

Universal security for randomness expansion from the spot-checking protocol

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Abstract

Colbeck (*Thesis*, 2006) proposed using Bell inequality violations to generate certified random numbers. While full quantum-security proofs have been given, it remains a major open problem to identify the broadest class of Bell inequalities and lowest performance requirements to achieve such security. In this paper, working within the broad class of *spot-checking protocols*, we prove exactly which Bell inequality violations can be used to achieve full security. Our result greatly improves the known noise tolerance for secure randomness expansion: for the commonly used CHSH game, full security was only known with a noise tolerance of 1.5%, and we improve this to 10.3%. We also generalize our results beyond Bell inequalities and give the first security proof for randomness expansion based on Kochen-Specker inequalities. The central technical contribution of the paper is a new uncertainty principle for the Schatten norm, which is based on the uniform convexity inequality of Ball, Carlen, and Lieb (*Inventiones mathematicae*, 115:463–482, 1994).

1 Introduction

Randomness is indispensable for modern day information processing. Secure generation of randomness is at the very foundation of modern cryptography: a message is secretive precisely when it is random to the adversary. But the generation of true randomness is challenging both theoretically and practically. A fundamental difficulty is the fact that there is no complete test of randomness — all randomness tests can be fooled by a deterministic generator. And indeed, current solutions have problems: for example, Heninger *et al.* [14] and Lenstra *et al.* [17] have found independently that a significant percentage of cryptographic keys can be broken due to the lack of entropy. One of NIST’s previous standards for pseudorandom number generation is widely believed to be backdoored [20].

More recently, RNGs based on quantum measurements have emerged in the market (e.g., IDQuantique’s Quantis). While a (close to) perfect implementation of certain measurements can theoretically guarantee randomness, current technology is still far from reaching that precision. An additional concern is, even if in the future the implementation technology is satisfactory, could there be backdoors in the generator inserted by a malicious party? It is difficult for the user, as a classical being, to directly verify the inner-working of the quantum device.

In his Ph.D. thesis [6], Colbeck formulated the problem of untrusted-device randomness expansion, and proposed a protocol based on non-local games. In Colbeck’s protocol, a nonlocal game is played repeatedly (using an initial random seed) on a multi-part quantum device. The proposed basis for security is that if the device exhibits a superclassical average score, then it must be exhibiting quantum, and therefore random, behavior. If a “success” event occurs in the protocol — if the average score is above a certain acceptance threshold — then the outputs produced by the device are assumed to be partially random, and a quantum-proof randomness extractor (e.g., [9]) is applied to produce a shorter string which represents the final output of the protocol.

The goal of the aforementioned protocol is that the final output be a uniformly random string, thus achieving *randomness expansion* — starting with initial random seed and obtaining a larger random output. Subsequent authors observed that one can minimize the size of the seed by skewing the input distribution used for the nonlocal game (e.g., by giving the device a fixed input string on most rounds).

Proving security for such a protocol requires showing that, when the protocol succeeds, there is a uniform lower bound $H_{min}^{\delta}(X | E) \geq C$ on the smooth min-entropy of X (the raw outputs of the device) conditioned on E (quantum information that may be possessed by an adversary). This bound must be proved to hold for all quantum devices compatible with the nonlocal game. Once this is established, it follows that any quantum-proof randomness extractor can be used to produce roughly C random bits [24]. (See [3, 4] for recent work on quantum-proof randomness extractors.)

Several authors proved security of Colbeck-type protocols against classical side information only [21, 12, 22, 7]. In a groundbreaking work, Vazirani and Vidick [29] proved full quantum security. Their protocol provided exponential randomness expansion, and was later extended to unbounded expansion [8]. The quantum protocol of [29] included an exact requirement on the performance of the device, and so it remained an open question whether quantum security could be proved for a robust (error-tolerant) protocol.

In [18], the present authors proved quantum security with error-tolerance, via a new set of techniques based on Renyi entropies. We proved other new features (cryptographic security, unit size quantum memory, nonzero bit-rate) and also proved, with Chung and Wu, that error-tolerant unbounded randomness expansion was possible ([5, 18, 13]). Our protocol used a class of n -player binary XOR games, and the noise tolerance was significant ($\geq 1.5\%$) but not proven to be optimal.

The motivating question for the current paper is this: what are the most general conditions under which randomness expansion can be proved with full quantum security? Answering this question is important for lowering the implementation requirements on randomness expansion protocols, and thus improving their practical value.

1.1 Central result

We wish to answer the following questions:

1. What devices can achieve secure randomness expansion?
2. What games can be used to achieve secure randomness expansion?
3. What is the largest noise tolerance for each game?

In this paper we will limit our focus in the following way: we will deal only with protocols in which the Bell inequality (or other device test) is the same on each round, and is applied using independently generated inputs. (This is a natural assumption, although we note that the literature includes protocols that do not satisfy this assumption, e.g., [29]. Answering questions 1-3 for such multi-stage protocols is another interesting avenue for research.)

Given that a protocol is using the same Bell inequality on each round, the only way to achieve superlinear randomness expansion is to adjust the input distribution for the Bell inequality so that a certain fixed input \bar{a} occurs with probability $(1 - q)$, where q is a parameter that we make small when the number of rounds (N) is large. (Without this requirement, the amount of seed needed to generate the random inputs would have to be at least linearly proportional to the number of extractable bits in the output.) This motivates a protocol which we import from [7], referred to here as the *spot-checking protocol*. A general version of the spot-checking protocol is shown in Figure 1. (In [7], this protocol was showed to be secure against classical adversaries.)

Let E be an external quantum system which may be entangled with the device D . Let A, X, T be classical registers denoting, respectively, the collected inputs of D , the collected outputs of D , and the bits (t_1, \dots, t_N) across all N rounds. Let $\Gamma = \Gamma_{AXTE}$ denote the joint state of these registers taken together, and let $\Gamma^s \leq \Gamma$ denote the subnormalized operator corresponding to the “success” event. If the normalization of Γ^s satisfies

$$H_{\min}(X | ATE) \geq y, \tag{1.1}$$

we say that Protocol R_{gen} has produced y extractable bits. If Γ_s is within trace-distance δ of an operator Γ' satisfying the same condition, or is within trace-distance δ of the zero state, then we say that Protocol R_{gen} has produced y extractable bits with soundness error δ .

For any nonlocal game G , let W_G be the supremum of all expected scores that can be achieved at G by compatible quantum devices, and let $W_{G,\bar{a}}$ denote the same supremum taken just over devices that give deterministic outputs on input \bar{a} . Our central result is summed up by the following theorem.

Theorem 1.1. *For any game G , there are functions $\pi: [0, W_G] \rightarrow \mathbb{R}$ and $\Delta: (0, 1]^2 \rightarrow \mathbb{R}$ such that the following hold.*

1. For any $b \in (0, 1]$, Protocol R_{gen} produces at least

$$N [\pi(\chi) - \Delta(b, q)] \tag{1.2}$$

extractable bits with soundness error $3 \cdot 2^{-bqN}$.

2. The function π is nonzero on the interval $(W_{G,\bar{n}}, W_G]$.
3. The function Δ tends to 0 as $(b, q) \rightarrow (0, 0)$.

Briefly stated, the theorem asserts that Protocol R_{gen} always achieves secure expansion provided that $\chi > W_{G,\bar{n}}$. The function Δ is an error term which vanishes when the test probability q and the soundness error term b are sufficiently small. Crucially, this function depends only on G and not on any other parameters in Protocol R_{gen} . The bound is therefore device-independent.

We note that the noise tolerance in this result is optimal: if the noise threshold χ were less than $W_{G,\bar{n}}$, then Protocol R_{gen} clearly could not produce superlinear expansion, since the device can always behave deterministically on generation rounds. We therefore have complete answers to questions 1–3 above. The upper limit for the noise tolerance for G is $W_{G,\bar{n}}$; the games that can be used in the spot-checking protocol are precisely those that have $W_{G,\bar{n}} < W_G$. And the devices that can be used in the spot-checking protocol are precisely those which exceed $W_{G,\bar{n}}$ for some G .

In a previous draft of this paper, we informally claimed that *all* superclassical devices can be used for exponential randomness expansion, but this was an overstatement of our results. Actually, there are superclassical devices D that do not exceed the threshold $W_{G,\bar{n}}$ for any G (see Appendix A). Such devices cannot be effectively used in the spot-checking protocol, but could potentially be used in other randomness expansion protocols, and studying their use remains an open problem.

Following a method used in [7], if we let $q = (\log^2 N)/N$ in Protocol R_{gen} , then it will require only $O_G(\log^3 N)$ bits of initial seed to approximate the input distribution TA , and $O_G(\log^2 N)$ bits of initial seed to perform randomness extraction on X once the protocol concludes [9]. Since the number of final random bits is $\Theta(N)$, exponential randomness expansion is achieved.

1.2 Beyond nonlocal games

Our proof of Theorem 1.1 builds on methods from our previous paper [18]. In that paper we focused on untrusted devices D that make measurements $\{P_{\mathbf{a}}^{\mathbf{x}}\}$ of the form

$$P_{\mathbf{a}}^{\mathbf{x}} = P_{1,a_1}^{x_1} \otimes \cdots \otimes P_{n,a_n}^{x_n}. \quad (1.3)$$

(where \mathbf{a} denotes an input sequence and \mathbf{x} denotes an output sequence). This tensor product form reflects the fact that D has n spatially separated components. In the present paper we considered whether this assumption can be replaced with a different assumption about the measurements of D .

