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Document Version

Final published version

Published in:

SIAM Journal on Financial Mathematics

DOI:

[10.1137/15M1011731](https://doi.org/10.1137/15M1011731)

Publication date:

2015

License

Unspecified

Citation for published version (APA):

Grbac, Z., Papapantoleon, A., Schoenmakers, J., & Skovmand, D. (2015). Affine LIBOR Models with Multiple Curves: Theory, Examples and Calibration. *SIAM Journal on Financial Mathematics*, 6(1), 984–1025. <https://doi.org/10.1137/15M1011731>

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Journal article (Publishers version)

This is the publishers version of an article published by *SIAM Journal on Financial Mathematics*, Vol. 6, Issue 1, Pages 984–1025.

DOI: <http://dx.doi.org/10.1137/15M1011731>

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Affine LIBOR Models with Multiple Curves: Theory, Examples and Calibration*

Zorana Grbac[†], Antonis Papapantoleon[‡], John Schoenmakers[§], and David Skovmand[¶]

Abstract. We introduce a multiple curve framework that combines tractable dynamics and semianalytic pricing formulas with positive interest rates and basis spreads. Negative rates and positive spreads can also be accommodated in this framework. The dynamics of overnight indexed swap and LIBOR rates are specified following the methodology of the affine LIBOR models and are driven by the wide and flexible class of affine processes. The affine property is preserved under forward measures, which allows us to derive Fourier pricing formulas for caps, swaptions, and basis swaptions. A model specification with dependent LIBOR rates is developed that allows for an efficient and accurate calibration to a system of caplet prices.

Key words. multiple curve models, LIBOR, OIS, basis spread, affine LIBOR models, caps, swaptions, basis swaptions, calibration

AMS subject classifications. 91G30, 91G20, 60G44

DOI. 10.1137/15M1011731

1. Introduction. The recent financial crisis has led to paradigm shifting events in interest rate markets because substantial spreads have appeared between rates that used to be closely matched; see Figure 1 for an illustration. We can observe, for example, that before the credit crunch the spread between the three-month LIBOR and the corresponding overnight indexed swap (OIS) rate was nonzero; however, it could be safely disregarded as negligible. The same is true for the three-month versus six-month basis swap spread. However, since August 2007 these spreads have been evolving randomly over time, are substantially too large to be neglected, and also depend on the tenor length. Therefore, the assumption of a single interest rate curve that could be used both for discounting and for generating future cash flows was seriously challenged, which led to the introduction of the so-called *multiple curve* interest rate models.

In the multiple curve framework, one curve is used for discounting purposes, where the usual choice is the OIS curve, and then as many LIBOR curves as market tenors (e.g., 1 month, 3 month, 6 month, and 1 year) are built for generating future cash flows. The difference

*Received by the editors March 10, 2015; accepted for publication (in revised form) August 10, 2015; published electronically October 20, 2015. All authors gratefully acknowledge the financial support from the DFG Research Center MATHEON, Projects E5 and E13.

<http://www.siam.org/journals/sifin/6/M101173.html>

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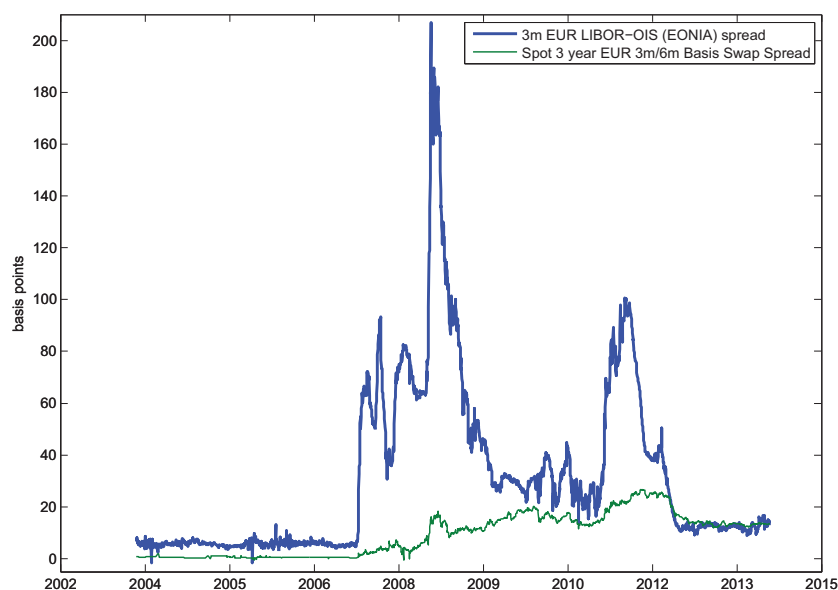


Figure 1. Spread development from January 2004 to April 2014.

between the OIS and each LIBOR rate is usually called basis spread or simply basis. There are several ways of modeling the curves and different definitions of the spread. One approach is to model the OIS and LIBOR rates directly which leads to tractable pricing formulas, but the sign of the spread is more difficult to control and may become negative. Another approach is to model the OIS and the spread directly and infer the dynamics of the LIBOR; this grants the positivity of the spread, but pricing formulas are generally less tractable. We refer to Mercurio (2010b, pp. 11–12) for a detailed discussion of the advantages and disadvantages of each approach. Moreover, there exist various definitions of the spread: an additive spread is used, e.g., by Mercurio (2010a), a multiplicative spread was proposed by Henrard (2010), while an instantaneous spread was used by Andersen and Piterbarg (2010); we refer to Mercurio and Xie (2012) for a discussion of the merits of each definition.

The literature on multiple curve models is growing rapidly and the different models proposed can be classified in one of the categories described above. Moreover, depending on the modeling approach, one can also distinguish between short rate models, Heath–Jarrow–Morton (HJM) models, and LIBOR market models (LMM) with multiple curves. The spreads appearing as modeling quantities in the short rate and the HJM models are, by the very nature of these models, instantaneous and given in additive form. We refer to Bianchetti and Morini (2013) for a detailed overview of several multiple curve models. In the short rate framework, we mention Kenyon (2010), Kijima, Tanaka, and Wong (2009), and Morino and Runggaldier (2014), where the additive short rate spread is modeled, which leads to multiplicative adjustments for interest rate derivative prices. HJM-type models have been proposed e.g., by Fujii, Shimada, and Takahashi (2011), Crépey, Grbac, and Nguyen (2012), Moreni and Pallavicini (2014), Crépey et al. (2015a), and Cuchiero, Fontana, and Gnoatto (2014). The models by

Mercurio (2009), Bianchetti (2010) (where an analogy with the cross-currency market has been exploited), and Henrard (2010) are developed in the LMM setup, while multiple curve extensions of the Flesaker and Hughston (1996) framework have been proposed in Nguyen and Seifried (2015) and Crépey et al. (2015b). Typically, multiple curve models address the issue of different interest rate curves under the same currency; however, the paper by Fujii, Shimada, and Takahashi (2011) studies a multiple curve model in a cross-currency setup. Filipović and Trolle (2013) offer a thorough econometric analysis of the multiple curve phenomena and decompose the spread into a credit risk and a liquidity risk component. In recent work, Gallitschke, Müller, and Seifried (2014) construct a structural model for interbank rates, which provides an endogenous explanation for the emergence of basis spreads.

Another important change due to the crisis is the emergence of significant counterparty risk in financial markets. In this paper, we consider the *clean valuation* of interest rate derivatives meaning that we do not take into account the default risk of the counterparties involved in a contract. As explained in Crépey et al. (2015a) and Morino and Runggaldier (2014), this is sufficient for calibration to market data which correspond to fully collateralized contracts. The price adjustments due to counterparty and funding risk for two particular counterparties can then be obtained on top of the clean prices; cf. Crépey et al. (2015a), Crépey et al. (2015b), and, in particular, Papapantoleon and Wardenga (2015) for computations in affine LIBOR models.

Let us also mention that there exist various other frameworks in the literature where different curves have been modeled simultaneously, for example, when dealing with cross-currency markets (cf., e.g., Amin and Jarrow (1991)) or when considering credit risk (cf., e.g., the book by Bielecki and Rutkowski (2002)). The models in the multiple curve world often draw inspiration from these frameworks.

The aim of this paper is to develop a multiple curve LIBOR model that combines tractable model dynamics and semianalytic pricing formulas with positive interest rates and basis spreads. The framework of the affine LIBOR models proposed by Keller-Ressel, Papapantoleon, and Teichmann (2013) turns out to be tailor-made for this task, since it allows us to model directly LIBOR rates that are greater than their OIS counterparts. In other words, the nonnegativity of spreads is automatically ensured. Simultaneously, the dynamics are driven by the wide and flexible class of affine processes. Similarly to the single curve case, the affine property is preserved under all forward measures, which leads to semianalytical pricing formulas for liquid interest rate derivatives. In particular, the pricing of caplets is as easy as in the single curve setup, while the model structure allows us to derive efficient and accurate approximations for the pricing of swaptions and basis swaptions using a linearization of the exercise boundary. In addition, the model offers adequate calibration results to a system of caplet prices for various strikes and maturities.

The paper is organized as follows: in section 2 we review the main properties of affine processes and the construction of ordered martingales greater than one. Section 3 introduces the multiple curve interest rate setting. The multiple curve affine LIBOR model is presented in section 4 and its main properties are discussed, in particular, the ability to produce positive rates and spreads and the analytical tractability (i.e., the preservation of the affine property). A model that allows for negative interest rates and positive spreads is also presented. In section 5 we study the connection between the class of affine LIBOR models and the class of

LMMs (driven by semimartingales). Sections 6 and 7 are devoted to the valuation of the most liquid interest rate derivatives such as swaps, caps, swaptions, and basis swaptions. In section 8 we construct a multiple curve affine LIBOR model where rates are driven by common and idiosyncratic factors and calibrate it market data. Moreover, we test numerically the accuracy of the swaption and basis swaption approximation formulas. Section 9 contains some concluding remarks and comments on future research. Finally, Appendix A provides an explicit formula for the terminal correlation between LIBOR rates.

2. Affine processes. This section provides a brief review of the main properties of affine processes and the construction of ordered martingales greater than one. More details and proofs can be found in Keller-Ressel, Papapantoleon, and Teichmann (2013) and the references therein.

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ denote a complete stochastic basis, where $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T]}$ and T denotes some finite time horizon. Consider a stochastic process X satisfying the following:

Assumption (A). Let $X = (X_t)_{t \in [0, T]}$ be a conservative, time-homogeneous, stochastically continuous Markov process with values in $D = \mathbb{R}_{\geq 0}^d$, and $(\mathbb{P}_x)_{x \in D}$ a family of probability measures on (Ω, \mathcal{F}) , such that $X_0 = x$, \mathbb{P}_x -almost surely for every $x \in D$. Setting

$$(2.1) \quad \mathcal{I}_T := \left\{ u \in \mathbb{R}^d : \mathbb{E}_x [e^{\langle u, X_T \rangle}] < \infty, \text{ for all } x \in D \right\},$$

we assume that

- (i) $0 \in \mathcal{I}_T^\circ$, where \mathcal{I}_T° denotes the interior of \mathcal{I}_T (with respect to the topology induced by the Euclidean norm on \mathbb{R}^d);
- (ii) the conditional moment generating function of X_t under \mathbb{P}_x has exponentially affine dependence on x ; that is, there exist deterministic functions $\phi_t(u) : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}$ and $\psi_t(u) : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}^d$ such that

$$(2.2) \quad \mathbb{E}_x [\exp \langle u, X_t \rangle] = \exp (\phi_t(u) + \langle \psi_t(u), x \rangle)$$

for all $(t, u, x) \in [0, T] \times \mathcal{I}_T \times D$.

Here $\langle \cdot, \cdot \rangle$ denotes the inner product on \mathbb{R}^d and \mathbb{E}_x the expectation with respect to \mathbb{P}_x . Moreover, it holds that $\mathcal{I}_T \subseteq \mathcal{I}_t$ for $t \leq T$; cf. Keller-Ressel and Mayerhofer (2015, Theorem 2.14). In other words, if $u \in \mathbb{R}^d$ is such that $\mathbb{E}_x [e^{\langle u, X_T \rangle}] < \infty$, then $\mathbb{E}_x [e^{\langle u, X_t \rangle}] < \infty$ for every $t \leq T$.

The functions ϕ and ψ satisfy the following system of ODEs, known as *generalized Riccati equations*:

$$(2.3a) \quad \frac{\partial}{\partial t} \phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0,$$

$$(2.3b) \quad \frac{\partial}{\partial t} \psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u,$$

for $(t, u) \in [0, T] \times \mathcal{I}_T$. The functions F and R are of Lévy–Khintchine form:

$$(2.4a) \quad F(u) = \langle b, u \rangle + \int_D (e^{\langle \xi, u \rangle} - 1) m(d\xi),$$

$$(2.4b) \quad R_i(u) = \langle \beta_i, u \rangle + \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \int_D (e^{\langle \xi, u \rangle} - 1 - \langle u, h_i(\xi) \rangle) \mu_i(d\xi),$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1 \leq i \leq d}$ are *admissible parameters* and $h_i : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}^d$ are suitable truncation functions. The functions ϕ and ψ also satisfy the semiflow equations

$$(2.5a) \quad \phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u)),$$

$$(2.5b) \quad \psi_{t+s}(u) = \psi_s(\psi_t(u))$$

for all $0 \leq t + s \leq T$ and $u \in \mathcal{I}_T$, with initial condition

$$(2.6) \quad \phi_0(u) = 0 \quad \text{and} \quad \psi_0(u) = u.$$

We refer to [Duffie, Filipović, and Schachermayer \(2003\)](#) for all the details.

The following definition will be used later, where $\mathbf{1} := (1, 1, \dots, 1)$.

Definition 2.1. *Let X be a process satisfying Assumption (A). Define*

$$(2.7) \quad \gamma_X := \sup_{u \in \mathcal{I}_T \cap \mathbb{R}_{>0}^d} \mathbb{E}_1[e^{\langle u, X_T \rangle}].$$

The quantity γ_X measures the ability of an affine process to fit the initial term structure of interest rates and equals infinity for several models used in mathematical finance, such as the Cox–Ingersoll–Ross process and Ornstein–Uhlenbeck models driven by subordinators; cf. [Keller-Ressel, Papapantoleon, and Teichmann \(2013, section 8\)](#).

An essential ingredient in affine LIBOR models is the construction of parametrized martingales which are greater than or equal to one and increasing in this parameter (see also [Papapantoleon \(2010\)](#)).

Lemma 2.2. *Consider an affine process X satisfying Assumption (A) and let $u \in \mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$. Then the process $M^u = (M_t^u)_{t \in [0, T]}$ with*

$$(2.8) \quad M_t^u = \exp(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle),$$

is a martingale, greater than or equal to one, and the mapping $u \mapsto M_t^u$ is increasing, for every $t \in [0, T]$.

Proof. Consider the random variable $Y_T^u := e^{\langle u, X_T \rangle}$. Since $u \in \mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$ we have that Y_T^u is greater than one and integrable. Then, from the Markov property of X , (2.2), and the tower property of conditional expectations we deduce that

$$(2.9) \quad M_t^u = \mathbb{E}[e^{\langle u, X_T \rangle} | \mathcal{F}_t] = \exp(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle)$$

is a martingale. Moreover, it is obvious that $M_t^u \geq 1$ for all $t \in [0, T]$, while the ordering

$$(2.10) \quad u \leq v \implies M_t^u \leq M_t^v \quad \forall t \in [0, T],$$

follows from the ordering of Y_T^u and the representation $M_t^u = \mathbb{E}[Y_T^u | \mathcal{F}_t]$. ■

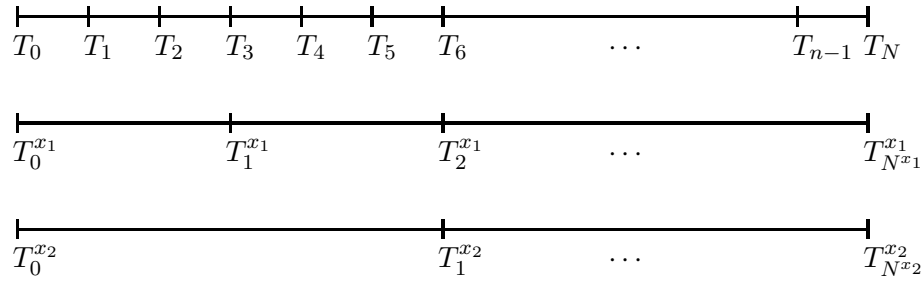


Figure 2. Illustration of different tenor structures.

3. A multiple curve LIBOR setting. We begin by introducing the notation and the main concepts of multiple curve LIBOR models. We will follow the approach introduced in Mercurio (2010a), which has become the industry standard in the meantime.

The fact that LIBOR-OIS spreads are now tenor dependent means that we cannot work with a single tenor structure any longer. Hence, we start with a discrete, equidistant time structure $\mathcal{T} = \{0 = T_0 < T_1 < \dots < T_N\}$, where $T_k, k \in \mathcal{K} := \{1, \dots, N\}$, denote the maturities of the assets traded in the market. Next, we consider different subsets of \mathcal{T} with equidistant time points, i.e., different tenor structures $\mathcal{T}^x = \{0 = T_0^x < T_1^x < \dots < T_{N^x}^x\}$, where $x \in \mathcal{X} := \{x_1, x_2, \dots, x_n\}$ is a label that indicates the tenor structure. Typically, we have $\mathcal{X} = \{1, 3, 6, 12\}$ months. We denote the tenor length by $\delta_x = T_k^x - T_{k-1}^x$ for every $x \in \mathcal{X}$. Let $\mathcal{K}^x := \{1, 2, \dots, N^x\}$ denote the collection of all subscripts related to the tenor structure \mathcal{T}^x . We assume that $\mathcal{T}^x \subseteq \mathcal{T}$ and $T_{N^x}^x = T_N$ for all $x \in \mathcal{X}$. A graphical illustration of a possible relation between the different tenor structures appears in Figure 2.

