-NMFG strategies resulting from backward stochastic differential equations (BSDEs) obtained by a stochastic maximum principle [35].

In this paper we extend the LQG model for major and minor (MM) agents [16] to the case of a nonlinear stochastic dynamic games formulation of controlled McKean-Vlasov (MV) type [20]. Specifically, we consider a large population dynamic game involving nonlinear stochastic dynamical systems with agents of the following mixed types: (i) a major agent, and (ii) a population of N minor agents where N is very large. The MM agents are coupled via both: (i) their individual nonlinear stochastic dynamics, and (ii) their individual finite time horizon nonlinear cost functions.

Applications of the major and minor formulation may be found in charging control of plug-in electric vehicles [51, 33], economic and social opinion models with an influential leader (e.g., [9]), and power markets involving large consumers and large utilities together with many domestic consumers represented by smart meter agents and possibly large numbers of renewable energy based generators [22].

A distinctive feature of the mixed agent MFG problem is that even asymptotically (as the population size N approaches infinity) the noise process of the major agent causes random fluctuation of the mean field behaviour of the minor agents [16, 35].

The main contributions of the paper are as follows:

- The overall asymptotic ($N \to \infty$) mean field game problem is decomposed into: (i) two non-standard Stochastic Optimal Control Problems (SOCPs) with random coefficient processes which yield forward adapted stochastic best response control processes determined from the solution of (backward in time) stochastic Hamilton-Jacobi-Bellman (SHJB) equations, and (ii) two stochastic coefficient McKean-Vlasov (SMV) equations which characterize the state of the major agent and the measure determining the mean field behaviour of the minor agents. This yields to a Stochastic Mean Field Game (SMFG) system which is in contrast to the deterministic mean field game system of the standard MFG problems with only minor agents.
- Existence and uniqueness of the solution to the SMFG system (SHJB and SMV equations) is established by a fixed point argument in the Wasserstein space of random probability measures.
- In the case that minor agents are coupled to the major agent only through their cost functions, the ϵ_N-Nash equilibrium property of the SMFG best responses is shown for a finite N population system where $\epsilon_N = O(1/\sqrt{N})$.
- As a particular but important case, the results of Nguyen and Huang [35] for major and minor agent MFG LQG systems with homogeneous population are retrieved in Appendix G in [37].
- Finally, the results of this paper are illustrated with a major and minor agent version of a game model of the synchronization of coupled nonlinear oscillators [50] (see Appendix H in [37]).

It is to be emphasized that the non-standard nature of the SOCPs in (i), which consists of the coupling through the SMV equations in (ii), arises from a distinct feature of the problem formulation. The source of this non-standard nature is the game structure whereby the minor agents are (through the Principle of Optimality) optimizing with respect to the future stochastic evolution of the major agent's state which is partly a result of that agent's future best response control actions. This

feature vanishes in the non-game theoretic setting of one controller with one cost function with respect to the trajectories of all the system components (the classical SOCPs), moreover it also vanishes in the infinite population limit of the standard ϵ-NMFG models with no major agent. This is true for both completely and partially observed SOCPs. The nonstandard feature of the SOCPs here give rise to the analysis of systems with (non necessarily Markovian) stochastic parameters. Here, as in [35, 52], the theory of BSDEs (see in particular [2, 43, 44, 45]) is used in the resulting stochastic dynamic game theory. More specifically, we utilize techniques from [44] which applies the Principle of Optimality to a stochastic nonlinear control problem with random coefficients; this leads to a formulation of a SHJB equation by use of (i) a semi-martingale representation for the corresponding stochastic value function, and (ii) the Itô-Kunita formula. An application of Peng results to portfolio-consumption optimization under habit formation in complete markets is studied in [10].

The organization of the paper is as follows. Section 2 is dedicated to the problem formulation. A McKean-Vlasov approximation for major and minor agent system is studied in Section 3. Section 4 presents a preliminary nonlinear SOCP with random parameters. The SMFG system of equations of the MM agents is given in Section 5, and the existence and uniqueness of its solution is established in Section 6. The ϵ-Nash equilibrium property of the resulting SMFG control laws is studied in Section 7. Finally, Section 8 concludes the paper.

1.1. Notation and Terminology. The following notation will be used throughout the paper. Let $\mathbb{R}^n$ denote the n-dimensional real Euclidean space with the standard Euclidean norm $|\cdot|$ and the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. The transpose of a vector (or matrix) x is denoted by x^T. $\mathrm{tr}(A)$ denotes the trace of a square matrix A. Let $\mathbb{R}^{n \times m}$ be the Hilbert space consisting of all $(n \times m)$-matrices with the inner product $< A, B >:= \mathrm{tr}(AB^T)$ and the norm $|A| :=< A, A >^{1/2}$. The set of non-negative real numbers is denoted by $\mathbb{R}_+$. $T \in [0, \infty)$ is reserved to denote the terminal time. The integer N is reserved to designate the population size of the minor agents. The superscript N for a process (such as state, control or cost function) is used to indicate the dependence on the population size N. We use the subscript 0 for the major agent $\mathcal{A}_0$ and an integer valued subscript for an individual minor agent $\{\mathcal{A}_i : 1 \leq i \leq N\}$. At time $t \geq 0$, (i) the states of agents $\mathcal{A}_0$ and $\mathcal{A}_i$ are respectively denoted by $z_0^N(t)$ and $z_i^N(t)$, $1 \leq i \leq N$, and (ii) for the system configuration of minor agents $(z_1^N(t), \cdots, z_N^N(t))$ the empirical distribution δ_t^N is defined as the normalized sum of Dirac's masses, i.e., $\delta_t^N := (1/N) \sum_{i=1}^N \delta_{z_i^N(t)}$ where $\delta_{(\cdot)}$ is the Dirac measure. $C(S)$ is the set of continuous functions and $C^k(S)$ the set of k-times continuously differentiable functions on S. The symbol ∂_t denotes the partial derivative with respect to variables t. We denote D_x and D_{xx}^2 as the gradient and Hessian operators with respect to the variable x. These are respectively denoted by ∂_x and ∂_{xx}^2 when applied to a function defined on a one-dimensional domain. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space. $\mathbb{E}$ denotes the expectation. The conditional expectation with respect to the σ-field $\mathcal{V}$ is denoted by $\mathbb{E}_\mathcal{V}$. For an Euclidean space H we denote by $L_\mathcal{G}^2([0, T]; H)$ the space of all $\{\mathcal{G}_t\}_{t \geq 0}$-adapted H-valued processes $f(t, \omega)$ such that $\mathbb{E} \int_0^T |f(t, \omega)|^2 dt < \infty$. We use the notation $(\mathbb{E}_\omega h)(z) := \int h(z, \omega) \mathbb{P}_\omega(d\omega)$ for any function $h(z, \omega)$ and sample point $\omega \in \Omega$. Finally, note that we may not display the dependence of random variables or stochastic processes on the sample point $\omega \in \Omega$.

2. Problem Formulation. We consider a dynamic game involving: (i) a major agent $\mathcal{A}_0$, and (ii) a population of N minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$ where N is very large. We assume homogenous minor agents although the modelling may be generalized to the case of multi-class heterogeneous minor agents [20, 16] (see [38]).

The dynamics of the agents are given by the following controlled Itô stochastic differential equations on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$:

$$(2.1) \quad dz_0^N(t) = \frac{1}{N} \sum_{j=1}^{N} f_0[t, z_0^N(t), u_0^N(t), z_j^N(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma_0[t, z_0^N(t), z_j^N(t)]dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,$$

$$(2.2) \quad dz_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} f[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma[t, z_i^N(t), z_0^N(t), z_j^N(t)]dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N,$$

with terminal time $T \in (0, \infty)$ where (i) $z_0^N : [0, T] \to \mathbb{R}^n$ is the state of the major agent $\mathcal{A}_0$ and $z_i^N : [0, T] \to \mathbb{R}^n$ is the state of the minor agent $\mathcal{A}_i$; (ii) $u_0^N : [0, T] \to U_0$ and $u_i^N : [0, T] \to U$ are respectively the control inputs of $\mathcal{A}_0$ and $\mathcal{A}_i$; (iii) $f_0 : [0, T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_0 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $f : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$; (iv) the set of initial states is given by $\{z_j^N(0) = z_j(0) : 0 \leq j \leq N\}$, and (v) the sequence $\{(w_j(t))_{t \geq 0} : 0 \leq j \leq N\}$ denotes $N + 1$ mutually independent standard Brownian motions in $\mathbb{R}^m$. We denote the filtration $\mathcal{F}_t$ as the σ-field generated by the initial states and the Brownian motions up to time t, i.e., $\mathcal{F}_t := \sigma\{z_j(0), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}$. We also set $\mathcal{F}_t^{w_0} = \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}$. These filtrations are augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$.

For $0 \leq j \leq N$ denote $u_{-j}^N := \{u_0^N, \cdots, u_{j-1}^N, u_{j+1}^N, \cdots, u_N^N\}$. The objective of each agent is to minimize its finite time horizon nonlinear cost function given by

$$(2.3) \qquad J_0^N(u_0^N; u_{-0}^N) := \mathbb{E} \int_0^T \Big(\frac{1}{N} \sum_{j=1}^{N} L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] \Big) dt,$$

$$(2.4) \qquad J_i^N(u_i^N; u_{-i}^N) := \mathbb{E} \int_0^T \Big(\frac{1}{N} \sum_{j=1}^{N} L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] \Big) dt,$$

for $1 \leq i \leq N$, where $L_0 : [0, T] \times \mathbb{R}^n \times U_0 \times \mathbb{R}^n \to \mathbb{R}_+$ and $L(z_i, u_i, z_0, x) : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ are the nonlinear cost-coupling functions of the major and minor agents. For $0 \leq j \leq N$, we indicate the dependence of J_j on u_j^N, u_{-j}^N and the population size N by $J_j^N(u_j^N; u_{-j}^N)$.

We note that in the modelling (2.1)-(2.4) the major agent $\mathcal{A}_0$ has a significant influence on minor agents while each minor agent has an asymptotically negligible impact on other agents in a large N population system. The major and minor agents are coupled via both: (i) their individual nonlinear stochastic dynamics (2.1)-(2.2), and (ii) their individual finite time horizon nonlinear cost functions (2.3)-(2.4).

We note that the coupling terms may be written as functionals of the empirical distribution $\delta_{(\cdot)}^N$ by the formula $\int_{\mathbb{R}^n} \phi(x) \delta_t^N(dx) = (1/N) \sum_{i=1}^N \phi(x_i(t))$ for a bounded continuous function ϕ in $\mathbb{R}^n$.

REMARK 2.1. *Under suitable conditions, the results of this paper may be adapted to deal with cost-couplings of the form:*

$$L_0[t, z_0^N(t), u_0^N(t), z_j^N(t), \frac{1}{N}\sum_{j=1}^N z_j^N(t)], \ \ L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t), \frac{1}{N}\sum_{j=1}^N z_j^N(t)],$$

in (2.3)-(2.4).

2.1. Assumptions. Let the empirical distribution of N minor agents' initial states be defined by $F_N(x) = (1/N) \sum_{i=1}^N 1_{\{\mathbb{E}z_i(0) < x\}}$, where $1_{\{\mathbb{E}z_i(0) < x\}} = 1$ if $\mathbb{E}z_i(0) < x$, and $1_{\{\mathbb{E}z_i(0) < x\}} = 0$ otherwise. We enunciate the following assumptions:

(**A1**) The initial states $\{z_j(0) : 0 \le j \le N\}$ are $\mathcal{F}_0$-adapted random variables mutually independent and independent of all Brownian motions $\{(w_j(t))_{t \ge 0} : 0 \le j \le N\}$, and there exists a constant k independent of N such that $\sup_{0 \le j \le N} \mathbb{E}|z_j(0)|^2 \le k < \infty$.

(**A2**) $\{F_N : N \ge 1\}$ converges to a probability distribution F weakly, i.e., for any bounded and continuous function ϕ on $\mathbb{R}^n$ we have $\lim_{N \to \infty} \int_{\mathbb{R}^n} \phi(x) dF_N(x) = \int_{\mathbb{R}^n} \phi(x) dF(x)$.

(**A3**) U_0 and U are compact metric spaces.

(**A4**) The functions $f_0[t, x, u, y]$, $\sigma_0[t, x, y]$, $f[t, x, u, y, z]$ and $\sigma[t, x, y, z]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x, y, z). In addition, their first order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in (y, z).

(**A5**) $f_0[t, x, u, y]$ and $f[t, x, u, y, z]$ are Lipschitz continuous in u.

(**A6**) $L_0[t, x, u, y]$ and $L[t, x, u, y, z]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x, y, z). In addition, their first order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in (y, z).

(**A7**) (*Non-degeneracy Assumption*) There exists a positive constant α such that

$$\sigma_0[t, x, y]\sigma_0^T[t, x, y] \ge \alpha I, \quad \sigma[t, x, y, z]\sigma^T(t, x, y, z) \ge \alpha I, \qquad \forall \ (t, x, y, z),$$

where σ_0 and σ are given in (2.1) and (2.2).

3. McKean-Vlasov Approximation for Mean Field Game Analysis. Motivated by the analysis in Section I.1 of [46] and in Section 8.1 of [20], we take a probabilistic approach to establish the following asymptotic properties: (i) The influence of any minor agent $\mathcal{A}_i$ on any other minor agent $\mathcal{A}_j$ is asymptotically negligible as the population size N goes to infinity, and (ii) In the limit, the effect of the mass of agents on a given minor agent $\mathcal{A}_i$ is that of the behaviour of a mass of predictable generic agents. This is in the form of a single mean field function in the LQG case [17, 19] or a predictable state probability distribution in the nonlinear case [20, 18, 28].

Let $\varphi_0(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \to U_0$ and $\varphi(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \to U$ be two arbitrary $\mathcal{F}_t^{w_0}$-measurable stochastic processes for which we introduce the following assumption:

(**H4**) $\varphi_0(\omega, t, x)$ and $\varphi(\omega, t, x)$ are Lipschitz continuous in x, and $\varphi_0(\omega, t, 0) \in L^2_{\mathcal{F}_t^{w_0}}([0, T]; U_0)$ and $\varphi(\omega, t, 0) \in L^2_{\mathcal{F}_t^{w_0}}([0, T]; U)$.

We assume that $\varphi_0(t,x) := \varphi_0(\omega, t, x)$ and $\varphi(t,x) := \varphi(\omega, t, x)$ are respectively used by the major and minor agents as their control laws in (2.1) and (2.2) (i.e., $u_0 = \varphi_0$ and $u_i = \varphi$ for $1 \le i \le N$). Then we have the following closed-loop equations with random coefficients:

$$d\hat{z}_0^N(t) = \frac{1}{N} \sum_{j=1}^{N} f_0[t, \hat{z}_0^N(t), \varphi_0(t, \hat{z}_0^N(t)), \hat{z}_j^N(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma_0[t, \hat{z}_0^N(t), \hat{z}_j^N(t)]dw_0(t), \quad \hat{z}_0^N(0) = z_0(0), \quad 0 \le t \le T,$$

$$d\hat{z}_i^N(t) = \frac{1}{N} \sum_{j=1}^{N} f[t, \hat{z}_i^N(t), \varphi(t, \hat{z}_i^N(t)), \hat{z}_0^N(t), \hat{z}_j^N(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma[t, \hat{z}_i^N(t), \hat{z}_0^N(t), \hat{z}_j^N(t)]dw_i(t), \quad \hat{z}_i^N(0) = z_i(0), \quad 1 \le i \le N.$$

Under (**A4**)-(**A5**) and (**H4**) there exists a unique solution $(z_0^N(\cdot), \cdots, z_N^N(\cdot))$ to the above system (see Theorem 6.16, Chapter 1 of [53], page 49).

We now introduce the McKean-Vlasov (MV) system

$$d\bar{z}_0(t) = f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \le t \le T,$$

$$d\bar{z}(t) = f[t, \bar{z}(t), \varphi(t, \bar{z}(t)), \bar{z}_0(t), \mu_t]dt + \sigma[t, \bar{z}(t), \bar{z}_0(t), \mu_t]dw(t),$$

with initial condition $(\bar{z}_0(0), \bar{z}(0))$, where for an arbitrary function $g \in C(\mathbb{R}^s)$ for appropriate s, and probability distribution μ_t in $\mathbb{R}^n$ we set

$$g[t, z, \varphi, z_0, \mu_t] = \int_{\mathbb{R}^n} g[t, z, \phi, z_0, x]\mu_t(dx),$$

when the indicated integral converges. In using the MV system it is assumed that the infinite population of minor agents can be modelled by the collection of sample paths of individual agents subject to their individual initial conditions and their individual Brownian sample paths.

In the above MV system $(\bar{z}_0(\cdot), \bar{z}(\cdot), \mu_{(\cdot)})$ is a "consistent solution" if $(\bar{z}_0(\cdot), \bar{z}(\cdot))$ is a solution to the above MV system, μ_t, $0 \le t \le T$, is the conditional law of $\bar{z}(t)$ given $\mathcal{F}_t^{w_0}$ (i.e., $\mu_t := \mathcal{L}(\bar{z}(t)|\mathcal{F}_t^{w_0})$).

Under (**A4**)-(**A5**) and (**H4**) it can be shown by a fixed point argument that there exists a unique solution $(\bar{z}_0(\cdot), \bar{z}(\cdot), \mu_{(\cdot)})$ to the above system (see Theorem 1.1 in [46] or Theorem 6.7 below).

We also introduce the equations

$$d\bar{z}_0(t) = f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \le t \le T,$$

$$d\bar{z}_i(t) = f[t, \bar{z}_i(t), \varphi(t, \bar{z}_i(t)), \bar{z}_0(t), \mu_t]dt + \sigma[t, \bar{z}_i(t), \bar{z}_0(t), \mu_t]dw_i(t), \quad 1 \le i \le N,$$

with initial conditions $\bar{z}_j(0) = z_j(0)$, $0 \le j \le N$, which can be viewed as N independent samples of the MV system above. We develop a decoupling result below such that each $\hat{z}_i^N$, $1 \le i \le N$, has the natural limit $\bar{z}_i$ in the infinite population limit (see Theorem 12 in [20]).

The proof of the following theorem, which is based on the Cauchy-Schwarz inequality, Gronwall's lemma and the conditional independence of minor agents given $\mathcal{F}_t^{w_0}$, is given in Appendix A in [37].

THEOREM 3.1. *[McKean-Vlasov Convergence Result] Assume (**A1**), (**A3**)-(**A5**) and (**H4**) hold. Then we have*

$$(3.1) \qquad \sup_{0 \le j \le N} \sup_{0 \le t \le T} \mathbb{E}|\hat{z}_j^N(t) - \bar{z}_j(t)| = O(1/\sqrt{N}),$$

where the right hand side may depend upon the terminal time T. $\qquad\qquad\square$

4. A Preliminary Nonlinear Stochastic Optimal Control Problem with Random Coefficients. Let $(W(t))_{t \ge 0}$ and $(B(t))_{t \ge 0}$ be mutually independent standard Brownian motions in $\mathbb{R}^m$, with $\mathcal{F}_t^{W,B} := \sigma\{W(s), B(s) : s \le t\}$ and $\mathcal{F}_t^W := \sigma\{W(s) : s \le t\}$ where both are augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$.

We now consider the following single agent nonlinear stochastic optimal control problem (SOCP) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$:

$$(4.1) \qquad dz(t, \omega) = f[t, \omega, z, u]dt + \sigma[t, \omega, z]dW(t) + \varsigma[t, \omega, z]dB(t), \quad 0 \le t \le T,$$

$$(4.2) \qquad \inf_{u \in \mathcal{U}} J(u) := \inf_{u \in \mathcal{U}} \mathbb{E}\Big[\int_0^T L[t, \omega, z(t), u(t)]dt \Big],$$

where the coefficients f, σ, ς and L are random depending on $\omega \in \Omega$ explicitly. In (4.1)-(4.2): (i) $z : [0, T] \times \Omega \to \mathbb{R}^n$ is the state of the agent with $\mathcal{F}_0^{W,B}$-adapted random initial state $z(0)$ such that $\mathbb{E}|z(0)|^2 < \infty$; (ii) $u : [0, T] \times \Omega \to U$ is the control input where U is a compact metric space; (iii) the functions $f : [0, T] \times \Omega \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma, \varsigma : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are $\mathcal{F}_t^W$-adapted stochastic processes; (iv) the admissible control set $\mathcal{U}$ is taken as $\mathcal{U} := \big\{u(\cdot) \in U : u(t)$ is adapted to σ-field $\mathcal{F}_t^{W,B}$ and $\mathbb{E}\int_0^T |u(t)|^2dt < \infty\big\}$. We introduce the following assumptions (see [44]).