Recent papers [15, 1, 28, 10] have analyzed randomness expansion protocols based on *contextuality*, rather than spatial separation. We were therefore motivated to generalize our proof to include contextual randomness expansion as well. We do this by defining a **contextual device** to be a device which accepts input sequences $\mathbf{a} = (a_1, \dots, a_m)$ (not necessarily of uniform length) and returns output sequences $\mathbf{x} = (x_1, \dots, x_m)$, with the only restriction that the measurements performed by the device have the form

$$P_{\mathbf{a}}^{\mathbf{x}} = P_{1,a_1}^{x_1} P_{2,a_2}^{x_2} \cdots P_{m,a_m}^{x_m}, \quad (1.4)$$

and the individual measurements $\{\{P_{1,a_i}^{x_i}\}_{x_i}\}_{i=1}^m$ are required to be simultaneously diagonalizable. (See Definition 3.) We prove security for such devices in parallel to spatially separated devices, thus showing that contextuality alone is sufficient to prove quantum-secure randomness expansion. This is also a new achievement.

1.3 Rate curves

An interesting unsolved problem left by our work is maximizing the rate curves π in Theorem 1.1. The rate curve proved in Theorem 6.3 is the following:

$$\pi_G(x) = \begin{cases} \frac{2(\log e)(x - W_{G,\bar{a}})^2}{r-1} & \text{if } x > W_{G,\bar{a}} \\ 0 & \text{otherwise,} \end{cases} \quad (1.5)$$

where r is the size of the total output alphabet for the device D . This is clearly not optimal: for example, we proved in [18] that the rate curve for the GHZ game is at least

$$x \mapsto 1 - 2H((1-x)/0.11) \quad (1.6)$$

for $x > 0.89$, where H denotes the Shannon entropy function, and this exceeds (1.5) for values of x near 1. Improving these curves is vital for maximizing the performance of quantum random number generation. This problem is distilled in Section 6 (in terms of the function $X \mapsto \text{Tr}[X^{1+\epsilon}]$) and can be studied as a separate problem.

Arguments:

G : A game with a distinguished input \bar{a} .

D : A quantum device compatible with G .

N : A positive integer (the **output length**).

q : A real number from the interval $(0, 1)$. (The **test probability**.)

χ : A real number from the interval $(0, 1)$. (The **score threshold**.)

Protocol R_{gen} :

1. Let c denote a real variable which we initially set to 0.
2. Choose a bit $t \in \{0, 1\}$ according to the distribution $(1 - q, q)$.
3. If $t = 1$ (“game round”), then the game G is played with D and the output is recorded. The score achieved is added to the variable c .
4. If $t = 0$ (“generation round”) then \bar{a} is given to D and the output is recorded.
5. Steps 2–4 are repeated $(N - 1)$ more times.
6. If $c < \chi q N$, then the protocol **aborts**. Otherwise, it **succeeds**.

Figure 1: Protocol R_{gen} (modified from [7])

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2 An overview of proof techniques

Our proof begins in the same setting as our previous paper [18]. Let $\rho: E \rightarrow E$ be a density matrix. Then, one way to measure the randomness of ρ is via the von Neumann entropy:

$$H(\rho) = -\text{Tr}[\rho \log \rho]. \tag{2.1}$$

This quantity measures, asymptotically, the number of random bits that can be extracted from $\rho^{\otimes m}$ as $m \rightarrow \infty$ [24, 27, 25]. This is not directly useful in the setting of untrusted-device cryptography, since there is typically no reliable way to produce independent copies of a single state. But consider instead the quantity

$$H_{1+\epsilon}(\rho) = -\frac{1}{\epsilon} \log \text{Tr}[\rho^{1+\epsilon}], \tag{2.2}$$

the Renyi entropy of the eigenvalues of ρ (which tends to $H(\rho)$ as $\epsilon \rightarrow 0$, although not uniformly). The smooth min-entropies of ρ satisfy

$$H_{min}^\epsilon(\rho) \geq H_{1+\epsilon}(\rho) - \frac{\log(1/\delta)}{\epsilon}. \quad (2.3)$$

Thus, function $H_{1+\epsilon}(\rho)$ can be used to provide lower bounds on the number of extractable bits in the cryptographic setting — provided that ϵ is not too small.

Thus, our goal becomes to prove in general that a device D that scores high at a nonlocal game G must exhibit a randomness in its output, as measured by the “ $(1 + \epsilon)$ -randomness” function $X \mapsto -\frac{1}{\epsilon} \log \text{Tr}(X^{1+\epsilon})$. However there is an apparent difficulty in proving such a direct relationship: the score of a device is defined in terms of the function $X \mapsto \text{Tr}(X)$, which cannot be uniformly determined from $X \mapsto \text{Tr}(X^{1+\epsilon})$. We therefore introduce in this paper the notion of the “ $(1 + \epsilon)$ -score” of the device D at the game G (see Definition 14), which is a quantity defined in terms of $\text{Tr}(X^{1+\epsilon})$ which tends to the ordinary expected score as $\epsilon \rightarrow 0$.

We prove in section 6 that the function π_G in 1.5 provides a universal lower bound on the $(1 + \epsilon)$ -randomness of a device D , on input \bar{a} , in terms of the $(1 + \epsilon)$ -score at the game G . (Thus, π_G is a “rate curve” for G .) For this we use the following uncertainty principle, which is the main new technical contribution in this paper (see section 5): Let $\|\cdot\|_{1+\epsilon}$ denote the $(1 + \epsilon)$ -Schatten matrix norm, which is defined by

$$\|Z\|_{1+\epsilon} = [\text{Tr}((Z^*Z)^{\frac{1+\epsilon}{2}})]^{\frac{1}{1+\epsilon}}. \quad (2.4)$$

Proposition 2.1. *For any finite dimensional Hilbert space V , any positive semidefinite operator $\tau: V \rightarrow V$ satisfying $\|\tau\|_{1+\epsilon} = 1$, and any binary projective measurement $\{R_0, R_1\}$ on V , the following holds. Let*

$$\tau' = R_0\tau R_0 + R_1\tau R_1. \quad (2.5)$$

Then

$$\|\tau'\|_{1+\epsilon} \leq 1 - (\epsilon/2) \|\tau - \tau'\|_{1+\epsilon}^2. \quad (2.6)$$

This principle can be understood as follows: if the measurement $\{R_0, R_1\}$ significantly disturbs the state τ (as measured by the Schatten norm) then the amount of randomness in the post-measurement state is significantly more than that of τ . The proof of this result is based on a known result [2] on the uniform convexity of $\|\cdot\|_{1+\epsilon}$. We also prove a similar result for non-binary measurements.

The proof of the rate curve then uses contrapositive reasoning: if input \bar{a} to device D does not produce much randomness, then the initial state of D must have already been close (under $\|\cdot\|_{1+\epsilon}$) to that of a device that gave predictable outputs on input \bar{a} , and therefore its $(1 + \epsilon)$ -score at G cannot be much larger than $W_{G,\bar{a}}$. With the rate curve established, we prove the final security result (Corollary 7.11) in section 7 via a simplified version of the methods used in [18].

Our proof in this paper is largely self-contained — the two primary outside results that we rely on are the uniform convexity result, and a theorem on the relationship between Renyi entropies and smooth min-entropy (Theorem 4.1).

3 Definitions and notation

3.1 Device models

For our purposes, a *quantum device component* is an object containing a quantum system Q which receives a classical input ($a \in \mathcal{A}$) at each use, performs a corresponding operation on an internal quantum system, and then produces a classical output ($x \in \mathcal{X}$). This process is repeated a

finite number of times. It is assumed that the device can maintain (quantum) memory in between rounds and may be entangled with other devices and external quantum systems. We can assume without loss of generality that the quantum operations performed on Q by such a component at each round are the same (since a device that uses different operations depending on previous inputs and outputs can always be simulated by one that maintains a transcript in memory). Also, using the Stinespring representation theorem, we can assume that the quantum operations performed consist of an \mathcal{X} -valued projective measurement on Q (which may depend on the input a) followed by a unitary automorphism of Q .

Definition 1. A quantum device with input alphabet \mathcal{A} and output alphabet \mathcal{X} consists of the following data.

1. A finite dimensional Hilbert space Q and a density operator $\phi: Q \rightarrow Q$.
2. For each $a \in \mathcal{A}$, a projective measurement $\{P_a^x\}_{x \in \mathcal{X}}$ on Q and a unitary automorphism $U_a: Q \rightarrow Q$.

Definition 2. A quantum device with r components is a quantum device satisfying the following additional conditions.

3. The space Q has the form $Q = Q_1 \otimes \cdots \otimes Q_r$ and the alphabets have the form $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_r$ and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_r$.
4. The projective measurements $\{P_a^x\}_x$ have the form

$$P_a^x = P_{a_1,1}^x \otimes \cdots \otimes P_{a_r,r}^x \quad (3.1)$$

where $\{P_{a_i,i}^x\}_{x \in \mathcal{X}}$ is a projective measurement on Q_i for each a_i, i .

These devices operate by first performing the projective measurement $\{P_a^x\}$ on Q and outputting the result, and then applying the unitary automorphism U_a . We note that there is no restriction placed on the unitary automorphisms U_a in Definition 2, which means that this model allows the components of the device D to communicate with one another in between uses.

We also make the following definition. For any finite set S , let $\text{Seq}(S)$ denote the set of non-repeating sequences of elements from S .

Definition 3. A contextual device is a quantum device (Definition 1) satisfying the following additional conditions.