Example 3.1. A natural construction of tenor structures is the following: let $\mathcal{T} = \{0 = T_0 < T_1 < \dots < T_N\}$ denote a discrete time structure, where $T_k = k\delta$ for $k = 1, \dots, N$ and $\delta > 0$. Let $\mathcal{X} = \{1 = x_1, x_2, \dots, x_n\} \subset \mathbb{N}$, where we assume that x_k is a divisor of N for all $k = 1, \dots, n$. Next, set for every $x \in \mathcal{X}$

$$T_k^x = k \cdot \delta \cdot x =: k\delta_x \quad \text{for } k = 1, \dots, N^x := N/x,$$

where obviously $T_k^x = T_{kx}$. Then, we can consider different subsets of \mathcal{T} , i.e., different tenor structures $\mathcal{T}^x = \{0 = T_0^x < T_1^x < \dots < T_{N^x}^x\}$, which satisfy by construction $\mathcal{T}^x \subset \mathcal{T}^{x_1} = \mathcal{T}$ and also $T_{N^x}^x = N^x \cdot \delta \cdot x = T_N$, for all $x \in \mathcal{X}$.

We consider the OIS curve as a discount curve, following the standard market practice of fully collateralized contracts. The market prices for caps and swaptions considered in the following for model calibration are indeed quoted under the assumption of full collateralization. A detailed discussion on the choice of the discount curve in the multiple curve setting can be found e.g., in Mercurio (2010a) and in Hull and White (2013). The discount factors $B(0, T)$ are stripped from market OIS rates and defined for every possible maturity $T \in \mathcal{T}$ via

$$T \mapsto B(0, T) = B^{OIS}(0, T).$$

We denote by $B(t, T)$ the discount factor, i.e., the price of a zero coupon bond, at time t for maturity T , which is assumed to coincide with the corresponding OIS-based zero coupon bond for the same maturity.

We also assume all our modeling objects live on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_N)$, where \mathbb{P}_N denotes the terminal forward measure, i.e., the martingale measure associated with the numeraire $B(\cdot, T_N)$. The corresponding expectation is denoted by \mathbb{E}_N . Then, we introduce forward measures \mathbb{P}_k^x associated with the numeraire $B(\cdot, T_k^x)$ for every pair (x, k) with $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. The corresponding expectation is denoted by \mathbb{E}_k^x . The forward measures \mathbb{P}_k^x are absolutely continuous with respect to \mathbb{P}_N , and defined in the usual way, i.e., via the Radon–Nikodym density

$$(3.1) \quad \frac{d\mathbb{P}_k^x}{d\mathbb{P}_N} = \frac{B(0, T_N)}{B(0, T_k^x)} \frac{1}{B(T_k^x, T_N)}.$$

Remark 3.2. Since $\mathcal{T}^x \subseteq \mathcal{T}$ there exists an $l \in \mathcal{K}$ and a $k \in \mathcal{K}^x$ such that $T_l = T_k^x$. Therefore, the corresponding numeraires and forward measures coincide, i.e., $B(\cdot, T_l) = B(\cdot, T_k^x)$ and $\mathbb{P}_l = \mathbb{P}_k^x$. See, again, Figure 2.

Next, we define the two rates that are the main modeling objects in the multiple curve LIBOR setting: the forward OIS rate and the forward LIBOR rate. We also define the additive and the multiplicative spread between these two rates. Let us denote by $L(T_{k-1}^x, T_k^x)$ the *spot LIBOR rate* at time T_{k-1}^x for the time interval $[T_{k-1}^x, T_k^x]$, which is an $\mathcal{F}_{T_{k-1}^x}$ -measurable random variable on the given stochastic basis.

Definition 3.3. The time- t forward OIS rate for the time interval $[T_{k-1}^x, T_k^x]$ is defined by

$$(3.2) \quad F_k^x(t) := \frac{1}{\delta_x} \left(\frac{B(t, T_{k-1}^x)}{B(t, T_k^x)} - 1 \right).$$

Definition 3.4. The time- t forward LIBOR rate for the time interval $[T_{k-1}^x, T_k^x]$ is defined by

$$(3.3) \quad L_k^x(t) = \mathbb{E}_k^x [L(T_{k-1}^x, T_k^x) | \mathcal{F}_t].$$

The forward LIBOR rate is the fixed rate that should be exchanged for the future spot LIBOR rate so that the forward rate agreement has zero initial value. Hence, this rate reflects the market expectations about the value of the future spot LIBOR rate. Notice that at time $t = T_{k-1}^x$ we have that

$$(3.4) \quad L_k^x(T_{k-1}^x) = \mathbb{E}_k^x [L(T_{k-1}^x, T_k^x) | \mathcal{F}_{T_{k-1}^x}] = L(T_{k-1}^x, T_k^x),$$

i.e., this rate coincides with the spot LIBOR rate at the corresponding tenor dates.

Remark 3.5. In the single curve setup, (3.2) is the definition of the forward LIBOR rate. However, in the multiple curve setup we have that

$$L(T_{k-1}^x, T_k^x) \neq \frac{1}{\delta_x} \left(\frac{1}{B(T_{k-1}^x, T_k^x)} - 1 \right),$$

hence the OIS and the LIBOR rates are no longer equal.

Definition 3.6. The spread between the LIBOR and the OIS rate is defined by

$$(3.5) \quad S_k^x(t) := L_k^x(t) - F_k^x(t).$$

Let us also provide an alternative definition of the spread based on a multiplicative, instead of an additive, decomposition.

Definition 3.7. *The multiplicative spread between the LIBOR and the OIS rate is defined by*

$$(3.6) \quad 1 + \delta_x R_k^x(t) := \frac{1 + \delta_x L_k^x(t)}{1 + \delta_x F_k^x(t)}.$$

A model for the dynamic evolution of the OIS and LIBOR rates, and thus also of their spread, should satisfy certain conditions which stem from economic reasoning, arbitrage requirements, and their respective definitions. These are, in general, consistent with market observations. We formulate them below as model requirements:

- (M1) $F_k^x(t) \geq 0$ and $F_k^x \in \mathcal{M}(\mathbb{P}_k^x)$ for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.
- (M2) $L_k^x(t) \geq 0$ and $L_k^x \in \mathcal{M}(\mathbb{P}_k^x)$ for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.
- (M3) $S_k^x(t) \geq 0$ and $S_k^x \in \mathcal{M}(\mathbb{P}_k^x)$ for all $x \in \mathcal{X}$, $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

Here $\mathcal{M}(\mathbb{P}_k^x)$ denotes the set of \mathbb{P}_k^x -martingales.

Remark 3.8. If the additive spread is positive, the multiplicative spread is also positive and vice versa.

4. The multiple curve affine LIBOR model. We describe next the affine LIBOR model for the multiple curve interest rate setting and analyze its main properties. In particular, we show that this model produces positive rates and spreads, i.e., it satisfies the modeling requirements (M1)–(M3) and is analytically tractable. OIS and LIBOR rates are modeled in the spirit of the affine LIBOR model introduced by Keller-Ressel, Papapantoleon, and Teichmann (2013).

Let X be an affine process defined on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P}_N)$, satisfying Assumption (A) and starting at the canonical value $\mathbf{1}$. Consider a fixed $x \in \mathcal{X}$ and the associated tenor structure \mathcal{T}^x . We construct two families of parametrized martingales following Lemma 2.2: take two sequences of vectors $(u_k^x)_{k \in \mathcal{K}^x}$ and $(v_k^x)_{k \in \mathcal{K}^x}$, and define the \mathbb{P}_N -martingales $M_t^{u_k^x}$ and $M_t^{v_k^x}$ via

$$(4.1) \quad M_t^{u_k^x} = \exp(\phi_{T_N-t}(u_k^x) + \langle \psi_{T_N-t}(u_k^x), X_t \rangle)$$

and

$$(4.2) \quad M_t^{v_k^x} = \exp(\phi_{T_N-t}(v_k^x) + \langle \psi_{T_N-t}(v_k^x), X_t \rangle).$$

The *multiple curve affine LIBOR model* postulates that the OIS and the LIBOR rates associated with the x -tenor evolve according to

$$(4.3) \quad 1 + \delta_x F_k^x(t) = \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} \quad \text{and} \quad 1 + \delta_x L_k^x(t) = \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}}$$

for every $k = 2, \dots, N_x$ and $t \in [0, T_{k-1}^x]$.

In the following three propositions, we show how to construct a multiple curve affine LIBOR model from any given initial term structure of OIS and LIBOR rates.

Proposition 4.1. Consider the time structure \mathcal{T} , let $B(0, T_l)$, $l \in \mathcal{K}$, be the initial term structure of nonnegative OIS discount factors and assume that

$$B(0, T_1) \geq \dots \geq B(0, T_N).$$

Then the following statements hold:

1. If $\gamma_X > B(0, T_1)/B(0, T_N)$, then there exists a decreasing sequence $u_1 \geq u_2 \geq \dots \geq u_N = 0$ in $\mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$ such that

$$(4.4) \quad M_0^{u_l} = \frac{B(0, T_l)}{B(0, T_N)} \quad \text{for all } l \in \mathcal{K}.$$

In particular, if $\gamma_X = \infty$, the multiple curve affine LIBOR model can fit any initial term structure of OIS rates.

2. If X is one dimensional, the sequence $(u_l)_{l \in \mathcal{K}}$ is unique.
3. If all initial OIS rates are positive, the sequence $(u_l)_{l \in \mathcal{K}}$ is strictly decreasing.

Proof. See Proposition 6.1 in Keller-Ressel, Papapantoleon, and Teichmann (2013). ■

After fitting the initial term structure of OIS discount factors, we want to fit the initial term structure of LIBOR rates, which is now tenor dependent. Thus, for each $k \in \mathcal{K}^x$, we set

$$(4.5) \quad u_k^x := u_l,$$

where $l \in \mathcal{K}$ is such that $T_l = T_k^x$; see Remark 3.2. In general, we have that $l = kT_1^x/T_1$, while in the setting of Example 3.1 we simply have $l = kx$, i.e., $u_k^x = u_{kx}$.

Proposition 4.2. Consider the setting of Proposition 4.1, the fixed $x \in \mathcal{X}$, and the corresponding tenor structure \mathcal{T}^x . Let $L_k^x(0)$, $k \in \mathcal{K}^x$, be the initial term structure of nonnegative LIBOR rates and assume that for every $k \in \mathcal{K}^x$

$$(4.6) \quad L_k^x(0) \geq \frac{1}{\delta_x} \left(\frac{B(0, T_{k-1}^x)}{B(0, T_k^x)} - 1 \right) = F_k^x(0).$$

The following statements hold:

1. If $\gamma_X > \max_{k \in \mathcal{K}^x} (1 + \delta_x L_k^x(0))B(0, T_k^x)/B(0, T_N^x)$, then a sequence $v_1^x, v_2^x, \dots, v_{N^x}^x = 0$ exists in $\mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$ such that $v_k^x \geq u_k^x$ and

$$(4.7) \quad M_0^{v_k^x} = (1 + \delta_x L_{k+1}^x(0))M_0^{u_{k+1}^x} \quad \text{for all } k \in \mathcal{K}^x \setminus \{N^x\}.$$

In particular, if $\gamma_X = \infty$, then the multiple curve affine LIBOR model can fit any initial term structure of LIBOR rates.

2. If X is one dimensional, the sequence $(v_k^x)_{k \in \mathcal{K}^x}$ is unique.
3. If all initial spreads are positive, then $v_k^x > u_k^x$ for all $k \in \mathcal{K}^x \setminus \{N^x\}$.

Proof. Similarly to the previous proposition, by fitting the initial LIBOR rates we obtain a sequence $(v_k^x)_{k \in \mathcal{K}^x}$ which satisfies (1)–(3). The inequality $v_k^x \geq u_k^x$ follows directly from (4.6). ■

Proposition 4.3. *Consider the setting of the previous propositions. Then we have:*

1. F_k^x and L_k^x are \mathbb{P}_k^x -martingales for every $k \in \mathcal{K}^x$.
2. $L_k^x(t) \geq F_k^x(t) \geq 0$ for every $k \in \mathcal{K}^x$, $t \in [0, T_{k-1}^x]$.

Proof. Since $M^{u_k^x}$ and $M^{v_k^x}$ are \mathbb{P}_N -martingales and the density process relating the measures \mathbb{P}_N and \mathbb{P}_k^x is provided by

$$(4.8) \quad \frac{d\mathbb{P}_k^x}{d\mathbb{P}_N} \Big|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_k^x)} \frac{B(t, T_k^x)}{B(t, T_N)} = \frac{M_t^{u_k^x}}{M_0^{u_k^x}},$$

we get from (4.3) that

$$(4.9) \quad 1 + \delta_x F_k^x \in \mathcal{M}(\mathbb{P}_k^x) \text{ because } (1 + \delta_x F_k^x)M^{u_k^x} = M^{u_{k-1}^x} \in \mathcal{M}(\mathbb{P}_N).$$

Similarly,

$$(4.10) \quad 1 + \delta_x L_k^x \in \mathcal{M}(\mathbb{P}_k^x) \text{ because } (1 + \delta_x L_k^x)M^{v_k^x} = M^{v_{k-1}^x} \in \mathcal{M}(\mathbb{P}_N).$$

The monotonicity of the sequence (u_k^x) together with (2.10) yields that $M^{u_{k-1}^x} \geq M^{u_k^x}$. Moreover, from the inequality $v_k^x \geq u_k^x$ together with (2.10) again, it follows that $M^{v_k^x} \geq M^{u_k^x}$ for all $k \in \mathcal{K}^x$. Hence,

$$1 + \delta_x L_k^x \geq 1 + \delta_x F_k^x \geq 1.$$

Therefore, the OIS rates, the LIBOR rates, and the corresponding spreads are nonnegative \mathbb{P}_k^x -martingales. ■

Remark 4.4. The above propositions provide the theoretical construction of affine LIBOR models with multiple curves, given initial term structures of OIS bond prices $B(0, T_k^x)$ and LIBOR rates $L_k^x(0)$, for any $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. The initial term structures determine the sequences (u_k^x) and (v_k^x) , but not in a unique way, as soon as the dimension of the driving process is strictly greater than one, which will typically be the case in applications. This provides plenty of freedom in the implementation of the model. For example, setting some components of the vectors (u_k^x) and (v_k^x) equal to zero allows us to exclude the corresponding components of the driving process and thus decide which components of the driving process X will affect the OIS rates, respectively, the LIBOR rates. Moreover, if the components of X are assumed mutually independent, one can create a factor model with common and idiosyncratic components for each OIS and LIBOR rate, as well as various other specific structures. In section 8.1 we present more details on this issue; see Remark 8.1 in particular.

Remark 4.5. Let us now look more closely at the relationship between the sequences (v_k^x) and (u_k^x) . Propositions 4.1 and 4.2 imply that $u_{k-1}^x \geq u_k^x$ and $v_k^x \geq u_k^x$ for all $k \in \mathcal{K}^x$. However, we do not know the ordering of v_k^x and u_{k-1}^x , or whether the sequence (v_k^x) is monotone or not. The market data for LIBOR spreads indicate that in a “normal” market situation $v_k^x \in [u_k^x, u_{k-1}^x]$. More precisely, on the one hand, we have $v_k^x \geq u_k^x$ because the LIBOR spreads are nonnegative. On the other hand, if $v_k^x > u_{k-1}^x$, then the LIBOR rate would be more than two times higher than the OIS rate spanning an interval twice as long, starting at the same date. This contradicts normal market behavior, hence, $v_k^x \in [u_k^x, u_{k-1}^x]$ and consequently the sequence (v_k^x) will also be decreasing. This ordering of the parameters (v_k^x) and (u_k^x) is illustrated in Figure 3 (top graph). However, the normal market situation

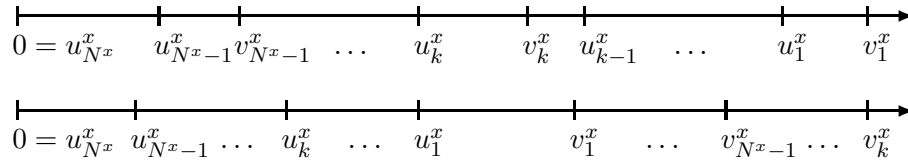


Figure 3. Two possible orderings of (u_k^x) and (v_k^x) .

alternates with an “extreme” situation, where the spread is higher than the OIS rate. In the bottom graph of Figure 3 we plot another possible ordering of the parameters (v_k^x) and (u_k^x) corresponding to such a case of very high spreads. Intuitively speaking, the value of the corresponding model spread depends on the distance between the parameters (v_k^x) and (u_k^x) , although in a nonlinear fashion.

The next result concerns an important property of the multiple curve affine LIBOR model, namely, its analytical tractability in the sense that the model structure is preserved under different forward measures. More precisely, the process X remains affine under any forward measure, although its “characteristics” become time dependent. We refer to Filipović (2005) for time-inhomogeneous affine processes. This property plays a crucial role in the derivation of tractable pricing formulas for interest rate derivatives in the forthcoming sections, since it entails that the law of any collection of LIBOR rates is known under any forward measure. The result below is presented in Keller-Ressel, Papapantoleon, and Teichmann (2013, cf. (6.14) and its proof), nevertheless, we include a short proof here for completeness. In section 5 we also provide an alternative proof for the case when X is an affine diffusion.

