

A SECOND ORDER ENSEMBLE TIMESTEPPING ALGORITHM FOR NATURAL CONVECTION

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Abstract. This paper presents an algorithm for calculating an ensemble of solutions to natural convection problems. The ensemble average is the most likely temperature distribution and its variance gives an estimate of prediction reliability. Solutions are calculated by solving two coupled linear systems, each involving a shared coefficient matrix, for multiple right-hand sides at each timestep. Storage requirements and computational costs to solve the system are thereby reduced. Moreover, this paper addresses a need for higher order methods to solve natural convection problems. Stability and convergence of the method are proven under a timestep condition involving fluctuations of the velocity. Numerical tests are provided which confirm the theoretical analyses.

1. Introduction. Ensemble calculations are essential in predictions of the most likely outcome of systems with uncertain data; for instance, weather forecasting [14] and ocean modeling [15]. Furthermore, they are finding application in an increasing number of fields, including turbulence [13], magnetohydrodynamics [18], and 3D printing [20]. Ensemble simulations classically involve J sequential, fine mesh runs or J parallel, coarse mesh runs of a given code. This leads to a competition between ensemble size and mesh density. We develop a linearly implicit timestepping method with shared coefficient matrices to address this issue. For such methods, it is more efficient in both storage and solution time to solve J linear systems with a shared coefficient matrix than with J different matrices. Prediction of thermal profiles is essential in many applications [1, 9, 17, 19]. Herein, we extend an earlier study [6] regarding first order timestepping algorithms for natural convection based on the pioneering work for isothermal flows of Jiang and Layton [7].

Consider natural convection within an enclosed cavity with zero wall thickness, see Figure 1 for a typical setup. Let $\Omega \subset \mathbb{R}^d$ ($d=2,3$) be a polyhedral domain with boundary $\partial\Omega$. The boundary is partitioned such that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2}$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$, $|\Gamma_1| > 0$, and $\Gamma_1 = \Gamma_H \cup \Gamma_N$. Given $u(x, 0; \omega_j) = u^0(x; \omega_j)$ and $T(x, 0; \omega_j) = T^0(x; \omega_j)$ for $j = 1, 2, \dots, J$, let $u(x, t; \omega_j) : \Omega \times (0, t^*] \rightarrow \mathbb{R}^d$, $p(x, t; \omega_j) : \Omega \times (0, t^*] \rightarrow \mathbb{R}$, and $T(x, t; \omega_j) : \Omega \times (0, t^*] \rightarrow \mathbb{R}$ satisfy

$$\begin{aligned} (1) \quad & u_t + u \cdot \nabla u - Pr \Delta u + \nabla p = Pr Ra \xi T + f \text{ in } \Omega, \\ (2) \quad & \nabla \cdot u = 0 \text{ in } \Omega, \\ (3) \quad & T_t + u \cdot \nabla T - \Delta T = \gamma \text{ in } \Omega, \\ (4) \quad & u = 0 \text{ on } \partial\Omega, \quad T = 1 \text{ on } \Gamma_N, \quad T = 0 \text{ on } \Gamma_H, \quad n \cdot \nabla T = 0 \text{ on } \Gamma_2, \end{aligned}$$

Here n denotes the usual outward normal, ξ denotes the unit vector in the direction of gravity, Pr is the Prandtl number, and Ra is the Rayleigh number. Further, f and γ are the body force and heat source, respectively.

Let $\langle u \rangle_e^n := \frac{1}{J} \sum_{j=1}^J (2u^n - u^{n-1})$ and $u'^n = 2u^n - u^{n-1} - \langle u \rangle_e^n$ be the extrapolated ensemble average and fluctuation; the ensemble average is denoted $\langle \cdot \rangle$. To present the idea, suppress the spatial discretization for the moment. We apply an implicit-explicit (IMEX) time-discretization to the system (1) - (4), while keeping the coefficient matrix independent of the ensemble members. This leads to the following timestepping method:

$$\begin{aligned} (5) \quad & \frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} + \langle u \rangle_e^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla (2u^n - u^{n-1}) - Pr \Delta u^{n+1} + \nabla p^{n+1} \\ & = Pr Ra \xi (2T^n - T^{n-1}) + f^{n+1}, \\ (6) \quad & \nabla \cdot u^{n+1} = 0, \\ (7) \quad & \frac{3T^{n+1} - 4T^n + T^{n-1}}{2\Delta t} + \langle u \rangle_e^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla (2T^n - T^{n-1}) - \Delta T^{n+1} = \gamma^{n+1}. \end{aligned}$$