(**H1**) $f[t, x, u]$ and $L[t, x, u]$ are a.s. continuous in (x, u) for each t, a.s. continuous in t for each (x, u), $f[t, 0, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$ and $L[t, 0, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}_+)$. In addition, they and all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(**H2**) $\sigma[t, x]$ and $\varsigma[t, x]$ are a.s. continuous in x for each t, a.s. continuous in t for each x and $\sigma[t, 0]$, $\varsigma[t, 0] \in L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^{n \times m})$. In addition, they and all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(**H3**) (*Non-degeneracy Assumption*) There exist non-negative constants α_1 and α_2 such that

$$\sigma[t, \omega, x]\sigma^T[t, \omega, x] \ge \alpha_1 I, \quad \varsigma[t, \omega, x]\varsigma^T(t, \omega, x) \ge \alpha_2 I, \quad a.s., \quad \forall(t, \omega, x),$$

where α_1 or α_2 (but not both) can be zero.

The value function for the SOCP (4.1)-(4.2) is defined by (see [44])

$$(4.3) \qquad \phi\big(t, x(t)\big) = \inf_{u \in \mathcal{U}} \mathbb{E}_{\mathcal{F}_t^W} \int_t^T L[s, \omega, z(s), u(s)]ds,$$

where $x(t)$ is the initial condition for the process $z(\cdot)$. We note that $\phi\big(t, x(t)\big)$ is an $\mathcal{F}_t^W$-adapted process which is sample path continuous a.s. under the assumptions (**H1**)-(**H2**). We assume that there exists an optimal control law $u^o \in \mathcal{U}$ such that

$$\phi\big(t, x(t)\big) = \mathbb{E}_{\mathcal{F}_t^W} \int_t^T L[s, \omega, x(s), u^o(s, \omega, x(s))]ds,$$

where $x(\cdot)$ is the closed-loop solution when the control law u^o is applied. By the Principle of Optimality, it can be shown that the process

$$(4.4) \qquad \zeta(t) := \phi\big(t, x(t)\big) + \int_0^t L[s, \omega, x(s), u^o(s, x(s))]ds,$$

is an $\{\mathcal{F}_t^W\}_{0 \le t \le T}$-martingale (see [3]). Next, by the martingale representation theorem (see Theorem 5.7, Chapter 1, [53]) along the optimal solution $x(\cdot)$ there exists an $\mathcal{F}_t^W$-adapted process $\psi(\cdot, x(\cdot))$ such that

$$(4.5) \qquad \zeta(t) = \phi\big(0, x(0)\big) + \int_0^t \psi^T(s, x(s))dW(s), \qquad t \in [0, T].$$

From (4.4)-(4.5) and the fact that $\phi(T, x(T)) = 0$, it follows that

$$\zeta(T) = \int_0^T L[s, \omega, x(s), u^o(s, x(s))]ds = \phi\big(0, x(0)\big) + \int_0^T \psi^T(s, x(s))dW(s),$$

which gives

$$(4.6) \qquad \phi(0, x(0)) = \int_0^T L[s, \omega, x(s), u^o(s, x(s))]ds - \int_0^T \psi^T(s, x(s))dW(s).$$

Hence, combining (4.4)-(4.6) yields

$$(4.7) \qquad \phi\big(t, x(t)\big) = \int_t^T L[s, \omega, x(s), u^o(s, x(s))]ds - \int_t^T \psi^T\big(s, x(s)\big)dW(s)$$

$$=: \int_t^T \Gamma\big(s, x(s)\big)ds - \int_t^T \psi^T\big(s, x(s)\big)dW(s), \quad t \in [0, T],$$

where $\phi\big(s, x(s)\big)$, $\Gamma\big(s, x(s)\big)$ and $\psi\big(s, x(s)\big)$ are $\mathcal{F}_s^W$-adapted stochastic processes (see the assumed semi-martingale representation form (3.5) in [44]).

Using the extended Itô-Kunita formula (see Appendix B in [37]) and the Principle of Optimality, Peng [44] showed that since $\phi(t, x)$ can be expressed in the semi-martingale form (4.7), and if $\phi(t, x)$, $\psi(t, x)$, $D_x \phi(t, x)$, $D_{xx}^2 \phi(t, x)$ and $D_x \psi(x, t)$ are a.s. continuous in (x, t), then the pair $\big(\phi(s, x), \psi(s, x)\big)$ satisfies the following backward in time stochastic Hamilton-Jacobi-Bellman (SHJB) equation:

$$(4.8) \qquad -d\phi(t, \omega, x) = \Big[H[t, \omega, x, D_x \phi(t, \omega, x)] + \big\langle \sigma[t, \omega, x], D_x \psi(t, \omega, x) \big\rangle$$

$$+ \frac{1}{2} \text{tr}\big(a[t, \omega, x]D_{xx}^2 \phi(t, \omega, x)\big) \Big]dt - \psi^T(t, \omega, x)dW(t, \omega), \quad \phi(T, x) = 0,$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$, $a[t, \omega, x] := \sigma[t, \omega, x]\sigma^T[t, \omega, x] + \varsigma[t, \omega, x]\varsigma^T[t, \omega, x]$, and the stochastic Hamiltonian $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$H[t, \omega, x, p] := \inf_{u \in \mathcal{U}} \big\{ \big\langle f[t, \omega, x, u], p \big\rangle + L[t, \omega, x, u] \big\}.$$

We note that the appearance of the term $\big\langle \sigma[t, \omega, x], D_x \psi(t, \omega, x) \big\rangle$ in equation (4.8) corresponds to the Brownian motion $W(\cdot)$ in the extended Itô-Kunita formula (B.1) for the composition of $\mathcal{F}_t^W$-adapted stochastic processes $\phi(t, \omega, x)$ and $z(t, \omega)$ given in (4.7) and (4.1), respectively.

The solution to the backward in time SHJB equation (4.8) is a unique forward in time $\mathcal{F}_t^W$-adapted pair $(\phi, \psi)(t, x) \equiv \big(\phi(t, \omega, x), \psi(t, \omega, x)\big)$ (see [44, 53]). We omit the proof of the following theorem which closely resembles that of Theorem 4.1 in [44].

THEOREM 4.1. *Assume (**H1**)-(**H3**) hold. Then the SHJB equation (4.8) has a unique solution* $(\phi(t, x), \psi(t, x))$ *in* $\big(L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}), L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^m)\big)$. □

The forward in time $\mathcal{F}_t^W$-adapted optimal control process of the SOCP (4.1)-(4.2) is given by (see [44])

$$(4.9) \qquad u^o(t, \omega, x) := \arg\inf_{u \in U} H^u[t, \omega, x, D_x\phi(t, \omega, x), u]$$

$$= \arg\inf_{u \in U} \big\{ \langle f[t, \omega, x, u], D_x\phi(t, \omega, x) \rangle + L[t, \omega, x, u] \big\}.$$

By a verification theorem approach, Peng [44] showed that if the unique solution $(\phi, \psi)(t, x)$ of the SHJB equation (4.8) satisfies:

(i) for each t, $(\phi, \psi)(t, \cdot)$ is a $C^2(\mathbb{R}^n)$ map from $\mathbb{R}^n$ into $\mathbb{R} \times \mathbb{R}^m$,

(ii) for each x, $(\phi, \psi)(t, x)$ and $(D_x\phi, D_{xx}^2\phi, D_x\psi)(t, x)$ are continuous F_t^W-adapted stochastic processes, then $\phi(x, t)$ coincides with the value function (4.3) of the SOCP (4.1)-(4.2).

5. The Major and Minor Agent Stochastic Mean Field Game System. In the formulation (2.1)-(2.4) all minor agents are reacting to the same major agent and hence the major agent has non-negligible influence on the mean field behaviour of the minor agents. In other words, the noise process of the major agent w_0 causes random fluctuation of the mean-field behaviour of the minor agents and makes it stochastic (see the discussion in Section 2 of [16] for the major and minor agent MFG LQG model).

In this section, we first construct two auxiliary stochastic optimal control problems (SOCP) with random coefficients for the major and a generic minor agent in Sections 5.1 and 5.2, respectively. Then, we present the stochastic mean field system for the major and minor agents game formulation (2.1)-(2.4) via the mean field game consistency condition in Section 5.3.

5.1. Stochastic Optimal Control Problem of the Major Agent. By the McKean-Vlasov convergence result in Theorem 3.1 which indicates that a single minor agent's statistical properties can effectively approximate the empirical distribution produced by all minor agents, we may approximate the empirical distribution of minor agents $\delta_{(\cdot)}^N$ with a stochastic probability measure $\mu_{(\cdot)}$ which depends on the noise process of the major agent w_0.

In this section, let $\mu_t(\omega)$, $0 \le t \le T$, be an exogenous nominal minor agent stochastic measure process such that $\mu_0(dx) := dF(x)$ where F is defined in (**A6.2**). Note that in Section 5.3 $\mu_t(\omega)$ will be characterized via the mean field game consistency condition as the random measure of minor agents' mean field behaviour.

We define the following SOCP (4.1)-(4.2) with $\mathcal{F}_t^{w_0}$-adapted random coefficients from the major agent's model (2.1) and (2.3) in the infinite population limit:

$$(5.1) \qquad dz_0(t) = f_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt + \sigma_0[t, z_0(t), \mu_t(\omega)]dw_0(t, \omega), \quad z_0(0),$$

$$(5.2) \qquad \inf_{u_0 \in \mathcal{U}_0} J_0(u_0) := \inf_{u_0 \in \mathcal{U}_0} \mathbb{E}\Big[\int_0^T L_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt \Big],$$

where we explicitly indicate the dependence of the random measure $\mu_{(\cdot)}$ on the sample point $\omega \in \Omega$.

Step I (*Major Agent's Stochastic Hamilton-Jacobi-Bellman (SHJB) Equation*):
The value function of the major agent's SOCP (5.1)-(5.2) is defined by

$$(5.3) \qquad \phi_0\big(t, x(t)\big) = \inf_{u_0 \in \mathcal{U}_0} \mathbb{E}_{\mathcal{F}_t^{w_0}} \int_t^T L_0[s, z_0(s), u_0(s), \mu_s(\omega)] ds,$$

where $x(t)$ is the initial condition for the process $z_0(s)$ (see (4.3)). As in Section 4, $\phi_0\big(t, x(t)\big)$ has the form (see (4.7))

$$\phi_0\big(t, x(t)\big) = \int_t^T \Gamma_0\big(s, x(s)\big) ds - \int_t^T \psi_0^T\big(s, x(s)\big) dw_0(s), \quad t \in [0, T],$$

where $\phi_0\big(s, x(s)\big)$, $\Gamma_0\big(s, x(s)\big)$ and $\psi_0\big(s, x(s)\big)$ are $\mathcal{F}_s^{w_0}$-adapted stochastic processes. If $\phi_0(t, x)$, $\psi_0(t, x)$, $D_x\phi_0(t, x)$, $D_{xx}^2\phi_0(t, x)$ and $D_x\psi_0(x, t)$ are a.s. continuous in (x, t), then the pair $\big(\phi_0(s, x), \psi_0(s, x)\big)$ satisfies the following stochastic Hamilton-Jacobi-Bellman (SHJB) equation:

$$(5.4) \qquad -d\phi_0(t, \omega, x) = \Big[H_0[t, \omega, x, D_x\phi_0(t, \omega, x)] + \big\langle \sigma_0[t, x, \mu_t(\omega)], D_x\psi_0(t, \omega, x) \big\rangle$$

$$+ \frac{1}{2} \text{tr}\big(a_0[t, \omega, x] D_{xx}^2\phi_0(t, \omega, x)\big) \Big] dt - \psi_0^T(t, \omega, x) dw_0(t, \omega), \quad \phi_0(T, x) = 0,$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$, $a_0[t, \omega, x] := \sigma_0[t, x, \mu_t(\omega)]\sigma_0^T[t, x, \mu_t(\omega)]$, and the stochastic Hamiltonian $H_0 : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$H_0[t, \omega, x, p] := \inf_{u \in \mathcal{U}_0} \big\{ \big\langle f_0[t, x, u, \mu_t(\omega)], p \big\rangle + L_0[t, x, u, \mu_t(\omega)] \big\}.$$

The solution to the backward in time SHJB equation (5.4) is a forward in time $\mathcal{F}_t^{w_0}$-adapted pair $\big(\phi_0(t, x), \psi_0(t, x)\big) \equiv \big(\phi_0(t, \omega, x), \psi_0(t, \omega, x)\big)$ (see [44]).

We note that the appearance of the term $\big\langle \sigma_0[t, x, \mu_t(\omega)], D_x\psi_0(t, \omega, x) \big\rangle$ in equation (5.4) corresponds to the major agent's Brownian motion $w_0(\cdot)$ in the extended Itô-Kunita formula (B.1) for the composition of $\mathcal{F}_t^{w_0}$-adapted processes $\phi_0(t, \omega, x)$ and $z_0(t, \omega)$ in (5.1).

The best response process of the major agent's SOCP (5.1)-(5.2) is given by

$$(5.5) \quad u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \le s \le T}) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, D_x\phi_0(t, \omega, x)]$$

$$\equiv \arg \inf_{u_0 \in U_0} \big\{ \big\langle f_0[t, x, u_0, \mu_t(\omega)], D_x\phi_0(t, \omega, x) \big\rangle + L_0[t, x, u_0, \mu_t(\omega)] \big\},$$

where the infimum exists a.s. here and in all analogous infimizations in the chapter due to the continuity of all functions appearing in $H_0^{u_0}$ and the compactness of U_0. It should be noted that the stochastic best response control u_0^o is a forward in time $\mathcal{F}_t^{w_0}$-adapted process which depends on the Brownian motion w_0 via the stochastic measure $\mu_t(\omega)$, $0 \le t \le T$. The notation in (5.5) indicates that u_0^o at time t depends upon the stochastic measure $\mu_s(\omega)$ on the whole interval $0 \le s \le T$.

Step II (*Major Agent's Stochastic Coefficient McKean-Vlasov (SMV) Equation*): By substituting the best response control process u_0^o (5.5) into the major agent's dynamics (5.1) we get the following stochastic McKean-Vlasov (SMV) dynamics with random coefficients:

$$(5.6) \qquad dz_0^o(t, \omega) = f_0[t, z_0^o, u_0^o(t, \omega, z_0^o), \mu_t(\omega)] dt + \sigma_0[t, z_0^o, \mu_t(\omega)] dw_0(t, \omega),$$

with $z_0^o(0) = z_0(0)$, where f_0 and σ_0 are random processes via the stochastic measure μ and u_0^o.

5.2. Stochastic Optimal Control Problem of the Generic Minor Agent.
As in Section 5.1 let μ_t, $0 \leq t \leq T$, be the exogenous nominal minor agent stochastic measure process approximating the empirical distribution produced by all minor agents in the infinite population limit such that $\mu_0(dx) = dF(x)$ where F is defined in (**A6.2**). We let $z_0^o(\cdot)$ be the solution to the major agent's SMV equation (5.6).

We define the following SOCP (4.1)-(4.2) with $\mathcal{F}_t^{w_0}$-adapted random coefficients from the ith generic minor agent's model (2.2), (2.4) in the infinite population limit:

$$(5.7) \qquad dz_i(t) = f[t, z_i(t), u_i(t), z_0^o(t, \omega), \mu_t(\omega)]dt + \sigma[t, z_i(t), z_0^o(t, \omega), \mu_t(\omega)]dw_i(t),$$

$$(5.8) \qquad \inf_{u_i \in \mathcal{U}} J_i(u_i) := \inf_{u_i \in \mathcal{U}} \mathbb{E}\Big[\int_0^T L[t, z_i(t), u_i(t), z_0^o(t, \omega), \mu_t(\omega)]dt \Big], \quad z_i(0),$$

where we explicitly indicate the dependence of the solution to the major agent's SMV equation $z_0^o(\cdot)$ and the nominal minor agent's random measure $\mu_{(\cdot)}$ on the sample point $\omega \in \Omega$.

Step I (*Generic Minor Agent's Stochastic Hamilton-Jacobi-Bellman (SHJB) Equation*):

The value function of the generic minor agent's SOCP (5.7)-(5.8) is defined by

$$(5.9) \qquad \phi_i\big(t, x(t)\big) = \inf_{u_i \in \mathcal{U}_0} \mathbb{E}_{\mathcal{F}_t^{w_0}} \int_t^T L[s, z_i(s), u_i(s), z_0^o(s, \omega), \mu_s(\omega)]ds,$$

where $x(t)$ is the initial condition for the process $z_i(\cdot)$. As in Section 4, $\phi_i\big(t, x(t)\big)$ has the form (see (4.7))

$$\phi_i\big(t, x(t)\big) = \int_t^T \Gamma_i\big(s, x(s)\big)ds - \int_t^T \psi_i^T\big(s, x(s)\big)dw_0(s), \quad t \in [0, T],$$

where $\phi_i\big(s, x(s)\big)$, $\Gamma_i\big(s, x(s)\big)$ and $\psi_i\big(s, x(s)\big)$ are $\mathcal{F}_s^{w_0}$-adapted stochastic processes. If $\phi_i(t, x)$, $\psi_i(t, x)$, $D_x\phi_i(t, x)$ and $D_{xx}^2\phi_i(t, x)$ are a.s. continuous in (x, t), then the pair $\big(\phi_i(s, x), \psi_i(s, x)\big)$ satisfies the following backward in time stochastic Hamilton-Jacobi-Bellman (SHJB) equation (see (4.8)):

$$(5.10) \quad -d\phi_i(t, \omega, x) = \Big[H[t, \omega, x, D_x\phi_i(t, \omega, x)] + \frac{1}{2}\text{tr}\big(a[t, \omega, x]D_{xx}^2\phi_i(t, \omega, x)\big)\Big]dt$$
$$- \psi_i^T(t, \omega, x)dw_0(t, \omega), \quad \phi_i(T, x) = 0,$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$, $a[t, \omega, x] := \sigma[t, x, z_0^o(t, \omega), \mu_t(\omega)]\sigma^T[t, x, z_0^o(t, \omega), \mu_t(\omega)]$, and the stochastic Hamiltonian $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$H[t, \omega, x, p] := \inf_{u \in \mathcal{U}} \big\{ \big\langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], p \big\rangle + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)] \big\}.$$

The solution to the backward in time SHJB equation (5.10) is a forward in time $\mathcal{F}_t^{w_0}$-adapted pair $\big(\phi_i(t, x), \psi_i(t, x)\big) \equiv \big(\phi_i(t, \omega, x), \psi_i(t, \omega, x)\big)$ (see [44]). We note that since the coefficients of the SOCP (5.7)-(5.8) are $\mathcal{F}_t^{w_0}$-adapted random processes we have the major agent's Brownian motion w_0 in (5.10) which allows us to seek for a forward in time adapted solution to the backward in time SHJB equation (5.10).

It is important to note that in (5.10) unlike the major agent's SHJB equation (5.4) we do not have the term $\big\langle \sigma[t, x, z_0^o(t, \omega), \mu_t(\omega)]D_x\psi_i(t, \omega, x)\big\rangle$ since the coefficients in the minor agent's model (5.7)-(5.8) are $\mathcal{F}_t^{w_0}$-adapted random processes depending

upon the major agent's Brownian motion (w_0) which is independent of the minor agent's Brownian motion (w_i) (see the extended Itô-Kunita formula (B.1)).

As in Section 5.1, the stochastic best response process of the minor agent's SOCP (5.7)-(5.8) is

(5.11)
$$u_i^o(t, \omega, x) \equiv u_i^o(t, x | \{z_0^o(s, \omega), \mu_s(\omega)\}_{0 \le s \le T}) := \arg \inf_{u \in U} H^u[t, \omega, x, u, D_x \phi_i(t, \omega, x)]$$
$$\equiv \arg \inf_{u \in U} \{\langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], D_x \phi_i(t, \omega, x)\rangle + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)]\},$$

where the infimum exists a.s. here and in all analogous infimizations in the chapter due to the continuity of all functions appearing in H^u and the compactness of U. It should be noted that the stochastic best response process of the generic minor agent u_i^o is a forward in time $\mathcal{F}_t^{w_0}$-adapted random process which depends on the Brownian motion w_0 via the major agent's state $z_0^o(t, \omega)$ and the stochastic measures $\mu_t(\omega)$, $0 \le t \le T$. The notation in (5.11) indicates that u_i^o at time t depends upon $z_0^o(s, \omega)$ and $\mu_s(\omega)$ on the whole interval $0 \le s \le T$.

Step II (*Minor Agent's Stochastic Coefficient McKean-Vlasov (SMV) and Stochastic Coefficient Fokker-Planck-Kolmogorov (SFPK) Equations*): By substituting the best response control process u_i^o (5.11) into the minor agent's dynamics (5.7) we get the following stochastic McKean-Vlasov (SMV) dynamics with random coefficients:

(5.12)
$$dz_i^o(t, \omega, \omega') = f[t, z_i^o, u_i^o(t, \omega, z_i), z_0^o(t, \omega), \mu_t(\omega)]dt$$
$$+ \sigma[t, z_i^o, z_0^o(t, \omega), \mu_t(\omega)]dw_i(t, \omega'), \quad z_i^o(0) = z_i(0),$$

where f and σ are random processes via z_0^o, μ, and the best response control process u_i^o which all depend on the Brownian motion of the major agent (w_0).