Proposition 4.6. *The process X is a time-inhomogeneous affine process under the measure \mathbb{P}_k^x for every $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. In particular*

$$(4.11) \quad \mathbb{E}_k^x [e^{\langle w, X_t \rangle}] = \exp \left(\phi_t^{k,x}(w) + \langle \psi_t^{k,x}(w), X_0 \rangle \right),$$

where

$$(4.12a) \quad \phi_t^{k,x}(w) := \phi_t(\psi_{T_N-t}(u_k^x) + w) - \phi_t(\psi_{T_N-t}(u_k^x)),$$

$$(4.12b) \quad \psi_t^{k,x}(w) := \psi_t(\psi_{T_N-t}(u_k^x) + w) - \psi_t(\psi_{T_N-t}(u_k^x))$$

for every $w \in \mathcal{I}^{k,x}$ with

$$(4.13) \quad \mathcal{I}^{k,x} := \left\{ w \in \mathbb{R}^d : \psi_{T_N-t}(u_k^x) + w \in \mathcal{I}_T \right\}.$$

Proof. Using the density process between the forward measures (see (4.8)), we have that

$$(4.14) \quad \begin{aligned} \mathbb{E}_k^x [e^{\langle w, X_t \rangle} | \mathcal{F}_s] &= \mathbb{E}_N \left[e^{\langle w, X_t \rangle} M_t^{u_k^x} / M_s^{u_k^x} | \mathcal{F}_s \right] \\ &= \mathbb{E}_N \left[\exp \left(\phi_{T_N-t}(u_k^x) + \langle \psi_{T_N-t}(u_k^x) + w, X_t \rangle \right) | \mathcal{F}_s \right] / M_s^{u_k^x} \\ &= \exp \left(\phi_{T_N-t}(u_k^x) - \phi_{T_N-s}(u_k^x) + \phi_{t-s}(\psi_{T_N-t}(u_k^x) + w) \right) \\ &\quad \times \exp \left\langle \psi_{t-s}(\psi_{T_N-t}(u_k^x) + w) - \psi_{T_N-s}(u_k^x), X_s \right\rangle, \end{aligned}$$

where the above expectation is finite for every $w \in \mathcal{I}^{k,x}$; recall (2.1). This shows that X is a time-inhomogeneous affine process under \mathbb{P}_k^x , while (4.11) follows by substituting $s = 0$ in (4.14) and using the flow equations (2.5). ■

Remark 4.7. The preservation of the affine property of the driving process under all forward measures is a stability property shared by all forward price models in which the process $1 + \delta_x F_k^x = \frac{B(\cdot, T_{k-1}^x)}{B(\cdot, T_k^x)}$ is modeled as a deterministic exponential transformation of the driving process. This is related to the density process of the measure change between subsequent forward measures given exactly as $\frac{d\mathbb{P}_{k-1}^x}{d\mathbb{P}_k^x} \Big|_{\mathcal{F}_t} = \frac{B(0, T_k^x)}{B(0, T_{k-1}^x)} \frac{B(t, T_{k-1}^x)}{B(t, T_k^x)} = \frac{1 + \delta_x F_k^x(t)}{1 + \delta_x F_k^x(0)}$ (see (4.8)), which is of the same exponential form, and guarantees that when performing a measure change the driving process remains in the same class. We refer to Eberlein and Kluge (2007) for an example of a forward price model driven by a time-inhomogeneous Lévy process under all forward measures. The models in the spirit of the LMM, where it is rather the forward rate F_k^x which is modeled as an exponential, do not possess this property. The measure change in these models yields the stochastic terms $\frac{\delta_x F_k^x}{1 + \delta_x F_k^x}$ appearing in the characteristics of the driving process (more precisely, in the drift and in the compensator of the random measure of jumps) under any forward measure different from the terminal one, which destroys the analytical tractability of the model. The tractability is often reestablished by freezing the value of these terms at their initial value $\frac{\delta_x F_k^x(0)}{1 + \delta_x F_k^x(0)}$ — an approximation referred to as *freezing the drift*. This approximation is widely known to be unreliable in many realistic settings; cf. Papapantoleon, Schoenmakers, and Skovmand (2012) and the references therein.

On the other hand, in LMMs the positivity of the rate F_k^x is ensured, which, in general, may not be the case in the forward price models. Due to their specific construction, the affine LIBOR models are able to reconcile both of these properties: the positivity of the rate F_k^x and the structure preservation for the driving process under all forward measures. We refer the interested reader to a detailed discussion on this issue in section 3 of Keller-Ressel, Papapantoleon, and Teichmann (2013). Finally, it should be emphasized that in the current market situation the observed OIS rates also have negative values, but this situation can easily be included in the affine LIBOR models; cf. section 4.1 below.

Remark 4.8 (single curve and deterministic spread). The multiple curve affine LIBOR model easily reduces to its single curve counterpart (cf. Keller-Ressel, Papapantoleon, and Teichmann (2013)) by setting $v_k^x = u_k^x$ for all $x \in \mathcal{X}$ and $k \in \mathcal{K}^x$. Another interesting question is whether the spread can be deterministic or, similarly, whether the LIBOR rate can be a deterministic transformation of the OIS rate.

Consider, for example, a 2-dimensional driving process $X = (X^1, X^2)$, where X^1 is an arbitrary affine process and X^2 the constant process (i.e., $X_t^2 \equiv X_0^2$). Then, by setting

$$u_{k-1}^x = (u_{1,k-1}^x, 0) \quad \text{and} \quad v_k^x = (u_{1,k-1}^x, v_{2,k-1}^x),$$

where $u_{1,k-1}^x, v_{2,k-1}^x > 0$ we arrive at

$$1 + \delta_x L_k^x(t) = (1 + \delta_x F_k^x(t)) e^{v_{2,k-1}^x \cdot X_0^2}.$$

Therefore, the LIBOR rate is a deterministic transformation of the OIS rate, although the spread S_k^x as defined in (3.5) is not deterministic. In that case, the multiplicative spread R_k^x defined in (3.6) is obviously deterministic.

4.1. A model with negative rates and positive spreads. The multiple curve affine LIBOR model produces positive rates and spreads, which is consistent with the typical market observations. However, in the current market environment, negative rates have been observed, while the spreads still remain positive. Negative interest rates (as well as spreads, if needed) can be easily accommodated in this setup by considering, for example, affine processes on \mathbb{R}^d instead of $\mathbb{R}_{\geq 0}^d$ or “shifted” positive affine processes, where $\text{supp}(X) \in [a, \infty)^d$ with $a < 0$.

In order to illustrate the flexibility of the affine LIBOR models, we provide below an explicit specification which allows for negative OIS rates, while still preserving the positive spreads. It is based on a particular choice of the driving process and suitable assumptions on the vectors u_k^x and v_k^x . Recall from Remark 4.4 that if the driving process is multidimensional, we have a certain freedom in the choice of the parameters u_k^x and v_k^x when fitting the initial term structure, that we shall exploit here.

Starting from the affine LIBOR model in (4.3), we have an expression for the OIS rates F_k^x and we will derive an expression for the multiplicative spreads R_k^x as defined in (3.6). We choose the multiplicative spreads as a more convenient quantity instead of the additive spreads S_k^x in (3.5), but obviously the additive spreads can easily be recovered from the multiplicative spreads and the OIS rates, and vice versa, by combining (3.5) and (3.6). Moreover, the two spreads always have the same sign, i.e., $R_k^x \geq 0$ if and only if $S_k^x \geq 0$.

The model specification below allows us, in addition, to ensure the monotonicity of the spreads with respect to the tenor length, which is also a feature typically observed in the markets. More precisely, this means that for any two tenors \mathcal{T}^{x_1} and \mathcal{T}^{x_2} such that $\mathcal{T}^{x_2} \subset \mathcal{T}^{x_1}$, i.e., such that $\delta_{x_1} \leq \delta_{x_2}$, the spreads have the following property: for all $k \in \mathcal{K}^{x_1}$ and $j \in \mathcal{K}^{x_2}$ such that $[T_{k-1}^{x_1}, T_k^{x_1}) \subset [T_{j-1}^{x_2}, T_j^{x_2})$ with $T_{k-1}^{x_1} = T_{j-1}^{x_2}$, we have

$$R_k^{x_1}(t) \leq R_j^{x_2}(t)$$

for all $t \leq T_{k-1}^{x_1}$. That is, the spreads are lower for shorter tenor lengths. For example, a 3-month spread is lower than a 6-month spread for a 6-month period starting at the same time as the 3-month period.

According to (4.3), the OIS rate F_k^x , for every $x \in \mathcal{X}$ and every $k \in \mathcal{K}^x$, is provided by

$$(4.15) \quad 1 + \delta_x F_k^x(t) = \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}},$$

where we note that $u_k^x = u_l$ for l such that $T_k^x = T_l$. This process is a \mathbb{P}_k^x -martingale by construction. The multiplicative spread $R_k^x(t)$ now takes the form

$$(4.16) \quad 1 + \delta_x R_k^x(t) = \frac{1 + \delta_x L_k^x(t)}{1 + \delta_x F_k^x(t)} = \frac{M_t^{v_{k-1}^x}}{M_t^{u_{k-1}^x}},$$

which is a \mathbb{P}_{k-1}^x -martingale by construction.

Let us now present a possible choice of the driving process which allows us to accommodate $F_k^x(t) \in \mathbb{R}$, while keeping $R_k^x(t) \in \mathbb{R}_{\geq 0}$, as well as ensuring the monotonicity of the spreads with respect to the tenor length. We assume that the initial term structure of forward OIS

rates $F_k^x(0) \in \mathbb{R}$ and of multiplicative spreads $R_k^x(0) \in \mathbb{R}_{\geq 0}$ are given for every fixed x and all $k \in \mathcal{K}^x$. Moreover, we assume that the initial spreads are monotone with respect to the tenor, i.e., for every two tenors x_1 and x_2 such that $\delta_{x_1} \leq \delta_{x_2}$, we have $R_k^{x_1}(0) \leq R_j^{x_2}(0)$ for all $k \in \mathcal{K}^{x_1}$ and $j \in \mathcal{K}^{x_2}$ with $T_{k-1}^{x_1} = T_{j-1}^{x_2}$.

In order to fix ideas, we shall consider only a 2-dimensional affine process $X = (X^1, X^2)$ on the state space $\mathbb{R} \times \mathbb{R}_{\geq 0}$ such that X^1 and X^2 are independent. The construction can easily be extended to d -dimensional affine processes on $\mathbb{R}^n \times \mathbb{R}_{\geq 0}^m$ with $n + m = d$, such that the first n components are independent of the last m components. The forward OIS rates will be driven by both components of the driving process X and for the spreads we shall use only the second component X^2 , which takes values in $\mathbb{R}_{\geq 0}$, to ensure the nonnegativity. This can be achieved by imposing appropriate assumptions on the parameters v_k^x . We split the construction into two steps.

Step 1. Given the initial term structure of forward OIS rates $F_k^x(0) \in \mathbb{R}$, for every fixed x and all $k \in \mathcal{K}^x$, we apply Proposition 4.1 and find a sequence $(u_k^x) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ such that the model (4.15) fits the initial term structure. Note that the u_k^x , $k \in \mathcal{K}^x$, do not have to be ordered and $F_k^x(t) \in \mathbb{R}$ for any t .

Step 2. Next, given the initial term structure of multiplicative spreads $R_k^x(0) \in \mathbb{R}_{\geq 0}$, for every fixed x and all $k \in \mathcal{K}^x$, we calculate the initial LIBOR rates $L_k^x(0)$ using (3.6). Applying Proposition 4.2 we can find a sequence $(v_k^x) \subset \mathbb{R} \times \mathbb{R}_{\geq 0}$ such that for each $k \in \mathcal{K}^x$, $v_{k-1}^x = (v_{1,k-1}^x, v_{2,k-1}^x)$ satisfies $v_{1,k-1}^x = u_{1,k-1}^x$ and the model (4.16) fits the initial term structure. Note that even though we fixed here the first component $v_{1,k-1}^x$ of each of the vectors v_{k-1}^x , Proposition 4.2 ensures that the initial term structure can be fitted using only the second components $v_{2,k-1}^x$. This yields

(4.17)

$$\begin{aligned} 1 + \delta_x R_k^x(t) &= \frac{M_t^{v_{k-1}^x}}{M_t^{u_{k-1}^x}} \\ &= \frac{\exp\left(\phi_{T_N-t}^1(u_{1,k-1}^x) + \phi_{T_N-t}^2(v_{2,k-1}^x) + \psi_{T_N-t}^1(u_{1,k-1}^x)X_t^1 + \psi_{T_N-t}^2(v_{2,k-1}^x)X_t^2\right)}{\exp\left(\phi_{T_N-t}^1(u_{1,k-1}^x) + \phi_{T_N-t}^2(u_{2,k-1}^x) + \psi_{T_N-t}^1(u_{1,k-1}^x)X_t^1 + \psi_{T_N-t}^2(u_{2,k-1}^x)X_t^2\right)} \\ &= \frac{\exp\left(\phi_{T_N-t}^2(v_{2,k-1}^x) + \psi_{T_N-t}^2(v_{2,k-1}^x)X_t^2\right)}{\exp\left(\phi_{T_N-t}^2(u_{2,k-1}^x) + \psi_{T_N-t}^2(u_{2,k-1}^x)X_t^2\right)}, \end{aligned}$$

due to the independence of X^1 and X^2 ; see Keller-Ressel (2008, Prop. 4.7). Therefore, R_k^x is driven only by X^2 and the fact that the initial values $R_k^x(0) \in \mathbb{R}_{\geq 0}$ implies that $v_{2,k-1}^x \geq u_{2,k-1}^x$ for all k . Consequently, we have $R_k^x(t) \in \mathbb{R}_{\geq 0}$ for all t , which follows immediately from (4.17).

Finally, it remains to show that the monotonicity of the initial spreads with respect to the tenor $R_k^{x_1}(0) \leq R_j^{x_2}(0)$ implies the monotonicity at all times t , $R_k^{x_1}(t) \leq R_j^{x_2}(t)$, for all x_1, x_2 and k, j as above. First note that $T_{k-1}^{x_1} = T_{j-1}^{x_2} = T_l$ implies $u_{k-1}^{x_1} = u_{j-1}^{x_2} = u_l$ and consequently

(4.18)
$$M_t^{u_{k-1}^{x_1}} = M_t^{u_{j-1}^{x_2}}.$$

Hence, $R_k^{x_1}(0) \leq R_j^{x_2}(0)$ implies that necessarily $M_0^{v_{k-1}^{x_1}} \leq M_0^{v_{j-1}^{x_2}}$ by (4.16). This in turn yields $v_{k-1}^{x_1} \leq v_{j-1}^{x_2}$ or, more precisely, $v_{2,k-1}^{x_1} \leq v_{2,j-1}^{x_2}$ since $v_{1,k-1}^{x_1} = u_{1,k-1}^{x_1} = u_{1,j-1}^{x_2} = v_{1,j-1}^{x_2}$. As a consequence, $R_k^{x_1}(t) \leq R_j^{x_2}(t)$ for all t , since

$$1 + \delta_{x_1} R_k^{x_1}(t) = \frac{M_t^{v_{k-1}^{x_1}}}{M_t^{u_{k-1}^{x_1}}} \leq \frac{M_t^{v_{j-1}^{x_2}}}{M_t^{u_{j-1}^{x_2}}} = 1 + \delta_{x_2} R_j^{x_2}(t)$$

due to (4.18).

5. Connection to LMMs. In this section, we will clarify the relationship between the affine LIBOR models and the “classical” LMMs (cf. Sandmann, Sondermann, and Miltersen (1995) and Brace, Gatarek, and Musiela (1997), and also Mercurio (2010a) for the extension of the LMMs to multiple curves). This relationship has not yet been investigated even in the single curve framework of Keller-Ressel, Papapantoleon, and Teichmann (2013). More precisely, we will embed the multiple curve affine LIBOR model (4.3) in the general semimartingale LMM of Jamshidian (1997) and derive the corresponding dynamics of OIS and LIBOR rates. We shall concentrate on affine diffusion processes for the sake of simplicity, in order to expose the ideas without too many technical details. The generalization to affine processes with jumps is straightforward and left to the interested reader.

An affine diffusion process on the state space $D = \mathbb{R}_{\geq 0}^d$ is the solution $X = X^x$ of the SDE

$$(5.1) \quad dX_t = (b + BX_t)dt + \sigma(X_t)dW_t^N, \quad X_0 = x,$$

where W^N is a d -dimensional \mathbb{P}_N -Brownian motion. The coefficients b , $B = (\beta_1, \dots, \beta_d)$, and σ have to satisfy the admissibility conditions for affine diffusions on $\mathbb{R}_{\geq 0}^d$; see Filipović (2009, Chap. 10). That is, the drift vectors satisfy

$$(5.2) \quad b \in \mathbb{R}_{\geq 0}^d, \quad \beta_{i(i)} \in \mathbb{R}, \quad \text{and} \quad \beta_{i(j)} \in \mathbb{R}_{\geq 0} \quad \text{for all } 1 \leq i, j \leq d, i \neq j,$$

where $\beta_{i(j)}$ denotes the j th element of the column vector β_i . Moreover, the diffusion matrix $\sigma : D \rightarrow \mathbb{R}^{d \times d}$ satisfies

$$(5.3) \quad \sigma(z)\sigma(z)^\top = \sum_{i=1}^d \alpha_i z_i \quad \text{for all } z \in D,$$

where the α_i are symmetric, positive semidefinite matrices for all $1 \leq i \leq d$, such that

$$(5.4) \quad \alpha_{i(ii)} \in \mathbb{R}_{\geq 0} \quad \text{and} \quad \alpha_{i(jk)} = 0 \quad \text{for all } 1 \leq i, j, k \leq d, i \neq j, k.$$

Here, $\alpha_{i(jk)}$ denotes the j, k th entry of the matrix α_i . Therefore, the affine diffusion process X is componentwise described by

$$(5.5) \quad dX_t^i = (b + BX_t)^i dt + \sqrt{X_t^i} \sigma_i dW_t^N$$

for all $i = 1, \dots, d$, where $\sigma_i = \sqrt{\alpha_{i(ii)}} \cdot e_i$ (with e_i the unit vector).

5.1. OIS dynamics. We start by computing the dynamics of OIS rates. As in the previous section, we consider a fixed $x \in \mathcal{X}$ and the associated tenor structure \mathcal{T}^x .

