

# Existence and Stability of Four-Vortex Collinear Relative Equilibria with Three Equal Vorticities

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## Abstract

We study collinear relative equilibria of the planar four-vortex problem where three of the four vortex strengths are identical. The  $S_3$  invariance obtained from the equality of vorticities is used to reduce the defining equations and organize the solutions into two distinct groups based on the ordering of the vortices along the line. The number and type of solutions are given, along with a discussion of the bifurcations that occur. The linear stability of all solutions is investigated rigorously and stable solutions are found to exist for cases where the vorticities have mixed signs. We employ a combination of analysis and computational algebraic geometry to prove our results.

**Key Words:** Relative equilibria,  $n$ -vortex problem, linear stability, symmetry

## 1 Introduction

The planar  $n$ -vortex problem is a Hamiltonian system describing the motion of  $n$  point vortices in the plane acting under a logarithmic potential function. It is a well-known model for approximating vorticity evolution in fluid dynamics [3, 20]. One of the most fruitful approaches to the problem is to study *stationary* configurations, solutions where the initial configuration of vortices is maintained throughout the motion. As explained by O’Neil [21], there are four possibilities: equilibria, relative equilibria (uniform rotations), rigidly translating configurations, and collapse configurations. Much attention has been given to relative equilibria since numerical simulations of certain physical processes (e.g., the eyewall of hurricanes [10, 14]) often produce rigidly rotating configurations of vortices. Analyzing the stability of relative equilibria improves our understanding of the local behavior of the flow; it also has some practical significance given the persistence of these solutions in numerical models of hurricane eyewalls. Other physical examples are provided in [5].

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There are many examples of stable relative equilibria in the planar  $n$ -vortex problem. Perhaps the most well known is the equilateral triangle solution, where three vortices of arbitrary circulations are placed at the vertices of an equilateral triangle. If the sum of the circulations does not vanish, then the triangle rotates rigidly about the center of vorticity. This periodic solution is linearly (and nonlinearly) stable provided that the *total vortex angular momentum*  $L = \sum_{i < j} \Gamma_i \Gamma_j$  is positive [29], where  $\Gamma_i \in \mathbb{R} - \{0\}$  represents the circulation or vorticity of the  $i$ th vortex. Other stable examples include the regular  $n$ -gon for  $4 \leq n \leq 7$  (equal-strength circulations required) [30, 12, 2, 8, 26]; the  $1 + n$ -gon for  $n \geq 3$  (a regular  $n$ -gon with an additional vortex at the center) [8]; the isosceles trapezoid [25]; a family of rhombus configurations [25]; and configurations with one “dominant” vortex and  $n$  small vortices encircling the larger one [6, 7]. In [4], Aref provides a comprehensive study of three-vortex collinear relative equilibria, finding linearly stable solutions for certain cases when the vortex strengths have mixed signs. The rhombus configuration studied in [25] and some particular solutions of the  $(1 + 3)$ -vortex problem discussed in [7] provide some additional examples of stable solutions with circulations of opposite signs.

Relative equilibria can be interpreted as critical points of the Hamiltonian  $H$  restricted to a level surface of the angular impulse  $I$ . This gives a promising topological viewpoint to approach the problem [23]. If all vortices have the same sign, then a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of  $H$  restricted to  $I = \text{constant}$  [25]. Moreover, because  $I$  is a conserved quantity, a technique of Dirichlet’s applies to show that any linearly stable relative equilibrium with same-signed circulations is also nonlinearly stable.

In this paper we apply methods from computational algebraic geometry to investigate the existence and stability of collinear relative equilibria in the four-vortex problem. To make the problem more tractable, we restrict to the case where three of the vortices are assumed to have the same circulation. Specifically, if  $\Gamma_i$  is the circulation of the  $i$ th vortex, then we assume that  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$  and  $\Gamma_4 = m$ , where  $m \in \mathbb{R} - \{0\}$  is treated as a parameter. Solutions to this problem come in groups of six due to the invariance that arises from permuting the three equal-strength vortices. We use this invariance to simplify the problem considerably, obtaining a complete classification of the number and type of solutions in terms of  $m$ . We also provide a straight-forward algorithm to rigorously find all solutions for a fixed  $m$ -value.

When counting the number of solutions, we follow the usual convention (inherited from the companion problem in celestial mechanics) of identifying solutions that are equivalent under rotation, scaling, or translation. In other words, we count equivalence classes of relative equilibria. In general, there are  $n!/2$  ways to arrange  $n$  vortices on a common line, where the factor of  $1/2$  occurs because configurations equivalent under a  $180^\circ$  rotation are identified. The 12 possible orderings in our setting are organized into two groups. Group I contains the 6 arrangements where the unequal vortex (vortex 4) is positioned exterior to the other three; Group II consists of the 6 orderings where vortex 4 is located between two equal-strength vortices. We show that for any  $m > -1/2$ , there are exactly 12 collinear relative equilibria, one for each possible ordering of the vortices. As  $m$  decreases through  $-1/2$ , the solutions in Group II disappear; there are precisely 6 solutions for each  $m \in (-1, -1/2]$ , one for each ordering in Group I. There are no collinear relative equilibria for  $m \leq -1$ .

We also consider the linear stability of the collinear relative equilibria in the planar setting. Due to the integrals and symmetry that naturally arise for any relative equilibrium, there are always four *trivial* eigenvalues  $0, 0, \pm i$  (after a suitable scaling). For the case  $n = 4$ , there are four nontrivial eigenvalues remaining that determine stability. We explain how the nontrivial eigenvalues can be computed from the trace  $T$  and determinant  $D$  of a particular  $2 \times 2$  matrix and provide useful formulas for  $T$  and  $D$

as well as conditions that guarantee linear stability. By applying these formulas and conditions to our specific problem, we are able to rigorously analyze the linear stability of all solutions in Groups I and II.

We show that the Group II solutions are always unstable, with two real pairs of nontrivial eigenvalues  $\pm\lambda_1, \pm\lambda_2$ . The Group I solutions go through two bifurcations, at  $m = m_c \approx -0.0175$  and  $m = m^* \approx -0.8564$ . These important parameter values are roots of a particular sixth-degree polynomial in  $m$  with integer coefficients. For  $m > m_c$ , the Group I solutions are unstable, with two real pairs of eigenvalues. At  $m = m_c$ , these pairs merge and then bifurcate into a complex quartet  $\pm\alpha \pm i\beta$  for  $m \in (m^*, m_c)$ . The Group I solutions are linearly stable for  $m \in (-1, m^*)$  and spectrally stable at  $m = m^*$ . The linear stability of the Group I solutions is somewhat surprising since four of the six solutions limit on a configuration with a pair of binary collisions as  $m \rightarrow -1^+$ .

This problem has recently been explored in [24], where the intent was to classify all relative equilibria, not just the collinear configurations. Unfortunately, there are some errors in this paper. For example, Theorem 4 claims the existence of two families of rhombus configurations. However, this violates the main theorem in [1], which states that a convex relative equilibrium is symmetric with respect to one diagonal if and only if the circulations of the vortices on the other diagonal are equal. To obtain a rhombus, there must be two pairs of equal-strength vortices, one pair for each diagonal (see Section 7.4 in [11] for the complete solution). If three vortices have equal circulations, then the only possible rhombus relative equilibrium is a square. In this article we treat the collinear case in much greater depth than in [24] and focus on the linear stability of solutions (the stability question is not considered in [24]).

Much of our work relies on the theory and computation of Gröbner bases and would not be feasible without the assistance of symbolic computing software. Computations were performed using Maple [15] and many results were checked numerically with Matlab [16]. The award-winning text by Cox, Little, and O’Shea [9] is an excellent reference for the theory and techniques used in this paper involving modern and computational algebraic geometry.

The paper is organized as follows. In the next section we introduce relative equilibria and provide the set up for our particular family of collinear configurations. We then explain how the solutions come in groups of six and use the invariance inherent in the problem to locate, count, and classify solutions in terms of the parameter  $m$ . In Section 3 we provide the relevant theory and techniques for studying the linear stability of relative equilibria in the planar  $n$ -vortex problem. Applying these ideas in our specific setting, we obtain reductions that reduce the stability problem to the calculation of two quantities,  $T$  and  $D$ . This leads to the discovery of linearly stable solutions and the bifurcation values that signify a change in eigenvalue structure.

## 2 Collinear Relative Equilibria with Three Equal Vorticities

We begin with some essential background. The planar  $n$ -vortex problem was first described as a Hamiltonian system by Kirchhoff [13]. Let  $z_i \in \mathbb{R}^2$  denote the position of the  $i$ th vortex and let  $r_{ij} = \|z_i - z_j\|$  represent the distance between the  $i$ th and  $j$ th vortices. The mutual distances  $r_{ij}$  are useful variables. The motion of the  $i$ th vortex is determined by

$$\Gamma_i \dot{z}_i = J \frac{\partial H}{\partial z_i} = J \sum_{j \neq i}^n \frac{\Gamma_i \Gamma_j}{r_{ij}^2} (z_j - z_i), \quad 1 \leq i \leq n, \quad (1)$$

where  $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  is the standard  $2 \times 2$  symplectic matrix and

$$H = - \sum_{i < j} \Gamma_i \Gamma_j \ln(r_{ij})$$

is the Hamiltonian function for the system. The *total circulation* of the system is  $\Gamma = \sum_i \Gamma_i$ , and as long as  $\Gamma \neq 0$ , the *center of vorticity*  $c = \frac{1}{\Gamma} \sum_i \Gamma_i z_i$  is well-defined.

## 2.1 General facts about relative equilibria

**Definition 2.1.** A *relative equilibrium* is a periodic solution of (1) where each vortex rotates about  $c$  with the same angular velocity  $\omega \neq 0$ . Specifically, we have

$$z_i(t) = c + e^{-\omega J t} (z_i(0) - c), \quad \text{for each } i \in \{1, \dots, n\}.$$

It is straight-forward to check that the mutual distances  $r_{ij}$  in a relative equilibrium are unchanged throughout the motion, so that the initial configuration of vortices is preserved. Upon substitution into system (1), we see that the initial positions of a relative equilibrium must satisfy the following system of algebraic equations

$$-\omega \Gamma_i (z_i - c) = \frac{\partial H}{\partial z_i} = \sum_{j \neq i}^n \frac{\Gamma_i \Gamma_j}{r_{ij}^2} (z_j - z_i), \quad i \in \{1, \dots, n\}. \quad (2)$$

Suppose that  $z_0 = (z_1(0), \dots, z_n(0))$  represents the initial positions of a relative equilibrium. Although it is a periodic solution, it is customary to treat a relative equilibrium as a point  $z_0 \in \mathbb{R}^{2n}$  (e.g., a fixed point in rotating coordinates). We will adopt this approach here. From equation (2), we see that any translation, scaling, or rotation of  $z_0$  leads to another relative equilibrium (perhaps with a different value of  $c$  or  $\omega$ ). Thus, relative equilibria are never isolated and it makes sense to consider them as members of an equivalence class  $[[z_0]]$ , where  $w_0 \sim z_0$  provided that  $w_0$  is obtained from  $z_0$  by translation, scaling, or rotation. The stability type of  $z_0$  is the same for all members of  $[[z_0]]$ . Reflections of  $z_0$  are also relative equilibria (e.g., multiplying the first coordinate of  $c$  and each  $z_i$  by  $-1$ ), but these will not be regarded as identical when counting solutions.

The quantity

$$I = \sum_{i=1}^n \Gamma_i \|z_i - c\|^2 = \frac{1}{\Gamma} \sum_{i < j} \Gamma_i \Gamma_j r_{ij}^2$$

can be regarded as a measure of the relative size of the system. It is known as the *angular impulse* with respect to the center of vorticity, the analog of the *moment of inertia* in the  $n$ -body problem. The angular impulse is an integral of motion for the planar  $n$ -vortex problem [20]. One important property of  $I$  is that relative equilibria (regarded as points in  $\mathbb{R}^{2n}$ ) are critical points of the Hamiltonian restricted to a level surface of  $I$ . This can be seen by rewriting system (2) as

$$\nabla H(z) + \frac{\omega}{2} \nabla I(z) = 0, \quad (3)$$

where  $\nabla$  is the usual gradient operator. Here we treat the constant  $\omega/2$  as a Lagrange multiplier. This gives a very useful topological approach to the study of relative equilibria. The main result of [25] is

that, for positive vorticities, a relative equilibrium is linearly stable if and only if it is a nondegenerate minimum of  $H$  restricted to  $I = \text{constant}$ . Using equation (3), it is straight-forward to derive the formula  $\omega = L/I$ .