Based on the McKean-Vlasov approximation in Section 3, the generic agent's statistical properties can effectively approximate the empirical distribution produced by all minor agents in a large population system. Hence, we obtain a new stochastic measure $\hat{\mu}_t(\omega)$ for the mean field behaviour of minor agents as the conditional law of the generic minor agent's process $z_i^o(t, \omega)$ given $\mathcal{F}_t^{w_0}$. We characterize $\hat{\mu}_t(\omega)$, $0 \le t \le T$, by $P(z_i^o(t, \omega) \le \alpha | \mathcal{F}_t^{w_0}) = \int_{-\infty}^{\alpha} \hat{\mu}(t, \omega, dx)$ a.s. for all $\alpha \in \mathbb{R}^n$ and $0 \le t \le T$, with $\hat{\mu}_0(dx) = \mu_0(dx) = dF(x)$ where F is defined in (**A6.2**).

An equivalent method to characterize the SMV of the generic minor agent is to express (5.12) in the form of stochastic Fokker-Planck-Kolmogorov (SFPK) equation with random coefficients:

(5.13)
$$d\hat{p}(t, \omega, x) = \Big(- \langle D_x, f[t, x, u_i^o(t, \omega, x), z_0^o(t, \omega), \mu_t(\omega)]\hat{p}(t, \omega, x)\rangle$$
$$+ \frac{1}{2}\text{tr}\langle D_{xx}^2, a[t, \omega, x]\hat{p}(t, \omega, x)\rangle\Big)dt, \quad \hat{p}(0, x) = p_0(x),$$

in $[0, T] \times \mathbb{R}^n$ where $p(t, \omega, x)$ is the conditional probability density of $z_i^o(t, \omega)$ given $\mathcal{F}_t^{w_0}$. By the the McKean-Vlasov approximation (see Section 3) it is possible to characterize the mean field behaviour of minor agents in terms of generic agent's density function $\hat{p}(t, \omega, x)$. The reason that the generic minor agent's FPK equation (5.13) does not include the Itô integral term with respect to w_i is due to the fact that $p(t, \omega, x)$ is the conditional probability density given $\mathcal{F}_t^{w_0}$, and the independence of the Brownian motions w_0 and w_i, $1 \le i \le N$.

The density function $\hat{p}(t, \omega, x)$ generates the random measure of the minor agent's mean field behaviour $\hat{\mu}_t(\omega)$ such that $\hat{\mu}(t, \omega, dx) = \hat{p}(t, \omega, x)dx$ (a.s.), $0 \le t \le T$.

We note that the major agent's SOCP (5.1)-(5.2) and minor agent's SOCP (5.7)-(5.8) may be written with respect to the random density $p(t, \omega, x)$ of the stochastic measure $\mu(t, \omega, dx)$ by $\mu(t, \omega, dx) = p(t, \omega, x)dx$ (a.s.), $0 \leq t \leq T$.

5.3. The Mean Field Game Consistency Condition. Based on the mean field game (MFG) or Nash certainty equivalence (NCE) consistency (see [20] and [28]), we close the "measure and control" mapping loop by setting $\hat{\mu}_t(\omega) = \mu_t(\omega)$ a.s., $0 \leq t \leq T$, or $\hat{p}(t, \omega, x) = p(t, \omega, x)$ a.s. for $(t, x) \in [0, T] \times \mathbb{R}^n$. The MFG consistency is demonstrated in: (i) the major agent's stochastic mean field game (SMFG) system

$$(5.14) \quad [\text{MFG-SHJB}] \quad -d\phi_0(t, \omega, x) = \Big[H_0[t, \omega, x, D_x\phi_0(t, \omega, x)]$$

$$+ \big\langle \sigma_0[t, x, \mu_t(\omega)], D_x\psi_0(t, \omega, x) \big\rangle + \frac{1}{2}\text{tr}\big(a_0[t, \omega, x]D_{xx}^2\phi_0(t, \omega, x)\big) \Big] dt$$

$$- \psi_0^T(t, \omega, x)dw_0(t, \omega), \quad \phi_0(T, x) = 0,$$

$$(5.15) \quad u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \leq s \leq T})$$

$$:= \arg\inf_{u_0 \in U_0} \big\{ \big\langle f_0[t, x, u_0, \mu_t(\omega)], D_x\phi_0(t, \omega, x) \big\rangle + L_0[t, x, u_0, \mu_t(\omega)] \big\},$$

$$(5.16) \quad [\text{MFG-SMV}] \quad dz_0^o(t, \omega) = f_0[t, z_0^o, u_0^o(t, \omega, z_0^o), \mu_t(\omega)]dt$$

$$+ \sigma_0[t, z_0^o, \mu_t(\omega)]dw_0(t, \omega), \quad z_0^o(0) = z_0(0),$$

together with (ii) the minor agents' SMF system

$$(5.17) \quad [\text{MFG-SHJB}] \quad -d\phi(t, \omega, x) = \Big[H[t, \omega, x, D_x\phi(t, \omega, x)]$$

$$+ \frac{1}{2}\text{tr}\big(a[t, \omega, x]D_{xx}^2\phi(t, \omega, x)\big) \Big] dt - \psi^T(t, \omega, x)dw_0(t, \omega), \quad \phi(T, x) = 0,$$

$$(5.18) \quad u^o(t, \omega, x) \equiv u^o(t, x | \{z_0^o(s, \omega), \mu_s(\omega)\}_{0 \leq s \leq T})$$

$$\equiv \arg\inf_{u \in U} \big\{ \big\langle f[t, x, u, z_0^o(t, \omega), \mu_t(\omega)], D_x\phi(t, \omega, x) \big\rangle + L[t, x, u, z_0^o(t, \omega), \mu_t(\omega)] \big\},$$

$$(5.19) \quad [\text{MFG-SMV}] \quad dz^o(t, \omega, \omega') = f[t, z^o, u^o(t, \omega, z^o), z_0^o(t, \omega), \mu_t(\omega)]dt$$

$$+ \sigma[t, z^o, z_0^o(t, \omega), \mu_t(\omega)]dw(t, \omega'),$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$, and $z^o(0)$ has the measure $\mu_0(dx) = dF(x)$ where F is defined in (**A6.2**). We note that in the minor agents' SMFG system (5.17)-(5.19) we dropped index i from the generic minor agent's equations (5.7)-(5.12). The Major and Minor (MM) agent SMFG system is given by (5.14)-(5.16) and (5.17)-(5.19).

The solution of the MM-SMFG system consists of 8-tuple $\mathcal{F}_t^{w_0}$-adapted random processes

$$\big(\phi_0(t, \omega, x), \psi_0(t, \omega, x), u_0^o(t, \omega, x), z_0^o(t, \omega), \phi(t, \omega, x), \psi(t, \omega, x), u^o(t, \omega, x), z^o(t, \omega)\big),$$

where $z^o(t, \omega)$ generates the conditional random law $\mu_t(\omega)$, i.e., $P(z^o(t, \omega) \leq \alpha | \mathcal{F}_t^{w_0}) = \int_{-\infty}^{\alpha} \mu_t(\omega, dx)$ for all $\alpha \in \mathbb{R}^n$ and $0 \leq t \leq T$. Note that the MM-SMFG equations (5.14)-(5.16) and (5.17)-(5.19) are coupled together through $z_0^o(\cdot)$ and $\mu_{(\cdot)}$.

We observe that the solution to the MM-SMFG system is a "stochastic mean field" in contrast to the deterministic mean field of the standard MFG problems with only minor agents considered in [20, 18, 26, 27, 28]. If the noise process of the major agent vanishes then the MM-SMFG system reduces to a deterministic MFG system (see (6)-(9) in [18]).

For the analysis of next section we denote $\mu_t^0(\omega)$, $0 \leq t \leq T$, as the unit mass random measure concentrated at $z_0^o(t, \omega)$ (i.e., $\mu_t^0(\omega) = \delta_{z_0^o(t, \omega)}$).

6. Existence and Uniqueness of Solutions to the Major and Minor Stochastic Mean Field Game System. In this section we establish existence and uniqueness for the solution of the joint major and minor (MM) agents' SMFG system (5.14)-(5.16) and (5.17)-(5.19). The analysis is based on providing sufficient conditions for a map that goes from the random measure of minor agents $\mu_{(\cdot)}(\omega)$ back to itself, through the equations (5.14)-(5.16) and (5.17)-(5.19), to be a contraction operator on the space of random probability measures (see the diagram below).

$$
\begin{array}{ccccc}
\mu_{(\cdot)}(\omega) & \xrightarrow{(5.14)} & \big(\phi_0(\cdot,\omega,x),\psi_0(\cdot,\omega,x)\big) & \xrightarrow{(5.15)} & u_0^o(\cdot,\omega,x) \\
\uparrow{\scriptstyle(5.19)} & & & & \downarrow{\scriptstyle(5.16)} \\
u^o(\cdot,\omega,x) & \xleftarrow{(5.18)} & \big(\phi(\cdot,\omega,x),\psi(\cdot,\omega,x)\big) & \xleftarrow{(5.17)} & \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t,\omega)}
\end{array}
$$

In this section we first introduce some preliminary material about the Wasserstein space of probability measures. Second, we analyze the SHJB and SMV equations of the major agent and minor agents in Sections 6.1 and 6.2, respectively. Third, the analysis of the joint major and minor agents' SMFG system is carried out in Section 6.3 where the main result is given in Theorem 6.12 which provides sufficient conditions for a contraction operator map that goes from the random measure of minor agents $\mu_{(\cdot)}(\omega)$ back to itself.

On the Banach space $C([0,T];\mathbb{R}^n)$ we define the metric $\rho_T(x,y) = \sup_{0 \le t \le T} |x(t) - y(t)|^2 \wedge 1$, where $\wedge$ denotes minimum. It can be shown that $C_\rho := \big(C([0,T];\mathbb{R}^n),\rho_T\big)$ forms a separable complete metric space (i.e., a Polish space). Let $\mathcal{M}(C_\rho)$ be the space of all Borel probability measures μ on $C([0,T];\mathbb{R}^n)$ such that $\int |x|^2 d\mu(x) < \infty$. We also denote $\mathcal{M}(C_\rho \times C_\rho)$ as the space of probability measures on the product space $C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$. As in [20] the process x is defined to be a generic random process with the sample space $C([0,T];\mathbb{R}^n)$, i.e., $x(t,\omega) = \omega(t)$ for $\omega \in C([0,T];\mathbb{R}^n)$.

Based on the metric ρ_T, we introduce the Wasserstein metric on $\mathcal{M}(C_\rho)$:

$$
D_T^\rho(\mu,\nu) = \inf_{\gamma \in \Pi(\mu,\nu)} \Big[\int_{C_\rho \times C_\rho} \rho_T(x(\omega_1),x(\omega_2))d\gamma(\omega_1,\omega_2) \Big]^{1/2},
$$

where $\Pi(\mu,\nu) \subset \mathcal{M}(C_\rho \times C_\rho)$ is the set of Borel probability measures γ such that $\gamma(A \times C([0,T];\mathbb{R}^n)) = \mu(A)$ and $\gamma(C([0,T];\mathbb{R}^n) \times A) = \nu(A)$ for any Borel set $A \in C([0,T];\mathbb{R}^n)$. The metric space $\mathcal{M}_\rho := \big(\mathcal{M}(C_\rho),D_T^\rho\big)$ is a Polish space since $C_\rho \equiv \big(C([0,T];\mathbb{R}^n),\rho_T\big)$ is a Polish space.

We also introduce the class $\mathcal{M}_\rho^\beta$ of stochastic measures in the space $\mathcal{M}_\rho$ with a.s. Hölder continuity of exponent β, $0 < \beta < 1$ (see Definition 3 in [20] for the non-stochastic case).

DEFINITION 6.1. *A stochastic probability measure $\mu_t(\omega)$, $0 \le t \le T$, in the space $\mathcal{M}_\rho$ is in $\mathcal{M}_\rho^\beta$ if μ is a.s. uniformly Hölder continuous with exponent $0 < \beta < 1$, i.e., there exists $\beta \in (0,1)$ and constant c such that for any bounded and Lipschitz continuos function ϕ on $\mathbb{R}^n$,*

$$
\Big| \int_{\mathbb{R}^n} \phi(x)\mu_t(\omega,dx) - \int_{\mathbb{R}^n} \phi(x)\mu_s(\omega,dx) \Big| \le c(\omega)|t-s|^\beta, \quad a.s.,
$$

for all $0 \le s < t \le T$, where c may depend upon the Lipschitz constant of ϕ and the sample point $\omega \in \Omega$.

As in [20], we may take μ_t, $0 \le t \le T$, to be a Dirac measure at any constant $x \in \mathbb{R}^n$ to show that the set $\mathcal{M}_\rho^\beta$ is nonempty. We introduce the following assumption.

(**A8**) For any $p \in \mathbb{R}^n$ and $\mu, \mu^0 := \delta_{z_0^\circ} \in \mathcal{M}_\rho^\beta$, the sets

$$S_0(t, \omega, x, p) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, p],$$

$$S(t, \omega, x, p) := \arg \inf_{u \in U} H^u[t, \omega, x, u, p],$$

where $H_0^{u_0}$ and H^u are respectively defined in (5.5) and (5.11), are singletons and the resulting u and u_0 as functions of $[t, \omega, x, p]$ are a.s. continuous in t, Lipschitz continuous in (x, p), uniformly with respect to t and $\mu, \mu^0 \in \mathcal{M}_\rho^\beta$. In addition, $u_0[t, \omega, 0, 0]$ and $u[t, \omega, 0, 0]$ are in the space $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$.

The first part of (**A8**) may be satisfied under suitable convexity conditions with respect to u_0 and u (see [20]).

6.1. Analysis of the Major Agent's SMFG System. Let $\mu_t(\omega)$, $0 \le t \le T$, be a fixed stochastic measure in the set $\mathcal{M}_\rho^\beta$ with $0 < \beta < 1$ such that $\mu_0(dx) := dF(x)$ where F is defined in (**A2**). Then, the functionals of $\mu_{(\cdot)}(\omega)$ in (5.1)-(5.2) become random functions which we write as

(6.1) $f_0^*[t, \omega, z_0, u_0] := f_0[t, z_0, u_0, \mu_t(\omega)], \quad \sigma_0^*[t, \omega, z_0] := \sigma_0[t, z_0, \mu_t(\omega)],$
$L_0^*[t, \omega, z_0, u_0] := L_0[t, z_0, u_0, \mu_t(\omega)].$

We have the following result which broadly follows Proposition 4 in [20].

PROPOSITION 6.2. *Assume (**A3**) holds for U_0. Let $\mu_t(\omega)$, $0 \le t \le T$, be a fixed stochastic measure in the set $\mathcal{M}_\rho^\beta$ with $0 < \beta < 1$. For f_0^*, σ_0^* and L_0^* defined in (6.1) it is the case that:*

(i) *Under (**A4**) for f_0 and σ_0, the functions $f_0^*[t, \omega, z_0, u_0]$ and $\sigma_0^*[t, \omega, z_0]$ and their first order derivatives (w.r.t z_0) are a.s. continuous and bounded on $[0, T] \times \mathbb{R}^n \times U_0$ and $[0, T] \times \mathbb{R}^n$. $f_0^*[t, \omega, z_0, u_0]$ and $\sigma_0^*[t, \omega, z_0]$ are a.s. Lipschitz continuous in z_0. In addition, $f_0^*[t, \omega, 0, 0]$ is in the space $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$ and $\sigma_0^*[t, \omega, 0]$ is in the space $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^{n \times m})$.*

(ii) *Under (**A5**) for f_0, the function $f_0^*[t, \omega, z_0, u_0]$ is a.s. Lipschitz continuous in $u_0 \in U_0$, i.e., there exist a constant $c > 0$ such that*

$$\sup_{t \in [0, T], z_0 \in \mathbb{R}^n} \left| f_0^*[t, \omega, z_0, u_0] - f_0^*[t, \omega, z_0, u_0'] \right| \le c(\omega) |u_0 - u_0'|, \quad (a.s.).$$

(iii) *Under (**A6**) for L_0, the function $L_0^*[t, \omega, z_0, u_0]$ and its first order derivative (w.r.t z_0) is a.s. continuous and bounded on $[0, T] \times \mathbb{R}^n \times U_0$. $L_0^*[t, \omega, z_0, u_0]$ is a.s. Lipschitz continuous in z_0. In addition, $L_0^*[t, \omega, 0, 0]$ is in the space $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}_+)$.*

(iv) *Under (**A8**) for $H_0^{u_0}$, the set of minimizers*

$$\arg \inf_{u_0 \in U_0} \left\{ \langle f_0^*[t, \omega, z_0, u_0], p \rangle + L_0^*[t, \omega, z_0, u_0] \right\},$$

is a singleton for any $p \in \mathbb{R}^n$, and the resulting u_0 as a function of $[t, \omega, z_0, p]$ is a.s. continuous in t, a.s. Lipschitz continuous in (z_0, p), uniformly with respect to t. In addition, $u_0[t, \omega, 0, 0]$ is in the space $L_{\mathcal{F}_t}^2([0, T]; \mathbb{R}^n)$.

Proof: (*i*) We only show the results for f_0^*, the analysis for σ_0^* is similar. For

$\omega \in \Omega$, we take (t, z, u) and (s, z', u') both from $[0, T] \times \mathbb{R}^n \times U_0$. We have

$$
\begin{aligned}
\left| f_0^*[t, \omega, z, u] - f_0^*[s, \omega, z', u'] \right| &\equiv \left| f_0[t, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)] \right| \\
&\leq \left| f_0[t, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)] \right| + \left| f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)] \right| \\
&\leq \left| f_0[t, z, u, \mu_t(\omega)] - f_0[s, z, u, \mu_t(\omega)] \right| + \left| f_0[s, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)] \right| \\
&\quad + \left| f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)] \right|.
\end{aligned}
$$

By (**A4**), $f_0[t, \omega, z, u]$ is continuous with respect to (t, z, u) and therefore

$$
\left| f_0[t, z, u, \mu_t(\omega)] - f_0[s, z, u, \mu_t(\omega)] \right| + \left| f_0[s, z, u, \mu_t(\omega)] - f_0[s, z', u', \mu_t(\omega)] \right| \to 0,
$$

as $|t - s| + |z - z'| + |u - u'| \to 0$. Since $\mu_{(\cdot)}(\omega)$ is in the set $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and by (**A4**) there exists a constant $k > 0$ independent of (s, z, u) such that

$$
\left| f_0[s, z, u, y] - f_0[s, z, u, y'] \right| \leq k|y - y'|,
$$

we get $\left| f_0[s, z', u', \mu_t(\omega)] - f_0[s, z', u', \mu_s(\omega)] \right| \to 0$ as $|t - s| \to 0$. This concludes the a.s. continuity of $f_0^*[t, \omega, z_0, u_0]$ on $[0, T] \times \mathbb{R}^n \times U_0$.

Using the Leibniz rule we have

$$
D_{z_0} f_0^*[t, \omega, z_0, u_0] = \int D_{z_0} f_0[t, z_0, u_0, x] \mu_t(\omega)(dx), \quad a.s.,
$$

where the partial derivative exists due to the boundedness of the first order derivative (w.r.t z_0) of f_0 by (**A4**). The a.s. continuity of $D_{z_0} f_0^*$ on $[0, T] \times \mathbb{R}^n \times U_0$ may be proved by a similar argument above for f_0^*. Other results of the Proposition follow directly from (**A4**).

(*ii*) This is a direct result of (**A5**).

(*iii*) The proofs are similar to the proofs for f_0^* in part (i).

(*iv*) This is a direct result of (**A8**) for S_0 using the measure $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$. □

Employing the results of Section 4, we analyze the SHJB equation (5.14) where the probability measure $\mu_{(\cdot)}(\omega)$ is in the set $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$.