Using the structure of the \mathbb{P}_N -martingale $M^{u_k^x}$ in (4.1), we have that

$$(5.6) \quad dM_t^{u_k^x} = M_t^{u_k^x} \psi_{T_N-t}(u_k^x) dX_t + (\dots) dt.$$

Hence, applying Itô's product rule to (4.3) and using (5.6) yields that

$$\begin{aligned} dF_k^x(t) &= \frac{1}{\delta_x} d \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} = \frac{1}{\delta_x} \frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} (\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)) dX_t + (\dots) dt \\ &= \frac{1}{\delta_x} (1 + \delta_x F_k^x(t)) (\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)) dX_t + (\dots) dt. \end{aligned}$$

Therefore, the OIS rates satisfy the following SDE

$$(5.7) \quad \frac{dF_k^x(t)}{F_k^x(t)} = \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} (\psi_{T_N-t}(u_{k-1}^x) - \psi_{T_N-t}(u_k^x)) dX_t + (\dots) dt$$

for all $k = 2, \dots, N^x$. Now, using the dynamics of the affine process X from (5.5) we arrive at

$$\begin{aligned} \frac{dF_k^x(t)}{F_k^x(t)} &= \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d (\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)) dX_t^i + (\dots) dt \\ &= \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d (\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)) \sqrt{X_t^i} \sigma_i dW_t^N + (\dots) dt \\ (5.8) \quad &=: \Gamma_{x,k}(t) dW_t^N + (\dots) dt, \end{aligned}$$

where we define the volatility structure

$$(5.9) \quad \Gamma_{x,k}(t) = \frac{1 + \delta_x F_k^x(t)}{\delta_x F_k^x(t)} \sum_{i=1}^d (\psi_{T_N-t}^i(u_{k-1}^x) - \psi_{T_N-t}^i(u_k^x)) \sqrt{X_t^i} \sigma_i \in \mathbb{R}_{\geq 0}^d.$$

On the other hand, we know from the general theory of discretely compounded forward rates (cf. Jamshidian (1997)) that the OIS rate should satisfy the following SDE under the terminal measure \mathbb{P}_N :

$$(5.10) \quad \frac{dF_k^x(t)}{F_k^x(t)} = - \sum_{l=k+1}^{N^x} \frac{\delta_x F_l^x(t)}{1 + \delta_x F_l^x(t)} \langle \Gamma_{x,l}(t), \Gamma_{x,k}(t) \rangle dt + \Gamma_{x,k}(t) dW_t^N,$$

for the volatility structure $\Gamma_{x,k}$ given in (5.9). Therefore, we get immediately that the \mathbb{P}_k^x -Brownian motion $W^{x,k}$ is related to the terminal Brownian motion W^N via the equality

$$\begin{aligned} W^{x,k} &:= W^N - \sum_{l=k+1}^{N^x} \int_0^\cdot \frac{\delta_x F_l^x(t)}{1 + \delta_x F_l^x(t)} \Gamma_{x,l}(t) dt \\ (5.11) \quad &= W^N - \sum_{l=k+1}^{N^x} \sum_{i=1}^d \int_0^\cdot (\psi_{T_N-t}^i(u_{l-1}^x) - \psi_{T_N-t}^i(u_l^x)) \sqrt{X_t^i} \sigma_i dt. \end{aligned}$$

Moreover, the dynamics of X under \mathbb{P}_k^x take the form

$$\begin{aligned}
 dX_t^i &= (b + BX_t)^i dt + \sqrt{X_t^i} \sigma_i dW_t^{x,k} \\
 &\quad + \sigma_i \sqrt{X_t^i} \sum_{l=k+1}^{N^x} \sum_{j=1}^d \left(\psi_{T_N-t}^j(u_{l-1}^x) - \psi_{T_N-t}^j(u_l^x) \right) \sqrt{X_t^j} \sigma_j dt \\
 &= \left(b_i + (BX_t)^i + \sum_{l=k+1}^{N^x} \left(\psi_{T_N-t}^i(u_{l-1}^x) - \psi_{T_N-t}^i(u_l^x) \right) X_t^i |\sigma_i|^2 \right) dt \\
 (5.12) \quad &\quad + \sqrt{X_t^i} \sigma_i dW_t^{x,k}
 \end{aligned}$$

for all $i = 1, \dots, d$. The last equation provides an alternative proof to Proposition 4.6 in the setting of affine diffusions, since it shows explicitly that X is a time-inhomogeneous affine diffusion process under \mathbb{P}_k^x . One should also note from (5.11), that the difference between the terminal and the forward Brownian motion does not depend on other forward rates as in classical LMMs. As mentioned in Remark 4.7, the same property is shared by forward price models.

Thus, we arrive at the following \mathbb{P}_k^x -dynamics for the OIS rates

$$(5.13) \quad \frac{dF_k^x(t)}{F_k^x(t)} = \Gamma_{x,k}(t) dW_t^{x,k}$$

with the volatility structure $\Gamma_{x,k}$ provided by (5.9). The structure of $\Gamma_{x,k}$ shows that there is a built-in shift in the model, whereas the volatility structure is determined by ψ and σ .

5.2. LIBOR dynamics. Next, we derive the dynamics of the LIBOR rates associated with the same tenor. Using (4.3), (4.2), and repeating the same steps as above, we obtain the following:

$$\begin{aligned}
 \frac{dL_k^x(t)}{L_k^x(t)} &= \frac{1}{\delta_x L_k^x(t)} d \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}} \\
 &= \frac{1}{\delta_x L_k^x(t)} \frac{M_t^{v_{k-1}^x}}{M_t^{u_k^x}} \left(\psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x) \right) dX_t + (\dots) dt \\
 &= \frac{1 + \delta_x L_k^x(t)}{\delta_x L_k^x(t)} \sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right) \sqrt{X_t^i} \sigma_i dW_t^N + (\dots) dt,
 \end{aligned}$$

for all $k = 2, \dots, N^x$. Similarly to (5.9) we introduce the volatility structure

$$(5.14) \quad \Lambda_{x,k}(t) := \frac{1 + \delta_x L_k^x(t)}{\delta_x L_k^x(t)} \sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right) \sqrt{X_t^i} \sigma_i \in \mathbb{R}_{\geq 0}^d,$$

and then obtain for L_k^x the following \mathbb{P}_k^x -dynamics

$$(5.15) \quad \frac{dL_k^x(t)}{L_k^x(t)} = \Lambda_{x,k}(t) dW_t^{x,k},$$

where $W^{x,k}$ is the \mathbb{P}_k^x -Brownian motion given by (5.11), while the dynamics of X are provided by (5.12).

5.3. Spread dynamics. Using that $S_k^x = L_k^x - F_k^x$, the dynamics of LIBOR and OIS rates under the forward measure \mathbb{P}_k^x in (5.13) and (5.15), as well as the structure of the volatilities in (5.9) and (5.14), after some straightforward calculations we arrive at

$$dS_k^x(t) = \left\{ S_k^x(t) \Upsilon_t(v_{k-1}^x, u_k^x) + \frac{1 + \delta_x F_k^x(t)}{\delta_x} \Upsilon_t(v_{k-1}^x, u_{k-1}^x) \right\} dW_t^{x,k},$$

where

$$(5.16) \quad \Upsilon_t(w, y) := \sum_{i=1}^d (\psi_{T_N-t}^i(w) - \psi_{T_N-t}^i(y)) \sqrt{X_t^i} \sigma_i.$$

5.4. Instantaneous correlations. The derivation of the SDEs that OIS and LIBOR rates satisfy allows us to quickly provide formulas for various quantities of interest, such as the instantaneous correlations between OIS and LIBOR rates or LIBOR rates with different maturities or tenors. We have, for example, that the instantaneous correlation between the LIBOR rates maturing at T_k^x and T_l^x is heuristically described by

$$\text{Corr}_t[L_k^x, L_l^x] = \frac{\frac{dL_k^x(t)}{L_k^x(t)} \cdot \frac{dL_l^x(t)}{L_l^x(t)}}{\sqrt{\frac{dL_k^x(t)}{L_k^x(t)} \cdot \frac{dL_k^x(t)}{L_k^x(t)} \cdot \frac{dL_l^x(t)}{L_l^x(t)} \cdot \frac{dL_l^x(t)}{L_l^x(t)}}};$$

therefore, we get that

$$\begin{aligned} \text{Corr}_t[L_k^x, L_l^x] &\stackrel{(5.15)}{=} \frac{\langle \Lambda_{x,k}, \Lambda_{x,l} \rangle}{|\Lambda_{x,k}| |\Lambda_{x,l}|} \\ &= \frac{\sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right) \left(\psi_{T_N-t}^i(v_{l-1}^x) - \psi_{T_N-t}^i(u_l^x) \right) X^i |\sigma_i|^2}{\sqrt{\sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{k-1}^x) - \psi_{T_N-t}^i(u_k^x) \right)^2 X^i |\sigma_i|^2} \sqrt{\sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{l-1}^x) - \psi_{T_N-t}^i(u_l^x) \right)^2 X^i |\sigma_i|^2}} \\ &\quad \times \frac{1}{\sqrt{\sum_{i=1}^d \left(\psi_{T_N-t}^i(v_{l-1}^x) - \psi_{T_N-t}^i(u_l^x) \right)^2 X^i |\sigma_i|^2}}. \end{aligned}$$

Similar expressions can be derived for other instantaneous correlations, e.g.,

$$\text{Corr}_t[F_k^x, L_k^x] \quad \text{or} \quad \text{Corr}_t[L_k^{x_1}, L_k^{x_2}].$$

Instantaneous correlations are important for describing the (instantaneous) interdependencies between different LIBOR rates. In the LMM, for instance, the rank of the instantaneous

correlation matrix determines the number of factors (e.g., Brownian motions) that is needed to drive the model. Explicit expressions for terminal correlations between LIBOR rates are provided in Appendix A.

6. Valuation of swaps and caps.

6.1. Interest rate and basis swaps. We start by presenting a fixed-for-floating payer interest rate swap on a notional amount normalized to 1, where fixed payments are exchanged for floating payments linked to the LIBOR rate. The LIBOR rate is set in advance and the payments are made in arrears, while we assume for simplicity that the timing and frequency of the payments of the floating leg coincides with those of the fixed leg. The swap is initiated at time $T_p^x \geq 0$, where $x \in \mathcal{X}$ and $p \in \mathcal{K}^x$. The collection of payment dates is denoted by $\mathcal{T}_{pq}^x := \{T_{p+1}^x < \dots < T_q^x\}$, and the fixed rate is denoted by K . Then, the time- t value of the swap, for $t \leq T_p^x$, is given by

$$\begin{aligned} \mathbb{S}_t(K, \mathcal{T}_{pq}^x) &= \sum_{k=p+1}^q \delta_x B(t, T_k^x) \mathbb{E}_k^x [L(T_{k-1}^x, T_k^x) - K | \mathcal{F}_t] \\ (6.1) \qquad \qquad \qquad &= \delta_x \sum_{k=p+1}^q B(t, T_k^x) (L_k^x(t) - K). \end{aligned}$$

Thus, the fair swap rate $K_t(\mathcal{T}_{pq}^x)$ is provided by

$$(6.2) \qquad \qquad \qquad K_t(\mathcal{T}_{pq}^x) = \frac{\sum_{k=p+1}^q B(t, T_k^x) L_k^x(t)}{\sum_{k=p+1}^q B(t, T_k^x)}.$$

Basis swaps are new products in interest rate markets, whose value reflects the discrepancy between the LIBOR rates of *different* tenors. A basis swap is a swap where two streams of floating payments linked to the LIBOR rates of different tenors are exchanged. For example, in a 3-month–6-month basis swap, a 3-month-LIBOR is paid (received) quarterly and a 6-month-LIBOR is received (paid) semiannually. We assume in what follows that both rates are set in advance and paid in arrears; of course, other conventions regarding the payments on the two legs of a basis swap also exist. A more detailed account on basis swaps can be found in Mercurio (2010b, section 5.2) or in Filipović and Trolle (2013, section 2.4 and Appendix F). Note that in the precrisis setup the value of such a product would have been zero at any time point, due to the no-arbitrage relation between the LIBOR rates of different tenors; see, e.g., Crépey, Grbac, and Nguyen (2012).

Let us consider a basis swap associated with two tenor structures denoted by $\mathcal{T}_{pq}^{x_1} := \{T_{p_1}^{x_1} < \dots < T_{q_1}^{x_1}\}$ and $\mathcal{T}_{pq}^{x_2} := \{T_{p_2}^{x_2} < \dots < T_{q_2}^{x_2}\}$, where $T_{p_1}^{x_1} = T_{p_2}^{x_2} \geq 0$, $T_{q_1}^{x_1} = T_{q_2}^{x_2}$, and $\mathcal{T}_{pq}^{x_2} \subset \mathcal{T}_{pq}^{x_1}$. The notional amount is again assumed to be 1 and the swap is initiated at time $T_{p_1}^{x_1}$, while the first payments are due at times $T_{p_1+1}^{x_1}$ and $T_{p_2+1}^{x_2}$, respectively. The basis swap spread is a fixed rate S which is added to the payments on the shorter tenor length. More precisely, for the x_1 -tenor, the floating interest rate $L(T_{i-1}^{x_1}, T_i^{x_1})$ at tenor date $T_i^{x_1}$ is replaced by $L(T_{i-1}^{x_1}, T_i^{x_1}) + S$ for every $i \in \{p_1 + 1, \dots, q_1\}$. The time- t value of such an agreement is given, for $0 \leq t \leq T_{p_1}^{x_1} = T_{p_2}^{x_2}$, by

$$\begin{aligned}
 (6.3) \quad \mathbb{BS}_t(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) &= \sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) \mathbb{E}_i^{x_2} [L(T_{i-1}^{x_2}, T_i^{x_2}) | \mathcal{F}_t] \\
 &\quad - \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) \mathbb{E}_i^{x_1} [L(T_{i-1}^{x_1}, T_i^{x_1}) + S | \mathcal{F}_t] \\
 &= \sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) L_i^{x_2}(t) - \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) (L_i^{x_1}(t) + S).
 \end{aligned}$$

We also want to compute the *fair basis swap spread* $S_t(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2})$. This is the spread that makes the value of the basis swap equal zero at time t , i.e., the fair spread is obtained by solving $\mathbb{BS}_t(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = 0$. We get that

$$(6.4) \quad S_t(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) = \frac{\sum_{i=p_2+1}^{q_2} \delta_{x_2} B(t, T_i^{x_2}) L_i^{x_2}(t) - \sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1}) L_i^{x_1}(t)}{\sum_{i=p_1+1}^{q_1} \delta_{x_1} B(t, T_i^{x_1})}.$$

The formulas for the fair swap rate and basis spread can be used to bootstrap the initial values of LIBOR rates from market data; see [Mercurio \(2010b\)](#), section 2.4).

6.2. Caps. The valuation of caplets, and thus caps, in the multiple curve affine LIBOR model is an easy task, which has complexity equal to the complexity of the valuation of caplets in the single curve affine LIBOR model; compare with Proposition 7.1 in [Keller-Ressel, Papapantoleon, and Teichmann \(2013\)](#). There are two reasons for this: on the one hand, the LIBOR rate is modeled directly (see (4.3)), as opposed to, e.g., [Mercurio \(2010a\)](#), where the LIBOR rate is modeled implicitly as the sum of the OIS rate and the spread. In our approach, the valuation of caplets remains a one-dimensional problem, while in the latter approach it becomes a “basket” option on the OIS rate and the spread. On the other hand, the driving process remains affine under any forward measure (cf. Proposition 4.6), which allows the application of Fourier methods for option pricing. In what follows we will derive semiexplicit pricing formulas for any multiple curve affine LIBOR model. Let us point out that we do not need to “freeze the drift” as is customary in LMMs with jumps (see Remark 4.7).

Proposition 6.1. *Consider an x -tenor caplet with strike K that pays out, at time T_k^x , $\delta_x(L(T_{k-1}^x, T_k^x) - K)^+$. The time-0 price is provided by*

$$(6.5) \quad \mathbb{C}_0(K, T_k^x) = \frac{B(0, T_k^x)}{2\pi} \int_{\mathbb{R}} K_x^{1-R+iw} \frac{\Theta_{\mathcal{W}_{k-1}^x}(R-iw)}{(R-iw)(R-1-iw)} dw$$

for $R \in (1, \infty) \cap \tilde{\mathcal{I}}^{k,x}$, assuming that $(1, \infty) \cap \tilde{\mathcal{I}}^{k,x} \neq \emptyset$, where $K_x = 1 + \delta_x K$, $\Theta_{\mathcal{W}_{k-1}^x}$ is given by (6.7), while the set $\tilde{\mathcal{I}}^{k,x}$ is defined as

$$\tilde{\mathcal{I}}^{k,x} = \left\{ z \in \mathbb{R} : (1-z)\psi_{T_N - T_{k-1}^x}(u_k^x) + z\psi_{T_N - T_{k-1}^x}(v_{k-1}^x) \in \mathcal{I}_T \right\}.$$

Proof. Using (3.3) and (4.3) the time-0 price of the caplet equals

$$\begin{aligned} \mathbb{C}_0(K, T_k^x) &= \delta_x B(0, T_k^x) \mathbb{E}_k^x [(L(T_{k-1}^x, T_k^x) - K)^+] \\ &= \delta_x B(0, T_k^x) \mathbb{E}_k^x [(L_k^x(T_{k-1}^x) - K)^+] \\ &= B(0, T_k^x) \mathbb{E}_k^x \left[\left(M_{T_{k-1}^x}^{v_{k-1}^x} / M_{T_{k-1}^x}^{u_k^x} - K_x \right)^+ \right] \\ &= B(0, T_k^x) \mathbb{E}_k^x \left[\left(e^{\mathcal{W}_{k-1}^x} - K_x \right)^+ \right], \end{aligned}$$

where

$$\begin{aligned} \mathcal{W}_{k-1}^x &= \log \left(M_{T_{k-1}^x}^{v_{k-1}^x} / M_{T_{k-1}^x}^{u_k^x} \right) \\ &= \phi_{T_N - T_{k-1}^x}(v_{k-1}^x) - \phi_{T_N - T_{k-1}^x}(u_k^x) \\ &\quad + \langle \psi_{T_N - T_{k-1}^x}(v_{k-1}^x) - \psi_{T_N - T_{k-1}^x}(u_k^x), X_{T_{k-1}^x} \rangle \\ (6.6) \quad &=: A + \langle B, X_{T_{k-1}^x} \rangle. \end{aligned}$$

Now, using Eberlein, Glau, and Papapantoleon (2010, Thm 2.2, Ex. 5.1), we arrive directly at (6.5), where $\Theta_{\mathcal{W}_{k-1}^x}$ denotes the \mathbb{P}_k^x -moment generating function of the random variable \mathcal{W}_{k-1}^x , i.e., for $z \in \tilde{\mathcal{I}}^{k,x}$,

$$\begin{aligned} \Theta_{\mathcal{W}_{k-1}^x}(z) &= \mathbb{E}_k^x [e^{z\mathcal{W}_{k-1}^x}] = \mathbb{E}_k^x [\exp(z(A + \langle B, X_{T_{k-1}^x} \rangle))] \\ (6.7) \quad &= \exp \left(zA + \phi_{T_{k-1}^x}^{k,x}(zB) + \langle \psi_{T_{k-1}^x}^{k,x}(zB), X_0 \rangle \right). \end{aligned}$$

The last equality follows from Proposition 4.6, noting that $z \in \tilde{\mathcal{I}}^{k,x}$ implies $zB \in \mathcal{I}^{k,x}$. \blacksquare

7. Valuation of swaptions and basis swaptions. This section is devoted to the pricing of options on interest rate and basis swaps, in other words, to the pricing of swaptions and basis swaptions. In the first part, we provide general expressions for the valuation of swaptions and basis swaptions making use of the structure of multiple curve affine LIBOR models. In the following two parts, we derive efficient and accurate approximations for the pricing of swaptions and basis swaptions by further utilizing the model properties, namely, the preservation of the affine structure under any forward measure, and applying the linear boundary approximation developed by Singleton and Umantsev (2002). Similarly to the pricing of caplets, here also we do not have to freeze the drift, while in special cases we can even derive closed or semiclosed form solutions (cf. Keller-Ressel, Papapantoleon, and Teichmann, (2013, section 8)).