3. There are finite sets \mathcal{B}, \mathcal{Y} such that $\mathcal{A} \subseteq \text{Seq}(\mathcal{B})$ and $\mathcal{X} = \text{Seq}(\mathcal{Y})$.
4. There are projective measurements $\{P_b^y\}_y$ for each $b \in \mathcal{B}$ such that for any $a = (b_1, \dots, b_m) \in \mathcal{A}$, the measurements $\{\{P_b^y\}_y \mid b \in \{b_1, \dots, b_m\}\}$ are simultaneously diagonalizable, and

$$P_a^x = P_{b_1}^{y_1} P_{b_2}^{y_2} \cdots P_{b_m}^{y_m} \quad (3.2)$$

for any m -length sequence $x = (y_1, \dots, y_m) \in \mathcal{X}$.

The following definition is convenient for some of the proofs that will follow.

Definition 4. An **abstract** quantum device is defined as in Definition 1, but with the assumption that ϕ is a density operator replaced by the weaker assumption that ϕ is a nonzero positive semidefinite operator.

We will use the following notation: if D is a device with initial state $\phi: Q \rightarrow Q$, and X is a positive semidefinite operator on Q , then

$$\phi_X := \sqrt{X}\phi\sqrt{X}. \quad (3.3)$$

and

$$\rho_X := \left(\sqrt{\phi}X\sqrt{\phi} \right)^\top. \quad (3.4)$$

The intuitions for these operators are as follows: if $\{M, \mathbb{I} - M\}$ is a binary measurement on Q , then ϕ_M represents the post-measurement state of Q for outcome M , and ρ_M represents the corresponding post-measurement state of a purifying system \overline{Q} for Q .

The operators ρ_X are crucial objects of study for establishing full security of a protocol involving the device D . The purifying system \overline{Q} represents the maximal amount of quantum information that an adversary could possess about the device D . We will refer to the operators ρ_X as the **adversary states** of D , and to the operators ϕ_X as the **device states** of D . Note that ρ_X has the same singular values as ϕ_X .

If an indexed sequence (z_1, z_2, \dots) is given, we use the boldface variable \mathbf{z} to denote the entire sequence. For any sequences $(a_1, \dots, a_n) \in \mathcal{A}^n$ and $(x_1, \dots, x_n) \in \mathcal{X}^n$, let

$$\phi_{\mathbf{a}}^x = M_n M_{n-1} \cdots M_1 \phi M_1^* \cdots M_{n-1}^* M_n^*, \quad (3.5)$$

$$\rho_{\mathbf{a}}^x = \left(\sqrt{\phi} M_1^* M_2^* \cdots M_n^* M_n \cdots M_2 M_1 \sqrt{\phi} \right)^\top, \quad (3.6)$$

where

$$M_j = U_{a_j} P_{a_j}^{x_j}. \quad (3.7)$$

These represent the device states and adversary states occurring for the input and output sequences \mathbf{a}, \mathbf{x} .

3.2 Games

We will state a general definition of a game. In this paper, we will frequently make use of a fixed input (\bar{a}) for the game, and so for convenience we make that choice of input part of the definition.

Definition 5. A *game* G consists of the following data.

1. A finite set \mathcal{A} (the **input alphabet**) with a distinguished element $\bar{a} \in \mathcal{A}$ and a probability distribution $p: \mathcal{A} \rightarrow [0, 1]$.
2. A finite set \mathcal{X} (the **output alphabet**).
3. A **scoring function** $H: \mathcal{A} \times \mathcal{X} \rightarrow [0, 1]$.

The game operates as follows: an input a is chosen according to the probability distribution p , and it is given to a device, which produces an output x . The function H is applied to (a, x) to obtain the score.

The following is a notational convenience: if $\mathbf{a} \in \mathcal{A}^n$ and $\mathbf{x} \in \mathcal{X}^n$, then let

$$p(\mathbf{a}) = p(a_1)p(a_2) \cdots p(a_n) \quad (3.8)$$

$$H(\mathbf{a}, \mathbf{x}) = H(a_1, x_1) + \dots + H(a_n, x_n). \quad (3.9)$$

Definition 6. An *unbounded game* is defined as in Definition 5, except that the function H maps to $[0, \infty)$.

Definition 7. A *nonlocal game* G with s players is a game in which the input and output alphabets have the form $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_s$ and $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_s$.

Definition 8. A *contextual game* G is a game in which the input and output alphabets satisfy $\mathcal{A} \subseteq \text{Seq}(\mathcal{B})$ and $\mathcal{X} = \text{Seq}(\mathcal{Y})$ for some finite sets \mathcal{B}, \mathcal{Y} .

A quantum device is **compatible** with a game if it has the same descriptor (nonlocal or contextual) and the input alphabet and output alphabet match those of the game (including their decompositions into Cartesian products or sequence-sets, as appropriate).

We can combine the above definitions as follows: if G is a game and D is a compatible device, then the expected score of D for G (on the first use) is given by

$$\sum_{a \in \mathcal{A}, x \in \mathcal{X}} p(a) H(a, x) \text{Tr}(P_a^x \phi). \quad (3.10)$$

Definition 9. Let G be a game. Then, the *quantum value* of G , denoted W_G , is the supremum of the expected scores for G taken over all devices compatible with G .

In order for a device to be useful for our purposes, it must generate a score at some nonlocal game which guarantees quantum behavior on a *fixed* input string (as in [7]). This motivates the following definitions.

Definition 10. Let D be an abstract quantum device and let $\bar{a} \in \mathcal{A}$ be a fixed input letter. Then, we say that D is **deterministic** on input \bar{a} if

$$\phi = P_{\bar{a}}^{\bar{x}} \phi P_{\bar{a}}^{\bar{x}} \quad (3.11)$$

for some output letter \bar{x} . We say that D is **classically predictable** on input \bar{a} if

$$\phi = \sum_{x \in \mathcal{X}} P_{\bar{a}}^x \phi P_{\bar{a}}^x. \quad (3.12)$$

Definition 11. If G is a game, then let $W_{G, \bar{a}}$ denote the supremum of the expected scores for G over all compatible devices D that have classically predictable outputs on input \bar{a} .

3.3 Device-independent vanishing error functions

It is necessary to be cautious about our use of asymptotic notation because ultimately we will want to assert that the bounds we prove on the rate of our protocol (including second-order terms) are truly device-independent. We adopt the following conventions for asymptotic notation: If we say,

“For all $x, y \in (0, 1]$, $F(x, y) \leq O(y)$.”

Then the coefficient and the range in the big- O expression must be independent of x . (The sentence above asserts that there is a function $G(y)$ satisfying $\lim_{y \rightarrow 0} G(y)/y < \infty$ such that $F(x, y) \leq G(y)$.) On the other hand, if we say

“Let $x \in (0, 1]$ be a real number. Then, for all $y \in (0, 1]$, $F(x, y) \leq O(y)$.”

Then it is understood that the coefficient and the range in the big- O expression may depend on x . (The sentences above assert that $\lim_{y \rightarrow 0} F(x, y)/y < \infty$ for all $x \in (0, 1]$.)

Equivalently, we may write

“For all $x, y \in (0, 1]$, $F(x, y) \leq O_x(y)$.”

Here the subscripted x indicates that dependence on x is allowed.

4 The functions $\|\cdot\|_{1+\epsilon}$ and $\langle \cdot \rangle_{1+\epsilon}$

We continue with preliminaries in this section. For any linear operator Z and any $\ell \geq 1$, the ℓ -Schatten norm is given by

$$\|Z\|_\ell = \text{Tr} \left[(Z^*Z)^{\frac{\ell}{2}} \right]^{\frac{1}{\ell}}. \quad (4.1)$$

(If W is positive semidefinite, then the ℓ -Schatten norm can be written more simply as $\langle W \rangle_\ell = \text{Tr} [W^\ell]^{\frac{1}{\ell}}$.) For convenience, we will also define the notation $\langle \cdot \rangle_\ell$ to mean the following simpler expression:

$$\langle Z \rangle_\ell = \text{Tr} \left[(Z^*Z)^{\frac{\ell}{2}} \right]. \quad (4.2)$$

We have $\langle Z \rangle_\ell = \|Z\|_\ell^\ell$. We will often be concerned with the functions $\langle W \rangle_{1+\epsilon}$ and $\|W\|_{1+\epsilon}$ for small ϵ .

A discussion of some relevant properties of the Schatten norm can be found in [23]. For our purposes, we will need the following properties of $\langle \cdot \rangle_{1+\epsilon}$ and $\|\cdot\|_{1+\epsilon}$. The functions are almost linear in the following sense: for positive semidefinite operators X, Y ,

$$(1 - O(\epsilon)) (\|X\|_{1+\epsilon} + \|Y\|_{1+\epsilon}) \leq \|X + Y\|_{1+\epsilon} \leq \|X\|_{1+\epsilon} + \|Y\|_{1+\epsilon} \quad (4.3)$$

$$\langle X \rangle_{1+\epsilon} + \langle Y \rangle_{1+\epsilon} \leq \langle X + Y \rangle_{1+\epsilon} \leq (1 + O(\epsilon)) (\langle X \rangle_{1+\epsilon} + \langle Y \rangle_{1+\epsilon}). \quad (4.4)$$

Also, the righthand inequalities in (4.3) and (4.4) hold also when X and Y are replaced by arbitrary linear operators (not necessarily positive semidefinite).

When A is a positive semidefinite operator on a space $V = V_1 \oplus \dots \oplus V_m$, and $A' = \sum_k P_k A P_k$, where P_k denotes projection on V_k , then

$$(1 - O_m(\epsilon)) \|A\|_{1+\epsilon} \leq \|A'\|_{1+\epsilon} \leq \|A\|_{1+\epsilon} \quad (4.5)$$

$$(1 - O_m(\epsilon)) \langle A \rangle_{1+\epsilon} \leq \langle A' \rangle_{1+\epsilon} \leq \langle A \rangle_{1+\epsilon}, \quad (4.6)$$

and the righthand inequalities both hold also when A is replaced by an arbitrary linear operator.