By lagging u' and using linear extrapolation for the coupling term ξT in the method, the fluid and thermal problems uncouple and each sub-problem contains a shared coefficient matrix for all ensemble members. In

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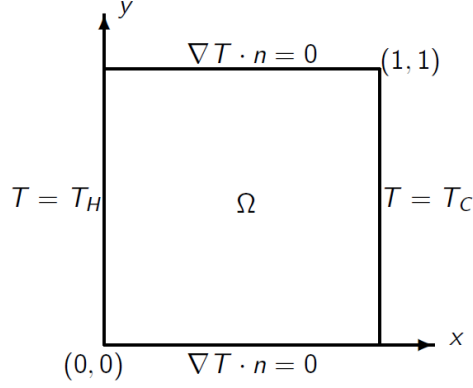


Fig. 1: Domain and boundary conditions for double pane window problem benchmark.

Section 2, we collect necessary mathematical tools. In Section 3, we present an algorithm based on (5) - (7) in the context of the finite element method. Stability and error analysis of the algorithm follow in Section 4. In particular, under a CFL-type condition, we prove the stability of the proposed algorithm in Theorem 4 and its convergence in Theorem 7. We end with numerical experiments and conclusions in Sections 5 and 6.

2. Mathematical Preliminaries. The $L^2(\Omega)$ inner product is (\cdot, \cdot) and the induced norm is $\|\cdot\|$. Define the Hilbert spaces,

$$\begin{aligned} X &:= H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d : v = 0 \text{ on } \partial\Omega\}, \quad Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : (1, q) = 0\}, \\ W &:= H^1(\Omega), \quad W_{\Gamma_1} := \{S \in W : S = 0 \text{ on } \Gamma_1\}, \quad V := \{v \in X : (q, \nabla \cdot v) = 0 \forall q \in Q\}, \end{aligned}$$

and $H^1(\Omega)$ norm $\|\cdot\|_1$. Moreover, we utilize the fractional order Hilbert space on the non-homogeneous Dirichlet boundary $H^{1/2}(\Gamma_N)$ with corresponding norm

$$\|R\|_{1/2, \Gamma_N} := \left(\int_{\Gamma_N} |R(s)|^2 ds + \int_{\Gamma_N} \int_{\Gamma_N} \frac{|R(s) - R(s')|^2}{|s - s'|^d} ds ds' \right)^{1/2}.$$

Let $\tau : \Omega \rightarrow \mathbb{R}$ be an extension of $T|_{\Gamma_N} = 1$ into the domain such that $\|\tau\|_1 \leq C_{tr} \|1\|_{1/2, \Gamma_N} = C_{tr} |\Gamma_N|^{1/2}$ for some $C_{tr} > 0$.

Remark: For natural convection within a unit square or cubic enclosure with a pair of differentially heated vertical walls, the linear conduction profile $\tau(x) = 1 - x_1$, where x_1 denotes the spatial coordinate in the horizontal direction, is such an extension satisfying: $\|\tau\|_1 \leq \frac{2\sqrt{3}}{3}$.

The explicitly skew-symmetric trilinear forms are denoted:

$$\begin{aligned} b(u, v, w) &= \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) \quad \forall u, v, w \in X, \\ b^*(u, T, S) &= \frac{1}{2}(u \cdot \nabla T, S) - \frac{1}{2}(u \cdot \nabla S, T) \quad \forall u \in X, \forall T, S \in W. \end{aligned}$$

They enjoy the following continuity results and properties.

LEMMA 1. *There are constants C_1, C_2, C_3, C_4, C_5 , and C_6 such that for all $u, v, w \in X$ and $T, S \in W$,*

$b(u, v, w)$ and $b^*(u, T, S)$ satisfy

$$\begin{aligned}
b(u, v, w) &= (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)v, w), \\
b^*(u, T, S) &= (u \cdot \nabla T, S) + \frac{1}{2}((\nabla \cdot u)T, S), \\
b(u, v, w) &\leq C_1 \|\nabla u\| \|\nabla v\| \|\nabla w\|, \\
b(u, v, w) &\leq C_2 \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \|\nabla w\|, \\
b^*(u, T, S) &\leq C_3 \|\nabla u\| \|\nabla T\| \|\nabla S\|, \\
b^*(u, T, S) &\leq C_4 \sqrt{\|u\| \|\nabla u\|} \|\nabla T\| \|\nabla S\|, \\
b(u, v, w) &\leq C_5 \|\nabla u\| \|\nabla v\| \sqrt{\|w\| \|\nabla w\|}, \\
b^*(u, T, S) &\leq C_6 \|\nabla u\| \|\nabla T\| \sqrt{\|S\| \|\nabla S\|}.
\end{aligned}$$