Let us consider first a payer swaption with strike rate K and exercise date T_p^x on a fixed-for-floating interest rate swap starting at T_p^x and maturing at T_q^x ; this was defined in section 6.1. A swaption can be regarded as a sequence of fixed payments $\delta_x(K_{T_p^x}(T_{pq}^x) - K)^+$ that are received at the payment dates T_{p+1}^x, \dots, T_q^x ; see Musiela and Rutkowski (2005, section 13.1.2,

p. 524). Here $K_{T_p^x}(\mathcal{T}_{pq}^x)$ is the swap rate of the underlying swap at time T_p^x ; cf. (6.2). Note that the classical transformation of a payer (resp., receiver) swaption into a put (resp., call) option on a coupon bond is not valid in the multiple curve setup, since LIBOR rates cannot be expressed in terms of zero coupon bonds; see Remark 3.5.

The value of the swaption at time $t \leq T_p^x$ is provided by

$$\begin{aligned} \mathbb{S}_t^+(K, \mathcal{T}_{pq}^x) &= B(t, T_p^x) \sum_{i=p+1}^q \delta_x \mathbb{E}_p^x \left[B(T_p^x, T_i^x) \left(K_{T_p^x}(\mathcal{T}_{pq}^x) - K \right)^+ \middle| \mathcal{F}_t \right] \\ &= B(t, T_p^x) \mathbb{E}_p^x \left[\left(\sum_{i=p+1}^q \delta_x L_i^x(T_p^x) B(T_p^x, T_i^x) - \sum_{i=p+1}^q \delta_x K B(T_p^x, T_i^x) \right)^+ \middle| \mathcal{F}_t \right] \end{aligned}$$

since the swap rate $K_{T_p^x}(\mathcal{T}_{pq}^x)$ is given by (6.2) for $t = T_p^x$. Using (3.2), (4.3), and a telescoping product, we get that

$$B(T_p^x, T_i^x) = \frac{B(T_p^x, T_i^x)}{B(T_p^x, T_{i-1}^x)} \frac{B(T_p^x, T_{i-1}^x)}{B(T_p^x, T_{i-2}^x)} \dots \frac{B(T_p^x, T_{p+1}^x)}{B(T_p^x, T_p^x)} = \frac{M_{T_p^x}^{u_i^x}}{M_{T_p^x}^{u_p^x}}.$$

Together with (4.3) for $L_i^x(T_p^x)$, this yields

$$\begin{aligned} \mathbb{S}_t^+(K, \mathcal{T}_{pq}^x) &= B(t, T_p^x) \mathbb{E}_p^x \left[\left(\sum_{i=p+1}^q \frac{M_{T_p^x}^{v_{i-1}^x}}{M_{T_p^x}^{u_p^x}} - \sum_{i=p+1}^q K_x \frac{M_{T_p^x}^{u_i^x}}{M_{T_p^x}^{u_p^x}} \right)^+ \middle| \mathcal{F}_t \right] \\ (7.1) \qquad &= B(t, T_N) \mathbb{E}_N \left[\left(\sum_{i=p+1}^q M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^q K_x M_{T_p^x}^{u_i^x} \right)^+ \middle| \mathcal{F}_t \right], \end{aligned}$$

where $K_x := 1 + \delta_x K$ and the second equality follows from the measure change from \mathbb{P}_p^x to \mathbb{P}_N as given in (4.8).

Next, we move on to the pricing of basis swaptions. A basis swaption is an option to enter a basis swap with spread S . We consider a basis swap as defined in section 6.1, which starts at $T_{p_1}^{x_1} = T_{p_2}^{x_2}$ and ends at $T_{q_1}^{x_1} = T_{q_2}^{x_2}$, while we assume that the exercise date is $T_{p_1}^{x_1}$. The payoff of a basis swap at time $T_{p_1}^{x_1}$ is given by (6.3) for $t = T_{p_1}^{x_1}$. Therefore, the price of a basis swaption at time $t \leq T_{p_1}^{x_1}$ is provided by

$$\begin{aligned} \mathbb{BS}_t^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) &= B(t, T_{p_1}^{x_1}) \mathbb{E}_{p_1}^{x_1} \left[\left(\sum_{i=p_2+1}^{q_2} \delta_{x_2} L_i^{x_2}(T_{p_2}^{x_2}) B(T_{p_2}^{x_2}, T_i^{x_2}) \right. \right. \\ &\qquad \left. \left. - \sum_{i=p_1+1}^{q_1} \delta_{x_1} (L_i^{x_1}(T_{p_1}^{x_1}) + S) B(T_{p_1}^{x_1}, T_i^{x_1}) \right)^+ \middle| \mathcal{F}_t \right]. \end{aligned}$$

Along the lines of the derivation for swaptions and using $M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} = M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}}$ (cf. (4.5)), we arrive at

$$\begin{aligned}
 (7.2) \quad & \mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) \\
 &= B(t, T_{p_1}^{x_1}) \mathbb{E}_{p_1}^{x_1} \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} / M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} / M_{T_{p_2}^{x_2}}^{u_{p_2}^{x_2}} \right) \right. \right. \\
 &\quad \left. \left. - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} / M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} / M_{T_{p_1}^{x_1}}^{u_{p_1}^{x_1}} \right) \right) \Big| \mathcal{F}_t \right]^+ \\
 &= B(t, T_N) \mathbb{E}_N \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \right) - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \right) \right) \Big| \mathcal{F}_t \right]^+,
 \end{aligned}$$

where $S_{x_1} := 1 - \delta_{x_1} S$.

7.1. Approximation formula for swaptions. We will now derive an efficient approximation formula for the pricing of swaptions. The main ingredients in this formula are the affine property of the driving process under forward measures and the linearization of the exercise boundary. Numerical results for this approximation will be reported in section 8.3.

We start by presenting some technical tools and assumptions that will be used in what follows. We define the probability measures $\overline{\mathbb{P}}_k^x$, for every $k \in \mathcal{K}^x$, via the Radon–Nikodym density

$$(7.3) \quad \frac{d\overline{\mathbb{P}}_k^x}{d\mathbb{P}_N} \Big|_{\mathcal{F}_t} = \frac{M_t^{v_k^x}}{M_0^{v_k^x}}.$$

The process X is obviously a time-inhomogeneous affine process under every $\overline{\mathbb{P}}_k^x$. More precisely, we have the following result which follows directly from Proposition 4.6.

Corollary 7.1. *The process X is a time-inhomogeneous affine process under the measure $\overline{\mathbb{P}}_k^x$ for every $x \in \mathcal{X}, k \in \mathcal{K}^x$, with*

$$(7.4) \quad \overline{\mathbb{E}}_k^x [e^{\langle w, X_t \rangle}] = \exp \left(\overline{\phi}_t^{k,x}(w) + \langle \overline{\psi}_t^{k,x}(w), X_0 \rangle \right),$$

where

$$(7.5a) \quad \overline{\phi}_t^{k,x}(w) := \phi_t(\psi_{T_N-t}(v_k^x) + w) - \phi_t(\psi_{T_N-t}(v_k^x)),$$

$$(7.5b) \quad \overline{\psi}_t^{k,x}(w) := \psi_t(\psi_{T_N-t}(v_k^x) + w) - \psi_t(\psi_{T_N-t}(v_k^x))$$

for every $w \in \overline{\mathcal{I}}^{k,x}$ with

$$(7.6) \quad \overline{\mathcal{I}}^{k,x} := \left\{ w \in \mathbb{R}^d : \psi_{T_N-t}(v_k^x) + w \in \mathcal{I}_T \right\}.$$

The price of a swaption is provided by (7.1), while for simplicity we shall consider the price at time $t = 0$ in what follows. We can rewrite (7.1) as

$$(7.7) \quad \begin{aligned} \mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N) \mathbb{E}_N \left[\left(\sum_{i=p+1}^q M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^q K_x M_{T_p^x}^{u_i^x} \right) \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] \\ &= B(0, T_N) \left(\sum_{i=p+1}^q \mathbb{E}_N \left[M_{T_p^x}^{v_{i-1}^x} \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] - K_x \sum_{i=p+1}^q \mathbb{E}_N \left[M_{T_p^x}^{u_i^x} \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] \right), \end{aligned}$$

where, recalling (4.1) and (4.2), we define the function $f : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}$ by

$$(7.8) \quad \begin{aligned} f(y) &= \sum_{i=p+1}^q \exp \left(\phi_{T_N - T_p^x}(v_{i-1}^x) + \langle \psi_{T_N - T_p^x}(v_{i-1}^x), y \rangle \right) \\ &\quad - \sum_{i=p+1}^q K_x \exp \left(\phi_{T_N - T_p^x}(u_i^x) + \langle \psi_{T_N - T_p^x}(u_i^x), y \rangle \right). \end{aligned}$$

This function determines the exercise boundary for the price of the swaption.

Now, since we cannot compute the characteristic function of $f(X_{T_p^x})$ explicitly, we will follow Singleton and Umantsev (2002) and approximate f by a linear function.

Approximation (S). We approximate

$$(7.9) \quad f(X_{T_p^x}) \approx \tilde{f}(X_{T_p^x}) := \mathcal{A} + \langle \mathcal{B}, X_{T_p^x} \rangle,$$

where the constants \mathcal{A}, \mathcal{B} are determined according to the linear regression procedure described in Singleton and Umantsev (2002, pp. 432–434). The line $\langle \mathcal{B}, X_{T_p^x} \rangle = -\mathcal{A}$ approximates the exercise boundary, hence, \mathcal{A} and \mathcal{B} are strike dependent.

The following assumption will be used for the pricing of swaptions and basis swaptions.

Assumption (CD). The cumulative distribution function of X_t is continuous for all $t \in [0, T_N]$.

Let $\Im(z)$ denote the imaginary part of a complex number $z \in \mathbb{C}$. Now, we state the main result of this subsection.

Proposition 7.2. *Assume that \mathcal{A}, \mathcal{B} are determined by Approximation (S) and that Assumption (CD) is satisfied. The time-0 price of a payer swaption with strike K , option maturity T_p^x , and swap maturity T_q^x , is approximated by*

$$(7.10) \quad \begin{aligned} \tilde{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N) \sum_{i=p+1}^q M_0^{v_{i-1}^x} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\xi}_{i-1}^x(z))}{z} dz \right] \\ &\quad - K_x \sum_{i=p+1}^q B(0, T_i^x) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\zeta}_i^x(z))}{z} dz \right], \end{aligned}$$

where $\tilde{\zeta}_i^x$ and $\tilde{\xi}_i^x$ are defined by (7.14) and (7.15), respectively.

Proof. Starting from the swaption price in (7.7) and using the relation between the terminal measure \mathbb{P}_N and the measures \mathbb{P}_k^x and $\overline{\mathbb{P}}_k^x$ in (4.8) and (7.3), we get that

$$(7.11) \quad \begin{aligned} \mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N) \sum_{i=p+1}^q M_0^{v_i^x-1} \overline{\mathbb{E}}_{i-1}^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] \\ &\quad - K_x \sum_{i=p+1}^q B(0, T_i^x) \mathbb{E}_i^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right]. \end{aligned}$$

In addition, from the inversion formula of Gil-Pelaez (1951) and using Assumption (CD), we get that

$$(7.12) \quad \mathbb{E}_i^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\zeta_i^x(z))}{z} dz,$$

$$(7.13) \quad \overline{\mathbb{E}}_i^x \left[\mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\xi_i^x(z))}{z} dz$$

for each $i \in \mathcal{K}^x$, where we define

$$\zeta_i^x(z) := \mathbb{E}_i^x \left[\exp(izf(X_{T_p^x})) \right] \quad \text{and} \quad \xi_i^x(z) := \overline{\mathbb{E}}_i^x \left[\exp(izf(X_{T_p^x})) \right].$$

However, the above characteristic functions cannot be computed explicitly, in general, thus we will linearize the exercise boundary as described by Approximation (S). That is, we approximate the unknown characteristic functions with ones that admit an explicit expression due to the affine property of X under the forward measures. Indeed, using Approximation (S), Proposition 4.6 and Corollary 7.1 we get that

$$(7.14) \quad \begin{aligned} \zeta_k^x(z) &\approx \tilde{\zeta}_k^x(z) := \mathbb{E}_k^x \left[\exp(iz\tilde{f}(X_{T_p^x})) \right] \\ &= \exp \left(iz\mathcal{A} + \phi_{T_p^x}^{k,x}(iz\mathcal{B}) + \langle \psi_{T_p^x}^{k,x}(iz\mathcal{B}), X_0 \rangle \right), \end{aligned}$$

$$(7.15) \quad \begin{aligned} \xi_k^x(z) &\approx \tilde{\xi}_k^x(z) := \overline{\mathbb{E}}_k^x \left[\exp(iz\tilde{f}(X_{T_p^x})) \right] \\ &= \exp \left(iz\mathcal{A} + \overline{\phi}_{T_p^x}^{k,x}(iz\mathcal{B}) + \langle \overline{\psi}_{T_p^x}^{k,x}(iz\mathcal{B}), X_0 \rangle \right). \end{aligned}$$

After inserting (7.12) and (7.13) into (7.11) and using (7.14) and (7.15) we arrive at the approximation formula for swaptions (7.10). ■

Remark 7.3. The pricing of swaptions is inherently a high-dimensional problem. The expectation in (7.1) corresponds to a d -dimensional integral, where d is the dimension of the driving process. However, the exercise boundary is nonlinear and hard to compute, in general. See, e.g., Brace, Gatarek, and Musiela (1997), Eberlein and Kluge (2006) or Keller-Ressel, Papapantoleon, and Teichmann (2013, section 7.2, section 8.3) for some exceptional cases that admit explicit solutions. Alternatively, one could express a swaption as a zero strike basket option written on $2(q-p)$ underlying assets and use Fourier methods for pricing; see Hubalek and Kallsen (2005) or Hurd and Zhou (2010). This leads to a $2(q-p)$ -dimensional numerical integration. Instead, the approximation derived in this section requires only the evaluation of $2(q-p)$ univariate integrals together with the computation of the constants \mathcal{A}, \mathcal{B} . This reduces the complexity of the problem considerably.

7.2. Approximation formula for basis swaptions. In this subsection, we derive an analogous approximate pricing formula for basis swaptions. Numerical results for this approximation will be reported in section 8.4.

Similarly to the case of swaptions, we can rewrite the time-0 price of a basis swaption (7.2) as follows:

$$\begin{aligned}
 & \mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) \\
 &= B(0, T_N) \left\{ \sum_{i=p_2+1}^{q_2} \left(\mathbb{E}_N \left[M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} \mathbf{1}_{\{g(X_{T_{p_2}^{x_2}}) \geq 0\}} \right] - \mathbb{E}_N \left[M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \mathbf{1}_{\{g(X_{T_{p_2}^{x_2}}) \geq 0\}} \right] \right) \right. \\
 (7.16) \quad & \left. - \sum_{i=p_1+1}^{q_1} \left(\mathbb{E}_N \left[M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} \mathbf{1}_{\{g(X_{T_{p_1}^{x_1}}) \geq 0\}} \right] - S_{x_1} \mathbb{E}_N \left[M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \mathbf{1}_{\{g(X_{T_{p_1}^{x_1}}) \geq 0\}} \right] \right) \right\},
 \end{aligned}$$

where we define the function $g : \mathbb{R}_{\geq 0}^d \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 g(y) &= \sum_{i=p_2+1}^{q_2} \exp(\phi_{T_N-T_{p_2}^{x_2}}(v_{i-1}^{x_2}) + \langle \psi_{T_N-T_{p_2}^{x_2}}(v_{i-1}^{x_2}), y \rangle) \\
 & - \sum_{i=p_2+1}^{q_2} \exp(\phi_{T_N-T_{p_2}^{x_2}}(u_i^{x_2}) + \langle \psi_{T_N-T_{p_2}^{x_2}}(u_i^{x_2}), y \rangle) \\
 & - \sum_{i=p_1+1}^{q_1} \exp(\phi_{T_N-T_{p_1}^{x_1}}(v_{i-1}^{x_1}) + \langle \psi_{T_N-T_{p_1}^{x_1}}(v_{i-1}^{x_1}), y \rangle) \\
 (7.17) \quad & + \sum_{i=p_1+1}^{q_1} S_{x_1} \exp(\phi_{T_N-T_{p_1}^{x_1}}(u_i^{x_1}) + \langle \psi_{T_N-T_{p_1}^{x_1}}(u_i^{x_1}), y \rangle),
 \end{aligned}$$

which determines the exercise boundary for the price of the basis swaption. This will be approximated by a linear function again following Singleton and Umantsev (2002).

Approximation (BS). We approximate

$$(7.18) \quad g(X_{T_{p_1}^{x_1}}) \approx \tilde{g}(X_{T_{p_1}^{x_1}}) := \mathcal{C} + \langle \mathcal{D}, X_{T_{p_1}^{x_1}} \rangle,$$

where \mathcal{C} and \mathcal{D} are determined via a linear regression.

Proposition 7.4. Assume that \mathcal{C}, \mathcal{D} are determined by Approximation (BS) and that Assumption (CD) is satisfied. The time-0 price of a basis swaption with spread S , option maturity $T_{p_1}^{x_1} = T_{p_2}^{x_2}$, and swap maturity $T_{q_1}^{x_1} = T_{q_2}^{x_2}$, is approximated by

$$\begin{aligned}
 \widetilde{\mathbb{BS}}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) &= B(0, T_N) \sum_{i=p_2+1}^{q_2} M_0^{v_{i-1}^{x_2}} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\xi}_{i-1}^{x_2}(z))}{z} dz \right] \\
 & - \sum_{i=p_2+1}^{q_2} B(0, T_i^{x_2}) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\zeta}_i^{x_2}(z))}{z} dz \right]
 \end{aligned}$$

$$(7.19) \quad \begin{aligned} & - B(0, T_N) \sum_{i=p_1+1}^{q_1} M_0^{v_{i-1}^{x_1}} \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\zeta}_{i-1}^{x_1}(z))}{z} dz \right] \\ & + S_{x_1} \sum_{i=p_1+1}^{q_1} B(0, T_i^{x_1}) \left[\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\Im(\tilde{\zeta}_i^{x_1}(z))}{z} dz \right], \end{aligned}$$

where $\tilde{\zeta}_i^{x_1}$ and $\tilde{\xi}_i^{x_1}$ are defined by (7.20) and (7.21) for $l = 1, 2$.