Unless otherwise specified, the domain of the variable ϵ will always be $(0, 1]$.

4.1 Relationship to extractable bits

Our motivation for studying the function $\langle \cdot \rangle_{1+\epsilon}$ is its relationship to quantum smooth min-entropy. The smooth min-entropy of a classical-quantum CQ state measures (asymptotically) the number of bits that can be extracted from C in the presence of an adversary who possesses Q [24].

Definition 12. Let CQ be a classical-quantum system whose state is Γ_{CQ} . Then, the min-entropy of C conditioned on Q is

$$H_{min}(C | Q) = \max_{\mathbb{I}_C \otimes \sigma \geq \alpha} [-\log \text{Tr}(\sigma)], \quad (4.7)$$

where σ varies over all positive semidefinite operators on Q that satisfy the given inequality. For any $\delta > 0$, the min-entropy of C conditioned on Q with smoothing parameter δ is

$$H_{min}^\delta(C | Q) = \max_{\|\Gamma' - \Gamma_{CQ}\|_1 \leq \delta} H_{min}(C | Q)_{\Gamma'}, \quad (4.8)$$

where Γ' varies over all classical-quantum positive semidefinite operators on CQ that satisfy the given inequality.

In this paper we will implicitly use the quantum Renyi divergence functions developed by [16, 19, 30], and surveyed recently by [26]. In order to conserve space, we will not introduce a full formalism for these functions here, but will just note the following intuition: Let CQ be a classical quantum system whose state is given by $\sum_c |c\rangle \langle c| \otimes \alpha_c$, and let $\alpha = \sum_c \alpha_c$. Then, the expression

$$-\frac{1}{\epsilon} \log \left(\sum_c \langle \alpha_c \rangle_{1+\epsilon} \right) \quad (4.9)$$

can be thought of as an absolute measure of the amount of randomness contained in CQ , while the related expression

$$-\frac{1}{\epsilon} \log \left(\sum_c \left\langle \alpha_c^{\frac{-\epsilon}{2+\epsilon}} \alpha_c \alpha_c^{\frac{-\epsilon}{2+\epsilon}} \right\rangle_{1+\epsilon} \right) \quad (4.10)$$

can be thought of as a measure of the amount of randomness in C conditioned on Q .¹ This intuition is supported by the following theorem about smooth min-entropy.

Theorem 4.1. Let Λ be a subnormalized operator on a bipartite system CQ of the form $\Lambda = \sum_c \alpha_c \otimes |c\rangle \langle c|$, and let σ be a density matrix on Q . Let

$$K = -\frac{1}{\epsilon} \log \left(\sum_c \left\langle \sigma_c^{\frac{-\epsilon}{2+\epsilon}} \alpha_c \sigma_c^{\frac{-\epsilon}{2+\epsilon}} \right\rangle_{1+\epsilon} \right) \quad (4.11)$$

Then, for any $\delta > 0$,

$$H_{min}^\delta(C | Q)_\Lambda \geq K - \frac{1 + 2 \log(1/\delta)}{\epsilon}. \quad (4.12)$$

Proof. Corollary D.8 in [18] proves the above (building on [27, 11]) with the extra assumption that $\text{Tr}(\Lambda) = 1$. The case $\text{Tr}(\Lambda) \leq 1$ follows by rescaling and using the fact that $1 + \epsilon \leq 2$. \square

(For a more detailed discussion of the relationship between smooth min-entropy and Renyi divergence, see section 6.4.1 of [26], which uses a different definition of $H_{min}^\delta(\cdot | \cdot)$.)

¹The latter quantity is, in formal terms, the negation of the Renyi divergence of the state of CQ relative to the state $\mathbb{I}_C \otimes \alpha$.

5 Randomness versus state disturbance

Our central goal is to prove the randomness of certain classical variables in the presence of quantum side information, using the Schatten norm as a metric. This section provides inductive steps for such proofs of randomness.

The basis for all of the results in this section is a known result on the *uniform convexity* of the Schatten norm, Theorem 1 of [2]. We state the following proposition, which is a special case of uniform convexity.

Proposition 5.1. *For any $\epsilon \in (0, 1]$, and any linear operators W and Z such that $\|W\|_{1+\epsilon} = \|Z\|_{1+\epsilon} = 1$,*

$$\left\| \frac{W+Z}{2} \right\|_{1+\epsilon} \leq 1 - \frac{\epsilon}{8} \|W-Z\|_{1+\epsilon}^2. \quad \square \quad (5.1)$$

Proof. Substituting $X = (W+Z)/2$, $Y = (W-Z)/2$, and $p = 1 + \epsilon$ into Theorem 1 of [2], we have

$$1 \geq \left\| \frac{W+Z}{2} \right\|_{1+\epsilon}^2 + \epsilon \left\| \frac{W-Z}{2} \right\|_{1+\epsilon}^2 \quad (5.2)$$

which implies $\|(W+Z)/2\|_{1+\epsilon}^2 \leq 1 - (\epsilon/4) \|W-Z\|_{1+\epsilon}^2$. The proof is completed by the fact that $\sqrt{1-x} \leq 1 - (x/2)$ for any $x \in [0, 1]$. \square

The next proposition compares the amount of randomness obtained from a measurement to the degree of disturbance in the state that is caused by the measurement.

Proposition 5.2. *For any finite dimensional Hilbert space V , any positive semidefinite operator $\tau: V \rightarrow V$ satisfying $\|\tau\|_{1+\epsilon} = 1$, and any binary projective measurement $\{R_0, R_1\}$ on V , the following holds. Let*

$$\tau' = R_0 \tau R_0 + R_1 \tau R_1. \quad (5.3)$$

Then

$$\|\tau'\|_{1+\epsilon} \leq 1 - (\epsilon/2) \|\tau - \tau'\|_{1+\epsilon}^2. \quad (5.4)$$

Proof. Choose a basis $\{e_1, \dots, e_n\}$ for V such that R_0 is the projector into the space spanned by e_1, \dots, e_m , and write τ in $(m, n-m)$ -block form:

$$\tau = \left[\begin{array}{c|c} T_{00} & T_{01} \\ \hline T_{10} & T_{11} \end{array} \right]. \quad (5.5)$$

Applying Proposition 5.1 with $W = \tau$ and

$$Z = \left[\begin{array}{c|c} T_{00} & -T_{01} \\ \hline -T_{10} & T_{11} \end{array} \right], \quad (5.6)$$

we obtain

$$\|\tau'\|_{1+\epsilon} = \|(W+Z)/2\|_{1+\epsilon} \quad (5.7)$$

$$\leq 1 - \frac{\epsilon}{8} \|W-Z\|_{1+\epsilon}^2 \quad (5.8)$$

$$= 1 - \frac{\epsilon}{8} \|2(\tau - \tau')\|_{1+\epsilon}^2 \quad (5.9)$$

$$= 1 - \frac{\epsilon}{2} \|\tau - \tau'\|_{1+\epsilon}^2, \quad (5.10)$$

as desired. \square

Proposition 5.3. For any finite dimensional Hilbert space V , any positive semidefinite operator $\tau: V \rightarrow V$ satisfying $\|\tau\|_{1+\epsilon} = 1$, and any binary projective measurement $\{R_0, R_1, \dots, R_n\}$ on V , the following holds. Let $\tau' = \sum_i P_i \tau P_i$. Then

$$\|\tau'\|_{1+\epsilon} \leq 1 - \frac{\epsilon}{2n} \|\tau - \tau'\|_{1+\epsilon}^2 + O_n(\epsilon^2). \quad (5.11)$$

Proof. For any $i \in \{1, \dots, n\}$, let

$$\tau_i = (P_0 \tau P_0 + \dots + P_{i-1} \tau P_{i-1}) + (P_i + \dots + P_n) \tau (P_i + \dots + P_n), \quad (5.12)$$

and let $\tau_0 = \tau$. Note that applying Proposition 5.2 to the measurement $\{R_n, \mathbb{I} - R_n\}$ and the state $\tau_{n-1} / \|\tau_{n-1}\|_{1+\epsilon}$ yields

$$\|\tau_n\|_{1+\epsilon} \leq \left[1 - \frac{\epsilon}{2} \left(\frac{\|\tau_n - \tau_{n-1}\|_{1+\epsilon}}{\|\tau_{n-1}\|_{1+\epsilon}} \right)^2 \right] \|\tau_{n-1}\|_{1+\epsilon} \quad (5.13)$$

$$\leq \left[1 - \frac{\epsilon}{2} \|\tau_n - \tau_{n-1}\|_{1+\epsilon}^2 \right] \|\tau_{n-1}\|_{1+\epsilon}. \quad (5.14)$$

By an inductive argument, we then have

$$\|\tau_n\|_{1+\epsilon} \leq \prod_{i=1}^n \left(1 - \frac{\epsilon}{2} \|\tau_i - \tau_{i-1}\|_{1+\epsilon}^2 \right) \cdot 1 \quad (5.15)$$

$$\leq 1 - \frac{\epsilon}{2} \sum_{i=1}^n \|\tau_i - \tau_{i-1}\|_{1+\epsilon}^2 + O_n(\epsilon^2) \quad (5.16)$$

$$\leq 1 - \frac{\epsilon}{2n} \left(\sum_{i=1}^n \|\tau_i - \tau_{i-1}\|_{1+\epsilon} \right)^2 + O_n(\epsilon^2) \quad (5.17)$$

$$\leq 1 - \frac{\epsilon}{2n} \|\tau_n - \tau_0\|_{1+\epsilon}^2 + O_n(\epsilon^2), \quad (5.18)$$

The operator τ_n is equal to τ' , and this completes the proof. \square

6 Rate curves

Our next goal is prove inequalities which relate the randomness generated by a device to its performance at a given game. First we state the following alternative version of Proposition 5.3, which follows easily from the fact that $\langle X \rangle_{1+\epsilon} = \|X\|_{1+\epsilon}^{1+\epsilon}$.