## 2.2 Defining equations

We now focus on four-vortex relative equilibria whose configurations are collinear, that is, all vortices lie on a common line. To make the problem tractable, we assume that three of the four vortex strengths are identical. Set  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$ , and  $\Gamma_4 = m$ , where  $m \in \mathbb{R}$  is a parameter. Without loss of generality, we take the positions of the relative equilibrium to be on the  $x$ -axis,  $z_i = (x_i, 0)$ , and translate and scale the configuration so that  $x_1 = -1$  and  $x_2 = 1$ . This produces a simpler system to solve than other approaches (e.g., setting  $c = 0$  and  $\omega = 1$ ). It also helps elucidate the inherent symmetries in the problem. In our set up, the center of vorticity  $c$  and angular vorticity  $\omega$  will vary, but the coordinates of the first two vortices will remain fixed (see Figure 1).

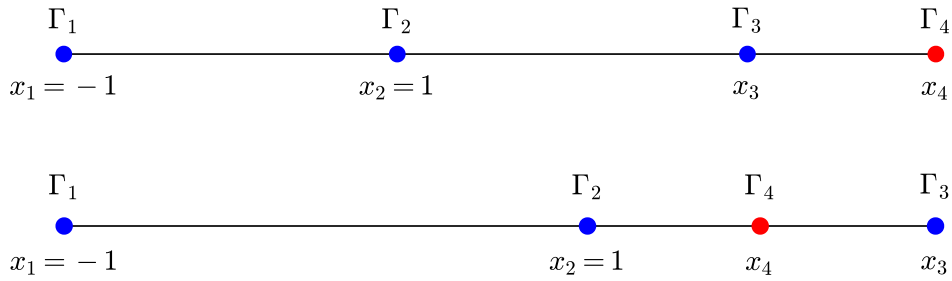


Figure 1: Two collinear relative equilibria, each with circulations  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1, \Gamma_4 = -0.25$ . The top configuration has ordering (1 2 3 4), with  $x_3 \approx 3.104$  and  $x_4 \approx 4.228$ , while the ordering of the bottom solution is (1 2 4 3), with  $x_3 \approx 2.328$  and  $x_4 \approx 1.659$ .

According to equation (2), a relative equilibrium in this form must satisfy the following system of equations:

$$\begin{aligned}
 \omega(-1 - c) + \frac{1}{2} + \frac{1}{x_3 + 1} + \frac{m}{x_4 + 1} &= 0, \\
 \omega(1 - c) - \frac{1}{2} + \frac{1}{x_3 - 1} + \frac{m}{x_4 - 1} &= 0, \\
 \omega(x_3 - c) - \frac{1}{x_3 + 1} - \frac{1}{x_3 - 1} + \frac{m}{x_4 - x_3} &= 0, \\
 \omega(x_4 - c) - \frac{1}{x_4 + 1} - \frac{1}{x_4 - 1} - \frac{1}{x_4 - x_3} &= 0.
 \end{aligned} \tag{4}$$

Let the numerators of the left-hand side of each equation above be denoted by  $f_1, f_2, f_3$ , and  $f_4$  respectively, and append the two polynomials

$$f_5 = u(x_4 - 1) - 1 \quad \text{and} \quad f_6 = v(x_4 + 1) - 1,$$

in order to eliminate solutions with collisions (i.e.,  $x_4 = \pm 1$ ). Let  $F$  be the polynomial ideal generated by  $f_1, \dots, f_6$  in  $\mathbb{Q}[\omega, c, u, v, x_4, x_3, m]$ . Computing a Gröbner basis of  $F$ , denoted by  $GB$ , with respect to the lexicographic order  $\omega > c > u > v > x_4 > x_3 > m$ , yields a basis with 26 elements. The first of these is a 12th-degree polynomial in  $x_3$  with coefficients in  $m$ , given by

$$\begin{aligned} P(x_3, m) = & (m+1)(2m+1)(m+2)^2 x_3^{12} - (32m^5 + 224m^4 + 635m^3 + 873m^2 + 576m + 144)x_3^{10} \\ & + (640m^5 + 4066m^4 + 10126m^3 + 12546m^2 + 7776m + 1944)x_3^8 \\ & - (3776m^5 + 23984m^4 + 60278m^3 + 75042m^2 + 46656m + 11664)x_3^6 \\ & + (5760m^5 + 40806m^4 + 115191m^3 + 158841m^2 + 104976m + 26244)x_3^4 \\ & - 27m^2(96m^3 + 464m^2 + 717m + 351)x_3^2 + 54m^4. \end{aligned}$$

Although  $P$  appears intimidating to analyze, we will use the equality of the vorticities and invariant group theory to factor it into four cubic polynomials in  $x_3$ .

Before performing this reduction, we repeatedly apply the Extension Theorem to insure that a zero of  $P$  can be extended to a full solution of system (4). The third term in  $GB$  is

$$Q(x_4, x_3, m) = 192m^3(4m+5)(x_3^2+3)x_4 + q_1(x_3, m),$$

where  $q_1$  is a polynomial in the variables  $x_3$  and  $m$ . If  $m \neq 0$  and  $m \neq -5/4$ , then the Extension Theorem applies to extend a zero of  $P$ , call it  $(x_3^*, m^*)$ , to a solution  $(x_4^*, x_3^*, m^*)$ . Moreover, since  $Q$  is linear in  $x_4$ , there is a unique such extension. The 20th term in  $GB$  is

$$(m+1)(c(m+3) - x_3 - mx_4),$$

which implies that

$$c = \frac{x_3 + mx_4}{m+3}$$

as long as  $m \neq -1, -3$ . As expected, this agrees with the formula for the center of vorticity in our set up. Similar arguments work to extend any zero of  $P$  uniquely to a solution of the full system (4). Note that we have not ruled out the case that  $x_3^* = x_4^*$ , a collision between the third and fourth vortices. We have proven the following lemma.

**Lemma 2.2.** *Fix an  $m \in \mathbb{R}$  with  $m \neq -3, -5/4, -1, 0$ . Then any solution  $x_3^*$  to  $P = 0$  can be extended uniquely to a full solution of system (4).*

### 2.3 Classifying solutions

Since vortices 1, 2, and 3 have the same vorticity, we can interchange their positions (a relabeling of the vortices) to create a new relative equilibrium. However, because  $x_1 = -1$  and  $x_2 = 1$  are always assumed, it is necessary to apply a scaling and translation in order to convert a relabeled solution into our specific coordinate system. This creates a map between solutions of system (4). To make these ideas precise, we will keep track of how the vortices are arranged under different permutations.

**Definition 2.3.** If vortices  $i, j, k$ , and  $l$  are positioned so that  $x_i < x_j < x_k < x_l$ , then the corresponding *ordering* is denoted  $(i \ j \ k \ l)$ .

To illustrate the inherent invariance in system (4), suppose that we have a relative equilibrium with coordinates

$$x = (x_1, x_2, x_3, x_4) = (-1, 1, a, b), \quad \text{with } 1 < a < b.$$

This corresponds to the ordering (1 2 3 4). Relabeling the vortices in the order (3 1 2 4) gives another relative equilibrium, but with coordinates

$$x' = (x'_1, x'_2, x'_3, x'_4) = (1, a, -1, b), \quad (5)$$

which does not match our setup. The linear map  $\phi(x_i) = \frac{2x_i - a - 1}{a - 1}$  satisfies  $\phi(1) = -1$  and  $\phi(a) = 1$ . Consequently, applying  $\phi$  to each entry in (5) will convert  $x'$  into the correct form. Because  $\phi$  is a scaling and translation, the resulting coordinate vector,

$$\phi(x') = \left( -1, 1, \frac{3+a}{1-a}, \frac{-2b+1+a}{1-a} \right),$$

is also a relative equilibrium, and its first two coordinates match our setup. This gives a new solution to system (4), one with ordering (3 1 2 4).

The above argument demonstrates an important invariance for the ideal  $F$ . It shows that for a fixed value of  $m$ , if  $(x_3, x_4) = (a, b)$  is a partial solution in the variety of  $F$ , then so is  $(x_3, x_4) = (\frac{3+a}{1-a}, \frac{-2b+1+a}{1-a})$ . Put another way, if  $\overline{F} = F \cap \mathbb{Q}[x_3, x_4, m]$  is an elimination ideal, then  $\overline{F}$  is invariant under the map

$$S(x_3, x_4) = \left( \frac{3+x_3}{1-x_3}, \frac{-2x_4+1+x_3}{1-x_3} \right), \quad (6)$$

after clearing denominators. (Here we can assume that  $x_3 \neq 1$  because  $x_3 = 1$  is a collision between vortices 2 and 3.) Another symmetry, which is easy to discern, arises by reflecting all four positions about the origin:

$$R(x_3, x_4) = (-x_3, -x_4). \quad (7)$$

However, this operation reverses the ordering of the vortices (e.g., ordering (1 2 3 4) maps to (4 3 2 1)), and thus requires that vortices 1 and 2 be interchanged in order to insure that  $x_1 = -1$  and  $x_2 = 1$  is maintained (e.g., ordering (4 3 2 1) becomes (4 3 1 2)). We can also consider the composition of  $R$  and  $S$  to generate additional invariants for  $\overline{F}$ . As expected, this yields a total of six invariants (including the identity) for  $\overline{F}$ , as  $R$  and  $S$  generate a group of order six that is isomorphic to  $S_3$ , the symmetric group on three symbols.

**Theorem 2.4.** *Let  $G$  be the group generated by the maps  $R$  and  $S$  under composition, where  $R$  and  $S$  are given by (7) and (6), respectively. Then  $G$  is isomorphic to  $S_3$  and the elimination ideal  $\overline{F} = F \cap \mathbb{Q}[x_3, x_4, m]$  arising from system (4) is invariant under  $G$ . Consequently, solutions to system (4) come in groups of six.*

**Proof:** As explained above,  $\overline{F}$  is invariant under both  $R$  and  $S$ . This was confirmed using Maple by checking that, for each polynomial  $p$  in a Gröbner basis of  $\overline{F}$ ,  $p(R(x_3, x_4))$  and the numerator of  $p(S(x_3, x_4))$  are also in  $\overline{F}$ . It follows that  $\overline{F}$  is also invariant under any composition of these maps.

Let  $e$  represent the identity function  $e(x_3, x_4) = (x_3, x_4)$ . We compute that  $R^2 = R \circ R = e$ ,  $S^3 = S \circ S \circ S = e$ , and  $(R \circ S)^2 = e$ . This is sufficient to show that  $G$  is isomorphic to  $S_3$ .

Since  $\overline{F}$  is invariant under  $G$  and the order of  $G$  is six, it follows that one solution in the variety of  $\overline{F}$  leads to five others. The only possible exception occurs when  $G$  has fixed points, that is, points

in  $\mathbb{R}^2$  that are mapped to the same place under different group transformations. A straight-forward calculation reveals that the only possible fixed points are  $(0, 0), (3, 1), (-3, -1), (-1, b)$ , and  $(1, b)$ , where  $b \in \mathbb{R}$  is arbitrary. However, each of these corresponds to a collision between two vortices, and is thus excluded. Therefore, the six solutions generated by  $G$  are distinct.  $\square$

Suppose that we have a relative equilibrium solution with  $(x_3, x_4) = (a, b)$ , where  $1 < a < b$ . This solution has ordering (1 2 3 4). Applying the transformations from  $G$  generates five additional solutions, each with a different ordering of the vortices. These solutions and their corresponding orderings are shown in the first two columns of Table 1 and will be denoted as *Group I*. Likewise, for a solution with  $1 < b < a$ , which corresponds to the ordering (1 2 4 3), there are five other solutions generated by  $G$  whose orderings are displayed in the third column of Table 1. These solutions will be referred to as *Group II*. The 12 orderings from the union of the two groups are the only allowable orderings because we have assumed that  $x_1 < x_2$ , thereby eliminating half of the 24 permutations in  $S_4$ . Note that each of the orderings in Group I have vortex 4 positioned exterior to the three equal-strength vortices, while for Group II, the fourth vortex always lies between two of the equal-strength vortices.

Coordinates $(x_3, x_4)$	Group I	Group II	Group Element
$(a, b)$	(1 2 3 4)	(1 2 4 3)	$e$
$(-a, -b)$	(4 3 1 2)	(3 4 1 2)	$R$
$\left(\frac{a+3}{a-1}, \frac{-2b+a+1}{a-1}\right)$	(4 1 2 3)	(1 4 2 3)	$R \circ S$
$\left(\frac{3-a}{a+1}, \frac{2b-a+1}{a+1}\right)$	(1 3 2 4)	(1 3 4 2)	$S \circ R$
$\left(\frac{3+a}{1-a}, \frac{-2b+a+1}{1-a}\right)$	(3 1 2 4)	(3 1 4 2)	$S$
$\left(\frac{a-3}{a+1}, \frac{-2b+a-1}{a+1}\right)$	(4 1 3 2)	(1 4 3 2)	$S^2 = S \circ S$

Table 1: **Invariants:** Since  $\Gamma_1 = \Gamma_2 = \Gamma_3$ , relative equilibria  $x = (-1, 1, x_3, x_4)$  come in groups of six, each with a different ordering of vortices. The positions of the third and fourth vortices (first column) are expressed in terms of one particular solution ( $x_3 = a, x_4 = b$ ). The orderings in Group I arise from a solution where  $1 < a < b$ , while those in Group II satisfy  $1 < b < a$ . The corresponding group element from  $G \simeq S_3$  is shown in the last column.