Proof. Starting from the expression for the basis swaption price given in (7.16), we follow the same steps as in the previous section: first, we use the relation between the terminal measure \mathbb{P}_N and the measures $\mathbb{P}_k^x, \overline{\mathbb{P}}_k^x$ to arrive at an expression similar to (7.11). Second, we approximate g by \tilde{g} in (7.18). Third, we define the approximate characteristic functions, which can be computed explicitly,

$$(7.20) \quad \begin{aligned} \tilde{\zeta}_i^{x_1}(z) & := \mathbb{E}_i^{x_1} [\exp(iz\tilde{g}(X_{T_{p_l}^{x_1}}))] \\ & = \exp \left(iz\mathcal{C} + \phi_{T_{p_l}^{x_1}}^{i, x_1}(iz\mathcal{D}) + \langle \psi_{T_{p_l}^{x_1}}^{i, x_1}(iz\mathcal{D}), X_0 \rangle \right), \end{aligned}$$

$$(7.21) \quad \begin{aligned} \tilde{\xi}_i^{x_1}(z) & := \overline{\mathbb{E}}_i^{x_1} [\exp(iz\tilde{g}(X_{T_{p_l}^{x_1}}))] \\ & = \exp \left(iz\mathcal{C} + \overline{\phi}_{T_{p_l}^{x_1}}^{i, x_1}(iz\mathcal{D}) + \langle \overline{\psi}_{T_{p_l}^{x_1}}^{i, x_1}(iz\mathcal{D}), X_0 \rangle \right) \end{aligned}$$

for $l = 1, 2$. Finally, putting all the pieces together we arrive at the approximation formula (7.19) for the price of a basis swaption. ■

8. Numerical examples and calibration. The aim of this section is twofold: on the one hand, we demonstrate how the multiple curve affine LIBOR model can be calibrated to market data and, on the other hand, we test the accuracy of the swaption and basis swaption approximation formulas. We start by discussing how to build a model which can simultaneously fit caplet volatilities when the options have different underlying tenors. Next, we test numerically the swaption and basis swaption approximation formulas (7.10) and (7.19) using the calibrated models and parameters. In the last subsection, we build a simple model and compute exact and approximate swaption and basis swaption prices in a setup which can be easily reproduced by interested readers.

8.1. A specification with dependent rates. There are numerous ways of constructing models and the trade-off is usually between parsimony and fitting ability. We have elected here a heavily parametrized approach that focuses on the fitting ability, as we believe it best demonstrates the utility of our model. In particular, we want to show that affine LIBOR models, which are driven by positive affine processes, can indeed be well calibrated to market data. Moreover, it is usually easier to move from a complex specification towards a simpler one, than the converse.

We provide below a model specification where LIBOR rates are driven by common and idiosyncratic factors which is suitable for sequential calibration to market data. The starting point is to revisit the expression for LIBOR rates in (4.3):

$$(8.1) \quad 1 + \delta_x L_k^x(t) = M_t^{v_{k-1}^x} / M_t^{u_k^x} = \exp \left(\phi_{T_N-t}(v_{k-1}^x) - \phi_{T_N-t}(u_k^x) + \langle \psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x), X_t \rangle \right).$$

According to Proposition 4.2, when the dimension of the driving process is greater than one, then the vectors v_{k-1}^x and u_k^x are not fully determined by the initial term structure. Therefore, we can navigate through different model specifications by altering the structure of the sequences (u_k^x) and (v_k^x) .

Remark 8.1. The following observation allows us to create an (exponential) linear factor structure for the LIBOR rates with as many common and idiosyncratic factors as desired. Consider an $\mathbb{R}_{\geq 0}^d$ -valued affine process

$$(8.2) \quad X = (X^1, \dots, X^d),$$

and denote the vectors $v_{k-1}^x, u_k^x \in \mathbb{R}_{\geq 0}^d$ by

$$(8.3) \quad v_{k-1}^x = (v_{1,k-1}^x, \dots, v_{d,k-1}^x) \quad \text{and} \quad u_k^x = (u_{1,k}^x, \dots, u_{d,k}^x).$$

Select a subset $\mathcal{J}_k \subset \{1, \dots, d\}$, set $v_{i,k-1}^x = u_{i,k}^x$ for all $i \in \mathcal{J}_k$, and assume that the $\{X^i\}_{i \in \mathcal{J}_k}$ are independent of $\{X^j\}_{j \in \{1, \dots, d\} \setminus \mathcal{J}_k}$. Then, it follows from (8.1) and Keller-Ressel (2008, Prop. 4.7) that L_k^x will also be independent of $\{X^i\}_{i \in \mathcal{J}_k}$ and will depend only on $\{X^j\}_{j \in \{1, \dots, d\} \setminus \mathcal{J}_k}$. The same observation also allows us to construct a model where different factors are used for driving the OIS and LIBOR rates; see also section 4.1.

Let $x_1, x_2 \in \mathcal{X}$ and consider the tenor structures $\mathcal{T}^{x_1}, \mathcal{T}^{x_2}$, where $\mathcal{T}^{x_2} \subset \mathcal{T}^{x_1}$. The dataset under consideration contains caplets maturing on M different dates for each tenor, where M is less than the number of tenor points in \mathcal{T}^{x_1} and \mathcal{T}^{x_2} . In other words, only M maturities are relevant for the calibration. The dynamics of OIS and LIBOR rates are driven by tuples of affine processes

$$(8.4) \quad dX_t^i = -\lambda_i(X_t^i - \theta_i)dt + 2\eta_i \sqrt{X_t^i} dW_t^i + dZ_t^i,$$

$$(8.5) \quad dX_t^c = -\lambda_c(X_t^c - \theta_c)dt + 2\eta_c \sqrt{X_t^c} dW_t^c,$$

for $i = 1, \dots, M$, where X^c denotes the common and X^i the idiosyncratic factor for the i th maturity. Here $X_0^i \in \mathbb{R}_{\geq 0}$, $\lambda_i, \theta_i, \eta_i \in \mathbb{R}_{\geq 0}$ for $i = c, 1, \dots, M$, and W^c, W^1, \dots, W^M , are independent Brownian motions. Moreover, the Z^i are independent compound Poisson processes with constant intensity ν_i and exponentially distributed jumps with mean values μ_i for $i = 1, \dots, M$. Therefore, the full process has dimension $M + 1$:

$$(8.6) \quad X = (X^c, X^1, \dots, X^M),$$

and the total number of process-specific parameters equals $5M + 3$. The affine processes X^c, X^1, \dots, X^M are mutually independent, hence, using Proposition 4.7 in Keller-Ressel

$$\begin{array}{rcl}
 \boxed{u_{\ell_1(M)}^{x_1}} & = & (\tilde{u}_{\ell_1(M)}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M)}^{x_1}) \\
 u_{\ell_1(M)-1}^{x_1} & = & (\tilde{u}_{\ell_1(M)-1}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M)-1}^{x_1}) \\
 u_{\ell_1(M)-2}^{x_1} & = & (\tilde{u}_{\ell_1(M)-2}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M)-2}^{x_1}) \\
 u_{\ell_1(M)-3}^{x_1} & = & (\tilde{u}_{\ell_1(M)-3}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \boxed{u_{\ell_1(M-1)}^{x_1}} & = & (\tilde{u}_{\ell_1(M-1)}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M-1)}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 u_{\ell_1(M-1)-1}^{x_1} & = & (\tilde{u}_{\ell_1(M-1)-1}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M-1)-1}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 u_{\ell_1(M-1)-2}^{x_1} & = & (\tilde{u}_{\ell_1(M-1)-2}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M-1)-2}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 u_{\ell_1(M-1)-3}^{x_1} & = & (\tilde{u}_{\ell_1(M-1)-3}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \boxed{u_{\ell_1(M-2)}^{x_1}} & = & (\tilde{u}_{\ell_1(M-2)}^{x_1} \quad 0 \quad \dots \quad 0 \quad \bar{u}_{\ell_1(M-2)}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \vdots & & \vdots \quad \vdots \quad \dots \quad \vdots \quad \vdots \quad \vdots \\
 \boxed{u_{\ell_1(1)}^{x_1}} & = & (\tilde{u}_{\ell_1(1)}^{x_1} \quad \bar{u}_{\ell_1(1)}^{x_1} \quad \bar{u}_{\ell_1(2)-3}^{x_1} \quad \dots \quad \bar{u}_{\ell_1(M-2)-3}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \vdots & & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 u_1^{x_1} & = & (\tilde{u}_1^{x_1} \quad \bar{u}_1^{x_1} \quad \bar{u}_{\ell_1(2)-3}^{x_1} \quad \dots \quad \bar{u}_{\ell_1(M-2)-3}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1})
 \end{array}$$

Figure 4. The sequence u^{x_1} encompasses the proposed “diagonal plus common” factor structure. In this particular example, $x_1 = 3$ months and caplets mature on entire years.

(2008), the functions $\phi_{(X^c, X^i)}$, respectively, $\psi_{(X^c, X^i)}$, are known in terms of the functions ϕ_{X^c} and ϕ_{X^i} , respectively, ψ_{X^c} and ψ_{X^i} , for all $i \in \{1, \dots, M\}$. The latter are provided, for example, by Grbac and Papapantoleon (2013, Ex. 2.3).

In order to create a “diagonal plus common” factor structure, where each rate for each tenor is driven by an idiosyncratic factor X^i and the common factor X^c , we will utilize Remark 8.1. We start from the longest maturity, which is driven by the idiosyncratic factor X^M and the common factor X^c . Then, at the next caplet maturity date we add the independent idiosyncratic factor X^{M-1} , while we cancel the contribution of X^M by “freezing” the values of u^{x_1} and v^{x_1} corresponding to that factor. The construction proceeds iteratively and the resulting structures for u^{x_1} and v^{x_1} are presented in Figures 4 and 5, where elements of u^{x_1} below a certain “diagonal” are “frozen” to the latest-set value and copied to v^{x_1} . These structures produce the desired feature that each rate is driven by an idiosyncratic and a common factor, while they do not violate inequalities $v_k^x \geq u_k^x \geq u_{k+1}^x$ that stem from Propositions 4.1 and 4.2. In Figures 4 and 5, $\ell_1(k) := k/\delta_{x_1}$ for $k = 1, \dots, M$, i.e., this function maps caplet maturities into tenor points. Moreover, all elements in these matrices are nonnegative and $u_{N^{x_1}}^{x_1} = v_{N^{x_1}}^{x_1} = 0 \in \mathbb{R}^{M+1}$.

The boxed elements are the only ones that matter in terms of pricing caplets when these are not available at every tenor date of \mathcal{T}^{x_1} . The impact of the common factor is determined by the difference between $\tilde{v}_{k-1}^{x_1}$ and $\tilde{u}_k^{x_1}$. If we set $\tilde{v}_{k-1}^{x_1} = \tilde{u}_k^{x_1}$, it follows from Remark 8.1 that $L_k^{x_1}$ will be independent of the common factor X^c and thus determined solely by the corresponding idiosyncratic factor X^i , with $k = \ell_1(i)$. If the values of $\tilde{v}_k^{x_1}$ and $\tilde{u}_k^{x_1}$ are fixed a priori, the remaining values $(\bar{u}_k^{x_1})_{k=1, \dots, N^{x_1}}$ and $(\bar{v}_k^{x_1})_{k=1, \dots, N^{x_1}}$ are determined uniquely by the initial term structure of OIS and LIBOR rates; see again Propositions 4.1 and 4.2. This model structure is consistent with $v_{k-1}^{x_1} \geq u_{k-1}^{x_1} \geq u_k^{x_1}$ if and only if the sequences \tilde{u}^{x_1} and \bar{u}^{x_1}

$$\begin{array}{rcl}
 v_{\ell_1(M)}^{x_1} & = & (\tilde{v}_{\ell_1(M)}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M)}^{x_1}) \\
 \boxed{v_{\ell_1(M)-1}^{x_1}} & = & (\tilde{v}_{\ell_1(M)-1}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M)-1}^{x_1}) \\
 v_{\ell_1(M)-2}^{x_1} & = & (\tilde{v}_{\ell_1(M)-2}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M)-2}^{x_1}) \\
 v_{\ell_1(M)-3}^{x_1} & = & (\tilde{v}_{\ell_1(M)-3}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M)-3}^{x_1}) \\
 v_{\ell_1(M-1)}^{x_1} & = & (\tilde{v}_{\ell_1(M-1)}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M-1)}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \boxed{v_{\ell_1(M-1)-1}^{x_1}} & = & (\tilde{v}_{\ell_1(M-1)-1}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M-1)-1}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 v_{\ell_1(M-1)-2}^{x_1} & = & (\tilde{v}_{\ell_1(M-1)-2}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M-1)-2}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 v_{\ell_1(M-1)-3}^{x_1} & = & (\tilde{v}_{\ell_1(M-1)-3}^{x_1} \quad 0 \quad \dots \quad 0 \quad 0 \quad \bar{v}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 v_{\ell_1(M-2)}^{x_1} & = & (\tilde{v}_{\ell_1(M-2)}^{x_1} \quad 0 \quad \dots \quad 0 \quad \bar{v}_{\ell_1(M-2)}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \boxed{v_{\ell_1(M-2)-1}^{x_1}} & = & (\tilde{v}_{\ell_1(M-2)-1}^{x_1} \quad 0 \quad \dots \quad 0 \quad \bar{v}_{\ell_1(M-2)-1}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \vdots & & \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \quad \vdots \\
 v_{\ell_1(1)}^{x_1} & = & (\tilde{v}_{\ell_1(1)}^{x_1} \quad \bar{v}_{\ell_1(1)}^{x_1} \quad \bar{u}_{\ell_1(2)-3}^{x_1} \quad \dots \quad \bar{u}_{\ell_1(M-2)-3}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \boxed{v_{\ell_1(1)-1}^{x_1}} & = & (\tilde{v}_{\ell_1(1)-1}^{x_1} \quad \bar{u}_{\ell_1(1)-3}^{x_1} \quad \bar{u}_{\ell_1(2)-3}^{x_1} \quad \dots \quad \bar{u}_{\ell_1(M-2)-3}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 \vdots & & \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 v_1^{x_1} & = & (\tilde{v}_1^{x_1} \quad \bar{u}_{\ell_1(1)-3}^{x_1} \quad \bar{u}_{\ell_1(2)-3}^{x_1} \quad \dots \quad \bar{u}_{\ell_1(M-2)-3}^{x_1} \quad \bar{u}_{\ell_1(M-1)-3}^{x_1} \quad \bar{u}_{\ell_1(M)-3}^{x_1})
 \end{array}$$

Figure 5. The sequence v^{x_1} is constructed analogously to u^{x_1} . In this particular example, $x_1 = 3$ months and caplets mature on entire years.

are decreasing, $\tilde{v}_k^{x_1} \geq \tilde{u}_k^{x_1}$ and $\bar{v}_k^{x_1} \geq \bar{u}_k^{x_1}$ for every $k = 1, \dots, N^{x_1}$. Moreover, this structure will be consistent with the normal market situation described in Remark 4.5 if, in addition, $\tilde{v}_k^{x_1} \in [\tilde{u}_k^{x_1}, \tilde{u}_{k-1}^{x_1}]$ and $\bar{v}_k^{x_1} \in [\bar{u}_k^{x_1}, \bar{u}_{k-1}^{x_1}]$ for every $k = 1, \dots, N^{x_1}$.

The corresponding matrices for the x_2 tenor are constructed in a similar manner. More precisely, u^{x_2} is constructed by simply copying the relevant rows from u^{x_1} . Simultaneously, for v^{x_2} the elements $(\tilde{v}_k^{x_2})_{k=0, \dots, N^{x_2}}$ are introduced in order to fit the x_2 -initial LIBOR term structure, as well as the elements $(\bar{v}_k^{x_2})_{k=0, \dots, N^{x_2}}$ which determine the role of the common factor. We present only four rows from these matrices in Figures 6 and 7, for the sake of brevity.

8.2. Calibration to caplet data. The data we use for calibration are from the EUR market on 27 May 2013 collected from Bloomberg. These market data correspond to fully collateralized contracts, hence they are considered “clean.” Bloomberg provides synthetic zero coupon bond prices for EURIBOR rates, as well as OIS rates constructed in a manner described in Akkara (2012). In our example, we will focus on the 3- and 6-month tenors only. The zero coupon bond prices are converted into zero coupon rates and plotted in Figure 8.