Proposition 6.1. For any finite dimensional Hilbert space V , any positive semidefinite operator $\tau: V \rightarrow V$ satisfying $\langle \tau \rangle_{1+\epsilon} = 1$, and any binary projective measurement $\{R_0, R_1, \dots, R_n\}$ on V , the following holds. Let $\tau' = \sum_i P_i \tau P_i$. Then

$$\langle \tau' \rangle_{1+\epsilon} \leq 1 - \frac{\epsilon}{2n} \langle \tau - \tau' \rangle_{1+\epsilon}^2 + O_n(\epsilon^2). \quad \square \quad (6.1)$$

6.1 The $(1 + \epsilon)$ -score of a game

Definition 13. Let $G = (p, H)$ be a game (with alphabets \mathcal{A}, \mathcal{X}), and let D be a compatible abstract device. Then, the **game operator** for D determined by G is

$$K := \sum_{\substack{a \in \mathcal{A} \\ x \in \mathcal{X}}} p(a) H(a, x) P_a^x. \quad (6.2)$$

where $\{\{P_a^x\}_x\}_a$ denote the measurements performed by D . Let the expressions ρ_G, ϕ_G denote the operators

$$\rho_G := (\sqrt{\rho}K\sqrt{\rho})^\top \quad \phi_G := \sqrt{K}\rho\sqrt{K}. \quad (6.3)$$

We note that since all of the terms $H(a, x)$ are assumed to be less than or equal to 1, it follows that the game operator always satisfies $K \leq \mathbb{I}$.

Some intuition for the expression ρ_G is as follows: Let Q' be a quantum system of the same dimension as the system Q inside D , and suppose that QQ' is in a pure state which purifies Q . Suppose that the game G is played, a score $h \in [0, 1]$ is obtained, and then a classical random variable $X \in \{P, F\}$ is set to be equal to P with probability h , and F with probability $(1 - h)$. Then, ρ_G is isomorphic to the subnormalized state of Q' corresponding to the event $X = P$.

Definition 14. Let $G = (p, H)$ be a game with input alphabet \mathcal{A} and output alphabet \mathcal{X} , and let D be a compatible abstract device. Let $\epsilon \in [0, 1]$. Then, the $(1 + \epsilon)$ -score of D (for G) is given by

$$W_G^\epsilon(D) := \frac{\langle \phi_G \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}}. \quad (6.4)$$

Let $W_G(D) = W_G^0(D)$. Note that if D is an ordinary quantum device this is simply the expected score of the device D at the game D .

Proposition 6.2. Let G be a game with distinguished input \bar{a} . Then for any compatible abstract device D whose output on input \bar{a} is classically predictable, we must have

$$W_G^\epsilon(D) \leq W_{G, \bar{a}} + O(\epsilon). \quad (6.5)$$

Proof. First consider the case where D behaves deterministically on input \bar{a} . Let

$$D = (Q, \phi, \{P_a^x\}, \{U_a\}). \quad (6.6)$$

We have $\text{Supp } \phi \subseteq \text{Supp } P_{\bar{a}}^{\bar{x}}$ for some \bar{x} . Let K denote the game operator for D, G .

The inequality

$$P_{\bar{a}}^{\bar{x}} K P_{\bar{a}}^{\bar{x}} \leq (W_{G, \bar{a}}) P_{\bar{a}}^{\bar{x}} \quad (6.7)$$

must hold, since if it did not, we could find a unit vector $v \in \text{Supp } P_{\bar{a}}^{\bar{x}}$ such that $vKv^* > W_{G, \bar{a}}$, which would mean the device

$$D' = (Q, vv^*, \{P_a^x\}, \{U_a\}) \quad (6.8)$$

breaks the score threshold $W_{G, \bar{a}}$ and gives deterministic output on \bar{a} , a contradiction. Therefore,

$$\langle \phi_G \rangle_{1+\epsilon} = \left\langle \sqrt{K}\phi\sqrt{K} \right\rangle_{1+\epsilon} \quad (6.9)$$

$$= \left\langle \sqrt{\phi}K\sqrt{\phi} \right\rangle_{1+\epsilon} \quad (6.10)$$

$$= \left\langle \sqrt{\phi}P_{\bar{a}}^{\bar{x}}K P_{\bar{a}}^{\bar{x}}\sqrt{\phi} \right\rangle_{1+\epsilon} \quad (6.11)$$

$$\leq W_{G, \bar{a}} \left\langle \sqrt{\phi}P_{\bar{a}}^{\bar{x}}\sqrt{\phi} \right\rangle_{1+\epsilon} \quad (6.12)$$

$$= W_{G, \bar{a}} \langle \phi \rangle_{1+\epsilon}, \quad (6.13)$$

as desired.

Now consider the case where D is classically predictable on input \bar{a} . In this case, the state of D is a convex combination of states that would give deterministic output on input \bar{a} , and thus the approximate linearity of $\langle \cdot \rangle_{1+\epsilon}$ (see (4.4)) yields the desired result. \square

When D is a quantum device and \bar{a} is an input letter, we measure the amount of randomness produced by D on input \bar{a} by comparing the values of the function $\langle \cdot \rangle_{1+\epsilon}$ applied to the premeasurement and postmeasurement states of D .

Definition 15. Let D be a quantum device and let \bar{a} be an input letter for D . Then, the $(1 + \epsilon)$ -randomness of D for input \bar{a} is the following quantity:

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{x \in \mathcal{X}} \langle \phi_{\bar{a}}^x \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}} \right). \quad (6.14)$$

Let G be a game with which D is compatible. Then, the $(1 + \epsilon)$ -randomness of D for the game G is the following quantity:

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{a \in \mathcal{A}, x \in \mathcal{X}} p(a) \langle \phi_a^x \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}} \right), \quad (6.15)$$

where p denotes the input distribution for G .

6.2 Universal rate curves for fixed input

Definition 16. Let G be a game, and let $\pi: [0, W_G] \rightarrow \mathbb{R}_{\geq 0}$ be a nondecreasing convex function which is differentiable on $(0, W_G)$. Then, π is a **rate curve for (G, \bar{a})** if for all compatible abstract devices D ,

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_x \langle \phi_{\bar{a}}^x \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}} \right) \geq \pi(W_G^\epsilon(D)) - O(\epsilon). \quad (6.16)$$

Definition 17. Let G be a game, and let $\pi: [0, W_G] \rightarrow \mathbb{R}_{\geq 0}$ be a nondecreasing convex function which is differentiable on $(0, W_G)$. Then, π is a **rate curve for G** if for all compatible abstract devices D ,

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{a,x} p(a) \langle \phi_a^x \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}} \right) \geq \pi(W_G^\epsilon(D)) - O(\epsilon). \quad (6.17)$$

Theorem 6.3. Let G be a game with output alphabet size $r \geq 2$, let \bar{a} be the distinguished input letter for G , and let $w = W_{G, \bar{a}}$. Then, the following function is a rate curve for (G, \bar{a}) .

$$\pi(x) = \begin{cases} \frac{2(\log e)(x-w)^2}{r-1} & \text{if } x > w \\ 0 & \text{otherwise.} \end{cases} \quad (6.18)$$

Proof. It suffices to give a proof for abstract devices D satisfying $\langle \phi \rangle_{1+\epsilon} = 1$ and $W_G^\epsilon(D) > w$, so we will assume those two conditions in what follows. Following previous notation, let $\phi' = \sum_x \phi_{\bar{a}}^x$, and let K denote the game operator. Let D' denote the device D with the operator ϕ replaced by ϕ' . We will prove the desired rate curve by comparing the distance between ϕ and ϕ' the difference between their $(1 + \epsilon)$ -scores.

Note that for any Hermitian operator X , the value of the operator

$$\left[\begin{array}{c|c} \sqrt{K}X\sqrt{K} & \sqrt{K}X\sqrt{\mathbb{I}-K} \\ \hline \sqrt{\mathbb{I}-K}X\sqrt{K} & \sqrt{\mathbb{I}-K}X\sqrt{\mathbb{I}-K} \end{array} \right] \quad (6.19)$$

under $\langle \cdot \rangle_{1+\epsilon}$ is the same as that of X , because the two operators are unitarily equivalent. Thus, using the discussion of inequalities (4.5–4.6), we have

$$\langle X \rangle_{1+\epsilon} \geq \langle \sqrt{K}X\sqrt{K} \rangle_{1+\epsilon} + \langle \sqrt{\mathbb{I}-K}X\sqrt{\mathbb{I}-K} \rangle_{1+\epsilon}. \quad (6.20)$$