**Remark 2.5.** Recall that  $S_3$  is isomorphic to  $D_3$ , the dihedral group of degree three. Since  $G \simeq S_3 \simeq D_3$ , the transformations  $S$  and  $S^2$  from Table 1 correspond to rotations, while the remaining non-identity elements represent reflections.



## 2.4 Using invariant group theory to find solutions

Based on the discussion in the previous section, we can apply invariant group theory to rigorously study the solutions to system (4) in terms of the parameter  $m$ . Let  $r_1 = x_3, r_2 = (x_3 - 3)/(x_3 + 1)$ , and  $r_3 = (3 + x_3)/(1 - x_3)$  denote the three values of  $x_3$  corresponding to the group elements  $e, S^2$ , and  $S$ , respectively (the rotations). The cubic polynomial with these three roots should be a factor of  $P(x_3, m)$ , the first polynomial in the lex Gröbner basis  $GB$  arising from system (4).

We introduce the coordinates  $\sigma, \tau$ , and  $\rho$ , defined by the elementary symmetric functions on the roots  $r_1, r_2$ , and  $r_3$ :

$$\sigma = r_1 + r_2 + r_3, \quad (8)$$

$$\tau = r_1 r_2 + r_1 r_3 + r_2 r_3, \quad (9)$$

$$\rho = r_1 r_2 r_3. \quad (10)$$

Consider the ideal in  $\mathbb{Q}[\sigma, \tau, \rho, x_3, m]$  generated by equations (8), (9), and (10) (after clearing denominators) and the twelfth-degree polynomial  $P(x_3, m)$ . Computing a lex Gröbner basis (denoted  $GB^*$ ) with respect to the ordering  $\tau < \sigma < x_3 < \rho < m$  yields a basis with four polynomials, the first two of which are

$$\begin{aligned} P_1 &= (m+1)(2m+1)(m+2)^2 \rho^4 - m^2(32m^3 + 152m^2 + 239m + 117)\rho^2 + 54m^4, \\ P_2 &= x_3^3 + \rho x_3^2 - 9x_3 - \rho. \end{aligned}$$

The polynomial  $P_2$  is just the expanded version of equation (10). The fact that  $P_1$  is even in  $\rho$  is expected from the reflection symmetry  $R$ . Note that  $(\rho, x_3) \mapsto (-\rho, -x_3)$  is a symmetry for  $P_2 = 0$ .

This Gröbner basis calculation effectively factors  $P$  into the product of four cubics in the form of  $P_2$ . Indeed, using Maple, we confirm that

$$P = (m+1)(2m+1)(m+2)^2 \prod_{i=1}^4 (x_3^3 + \rho_i x_3^2 - 9x_3 - \rho_i),$$

where  $\rho_i, i \in \{1, 2, 3, 4\}$ , are the four roots of  $P_1$  (each a function of  $m$ ).

We now analyze the roots of the quartic  $P_1$  as a function of  $m$ .

**Lemma 2.6.** *The roots  $\rho_i(m)$  of  $P_1$  satisfy the following properties:*

- For  $m > -1/2$  and  $m \neq 0$ ,  $P_1$  has four real roots.
- For  $-1 < m \leq -1/2$ ,  $P_1$  has exactly two real roots.
- At  $m = 0$ ,  $P_1$  has precisely one real root at zero of multiplicity four.
- At  $m = -1/2$ ,  $P_1$  reduces to a quadratic function with two real roots at  $\pm\sqrt{3/7}$ .
- For  $m \leq -1$ ,  $P_1$  has no real roots.

**Proof:** Introduce the variable  $\xi = \rho^2$ . The results follow by treating  $P_1$  as a quadratic function of  $\xi$  with coefficients in  $\mathbb{Q}[m]$ . The discriminant of  $P_1(\xi)$  is given by

$$\Delta_1 = m^4(4m + 5)^2(64m^4 + 448m^3 + 1153m^2 + 1278m + 513),$$

which is clearly positive for  $m > 0$ . Using Sturm's Theorem [28], it is straight-forward to check that  $\Delta_1 > 0$  for  $-1 < m < 0$  as well. Therefore, the roots of  $P_1(\xi)$  are real for  $m > -1$ .

If  $m > -1/2$  and  $m \neq 0$ , the leading coefficient and constant term of  $P_1(\xi)$  are positive, while the middle term has a negative coefficient. Since the roots are real, Descartes' Rule of Signs shows that  $P_1(\xi)$  has two positive roots, which implies that  $P_1(\rho)$  has four real roots of the form  $\pm\rho_1(m), \pm\rho_2(m)$ .

The leading coefficient of  $P_1$  becomes negative for  $-1 < m < -1/2$ , while the middle coefficient flips sign at  $m \approx -0.942$ . Thus, the sign pattern for the coefficients of  $P_1(\xi)$  is either  $- - +$  or  $- + +$ . In either case there is just one sign change, so Descartes' Rule implies that  $P_1(\xi)$  has only one positive root. Thus, for  $-1 < m < -1/2$ ,  $P_1(\rho)$  has precisely two real roots of the form  $\pm\rho_1(m)$ .

For  $m < -1$ , all three coefficients are positive so we have  $P_1(\xi) > 0$  for  $\xi \geq 0$ . Consequently,  $P_1(\rho)$  has no real roots. The remaining facts listed for the specific  $m$ -values  $0, -1/2$ , and  $-1$  are easily confirmed.  $\square$

**Lemma 2.7.** *The roots of  $P_2(x_3) = x_3^3 + \rho x_3^2 - 9x_3 - \rho$  are real and distinct for any  $\rho \in \mathbb{R}$ . If  $r_1 = a$  is a root, then the other two roots are given by*

$$r_2 = \frac{a - 3}{a + 1} \quad \text{and} \quad r_3 = \frac{3 + a}{1 - a}.$$

*Let  $r_1 = a$  denote the largest root. If  $\rho > 0$ , then the roots satisfy  $1 < r_1 < 3, -1 < r_2 < 0$ , and  $r_3 < -3$ . If  $\rho < 0$ , then the roots satisfy  $r_1 > 3, 0 < r_2 < 1$ , and  $-3 < r_3 < -1$ .*

**Proof:** The discriminant of  $P_2$  with respect to  $x_3$  is  $4(27 + \rho^2)^2$ , which is always positive. Consequently, the roots of  $P_2$  are always real. If  $a$  is a root of  $P_2$ , then we have  $\rho = (a^3 - 9a)/(1 - a^2) = r_1 r_2 r_3$ , as expected. Then, it is straight-forward to check that  $P_2$  factors as  $(x - r_1)(x - r_2)(x - r_3)$ .

Next we note that  $P_2(1) = -8$  and  $P_2(3) = 8\rho$ . By the Intermediate Value Theorem, we have a root  $a$  satisfying  $1 < a < 3$  if  $\rho > 0$ , or  $a > 3$  if  $\rho < 0$ . In the first case, we see that  $-1 < r_2 < 0$  and  $r_3 < -3$  by straight-forward algebra. This also serves to show that  $r_1 = a$  is the largest root. For the case  $\rho < 0$  and  $a > 3$ , we have  $0 < r_2 < 1$  and  $-3 < r_3 < -1$ , as desired.  $\square$

Lemma 2.7 is important because it provides specific information on the location of the third vortex without having to work with the complicated expressions that arise from Cardano's cubic formula. Note that if  $\rho$  is a complex number, then the roots of  $P_2$  must also be complex.

**Algorithm for computing solutions:** Applying Lemma 2.7 and the reductions outlined above, we have the following algorithm for computing the positions of all relative equilibria for a fixed value of  $m$ . In theory, the calculations are exact because they only require solving, in order, a quadratic, cubic, and linear equation.

1. Compute the real roots  $\rho_i$  of the even quartic  $P_1$ .
2. For each real value of  $\rho_i$ , substitute into the cubic  $P_2$  and find the largest root to obtain  $x_3$ .

3. Substitute  $x_3$  and  $m$  into  $Q$  and solve  $Q = 0$  for  $x_4$ . (Recall that  $Q$  is linear in  $x_4$ .)
4. Two additional solutions for  $(x_3, x_4)$  are obtained by using the formulas in the bottom two rows of Table 1.

**Remark 2.8.** 1. Each choice of  $\rho$  leads to three distinct solutions with different orderings. By symmetry, using both  $\rho$  and  $-\rho$  yields six solutions that correspond to six orderings in a particular group (either Group I or Group II). Thus, two positive roots of the quadratic  $P_1(\xi)$  will generate 12 solutions, while one positive root leads to six solutions. If  $P_1(\xi)$  has no positive roots, then there are no solutions.

2. The remaining two polynomials in the Gröbner basis  $GB^*$  reveal some peculiar properties of solutions. The third polynomial in  $GB^*$  is simply  $\sigma + \rho$ , which implies that the sum of the roots of  $P_2$  is the negative of the product of the roots. The remaining entry in  $GB^*$  is just  $\tau + 9$ , which reveals that the symmetric product of the three roots is always equal to  $-9$ . These facts can also be verified by examining the coefficients of  $P_2$  and are apparently an artifact of our special choice of coordinates  $x_1 = -1, x_2 = 1$ .

Next we demonstrate our algorithm for finding all relative equilibria solutions in two important cases.

**Example 2.9. The case  $m = 1$ .**

If all four vortices have the same strength  $\Gamma_i = 1$ , then the four roots of  $P_1$  are  $\rho_i = \pm(\sqrt{3} \pm \sqrt{2})$ . Taking  $\rho = \sqrt{3} - \sqrt{2}$ , the three roots of  $P_2$  are

$$-\sqrt{3} - \sqrt{2} \approx -3.146, \quad 1 + \sqrt{2} - \sqrt{6} \approx -0.035, \quad \text{and} \quad -1 + \sqrt{2} + \sqrt{6} \approx 2.864. \quad (11)$$

Notice that the sum and product of these roots equals  $-\rho$  and  $\rho$ , respectively, in accordance with part 2 of Remark 2.8. Since  $r_{12} = 2$ , we expect the solution with ordering (1 2 3 4) to have  $r_{34} = 2$  by symmetry. Choosing  $x_3 = -1 + \sqrt{2} + \sqrt{6}$  and  $x_4 = 1 + \sqrt{2} + \sqrt{6} \approx 4.864$  gives the desired solution. After computing the corresponding values of  $c$  and  $\omega$ , this solution was confirmed by substituting it into the Gröbner basis  $GB$  as well as into system (4). The center of vorticity is  $c = (\sqrt{2} + \sqrt{6})/2 \approx 1.932$  and the angular velocity is  $\omega = 3/(6 + 2\sqrt{3}) \approx 0.317$ . The other two roots of  $P_2$  shown in equation (11) yield solutions with orderings (3 1 2 4) and (4 1 3 2), with  $x_4$  coordinates  $\sqrt{3} + \sqrt{2}$  and  $-1 + \sqrt{2} - \sqrt{6}$ , respectively. These solutions concur with those obtained by using the symmetry transformations indicated on the bottom two rows of Table 1.

If we choose  $\rho = -\sqrt{3} - \sqrt{2}$  instead, then we obtain  $x_3 = 1 + \sqrt{2} + \sqrt{6} \approx 4.864$  as the largest root of  $P_2$ . Then  $x_4 = -1 + \sqrt{2} + \sqrt{6} \approx 2.864$  gives the coordinate of the fourth vortex. This solution corresponds to ordering (1 2 4 3). The other two roots of  $P_2$  lead to solutions with orderings (3 1 4 2) and (1 4 3 2).

The remaining two values of  $\rho$  lead to six other solutions corresponding to the orderings in rows 2, 3, and 4 of Table 1. All 12 solutions are symmetric, with the distance between the first pair of vortices equal to the distance between the second pair (i.e., for the ordering  $(i j k l)$ , we have  $r_{ij} = r_{kl}$ ). The ratio of this distance over the distance between the inner pair of vortices is always  $(\sqrt{3} + \sqrt{2} - 1)/2 \approx 1.073$ . All 12 solutions are geometrically equivalent. These results agree with those given in Section 5.1 of [11].  $\square$

**Example 2.10. The case  $m = 0$ .**

The solutions for the case  $\Gamma_4 = 0$  are relative equilibria of the restricted four-vortex problem. The three equal-strength vortices are akin to the large masses (called *primaries*) in the celestial mechanics setting. It is known that the primaries must form a relative equilibrium on their own [31]. Thus, based on symmetry, we expect the first three vortices to be equally spaced.