Cap prices are converted into caplet implied volatilities using the algorithm described in Levin (2012). The implied volatility is calculated using OIS discounting when inverting the Black (1976) formula. The caplet data we have at our disposal correspond to 3- and 6-month tenor structures. More precisely, in the EUR market caps written on the 3-month tenor are quoted only up to a maturity of 2 years, while 6-month tenor caps are quoted from maturity 3 years and onwards. Moreover, we have option prices only for the maturities corresponding to entire years and not for every tenor point. We have a grid of 14 strikes ranging from 1% to 10% as quoted in Bloomberg. We calibrate to caplet data for maturities

$$\begin{aligned}
 u_{\ell_2(M)}^{x_2} &= (\tilde{u}_{\ell_1(M)}^{x_1} & 0 & \dots & 0 & 0 & \bar{u}_{\ell_1(M)}^{x_1}) \\
 u_{\ell_2(M)-1}^{x_2} &= (\tilde{u}_{\ell_1(M)-2}^{x_1} & 0 & \dots & 0 & 0 & \bar{u}_{\ell_1(M)-2}^{x_1}) \\
 u_{\ell_2(M-1)}^{x_2} &= (\tilde{u}_{\ell_1(M-1)}^{x_1} & 0 & \dots & 0 & \bar{u}_{\ell_1(M-1)}^{x_1} & \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 u_{\ell_2(M-1)-1}^{x_2} &= (\tilde{u}_{\ell_1(M-1)-2}^{x_1} & 0 & \dots & 0 & \bar{u}_{\ell_1(M-1)-2}^{x_1} & \bar{u}_{\ell_1(M)-3}^{x_1})
 \end{aligned}$$

Figure 6. The first four rows of u^{x_2} . In this particular example, $x_2 = 6$ months and caplets mature on entire years.

$$\begin{aligned}
 v_{\ell_2(M)}^{x_2} &= (\tilde{v}_{\ell_2(M)}^{x_2} & 0 & \dots & 0 & 0 & \bar{v}_{\ell_2(M)}^{x_2}) \\
 v_{\ell_2(M)-1}^{x_2} &= (\tilde{v}_{\ell_2(M)-1}^{x_2} & 0 & \dots & 0 & 0 & \bar{v}_{\ell_2(M)-1}^{x_2}) \\
 v_{\ell_2(M-1)}^{x_2} &= (\tilde{v}_{\ell_2(M-1)}^{x_2} & 0 & \dots & 0 & \bar{v}_{\ell_2(M-1)}^{x_2} & \bar{u}_{\ell_1(M)-3}^{x_1}) \\
 v_{\ell_2(M-1)-1}^{x_2} &= (\tilde{v}_{\ell_2(M-1)-1}^{x_2} & 0 & \dots & 0 & \bar{v}_{\ell_2(M-1)-1}^{x_2} & \bar{u}_{\ell_1(M)-3}^{x_1})
 \end{aligned}$$

Figure 7. The first four rows of v^{x_2} . In this particular example, $x_2 = 6$ months and caplets mature on entire years.

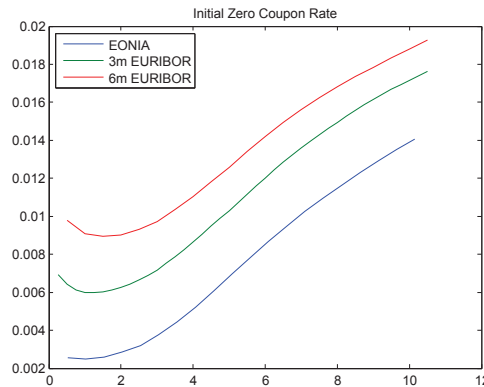


Figure 8. Zero coupon rates, EUR market, 27 May 2013.

1, 2, ..., 10 years and the OIS zero coupon bond $B(\cdot, 10.5)$ defines the terminal measure.¹ We fix in advance the values of the parameters (\tilde{u}^{x_1}) , (\tilde{u}^{x_2}) , (\tilde{v}^{x_1}) , and (\tilde{v}^{x_2}) , as well as the parameters of the process X^c . The impact of X^c is determined by the spread between $\tilde{u}_{\ell_1(i)}^{x_1}$ and $\tilde{v}_{\ell_1(i)-1}^{x_1}$, and $\tilde{u}_{\ell_2(i)}^{x_2}$ and $\tilde{v}_{\ell_2(i)-1}^{x_2}$ for the 3-month and 6-month tenor caplets, respectively, and we will simplify by setting $\tilde{u}_1^{x_1} = \dots = \tilde{u}_{N^{x_1}-1}^{x_1} = u_c$ constant. The constant u_c , along with $\tilde{v}_1^{x_1}, \dots, \tilde{v}_{N^{x_1}}^{x_1}$ and $\tilde{v}_1^{x_2}, \dots, \tilde{v}_{N^{x_2}}^{x_2}$ are not identified by the initial term structures and have to be determined in some other manner (e.g., by calibration to swaptions or basis swaptions). They also cannot be chosen completely freely and one has to validate that the values of $u_k^{x_1}, u_k^{x_2}$ and $v_k^{x_1}, v_k^{x_2}$ stemming from these procedures satisfy the necessary inequalities, i.e., $v_{k-1}^x \geq u_{k-1}^x \geq u_k^x$. Having this in mind, we chose these values in a manner such that

¹We found that the model performs slightly better in calibration using this numeraire than when choosing 10 years as the terminal maturity.

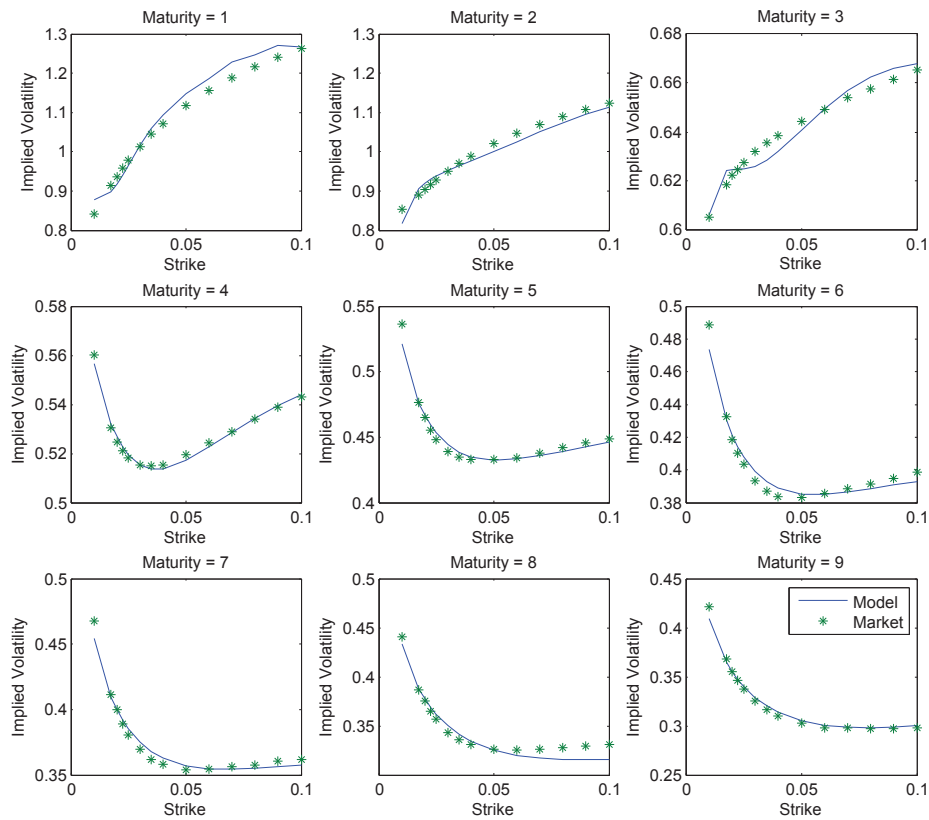


Figure 9. Market and model implied volatility for caplets written on 3- (1–2-year maturity) and 6- (2–9-year maturity) month tenor EURIBOR.

X^c accounts for approximately 50% of the total variance of LIBOR rates from maturities 4 until 10, and about 10% of the total variance for maturities 1 until 3. We have verified through experimentation that this adhoc choice of dependence structure does not have a qualitative impact on the results of the following sections. Alternatively, these parameters could be calibrated to derivatives such as swaptions, basis swaptions, or other derivatives partly determined by the dependence structure of the LIBOR rates. However, since interest rate derivative markets remain highly segmented and joint calibration of caplets and swaptions is a perennial challenge (see, e.g., [Brigo and Mercurio \(2006\)](#) or [Ladkau, Schoenmakers, and Zhang \(2013\)](#)), we will leave this issue for future research.

The model construction summarized in Figures 4–7 has the advantage that caplets can be calibrated sequentially one maturity at a time starting at the longest maturity and then moving backwards. In the calibration procedure we fit the parameters of each idiosyncratic process X^i to caplet prices with maturity T_i^x while simultaneously choosing $u_i^{x_1}, u_i^{x_2}$ and $v_i^{x_1}, v_i^{x_2}$ to match the corresponding values of the initial OIS and LIBOR rates. Caplets are priced using formula (6.5) and the parameters are found using standard least-squares minimization between market and model implied volatility. The results² from fitting the caplets are shown in Figure 9. We

²All calibrated and chosen parameter values as well as the calibrated matrices u^{x_j}, v^{x_j} for $j = 1, 2$ are available from the authors upon request.

can observe that the model performs very well for different types of volatility smiles across the whole term structure, with only minor problems for extreme strikes in maturities 1-3. These problems are however mainly cosmetic in nature as these prices and more importantly the deltas for these contracts are very close to zero anyway, making any model error in this region economically insignificant.

8.3. Swaption price approximation. The next two sections are devoted to numerically testing the validity of the swaption and basis swaption price approximation formulas derived in sections 7.1 and 7.2. We will run a Monte Carlo (MC) study comparing the true price with the linear boundary approximation formula. The model parameters used stem from the calibration to the market data described in the previous section.

Let us denote the true and the approximate prices as follows:

$$\begin{aligned} \mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N) \mathbb{E}_N \left[\left(\sum_{i=p+1}^q M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^q K_x M_{T_p^x}^{u_i^x} \right) \mathbf{1}_{\{f(X_{T_p^x}) \geq 0\}} \right], \\ \tilde{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N) \mathbb{E}_N \left[\left(\sum_{i=p+1}^q M_{T_p^x}^{v_{i-1}^x} - \sum_{i=p+1}^q K_x M_{T_p^x}^{u_i^x} \right) \mathbf{1}_{\{\tilde{f}(X_{T_p^x}) \geq 0\}} \right], \end{aligned}$$

where f and \tilde{f} were defined in (7.8) and (7.9), respectively. The MC estimator³ of $\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x)$ is denoted by $\hat{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x)$ and we will refer to it as the “true price.” Instead of computing $\tilde{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x)$ using Fourier methods, we will form another MC estimator $\hat{\tilde{\mathbb{S}}}_0^+(K, \mathcal{T}_{pq}^x)$. This has the advantage that, when the same realizations are used to calculate both MC estimators, the difference $\hat{\mathbb{S}}_0^+(K, \mathcal{T}_{pq}^x) - \hat{\tilde{\mathbb{S}}}_0^+(K, \mathcal{T}_{pq}^x)$ will be an estimate of the error induced by the linear boundary approximation which is minimally affected by simulation bias.

Swaption prices vary considerably across strike and maturity, thus we will express the difference between the true and the approximate price in terms of implied volatility (using OIS discounting), which better demonstrates the economic significance of any potential errors. We price swaptions on three different underlying swaps. The results for the 3-month underlying tenor are exhibited in Figure 10. The corresponding results for the 6-month tenor swaptions have errors which are approximately one-half the level in the graphs shown here and have been omitted for brevity.

On the left-hand side of Figure 10, implied volatility levels are plotted for the true and the approximate prices. The strikes are chosen to range from 60% to 200% of the spot value of the underlying fair swap rate, which is the normal range the products are quoted. The right-hand side shows the difference between the two implied volatilities in basis points (i.e., multiplied by 10^4). As was also documented in [Schrager and Pelsser \(2006\)](#), the errors of the approximation usually increase with the number of payments in the underlying swap. This is also the case here, however, the level of the errors is in all cases very low. In normal markets,

³We construct the MC estimate using 5 million paths of X with 10 discretization steps per year. In each discretization step the continuous part is simulated using the algorithm in [Glasserman \(2003, section 3.4.1\)](#) while the jump part is handled using [Glasserman \(2003, pp. 137–139\)](#) with jump size distribution changed from log-normal to exponential.

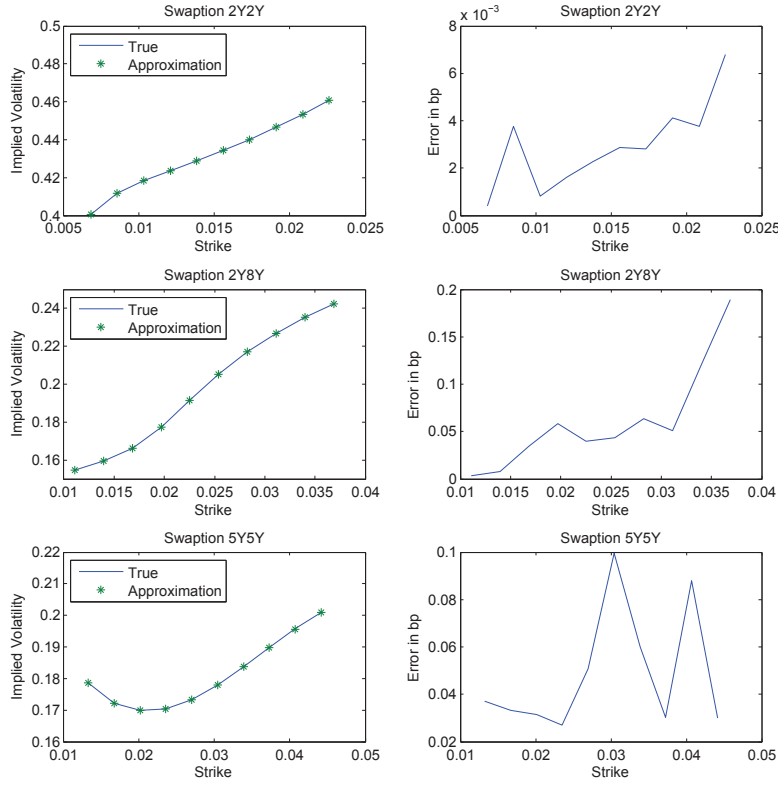


Figure 10. Implied volatility and absolute errors for 3-month swaptions.

bid-ask spreads typically range from 10 to 300 basis points (at the at-the-money level), thus, even the highest errors are too small to be of any economic significance. This is true even in the case of the 2Y8Y swaption which contains 32 payments.

8.4. Basis swaption price approximation. In order to test the approximation formula for basis swaptions, we will follow the same methodology as in the previous subsection. That is, we calculate MC estimators for the following two expectations:

$$\begin{aligned} \mathbb{BS}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) &= B(0, T_N) \mathbb{E}_N \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \right) \right. \right. \\ &\quad \left. \left. - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \right) \right) \mathbf{1}_{\{g(X_{T_{p_2}}) \geq 0\}} \right], \\ \widetilde{\mathbb{BS}}_0^+(S, \mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}) &= B(0, T_N) \mathbb{E}_N \left[\left(\sum_{i=p_2+1}^{q_2} \left(M_{T_{p_2}^{x_2}}^{v_{i-1}^{x_2}} - M_{T_{p_2}^{x_2}}^{u_i^{x_2}} \right) \right. \right. \\ &\quad \left. \left. - \sum_{i=p_1+1}^{q_1} \left(M_{T_{p_1}^{x_1}}^{v_{i-1}^{x_1}} - S_{x_1} M_{T_{p_1}^{x_1}}^{u_i^{x_1}} \right) \right) \mathbf{1}_{\{\tilde{g}(X_{T_{p_2}}) \geq 0\}} \right], \end{aligned}$$

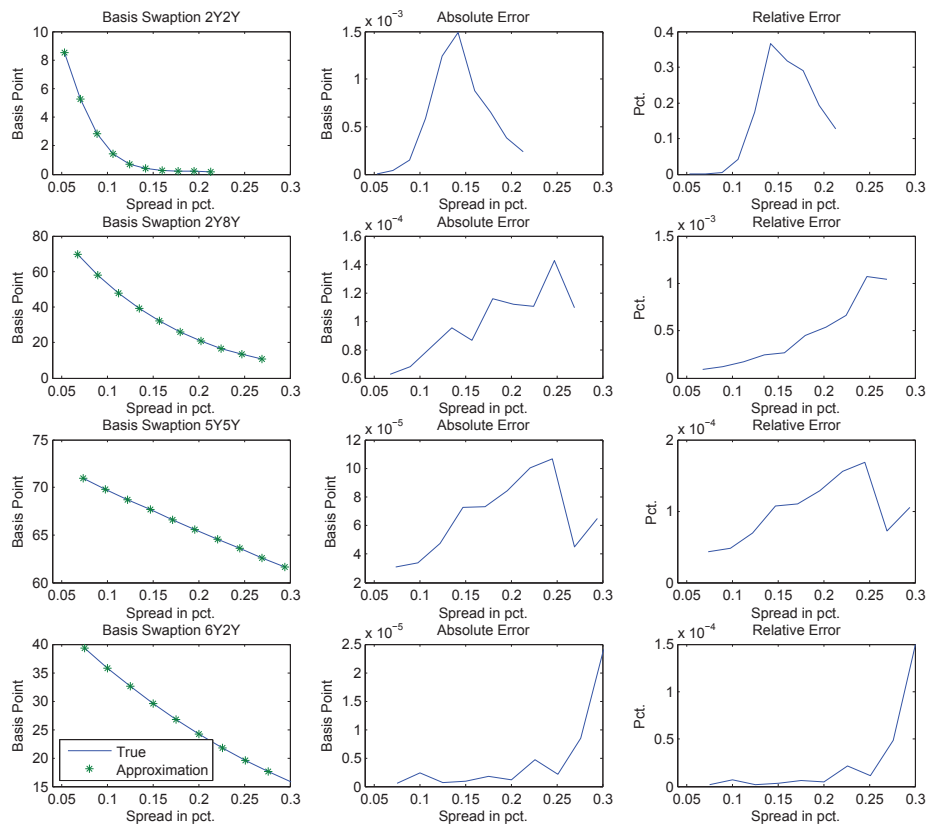


Figure 11. Prices in basis points, absolute and relative errors for 3-month–6-month basis swaptions.

where $S_{x_1} = 1 - \delta_{x_1}S$, while g and \tilde{g} were defined in (7.17) and (7.18). Using the same realizations, we plot the level, and the absolute and relative differences between both prices measured in basis points as a function of the spread for three different underlying basis swaps. The spreads are chosen to range from 50% to 200% of the at-the-money level, i.e., the spread that sets the underlying basis swap to a value of zero (see again (6.4)),

$$S_{ATM} := S_0(\mathcal{T}_{pq}^{x_1}, \mathcal{T}_{pq}^{x_2}).$$

The numerical results can be seen in Figure 11. We have chosen these maturities to be representative of two general patterns. The first is that the errors tend to increase with the length of the basis swap, which is exemplified by comparing errors for the 2Y8Y, 5Y5Y, and 6Y2Y contracts. The second pattern relates to when the majority of payments in the contract are paid out. We can notice that the errors for the 2Y2Y contract are much larger than for the corresponding 6Y2Y, even though both contain the same number of payments. Furthermore, we can also see that the 2Y2Y contract has larger errors than the 2Y8Y even though both have the same maturity and the latter has more payments. This anomalous result can be explained by the convexity of the term structure of interest rates. In Figure 8 we can notice that the majority of the payments of the 2Y2Y contract fall in a particularly curved region of the term structure. This will result in an exercise boundary which is also more nonlinear,

thus leading to the relative deterioration of the linear boundary approximation. However, it must be emphasized that the errors are still at a level easily deemed economically insignificant, with a maximum relative error of 0.4% in a spread region where the price levels are particularly low.