THEOREM 6.3. *Assume (**A3**)-(**A7**) for U_0, f_0, σ_0 and L_0 hold, and the probability measure $\mu_{(\cdot)}(\omega)$ is in the set $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$. Then the SHJB equation for the major agent (5.14) has a unique solution $(\phi_0(t, x), \psi_0(t, x))$ in $\left(L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}), L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^m) \right)$.*
Proof: Proposition 6.2 indicates that the SOCP of the major agent (5.1)-(5.2) satisfies the Assumptions (**H1**)-(**H3**) of Section 4 with $\varsigma[t, x] = 0$. The result follows directly from Theorem 4.1. □

Let $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, be given. We assume that the unique solution $(\phi_0, \psi_0)(t, x)$ to the SHJB equation (5.14) satisfies the regularity properties: (i) for each t, $(\phi_0, \psi_0)(t, x)$ is a $C^2(\mathbb{R}^n)$ map from $\mathbb{R}^n$ into $\mathbb{R} \times \mathbb{R}^m$, (ii) for each x, (ϕ_0, ψ_0) and $(D_x \phi_0, D_{xx}^2 \phi_0, D_x \psi_0)$ are continuous F_t^W-adapted stochastic processes. Then, $\phi_0(x, t)$ coincides with the value function (5.3) [44], and under (**A8**) for $H_0^{u_0}$ we get the best response control process (5.5):

(6.2) $u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \leq s \leq T}) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t, \omega, x, u_0, D_x \phi_0(t, \omega, x)],$

where $(t, x) \in [0, T] \times \mathbb{R}^n$.

We introduce the following assumption (see (H6) in [20]).

(**A9**) For any $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, the best response control $u_0^o(t, \omega, x)$ is a.s. continuous in (t, x) and a.s. Lipschitz continuous in x.

We denote $C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; H)$ be the class of a.s. continuous functions from $[0, T] \times \Omega \times \mathbb{R}^n$ to H, which are a.s. Lipschitz continuous in x [20]. We introduce the following well-defined map:

$$(6.3) \qquad \Upsilon_0^{\text{SHJB}} : M_\rho^\beta \longrightarrow C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0), \qquad 0 < \beta < 1,$$

$$\Upsilon_0^{\text{SHJB}}\big(\mu_{(\cdot)}(\omega)\big) = u_0^o(t, \omega, x) \equiv u_0^o(t, x | \{\mu_s(\omega)\}_{0 \le s \le T}).$$

We now analyze the major agent's SMV equation (5.16) with $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ where $0 < \beta < 1$, and $u_0^o(t, \omega, x) \in C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$ be given in (6.2).

THEOREM 6.4. *Assume (**A3**)-(**A7**) for U_0, f_0 and σ_0, and (**A9**) hold. Let $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ where $0 < \beta < 1$, and $u_0^o(t, \omega, x)$ be given in (6.2). Then, there exists a unique solution z_0^o on $[0, T] \times \Omega$ to the major agent's SMV equation (5.16).*

Proof: Proposition 6.2 indicates that the major agent's SMV equation (5.16) satisfies the Assumption (**RC**) in [53], page 49. The result follows directly from Theorem 6.16, Chapter 1 of [53], page 49. □

THEOREM 6.5. *Assume (**A3**)-(**A7**) for U_0, f_0 and σ_0, and (**A9**) hold. Let $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$ where $0 < \beta < 1$, and $u_0^o(t, \omega, x)$ be given in (6.2). Then, the probability measure $\mu_{(\cdot)}^0(\omega)$ as the unit mass measure concentrated at $z_0^o(t, \omega)$ (i.e., $\mu_t^0(\omega) = \delta_{z_0^o(t,\omega)}$) which is obtained from the major agent's SMV equation (5.16) is in the class $\mathcal{M}_\rho^\gamma$ where $0 < \gamma < 1/2$.*

Proof: We take $0 \le s < t \le T$. Since $\mu_t^0(\omega) = \delta_{z_0^o(t,\omega)}$, for any bounded and Lipschitz continuos function ϕ on $\mathbb{R}^n$ with a Lipschitz constant $K > 0$, we have

$$\mathbb{E}\Big| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \Big| = \mathbb{E}\big| \phi(z_0^o(t, \omega)) - \phi(z_0^o(s, \omega)) \big|$$

$$\le K \, \mathbb{E}\big| z_0^o(t, \omega) - z_0^o(s, \omega) \big|.$$

On the other hand, Theorem 6.4 indicates that there exists a unique solution to the SMV equation (5.16) such that

$$z_0^o(t, \omega) - z_0^o(s, \omega) = \int_s^t f_0[\tau, z_0^o, u_0^o, \mu_\tau(\omega)] d\tau + \int_s^t \sigma_0[\tau, z_0^o, \mu_\tau(\omega)] dw_0(\tau).$$

Boundedness of f_0 and σ_0 (see (**A4**)), the Cauchy-Schwarz inequality and the property of Itô integral yield

$$\mathbb{E}\big| z_0^o(t, \omega) - z_0^o(s, \omega) \big|^2 \le 2 C_1^2 |t - s|^2 + 2 C_2^2 |t - s|,$$

where C_1 and C_2 are upper bounds for f_0 and σ_0, respectively. Hence,

$$\mathbb{E}\Big| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \Big| \le \sqrt{2} K \big(C_1 |t - s| + C_2 |t - s|^{1/2} \big)$$

$$\le \sqrt{2} K (C_1 \sqrt{T} + C_2) |t - s|^{1/2}.$$

By Kolmogorov's Theorem (Theorem 18.19, Page 266, [24]), for each $0 < \gamma < 1/2$, $T > 0$, and almost every $\omega \in \Omega$, there exists a constant $c(\omega, \gamma, K, T)$ such that

$$\Big| \int_{\mathbb{R}^n} \phi(x) \mu_t^0(\omega, dx) - \int_{\mathbb{R}^n} \phi(x) \mu_s^0(\omega, dx) \Big| \le c(\omega, \gamma, K, T) |t - s|^\gamma,$$

for all $0 \leq s < t \leq T$. Hence, $\mu_{(\cdot)}^0(\omega)$ is in the class $\mathcal{M}_\rho^\gamma$ where $0 < \gamma < 1/2$. $\square$

By Theorems 6.4 and 6.5 we may now introduce the following well-defined map:

(6.4)
$$\Upsilon_0{}^{\mathrm{SMV}} : M_\rho^\beta \times C_{Lip(x)}([0,T] \times \Omega \times \mathbb{R}^n; U_0) \longrightarrow M_\rho^\gamma, \quad 0 < \beta < 1,\ 0 < \gamma < 1/2,$$
$$\Upsilon_0{}^{\mathrm{SMV}}\big(\mu_{(\cdot)}(\omega), u_0^o(t,\omega,x)\big) = \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t,\omega)}.$$

6.2. Analysis of the Minor Agents' SMFG System. Let $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, be the fixed stochastic measure assumed in Section 6.1. In this section we assume that $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, is the unit mass random measure concentrated at $z_0^o(\cdot, \omega)$ (i.e., $\mu_t^0(\omega) = \delta_{z_0^o(t,\omega)}$) obtained from the composite map:

(6.5) $$\Upsilon_0 : M_\rho^\beta \longrightarrow M_\rho^\gamma, \qquad 0 < \beta < 1,\ 0 < \gamma < 1/2,$$
$$\Upsilon_0\big(\mu_{(\cdot)}(\omega)\big) := \Upsilon_0{}^{\mathrm{SMV}}\Big(\mu_{(\cdot)}(\omega), \Upsilon_0{}^{\mathrm{SHJB}}\big(\mu_{(\cdot)}(\omega)\big)\Big) = \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t,\omega)},$$

where $\Upsilon_0{}^{\mathrm{SHJB}}$ and $\Upsilon_0{}^{\mathrm{SMV}}$ are given in (6.3) and (6.4), respectively.

Following arguments exactly parallel to those used in Section 6.1, we analyze the SHJB equation (5.17) where the probability measures $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$ and $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$.

THEOREM 6.6. *Assume (**A3**)-(**A7**) for U, f, σ and L hold, and $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$ and $\mu_{(\cdot)}^0(\omega)$ is in the set $\mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$. Then the SHJB equation for the generic minor agent (5.10) has a unique solution $(\phi_i(t,x), \psi_i(t,x))$ in $\big(L_{\mathcal{F}_t}^2([0,T]; \mathbb{R}), L_{\mathcal{F}_t}^2([0,T]; \mathbb{R}^m)\big)$.*

Proof: A similar argument to Proposition 6.2 for the generic minor agent (see Proposition C.1 in [37]) indicates that the SOCP of the generic minor agent (5.7)-(5.8) satisfies the Assumptions (**H1**)-(**H3**) of Section 4 with $\sigma[t,x] = 0$. The result follows directly from Theorem 4.1. $\square$

For the probability measure $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, we assume that the unique solution $(\phi_i, \psi_i)(t,x)$ to the SHJB equation (5.10) satisfies the regularity properties: (i) for each t, $(\phi_i, \psi_i)(t,x)$ is a $C^2(\mathbb{R}^n)$ map from $\mathbb{R}^n$ into $\mathbb{R} \times \mathbb{R}^m$, (ii) for each x, (ϕ_i, ψ_i) and $(D_x\phi_i, D_{xx}^2\phi_i, D_x\psi_i)$ are continuous F_t^W-adapted stochastic processes. Then, $\phi_i(x,t)$ coincides with the value function (5.9) [44], and under (**A8**) for H^u we get the best response control process (5.11):

(6.6) $$u_i^o(t,\omega,x) \equiv u_i^o(t,x | \{\mu_s^0(\omega), \mu_s(\omega)\}_{0 \leq s \leq T})$$
$$:= \arg\inf_{u_i \in U} H^u[t,\omega,x,u_i, D_x\phi_i(t,\omega,x)],$$

where $(t,x) \in [0,T] \times \mathbb{R}^n$.

We introduce the following assumption (see (**A9**) or (H6) in [20]).

(**A10**) For any $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, the best response control process $u_i^o(t,\omega,x)$ is a.s. continuous in (t,x) and a.s. Lipschitz continuous in x.

We introduce the following well-defined map for the generic minor agent i:

(6.7) $\Upsilon_i{}^{\mathrm{SHJB}}: M_\rho^\beta \times M_\rho^\gamma \longrightarrow C_{\mathrm{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U), \quad 0 < \beta < 1,\ 0 < \gamma < 1/2,$

$\quad \Upsilon_i{}^{\mathrm{SHJB}}\big(\mu_{(\cdot)}(\omega), \mu_{(\cdot)}^0(\omega)\big) = u_i^o(t,\omega,x) \equiv u_i^o(t,x | \{\mu_s^0(\omega), \mu_s(\omega)\}_{0 \leq s \leq T}).$

For given probability measure $\mu^0_{(\cdot)}(\omega) \in \mathcal{M}^\gamma_\rho$, $0 < \gamma < 1/2$, we analyze the generic minor agent's SMV equation (5.12):

$$(6.8) \qquad dz^o_i(t, \omega, \omega') = f[t, z^o_i, u^o_i(t, \omega, z^o_i), \mu^0_t(\omega), \mu_t(\omega)]dt$$
$$+ \sigma[t, z^o_i, \mu^0_t(\omega), \mu_t(\omega)]dw_i(t, \omega'), \quad z^o_i(0) = z_i(0),$$

where $u^o_i(t, \omega, x) \in C_{\text{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U)$ is given in (6.6). We call the pair $(z^o_i(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$ a consistent solution of the generic minor agent's SMV equation (6.8) if $(z^o_i(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$ solves (6.8) and $\mu_{(\cdot)}(\omega)$ be the the law of the process $z^o_i(\cdot, \omega, \omega')$, i.e., $\mu_{(\cdot)} = \mathcal{L}(z^o_i(\cdot, \omega, \omega'))$. We define Λ as the map which associates to $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1/2$, the law of the process $z^o_i(\cdot, \omega, \omega')$ in (6.8):

$$(6.9) \quad z^o_i(t, \omega, \omega') = z^o_i(0) + \int_0^t \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f[s, z^o_i, u^o_i, y, z]d\mu^0_s(\omega)(y)d\mu_s(\omega)(z) \Big) ds$$
$$+ \int_0^t \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma[s, z^o_i, y, z]d\mu^0_s(\omega)(y)d\mu_s(\omega)(z) \Big) dw_i(s, \omega'),$$

where we observe that the law Λ depends on the sample point $\omega \in \Omega$.

We now show that there exists a unique $\mu_{(\cdot)}(\omega) \in \mathcal{M}^\beta_\rho$, $0 < \beta < 1$, such that $\mu(\omega) = \Lambda\big(\mu(\omega)\big)$. The proof of the following theorem, which is given in Appendix D in [37], is based upon a fixed point argument with random parameters (see Theorem 6 in [20] and Theorem 1.1 in [46] for the standard fixed point argument).

THEOREM 6.7. *Assume (**A3**)-(**A7**) for U, f and σ, and (**A10**) hold. Let $\mu^0_{(\cdot)}(\omega)$ be in the set $\mathcal{M}^\gamma_\rho$ where $0 < \gamma < 1/2$, and $u^o_i(t, \omega, x)$ be given in (6.6). Then, there exists a unique consistent solution pair $(z^o_i(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$ to the generic minor agent's SMV equation (6.8) where $\mu_{(\cdot)}(\omega) = \mathcal{L}\big(z^o_i(\cdot, \omega, \omega')\big)$.* $\square$

THEOREM 6.8. *Assume (**A3**)-(**A7**) for U, f and σ, and (**A10**) hold. Let $\mu^0_{(\cdot)}(\omega)$ be in the set $\mathcal{M}^\gamma_\rho$ where $0 < \gamma < 1/2$. For given $u^o_i(t, \omega, x)$ in (6.6), let $(z^o_i(\cdot, \omega, \omega'), \mu_{(\cdot)}(\omega))$ be the consistent solution pair of the SMV equation (6.8). Then, the probability measure $\mu_{(\cdot)}(\omega)$ is in the class $\mathcal{M}^\beta_\rho$ where $0 < \beta < 1$.*

Proof: We take $0 \le s < t \le T$. For any bounded and Lipschitz continuos function ϕ on $\mathbb{R}^n$ with a Lipschitz constant $K > 0$, we have

$$\mathbb{E}\Big| \int_{\mathbb{R}^n} \phi(x)\mu_t(\omega, dx) - \int_{\mathbb{R}^n} \phi(x)\mu_s(\omega, dx) \Big| = \mathbb{E}\big| \mathbb{E}_\omega \big(\phi(z^o_i(t, \omega, \omega')) - \phi(z^o_i(s, \omega, \omega')) \big) \big|$$
$$\le K \, \mathbb{E}\big| \mathbb{E}_\omega \big(z^o_i(t, \omega, \omega') - z^o_i(s, \omega, \omega') \big) \big|.$$

On the other hand, Theorem 6.7 indicates that there exists a unique solution to the SMV equation (6.8) such that

$$\mathbb{E}_\omega \big(z^o_i(t, \omega, \omega') - z^o_i(s, \omega, \omega') \big) = \int_s^t f[\tau, z^o_i, u^i_0, \mu^0_\tau(\omega), \mu_\tau(\omega)]d\tau,$$

where we note that $\mathbb{E}_\omega \int_0^t \sigma[\tau, z^o_i, \mu^0_\tau(\omega), \mu_\tau(\omega)]dw_i(\tau, \omega') = 0$ for $0 \le t \le T$. Boundedness of f (see (**A4**)) yields

$$\mathbb{E}\big| \mathbb{E}_\omega \big(z^o_i(t, \omega, \omega') - z^o_i(s, \omega, \omega') \big) \big| \le C_1|t - s|,$$

where C_1 is the upper bound for f.

By Kolmogorov's Theorem (Theorem 18.19, [24], Page 266), for each $0 < \gamma < 1$, $T > 0$, and almost every $\omega \in \Omega$, there exists a constant $c(\omega, \gamma, K, T)$ such that

$$\Big| \int_{\mathbb{R}^n} \phi(x)\mu_t(\omega, dx) - \int_{\mathbb{R}^n} \phi(x)\mu_s(\omega, dx) \Big| \le c(\omega, \gamma, K, T)|t - s|^\gamma,$$

for all $0 \le s < t \le T$. Hence, $\mu_{(\cdot)}(\omega)$ is in the class $\mathcal{M}_\rho^\beta$ where $0 < \beta < 1$. $\quad\square$

By Theorems 6.7 and 6.8 we may now introduce the following well-defined map:

(6.10)
$$\Upsilon_i^{\mathrm{SMV}} : M_\rho^\beta \times M_\rho^\gamma \times C_{\mathrm{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0) \longrightarrow M_\rho^\beta, \ 0 < \beta < 1, \ 0 < \gamma < 1/2,$$
$$\Upsilon_i^{\mathrm{SMV}}\big(\mu_{(\cdot)}(\omega), \mu_{(\cdot)}^0(\omega), u_i^o(t, \omega, x)\big) = \mu_{(\cdot)}(\omega).$$

6.3. Analysis of the Joint Major and Minor Agents' SMFG System. Based on the analysis of Sections 6.1 and 6.2 we obtain the following well-defined map:

(6.11)
$$\Upsilon : M_\rho^\beta \longrightarrow M_\rho^\beta, \qquad 0 < \beta < 1,$$
$$\Upsilon\big(\mu_{(\cdot)}(\omega)\big) = \Upsilon_i^{\mathrm{SMV}}\Big(\mu_{(\cdot)}(\omega), \Upsilon_0\big(\mu_{(\cdot)}(\omega)\big), \Upsilon_i^{\mathrm{SHJB}}\big(\mu_{(\cdot)}(\omega)\big), \Upsilon_0\big(\mu_{(\cdot)}(\omega)\big)\big)\Big) = \mu_{(\cdot)}(\omega),$$

which is the composition of the maps Υ_0, $\Upsilon_i^{\mathrm{SHJB}}$ and $\Upsilon_i^{\mathrm{SMV}}$ introduced in (6.5), (6.7) and (6.10), respectively. Subsequently, the problem of existence and uniqueness of solution to the MM SMV system (5.14)-(5.16) and (5.17)-(5.19) is translated into a fixed point problem with random parameters for the map Υ on the Polish space $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$.

We introduce the following assumption without which one needs to work with the "expectation" of the Wasserstein metric $D_{(\cdot)}^\rho$ of stochastic measure.

(**A11**) We assume that the diffusion coefficient of the major agent σ_0 in (2.1) does not depend on its own state z_0^N and the states of the minor agents z_i^N, $1 \le i \le N$.

The proof of the following lemma is given in Appendix E in [37].

LEMMA 6.9. *(i) Assume (**A3**)-(**A7**) for U_0, f_0 and σ_0, and (**A11**) hold. Let $\mu_{(\cdot)}(\omega)$ be in the set $\mathcal{M}_\rho^\beta$ where $0 < \beta < 1$. Then, for given $u_0, u_0' \in C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$ there exists a constant c_0 such that*

(6.12) $\quad \Big(D_T^\rho\big(\mu^0(\omega), \nu^0(\omega)\big) \Big)^2 \le c_0 \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \big| u_0(t, \omega, x) - u_0'(t, \omega, x) \big|^2, \qquad a.s.,$

where $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, are induced by the map Υ_0^{SMV} in (6.4) using the two control processes u_0 and u_0', respectively.
*(ii) Assume (**A3**)-(**A7**) for U_0, f_0 and σ_0, and (**A11**) hold. Let u_0^o be in the space $C_{Lip(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$. Then, for given $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, there exists a constant c_1 such that*

(6.13) $\qquad \Big(D_T^\rho\big(\mu^0(\omega), \nu^0(\omega)\big) \Big)^2 \le c_1 \Big(D_T^\rho\big(\mu(\omega), \nu(\omega)\big) \Big)^2, \qquad a.s.,$

where $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, are induced by the map Υ_0^{SMV} in (6.4) using the stochastic measures $\mu(\omega)$ and $\nu(\omega)$, respectively.
*(iii) Assume (**A3**)-(**A7**) for U, f and σ hold. Let $\mu_{(\cdot)}^0(\omega)$ be in the set $\mathcal{M}_\rho^\gamma$ where*

$0 < \gamma < 1/2$. *Then, for given* $u, u' \in C_{Lip(x)}([0,T] \times \Omega \times \mathbb{R}^n; U)$ *there exists a constant* c_2 *such that*

$$(6.14) \quad \left(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\right)^2 \leq c_2 \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \left|u(t,\omega,x) - u'(t,\omega,x)\right|^2, \qquad a.s.,$$

where $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, *are induced by the map* Υ_i^{SMV} *in (6.10) using the two control processes* u *and* u', *respectively.*

*(iv) Assume (**A3**)-(**A7**) for* U, f *and* σ *hold. Let* u_i^o *be in the space* $C_{Lip(x)}([0,T] \times \Omega \times \mathbb{R}^n; U)$. *Then, for given* $\mu^0(\omega), \nu^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, *there exists a constant* c_3 *such that*

$$(6.15) \quad \left(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\right)^2 \leq c_3 \left(D_T^\rho\big(\mu^0(\omega), \nu^0(\omega)\big)\right)^2, \qquad a.s.,$$

where $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, *are induced by the map* Υ_i^{SMV} *in (6.10) using the stochastic measures* $\mu^0(\omega)$ *and* $\nu^0(\omega)$, *respectively.* $\qquad\square$

We define the Gâteaux derivative of the function $F(t,x,\mu)$ with respect to the measure $\mu(y)$ as [23]

$$\partial_{\mu(y)} F(t,x,\mu) = \lim_{\epsilon \to 0} \frac{F(t,x,\mu + \epsilon\delta(y)) - F(t,x,\mu)}{\epsilon},$$

where δ is the Dirac delta function. We introduce the following assumptions:

(**A12**) (i) In (5.1)-(5.2) the Gâteaux derivative of f_0, σ_0 and L_0 with respect to μ exist, are $C^\infty(\mathbb{R}^n)$ and a.s. uniformly bounded. (ii) In (5.7)-(5.8) the partial derivatives of f, σ and L with respect to μ^0 and μ exist, are $C^\infty(\mathbb{R}^n)$ and a.s. uniformly bounded.