Setting  $m = 0$  in  $P_1$  gives  $\rho_i = 0 \forall i$ . The cubic  $P_2$  reduces to  $x_3^3 - 9x_3$  and thus  $P(x_3, m)$  factors as

$$P(x_3, 0) = 4x_3^4(x_3 - 3)^4(x_3 + 3)^4,$$

with roots  $x_3 = -3, 0, 3$  each repeated four times. As expected, each of the three possible values for  $x_3$  yield an equally-spaced configuration for the equal-strength vortices.

The repeated roots and the fact that Lemma 2.2 does not apply when  $m = 0$  suggest a bifurcation. Surprisingly, this does *not* happen: there are still 12 different solutions, one for each possible ordering in Table 1. While the first nine elements of the Gröbner basis  $GB$ , including  $Q$ , vanish entirely at  $m = 0$  and  $x_3 = 3$ , the tenth element yields a quartic polynomial in  $x_4$  with four distinct real roots. The same feature occurs if  $x_3 = -3$  or  $x_3 = 0$ . We obtain 12 solutions given by

$$(x_3, x_4) = \left( a_i, \frac{a_i}{3} \pm \frac{(a_i^2 + 9)\sqrt{54 \pm 6\sqrt{57}}}{54} \right),$$

where  $a_i = -3, 0$ , or  $3$ , and all four sign combinations occur. These solutions were checked to insure that each satisfied system (4). □

We now have enough information to count and classify all solutions in terms of the parameter  $m$ .

**Theorem 2.11.** *The number and type of four-vortex collinear relative equilibria with circulations  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$  and  $\Gamma_4 = m$  are given as follows:*

- (i) *If  $m > -1/2$ , then there are 12 relative equilibria, one for each possible ordering of the vortices (both Groups I and II are realized);*
- (ii) *If  $-1 < m \leq -1/2$ , there are 6 relative equilibria, one for each ordering in Group I;*
- (iii) *If  $m \leq -1$ , there are no solutions.*

**Proof:** First, we compute the discriminant of  $P(x_3, m)$  as a polynomial in  $x_3$ , and find that for  $m > -1$ , the discriminant vanishes only if  $m = -1/2$  or  $m = 0$ . Consequently, the roots of  $P$  are distinct for  $m > -1$  except in these two special cases.

(i) Suppose that  $m$  is fixed with  $m > -1/2$  and  $m \neq 0$ . Lemmas 2.6 and 2.7 combine to show that  $P$  has 12 distinct roots, and Lemma 2.2 implies that each of these  $x_3$ -values can be extended to a full solution of system (4). Thus there are 12 relative equilibria. We now show that each possible ordering in Table 1 is realized.

Recall from Example 2.9 that for the case  $m = 1$ , choosing  $\rho = \sqrt{3} - \sqrt{2}$  (the smaller positive root of  $P_1$ ) leads to a solution with  $1 < x_3 < x_4$  (ordering (1 2 3 4)). This solution can be continued

analytically as  $m$  varies away from 1 by following the solution corresponding to the smaller positive root of  $P_1$  and the largest root of  $P_2$ . By continuity, the only way for the ordering (1 2 3 4) to disappear is for there to be a collision of vortices, either with  $x_3 = x_2 = 1$  or  $x_3 = x_4$ , at some particular  $m$ -value. The first of these possibilities is ruled out by Lemma 2.7. The second possibility is eliminated by taking the third and tenth polynomials in  $GB$  and making the substitution  $x_4 = x_3$ . Computing a Gröbner basis for these two polynomials, along with  $P_1$  and  $P_2$ , produces the polynomial 1. Consequently, there are no solutions in the variety of  $F$  with  $x_3 = x_4$ . We note that neither  $x_3$  nor  $x_4$  can become infinite for a particular  $m$ -value when  $m > -1/2$ . This follows from Lemmas 2.6 and 2.7 and from the fact that  $Q$  is linear in  $x_4$ .

Some care must be taken to continue the solution with ordering (1 2 3 4) to  $m < 0$  because  $\rho_i = 0 \forall i$  at  $m = 0$  and  $P$  has repeated roots. However, as explained in Example 2.10, there are 12 different solutions for the case  $m = 0$ , one for each possible ordering. A similar calculation to the one outlined in Example 2.9 shows that for the case  $m = -1/4$ , there are also 12 solutions, one for each ordering. In this case, we take  $\rho$  to be the smaller (in absolute value) *negative* root of  $P_1$  in order to obtain the solution with  $1 < x_3 < x_4$ . As explained above, this solution can be continued throughout the interval  $-1/2 < m < 0$  because  $x_3 = x_2 = 1$  and  $x_3 = x_4$  are impossible. Thus, the solution with ordering (1 2 3 4) varies continuously as  $m$  decreases through 0, as the corresponding choice for  $\rho$  transitions from the smallest positive root of  $P_1$  to the smallest negative root of  $P_1$ , passing through  $\rho = 0$  at  $m = 0$ . Applying Theorem 2.4, we have shown that the six orderings from Group I are realized for any  $m > -1/2$ . A plot of the solution curve in the  $x_3x_4$ -plane that corresponds to the ordering (1 2 3 4) for  $-1/2 \leq m \leq 10$  is shown to the left in Figure 2.

The argument for the ordering (1 2 4 3) and its five cousins is similar. This time we follow the larger (in absolute value) negative root of  $P_1$  for  $m > 0$  because  $\rho = -\sqrt{3} - \sqrt{2}$  corresponds to the solution with  $1 < x_4 < x_3$  when  $m = 1$ . For  $-1/2 < m < 0$ , we follow the larger positive root of  $P_1$ , making a continuous transition through  $\rho = 0$  at  $m = 0$ . We know the solution satisfying  $1 < x_4 < x_3$  persists for all  $m > -1/2$  because the only possible collisions are at  $x_4 = x_3$  and  $x_4 = 1$ . The first of these was eliminated by the Gröbner basis calculation mentioned above, while the second is impossible because  $f_5 = u(x_4 - 1) - 1$  was included in the original calculation of  $GB$ . By Theorem 2.4, it follows that all six orderings from Group II are realized. A plot of the solution curve in the  $x_3x_4$ -plane that corresponds to the ordering (1 2 4 3) for  $-1/2 < m \leq 10$  is shown on the right in Figure 2. Although the curve appears to be linear, it is not. This completes the proof of item (i).

**(ii)** A bifurcation occurs at  $m = -1/2$  as the quartic  $P_1$  becomes a quadratic with roots at  $\pm\sqrt{3/7}$ . Here,  $P$  reduces to a tenth degree polynomial with repeated roots at  $-1$  and  $1$ , each with multiplicity two. These correspond to collisions between  $x_3$  and  $x_1$ , or  $x_3$  and  $x_2$ , respectively. The remaining six roots of  $P$  give six relative equilibria with the orderings from Group I. This can be shown rigorously by calculating the largest root of  $P_2$  when  $\rho = -\sqrt{3/7}$  and then finding  $x_4$  from  $Q$ . We find that  $1 < x_3 < x_4$ , so this solution has ordering (1 2 3 4). Theorem 2.4 then yields the remaining five solutions from Group I.

For  $-1 < m < -1/2$ , Lemmas 2.6 and 2.7 combine to show that  $P$  has six distinct real roots and six complex roots. By Lemma 2.2, the six real roots can be extended to a full solution of system (4). As with case (i), rigorously justifying that the six solutions belong to the orderings from Group I involves picking a sample test case (we choose  $m = -3/4$ ) and showing that the solution with ordering (1 2 3 4) persists for all  $m$  in the open interval  $(-1, -1/2)$ . Here we follow the negative root of  $P_1$ . The argument is similar to that used in case (i).

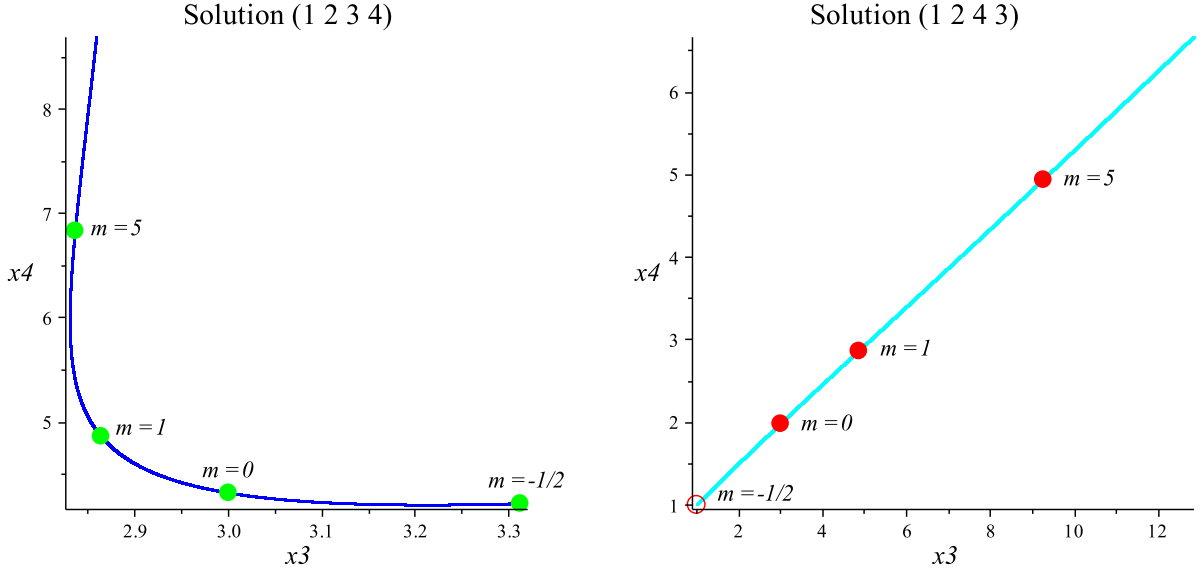


Figure 2: A plot of  $x_4$  versus  $x_3$  for the collinear relative equilibrium with ordering (1 2 3 4) (left) and (1 2 4 3) (right) as the parameter  $m$  varies from  $-1/2$  to 10. Note the continuous transition as  $m$  flips sign.

(iii) If  $m \leq -1$ , then all of the roots of  $P_1$  are complex. Applying Lemma 2.7 shows that  $P_2$  has no real roots, so there are no solutions.  $\square$

**Remark 2.12.** For  $m > 0$ , the existence of a unique collinear relative equilibrium for each possible ordering is a consequence of a well-known result from the Newtonian  $n$ -body problem due to Moulton [19]. For any choice of positive masses, there are exactly  $n!/2$  collinear relative equilibria, one for each possible ordering (see Section 2.1.5 of [17] or Section 2.9 of [18]). The result generalizes to the vortex setting as long as the circulations are positive [23, 21]. For the case  $-1 < m < 0$ , a result due to O’Neil implies that there are at least six solutions (Theorem 6.2.1 in [21]).

## 2.5 Bifurcations

We now discuss the bifurcations at  $m = -1/2$  and  $m = -1$  in greater detail, focusing on the behavior of those solutions which disappear after the bifurcation. As  $m$  decreases toward  $-1/2$ , four of the solutions with orderings from Group II head toward triple collision, while the remaining two orderings have the third vortex escaping to  $\pm\infty$ . To see this, note that two of the roots of  $P_1$  are heading off to  $\pm\infty$  as  $m \rightarrow -1/2^+$ . Focusing on the ordering (1 2 4 3), we track the solution corresponding to  $\rho \rightarrow +\infty$ . Since  $\rho$  is positive, we have  $1 < x_4 < x_3 < 3$  for this particular solution. Moreover, since

$$x_3^3 + \rho x_3^2 - 9x_3 - \rho = 0 \implies x_3^2 - 1 = \frac{x_3(9 - x_3^2)}{\rho},$$

we see that  $x_3 \rightarrow 1^+$  as  $\rho \rightarrow \infty$ . By the Squeeze Theorem, we also have that  $x_4 \rightarrow 1^+$ , and thus the limiting configuration has a triple collision between vortices 2, 3, and 4. To track the other five solutions from Group II, we use the formulas in Table 1 and take limits as  $a \rightarrow 1^+$  and  $b \rightarrow 1^+$ . The

solution with ordering (1 3 4 2) also limits on triple collision between vortices 2, 3, and 4. Orderings (3 4 1 2) and (1 4 3 2) limit on triple collision between vortices 1, 3, and 4. The solutions with orderings (1 4 2 3) and (3 1 4 2) have  $x_3 \rightarrow \infty$  and  $x_3 \rightarrow -\infty$ , respectively, but the  $x_4$ -coordinate takes the form 0/0.