Proof. See Lemma 1 on p. 2 of [6]. □

The weak formulation of system (1) - (4) is: Find $u : [0, t^*] \rightarrow X$, $p : [0, t^*] \rightarrow Q$, $T : [0, t^*] \rightarrow W$ for a.e. $t \in (0, t^*]$ satisfying for $j = 1, \dots, J$:

$$(8) \quad (u_t, v) + b(u, u, v) + Pr(\nabla u, \nabla v) - (p, \nabla \cdot v) = PrRa(\xi T, v) + (f, v) \quad \forall v \in X,$$

$$(9) \quad (q, \nabla \cdot u) = 0 \quad \forall q \in Q,$$

$$(10) \quad (T_t, S) + b^*(u, T, S) + (\nabla T, \nabla S) = (\gamma, S) \quad \forall S \in W_{\Gamma_1}.$$

2.1. Finite Element Preliminaries. Consider a regular, quasi-uniform mesh $\Omega_h = \{K\}$ of Ω with maximum triangle diameter length h . Let $X_h \subset X$, $Q_h \subset Q$, $\hat{W}_h = (W_h, W_{\Gamma_1, h}) \subset (W, W_{\Gamma_1}) = \hat{W}$ be conforming finite element spaces consisting of continuous piecewise polynomials of degrees j , l , and j , respectively. Moreover, assume they satisfy the following approximation properties $\forall 1 \leq j, l \leq k, m$:

$$(11) \quad \inf_{v_h \in X_h} \left\{ \|u - v_h\| + h \|\nabla(u - v_h)\| \right\} \leq Ch^{k+1} |u|_{k+1},$$

$$(12) \quad \inf_{q_h \in Q_h} \|p - q_h\| \leq Ch^m |p|_m,$$

$$(13) \quad \inf_{S_h \in \hat{W}_h} \left\{ \|T - S_h\| + h \|\nabla(T - S_h)\| \right\} \leq Ch^{k+1} |T|_{k+1},$$

for all $u \in X \cap H^{k+1}(\Omega)^d$, $p \in Q \cap H^m(\Omega)$, and $T \in \hat{W} \cap H^{k+1}(\Omega)$. Furthermore, we consider those spaces for which the discrete inf-sup condition is satisfied,

$$(14) \quad \inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\| \|\nabla v_h\|} \geq \beta > 0,$$

where β is independent of h . Examples include the MINI-element, Taylor-Hood, and non-conforming Crouzeix-Raviart elements [8]. The space of discretely divergence free functions is defined by

$$V_h := \{v_h \in X_h : (q_h, \nabla \cdot v_h) = 0, \forall q_h \in Q_h\}.$$

The space V_h^* , dual to V_h , is endowed with the following dual norm

$$\|w\|_{V_h^*} := \sup_{v_h \in V_h} \frac{(w, v_h)}{\|\nabla v_h\|}.$$

The discrete inf-sup condition implies that we may approximate functions in V well by functions in V_h ,

LEMMA 2. *Suppose the discrete inf-sup condition (14) holds, then for any $v \in V$*

$$\inf_{v_h \in V_h} \|\nabla(v - v_h)\| \leq C(\beta) \inf_{v_h \in X_h} \|\nabla(v - v_h)\|.$$

Proof. See Chapter 2, Theorem 1.1 on p. 59 of [10]. \square

We will also assume that the mesh and finite element spaces satisfy the standard inverse inequality [5]:

$$\|\nabla\chi_{1,2}\| \leq C_{inv,1,2}h^{-1}\|\chi_{1,2}\| \quad \forall\chi_1 \in X_h, \forall\chi_2 \in W_{\Gamma_1,h},$$

where $C_{inv,1,2}$ depends on the minimum angle α_{min} in the triangulation. A discrete Gronwall inequality will play a role in the upcoming analysis.