The proof of the following lemma is based on the sensitivity analysis of the SHJB equations (5.14) and (5.17) to the stochastic measures $\mu_{(\cdot)}(\omega)$ and $\mu_{(\cdot)}^0(\omega)$ developed in Appendix F in [37] (see also Section 6 in [23]).

LEMMA 6.10. *(i) Assume (**A3**)-(**A7**) for* U_0, f_0, σ_0, L_0, *and (**A12**)-(i) hold. Let* $(\phi_0(t,x), \psi_0(t,x))$ *be the unique solution pair to (5.14) which is* $C^\infty(\mathbb{R}^n)$ *and is a.s. uniformly bounded. In addition, we assume (**A8**) holds for* S_0 *and the resulting* u_0 *is also a.s. Lipschitz continuous in* μ. *Then, for* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$ *in the set* $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, *there exists a constant* c_4 *such that*

$$(6.16) \quad \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \left|u_0(t,\omega,x) - u_0'(t,\omega,x)\right|^2 \leq c_4 \left(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\right)^2, \qquad a.s.,$$

where $u_0, u_0' \in C_{Lip(x)}([0,T] \times \Omega \times \mathbb{R}^n; U_0)$ *are induced by the map* Υ_0^{SHJB} *in (6.3) using two stochastic measures* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$, *respectively.*

*(ii) Assume (**A3**)-(**A7**) for* U, f, σ, L, *and (**A12**)-(ii) hold. Let* $(\phi(t,x), \psi(t,x))$ *be the unique solution pair to (5.17) which is* $C^\infty(\mathbb{R}^n)$ *and is a.s. uniformly bounded. In addition, we assume (**A8**) holds for* S *and the resulting* u *is also a.s. Lipschitz continuous in* μ. *Then, for* $\mu_{(\cdot)}^0(\omega) \in \mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, *and* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$ *in the set* $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, *there exists a constant* c_5 *such that*

$$(6.17) \quad \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \left|u(t,\omega,x) - u'(t,\omega,x)\right|^2 \leq c_5 \left(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\right)^2, \qquad a.s.,$$

where $u, u' \in C_{Lip(x)}([0,T] \times \Omega \times \mathbb{R}^n; U)$ *are induced by the map* Υ_i^{SHJB} *in (6.7) using two stochastic measures* $\mu_{(\cdot)}(\omega)$ *and* $\nu_{(\cdot)}(\omega)$, *respectively.*

(iii) Assume (**A3**)*-*(**A7**) *for* U, f, σ, L, *and* (**A12**)*-(ii) hold. Let* $(\phi(t,x), \psi(t,x))$ *be the unique solution pair to (5.17) which is* $C^\infty(\mathbb{R}^n)$ *and is a.s. uniformly bounded. In addition, we assume* (**A8**) *holds for* S *and the resulting* u *is also a.s. Lipschitz continuous in* μ^0. *Then, for* $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, *and* $\mu_{(\cdot)}^0(\omega)$ *and* $\nu_{(\cdot)}^0(\omega)$ *in the set* $\mathcal{M}_\rho^\gamma$, $0 < \gamma < 1/2$, *there exists a constant* c_6 *such that*

$$(6.18) \qquad \sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \left| u(t,\omega,x) - u'(t,\omega,x) \right|^2 \leq c_6 \Big(D_T^\rho \big(\mu^0(\omega), \nu^0(\omega) \big) \Big)^2, \qquad a.s.,$$

where $u, u' \in C_{Lip(x)}([0,T]\times\Omega\times\mathbb{R}^n; U)$ *are induced by the map* Υ_i^{SHJB} *in (6.7) using the two stochastic measures* $\mu_{(\cdot)}^0(\omega)$ *and* $\nu_{(\cdot)}^0(\omega)$, *respectively.*

Proof: (i) Assumption (**A8**) for S_0 together with the fact that the resulting u_0 in (**A8**) is also a.s. Lipschitz continuous in μ yields

$$(6.19) \qquad \begin{aligned} |u_0(t,\omega,x) - u_0'(t,\omega,x)| &\leq k_1 D_t^\rho \big(\mu(\omega), \nu(\omega) \big) \\ &\quad + k_2 |D_x\phi_0^\mu(t,\omega,x) - D_x\phi_0^\nu(t,\omega,x)|, \end{aligned}$$

with positive constants k_1, k_2, where we indicate the dependence of ϕ_0 on measures μ and ν by ϕ_0^μ and ϕ_0^ν, respectively.

We consider the Gâteaux derivative of ϕ_0 with respect to the measure μ. The assumptions of the theorem imply that the conditions for Proposition F.1 in [37] hold. Therefore, Proposition F.1 in [37] concludes that the Gâteaux derivative of $D_x\phi_0$ with respect to measure μ is a.s. uniformly bounded. This together with the mean value theorem yields

$$(6.20) \qquad |D_x\phi_0^\mu(t,\omega,x) - D_x\phi_0^\nu(t,\omega,x)| \leq k_3 D_t^\rho \big(\mu(\omega), \nu(\omega) \big),$$

with positive constant k_3. (6.19) and (6.20) give

$$|u_0(t,\omega,x) - u_0'(t,\omega,x)| \leq k D_t^\rho \big(\mu(\omega), \nu(\omega) \big),$$

with $k := k_1 + k_2 k_3$, which yields the result.   □

REMARK 6.11. *In the standard mean field game model of [20] a similar condition to (6.16)-(6.18) is taken as an assumption (see the feedback regularity condition (37) in [20]). Following the argument in Section 7.1 of [20], one can show that the inequalities (6.16)-(6.18) hold in the linear-quadratic-Gaussian (LQG) model with Lipschitz continuous nonlinear couplings.*

We recall the map Υ given in (6.11) which is the composition of the maps Υ_0, Υ_i^{SHJB} and Υ_i^{SMV} introduced in (6.5), (6.7) and (6.10), respectively (see the diagram below).

$$\begin{array}{ccc} \mu_{(\cdot)}(\omega) & \xrightarrow{\Upsilon_0^{SHJB}} & u_0^o(\cdot,\omega,x) \\ \uparrow\Upsilon_i^{SMV} & & \downarrow\Upsilon_0^{SMV} \\ u^o(\cdot,\omega,x) & \xleftarrow{\Upsilon_i^{SHJB}} & \mu_{(\cdot)}^0(\omega) \equiv \delta_{z_0^o(t,\omega)} \end{array}$$

THEOREM 6.12. (**Main Result**) *Let the assumptions of both Lemma 6.9 and Lemma 6.10 hold. If the constants* $\{c_i : 0 \leq i \leq 6\}$ *for (6.12)-(6.15) and (6.16)-(6.18) satisfy the gain condition*

$$\max \{ c_2 c_5, c_2 c_6 c_0, c_2 c_6 c_1, c_3 c_1, c_3 c_0 c_4 \} < 1,$$

then there exists a unique solution for the map Υ, and hence a unique solution to the MM-SMFG system (5.14)-(5.16) and (5.17)-(5.19).

   Proof: The result follows from the Banach fixed point theorem for the map Υ given in (6.11) on the Polish space $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$. We note that the gain condition ensures that Υ is a contraction. $\qquad\square$

   As in the classical FBSDEs, the gain condition in Theorem 6.12 is expected to hold for short time-horizon T. Another approach to the solution existence of the MM-SMFG system (5.14)-(5.16) and (5.17)-(5.19) is Schauder's fixed point argument which is the topic of future work.

7. ϵ-Nash Equilibrium Property of the SMFG Control Laws. We let

$$\big(\phi_0(t,\omega,x), \psi_0(t,\omega,x), u_0^o(t,\omega,x), z_0^o(t,\omega), \phi(t,\omega,x), \psi(t,\omega,x), u^o(t,\omega,x), z^o(t,\omega)\big),$$

be the unique solution of the MM-SMFG system (5.14)-(5.16) and (5.17)-(5.19) such that SMFG best response $u_0^o(t,\omega,x)$ and $u^o(t,\omega,x)$ are a.s. continuous in (t,x) and a.s. Lipschitz continuous in x.

   We now apply the SMFG best responses $u_0^o(t,\omega,x)$ and $u^o(t,\omega,x)$ into a finite $N+1$ major and minor population (2.1)-(2.2). This yields the following closed loop individual dynamics:

$$(7.1)\quad dz_0^{o,N}(t) = \frac{1}{N}\sum_{j=1}^N f_0[t, z_0^{o,N}(t), u_0^o(t, z_0^{o,N}(t)), z_j^{o,N}(t)]dt$$

$$+ \frac{1}{N}\sum_{j=1}^N \sigma_0[t, z_0^{o,N}(t), z_j^{o,N}(t)]dw_0(t), \quad z_0^{o,N}(0) = z_0(0),\ 0 \le t \le T,$$

$$(7.2)\quad dz_i^{o,N}(t) = \frac{1}{N}\sum_{j=1}^N f[t, z_i^{o,N}(t), u^o(t, z_i^{o,N}(t)), z_0^{o,N}(t), z_j^{o,N}(t)]dt$$

$$+ \frac{1}{N}\sum_{j=1}^N \sigma[t, z_i^{o,N}(t), z_0^{o,N}(t), z_j^{o,N}(t)]dw_i(t), \quad z_i^{o,N}(0) = z_i(0),\ 1 \le i \le N,$$

   We set the admissible control set of agent $\mathcal{A}_j$, $0 \le j \le N$, as

$$\mathcal{U}_j = \Big\{ u_j(\cdot,\omega) := u_j\big(\cdot, \omega, z_0(\cdot,\omega), \cdots, z_N(\cdot,\omega)\big) \in C_{\text{Lip}(z_0,\cdots,z_N)} : u_j(t,\omega) \text{ is a}$$

$$\mathcal{F}_t^{w_0}\text{-measurable process adapted to sigma-field } \sigma\big\{z_i(\tau,\omega) : 0 \le i \le N, 0 \le \tau \le t\big\}$$

$$\text{such that } \mathbb{E}\int_0^T |u_j(t,\omega)|^2 dt < \infty\Big\}.$$

We note that $\mathcal{U}_j$, $0 \le j \le N$, are the full information admissible control which are not restricted to be decentralized.

   DEFINITION 7.1. *Given $\epsilon > 0$, the admissible control laws $(u_0^o, \cdots, u_N^o)$ for $N+1$ agents generates an ϵ-Nash equilibrium with respect to the costs J_j^N, $0 \le j \le N$, if $J_j^N(u_j^o; u_{-j}^o) - \epsilon \le \inf_{u_j \in \mathcal{U}_j} J_j^N(u_j; u_{-j}^o) \le J_j^N(u_j^o; u_{-j}^o)$, for any $0 \le j \le N$.* $\qquad\square$

   We now show that the SMFG best responses for a finite $N+1$ major and minor population system (7.1)-(7.2) is an ϵ-Nash equilibrium with respect to the cost functions (2.3)-(2.4) in the case that minor agents are coupled to the major agent only through their cost functions (see the MM-MFG LQG model in [35]).

(**A13**) Assume the functions f and σ in (2.2) (and hence in (7.2)) do not contain the state of major agent z_0^N.

Note that in the case of assumption (**A13**) the major agent $\mathcal{A}_0$ has non-negligible influence on the minor agents through their cost functions (2.4). An analysis based on the anticipative variational calculations used in the MM-MFG LQG case [36] is required for establishing the ϵ-Nash equilibrium property of the SMFG best responses in the general case. This extension is currently under investigation and will be reported in future work.

THEOREM 7.2. *Assume (**A1**)-(**A6**) and (**A13**) hold, and there exists a unique solution to the MM-SMFG system (5.14)-(5.16) and (5.17)-(5.19) such that the SMFG best response control processes $u_0^o(t, \omega, x)$ and $u^o(t, \omega, x)$ are a.s. continuous in (t, x) and a.s. Lipschitz continuous in x. Then $(u_0^o, u_1^o, \cdots, u_N^o)$ where $u_i^o \equiv u^o$, $1 \le i \le N$, generates an $O(\epsilon_N + 1/\sqrt{N})$-Nash equilibrium with respect to the cost functions (2.3)-(2.4) such that $\lim_{N \to \infty} \epsilon_N = 0$.*

Proof: Under (**A13**) we have the the following closed loop individual dynamics under the SMFG best response control processes:

$$dz_0^{o,N}(t) = \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^{o,N}(t), u_0^o(t, z_0^{o,N}(t)), z_j^{o,N}(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^{o,N}(t), z_j^{o,N}(t)]dw_0(t), \quad z_0^{o,N}(0) = z_0(0), \ 0 \le t \le T,$$

$$dz_i^{o,N}(t) = \frac{1}{N} \sum_{j=1}^N f[t, z_i^{o,N}(t), u^o(t, z_i^{o,N}(t)), z_j^{o,N}(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^N \sigma[t, z_i^{o,N}(t), z_j^{o,N}(t)]dw_i(t), \quad z_i^{o,N}(0) = z_i(0), \ 1 \le i \le N.$$

We also introduce the associated Mckean-Vlasov (MV) system

$$(7.3) \qquad \begin{aligned} dz_0^o(t) &= f_0[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t]dt + \sigma_0[t, z_0^o(t), \mu_t]dw_0(t), \\ dz_i^o(t) &= f[t, z_i^o(t), u^o(t, z_i^o), \mu_t]dt + \sigma[t, z_i^o, \mu_t]dw_i(t), \end{aligned}$$

with the initial condition $z_j^o(0) = z_j(0), 0 \le j \le N$. In the above MV equation $\mu_t, 0 \le t \le T$, is the conditional law of $z_i^o(t), 1 \le i \le N$, given $\mathcal{F}_t^{w_0}$ (i.e., $\mu_t := \mathcal{L}(z_i^o(t)|\mathcal{F}_t^{w_0})$, $1 \le i \le N$). Theorem 3.1 implies that

$$(7.4) \qquad \sup_{0 \le j \le N} \sup_{0 \le t \le T} \mathbb{E}|z_j^{o,N}(t) - z_j^o(t)| = O(1/\sqrt{N}),$$

where the right hand side may depend upon the terminal time T.

Let $z(0) = \int_{\mathbb{R}^n} x dF(x)$ be the mean value of the minor agents' initial states (see (**A2**)). We denote

$$(\epsilon_N)^2 = \left| \int_{\mathbb{R}^N} x^T x dF_N(x) - 2z^T(0) \int_{\mathbb{R}^N} x dF_N(x) + z^T(0)z(0) \right|.$$

It is evident from (**A2**) that $\lim_{N \to \infty} \epsilon_N = 0$. To prove the ϵ-Nash equilibrium property we consider two cases as follows.

Case I (strategy change for the major agent $\mathcal{A}_0$): While the minor agents are using the SMFG best response control law $u^0(t, \omega, x)$, a strategy change from $u_0^0(t, \omega, x)$ to the $\mathcal{F}_t^{w_0}$-adapted process $u_0\left(t, \omega, x, z_{-0}^{o,N}(t, \omega)\right) \in \mathcal{U}_0$ for the major agent yields

$$dz_0^N(t) = \frac{1}{N} \sum_{j=1}^{N} f_0[t, z_0^N(t), u_0\left(t, z_0^N(t), z_{-0}^{o,N}(t)\right), z_j^{o,N}(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma_0[t, z_0^N(t), z_j^{o,N}(t)]dw_0(t), \quad z_0^N(0) = z_0(0), \ 0 \le t \le T,$$

where $z_{-0}^{o,N} \equiv (z_1^{o,N}, \cdots, z_N^{o,N})$. Since minor agents are coupled to the major agent only through their cost functions (see (**A13**)) the strategy change of the major agent does not affect the the minor agents' states $z_i^{o,N}$ and z_i^o, $1 \le i \le N$, above.

Let $\hat{z}_0^N(\cdot)$ be the solution to

$$d\hat{z}_0^N(t) = \frac{1}{N} \sum_{j=1}^{N} f_0[t, \hat{z}_0^N(t), u_0\left(t, \hat{z}_0^N(t), z_{-0}^o(t)\right), z_j^o(t)]dt$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \sigma_0[t, \hat{z}_0^N(t), z_j^o(t)]dw_0(t), \quad \hat{z}_0^N(0) = z_0(0), \ 0 \le t \le T,$$

where $z_{-0}^o \equiv (z_1^o, \cdots, z_N^o)$ is given by the MV system above. Theorem 3.1 and the Gronwall's lemma imply that

(7.5) $$\sup_{0 \le t \le T} \mathbb{E}|z_0^N(t) - \hat{z}_0^N(t)| = O(1/\sqrt{N}).$$

We also introduce

$$d\hat{z}_0(t) = f_0[t, \hat{z}_0(t), u_0(t, \hat{z}_0(t), z_{-0}^o(t)), \mu_t]dt + \sigma_0[t, \hat{z}_0(t), \mu_t]dw_0(t),$$

with initial condition $\hat{z}_0(0) = z_0(0)$, where $\mu_{(\cdot)}$ is the minor agents' measure given by the MV system above. Again, by Theorem 3.1 and the Gronwall's lemma It can be shown that

(7.6) $$\sup_{0 \le t \le T} \mathbb{E}|\hat{z}_0^N(t) - \hat{z}_0(t)| = O(1/\sqrt{N}).$$

(**A3**), (**A6**), (7.4)-(7.6) and Theorem 3.1 yield

(7.7) $$J_0^N(u_0; u_{-0}^o) \equiv \mathbb{E} \int_0^T \left((1/N) \sum_{j=1}^{N} L_0\left[t, z_0^N(t), u_0(t, z_0^N, z_{-0}^{o,N}), z_j^{o,N}(t)\right]\right) dt$$

$$\overset{(7.4)}{\ge} \mathbb{E} \int_0^T \left((1/N) \sum_{j=1}^{N} L_0\left[t, z_0^N(t), u_0(t, z_0^N, z_{-0}^o), z_j^o(t)\right]\right) dt - O(\epsilon_N + 1/\sqrt{N})$$

$$\overset{(7.5)}{\ge} \mathbb{E} \int_0^T \left((1/N) \sum_{j=1}^{N} L_0\left[t, \hat{z}_0^N(t), u_0(t, \hat{z}_0^N, z_{-0}^o), z_j^o(t)\right]\right) dt - O(\epsilon_N + 1/\sqrt{N})$$

$$\overset{(7.6)}{\ge} \mathbb{E} \int_0^T \left((1/N) \sum_{j=1}^{N} L_0\left[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), z_j^o(t)\right]\right) dt - O(\epsilon_N + 1/\sqrt{N})$$

$$\overset{(3.1)}{\ge} \mathbb{E} \int_0^T L_0\left[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), \mu_t\right] dt - O(\epsilon_N + 1/\sqrt{N}),$$

where the appearance of the ϵ_N term in the first inequality of (7.7) is due to the fact that here the sequence of minor agents' initials $\{z_j^o(0) : 1 \leq j \leq N\}$ in the SMV system (7.3) is generated by independent randomized observations on the distribution F given in (**A2**).

Furthermore, by the construction of the major agent's SMFG system (5.14)-(5.16) (see the major agent's SOCP (5.1)-(5.2)) we have

$$(7.8) \qquad \mathbb{E} \int_0^T L_0\big[t, \hat{z}_0(t), u_0(t, \hat{z}_0, z_{-0}^o), \mu_t\big] dt \geq \mathbb{E} \int_0^T L_0\big[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t\big] dt.$$

But, Theorem 3.1 and (7.4) imply

(7.9)

$$\mathbb{E} \int_0^T L_0\big[t, z_0^o(t), u_0^o(t, z_0^o), \mu_t\big] dt$$

$$\overset{(3.1)}{\geq} \mathbb{E} \int_0^T \Big((1/N) \sum_{j=1}^N L_0\big[t, z_0^o(t), u_0(t, z_0^o), z_j^o(t)\big]\Big) dt - O(\epsilon_N + 1/\sqrt{N})$$

$$\overset{(7.4)}{\geq} \mathbb{E} \int_0^T \Big((1/N) \sum_{j=1}^N L_0\big[t, z_0^{o,N}(t), u_0(t, z_0^{o,N}), z_j^{o,N}(t)\big]\Big) dt - O(\epsilon_N + 1/\sqrt{N})$$

$$\equiv J_0^N(u_0^o; u_{-0}^o) - O(\epsilon_N + 1/\sqrt{N}).$$

It follows from (7.7)-(7.9) that $J_0^N(u_0^o; u_{-0}^o) - O(\epsilon_N + 1/\sqrt{N}) \leq \inf_{u_0 \in \mathcal{U}_0} J_0^N(u_0; u_{-0}^o)$.