To determine the fate of the fourth vortex for the orderings (1 4 2 3) and (3 1 4 2), we first compute an asymptotic expansion for  $x_3$  and  $x_4$  corresponding to the ordering (1 2 4 3). Introduce the small parameter  $\epsilon$  by setting  $m = -1/2 + \epsilon^2$ , and let  $\kappa = 1/\rho$  be a new variable. As  $\epsilon \rightarrow 0$ , we have  $\kappa \rightarrow 0$ , while  $x_3$  and  $x_4$  each approach 1. Rewriting  $P_1 = 0$  and  $P_2 = 0$  with  $\kappa$ , we can expand  $\kappa$  and  $x_3$  in powers of  $\epsilon$ . This, in turn, leads to an expansion for  $x_4$ . We find that

$$\begin{aligned} \kappa &= \frac{\sqrt{14}}{7} \epsilon + \frac{289}{1029} \sqrt{14} \epsilon^3 + \frac{24373}{43218} \sqrt{14} \epsilon^5 + \mathcal{O}(\epsilon^7), \\ x_3 &= 1 + \frac{4}{7} \sqrt{14} \epsilon + \frac{8}{7} \epsilon^2 - \frac{20}{1029} \sqrt{14} \epsilon^3 - \frac{416}{1029} \epsilon^4 + \mathcal{O}(\epsilon^5), \end{aligned} \tag{12}$$

$$x_4 = 1 + \frac{2}{7} \sqrt{14} \epsilon + \frac{4}{7} \epsilon^2 - \frac{10}{1029} \sqrt{14} \epsilon^3 - \frac{320}{1029} \epsilon^4 + \mathcal{O}(\epsilon^5) \tag{13}$$

are expansions for the solution with ordering (1 2 4 3) near  $m = -1/2$ . Using these expansions, we have

$$\frac{-2x_4 + x_3 + 1}{x_3 - 1} = \frac{4}{147} \sqrt{14} \epsilon^3 + \mathcal{O}(\epsilon^4),$$

which implies that  $x_4 \rightarrow 0$  as  $m \rightarrow -1/2^+$  for both solutions with orderings (1 4 2 3) and (3 1 4 2). Notice that  $x_4 - 1 \approx (1/2)(x_3 - 1)$ , an observation which supports the nearly linear relationship shown in the right-hand graph of Figure 2. Finally, the expansions given above are perfectly valid for  $\epsilon < 0$  as well. In this case, they correspond to the solution with ordering (1 3 4 2) near  $m = -1/2$ .

As  $m$  decreases through  $-1$ , the solutions with orderings from Group I vanish; however, in this case, four of the limiting configurations end with a pair of binary collisions. As  $m \rightarrow -1^+$ , the remaining real roots of  $P_1$  are heading off to  $\pm\infty$ . Focusing on the ordering (1 2 3 4), we track the solution corresponding to  $\rho \rightarrow -\infty$ . By Lemma 2.7 we have  $3 < x_3 < x_4$  for this solution. Since  $P_2(-\rho) = 8\rho < 0$  and the leading coefficient of  $P_2$  is positive, we see that  $P_2$  has a root larger than  $-\rho$ . In fact, for  $m$  close to  $-1$ ,  $x_3 = -\rho$  is an excellent approximation to this root. Thus, for the ordering (1 2 3 4),  $x_3 \rightarrow \infty$ , which implies  $x_4 \rightarrow \infty$  as well. A similar fate occurs for the solution with ordering (4 3 1 2), except that here,  $x_3$  and  $x_4$  approach  $-\infty$  as  $m \rightarrow -1^+$ .

For the remaining four orderings in Group I, the formulas in Table 1 show that the  $x_3$ -coordinate is approaching 1 or  $-1$ , while the  $x_4$ -coordinate is an indeterminate form. Substituting  $m = -1$  and  $x_3 = 1$  into  $Q$  quickly yields  $x_4 = -1$ , while inserting  $m = -1$  and  $x_3 = -1$  into  $Q$  gives  $x_4 = 1$ . Thus, as  $m \rightarrow -1^+$ , the solutions with orderings (4 1 2 3) and (4 1 3 2) have vortex four approaching vortex one, and vortex three approaching vortex two. For the orderings (1 3 2 4) and (3 1 2 4), the opposite collisions occur, with vortex three approaching vortex one, and vortex four approaching vortex two. A plot of the solution curve in the  $x_3x_4$ -plane that corresponds to the ordering (1 3 2 4) for  $-1 < m \leq -1/2$  is shown in Figure 3. We will give an asymptotic expansion about  $m = -1$  for this solution in Section 3.3.

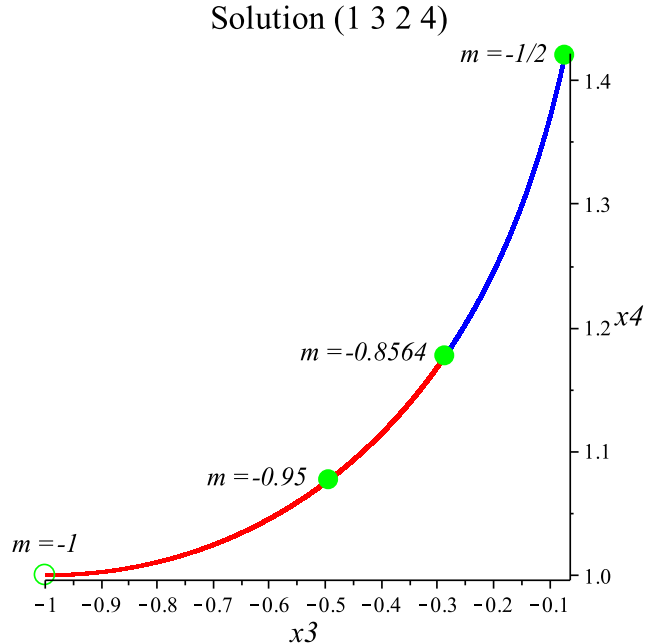


Figure 3: A plot of  $x_4$  versus  $x_3$  for the collinear relative equilibrium with ordering (1 3 2 4) as  $m$  ranges from  $-1/2$  to  $-1$ . The red portion of the curve corresponds to linearly stable configurations, while the blue portion represents unstable solutions. As  $m \rightarrow -1^+$ , vortex three approaches vortex one and vortex four collides with vortex two.

### 3 Linear Stability of Solutions

We now turn to the linear stability of the collinear relative equilibria found in Section 2, investigating the eigenvalues as the parameter  $m$  varies. Through our analysis, we discover an important polynomial,

$$\Psi = 64m^6 + 320m^5 + 96m^4 - 220m^3 + 505m^2 + 522m + 9,$$

whose roots include two new bifurcation values. Using Sturm's Theorem,  $\Psi$  has four real roots, all of which are negative, and precisely two real roots between  $-1$  and  $0$ . The root closest to  $-1$  is  $m^* \approx -0.8564136$ . Note that  $-6/7$  is a fairly good approximation to this root; it is the second convergent in the continued fraction expansion for  $m^*$ . The root  $m_c \approx -0.0175413$  will also be significant. We will prove that the collinear relative equilibria in Group I are linearly stable for  $-1 < m < m^*$ . For all other  $m$ -values, the relative equilibria in both groups are unstable.

#### 3.1 Background and a useful lemma

We first review some key definitions and properties concerning the linear stability of a relative equilibrium  $z_0$  in the planar  $n$ -vortex problem. We follow the approach and setup described in [25]. The natural setting for determining the stability of  $z_0$  is to change to rotating coordinates and treat  $z_0$  as a rest point of the corresponding flow. We will assume that  $z_0$  has been translated so that its center of vorticity  $c$  is located at the origin.

Denote  $M = \text{diag}\{\Gamma_1, \Gamma_1, \dots, \Gamma_n, \Gamma_n\}$  as the  $2n \times 2n$  matrix of circulations, and let  $K$  be the  $2n \times 2n$



block diagonal matrix containing  $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  on the diagonal. The matrix that determines the linear stability of a relative equilibrium  $z_0$  with angular velocity  $\omega$  is given by

$$B = K(M^{-1}D^2H(z_0) + \omega I),$$

where  $D^2H(z_0)$  is the Hessian of the Hamiltonian evaluated at  $z_0$  and  $I$  is the  $2n \times 2n$  identity matrix. Since we are working with a Hamiltonian system, the eigenvalues of  $B$  come in pairs  $\pm\lambda$ . For a solution to be linearly stable, the eigenvalues must lie on the imaginary axis.

One important property of the Hessian is that it anti-commutes with  $K$ , that is,

$$D^2H(z)K = -KD^2H(z). \quad (14)$$

From this, it is straight-forward to see that the characteristic polynomial of  $M^{-1}D^2H(z_0)$  is even. In addition, if  $v$  is an eigenvector of  $M^{-1}D^2H(z_0)$  with eigenvalue  $\mu$ , then  $Kv$  is also an eigenvector with eigenvalue  $-\mu$  (see Lemma 2.4 in [25]). This fact cuts the dimension of the problem in half.

In order to compare the collinear relative equilibria within a particular group, it is easier to work with the *scaled stability matrix*

$$\omega^{-1}B = K(\omega^{-1}M^{-1}D^2H(z_0) + I). \quad (15)$$

This scaling has no effect on the stability of  $z_0$  because the characteristic polynomial of  $B$  is even. We will refer to the eigenvalues of  $\omega^{-1}B$  as *normalized eigenvalues*. The following lemma explains how to compute the normalized eigenvalues from the eigenvalues of  $\omega^{-1}M^{-1}D^2H(z_0)$ .

**Lemma 3.1.** *Let  $p(\lambda)$  denote the characteristic polynomial of the scaled stability matrix  $\omega^{-1}B$ .*

- (i) *Suppose that  $v$  is a real eigenvector of  $\omega^{-1}M^{-1}D^2H(z_0)$  with eigenvalue  $\mu$ . Then  $\{v, Kv\}$  is a real invariant subspace of  $\omega^{-1}B$  and the restriction of  $\omega^{-1}B$  to  $\{v, Kv\}$  is*

$$\begin{bmatrix} 0 & \mu - 1 \\ \mu + 1 & 0 \end{bmatrix}. \quad (16)$$

*Consequently,  $p(\lambda)$  has a quadratic factor of the form  $\lambda^2 + 1 - \mu^2$ .*

- (ii) *Suppose that  $v = v_1 + iv_2$  is a complex eigenvector of  $\omega^{-1}M^{-1}D^2H(z_0)$  with complex eigenvalue  $\mu = \alpha + i\beta$ . Then  $\{v_1, v_2, Kv_1, Kv_2\}$  is a real invariant subspace of  $\omega^{-1}B$  and the restriction of  $\omega^{-1}B$  to this space is*

$$\begin{bmatrix} 0 & 0 & \alpha - 1 & \beta \\ 0 & 0 & -\beta & \alpha - 1 \\ \alpha + 1 & \beta & 0 & 0 \\ -\beta & \alpha + 1 & 0 & 0 \end{bmatrix}. \quad (17)$$

*Consequently,  $p(\lambda)$  has a quartic factor of the form  $(\lambda^2 + 1 - \mu^2)(\lambda^2 + 1 - \bar{\mu}^2)$ , where  $\bar{\mu} = \alpha - i\beta$ .*

**Proof:** (i) Since  $\omega^{-1}M^{-1}D^2H(z_0)v = \mu v$ , we have  $\omega^{-1}M^{-1}D^2H(z_0)Kv = -\mu Kv$  by equation (14). This implies that  $\omega^{-1}Bv = (\mu + 1)Kv$  and  $\omega^{-1}B(Kv) = (\mu - 1)v$ , verifying matrix (16). The characteristic polynomial of matrix (16) is  $\lambda^2 + 1 - \mu^2$  and therefore, this quadratic is a factor of  $p(\lambda)$ .

(ii) If  $v_1 + iv_2$  is a complex eigenvector with eigenvalue  $\alpha + i\beta$ , then we have  $\omega^{-1}M^{-1}D^2H(z_0)v_1 = \alpha v_1 - \beta v_2$  and  $\omega^{-1}M^{-1}D^2H(z_0)v_2 = \alpha v_2 + \beta v_1$ . Using equation (14), this implies that  $\omega^{-1}Bv_1 = (\alpha + 1)Kv_1 - \beta Kv_2$ ,  $\omega^{-1}Bv_2 = \beta Kv_1 + (\alpha + 1)Kv_2$ ,  $\omega^{-1}B(Kv_1) = (\alpha - 1)v_1 - \beta v_2$ , and  $\omega^{-1}B(Kv_2) = \beta v_1 + (\alpha - 1)v_2$ , which confirms matrix (17). The characteristic polynomial of matrix (17) is  $(\lambda^2 + 1 - \mu^2)(\lambda^2 + 1 - \bar{\mu}^2)$  and hence, this quartic is a factor of  $p(\lambda)$ .  $\square$

Due to the conserved quantities of the  $n$ -vortex problem, any relative equilibrium will have the four normalized eigenvalues  $0, 0, \pm i$ . We call these eigenvalues *trivial*. The eigenvalues  $\pm i$  arise from the center of vorticity integral and can be derived by noting that the vector  $[1, 0, 1, 0, \dots, 1, 0]$  is in the kernel of  $\omega^{-1}M^{-1}D^2H(z_0)$ . The two zero eigenvalues appear because relative equilibria are not isolated rest points. For any relative equilibrium  $z_0$ , the vector  $Kz_0$  is in the kernel of  $\omega^{-1}B$ . This vector is tangent to the periodic orbit determined by  $z_0$  at  $t = 0$ . This follows from the identity

$$\omega^{-1}M^{-1}D^2H(z_0)z_0 = z_0, \quad (18)$$

and part (i) of Lemma 3.1. Thus, in the full phase space, a relative equilibrium is always degenerate.