LEMMA 3. (*Discrete Gronwall Lemma*). Let Δt , H , a_n , b_n , c_n , and d_n be finite nonnegative numbers for $n \geq 0$ such that for $N \geq 1$

$$a_N + \Delta t \sum_0^N b_n \leq \Delta t \sum_0^{N-1} d_n a_n + \Delta t \sum_0^N c_n + H,$$

then for all $\Delta t > 0$ and $N \geq 1$

$$a_N + \Delta t \sum_0^N b_n \leq \exp\left(\Delta t \sum_0^{N-1} d_n\right) \left(\Delta t \sum_0^N c_n + H\right).$$

Proof. See Lemma 5.1 on p. 369 of [12]. \square

Lastly, the discrete time analysis will utilize the following norms $\forall 1 \leq k \leq \infty$:

$$\|v\|_{\infty,k} := \max_{0 \leq n \leq N} \|v^n\|_k, \quad \|v\|_{p,k} := \left(\Delta t \sum_{n=0}^N \|v^n\|_k^p\right)^{1/p}.$$

3. Numerical Scheme. Denote the fully discrete solutions by u_h^n , p_h^n , and T_h^n at time levels $t^n = n\Delta t$, $n = 1, 2, \dots, N$, and $t^* = N\Delta t$. Given $(u_h^{n-1}, p_h^{n-1}, T_h^{n-1})$ and $(u_h^n, p_h^n, T_h^n) \in (X_h, Q_h, W_h)$, find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = 1, 2, \dots, N$, the fully discrete approximation of (1) - (4)

$$(15) \quad \left(\frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t}, v_h\right) + b(\langle u_h \rangle_e^n, u_h^{n+1}, v_h) + b(u_h^n, 2u_h^n - u_h^{n-1}, v_h) \\ + Pr(\nabla u_h^{n+1}, \nabla v_h) - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\xi(2T_h^n - T_h^{n-1}), v_h) + (f^{n+1}, v_h) \quad \forall v_h \in X_h,$$

$$(16) \quad (q_h, \nabla \cdot u_h^{n+1}) = 0 \quad \forall q_h \in Q_h,$$

$$(17) \quad \left(\frac{3T_h^{n+1} - 4T_h^n + T_h^{n-1}}{2\Delta t}, S_h\right) + b^*(\langle u_h \rangle_e^n, T_h^{n+1}, S_h) + b^*(u_h^n, 2T_h^n - T_h^{n-1}, S_h) \\ + (\nabla T_h^{n+1}, \nabla S_h) = (\gamma^{n+1}, S_h) \quad \forall S_h \in W_{\Gamma_1,h}.$$

Remark: To ensure second order accuracy of the method, the first iterate should be computed with a second order method such as the trapezoidal rule.

Remark: The treatment of the nonlinear terms in the time discretization (5) - (7) leads to a shared coefficient matrix, in the above, independent of the ensemble members.

4. Numerical Analysis of the Ensemble Algorithm. We present stability results for the aforementioned algorithm under the following timestep condition:

$$(18) \quad \frac{C_{\dagger}\Delta t}{h} \max_{1 \leq j \leq J} \|\nabla u_h^n\|^2 \leq 1,$$

where $C_{\dagger} \equiv C_{\dagger}(|\Omega|, \alpha_{min}, Pr)$. In Theorem 4, the nonlinear stability of the velocity, temperature, and pressure approximations are proven under condition 18 for the the scheme (15) - (17).

4.1. Stability Analysis.

THEOREM 4. *Suppose $f \in L^\infty(0, t^*; H^{-1}(\Omega)^d)$, $\gamma \in L^\infty(0, t^*; H^{-1}(\Omega))$. If the scheme (15) - (17) satisfies Condition 18, then*

$$\begin{aligned}
(19) \quad & \frac{1}{2} \|T_h^N\|^2 + \frac{1}{2} \|2T_h^N - T_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} \|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2 \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla T_h^{n+1}\|^2 + Pr\Delta t \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \\
& \leq \exp(Ct^*) \left\{ \Delta t \sum_{n=1}^{N-1} \left(\frac{6}{Pr} \|f^{n+1}\|_{-1}^2 + 4\|\gamma^{n+1}\|_{-1}^2 + 8C_I^2 C_{tr}^2 |\Gamma_N| \right. \right. \\
& \left. \left. + 6PrRa^2 C_{PF,1}^2 C_I^2 C_{tr}^2 |\Gamma_N| \right) + 2\|T_h^1\|^2 + 2\|2T_h^1 - T_h^0\|^2 + \|u_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 \right\} \\
& \quad + C_I^2 C_{tr}^2 |\Gamma_N| \left(2 + t^* + 4\exp(Ct^*) \right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(20) \quad & \beta\Delta t \sum_{n=1}^{N-1} \|p_h^{n+1}\| \leq 2 \left\{ C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| + \frac{2C_1 h}{C_\dagger} \left(\|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right) \right. \\
& \left. + Pr\Delta t \|\nabla u_h^{n+1}\| + 2PrRaC_{PF,1}\Delta t \left(\|2T_h^n - T_h^{n-1}\| + \frac{3C_I C_{tr} |\Gamma_N|^{1/2}}{2} \right) + \Delta t \|f^{n+1}\|_{-1} \right\}.
\end{aligned}$$