Case II (strategy change for the minor agents): Without loss of generality, we assume that the first minor agent changes its MF best response control strategy $u^o(t, \omega, x)$ to $u_1\big(t, \omega, x, z_{-1}(t, \omega)\big) \in \mathcal{U}_1$. This leads to

$$dz_0^N(t) = \frac{1}{N} \sum_{j=1}^N f_0[t, z_0^N, u_0^o(t, z_0^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma_0[t, z_0^N, z_j^N] dw_0(t),$$

$$dz_1^N(t) = \frac{1}{N} \sum_{j=1}^N f[t, z_1^N, u_1(t, z_1^N, z_{-1}^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_1^N, z_j^N] dw_1(t),$$

$$dz_2^N(t) = \frac{1}{N} \sum_{j=1}^N f[t, z_2^N, u^o(t, z_2^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_2^N, z_j^N] dw_2(t),$$

$$\vdots$$

$$dz_N^N(t) = \frac{1}{N} \sum_{j=1}^N f[t, z_N^N, u^o(t, z_N^N), z_j^N] dt + \frac{1}{N} \sum_{j=1}^N \sigma[t, z_N^N, z_j^N] dw_N(t).$$

By the same argument as in proving Theorem 3.1 (see Appendix B in [37]) it can be shown that

$$\sup_{j=0,2,\cdots,N} \sup_{0 \leq t \leq T} \mathbb{E}|z_j^{o,N}(t) - z_j^N(t)| = O(1/\sqrt{N}),$$

$$\sup_{j=0,2,\cdots,N} \sup_{0 \leq t \leq T} \mathbb{E}|z_j^o(t) - z_j^N(t)| = O(1/\sqrt{N}).$$

Let $\hat{z}_1^N(\cdot)$ be the solution to

$$d\hat{z}_1^N(t) = \frac{1}{N}\sum_{j=1}^N f[t, \hat{z}_1^N(t), u_1\big(t, \hat{z}_1^N(t), z_{-1}^o(t)\big), z_j^o(t)]dt$$

$$+ \frac{1}{N}\sum_{j=1}^N \sigma[t, \hat{z}_1^N(t), z_j^o(t)]dw_1(t), \quad \hat{z}_1^N(0) = z_1(0), \ 0 \le t \le T,$$

where $z_{-1}^o \equiv (z_1^o, \cdots, z_N^o)$ is given by the MV system above. Theorem 3.1 and the Gronwall's lemma implies that

(7.10) $$\sup_{0 \le t \le T} \mathbb{E}|z_1^N(t) - \hat{z}_1^N(t)| = O(1/\sqrt{N}).$$

We also introduce

$$d\hat{z}_1(t) = f[t, \hat{z}_1(t), u_1(t, \hat{z}_1(t), z_{-1}^o(t)), \mu_t]dt + \sigma[t, \hat{z}_1(t), \mu_t]dw_1(t),$$

with initial condition $\hat{z}_1(0) = z_1(0)$, where $\mu_{(\cdot)}$ is the minor agents' measure given by the MV system above. Again, by Theorem 3.1 and the Gronwall's lemma It can be shown that

(7.11) $$\sup_{0 \le t \le T} \mathbb{E}|\hat{z}_1^N(t) - \hat{z}_1(t)| = O(1/\sqrt{N}).$$

Using (7.4) and (7.10)-(7.11), and by the same argument as in (7.7)-(7.9) we can show that $J_1^N(u_1^o; u_{-1}^o) - O(\epsilon_N + 1/\sqrt{N}) \le \inf_{u \in \mathcal{U}_1} J_1^N(u_1; u_{-1}^o)$. $\square$

8. Conclusion. This paper studies a stochastic mean field game (SMFG) system for a class of dynamic games involving nonlinear stochastic dynamical systems with major and minor (MM) agents. The SMFG system consists of coupled (i) backward in time stochastic Hamilton-Jacobi-Bellman (SHJB) equations, and (ii) forward in time stochastic McKean-Vlasov (SMV) or stochastic Fokker-Planck-Kolmogorov (SFPK) equations. Existence and uniqueness of the solution to the MM-SMFG system is established by a fixed point argument in the Wasserstein space of random probability measures. In the case that minor agents are coupled to the major agent only through their cost functions, the ϵ_N-Nash equilibrium property of the SMFG best responses is shown for a finite N population system where $\epsilon_N = O(1/\sqrt{N})$. As a particular but important case, the results of Nguyen and Huang [35] for MM-SMFG linear-quadratic-Gaussian (LQG) systems with homogeneous population are retrieved, and, in addition, the results of this paper are illustrated with a major and minor agent version of a game model of the synchronization of coupled nonlinear oscillators (see Appendices G and H in [37]).

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Appendices[1]

Appendix A: Proof of Theorem 3.1 (McKean-Vlasov Convergence Result). We will show

$$\sup_{0 \le j \le N} \sup_{0 \le t \le T} \mathbb{E}|\hat{z}_j^N(t) - \bar{z}_j(t)|^2 = O(1/N),$$

which implies the result of the theorem by the Cauchy-Schwarz inequality. First by the inequality $(x+y)^2 \le 2x^2 + 2y^2$, we have

$$\mathbb{E}|\hat{z}_0^N(t) - \bar{z}_0(t)|^2$$

$$\le 2\mathbb{E}\Big|\int_0^t \Big(\frac{1}{N}\sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]\Big) ds\Big|^2$$

$$+ 2\mathbb{E}\Big|\int_0^t \Big(\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \hat{z}_0^N, \hat{z}_j^N] - \sigma_0[s, \bar{z}_0, \mu_s]\Big) dw_0(s)\Big|^2.$$

By the Cauchy-Schwarz inequality and the properties of Itô integrals we then obtain

(A.1) $\mathbb{E}|\hat{z}_0^N(t) - \bar{z}_0(t)|^2$

$$\le 2t\mathbb{E}\Big(\int_0^t \Big|\frac{1}{N}\sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]\Big|^2 ds\Big)$$

$$+ 2\mathbb{E}\Big(\int_0^t \Big|\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \hat{z}_0^N, \hat{z}_j^N] - \sigma_0[s, \bar{z}_0, \mu_s]\Big|^2 ds\Big).$$

Clearly,

(A.2) $\dfrac{1}{N}\sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]$

$$= \Big(\frac{1}{N}\sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - \frac{1}{N}\sum_{j=1}^N f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \hat{z}_j^N]\Big)$$

$$+ \Big(\frac{1}{N}\sum_{j=1}^N f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \hat{z}_j^N] - \frac{1}{N}\sum_{j=1}^N f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \bar{z}_j]\Big)$$

$$+ \Big(\frac{1}{N}\sum_{j=1}^N f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \bar{z}_j] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]\Big),$$

and

$$\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \hat{z}_0^N, \hat{z}_j^N] - \sigma_0[s, \bar{z}_0, \mu_s] = \Big(\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \hat{z}_0^N, \hat{z}_j^N] - \frac{1}{N}\sum_{j=1}^N \sigma_0[s, \bar{z}_0, \hat{z}_j^N]\Big)$$

$$+ \Big(\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \bar{z}_0, \hat{z}_j^N] - \frac{1}{N}\sum_{j=1}^N \sigma_0[s, \bar{z}_0, \bar{z}_j]\Big) + \Big(\frac{1}{N}\sum_{j=1}^N \sigma_0[s, \bar{z}_0, \bar{z}_j] - \sigma_0[s, \bar{z}_0, \mu_s]\Big).$$

Applying the inequality $(x+y+z)^2 \leq 3(x^2+y^2+z^2)$, and the Lipschitz continuity conditions of f_0 and φ_0 to (A.2) we obtain

(A.3) $\mathbb{E}\Big(\int_0^t \Big| \frac{1}{N} \sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s] \Big|^2 ds$

$$\leq 3C \int_0^t \mathbb{E}\big|\hat{z}_0^N(s) - \bar{z}_0(s)\big|^2 ds + 3C \int_0^t \mathbb{E}\big|\frac{1}{N} \sum_{j=1}^N \hat{z}_j^N(s) - \bar{z}_j(s)\big|^2 ds$$

$$+ 3C \int_0^t \mathbb{E}\big|\frac{1}{N} \sum_{j=1}^N f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \bar{z}_j] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]\big|^2 ds,$$

where $C > 0$ is a constant independent of N. Due to the centring of $g_s[s, \bar{z}_0, x] := f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), x] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s]$ with respect to x and the independence of $\bar{z}_j$ and $\bar{z}_{j'}$ when $j \neq j'$, there are no cross terms in the expansion of the last term in (A.3), i.e., $\mathbb{E}\big(g_s[s, \bar{z}_0, \bar{z}_j]g_s[s, \bar{z}_0, \bar{z}_{j'}]\big) = \mathbb{E}\mathbb{E}_{\mathcal{F}_t^{w_0}}\big(g_s[s, \bar{z}_0, \bar{z}_j]g_s[s, \bar{z}_0, \bar{z}_{j'}]\big) = 0$ for $j \neq j'$ (see [46], Page 175). This property together with (A.3), the boundedness of f_0 and the inequality $(\sum_{i=1}^N x_i)^2 \leq N \sum_{i=1}^N x_i^2$ yields

(A.4) $\mathbb{E}\Big(\int_0^t \Big| \frac{1}{N} \sum_{j=1}^N f_0[s, \hat{z}_0^N, \varphi_0(s, \hat{z}_0^N), \hat{z}_j^N] - f_0[s, \bar{z}_0, \varphi_0(s, \bar{z}_0), \mu_s] \Big|^2 ds$

$$\leq 3C \int_0^t \mathbb{E}\big|\hat{z}_0^N(s) - \bar{z}_0(s)\big|^2 ds + \frac{3C}{N} \int_0^t \sum_{j=1}^N \mathbb{E}|\hat{z}_j^N(s) - \bar{z}_j(s)|^2 ds + \frac{k_1(t)}{N},$$

where $k_1(t) \geq 0$ is an increasing function independent of N. Similarly, for the second term on the right hand side of (A.1) we have

(A.5) $\mathbb{E}\Big(\int_0^t \Big| \frac{1}{N} \sum_{j=1}^N \sigma_0[s, \hat{z}_0^N, \hat{z}_j^N] - \sigma_0[s, \bar{z}_0, \mu_s] \Big|^2 ds\Big)$

$$\leq 3C \int_0^t \mathbb{E}|\hat{z}_0^N(s) - \bar{z}_0(s)|^2 ds + \frac{3C}{N} \int_0^t \sum_{j=1}^N \mathbb{E}|\hat{z}_j^N(s) - \bar{z}_j(s)|^2 ds + \frac{k_1(t)}{N}.$$

The inequalities (A.1), (A.4) and (A.5) imply that

(A.6) $\sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_0^N(t) - \bar{z}_0(t)|^2 \leq 6C(T+1) \int_0^T \mathbb{E}|\hat{z}_0^N(s) - \bar{z}_0(s)|^2 ds$

$$+ \frac{6C(T+1)}{N} \int_0^T \sum_{j=1}^N \mathbb{E}|\hat{z}_j^N(s) - \bar{z}_j(s)|^2 ds + \frac{2(T+1)k_1(T)}{N}.$$

Second, by taking a similar approach for the ith minor agent $(1 \leq i \leq N)$ we get

(A.7) $\sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_i^N(t) - \bar{z}_i(t)|^2 \leq 8C(T+1) \int_0^T \mathbb{E}|\hat{z}_i^N(s) - \bar{z}_i(s)|^2 ds + \frac{k(T)}{N}$

$$+ 8C(T+1)\Big(\int_0^T \mathbb{E}|\hat{z}_0^N(s) - \bar{z}_0(s)|^2 ds + \frac{1}{N} \int_0^T \sum_{j=1}^N \mathbb{E}|\hat{z}_j^N(s) - \bar{z}_j(s)|^2 ds\Big),$$

where $k(T) > 0$ is independent of N.

The inequalities (A.6) and (A.7) yield

$$
(A.8) \qquad g^N(T) := \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_0^N(t) - \bar{z}_0(t)|^2 + \frac{1}{N} \sum_{j=1}^N \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_j^N(t) - \bar{z}_j(t)|^2
$$

$$
\leq 22C(T+1) \int_0^T \left(\mathbb{E}|\hat{z}_0^N(s) - \bar{z}_0(s)|^2 + \frac{1}{N} \sum_{j=1}^N \mathbb{E}|\hat{z}_j^N(s) - \bar{z}_j(s)|^2 \right) ds
$$

$$
+ \frac{k_0(T) + k(T)}{N} \leq 22C(T+1) \int_0^T g(s)ds + \frac{k_0(T) + k(T)}{N}.
$$

It follows from Gronwall's Lemma that

$$
(A.9) \qquad g^N(T) \leq \frac{k_0(T) + k(T)}{N} \Big(\exp \big(22C(T+1)T\big) \Big) = O(1/N),
$$

where the right hand side may only depend upon the terminal time T. This yields

$$
\sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_0^N(t) - \bar{z}_0(t)|^2 = O(1/N).
$$

The inequalities (A.7) and (A.9) combined with Gronwall's Lemma imply that

$$
\sup_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \mathbb{E}|\hat{z}_i^N(t) - \bar{z}_i(t)|^2 = O(1/N).
$$

This completes the proof.   □

Appendix B: Extended Itô-Kunita Formula. We recall an extended version of the Itô-Kunita formula [25] for the composition of stochastic processes (see Theorem 2.3 in [44]).

THEOREM B.1. *Let $\phi(t,x)$ be a stochastic process a.s. continuous in (t,x) such that (i) for each t, $\phi(t, \cdot)$ is a $C^2(\mathbb{R}^n)$ map a.s., (ii) for each x, $\phi(\cdot, x)$ is a continuous semi-martingale represented by*

$$
d\phi(t,x) = -\Gamma(t,x)dt + \sum_{k=1}^m \psi_k(t,x)dW_k(t), \qquad (t,x) \in [0,T] \times \mathbb{R}^n,
$$

where $\Gamma(t,x)$ and $\psi_k(t,x)$, $1 \leq k \leq m$, are $\mathcal{F}_t^W$-adapted stochastic processes which are continuous in (t,x) a.s., such that for each t, $\Gamma(t, \cdot)$ is a $C^1(\mathbb{R}^n)$ map a.s., and $\psi_k(t, \cdot)$, $1 \leq k \leq m$, are $C^2(\mathbb{R}^n)$ maps (a.s.).

Let $x(\cdot) = \big(x^1(\cdot), \cdots, x^n(\cdot)\big)$ be a continuous semi-martingale of the form

$$
dx^i(t) = f_i(t)dt + \sum_{k=1}^m \sigma_{ik}(t)dW_k(t) + \sum_{k=1}^m \varsigma_{ik}(t)dB_k(t), \quad 1 \leq i \leq n,
$$

where f_i, $\sigma_i = (\sigma_{i1}, \cdots, \sigma_{im})$ and $\varsigma_i = (\varsigma_{i1}, \cdots, \varsigma_{im})$, $1 \leq i \leq n$, are $\mathcal{F}_t^W$-adapted stochastic processes such that (i) f_i is an integrable process a.s., and (ii) σ_i and ς_i are square integrable processes (a.s.).

Then the composition map $\phi(\cdot, x(\cdot))$ is also a continuous semi-martingale which has the form

$$(\text{B.1}) \quad d\phi\big(t, x(t)\big) = -\Gamma\big(t, x(t)\big)dt + \sum_{k=1}^{m} \psi_k\big(t, x(t)\big)dW_k(t) + \sum_{i=1}^{n} \partial_{x_i}\phi\big(t, x(t)\big)f_i(t)dt$$

$$+ \sum_{i=1}^{n}\sum_{k=1}^{m} \partial_{x_i}\phi\big(t, x(t)\big)\sigma_{ik}(t)dW_k(t) + \sum_{i=1}^{n}\sum_{k=1}^{m} \partial_{x_i}\phi\big(t, x(t)\big)\varsigma_{ik}(t)dB_k(t)$$

$$+ \sum_{i=1}^{n}\sum_{k=1}^{m} \partial_{x_i}\psi_k\big(t, x(t)\big)\sigma_{ik}(t)dt + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} \partial^2_{x_i x_j}\phi\big(t, x(t)\big)\sigma_{ik}(t)\sigma_{jk}(t)dt$$

$$+ \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} \partial^2_{x_i x_j}\phi\big(t, x(t)\big)\varsigma_{ik}(t)\varsigma_{jk}(t)dt.$$

$\square$

Appendix C. We may write the functionals of $\mu^0_{(\cdot)}(\omega)$ and $\mu_{(\cdot)}(\omega)$ in (5.7)-(5.8) as random functions:

$$(\text{C.1}) \quad f^*[t, \omega, z_i, u_i] := f[t, z_i, u_i, \mu^0_t(\omega), \mu_t(\omega)], \quad \sigma^*[t, \omega, z_i] := \sigma[t, z_i, \mu^0_t(\omega), \mu_t(\omega)],$$

$$L^*[t, \omega, z_i, u_i] := L[t, z_i, u_i, \mu^0_t(\omega), \mu_t(\omega)].$$

We have the following proposition where its proof closely resembles that of Proposition 6.2 (see Proposition 4 in [20]).

PROPOSITION C.1. *Assume* (**A3**) *holds for* U. *Let* $\mu_t(\omega)$, $0 \le t \le T$, *be a fixed stochastic measure in the set* $\mathcal{M}^\beta_\rho$ *with* $0 < \beta < 1$, *and* $\mu^0_{(\cdot)}(\omega) = \Upsilon_0\big(\mu_{(\cdot)}(\omega)\big) \in \mathcal{M}^\gamma_\rho$, $0 < \gamma < 1/2$, *be the obtained probability measure of the major agent in Section 6.1. For* f^*, σ^* *and* L^* *defined in* (C.1) *we have:*

(i) *Under* (**A4**) *for* f *and* σ, *the functions* $f^*[t, \omega, z_i, u_i]$ *and* $\sigma^*[t, \omega, z_i]$ *and their first order derivatives (w.r.t* z_i) *are a.s. continuous and bounded on* $[0, T] \times \mathbb{R}^n \times U$ *and* $[0, T] \times \mathbb{R}^n$. $f^*[t, \omega, z_i, u_i]$ *and* $\sigma^*[t, \omega, z_i]$ *are a.s. Lipschitz continuous in* z_i. *In addition,* $f^*[t, \omega, 0, 0]$ *is in the space* $L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^n)$ *and* $\sigma^*[t, \omega, 0]$ *is in the space* $L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^{n \times m})$.

(ii) *Under* (**A5**) *for* f, *the function* $f^*[t, \omega, z_i, u_i]$ *is a.s. Lipschitz continuous in* $u_i \in U$, *i.e., there exist a constant* $c > 0$ *such that*

$$\sup_{t \in [0,T], z_i \in \mathbb{R}^n} \big| f^*[t, \omega, z_i, u_i] - f^*[t, \omega, z_i, u'_i]\big| \le c(\omega)|u_i - u'_i|, \quad (a.s.).$$

(iii) *Under* (**A6**) *for* L, *the function* $L^*[t, \omega, z_i, u_i]$ *and its first order derivative (w.r.t* z_i) *is a.s. continuous and bounded on* $[0, T] \times \mathbb{R}^n \times U$. *It is a.s. Lipschitz continuous in* z_i. *In addition,* $L^*[t, \omega, 0, 0] \in L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}_+)$.

(iv) *Under* (**A8**) *for* H^u, *the set of minimizers*

$$\arg\inf_{u_i \in U} \big\{ \langle f^*[t, \omega, z_i, u_i], p \rangle + L^*[t, \omega, z_i, u_i] \big\},$$

is a singleton for any $p \in \mathbb{R}^n$, *and the resulting* u_i *as a function of* $[t, \omega, z_i, p]$ *is a.s. continuous in* t, *a.s. Lipschitz continuous in* (z_i, p), *uniformly with respect to* t. *In addition,* $u_i[t, \omega, 0, 0]$ *is in the space* $L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^n)$.

$\square$

Appendix D: Proof of Theorems 6.7. Let $\omega \in \Omega$ be fixed. For given probability measure $\mu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$, we can show that the law of the process $z_i^o(\cdot, \omega, \omega')$ given in (6.9), $\Lambda\big(z_i^o(\cdot, \omega, \omega')\big)$, belongs to $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$ (see Theorem 6.8).