Applying this for $X = \phi - \phi'$, followed by the general inequality

$$\langle Y - Z \rangle_{1+\epsilon} \geq (1 + O(\epsilon)) |\langle Y \rangle_{1+\epsilon} - \langle Z \rangle_{1+\epsilon}|, \quad (6.21)$$

yields

$$\langle \phi - \phi' \rangle_{1+\epsilon} \geq \langle \sqrt{K}\phi\sqrt{K} \rangle_{1+\epsilon} - \langle \sqrt{K}\phi'\sqrt{K} \rangle_{1+\epsilon} \quad (6.22)$$

$$+ \langle \sqrt{\mathbb{I}-K}\phi'\sqrt{\mathbb{I}-K} \rangle_{1+\epsilon} - \langle \sqrt{\mathbb{I}-K}\phi\sqrt{\mathbb{I}-K} \rangle_{1+\epsilon} - O(\epsilon) \quad (6.23)$$

$$\geq W_G^\epsilon(D) - W_G^\epsilon(D') + [1 - W_G^\epsilon(D')] - [1 - W_G^\epsilon(D)] - O(\epsilon) \quad (6.24)$$

$$= 2[W_G^\epsilon(D) - W_G^\epsilon(D')] - O(\epsilon). \quad (6.25)$$

Note that by Proposition 6.2, we have

$$W_G^\epsilon(D') \leq (w + O(\epsilon)) \langle \phi' \rangle_{1+\epsilon} \quad (6.26)$$

$$\leq (w + O(\epsilon)) \langle \phi \rangle_{1+\epsilon} \quad (6.27)$$

$$= (w + O(\epsilon)), \quad (6.28)$$

and thus (6.25) implies

$$\langle \phi - \phi' \rangle_{1+\epsilon} \geq 2[W_G^\epsilon(D) - w] - O(\epsilon). \quad (6.29)$$

Substituting this relationship into Proposition 5.3 yields

$$\langle \phi' \rangle_{1+\epsilon} \leq 1 - \frac{2\epsilon}{r-1} (W_G(D) - w)^2 + O(\epsilon^2), \quad (6.30)$$

which implies the desired rate curve. \square

7 A general security proof

In this section we provide a general proof of security for Protocol R_{gen} in Figure 1. The method of proof is a generalization of that of our previous paper on this topic [18].

Note that for any device D , each device-state ϕ_a^x has the same singular values as the corresponding adversary-state ρ_a^x , and so any calculation involving the singular values of the first can be rewritten in terms of the second, and vice versa. Since this section is concerned with proving security against an adversary, we will focus on expressions involving the adversary states.

7.1 The weighted $(1 + \epsilon)$ -randomness function

If G is an unbounded game, and D is an abstract device compatible with G , then let $R_G^\epsilon(D)$ denote the $(1 + \epsilon)$ -randomness of D for G (Definition 15):

$$R_G^\epsilon(D) = -\frac{1}{\epsilon} \log \left(\frac{\sum_{a,x} p(a) \langle \phi_a^x \rangle_{1+\epsilon}}{\langle \phi \rangle_{1+\epsilon}} \right). \quad (7.1)$$

Equivalently,

$$R_G^\epsilon(D) = -\frac{1}{\epsilon} \log \left(\frac{\sum_{a,x} p(a) \langle \rho_a^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right). \quad (7.2)$$

Also if a denotes an input for G , let $R_a^\epsilon(D)$ denote the $(1 + \epsilon)$ -randomness of D on input \bar{a} :

$$R_a^\epsilon(D) = -\frac{1}{\epsilon} \log \left(\frac{\sum_x \langle \rho_a^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right). \quad (7.3)$$

For $s \in \mathbb{R}$, let $R_G^{\epsilon,s}(D)$ denote the expression

$$R_G^{\epsilon,s}(D) = -\frac{1}{\epsilon} \log \left(\frac{\sum_{a,x} p(a) 2^{\epsilon s H(a,x)} \langle \rho_a^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right), \quad (7.4)$$

which we will call the $(1 + \epsilon)$ -randomness of D for G , weighted by s . (This quantity is central to the inductive argument in our security proof.)

Proposition 7.1. *Let G be a game and let $s \in \mathbb{R}$. Then, for all compatible devices D ,*

$$R_G^{\epsilon,s}(D) \geq R_G^\epsilon(D) - s W_G^\epsilon(D) - O(\epsilon). \quad (7.5)$$

Proof. For simplicity, we prove the result for abstract devices D satisfying $\langle \rho \rangle_{1+\epsilon} = 1$. (The general case then follows easily.) We have the following, using the fact that $2^t = 1 + (\ln 2)t + O(t^2)$.

$$\sum_{a,x} p(a) 2^{\epsilon s H(a,x)} \langle \rho_a^x \rangle_{1+\epsilon} \quad (7.6)$$

$$= \sum_{a,x} p(a) \langle \rho_a^x \rangle_{1+\epsilon} + \sum_{a,x} p(a) (2^{\epsilon s H(a,x)} - 1) \langle \rho_a^x \rangle_{1+\epsilon} \quad (7.7)$$

$$\leq \sum_{a,x} p(a) \langle \rho_a^x \rangle_{1+\epsilon} + \sum_{a,x} p(a) (\ln 2) \epsilon s H(a,x) \langle \rho_a^x \rangle_{1+\epsilon} + O(\epsilon^2) \quad (7.8)$$

$$\leq [1 - \epsilon (\ln 2) R_G^\epsilon(D)] + \epsilon s (\ln 2) W_G^\epsilon(D) + O(\epsilon^2). \quad (7.9)$$

Applying the function $-\frac{1}{\epsilon} \log(\cdot)$ to both sides yields the desired result. \square

It is desirable to have a lower bound on the weighted randomness quantity that is device-independent. For this, we will use the following principle. Suppose that $R: [a, b] \rightarrow \mathbb{R}$ is a differentiable convex function, and we wish to compute $\min_{x \in [a,b]} R(x)$. Then, there are three possibilities:

1. $R'(t) > 0$ for all t , in which case $\min_x R(x) = R(a)$.
2. $R'(t) < 0$ for all t , in which case $\min_x R(x) = R(b)$.
3. There exists t_0 such that $R'(t_0) = 0$, in which case $\min_x R(x) = R(t_0)$.

The next proposition asserts a bound on the weighted randomness of a device, in terms of a rate curve π . In order to facilitate the use of the aforementioned principle, we will choose a weighting term in the form $\pi'(x)$ (where π' denotes the derivative of π).

Proposition 7.2. *Let G be a game, and let π be a rate curve for G . Then, for all compatible devices D and all $r \in (0, W_G)$,*

$$R_G^{\epsilon, \pi'(r)}(D) \geq \pi(r) - \pi'(r)r - O(\epsilon). \quad (7.10)$$

Proof. We have the following:

$$R_G^{\epsilon, \pi'(r)}(D) \geq R_G^\epsilon(D) - \pi'(r)W_G^\epsilon(D) - O(\epsilon) \quad (7.11)$$

$$\geq \pi(W_G^\epsilon(D)) - \pi'(r)W_G^\epsilon(D) - O(\epsilon) \quad (7.12)$$

$$\geq \min_{t \in [0, W_G]} [\pi(t) - \pi'(r)t] - O(\epsilon) \quad (7.13)$$

$$= \pi(r) - \pi'(r)r - O(\epsilon), \quad (7.14)$$

as desired. \square

7.2 The modified game G_q

To analyze Protocol R_{gen} , it is helpful to define a new unbounded game G_q , with input alphabet $\mathcal{I} := \{0, 1\} \times \mathcal{A}$ and output alphabet \mathcal{X} , which represents the procedure used in Protocol \bar{R} . Let

$$p_q((t, a)) = \begin{cases} q \cdot p(a, x) & \text{if } t = 1, \\ (1 - q) & \text{if } t = 0 \text{ and } a = \bar{a}, \\ 0 & \text{if } t = 1 \text{ and } a \neq \bar{a}. \end{cases} \quad (7.15)$$

$$H_q((t, a), x) = \begin{cases} H(a, x)/q & \text{if } t = 1, \\ 0 & \text{if } t = 0. \end{cases} \quad (7.16)$$

(The inclusion of the denominator q in (7.16) can be thought of as compensation for the fact that game rounds only occur with frequency $1/q$.) Also let $\rho_i^x = \rho_a^x$, where $\mathbf{i} = ((t_1, a_1), \dots, (t_n, a_n))$, and define $W_{G_q}^\epsilon(D)$, $R_{G_q}^\epsilon(D)$ and $R_{G_q}^{\epsilon, s}(D)$ via these states. We assert the following, which relates the difference between the weighted and unweighted randomness of G_q to the score of the original game G .

Proposition 7.3. *Let G be a game and let $s \in \mathbb{R}$. Then, for all compatible devices D , all $q \in (0, 1)$, and all $\epsilon \in (0, 1]$,*

$$R_{G_q}^{\epsilon, s}(D) \geq R_{G_q}^\epsilon(D) - sW_{G_q}^\epsilon(D) - O(\epsilon/q). \quad (7.17)$$

Proof. The proof is similar to that of Proposition 7.1, but with a little more care taken with the error term. We must take into account the fact that the scores in the game $H_q(a, x)$ grow as $\Theta(1/q)$ as $q \rightarrow 0$.