Proof. Let $T_h^{n+1} = \theta_h^{n+1} + I_h\tau$, where $I_h\tau$ is an interpolant of τ satisfying $\|I_h\tau\|_1 \leq C_I \|\tau\|_1$. Add equations (15) and (17), let $(v_h, q_h, S_h) = (u_h^{n+1}, p_h^{n+1}, \theta_h^{n+1}) \in (V_h, Q_h, W_{\Gamma_1, h})$ and use the polarization identity. Then,

$$\begin{aligned}
(21) \quad & \frac{1}{4\Delta t} \left\{ \|\theta_h^{n+1}\|^2 + \|2\theta_h^{n+1} - \theta_h^n\|^2 \right\} - \frac{1}{4\Delta t} \left\{ \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 \right\} + \frac{1}{4\Delta t} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\
& + \frac{1}{4\Delta t} \left\{ \|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 \right\} - \frac{1}{4\Delta t} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 \right\} + \frac{1}{4\Delta t} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\
& \quad + \|\nabla \theta_h^{n+1}\|^2 + (\nabla I_h\tau, \nabla \theta_h^{n+1}) + Pr\|\nabla u_h^{n+1}\|^2 + b(u_h^n, 2u_h^n - u_h^{n-1}, u_h^{n+1}) \\
& \quad + b^*(u_h^n, 2\theta_h^n - \theta_h^{n-1}, \theta_h^{n+1}) = PrRa(\gamma(2\theta_h^n - \theta_h^{n-1} + I_h\tau), u_h^{n+1}) - b^*(u_h^n, I_h\tau, \theta_h^{n+1}) \\
& \quad \quad \quad + (f^{n+1}, u_h^{n+1}) + (\gamma^{n+1}, T_h^{n+1}).
\end{aligned}$$

Multiply by Δt , consider $-\Delta t(\nabla I_h\tau, \nabla \theta_h^{n+1})$ and $\Delta t PrRa(\xi I_h\tau, u_h^{n+1})$. Use the Cauchy-Schwarz-Young inequality, interpolation estimates and note that $\|\xi\| = 1$,

$$\begin{aligned}
(22) \quad & -\Delta t(\nabla I_h\tau, \nabla \theta_h^{n+1}) \leq 2\Delta t \|I_h\tau\|_1^2 + \frac{\Delta t}{8} \|\nabla \theta_h^{n+1}\|^2 \leq 2C_I^2 \Delta t \|\tau\|_1^2 + \frac{\Delta t}{8} \|\nabla \theta_h^{n+1}\|^2 \\
& \leq 2C_I^2 C_{tr}^2 |\Gamma_N| \Delta t + \frac{\Delta t}{8} \|\nabla \theta_h^{n+1}\|^2,
\end{aligned}$$

$$(23) \quad \Delta t PrRa(\xi I_h\tau, u_h^{n+1}) \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2 C_I^2 C_{tr}^2 |\Gamma_N|}{2\epsilon_2} + \frac{\Delta t \epsilon_2}{2} \|\nabla u_h^{n+1}\|^2.$$