We take $\mu_{(\cdot)}(\omega)$, $\nu_{(\cdot)}(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$. Let $z_i^o(\cdot, \omega, \omega')$ be defined by (6.9), and similarly $x_i^o(\cdot, \omega, \omega')$ be defined by (6.9) after replacing $\mu_{(\cdot)}(\omega)$ by $\nu_{(\cdot)}(\omega)$. We have

(D.1)
$$\mathbb{E}_{\mathcal{F}_t^{w_0}} \sup_{0 \le s \le t} \big|z_i^o(s, \omega) - x_i^o(s, \omega)\big|^2$$

$$\le 2t \int_0^t \Big| \int_{\mathbb{R}^n \times \mathbb{R}^n} f[s, z_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z)$$

$$- \int_{\mathbb{R}^n \times \mathbb{R}^n} f[s, x_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\nu_s(\omega)(z) \Big|^2 ds$$

$$+ 2 \int_0^t \Big| \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma[s, z_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z)$$

$$- \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma[s, x_i^o, y, z] d\mu_s^0(\omega)(y) d\nu_s(\omega)(z) \Big|^2 ds.$$

But,

$$\Big| \int f[s, z_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z) - \int f[s, x_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\nu_s(\omega)(z) \Big|^2$$

$$\le 2C \Big(|z_i^o(s) - x_i^o(s)|^2 + \int_{C_\rho \times C_\rho} |z_s(\omega_1) - z_s(\omega_2)|^2 d\gamma(\omega_1, \omega_2) \Big),$$

where C is obtained from the boundedness and Lipschitz continuity of both f and u^o, and $\gamma \in \mathcal{M}(C_\rho \times C_\rho)$ is any coupling of μ and ν where $\gamma(A \times C([0, T]; \mathbb{R}^n)) = \mu(A)$ and $\gamma(C([0, T]; \mathbb{R}^n) \times A) = \nu(A)$ for any Borel set $A \in C([0, T]; \mathbb{R}^n)$. Taking the infimum over all such γ couplings and then using the definition of metrics $\rho_{(\cdot)}$ and $D_{(\cdot)}^\rho$ yields

(D.2)
$$\Big| \int f[s, z_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z) - \int f[s, x_i^o, u_i^o, y, z] d\mu_s^0(\omega)(y) d\nu_s(\omega)(z) \Big|^2$$

$$\le 2C \Big(\rho_s\big(z_i^o(s), x_i^o(s)\big) + \big(D_s^\rho(\mu, \nu)\big)^2 \Big).$$

Similarly we have

(D.3)
$$\Big| \int \sigma[s, z_i^o, y, z] d\mu_s^0(\omega)(y) d\mu_s(\omega)(z) - \int \sigma[s, x_i^o, y, z] d\mu_s^0(\omega)(y) d\nu_s(\omega)(z) \Big|^2$$

$$\le 2C_1 \Big(\rho_s\big(z_i^o(s), x_i^o(s)\big) + \big(D_s^\rho(\mu, \nu)\big)^2 \Big),$$

where C_1 is obtained from the boundedness and Lipschitz continuity of both σ.

It follows from (D.1)-(D.3) that

(D.4)
$$\mathbb{E}_{\mathcal{F}_t^{w_0}} \rho_t\big(z_i^o(\omega), x_i^o(\omega)\big) \equiv \mathbb{E}_{\mathcal{F}_t^{w_0}} \sup_{0 \le s \le t} \big|z_i^o(s, \omega) - x_i^o(s, \omega)\big|^2 \wedge 1$$

$$\le 2(Ct + C_1) \int_0^t \Big(\rho_s\big(z_i^o(\omega), x_i^o(\omega)\big) + \big(D_s^\rho(\mu(\omega), \nu(\omega))\big)^2 \Big) ds,$$

which by Gronwall's lemma yields

$$\mathbb{E}_{\mathcal{F}_t^{w_0}} \rho_t\big(z_i^o(\omega), x_i^o(\omega)\big) \le 2(CT + C_1) \exp\big(2(CT + C_1)\big) \int_0^t \Big(D_s^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 ds.$$

This together with the definition of the Wasserstein metric $D_{(\cdot)}^\rho$ leads to the contraction inequality:

$$\Big(D_t^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 \le 2(CT + C_1) \exp\big(2(CT + C_1)\big) \int_0^t \Big(D_s^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 ds.$$

By following a similar argument as in [46] (Theorem 1.1), we can show that $\{\Lambda^k(\mu(\omega)) : k \ge 1\}$ forms a Cauchy sequence a.s. in the complete metric space $\mathcal{M}_\rho^\beta$, $0 < \beta < 1$, and converges a.s. to a unique (a.s.) fixed point of Λ. $\square$

Appendix E: Proof of Lemma 6.9. (i) (5.16) gives

$$z_0(s, \omega) = z_0(0) + \int_0^s \Big(\int_{\mathbb{R}^n} f_0[\tau, z_0, u_0, y] d\mu_\tau(\omega)(y) \Big) d\tau + \int_0^s \sigma_0[\tau] dw_0(\tau, \omega),$$

$$z_0'(s, \omega) = z_0(0) + \int_0^s \Big(\int_{\mathbb{R}^n} f_0[\tau, z_0', u_0', y] d\mu_\tau(\omega)(y) \Big) d\tau + \int_0^s \sigma_0[\tau] dw_0(\tau, \omega),$$

corresponding to the control processes u_0 and u_0' in $C_{\mathrm{Lip}(x)}([0, T] \times \Omega \times \mathbb{R}^n; U_0)$. By the Lipschitz continuity of f_0 (see (**A4**) and (**A5**)) there are positive constants C_0 and C_1 such that

$$|z_0(s, \omega) - z_0'(s, \omega)|^2 \le 2C_0 s \int_0^s |z_0(\tau, \omega) - z_0'(\tau, \omega)|^2 d\tau$$
$$+ 2C_1 s^2 \sup_{(t,x) \in [0,T] \times \mathbb{R}^n} \big|u_0(t, \omega, x) - u_0'(t, \omega, x)\big|^2.$$

The Gronwall's lemma yields

$$\rho_t\big(z_0(\omega), z_0'(\omega)\big) \le 2C_1 t^2 \exp(2C_0 t) \sup_{t,x} \big|u_0(t, \omega, x) - u_0'(t, \omega, x)\big|^2.$$

This together with the fact that $\mu_t^0(\omega) = \delta_{z_0(t,\omega)}$ and $\nu_t^0(\omega) = \delta_{z_0'(t,\omega)}$, and the definition of the Wasserstein metric $D_{(\cdot)}^\rho$ leads to (6.12) where $c_0 := 2C_1 T^2 \exp(2C_0 T)$.

(ii) We have

$$z_0(s, \omega) = z_0(0) + \int_0^s \Big(\int_{\mathbb{R}^n} f_0[\tau, z_0, u_0^o, y] d\mu_\tau(\omega)(y) \Big) d\tau + \int_0^s \sigma_0[\tau] dw_0(\tau, \omega),$$

$$z_0'(s, \omega) = z_0(0) + \int_0^s \Big(\int_{\mathbb{R}^n} f_0[\tau, z_0', u_0^o, y] d\nu_\tau(\omega)(y) \Big) d\tau + \int_0^s \sigma_0[\tau] dw_0(\tau, \omega),$$

corresponding to the stochastic measures $\mu(\omega), \nu(\omega) \in \mathcal{M}_\rho^\beta$, $0 < \beta < 1$. By the Lipschitz continuity of f_0 (see (**A4**) and (**A5**)) and u_0^o there are positive constants C_0 and C_1 such that

$$|z_0(s, \omega) - z_0'(s, \omega)|^2 \le 2C_0 s \int_0^s |z_0(\tau, \omega) - z_0'(\tau, \omega)|^2 d\tau$$
$$+ 2C_1 s^2 \Big(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2.$$

The Gronwall's lemma yields

$$\rho_t\big(z_0(\omega), z_0'(\omega)\big) \leq 2C_1 t^2 \exp(2C_0 t)\Big(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2.$$

This together with the fact that $\mu_t^0(\omega) = \delta_{z_0(t,\omega)}$ and $\nu_t^0(\omega) = \delta_{z_0'(t,\omega)}$, and the definition of the Wasserstein metric $D_{(\cdot)}^\rho$ leads to (6.13) where $c_1 := 2C_1 T^2 \exp(2C_0 T)$.

    (iii) (5.19) gives

$$z_i(s,\omega,\omega') = z_i(0) + \int_0^t \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f[s, z_i, u, y, z]d\mu_s^0(\omega)(y)d\mu_s(\omega)(z)\Big)ds$$

$$+ \int_0^t \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma[s, z_i, y, z]d\mu_s^0(\omega)(y)d\mu_s(\omega)(z)\Big)dw_i(s,\omega'),$$

$$z_i'(s,\omega,\omega') = z_i(0) + \int_0^t \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f[s, z_i', u', y, z]d\mu_s^0(\omega)(y)d\nu_s(\omega)(z)\Big)ds$$

$$+ \int_0^t \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma[s, z_i', y, z]d\mu_s^0(\omega)(y)d\nu_s(\omega)(z)\Big)dw_i(s,\omega'),$$

corresponding to the control processes u and u' in $C_{\mathrm{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U)$. By the Lipschitz continuity of f and σ (see (**A4**) and (**A5**)) there are positive constants C_0, C_1 and C_2 such that

$$\mathbb{E}_\omega |z_i(s,\omega,\omega') - z_i'(s,\omega,\omega')|^2 \leq 2(3C_0 s + 2C_1)\mathbb{E}_\omega \int_0^s |z_0(\tau,\omega) - z_0'(\tau,\omega)|^2 d\tau$$

$$+ 2(3C_0 s + 2C_1)\mathbb{E}_\omega \int_0^s \Big(D_\tau^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 d\tau$$

$$+ 6C_2 s^2 \sup_{t,x} \mathbb{E}_\omega \big|u(t,\omega,x) - u'(t,\omega,x)\big|^2.$$

The Gronwall's lemma yields

$$\rho_t\big(z_i(s,\omega), z_i'(s,\omega)\big) \leq 2(3C_0 t + 2C_1)\exp\big(2(3C_0 t + 2C_1)\big)\int_0^t \Big(D_\tau^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 d\tau$$

$$+ 6C_2 t^2 \exp\big(2(3C_0 t + 2C_1)\big)\sup_{t,x}\big|u(t,\omega,x) - u'(t,\omega,x)\big|^2.$$

This together with the definition of the Wasserstein metric $D_{(\cdot)}^\rho$ leads to

$$\Big(D_T^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 \leq K(T)\int_0^T \Big(D_\tau^\rho\big(\mu(\omega), \nu(\omega)\big)\Big)^2 d\tau$$

$$+ K'(T)\sup_{t,x}\big|u(t,\omega,x) - u'(t,\omega,x)\big|^2,$$

where $K(T) := 2(3C_0 T + 2C_1)\exp\big(2(3C_0 T + 2C_1)\big)$ and $K'(T) := 6C_2 T^2 \exp\big(2(3C_0 T + 2C_1)\big)$. Applying the Gronwall's lemma gives (6.14) with $c_2 := K'(T)\exp(K(T))$.

    (iv) The proof of this part closely resembles that of Part (iii).   □

    Appendix F: The Sensitivity Analysis of the SHJB Equations. In this section we study the sensitivity of the major and minor agents' SHJB equations (5.14) and (5.17) to the stochastic measures $\mu_{(\cdot)}(\omega)$ and $\mu_{(\cdot)}^0(\omega)$ in order to show the feedback

regularity conditions. The analysis of this section is based on the framework of Section 6 of [23].

First we consider a family of stochastic optimal control problems (SOCP) (4.1)-(4.2) parameterized by $\alpha \in \mathbb{R}$. In this α-parameterized formulation called $(\text{SOCP})_\alpha$: (i) the dynamics of the states $z^\alpha(t, \omega)$, denoted by $(4.1)_\alpha$, are of the form (4.1) with $f[t, \omega, z, u]$, $\sigma[t, \omega, z]$ and $\varsigma[t, \omega, z]$ replaced by $f^\alpha[t, \omega, z^\alpha, u^\alpha]$, $\sigma^\alpha[t, \omega, z^\alpha]$ and $\varsigma^\alpha[t, \omega, z^\alpha]$, respectively, and (ii) the cost functions $J^\alpha(u^\alpha)$, denoted by $(4.2)_\alpha$, are of the form (4.2) with $L[t, \omega, z, u]$ replaced by $L^\alpha[t, \omega, z^\alpha, u^\alpha]$.

The value functions $\phi^\alpha(\cdot, x(\cdot))$ correspond to the $(\text{SOCP})_\alpha$ are defined similar to (4.3) with $L[t, \omega, z, u]$ replaced by $L^\alpha[t, \omega, z^\alpha, u^\alpha]$. Based on [44] we shall restrict to the case where $\phi^\alpha(\cdot, x(\cdot))$ are semi-martingales of the form (4.7) with $\Gamma(\cdot, x(\cdot))$ and $\psi(\cdot, x(\cdot))$ are replaced by $\Gamma^\alpha(\cdot, x(\cdot))$ and $\psi^\alpha(\cdot, x(\cdot))$, respectively.

If the α-parameterized family of processes $\phi^\alpha(t, x)$, $\Gamma^\alpha(t, x)$ and $\psi^\alpha(t, x)$ are a.s. continuous in (x, t) and are smooth enough with respect to x, then by using the analysis in [44] we can show that the pairs $\big(\phi^\alpha(s, x), \psi^\alpha(s, x)\big)$ satisfy the following backward in time α-parameterized stochastic Hamilton-Jacobi-Bellman $(\text{SHJB})_\alpha$ equations:

(F.1) $\quad - d\phi^\alpha(t, \omega, x) = \Big[H^\alpha[t, \omega, x, D_x\phi^\alpha(t, \omega, x)] + \big\langle \sigma^\alpha[t, \omega, x], D_x\psi^\alpha(t, \omega, x) \big\rangle$

$$+ \frac{1}{2}\text{tr}\big(a^\alpha[t, \omega, x]D^2_{xx}\phi^\alpha(t, \omega, x)\big)\Big]dt - (\psi^\alpha)^T(t, \omega, x)dW(t, \omega), \ \phi^\alpha(T, x) = 0,$$

where $a^\alpha[t, \omega, x] := \sigma^\alpha[t, \omega, x]\big(\sigma^\alpha[t, \omega, x]\big)^T + \varsigma^\alpha[t, \omega, x]\big(\varsigma^\alpha[t, \omega, x]\big)^T$, and the stochastic Hamiltonians $H^\alpha : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are given by

$$H^\alpha[t, \omega, x, p] := \inf_{u^\alpha \in \mathcal{U}} \big\{ \big\langle f^\alpha[t, \omega, x, u], p \big\rangle + L^\alpha[t, \omega, x, u] \big\}.$$

Suppose the assumptions (**H1**)-(**H3**) hold for $(f^\alpha, L^\alpha, \sigma^\alpha, \varsigma^\alpha)$. Then the $(\text{SHJB})_\alpha$ equations (F.1) have unique solutions (see Theorem 4.1 or Theorem 4.1 in [44]):

$$(\phi^\alpha(t, x), \psi^\alpha(t, x)) \in \big(L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}), L^2_{\mathcal{F}_t}([0, T]; \mathbb{R}^m)\big), \qquad \forall \alpha \in \mathbb{R}.$$

The forward in time $\mathcal{F}^W_t$-adapted optimal control processes of the $(\text{SOCP})_\alpha$ $(4.1)_\alpha$-$(4.2)_\alpha$ are given by (see [44])

(F.2) $\quad u^{\alpha,o}(t, \omega, x) := \arg\inf_{u^\alpha \in U} H^{\alpha,u}[t, \omega, x, D_x\phi^\alpha(t, \omega, x), u^\alpha]$

$$= \arg\inf_{u^\alpha \in U} \big\{ \big\langle f^\alpha[t, \omega, x, u^\alpha], D_x\phi^\alpha(t, \omega, x) \big\rangle + L^\alpha[t, \omega, x, u^\alpha] \big\}.$$

We set

$$g^\alpha[t, \omega, x, \phi^\alpha(t, \omega, x), \psi^\alpha(t, \omega, x)] := H^\alpha[t, \omega, x, D_x\phi^\alpha(t, \omega, x)]$$
$$+ \big\langle \sigma^\alpha[t, \omega, x], D_x\psi^\alpha(t, \omega, x) \big\rangle,$$
$$A^\alpha(t, \omega, x)(\cdot) := \frac{1}{2}\text{tr}\big(a^\alpha[t, \omega, x]D^2_{xx}(\cdot)\big),$$

where A^α in $[0, T] \times \Omega \times \mathbb{R}^n$ is an operator on $C^2(\mathbb{R}^n)$. We may now rewrite the backward in time α-parameterized $(\text{SHJB})_\alpha$ equations (F.1) as

(F.3) $\quad d\phi^\alpha(t, \omega, x) + A^\alpha(t, \omega, x)\big(\phi^\alpha(t, \omega, x)\big)dt$

$$= -g^\alpha[t, \omega, x, \phi^\alpha(t, \omega, x), \psi^\alpha(t, \omega, x)]dt + (\psi^\alpha)^T(t, \omega, x)dW(t, \omega),$$

with $\phi^\alpha(T, x) = 0$.

At this point we introduce the mild form of (F.3) because this form is more suitable for the sensitivity analysis of this section. We note that it is sufficient to consider the mild solution in the analysis of existence and uniqueness of solutions to the SMFG system.

If the pair $(\phi^\alpha(t, x), \psi^\alpha(t, x))$ is a smooth solution to (F.3) that satisfies the following mild form by a Duhamel Principle [23]:

(F.4)
$$\phi^\alpha(t, \omega, x) = \int_t^T \exp\Big(\int_t^s A^\alpha(\tau, \omega, x) d\tau \Big) \Big(g^\alpha[s, \omega, x, \phi^\alpha(s, \omega, x), \psi^\alpha(s, \omega, x)] \Big) ds$$
$$- \int_t^T \exp\Big(\int_t^s A^\alpha(\tau, \omega, x) d\tau \Big) \Big((\psi^\alpha)^T(s, \omega, x) \Big) dW(s, \omega).$$

We define the operators:

$$\Phi^\alpha(t, s, \omega, x)(\cdot) = \exp\Big(\int_t^s A^\alpha(\tau, \omega, x)(\cdot) d\tau \Big) \equiv \exp\Big(\int_t^s \frac{1}{2} \mathrm{tr}\big(a^\alpha[\tau, \omega, x] D_{xx}^2(\cdot) \big) d\tau,$$

$$\Psi^\alpha(t, s, \omega, x)(\cdot) = \int_t^s \partial_\alpha A^\alpha(\tau, \omega, x)(\cdot) d\tau \equiv \int_t^s \frac{1}{2} \mathrm{tr}\big(\partial_\alpha a^\alpha[\tau, \omega, x] D_{xx}^2(\cdot) \big) d\tau,$$

in $[0, T] \times \Omega \times \mathbb{R}^n$ which are maps on $C^\infty(\mathbb{R}^n)$ and $C^2(\mathbb{R}^n)$, respectively.