Again, it suffices to prove the result for abstract devices D satisfying $\langle \rho \rangle_{1+\epsilon} = 1$. Then,

$$\sum_{i, x} p_q(i) 2^{\epsilon s H_q(i, x)} \langle \rho_i^x \rangle_{1+\epsilon} \quad (7.18)$$

$$= \sum_{i, x} p_q(i) \langle \rho_i^x \rangle_{1+\epsilon} + \sum_{i, x} p_q(i) (2^{\epsilon s H_q(i, x)} - 1) \langle \rho_i^x \rangle_{1+\epsilon} \quad (7.19)$$

$$= \sum_{i, x} p_q(i) \langle \rho_a^x \rangle_{1+\epsilon} + q \sum_{a, x} p(a) (2^{\epsilon s H(a, x)/q} - 1) \langle \rho_a^x \rangle_{1+\epsilon}. \quad (7.20)$$

$$\leq \sum_{i, x} p_q(a) \langle \rho_i^x \rangle_{1+\epsilon} + q[\epsilon s (\ln 2) W_G^\epsilon(D)/q + O((\epsilon/q)^2)] \quad (7.21)$$

$$\leq 1 - \epsilon (\ln 2) R_{G_q}^\epsilon(D) + \epsilon s (\ln 2) W_G^\epsilon(D) + O(\epsilon^2/q). \quad (7.22)$$

Applying the function $-\frac{1}{\epsilon} \log(\cdot)$ to both sides of the bound above yields the result. \square

Proposition 7.4. *Let G be a game. Then, for all compatible devices D , all $q \in (0, 1)$, and all $\epsilon \in (0, 1]$,*

$$R_{G,q}^\epsilon(D) \geq R_{\bar{a}}^\epsilon(D) - O(q). \quad (7.23)$$

Proof. Observe the following:

$$\left(\frac{\sum_i \langle \rho_i^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) = (1-q) \left(\frac{\sum_x \langle \rho_{\bar{a}}^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) + q \left(\frac{\sum_{a,x} p(a) \langle \rho_a^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) \quad (7.24)$$

$$\leq (1-q) \left(\frac{\sum_x \langle \rho_{\bar{a}}^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) + q \quad (7.25)$$

$$\leq (1 + O(q\epsilon)) \left(\frac{\sum_x \langle \rho_{\bar{a}}^x \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right). \quad (7.26)$$

Applying the function $-\frac{1}{\epsilon} \log(\cdot)$ to both sides yields the desired result. \square

Combining Propositions 7.3 and 7.4, we have the following.

Proposition 7.5. *Let G be a game, and let $s \in \mathbb{R}$. Then, for all compatible devices D , all $q \in (0, 1)$, and all $\epsilon \in (0, 1]$,*

$$R_{G,q}^{\epsilon,s}(D) \geq R_{\bar{a}}^\epsilon(D) - sW_G^\epsilon(D) - O(q + \epsilon/q). \quad \square \quad (7.27)$$

The next proposition now follows by the same reasoning that was used to prove Proposition 7.2.

Proposition 7.6. *Let G be a game, and let π be a rate curve for (G, \bar{a}) . Then, for all compatible devices D , and all $r \in (0, W_G)$, $q \in (0, 1)$ and $\epsilon \in (0, 1]$,*

$$R_{G,q}^{\epsilon,\pi'(r)}(D) \geq \pi(r) - \pi'(r)r - O(q + \epsilon/q). \quad \square \quad (7.28)$$

7.3 The randomness of the success state of Protocol R_{gen}

Let E be a quantum system which purifies the device D at the beginning of Protocol R_{gen} . We use the following notation: let X denote a classical register containing the output sequence (x_1, \dots, x_N) at the conclusion of Protocol R_{gen} , let I denote a classical register containing the sequence

$$((t_1, a_1), \dots, (t_N, a_N)),$$

and let Γ_{IXE} denote the joint state of IXE at the conclusion of Protocol R_{gen} . Let $s \subseteq \mathcal{I}^N \times \mathcal{X}^N$ denote the event that Protocol R_{gen} succeeds. We can write the corresponding state as

$$\Gamma_{IXE}^s = \sum_{(\mathbf{i}, \mathbf{x}) \in s} |\mathbf{i}\mathbf{x}\rangle \langle \mathbf{i}\mathbf{x}| \otimes \rho_{\mathbf{i}}^{\mathbf{x}}. \quad (7.29)$$

The next theorem addresses the $(1 + \epsilon)$ -randomness of the success state of Protocol R_{gen} .

Theorem 7.7. *Fix a game G and a rate curve π for (G, \bar{a}) . Then,*

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{(\mathbf{i}, \mathbf{x}) \in s} p_q(\mathbf{i}) \langle \rho_{\mathbf{i}}^{\mathbf{x}} \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) \geq N [\pi(\chi) - O(q + \epsilon/q)]. \quad (7.30)$$

Proof. Applying Proposition 7.6 with induction on N , we find that

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{\mathbf{i}, \mathbf{x}} p_q(\mathbf{i}) 2^{\epsilon \pi'(\chi) H_q(\mathbf{a}, \mathbf{x})} \langle \rho_{\mathbf{i}}^{\mathbf{x}} \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) \geq N[\pi(\chi) - \pi'(\chi)\chi - O(q + \epsilon/q)]. \quad (7.31)$$

The event s is determined by the inequality $H_q(\mathbf{i}, \mathbf{x}) \geq \chi N$. The inequality above is thus preserved when we drop all terms corresponding to $(\mathbf{i}, \mathbf{x}) \notin s$ from the summation, and in those that remain, replace the subterm $H_q(\mathbf{i}, \mathbf{x})$ with χN :

$$-\frac{1}{\epsilon} \log \left(\frac{\sum_{(\mathbf{i}, \mathbf{x}) \in s} p_q(\mathbf{i}) 2^{\epsilon \pi'(\chi) \chi N} \langle \rho_{\mathbf{i}}^{\mathbf{x}} \rangle_{1+\epsilon}}{\langle \rho \rangle_{1+\epsilon}} \right) \geq N[\pi(\chi) - \pi'(\chi)\chi - O(q + \epsilon/q)]. \quad (7.32)$$

Adding $N\pi'(\chi)\chi$ to both sides yields the desired result. \square

7.4 Extractable bits

Our goal is now to use the content of subsection 4.1 to prove a lower bound on the extractable bits in the output of Protocol R_{gen} . First we will obtain a randomness lower bound for adversary states in the form shown on the right side of (4.11). This is done by applying Theorem 7.7 to a class of modified devices.

Proposition 7.8. *Fix a game G and a rate curve π for (G, \bar{a}) . Then,*

$$-\frac{1}{\epsilon} \log \left(\sum_{(\mathbf{i}, \mathbf{x}) \in s} p_q(\mathbf{i}) \langle \rho^{\frac{-\epsilon}{2+2\epsilon}} \rho_{\mathbf{i}}^{\mathbf{x}} \rho^{\frac{-\epsilon}{2+2\epsilon}} \rangle_{1+\epsilon} \right) \geq N[\pi(\chi) - O(q + \epsilon/q)]. \quad (7.33)$$

Proof. Let D_ϵ be the abstract device that arises from D by replacing the initial state ϕ with the operator $\phi^{\frac{1}{1+\epsilon}}$. Then, (using the notation of subsection 3.1) the adversary state of D_ϵ for any \mathbf{i}, \mathbf{x} is isomorphic to

$$\left(\phi^{\frac{1}{2+2\epsilon}} M_1^* M_2^* \cdots M_n^* M_n \cdots M_2 M_1 \phi^{\frac{1}{2+2\epsilon}} \right)^\top = \rho^{\frac{-\epsilon}{2+2\epsilon}} \rho_{\mathbf{i}}^{\mathbf{x}} \rho^{\frac{-\epsilon}{2+2\epsilon}}, \quad (7.34)$$

where $M_j = U_{a_j} P_{a_j}^{x_j}$. Applying Theorem 7.7 to D_ϵ (using the observation that $\langle \rho^{1/(1+\epsilon)} \rangle_{1+\epsilon} = \text{Tr}(\rho) = 1$) yields the desired result. \square

We note that the quantity on the lefthand side of (7.33) can be expressed in the form of (4.11), with $\Lambda = \Gamma_{XIE}^s$ and

$$\sigma = \left(\sum_{\mathbf{i}} p_q(\mathbf{i}) |\mathbf{i}\rangle \langle \mathbf{i}| \right) \otimes \rho. \quad (7.35)$$

We are now ready to prove our main security results.

Theorem 7.9. *Fix a game G and a rate curve π for (G, \bar{a}) . Then, for any $\delta > 0$,*

$$\frac{H_{min}^\delta(X | IE)_{\Gamma^s}}{N} \geq \pi(\chi) - O\left(q + \sqrt{\frac{\log(2/\delta^2)}{qN}}\right). \quad (7.36)$$

Proof. Combining Theorem 4.1 and Proposition 7.8 yields

$$\frac{H_{min}^\delta(X | IE)_{\Gamma^s}}{N} \geq \pi(\chi) - O\left(q + \epsilon/q + \frac{\log(2/\delta^2)}{N\epsilon}\right). \quad (7.37)$$

for all $\epsilon \in (0, 1]$. We can minimize the big- O expression in (7.37) by setting ϵ so that the second and third terms become equal. Let

$$\epsilon = \min\left\{1, \sqrt{\frac{q \log(2/\delta^2)}{N}}\right\}, \quad (7.38)$$

and the desired result holds. \square

Corollary 7.10. *Fix a game G and a rate curve π for (G, \bar{a}) . Then, for any $b > 0$,*

$$\frac{H_{min}^{\sqrt{2} \cdot 2^{-bqN}}(X | IE)_{\Gamma^s}}{N} \geq \pi(\chi) - o(1), \quad (7.39)$$

where $o(1)$ denotes a function of (q, b) that tends to zero as $(q, b) \rightarrow (0, 0)$.

Proof. This is immediate from Theorem 7.9 by substitution. \square

For the final security statement, we use the terminology of extractable bits discussed in subsection 1.1.

Corollary 7.11. *Fix a game G and a rate curve π for (G, \bar{a}) . Then for any $b > 0$, Protocol R_{gen} produces at least*

$$N[\pi(\chi) - o(1)] \quad (7.40)$$

extractable bits with soundness error $3 \cdot 2^{-bqN}$, where $o(1)$ denotes a function of (q, b) (which can depend on G, π) that vanishes as $(q, b) \rightarrow (0, 0)$.