Use the Cauchy-Schwarz-Young inequality on $\Delta t(\gamma^{n+1}, \theta_h^n)$, $\Delta t PrRa(\xi(2\theta_h^n - \theta_h^{n-1}), u_h^{n+1})$, and $\Delta t(f^{n+1}, u_h^{n+1})$. Then,

$$(24) \quad \Delta t(\gamma^{n+1}, \theta_h^{n+1}) \leq 2\Delta t \|\gamma^{n+1}\|_{-1}^2 + \frac{\Delta t}{8} \|\nabla \theta_h^{n+1}\|^2,$$

$$(25) \quad \Delta t PrRa(\xi(2\theta_h^n - \theta_h^{n-1}), u_h^{n+1}) \leq \frac{\Delta t Pr^2 Ra^2 C_{PF,1}^2}{2\epsilon_1} \|2\theta_h^n - \theta_h^{n-1}\|^2 + \frac{\Delta t \epsilon_1}{2} \|\nabla u_h^{n+1}\|^2,$$

$$(26) \quad \Delta t(f^{n+1}, u_h^{n+1}) \leq \frac{\Delta t}{2\epsilon_3} \|f^{n+1}\|_{-1}^2 + \frac{\Delta t \epsilon_3}{2} \|\nabla u_h^{n+1}\|^2.$$

Consider $-\Delta tb^*(u_h^n, 2\theta_h^n - \theta_h^{n-1}, \theta_h^{n+1})$ and $\Delta tb(u_h^n, 2u_h^n - u_h^{n-1}, u_h^{n+1})$. Use skew-symmetry, Lemma 1, the inverse inequality, and the Cauchy-Schwarz-Young inequality. Then,

(27)

$$\begin{aligned}
-\Delta tb^*(u_h^n, 2\theta_h^n - \theta_h^{n-1}, \theta_h^{n+1}) &= -\Delta tb^*(u_h^n, \theta_h^{n+1}, \theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}) \\
&\leq \Delta t C_6 \|\nabla u_h^n\| \|\nabla \theta_h^{n+1}\| \sqrt{\|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\| \|\nabla(\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1})\|} \\
&\leq \frac{\Delta t C_6 C_{inv,2}^{1/2}}{h^{1/2}} \|\nabla u_h^n\| \|\nabla \theta_h^{n+1}\| \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\| \\
&\leq \frac{2\Delta t^2 C_6^2 C_{inv,2}}{h} \|\nabla u_h^n\|^2 \|\nabla \theta_h^{n+1}\|^2 + \frac{1}{8} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2, \\
-\Delta tb(u_h^n, 2u_h^n - u_h^{n-1}, u_h^{n+1}) &\leq \frac{2\Delta t^2 C_5^2 C_{inv,1}}{h} \|\nabla u_h^n\|^2 \|\nabla u_h^{n+1}\|^2 \\
&\quad + \frac{1}{8} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2.
\end{aligned} \tag{28}$$

Use the Cauchy-Schwarz-Young and Poincaré-Friedrichs inequalities on $-\Delta tb^*(u_h^n, I_h \tau, \theta_h^{n+1})$,

$$\begin{aligned}
-\Delta tb^*(u_h^n, I_h \tau, \theta_h^{n+1}) &\leq \frac{1}{2} \|u_h^n \cdot \nabla I_h \tau\| \|\theta_h^{n+1}\| + \frac{1}{2} \|u_h^n \cdot \nabla \theta_h^{n+1}\| \|I_h \tau\| \\
&\leq \frac{(1 + C_{PF_2} C_I^2) C_{tr}^2 |\Gamma_N| \Delta t}{4\epsilon_4} \|u_h^n\|^2 + \frac{\epsilon_4}{4} \|\nabla \theta_h^{n+1}\|^2.
\end{aligned} \tag{29}$$

Let $\epsilon_1 = \epsilon_2 = \epsilon_3 = Pr/3$ and $\epsilon_4 = 1$. Using (22) - (29) in (21) leads to

$$\begin{aligned}
(30) \quad &\frac{1}{4} \left\{ \|\theta_h^{n+1}\|^2 + \|2\theta_h^{n+1} - \theta_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 \right\} + \frac{1}{8} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\
&\quad + \frac{1}{4} \left\{ \|u_h^{n+1}\|^2 + \|2u_h^{n+1} - u_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 \right\} + \frac{1}{8} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 \\
&\quad + \frac{\Delta t}{4} \|\nabla \theta_h^{n+1}\|^2 + \frac{Pr\Delta t}{4} \|\nabla u_h^{n+1}\|^2 + \frac{\Delta t}{2} \|\nabla \theta_h^{n+1}\|^2 \left\{ 1 - \frac{4\Delta t C_6^2 C_{inv,2}}{h} \|\nabla u_h^n\|^2 \right\} \\
&\quad + \frac{Pr\Delta t}{2} \|\nabla u_h^{n+1}\|^2 \left\{ 1 - \frac{4\Delta t C_5^2 C_{inv,1}}{Prh} \|\nabla u_h^n\|^2 \right\} \leq \frac{3\Delta t Pr Ra^2 C_{PF,1}^2}{2} \|2\theta_h^n - \theta_h^{n-1}\|^2 \\
&\quad + \frac{(1 + C_{PF_2} C_I^2) C_{tr}^2 |\Gamma_N| \Delta t}{4} \|u_h^n\|^2 + 2C_I^2 C_{tr}^2 |\Gamma_N| \Delta t + \frac{3\Delta t Pr Ra^2 C_{PF,1}^2 C_I^2 C_{tr}^2 |\Gamma_N|}{2} \\
&\quad \quad \quad + \frac{3\Delta t}{2Pr} \|f^{n+1}\|_{-1}^2 + \Delta t \|\gamma^{n+1}\|_{-1}^2.
\end{aligned}$$