Differentiating (F.4) with respect to α gives

(F.5) $\qquad \partial_\alpha \phi^\alpha(t, \omega, x) = \int_t^T \big(\Phi^\alpha(t, s, \omega, x) \big) \big(\Psi^\alpha(t, s, \omega, x) \big)$

$$\big(g^\alpha[s, \omega, x, \phi^\alpha(s, \omega, x), \psi^\alpha(s, \omega, x)] \big) ds$$

$$+ \int_t^T \big(\Phi^\alpha(t, s, \omega, x) \big) \big(\partial_\alpha g^\alpha[s, \omega, x, \phi^\alpha(s, \omega, x), \psi^\alpha(s, \omega, x)] \big) ds$$

$$- \int_t^T \big(\Phi^\alpha(t, s, \omega, x) \big) \big(\Psi^\alpha(t, s, \omega, x) \big) \big((\psi^\alpha)^T(s, \omega, x) \big) dW(s, \omega)$$

$$- \int_t^T \big(\Phi^\alpha(t, s, \omega, x) \big) \big((\partial_\alpha \psi^\alpha)^T(s, \omega, x) \big) dW(s, \omega),$$

where

$$\partial_\alpha g^\alpha[t, \omega, x, \phi^\alpha(t, \omega, x), \psi^\alpha(t, \omega, x)] \equiv \partial_\alpha H^\alpha[t, \omega, x, D_x \phi^\alpha(t, \omega, x)]$$
$$+ \partial_p H^\alpha[t, \omega, x, D_x \phi^\alpha(t, \omega, x)] D_x \big(\partial_\alpha \phi^\alpha(t, \omega, x) \big)$$
$$+ \big\langle \partial_\alpha \sigma^\alpha[t, \omega, x], D_x \psi^\alpha(t, \omega, x) \big\rangle + \big\langle \sigma^\alpha[t, \omega, x], D_x \big(\partial_\alpha \psi^\alpha(t, \omega, x) \big) \big\rangle.$$

We may rewrite (F.5) as

$$
\begin{aligned}
\text{(F.6)} \qquad \partial_\alpha \phi^\alpha(t,\omega,x) = & \int_t^T \big(\Phi^\alpha(t,s,\omega,x)\big) A_1^\alpha(s,\omega,x)\big(\partial_\alpha \phi^\alpha(t,\omega,x)\big) ds \\
& + \int_t^T \big(\Phi^\alpha(t,s,\omega,x)\big)\Big(h_1^\alpha[t,s,\omega,x,\partial_\alpha\psi^\alpha]\Big) ds \\
& - \int_t^T \big(\Phi^\alpha(t,s,\omega,x)\big)\Big((\partial_\alpha\psi^\alpha)^T(s,\omega,x)\Big) dW(s,\omega), \\
& - \int_t^T \big(\Phi^\alpha(t,s,\omega,x)\big)\Big(h_2^\alpha[t,s,\omega,x]\Big) dW(s,\omega),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^\alpha(s,\omega,x)(\cdot) := & \; \partial_p H^\alpha[s,\omega,x,D_x\phi^\alpha(s,\omega,x)] D_x(\cdot), \\
h_1^\alpha[t,s,\omega,x,\partial_\alpha\psi^\alpha] := & \; \big(\Psi^\alpha(t,s,\omega,x)\big)\Big(g^\alpha[s,\omega,x,\phi^\alpha(s,\omega,x),\psi^\alpha(s,\omega,x)]\Big) \\
& + \partial_\alpha H^\alpha[s,\omega,x,D_x\phi^\alpha(s,\omega,x)] + \big\langle \partial_\alpha\sigma^\alpha[s,\omega,x], D_x\psi^\alpha(s,\omega,x)\big\rangle \\
& + \big\langle \sigma^\alpha[s,\omega,x], D_x\big(\partial_\alpha\psi^\alpha\big)\big\rangle, \\
h_2^\alpha[t,s,\omega,x] := & \; \big(\Psi^\alpha(t,s,\omega,x)\big)\Big((\psi^\alpha)^T(s,\omega,x)\Big).
\end{aligned}
$$

We introduce the following assumption:

(**H5**) $\partial_\alpha f^\alpha[t,x,u]$, $\partial_\alpha L^\alpha[t,x,u]$, $\partial_\alpha \sigma^\alpha[t,x]$ and $\partial_\alpha \varsigma^\alpha[t,x]$ exist and are $C^\infty(\mathbb{R}^n)$. Assume (**H1**)-(**H3**) hold where (f,L,σ,ς) are replaced by $(\partial_\alpha f^\alpha, \partial_\alpha L^\alpha, \partial_\alpha \sigma^\alpha, \partial_\alpha \varsigma^\alpha)$, and all the boundedness assumptions are uniformly.

PROPOSITION F.1. *Assume (**H11**)-(**H3**) hold for $(f^\alpha, L^\alpha, \sigma^\alpha, \varsigma^\alpha)$. Let the pair $(\phi^\alpha(t,x), \psi^\alpha(t,x))$ be the unique solution to (F.1) which are $C^\infty(\mathbb{R}^n)$ and a.s. uniformly bounded. In addition, we assume (**H5**) holds. Then, the equation (F.5) has a unique solution*

$$
(\partial_\alpha \phi(t,x), \partial_\alpha \psi(t,x)) \in \big(L_{\mathcal{F}_t}^2([0,T];\mathbb{R}), L_{\mathcal{F}_t}^2([0,T];\mathbb{R}^m)\big)
$$

such that $\sup_{0\le t\le T} |D_x\partial_\alpha\phi(t,\cdot)| < \infty$ *(a.s.).*

Proof: The proof of existence and uniqueness of solution to (F.6) follows from Theorem 4.1 in [15] (see the proof of Theorem 4.1 in [44], see also [30, 32, 14] or Chapter 5 of [31]). By taking the conditional expectation $\mathbb{E}_{\mathcal{F}_t^{w_0}}$ of the square of both sides of (F.6) and the boundedness assumptions in the theorem, one can show $\sup_{0\le t\le T} |\partial_\alpha\phi(t,\cdot)| < \infty$ (a.s.) (see the proof of Theorem 2.1 in [44]). Using this in equation (F.6) implies the boundedness of $D_x\partial_\alpha\phi(t,\cdot)$. $\qquad\square$

Appendix G: The Major and Minor (MM) SMFG Linear-Qudratic-Gaussian (LQG) System. We consider the MM LQG dynamic game problem of

[16]. In this case all functions in (2.1)-(2.4) are given by (see Remark 2.1)

$$f_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] = A_0 z_0^N(t) + B_0 u_0^N(t) + F_0 z_j^N(t),$$

$$f[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] = A z_i^N(t) + B u_i^N(t) + F z_j^N(t) + G z_0^N(t),$$

$$\sigma_0[t, z_0^N(t), z_j^N(t)] = S_0, \qquad\qquad \sigma[t, z_i^N(t), z_0^N(t), z_j^N(t)] = S,$$

$$L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] = \left[z_0^N(t) - \left(H_0 \big(\frac{1}{N} \sum_{j=1}^N z_j^N(t)\big) + \eta_0 \right) \right]^T Q_0$$

$$\times \left[z_0^N(t) - \left(H_0 \big(\frac{1}{N} \sum_{j=1}^N z_j^N(t)\big) + \eta_0 \right) \right]^T + (u_0^N(t))^T R_0 u_0^N(t),$$

$$L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] = \left[z_i^N(t) - \left(H z_0^N(t) + \hat{H} \big(\frac{1}{N} \sum_{j=1}^N z_j^N(t)\big) + \eta \right) \right]^T Q$$

$$\times \left[z_i^N(t) - \left(H z_0^N(t) + \hat{H} \big(\frac{1}{N} \sum_{j=1}^N z_j^N(t)\big) + \eta \right) \right] + (u_i^N(t))^T R u_i^N(t),$$

with the deterministic constant matrices: (i) $A_0, F_0, A, F, G, H_0, H$ and $\hat{H}$ in $\mathbb{R}^{n \times n}$, (ii) B_0 and B in $\mathbb{R}^{n \times k}$, (iii) S_0 and S in $\mathbb{R}^{n \times m}$, (iv) the symmetric nonnegative definite matrices Q_0 and Q in $\mathbb{R}^{n \times n}$, (v) the symmetric positive definite matrices R_0 and R in $\mathbb{R}^{k \times k}$, and the deterministic constant vectors η and η_0 are in $\mathbb{R}^n$.

In this formulation the major agent's SMFG system (5.14)-(5.16) is of the form

$$(G.1) - d\phi_0(t, \omega, x) = \left[\left\langle A_0 x - \frac{1}{4} B_0 R_0^{-1} B_0^T D_x \phi_0(t, \omega, x) + F_0 z^o(t, \omega), D_x \phi_0(t, \omega, x) \right\rangle \right.$$

$$+ \left\langle x - (H_0 z^o(t, \omega) + \eta_0), Q_0 \big(x - (H_0 z^o(t, \omega) + \eta_0) \big) \right\rangle$$

$$\left. + \left\langle S_0, D_x \psi_0(t, \omega, x) \right\rangle + \frac{1}{2} \mathrm{tr}\big((S_0^T S_0) D_{xx}^2 \phi_0(t, \omega, x) \big) \right] dt$$

$$- \psi_0^T(t, \omega, x) dw_0(t, \omega), \qquad\qquad \phi_0(T, x) = 0,$$

$$(G.2) \quad u_0^o(t, \omega, x) = -\frac{1}{2} R_0^{-1} B_0^T D_x \phi_0(t, \omega, x),$$

$$(G.3) \quad dz_0^o(t, \omega) = \left[A_0 z_0^o(t, \omega) + B_0 u_0^o(t, \omega, z_0^o) + F_0 z^o(t, \omega) \right] dt$$

$$+ S_0 dw_0(t, \omega), \qquad\qquad z_0^o(0) = z_0(0),$$

and the minor agents' SMFG system (5.17)-(5.19) is given by

$$(G.4) - d\phi(t, \omega, x) = \left[\left\langle A x - \frac{1}{4} B R^{-1} B^T D_x \phi(t, \omega, x) + F x + G z_0^o(t, \omega), D_x \phi(t, \omega, x) \right\rangle \right.$$

$$+ \left\langle x - (H z_0^o(t, \omega) + \hat{H} x + \eta), Q \big(x - (H z_0^o(t, \omega) + \hat{H} x + \eta) \big) \right\rangle$$

$$\left. + \frac{1}{2} \mathrm{tr}\big((S^T S) D_{xx}^2 \phi(t, \omega, x) \big) \right] dt - \psi^T(t, \omega, x) dw(t, \omega), \quad \phi_0(T, x) = 0,$$

$$(G.5) \quad u^o(t, \omega, x) = -\frac{1}{2} R^{-1} B^T D_x \phi(t, \omega, x),$$

$$(G.6) \quad dz^o(t, \omega) = \left[A z^o(t, \omega) + B u^o(t, \omega, z^o) + F_0 z^o(t, \omega) + G z_0^o(t, \omega) \right] dt$$

$$+ S dw(t, \omega), \qquad\qquad z_0^o(0) = z_0(0).$$

Let $\Pi_0(\cdot) \geq 0$ be the unique solution of the deterministic Riccati equation

$$\partial_t \Pi_0(t) + \Pi_0(t)A_0 + A_0^T \Pi_0(t) - \Pi_0(t)B_0 R_0^{-1} B_0^T \Pi_0(t) + Q_0 = 0, \quad \Pi_0(T) = 0.$$

We denote $\mathbb{A}_0(\cdot) = A_0 - B_0 R_0^{-1} B_0^T \Pi_0(\cdot)$. It can be verified that the pair $(\phi_0, \psi_0)(t, \omega, x)$ in (5.14) is given by

$$\phi_0(t, \omega, x) = x^T \Pi_0(t)x + 2x^T s_0(t, \omega) + g_0(t, \omega),$$
$$\psi_0^T(t, \omega, x) = 2x^T q_0(t, \omega) + h_0(t, \omega),$$

where $(s_0, q_0)(t, \omega)$ and $(g_0, h_0)(t, \omega)$ are unique solutions of the following Backward Stochastic Differential Equations (BSDEs):

$$-ds_0(t, \omega) = \left[\mathbb{A}_0^T(t)s_0(t, \omega) + \left(\Pi_0(t)F_0 - Q_0 H_0 \right) z^o(t, \omega) - Q_0 \eta_0 \right] dt$$
$$- q_0(t, \omega)dw_0(t, \omega), \qquad\qquad s_0(T) = 0,$$
$$-dg_0(t, \omega) = \left[-s_0^T(t, \omega)B_0 R_0^{-1} B_0^T s_0(t, \omega) + 2F_0 z^o(t, \omega) + 2 \operatorname{tr}\left(S_0^T q_0(t, \omega) \right) \right.$$
$$\left. + \left(H_0 z^o(t, \omega) + \eta_0 \right)^T Q_0 \left(H_0 z^o(t, \omega) + \eta_0 \right) + \operatorname{tr}\left(S_0^T S_0 \Pi_0(t) \right) \right] dt$$
$$- h_0(t, \omega)dw_0(t, \omega), \qquad\qquad g_0(T) = 0.$$

We may now express the major agent's SMFG LQG system (G.1)-(G.3) in the following form:

$$-ds_0(t, \omega) = \left[\mathbb{A}_0^T(t)s_0(t, \omega) + \left(\Pi_0(t)F_0 - Q_0 H_0 \right) z^o(t, \omega) - Q_0 \eta_0 \right] dt$$
$$- q_0(t, \omega)dw_0(t, \omega), \qquad\qquad s_0(T) = 0,$$
$$u_0^o(t, \omega) = -R_0^{-1} B_0^T \left(\Pi_0(t)z_0^o(t, \omega) + s_0(t, \omega) \right),$$
$$dz_0^o(t, \omega) = \left[\mathbb{A}_0(t)z_0^o(t, \omega) - B_0 R_0^{-1} B_0^T \Pi_0(t)s_0(t, \omega) + F_0 z^o(t, \omega) \right] dt$$
$$+ S_0 dw_0(t, \omega), \qquad\qquad z_0^o(0) = z_0(0),$$

where $z^o(t, \omega)$ is the mean field behaviour of the minor agents (see the minor agents' SMFG LQG system below).

In a similar way, let $\Pi(\cdot) \geq 0$ be the unique solution of the deterministic Riccati equation

$$\partial_t \Pi(t) + \Pi(t)A + A^T \Pi(t) - \Pi(t)B R^{-1} B^T \Pi(t) + Q = 0, \quad \Pi(T) = 0.$$

We denote $\mathbb{A}(\cdot) = A - B R^{-1} B^T \Pi(\cdot)$. It can be verified that the pair $(\phi, \psi)(t, \omega, x)$ in (5.17) is given by

$$\phi(t, \omega, x) = x^T \Pi(t)x + 2x^T s(t, \omega) + g(t, \omega),$$
$$\psi^T(t, \omega, x) = 2x^T q(t, \omega) + h(t, \omega),$$

where $(s,q)(t,\omega)$ and $(g,h)(t,\omega)$ are unique solutions of the following BSDEs:

$$
\begin{aligned}
-ds(t,\omega) = \Big[& \mathbb{A}^T(t)s(t,\omega) + \big(\Pi(t)F - Q\hat{H}\big)z^o(t,\omega) + \big(\Pi(t)G - QH\big)z_0^o(t,\omega) \\
& - Q\eta \Big]dt - q(t,\omega)dw_0(t,\omega), \qquad\qquad\qquad s(T) = 0,
\end{aligned}
$$

$$
\begin{aligned}
-dg(t,\omega) = \Big[& -s^T(t,\omega)BR^{-1}B^T s(t,\omega) + 2Fz^o(t,\omega) + 2Gz_0^o(t,\omega) \\
& + \big(\hat{H}z^0(t,\omega) + Hz_0^o(t,\omega) + \eta\big)^T Q_0\big(\hat{H}z^0(t,\omega) + Hz_0^o(t,\omega) + \eta\big) \\
& + \mathrm{tr}\big(S^T S\Pi(t)\big)\Big]dt - h(t,\omega)dw_0(t,\omega), \qquad\quad g(T) = 0.
\end{aligned}
$$

We may now express the minor agents' SMFG LQG system (G.4)-(G.6) in the following form:

$$
\begin{aligned}
-ds(t,\omega) = \Big[& \mathbb{A}^T(t)s(t,\omega) + \big(\Pi(t)F - Q\hat{H}\big)z^o(t,\omega) + \big(\Pi(t)G - QH\big)z_0^o(t,\omega) \\
& - Q\eta \Big]dt - q(t,\omega)dw_0(t,\omega), \qquad\qquad\qquad s(T) = 0,
\end{aligned}
$$

$$
u^o(t,\omega) = -R^{-1}B^T\big(\Pi(t)z^o(t,\omega) + s(t,\omega)\big),
$$

$$
\begin{aligned}
dz^o(t,\omega) = \Big[& \big(\mathbb{A}(t) + F\big)z^o(t,\omega) - BR^{-1}B^T\Pi(t)s(t,\omega) + Gz_0^o(t,\omega)\Big]dt \\
& + Sdw(t,\omega), \qquad\qquad\qquad\qquad\qquad\qquad z^o(0) = z(0).
\end{aligned}
$$

So we retrieve the MM-SMFG system for LQG dynamic games model of [35] for minor agents with uniform parameters (see equations (10)-(11) and (22)-(23) in [35], see also [16]). The reader is referred to [35] for an explicit representation of a solution to the SMFG LQG system under some appropriate conditions.

We note that key assumption for solution existence and uniqueness of MM-SMFG system is that all drift and cost functions and their derivatives are bounded (see Section 2.1) which clearly does not hold for the MM-SMFG LQG problem (as in classical LQG control). In this case, a generalized Four-Step Scheme (see Section 5.2 in Chapter 7 of [53]) seems to give not only weaker general conditions but also presents explicit solutions to the MM-SMFG LQG case. This is currently under investigation and will be reported in future work.

Appendix H: A Nonlinear Example. In this section we present a major and minor version of the synchronization of coupled nonlinear oscillators game model [50]. Consider a population of $N + 1$ oscillators with dynamics

$$
\text{(H.1)} \qquad d\theta_j^N(t) = u_j^N(t)dt + \sigma dw_j(t) \pmod{2\pi} \qquad 0 \le j \le N, \quad t \ge 0,
$$

where $\theta_j(t) \in [0, 2\pi]$ is the phase of the jth oscillator at time t, $u_j(\cdot)$ is the control input, σ is a non-negative scalar, and $\{w_j : 0 \le j \le N\}$ denotes a sequence of independent standard scalar Wiener processes (see [50]). It is assumed that the initial states $\{\theta_j(0)\}$ are chosen independently on $[0, 2\pi]$. The objective of the jth oscillator

is to minimize its own cost function

(H.2) $J_0^N(u_0^N, u_{-0}^N) := \mathbb{E} \int_0^T \Big(\frac{1}{N} \sum_{k=1}^N \sin^2 \big[\theta_0^N(t) - \theta_k^N(t)\big] + r\big(u_0^N(t)\big)^2 \Big) dt,$

(H.3) $J_i^N(u_i^N, u_{-i}^N) := \mathbb{E} \int_0^T \Big(\frac{1}{N} \sum_{k=1}^N \sin^2 \big[\theta_i^N(t) - \big(\lambda \theta_0^N(t) + (1-\lambda)\theta_k^N(t)\big)\big]$

$$+ r\big(u_i^N(t)\big)^2 \Big) dt, \qquad\qquad 1 \le i \le N,$$

where r is a positive scalar and $\lambda \in (0,1)$.

Similar arguments in previous section yield the following major agent's SMFG system (5.14)-(5.16):

$$- d\phi_0(t,\omega,x) = \Big[-\frac{1}{4r}(\partial_x \phi_0(t,\omega,x))^2 + m_0(t,\omega,x) + \sigma \partial_x \psi_0(t,\omega,x)$$

$$+ \frac{\sigma^2}{2} \partial_{xx}^2 \phi_0(t,\omega,x) \Big] dt - \psi_0(t,\omega,x) dw_0(t,\omega), \qquad \phi_0(T,x) = 0,$$

$$u_0^o(t,\omega,x) = -\frac{1}{2r} \partial_x \phi_0(t,\omega,x),$$

$$dp_s^0(t,\omega,x) = \Big[\frac{1}{2r} \partial_x \Big((\partial_x \phi_0(t,\omega,x)) p_s^0(t,\omega,x) \Big) + \frac{\sigma^2}{2} \partial_{xx}^2 p_s^0(t,\omega,x) \Big] dt$$

$$- \sigma \partial_x p_s^0(t,\omega,x) dw_0(t,\omega), \qquad\qquad p_s^0(s,x) = \delta_{\theta_0^o(s)}(dx),$$

$$m_0(t,\omega,x) = \int_0^{2\pi} \sin^2(x - \theta) p(t,\omega,\theta) d\theta,$$

where $m_0(t,\omega,x)$ is called the infinite population cost-coupling of the major agent, and $\theta_0^o(\cdot)$ is the solution of the closed-loop equation

$$d\theta_0^o(t) = u_0^o(t,\theta_0^o(t)) dt + \sigma dw_0(t) \pmod{2\pi} \qquad\qquad t \ge 0.$$

In a similar way, the minor agents' SMFG system (5.13) and (5.17)-(5.18) is given by

$$- d\phi(t,\omega,x) = \Big[-\frac{1}{4r}(\partial_x \phi(t,\omega,x))^2 + m(t,\omega,x) + \frac{\sigma^2}{2} \partial_{xx}^2 \phi(t,\omega,x) \Big] dt$$

$$- \psi(t,\omega,x) dw(t,\omega), \qquad\qquad \phi(T,x) = 0,$$

$$u^o(t,\omega,x) = -\frac{1}{2r} \partial_x \phi(t,\omega,x),$$

$$dp(t,\omega,x) = \Big[\frac{1}{2r} \partial_x \Big((\partial_x \phi(t,\omega,x)) p(t,\omega,x) \Big) + \frac{\sigma^2}{2} \partial_{xx}^2 p(t,\omega,x) \Big] dt, \quad p(0,x)$$

$$m(t,\omega,x) = \int_0^{2\pi} \int_0^{2\pi} \sin^2\big(x - (\lambda \theta_0 + (1-\lambda)\theta)\big) p_0^0(t,\omega,\theta_0) p(t,\omega,\theta) d\theta_0 d\theta,$$

where $m(t,\omega,x)$ is called the infinite population cost-coupling of the major agent. The reader is referred to the deterministic mean field system (14a)-(14c) in [50] for the synchronization of coupled nonlinear oscillators game model with only minor agents.