One method for dealing with the issues that arise from the symmetries of the problem is to work in a reduced phase space (e.g., quotienting out the rotational symmetry). However, it is typically easier to make the computations in  $\mathbb{R}^{2n}$  and then define linear stability by restricting to the appropriate subspace. This is the approach we follow here.

Let  $V = \text{span}\{z_0, Kz_0\}$  and denote  $V^\perp$  as the  $M$ -orthogonal complement of  $V$ , that is,

$$V^\perp = \{w \in \mathbb{R}^{2n} : w^T M v = 0 \quad \forall v \in V\}.$$

The invariant subspace  $V$  accounts for the two zero eigenvalues; the vector space  $V^\perp$  has dimension  $2n - 2$  and is invariant under  $\omega^{-1}B$ . We also have that  $V \cap V^\perp = \{0\}$  provided  $L \neq 0$ . This motivates the following definition for linear stability.

**Definition 3.2.** A relative equilibrium  $z_0$  always has the four trivial normalized eigenvalues  $0, 0, \pm i$ . We call  $z_0$  *nondegenerate* if the remaining  $2n - 4$  eigenvalues are nonzero. A nondegenerate relative equilibrium is *spectrally stable* if the nontrivial eigenvalues lie on the imaginary axis, and *linearly stable* if, in addition, the restriction of the scaled stability matrix  $\omega^{-1}B$  to  $V^\perp$  has a block-diagonal Jordan form with blocks  $\begin{bmatrix} 0 & \beta_i \\ -\beta_i & 0 \end{bmatrix}$ .

As noted in [25], if  $\Gamma_i > 0 \forall i$ , then  $\omega^{-1}M^{-1}D^2H(z_0)$  is symmetric with respect to an  $M$ -orthonormal basis and has a full set of linearly independent real eigenvectors. Consequently, part (i) of Lemma 3.1 applies repeatedly and the characteristic polynomial factors as

$$p(\lambda) = \lambda^2(\lambda^2 + 1) \prod_{j=1}^{n-2} (\lambda^2 + 1 - \mu_j^2),$$

where the  $\mu_j$  are the nontrivial eigenvalues of  $\omega^{-1}M^{-1}D^2H(z_0)$ . It follows that the relative equilibrium is linearly stable if and only if  $|\mu_j| < 1 \forall j$ .

If the circulations  $\Gamma_i$  are of mixed sign, then  $\omega^{-1}M^{-1}D^2H(z_0)$  may have complex eigenvalues, leading to quartic factors of the characteristic polynomial, as explained in part (ii) of Lemma 3.1. This is the case for certain values of  $m < 0$  in our problem. Note that when  $\mu_j \in \mathbb{C} - \mathbb{R}$ , the corresponding eigenvalues of the relative equilibrium form a complex quartet  $\pm\alpha' \pm i\beta'$ . This implies instability unless  $\alpha' = 0$ , which only occurs when  $\text{Re}(\mu_j) = 0$ .

### 3.2 Finding the nontrivial eigenvalues of a collinear relative equilibrium

We now focus on the stability of a collinear relative equilibrium  $z_0 = (z_1, z_2, \dots, z_n) \in \mathbb{R}^{2n}$ , where  $z_i = (x_i, 0) \forall i$ . Rearranging the coordinates from  $(x_1, y_1, x_2, y_2, \dots, x_n, y_n)$  to  $(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n)$ , we find that  $\omega^{-1}M^{-1}D^2H(z_0)$  takes the special form

$$\omega^{-1}M^{-1}D^2H(z_0) = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix},$$

where  $A$  is an  $n \times n$  matrix with entries  $A_{ij} = -\omega^{-1}\Gamma_j r_{ij}^{-2}$  if  $i \neq j$  and  $A_{ii} = -\sum_{j \neq i} A_{ij}$ . This reduces our calculations from a  $2n$ -dimensional vector space to an  $n$ -dimensional one.

For the case  $n = 4$ , we have

$$A = \omega^{-1} \begin{bmatrix} \frac{\Gamma_2}{r_{12}^2} + \frac{\Gamma_3}{r_{13}^2} + \frac{\Gamma_4}{r_{14}^2} & -\frac{\Gamma_2}{r_{12}^2} & -\frac{\Gamma_3}{r_{13}^2} & -\frac{\Gamma_4}{r_{14}^2} \\ -\frac{\Gamma_1}{r_{12}^2} & \frac{\Gamma_1}{r_{12}^2} + \frac{\Gamma_3}{r_{23}^2} + \frac{\Gamma_4}{r_{24}^2} & -\frac{\Gamma_3}{r_{23}^2} & -\frac{\Gamma_4}{r_{24}^2} \\ -\frac{\Gamma_1}{r_{13}^2} & -\frac{\Gamma_2}{r_{23}^2} & \frac{\Gamma_1}{r_{13}^2} + \frac{\Gamma_2}{r_{23}^2} + \frac{\Gamma_4}{r_{34}^2} & -\frac{\Gamma_4}{r_{34}^2} \\ -\frac{\Gamma_1}{r_{14}^2} & -\frac{\Gamma_2}{r_{24}^2} & -\frac{\Gamma_3}{r_{34}^2} & \frac{\Gamma_1}{r_{14}^2} + \frac{\Gamma_2}{r_{24}^2} + \frac{\Gamma_3}{r_{34}^2} \end{bmatrix}.$$

Note that the vectors  $s = [1, 1, 1, 1]^T$  and  $x = [x_1, x_2, x_3, x_4]^T$  are eigenvectors of  $A$  with eigenvalues 0 and 1, respectively. These are the two trivial eigenvalues of  $A$  arising from the center of vorticity integral and the rotational symmetry. The remaining two eigenvalues of  $A$  determine the linear stability of  $z_0$ . Specifically, applying both parts of Lemma 3.1, the normalized nontrivial eigenvalues of  $z_0$  are given by

$$\lambda_1 = \pm\sqrt{\mu_1^2 - 1} \quad \text{and} \quad \lambda_2 = \pm\sqrt{\mu_2^2 - 1}, \quad (19)$$

where  $\mu_1$  and  $\mu_2$  are the nontrivial eigenvalues of  $A$ .

Let  $W = \text{span}\{s, x\}$  and let  $W^\perp$  denote the  $M$ -orthogonal complement of  $W$  where  $M = \text{diag}\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ . The subspace  $W^\perp$  is invariant under  $A$ . To find  $\mu_1$  and  $\mu_2$ , we compute the restriction of  $A$  to  $W^\perp$ . It is straight-forward to check that the two vectors

$$\begin{aligned} w_1 &= [\Gamma_2\Gamma_3(x_3 - x_2), \Gamma_1\Gamma_3(x_1 - x_3), \Gamma_1\Gamma_2(x_2 - x_1), 0]^T, \\ w_2 &= [\Gamma_2\Gamma_4(x_4 - x_2), \Gamma_1\Gamma_4(x_1 - x_4), 0, \Gamma_1\Gamma_2(x_2 - x_1)]^T \end{aligned}$$

form a basis for  $W^\perp$ , as  $w_i^T Ms = 0$  and  $w_i^T Mx = 0$  for each  $i$ . However, it is not an  $M$ -orthogonal basis since  $w_1^T Mw_2 \neq 0$ .

Let  $C$  denote the restriction of  $A$  to  $W^\perp$  and write

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

To find the entries of  $C$ , note that  $C_{11}w_1 + C_{21}w_2 = [*, *, C_{11}\Gamma_1\Gamma_2(x_2 - x_1), C_{21}\Gamma_1\Gamma_2(x_2 - x_1)]^T$ . Thus, we see that  $C_{11}\Gamma_1\Gamma_2(x_2 - x_1)$  and  $C_{21}\Gamma_1\Gamma_2(x_2 - x_1)$  are equal to the third and fourth coordinates,

respectively, of the vector  $Aw_1$ . A similar fact applies for  $C_{12}$  and  $C_{22}$ . After some computation, we find that

$$\begin{aligned} C_{11} &= \omega^{-1} \left[ \frac{\Gamma_1 + \Gamma_3}{r_{13}^2} + \frac{\Gamma_2 + \Gamma_3}{r_{23}^2} + \frac{\Gamma_4}{r_{34}^2} + \frac{\Gamma_3}{(x_3 - x_1)(x_3 - x_2)} \right], \\ C_{22} &= \omega^{-1} \left[ \frac{\Gamma_1 + \Gamma_4}{r_{14}^2} + \frac{\Gamma_2 + \Gamma_4}{r_{24}^2} + \frac{\Gamma_3}{r_{34}^2} + \frac{\Gamma_4}{(x_4 - x_1)(x_4 - x_2)} \right], \\ C_{21} &= -\omega^{-1} \frac{\Gamma_3}{x_2 - x_1} \left[ \frac{x_3 - x_2}{r_{14}^2} + \frac{x_1 - x_3}{r_{24}^2} + \frac{x_2 - x_1}{r_{34}^2} \right], \\ C_{12} &= -\omega^{-1} \frac{\Gamma_4}{x_2 - x_1} \left[ \frac{x_4 - x_2}{r_{13}^2} + \frac{x_1 - x_4}{r_{23}^2} + \frac{x_2 - x_1}{r_{34}^2} \right]. \end{aligned}$$

The nontrivial eigenvalues of  $A$ ,  $\mu_1$  and  $\mu_2$ , are equivalent to the eigenvalues of  $C$ . They are easily expressed in terms of the trace and determinant of  $C$ . The quantity  $\delta$ , defined by

$$\delta = \frac{(\Gamma_1 + \Gamma_2)(\Gamma_3 + \Gamma_4)}{r_{12}^2 r_{34}^2} + \frac{(\Gamma_1 + \Gamma_3)(\Gamma_2 + \Gamma_4)}{r_{13}^2 r_{24}^2} + \frac{(\Gamma_1 + \Gamma_4)(\Gamma_2 + \Gamma_3)}{r_{14}^2 r_{23}^2} + \sum_{i=1}^4 \sum_{\substack{j < k \\ j, k \neq i}}^4 \frac{\Gamma_i(\Gamma_i + \Gamma_j + \Gamma_k)}{r_{ij}^2 r_{ik}^2},$$

is important in the computation of the determinant of  $C$ .

**Lemma 3.3.** *Let  $T$  and  $D$  denote the trace and determinant, respectively, of  $C$ . We have*

$$T = \omega^{-1} \sum_{i < j}^4 \frac{\Gamma_i + \Gamma_j}{r_{ij}^2} - 1, \quad (20)$$

$$D = -T + \omega^{-2} \delta. \quad (21)$$

**Proof:** By definition, if  $(x_1, x_2, x_3, x_4)$  are the coordinates of a collinear relative equilibrium with center of vorticity at the origin, then equation (2) implies that

$$\frac{\Gamma_2}{x_2 - x_1} + \frac{\Gamma_3}{x_3 - x_1} + \frac{\Gamma_4}{x_4 - x_1} + \omega x_1 = 0, \quad (22)$$

$$\frac{\Gamma_1}{x_1 - x_2} + \frac{\Gamma_3}{x_3 - x_2} + \frac{\Gamma_4}{x_4 - x_2} + \omega x_2 = 0. \quad (23)$$

Subtracting equation (23) from equation (22) and multiplying through by  $1/(x_2 - x_1)$  gives

$$\omega = \frac{\Gamma_1 + \Gamma_2}{r_{12}^2} - \frac{\Gamma_3}{(x_3 - x_1)(x_3 - x_2)} - \frac{\Gamma_4}{(x_4 - x_1)(x_4 - x_2)}.$$

It then follows that

$$T = C_{11} + C_{22} = \omega^{-1} \sum_{i < j}^4 \frac{\Gamma_i + \Gamma_j}{r_{ij}^2} - 1.$$

Alternatively, recall that the trace of a matrix is equal to the sum of its eigenvalues. Applying this fact to both matrices  $A$  and  $C$  gives

$$1 + T = 0 + 1 + \mu_1 + \mu_2 = \text{Tr}(A) = \omega^{-1} \sum_{i < j}^4 \frac{\Gamma_i + \Gamma_j}{r_{ij}^2},$$

which gives an alternative proof of formula (20).