Proof. Find a classical-quantum operator $\Gamma' \geq 0$ with $\|\Gamma' - \Gamma^s\|_1 \leq \sqrt{2} \cdot 2^{-bqN}$ whose min-entropy $H_{min}(X | IE)_{\Gamma'}$ is equal to the numerator in 7.39. If $\text{Tr}(\Gamma^s) \leq 3 \cdot 2^{-bqN}$, then the assertion of the corollary is trivial. If not, then $\text{Tr}(\Gamma') > 2^{-bqN}$, and

$$\frac{H_{min}(X | IE)_{\Gamma'/\text{Tr}(\Gamma')}}{N} \geq \pi(\chi) - o(1) + \frac{\log \text{Tr}(\Gamma')}{N}, \quad (7.41)$$

$$\geq \pi(\chi) - o(1), \quad (7.42)$$

as desired. \square

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A Not all superclassical distributions are randomness generating

The Magic Square game M is a two player game defined as follows:

1. The input and output alphabets are $\{0, 1, 2\}$ and $\{0, 1\}^3$, respectively for each player.
2. The input probability distribution is uniform.
3. The game is won iff the inputs a_1 and a_2 and outputs $(x_1^{(0)}, x_1^{(1)}, x_1^{(2)})$ and $(x_2^{(0)}, x_2^{(1)}, x_2^{(2)})$ satisfy all of the following

$$x_1^{(0)} \oplus x_1^{(1)} \oplus x_1^{(2)} = 0 \quad (\text{A.1})$$

$$x_2^{(0)} \oplus x_2^{(1)} \oplus x_2^{(2)} = 1 \quad (\text{A.2})$$

$$x_1^{(a_1)} = x_2^{(a_2)}. \quad (\text{A.3})$$

The reason for the name “magic square” is this: the game is for the first player to fill in one row of bits in a 3×3 square while the second player is asked to fill in one column. The requirements are that the row has even parity, the column has odd parity, and the overlapping bits agree. The maximal classical score for this game is $8/9 \approx 0.888$.

We construct a device for this game. Let $Q = Q_1 \otimes Q_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$, let α be the projector of Q onto the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let the measurement strategy $\{\{P_{1,a}^x\}_x\}_a$ for the first player be given by

$$P_{1,0}^{000} = \mathbb{I} \quad (\text{A.4})$$

$$P_{1,1}^{000} = |0\rangle\langle 0| \quad (\text{A.5})$$

$$P_{1,1}^{011} = |\pi/2\rangle\langle \pi/2| \quad (\text{A.6})$$

$$P_{1,2}^{101} = |-\pi/4\rangle\langle -\pi/4|, \quad (\text{A.7})$$

$$P_{1,2}^{110} = |\pi/4\rangle\langle \pi/4|, \quad (\text{A.8})$$

where we have used the notation $|\theta\rangle = (\cos \theta)|0\rangle + (\sin \theta)|1\rangle$. Let the measurement strategy for the second player be given by

$$P_{2,0}^{001} = \mathbb{I} \quad (\text{A.9})$$

$$P_{2,1}^{001} = |\pi/8\rangle\langle \pi/8| \quad (\text{A.10})$$

$$P_{2,1}^{010} = |5\pi/8\rangle\langle 5\pi/8| \quad (\text{A.11})$$

$$P_{2,2}^{001} = |-\pi/8\rangle\langle -\pi/8|, \quad (\text{A.12})$$

$$P_{2,2}^{010} = |3\pi/8\rangle\langle 3\pi/8|. \quad (\text{A.13})$$

The winning probability is 1 when the inputs are such that $(a_1 = 0) \vee (a_2 = 0)$, and the winning probability is $\frac{1}{2} + \frac{\sqrt{2}}{4}$ otherwise. (In the latter case the players are essentially conducting an optimal strategy for the CHSH game.) Thus the average winning probability across all inputs is

$$(5/9) + (4/9) \left(\frac{1}{2} + \frac{\sqrt{2}}{4} \right) \approx 0.934. \quad (\text{A.14})$$

Similarly, for any output pair (\bar{x}_1, \bar{x}_2) satisfying the conditions

$$\bar{x}_1^{(0)} \oplus \bar{x}_1^{(1)} \oplus \bar{x}_1^{(2)} = 0 \quad (\text{A.15})$$

$$\bar{x}_2^{(0)} \oplus \bar{x}_2^{(1)} \oplus \bar{x}_2^{(2)} = 1 \quad (\text{A.16})$$

$$\bar{x}_1^{(0)} = \bar{x}_2^{(0)}. \quad (\text{A.17})$$

construct an analogous device, which we denote $E^{\bar{x}_1, \bar{x}_2}$, which always generates output (\bar{x}_1, \bar{x}_2) on input $(0, 0)$ and wins with overall probability ≈ 0.934 . Let E be a quantum device which chooses an output pair (\bar{x}_1, \bar{x}_2) according to a uniform distribution among pairs satisfying (A.15–A.17) and then behaves identically to $E^{\bar{x}_1, \bar{x}_2}$. (Note that if an adversary possesses knowledge of the choice of (\bar{x}_1, \bar{x}_2) , the outputs of such a device on input $(0, 0)$ will be fully predictable to her.)

For each pair $(\bar{a}_1, \bar{a}_2) \in \{1, 2, 3\}^2$, it is possible to construct a completely classical two-part classical device $D^{\bar{a}_1, \bar{a}_2}$ which performs as follows: if the input pair is anything other than (\bar{a}_1, \bar{a}_2) , the device generates a uniform distribution over all pairs of sequences satisfying (A.1–A.3), and if the input pair is (\bar{a}_1, \bar{a}_2) , the device generates a uniform distribution over all pairs of sequences that satisfy (A.1–A.2) but do not satisfy (A.3). Let $S = \{(\bar{a}_1, \bar{a}_2) \mid (\bar{a}_1 = 0) \vee (\bar{a}_2 = 0)\}$, and let D^S denote a classical device which, on any input, chooses an element $(\bar{a}_1, \bar{a}_2) \in S$ uniformly at random and behaves as $D^{(\bar{a}_1, \bar{a}_2)}$.

The device D^S loses the Magic Square game with probability $1/5 = 0.2$ if its given input pair is in S , and loses with probability 0 if its given input pair is not in S . The device E constructed above loses with probability 0 if the given input pair is in S , and loses with probability $\beta := (\frac{1}{2} - \frac{\sqrt{2}}{4}) \approx 0.146$ otherwise. Let D be a two-part quantum device which behaves identically to D^S with probability $\beta/(0.2 + \beta)$, and behaves identically to E with probability $0.2/(0.2 + \beta)$. Then, the losing probability for D on any input pair is

$$\frac{0.2\beta}{0.2 + \beta} \approx 0.0845. \quad (\text{A.18})$$

By construction, the output of D on input $(0, 0)$ is fully predictable to an adversary who possess appropriate classical information. On the other hand, by the symmetry of the conditional input-output distribution of D , it is possible to take any input pair (\bar{a}_1, \bar{a}_2) and construct a simulation of D for which the output of (\bar{a}_1, \bar{a}_2) is fully predictable. Therefore, even though D achieves a superclassical score at the Magic Square game, it is not randomness generating.

(See [15] for a related calculation involving the magic square game.)

B The noise tolerance for the CHSH game

Let *CHSH* denote the two-player game in which all alphabets \mathcal{A}_i and \mathcal{X}_i are equal to $\{0, 1\}$, the input distribution is uniform, and

$$H(\mathbf{a}, \mathbf{x}) = \neg(x_1 \oplus x_2 \oplus (a_1 \wedge a_2)). \quad (\text{B.1})$$

Proposition B.1. *The quantity $W_{\text{CHSH}, 00}$ is equal to $3/4$.*

Proof. Let D be a device compatible with the CHSH game which gives deterministic outputs on input 00. We wish to prove that the losing probability of D is at least $1/4$. Let A_1, A_2, X_1, X_2 denote random variables representing the inputs and output of D .

Note that for any binary random variables X, Y , the probability of the event $X \neq Y$ is at least $|\mathbf{P}(X = 0) - \mathbf{P}(Y = 0)|$ and the probability of the event $X = Y$ is at least $|\mathbf{P}(X = 0) - 1 + \mathbf{P}(Y = 0)|$. Observe the following inequalities, where we use the shorthand $p_u^v = \mathbf{P}(X_1 = v \mid A_1 = u)$ and $q_u^v = \mathbf{P}(X_2 = v \mid A_2 = u)$:

$$\begin{aligned} \mathbf{P}_D(X_1 \oplus X_2 \neq A_1 \wedge A_2) &= \frac{1}{4} [\mathbf{P}_D(X_1 \neq X_2 \mid A = (0, 0)) + \mathbf{P}_D(X_1 \neq X_2 \mid A = (0, 1)) \\ &\quad + \mathbf{P}_D(X_1 \neq X_2 \mid A = (1, 0)) + \mathbf{P}_D(X_1 = X_2 \mid A = (1, 1))] \\ &\geq |p_0^0 - q_0^0| + |p_0^0 - q_1^0| + |q_0^0 - p_1^0| + |q_1^0 - 1 + p_1^0| \\ &\geq \frac{1}{4} |2p_0^0 - 1| \end{aligned}$$

where we used the triangle inequality in the last step. The final quantity is equal to $1/4$, and this completes the proof. \square

Thus $W_{CHSH,00} = 0.75$, while $W_{CHSH} = \frac{1}{2} + \frac{\sqrt{2}}{4} \approx 0.853$. Therefore the noise tolerance provided by Theorem 1.1 for the CHSH game is approximately 10.3%.

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