Remark 8.2. The approximative formulas from Propositions 7.2 and 7.4 can be used for calibration to swaptions and basis swaptions. Error bounds for these approximations are not available in closed form and thus accuracy in the entire parameter space cannot be guaranteed. Any desired accuracy in pricing is achievable using MC methods, which means that the accuracy of the approximate formula can always be validated numerically. However, when performing a full calibration, which may require several hundreds of iterations in order to achieve convergence in a numerical optimization procedure, MC methods are slow in comparison to the analytical approximation. Thus, in order to calibrate to swaption prices, one would prefer the approximation to the MC method. Then one can perform a MC simulation (just one) to validate that the approximation is also correct for the parameters found by the numerical optimization. Let us also mention that in a typical calibration procedure an acceptable error is around 2%, well above the error of the approximative formulas, else the risk of overfitting the data is present.

8.5. A simple example. The purpose of this section is to provide a simple example to help the reader’s intuition regarding the numerical implementation of the model. We present a fully constructed and more manageable numerical toy example of fitting the model parameters u_k^x and v_k^x to the initial term structures, which can be reproduced by the reader himself/herself as opposed to the full calibration example in sections 8.3 and 8.4. Moreover, we show in this simple setting how Approximations (S) and (BS) are computed.

We start by choosing a simple two factor model $X = (X^1, X^2)$ with

$$(8.7) \quad dX_t^i = -\lambda_i(X_t^i - \theta_i)dt + 2\eta_i\sqrt{X_t^i}dW_t^i + dZ_t^i, \quad i = 1, 2,$$

where we set

i	X_0^i	λ_i	θ_i	η_i	ν_i	μ_i
1	0.5000	0.1000	1.5300	0.2660	0	0
2	9.4531	0.0407	0.0591	0.4640	0.0074	0.2499

The initial term structures are constructed from a Nelson–Siegel parametrization of the zero coupon rate $R(T)$

$$(8.8) \quad R(T) = \beta_0 + \beta_1 \frac{1 - e^{-\gamma T}}{\gamma T} + \beta_2 \left(\frac{1 - e^{-\gamma T}}{\gamma T} - e^{-\gamma T} \right).$$

We limit ourselves to two tenors, x_1 corresponding to 3 months and x_2 corresponding to 6 months. We construct the initial curves from the following parameters:

Curve	β_0	β_1	β_2	γ
OIS	0.0003	0.01	0.07	0.06
3m	0.0032	0.01	0.07	0.06
6m	0.0050	0.01	0.07	0.06

In particular, we use (8.8) to construct the initial 3- and 6-month LIBOR curves via the expression

$$L_k^x(0) = \frac{1}{\delta_x} \left(\frac{\exp(-R^x(T_{k-1})T_{k-1})}{\exp(-R^x(T_k)T_k)} - 1 \right)$$

for $x = 3$ months and 6 months, and a third one to construct an initial OIS curve consistent with the system

$$B(0, T_k) = \exp(-R^{OIS}(T_k)T_k).$$

Moreover, we construct the matrices u^{xj} and v^{xj} in the following simple manner:

$$(8.9) \quad u_k^{x1} = (u_c \quad \bar{u}_k^{x1}), \quad k = 1, \dots, N^{x1},$$

$$(8.10) \quad u_k^{x2} = u_{k\delta_{x2}/\delta_{x1}}^{x1}, \quad k = 1, \dots, N^{x2},$$

$$(8.11) \quad v_k^{x1} = (\tilde{v}_c^{x1} \quad \bar{v}_k^{x1}), \quad k = 0, \dots, N^{x1} - 1,$$

$$(8.12) \quad v_k^{x2} = (\tilde{v}_c^{x2} \quad \bar{v}_k^{x2}), \quad k = 0, \dots, N^{x2} - 1,$$

and $u_{N^{x1}}^{x1} = u_{N^{x2}}^{x2} = 0$, where $\bar{u}_k^{xj}, \bar{v}_k^{xj} \in \mathbb{R}_{\geq 0}$ for $j = 1, 2$. The bond $B(\cdot, 4.5)$ defines the terminal measure, thus $N^{x1} = 18$ and $N^{x2} = 9$. We set $u_c = 0.0065$, $\tilde{v}_c^{x1} = 0.007$, and $\tilde{v}_c^{x2} = 0.0075$. The remaining values can then be determined uniquely using (4.4) and (4.6), i.e., by fitting the initial term structures. We get that

k	u_k^{x1}	v_k^{x1}	u_k^{x2}	v_k^{x2}
0	-	0.008966	-	0.009035
1	0.008638	0.008641	0.008286	0.008358
2	0.008286	0.008289	0.007505	0.007577
3	0.007908	0.007911	0.006625	0.006697
4	0.007505	0.007507	0.005652	0.005725
5	0.007077	0.007079	0.004591	0.004664
6	0.006625	0.006627	0.003447	0.003520
7	0.006150	0.006152	0.002225	0.002298
8	0.005652	0.005654	0.000929	0.001003
9	0.005132	0.005135	0	-
10	0.004591	0.004594		
11	0.004029	0.004032		
12	0.003447	0.003450		
13	0.002847	0.002848		
14	0.002225	0.002228		
15	0.001586	0.001589		
16	0.000929	0.000932		
17	0.000254	0.000257		
18	0	-		

We can observe that all sequences u^{xj}, v^{xj} for $j = 1, 2$ are decreasing, which corresponds to the normal market situation; see, again, Remark 4.5.

8.5.1. Swaption approximation. Let us consider a 2Y2Y swaption on 3-month LIBOR rates, i.e., a swaption in the notation of section 7.1 with $p = 8$ and $q = 16$. We run an MC study equivalent to the one in section 8.3 and the results are reported for four different strikes:

Strike (K)	\hat{S}_0^+	Error	IV (%)	IV Error	\mathcal{A}	\mathcal{B}
0.013238	176.17	2.06e-08	30.38	2.326e-10	-5.5403	(1.1596 1)
0.023535	52.214	4.31e-08	26.78	1.818e-10	-10.2982	(1.1605 1)
0.033831	9.7898	4.09e-08	24.82	2.971e-10	-15.0481	(1.1615 1)
0.044128	1.4016	7.90e-09	23.72	2.016e-10	-19.7899	(1.1625 1)

where

- $\hat{S}_0^+ := \hat{S}_0^+(K, \mathcal{T}_{8,16}^{x_1})$ and IV denote the MC estimator of the price (in basis points) and the implied volatility (with OIS discounting) using the true exercise boundary defined in (7.8).
- Error := $|\hat{S}_0^+(K, \mathcal{T}_{8,16}^{x_1}) - \hat{S}_0^+(K, \mathcal{T}_{8,16}^{x_1})|$, where $\hat{S}_0^+(K, \mathcal{T}_{8,16}^{x_1})$ denotes the MC estimator of the price (in basis points) using the approximate exercise boundary defined in (7.9).
- IV Error = $|\text{IV} - \widetilde{\text{IV}}|$, where $\widetilde{\text{IV}}$ denotes the implied volatility (with OIS discounting) calculated from $\hat{S}_0^+(K, \mathcal{T}_{8,16}^{x_1})$.
- $\mathcal{A} \in \mathbb{R}$ and $\mathcal{B} \in \mathbb{R}^2$ determine the linear approximation to the exercise boundary defined by the function f in (7.8):

$$f(y) \approx \mathcal{A} + \langle \mathcal{B}, y \rangle.$$

Applying the procedure in Singleton and Umantsev (2002, pp. 432–434), we first calculate the upper and lower quantiles for $X_2^{(1)}$ using Gaussian approximations for speed. We solve for x_l and x_u in

$$f\left(\left[q_{X_2^{(1)}}(0.05), x_l\right]\right) = 0 \quad \text{and} \quad f\left(\left[q_{X_2^{(1)}}(0.95), x_u\right]\right) = 0.$$

Then, \mathcal{A} and \mathcal{B} are computed by fitting the straight line

$$\mathcal{A} + \langle \mathcal{B}, y \rangle = 0$$

through the two points $y_l = \left[q_{X_2^{(1)}}(0.05), x_l\right]$ and $y_u = \left[q_{X_2^{(1)}}(0.95), x_u\right]$.

8.5.2. Basis swaption approximation. Let us also consider a 2Y2Y basis swaption. This is an option to enter into a basis swap paying 3-month LIBOR plus spread S and receiving 6-month LIBOR, which starts at year 2 and ends at year 4. Once again we conduct an MC study equivalent to section 8.4, and get that

Spread (S)	$\mathbb{B}\hat{S}_0^+$	Price error	\mathcal{C}	\mathcal{D}
0.0010945	13.778	2.103e-06	-7.7191	(1 5.7514)
0.0019458	3.7972	4.784e-05	-14.0029	(1 5.7694)
0.0027971	0.64406	9.364e-05	-20.2158	(1 5.7868)
0.0036484	0.080951	5.852e-05	-26.3597	(1 5.8037)

where

- $\widehat{\mathbb{B}}\mathbb{S}_0^+ := \widehat{\mathbb{B}}\mathbb{S}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})$ denotes the MC estimator of the price (in basis points) using the true exercise boundary defined in (7.17).
- Price error := $|\widehat{\mathbb{B}}\mathbb{S}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2}) - \widehat{\widehat{\mathbb{B}}}\mathbb{S}_0^+(S, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})|$, where $\widehat{\widehat{\mathbb{B}}}\mathbb{S}_0^+(K, \mathcal{T}_{8,16}^{x_1}, \mathcal{T}_{4,8}^{x_2})$ similarly denotes the MC estimator of the price (in basis points) using the approximate exercise boundary defined in (7.18).
- $\mathcal{C} \in \mathbb{R}$ and $\mathcal{D} \in \mathbb{R}^2$ determine the linear approximation to the exercise boundary defined by the function g in (7.17):

$$g(y) \approx \mathcal{C} + \langle \mathcal{D}, y \rangle.$$

Applying again the same procedure, we first calculate the upper and lower quantiles for $X_2^{(1)}$ and solve for \tilde{x}_l and \tilde{x}_u in

$$g\left(\left[q_{X_2^{(1)}}(0.05), \tilde{x}_l\right]\right) = 0 \quad \text{and} \quad g\left(\left[q_{X_2^{(1)}}(0.95), \tilde{x}_u\right]\right) = 0.$$

Then, \mathcal{C} and \mathcal{D} are computed by fitting the straight line

$$\mathcal{C} + \langle \mathcal{D}, \tilde{y} \rangle = 0$$

through the two points $\tilde{y}_l = \left[q_{X_2^{(1)}}(0.05), \tilde{x}_l\right]$ and $\tilde{y}_u = \left[q_{X_2^{(1)}}(0.95), \tilde{x}_u\right]$.

These simple examples highlight once again the accuracy of the linear boundary approximations developed in sections 7.1 and 7.2.

9. Concluding remarks and future research. Finally, let us conclude with some remarks that further highlight the merits of the affine LIBOR models and some topics for future research. Consider the following exotic product: a loan with respect to a 1\$ notional over a monthly tenor structure $\mathcal{T} = \{0 = T_0 < T_1 < \dots < T_N\}$ with optional interest payments due to the following scheme: at time $t = 0$, the product holder may contract to settle the first interest payment either after 1, 3, or 6 months (as long as the maturity T_N is not exceeded). Next, at the first settlement date, the holder may choose again either the 1-, 3-, or 6-month LIBOR to be settled 1, 3 or 6 months later (while not exceeding T_N). She/He continues until the last payment is settled at T_N and the notional is paid back. Clearly, the value of this product at $t = 0$ in the single curve (precrisis) LIBOR world would be simply zero. However, in the multiple curve world the pricing of this product is highly nontrivial. In particular, such an evaluation would involve the dynamics of any LIBOR rate over the periods $[T_i, T_j]$, $0 \leq i \leq j \leq N$, where $T_j - T_i$ equals 1, 3 or 6 months. As a matter of fact, the affine LIBOR model with multiple curves presented in this paper is tailor made for this problem as it produces “internally consistent” LIBOR and OIS rates over any subtenor structure. This means that for all subtenor structures the rates have the same type of dynamics and the driving process remains affine under any forward measure. To the best of our knowledge, this is the only multiple curve LIBOR model in the literature that naturally produces this consistency across all different tenors. The full details of the pricing of this product are, however, beyond the scope of this article.

The property of internal consistency is beneficial already in the single curve LIBOR models. More precisely, the dynamics of LIBOR rates in the classical LMMs are specified by setting a “natural” volatility structure of a LIBOR system based on a particular tenor structure. As a consequence, the volatilities of the LIBOR rates spanning, e.g., a double period length are immediately hard to determine, as they contain the LIBOR rates of the shorter period. On the contrary, in the single curve affine LIBOR models the dynamics of the LIBOR are specified via ratios of martingales that are connected with different underlying tenors, thus one has simultaneously specified the dynamics of all possible LIBOR rates in an internally consistent way.

However, the other side of this coin is that a proper choice of the driving affine process, and the effective calibration of the affine LIBOR models entailed, are far from trivial. In fact these issues require the development of new approaches and thus provide a new strand of research on its own. Therefore the calibration experiments in this paper are to be considered preliminary and merely to demonstrate the potential flexibility of the affine LIBOR model with multiple curves.

Appendix A. Terminal correlations.

This appendix is devoted to the computation of terminal correlations. The expression “terminal correlation” is used in the same sense as in [Brigo and Mercurio \(2006, section 6.6\)](#), i.e., it summarizes the degree of dependence between two LIBOR rates at a fixed, terminal time point. Here the driving process is a general affine process and not just an affine diffusion as in section 5.4.

We start by introducing some shorthand notation:

$$\begin{aligned} \Phi_k^x(t) &:= \phi_{T_N-t}(v_{k-1}^x) - \phi_{T_N-t}(u_k^x), \\ \Psi_k^x(t) &:= \psi_{T_N-t}(v_{k-1}^x) - \psi_{T_N-t}(u_k^x), \\ \Phi_{k_1, k_2}^{x_1, x_2}(t) &:= \Phi_{k_1}^{x_1}(t) + \Phi_{k_2}^{x_2}(t), \\ \Psi_{k_1, k_2}^{x_1, x_2}(t) &:= \Psi_{k_1}^{x_1}(t) + \Psi_{k_2}^{x_2}(t), \end{aligned}$$

where $k \in \mathcal{K}^x$ and $k_l \in \mathcal{K}^{x_l}$ for $l = 1, 2$. Then, we have from (4.3) that

$$(A.1) \quad 1 + \delta_{x_l} L_{k_l}^{x_l}(T_i) = M_{T_i}^{v_{k_l-1}^{x_l}} / M_{T_i}^{u_{k_l}^{x_l}} = \exp \left(\Phi_{k_l}^{x_l}(T_i) + \langle \Psi_{k_l}^{x_l}(T_i), X_{T_i} \rangle \right)$$

for $l = 1, 2$ and $T_i \leq T_{k_1-1}^{x_1} \vee T_{k_2-1}^{x_2}$. We also denote the moment generating function of X_{T_i} under the measure \mathbb{P}_N as follows:

$$(A.2) \quad \Theta_{T_i}(z) = \mathbb{E}_N [e^{\langle z, X_{T_i} \rangle}] = \exp \left(\phi_{T_i}(z) + \langle \psi_{T_i}(z), X_0 \rangle \right).$$

Therefore we get that

$$(A.3) \quad \mathbb{E}_N \left[M_{T_i}^{v_{k-1}^x} / M_{T_i}^{u_k^x} \right] = e^{\Phi_k^x(T_i)} \Theta_{T_i}(\Psi_k^x(T_i)),$$

$$(A.4) \quad \mathbb{E}_N \left[\left(M_{T_i}^{v_{k-1}^x} / M_{T_i}^{u_k^x} \right)^2 \right] = e^{2\Phi_k^x(T_i)} \Theta_{T_i}(2\Psi_k^x(T_i)),$$

$$(A.5) \quad \mathbb{E}_N \left[M_{T_i}^{v_{k_1-1}^{x_1}} / M_{T_i}^{u_{k_1}^{x_1}} \cdot M_{T_i}^{v_{k_2-1}^{x_2}} / M_{T_i}^{u_{k_2}^{x_2}} \right] = e^{\Phi_{k_1, k_2}^{x_1, x_2}(T_i)} \Theta_{T_i}(\Psi_{k_1, k_2}^{x_1, x_2}(T_i)).$$

The formula for terminal correlations follows after inserting the expressions above in the definition of correlation and doing some tedious, but straightforward, computations:

$$\begin{aligned} \text{Corr}_{T_i}[L_{k_1}^{x_1}, L_{k_2}^{x_2}] &\stackrel{(A.1)}{=} \text{Corr} \left[M_{T_i}^{v_{k_1-1}^{x_1}} / M_{T_i}^{u_{k_1}^{x_1}}, M_{T_i}^{v_{k_2-1}^{x_2}} / M_{T_i}^{u_{k_2}^{x_2}} \right] \\ &= \frac{\Theta_{T_i}(\Psi_{k_1, k_2}^{x_1, x_2}(T_i)) - \Theta_{T_i}(\Psi_{k_1}^{x_1}(T_i))\Theta_{T_i}(\Psi_{k_2}^{x_2}(T_i))}{\sqrt{\Theta_{T_i}(2\Psi_{k_1}^{x_1}(T_i)) - \Theta_{T_i}(\Psi_{k_1}^{x_1}(T_i))^2} \sqrt{\Theta_{T_i}(2\Psi_{k_2}^{x_2}(T_i)) - \Theta_{T_i}(\Psi_{k_2}^{x_2}(T_i))^2}}. \end{aligned}$$

Acknowledgments. We are grateful to Fabio Mercurio and Steven Shreve for valuable discussions and suggestions. We also thank seminar participants at Cass Business School, Imperial College London, the London Mathematical Finance Seminar, Carnegie Mellon University, and the University of Padova for their comments.

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