Computing the determinant  $D$  from the entries of  $C$  gives a messy expression. A more useful formula can be obtained by utilizing the fact that the sum of the product of all pairs of eigenvalues of  $A$  is equal to half the quantity  $(\text{Tr}(A))^2 - \text{Tr}(A^2)$ . This yields

$$0 \cdot 1 + 0 \cdot \mu_1 + 0 \cdot \mu_2 + 1 \cdot \mu_1 + 1 \cdot \mu_2 + \mu_1 \cdot \mu_2 = \frac{1}{2} \left[ \omega^{-2} \left( \sum_{i < j}^4 \frac{\Gamma_i + \Gamma_j}{r_{ij}^2} \right)^2 - \text{Tr}(A^2) \right],$$

which, after some calculation, gives

$$T + D = \omega^{-2} \delta.$$

□

**Remark 3.4.** Recall that the angular velocity of a relative equilibrium is given by  $\omega = L/I$ , where  $L = \sum_{i < j} \Gamma_i \Gamma_j$  and  $I = (1/\Gamma) \sum_{i < j} \Gamma_i \Gamma_j r_{ij}^2$ . It follows from formulas (20) and (21), that  $T$  and  $D$  depend only on the circulations  $\Gamma_i$  and the mutual distances  $r_{ij}$ . Thus, as we would expect, the stability of a relative equilibrium is unaffected by translation, and we may retain our original coordinates (e.g.,  $x_1 = -1, x_2 = 1$ ) when calculating  $T$  and  $D$ , rather than shifting the configuration so that the center of vorticity is at the origin.

**Theorem 3.5.** *The nontrivial eigenvalues of  $A$  are the roots of  $\lambda^2 - T\lambda + D$ , where  $T$  and  $D$  are given by (20) and (21), respectively. They are identical for any solution within a particular group of orderings. Sufficient conditions for linear stability of the relative equilibrium are*

$$\text{(i)} \quad -2 < T < 2, \quad \text{(ii)} \quad D < T^2/4, \quad \text{(iii)} \quad D > T - 1, \quad \text{and} \quad \text{(iv)} \quad D > -T - 1.$$

**Proof:** As explained above, the nontrivial eigenvalues of  $A$  are equivalent to the eigenvalues of  $C$ , and the characteristic polynomial of  $C$  is  $\lambda^2 - T\lambda + D$ . It is straight-forward to check that  $T$  and  $D$  are invariant under the maps  $S$  and  $R$  defined in equations (6) and (7), respectively. This was also confirmed using Maple. It follows that  $T$  and  $D$  are invariant under the group  $G$  and thus, the nontrivial normalized eigenvalues are identical for all six solutions in a given group of orderings.

If  $\mu_i \in \mathbb{R}$ , then formula (19) shows that  $|\mu_1| < 1$  and  $|\mu_2| < 1$  are both required for stability. This is guaranteed if conditions (i) through (iv) are satisfied (see Figure 4). Note that linear stability follows as well because condition (ii) insures that  $\mu_1 \neq \mu_2$ , so there are no repeated nontrivial eigenvalues. □

**Remark 3.6.** In addition to the blue region in Figure 4, solutions are also linearly stable along the positive  $D$ -axis ( $T = 0, D > 0$ ). In this case the eigenvalues of  $C$  are pure imaginary and the nontrivial eigenvalues of the relative equilibrium are  $\pm i\sqrt{1 + D}, \pm i\sqrt{1 + D}$ . Using matrix (17), it is straight-forward to show that the Jordan form of  $\omega^{-1}B$  has no off-diagonal blocks, and thus the solution is linearly stable.

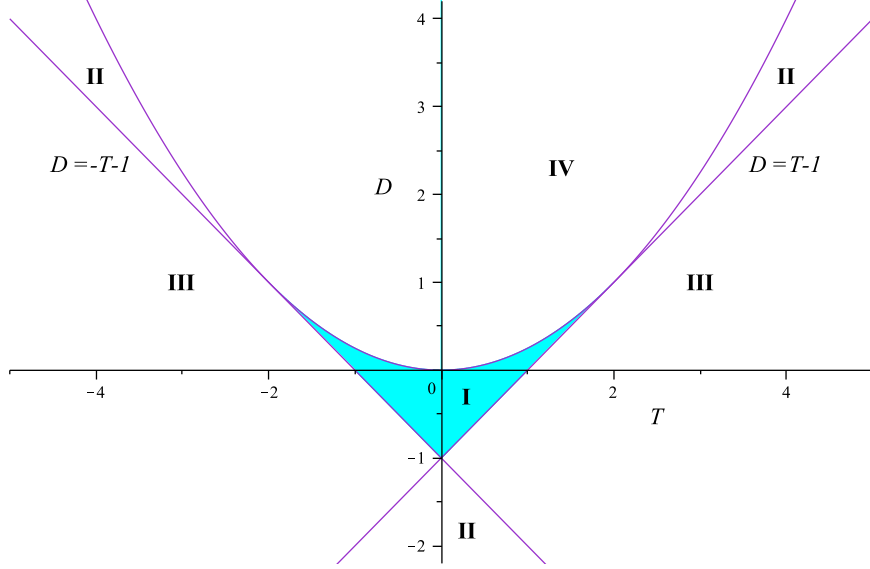


Figure 4: The stability diagram in terms of the trace  $T$  and determinant  $D$  of  $C$ . The light blue region corresponds to linearly stable solutions. Zero eigenvalues occur on the bifurcation lines  $D = T - 1$  or  $D = -T - 1$ , while repeated eigenvalues arise on the parabola  $D = T^2/4$ . In region II, there are two real pairs of eigenvalues  $\pm\lambda_1, \pm\lambda_2$ ; in region III, there is a real pair  $\pm\lambda_1$  and a pure imaginary pair  $\pm i\lambda_2$  of eigenvalues; in region IV, the eigenvalues form a complex quartuplet  $\pm\alpha' \pm i\beta'$ .

**Example 3.7. The case  $m = 1$ .**

Recall from Example 2.9 that in the case of equal-strength vortices, the solution with ordering (1 2 3 4) has positions  $(x_1, x_2, x_3, x_4) = (-1, 1, -1 + \sqrt{2} + \sqrt{6}, 1 + \sqrt{2} + \sqrt{6})$ . This gives  $T = 5$  and  $D = 6$ , so the nontrivial eigenvalues of  $A$  are  $\mu_1 = 2$  and  $\mu_2 = 3$  and the relative equilibrium is unstable. The same result holds for the ordering (1 2 4 3). By formula (19), the nontrivial normalized eigenvalues for either group are  $\pm\sqrt{3}$  and  $\pm 2\sqrt{2}$ .

**Example 3.8. The case  $m = 0$ .**

If  $\Gamma_4 = 0$ , we found in Example 2.10 that the solution with ordering (1 2 3 4) has positions  $(x_1, x_2, x_3, x_4) = (-1, 1, 3, 1 + (1/3)\sqrt{54 + 6\sqrt{57}})$ . From these values, we find that  $\mu_1 = 2$  and  $\mu_2 = (15 - \sqrt{57})/4$ , so the relative equilibrium is unstable. The nontrivial normalized eigenvalues are  $\pm\sqrt{3} \approx \pm 1.732$  and  $\pm(1/4)\sqrt{266 - 30\sqrt{57}} \approx \pm 1.571$ . A similar result holds for the solution with ordering (1 2 4 3), except that the eigenvalues are much further apart. The nontrivial normalized eigenvalues for this solution are  $\pm\sqrt{3}$  and  $\pm(1/4)\sqrt{266 + 30\sqrt{57}} \approx \pm 5.548$ .

### 3.3 Stability, bifurcations, and eigenvalue structure

We now apply Theorem 3.5 and formulas (20) and (21) to investigate the linear stability of our two families of relative equilibria as  $m$  varies. Recall from Theorem 2.11 that there are two groups of solutions varying continuously in  $m$ : Group I exists for all  $m > -1$  and Group II exists for all  $m > -1/2$ . Moreover, for a fixed  $m$ , the values of  $x_3$  and  $x_4$ , and thereby the trace  $T$  and determinant  $D$ , can be found analytically by working with the roots of a cubic equation.

We first compute an asymptotic expansion for the solution with ordering (1 3 2 4) about  $m = -1$ . Recall that as  $m \rightarrow -1^+$ , the limiting configuration for this specific ordering contains a pair of binary collisions, that is,  $x_3 \rightarrow -1$  and  $x_4 \rightarrow 1$  (see Figure 3). Following the same approach used in Section 2.5, we let  $m = -1 + \epsilon^2$ , where  $\epsilon$  is a small positive parameter, and repeatedly solve the equations  $P_1 = 0$ ,  $P_2 = 0$ , and  $Q = 0$  to obtain the expansions

$$x_3 = -1 + 2\sqrt{2}\epsilon - 2\epsilon^2 - 3\sqrt{2}\epsilon^3 + 7\epsilon^4 + \dots - \frac{71179}{16}\sqrt{2}\epsilon^{11} + \frac{226695}{16}\epsilon^{12} + \mathcal{O}(\epsilon^{13}), \quad (24)$$

$$x_4 = 1 + 2\epsilon^2 - \sqrt{2}\epsilon^3 - 5\epsilon^4 + \frac{9}{2}\sqrt{2}\epsilon^5 + \dots - \frac{24863}{16}\sqrt{2}\epsilon^{11} - \frac{192805}{16}\epsilon^{12} + \mathcal{O}(\epsilon^{13}). \quad (25)$$

Substituting these expressions into the formula  $\omega = L/I$  shows that  $1/\omega = 4\epsilon^2 + \mathcal{O}(\epsilon^3)$ , which implies that the angular velocity becomes infinite as the vortices approach collision. Using formulas (20) and (21), we find the following expansions for  $T$  and  $D$ :

$$T = 1 + 6\epsilon^2 - 12\epsilon^4 + 60\epsilon^6 - 426\epsilon^8 + \mathcal{O}(\epsilon^{10}), \quad (26)$$

$$D = 6\epsilon^2 - 12\epsilon^4 + 78\epsilon^6 - 588\epsilon^8 + \frac{9453}{2}\epsilon^{10} + \mathcal{O}(\epsilon^{12}). \quad (27)$$

The fact that each series contains only even powers of  $\epsilon$  is a consequence of the invariance described in Theorem 3.5. Choosing  $\epsilon < 0$  in formulas (24) and (25) is perfectly valid; in fact, it provides an expansion for the solution with ordering (3 1 2 4), which is also a member of the Group I orderings. Since  $T$  and  $D$  are invariant within a specific group, they must be even functions of the parameter  $\epsilon$ .

Recall that  $m^* \approx -0.8564136$  and  $m_c \approx -0.0175413$  are important roots of the polynomial

$$\Psi(m) = 64m^6 + 320m^5 + 96m^4 - 220m^3 + 505m^2 + 522m + 9.$$

**Theorem 3.9.** *The linear stability and nontrivial eigenvalue structure for the four-vortex collinear relative equilibria with circulations  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 1$  and  $\Gamma_4 = m$  are as follows:*

- (i) *The solutions from Group I are linearly stable for  $-1 < m < m^*$ , spectrally stable at  $m = m^*$ , and unstable for  $m > m^*$ .*
- (ii) *For  $m > m_c$ , the nontrivial eigenvalues for the Group I solutions consist of two real pairs, and at  $m = m_c$ , these pairs merge to form a real pair with multiplicity two. As  $m \rightarrow \infty$ , the normalized Group I eigenvalues approach  $\pm 2\sqrt{6}$  and  $\pm 2\sqrt{2}$ . For  $m^* < m < m_c$ , the nontrivial eigenvalues form a complex quartuplet  $\pm\alpha \pm i\beta$ . As  $m \rightarrow -1^+$ , the nontrivial normalized eigenvalues approach  $0, 0, \pm i$ .*
- (iii) *The solutions from Group II ( $m > -1/2$ ) are always unstable with two real pairs of nontrivial normalized eigenvalues  $\pm\lambda_1, \pm\lambda_2$ . As  $m \rightarrow -1/2^+$ ,  $\lambda_1 \rightarrow \infty$  and  $\lambda_2 \rightarrow 2\sqrt{14}/5$ . As  $m \rightarrow \infty$ ,  $\lambda_1 \rightarrow 2\sqrt{2}$  and  $\lambda_2 \rightarrow 0$ .*

**Proof:** We begin by focusing on the solutions with the orderings in Group I. By Theorem 3.5 we can restrict our attention to one particular solution from this group. Since the solution varies continuously in  $m$  ( $m > -1$ ), so do the values of  $T = T(m)$  and  $D = D(m)$  that govern stability. Figure 5 shows a

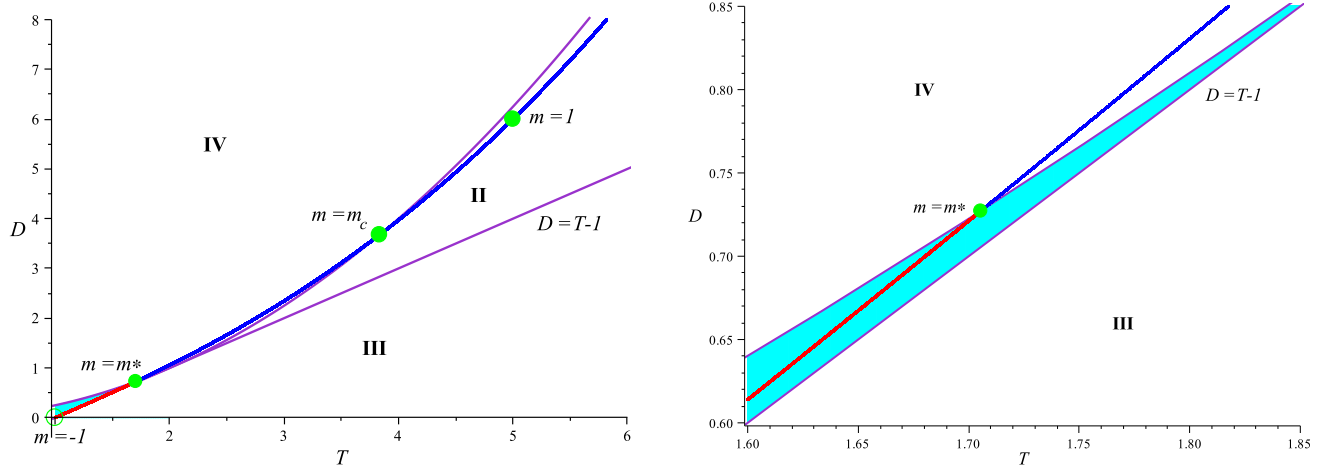


Figure 5: The trace-determinant curve as  $m$  varies for the Group I solutions (red is stable, blue is unstable). For  $m > m_c$ , the curve lies in region II and the solutions have two real pairs of eigenvalues. At  $m = m_c \approx -0.0175$ , the eigenvalues bifurcate into a complex quartuplet as the curve enters region IV. As  $m$  decreases past  $m^* \approx -0.8564$ , the eigenvalues bifurcate again, forming two pairs of pure imaginary eigenvalues. The solutions are linearly stable for  $-1 < m < m^*$ . A zoom of the figure near the key bifurcation value  $m^*$  is shown to the right.

plot of  $D$  versus  $T$  as  $m$  varies, including two bifurcations at  $m = m_c$  and  $m = m^*$ . Our intent is to justify this picture rigorously.

Using the asymptotic expansion for the solution with ordering (1 3 2 4), we find from equations (26) and (27) that  $T$  and  $D$  are approaching 1 and 0, respectively, as  $m \rightarrow -1^+$ . By formula (19), the nontrivial normalized eigenvalues are limiting on  $0, 0, \pm i$ . We also find that

$$D - (T - 1) = 18\epsilon^6 - 162\epsilon^8 + \mathcal{O}(\epsilon^{10}),$$

so that  $D > T - 1$  for  $m$  sufficiently close to  $-1$ . Since the other three conditions in Theorem 3.5 are also satisfied, the solution is linearly stable.

This proves that the trace-determinant curve for the Group I solutions lies in the stability region for  $m$  sufficiently close to  $-1$ . At  $m = 1$ , this curve reaches the point  $(T = 5, D = 6)$  in the unstable region II (two real pairs of eigenvalues). To ascertain how stability is lost, we search for bifurcations, that is, we look for intersections between the trace-determinant curve and the boundaries of the stability region.

Adding the polynomial obtained from the numerator of  $T^2/4 - D$  to our defining system of equations  $\{f_1, f_2, \dots, f_6\}$  yields a system of polynomials whose solutions contain those relative equilibria with repeated eigenvalues. Fortunately, it is possible to compute a lex Gröbner basis for this augmented system and eliminate all variables except for the parameter  $m$ . The first polynomial in this basis is  $(m + 1)^2(m + 3)^2\Psi(m)$ . Therefore, repeated eigenvalues may only occur if  $m = m_c$  or  $m = m^*$ .

Applying back-substitution into the Gröbner basis, we find that there are six solutions corresponding to the orderings in Group I at both  $m = m_c$  and  $m = m^*$ . At  $m = m_c$ , the solutions have a  $T$ -value larger than 2 ( $T \approx 3.8344$ ) and are therefore unstable. On the other hand, the value of  $T$  for the solutions at  $m = m^*$  is  $T^* \approx 1.7054$ . Applying formula (19), the Group I relative equilibria are spectrally stable with repeated nontrivial normalized eigenvalues  $\pm i\sqrt{1 - (T^*/2)^2}$ . The relative



equilibria at  $m = m^*$  are not linearly stable because the matrix  $C$  is not a scalar multiple of the identity matrix. This was confirmed by appending the numerator of  $C_{21}$  to the augmented system described above and computing a Gröbner basis. Since the polynomial 1 was obtained, the value of  $C_{21}$  at either bifurcation is nonzero.

Computing the trace and determinant for the Group I solution at  $m = -1/2$  ( $\rho = -\sqrt{3/7}$ ), we find that  $D > T^2/4$  and thus the nontrivial eigenvalues  $\mu_1$  and  $\mu_2$  are complex. It follows that the trace-determinant curve for the Group I solution lies in region IV for  $m^* < m < m_c$ . By part **(ii)** of Lemma 3.1, the normalized eigenvalues form a complex quartuplet  $\pm\alpha \pm i\beta$  for these  $m$ -values.

Next, we add the numerator of  $D - (T - 1)$  to the system  $\{f_1, f_2, \dots, f_6\}$  and compute a lex Gröbner basis for this augmented system. The first term in the Gröbner basis is simply  $(m + 1)^2(m + 3)^2$ . The  $m + 1$  term is expected because  $T = 1$  and  $D = 0$  are the limiting values as  $m \rightarrow -1^+$ . The fact that there are no other roots for  $m > -1$  shows that  $D > T - 1$  for all solutions (using continuity) because the values at  $m = 1$  satisfy this inequality (see Example 3.7). This is true for solutions from either Group I or II. A similar Gröbner basis calculation shows that  $T > 1$  for all solutions. It follows that the trace-determinant curve lies in the first quadrant and never in region III. Moreover, the only possible bifurcations occur at  $m = m_c$  and  $m = m^*$ , where the curve crosses the repeated root parabola  $D = T^2/4$ . Thus, we have shown that the trace-determinant curve for the Group I solutions lies in the stability region for  $-1 < m < m^*$ , in region IV for  $m^* < m < m_c$ , and in region II for  $m > m_c$ .

To determine the fate of the Group I normalized eigenvalues as  $m \rightarrow \infty$ , we compute an asymptotic expansion for the solution with ordering (1 2 3 4). Setting  $m = 1/\epsilon^2$  and treating  $\epsilon$  as a small parameter, we find

$$\begin{aligned} x_3 &= 3 - \frac{\sqrt{3}}{3}\epsilon + \frac{1}{12}\epsilon^2 + \frac{1025}{1296}\sqrt{3}\epsilon^3 - \frac{2059}{5184}\epsilon^4 + \mathcal{O}(\epsilon^5), \\ x_4 &= \frac{4}{3}\sqrt{3}\epsilon^{-1} + \frac{2}{3} + \frac{35}{27}\sqrt{3}\epsilon + \frac{307}{648}\epsilon^2 + \mathcal{O}(\epsilon^3), \\ T &= 8 - \frac{45}{4}\epsilon^2 + \mathcal{O}(\epsilon^4), \\ D &= 15 - \frac{171}{4}\epsilon^2 + \mathcal{O}(\epsilon^4). \end{aligned}$$

Thus,  $\mu_1 = (T + \sqrt{T^2 - 4D})/2$  approaches 5 and  $\mu_2 = (T - \sqrt{T^2 - 4D})/2$  approaches 3 as  $m \rightarrow \infty$ . The limiting behavior of the nontrivial normalized eigenvalues now follows from formula (19). We note that the solution with this particular ordering limits on a configuration with the three equal-strength vortices equally spaced ( $r_{12} = r_{23} = 2$ ) and the fourth vortex infinitely far away. This completes the proof of items **(i)** and **(ii)** of the theorem.

The Gröbner basis calculations above show that the Group II solutions never bifurcate. Since we also have  $T = 5$  and  $D = 6$  at  $m = 1$  for this group of orderings, it follows that the trace-determinant curve for the Group II solutions is always contained in region II. Consequently, the nontrivial eigenvalues form two real pairs  $\pm\lambda_1, \pm\lambda_2$  for all  $m > -1/2$ . Using the asymptotic expansions (12) and (13) for the Group II solution (1 2 4 3), we find that

$$T = \frac{21}{10}\epsilon^{-2} + \frac{578}{175} + \mathcal{O}(\epsilon^2) \quad \text{and} \quad D = \frac{189}{50}\epsilon^{-2} + \frac{3459}{875} + \mathcal{O}(\epsilon^2)$$

are expansions for the trace and determinant of the Group II solutions for  $m$  close to  $-1/2$ . It follows

that  $\mu_1 \approx (21/10)\epsilon^{-2}$  approaches  $\infty$ , while  $\mu_2$  approaches  $9/5$  as  $m \rightarrow -1/2^+$ . The limiting behavior of the nontrivial normalized eigenvalues now follows from formula (19).

To determine the fate of the Group II normalized eigenvalues as  $m \rightarrow \infty$ , we compute an asymptotic expansion for the solution with ordering (1 4 3 2). Setting  $m = 1/\epsilon^2$  and treating  $\epsilon$  as a small parameter, we find

$$\begin{aligned} x_3 &= 1 - \epsilon + \frac{1}{4}\epsilon^2 + \frac{1}{16}\epsilon^3 - \frac{3}{64}\epsilon^4 + \mathcal{O}(\epsilon^5), \\ x_4 &= -\frac{1}{4}\epsilon - \frac{3}{8}\epsilon^2 + \frac{1}{8}\epsilon^3 + \mathcal{O}(\epsilon^4), \\ T &= 4 + \frac{3}{4}\epsilon^2 + \mathcal{O}(\epsilon^4), \\ D &= 3 + \frac{21}{4}\epsilon^2 + \mathcal{O}(\epsilon^4). \end{aligned}$$

It follows that  $\mu_1 \rightarrow 3$  and  $\mu_2 \rightarrow 1^+$  as  $m \rightarrow \infty$ . The limiting behavior of the nontrivial normalized eigenvalues now follows from formula (19). We note that the solution with this particular ordering limits on a configuration containing a collision between vortices 2 and 3, with the fourth vortex located in the middle of vortices 1 and 2 ( $r_{14} = r_{24} = 1$ ). This completes the proof of item (iii).  $\square$

- Remark 3.10.**
1. Using Gröbner bases, it is possible to express the  $x_3$ -coordinate at  $m = m^*$  as the root of an even 36th-degree polynomial in one variable with integer coefficients. The same is true for the  $x_4$ -coordinate (same degree, different polynomial).
  2. The fact that both the Group I and II collinear relative equilibria are unstable for  $m > 0$  agrees with Corollary 3.5 in [25]. In general, any collinear relative equilibrium of  $n$  vortices, where all circulations have the same sign, is always unstable and has  $n - 2$  nontrivial real eigenvalue pairs  $\pm\lambda_j$ . This follows by generalizing a clever argument of Conley's from the collinear  $n$ -body setting (see Pacella [22] or Moeckel [18] for details in the  $n$ -body case).
  3. Recall that relative equilibria are critical points of the Hamiltonian  $H$  restricted to a level surface of the angular impulse  $I$ . Numerical calculations in Matlab indicate that all of our solutions (both stable and unstable) are saddles (the Morse index is always 2, except for  $m = 0$ ). Thus, in contrast to the case of same-signed circulations, with mixed signs it is possible for a saddle to be linearly stable. A similar observation, using a modified potential function, was also made in [7].

## 4 Conclusion

We have used ideas from modern and computational algebraic geometry to rigorously study the collinear relative equilibria in the four-vortex problem where three circulation strengths are assumed identical. Exploiting the  $S_3$  invariance in the problem, we simplified the defining equations and obtained a specific count on the number and type of solutions in terms of the fourth vorticity  $\Gamma_4 = m$ . The linear stability of solutions in the full plane was investigated and stable solutions were discovered for  $m$  negative. Reductions were made to simplify the stability calculations and useful formulas

were derived that apply to any four-vortex collinear relative equilibrium. Asymptotic expansions were computed to rigorously justify the behavior of solutions near collision. Gröbner bases were used to locate key bifurcation values. It is hoped that the reductions employed here involving symmetry and invariant group theory will prove useful in similar problems.

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