Use the timestep condition 18, multiply by 4, and add both $\|2u_h^n - u_h^{n-1}\|^2$ and $\|\theta_h^n\|^2$ to the r.h.s. Taking a maximum over constants in the first two terms and the added terms on the r.h.s. and summing from $n = 1$ to $n = N - 1$ leads to,

$$\begin{aligned}
(31) \quad &\|\theta_h^N\|^2 + \|2\theta_h^N - \theta_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\
&\quad + \frac{1}{2} \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \Delta t \sum_{n=1}^{N-1} \|\nabla \theta_h^{n+1}\|^2 + Pr\Delta t \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \\
&\quad \leq C\Delta t \sum_{n=1}^{N-1} \left\{ \|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 + \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2 \right\} \\
&\quad + \Delta t \sum_{n=1}^{N-1} \left\{ \frac{6}{Pr} \|f^{n+1}\|_{-1}^2 + 4\|\gamma^{n+1}\|_{-1}^2 + 8C_I^2 C_{tr}^2 |\Gamma_N| + 6Pr Ra^2 C_{PF,1}^2 C_I^2 C_{tr}^2 |\Gamma_N| Pr \right\} \\
&\quad \quad \quad + \|\theta_h^1\|^2 + \|2\theta_h^1 - \theta_h^0\|^2 + \|u_h^1\|^2 + \|2u_h^1 - u_h^0\|^2.
\end{aligned}$$

Apply Lemma 3. Then,

$$\begin{aligned}
(32) \quad & \|\theta_h^N\|^2 + \|2\theta_h^N - \theta_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 \\
& + \frac{1}{2} \sum_{n=1}^{N-1} \|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \Delta t \sum_{n=1}^{N-1} \|\nabla \theta_h^{n+1}\|^2 + Pr \Delta t \sum_{n=1}^{N-1} \|\nabla u_h^{n+1}\|^2 \\
& \leq \exp(Ct^*) \left\{ \Delta t \sum_{n=1}^{N-1} \left(\frac{6}{Pr} \|f^{n+1}\|_{-1}^2 + 4\|\gamma^{n+1}\|_{-1}^2 + 8C_I^2 C_{tr}^2 |\Gamma_N| + 6PrRa^2 C_{PF,1}^2 C_I^2 C_{tr}^2 |\Gamma_N| \right) \right. \\
& \quad \left. + \|\theta_h^1\|^2 + \|2\theta_h^1 - \theta_h^0\|^2 + \|u_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 \right\}.
\end{aligned}$$

The result follows by recalling the identity $T_h^{n+1} = \theta_h^{n+1} - I_h \tau$ and applying the triangle inequality. Thus, numerical approximations of velocity and temperature are stable. We now prove stability of the pressure approximation. We first form an estimate for the discrete time derivative term. Consider (15), isolate $(\frac{3u_h^{n+1} - 4u_h^n + u_h^{n-1}}{2\Delta t}, v_h)$, let $0 \neq v_h \in V_h$, and multiply by Δt . Then,

$$\begin{aligned}
(33) \quad & \frac{1}{2} (3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) = -\Delta t b(\langle u_h \rangle_e^n, u_h^{n+1}, v_h) - \Delta t b(u_h^n, 2u_h^n - u_h^{n-1}, v_h) \\
& \quad - \Delta t Pr(\nabla u_h^{n+1}, \nabla v_h) + \Delta t Pr Ra(\xi(2\theta_h^n - \theta_h^{n-1} + I_h \tau), v_h) + \Delta t (f^{n+1}, v_h).
\end{aligned}$$

Applying Lemma 1 to the skew-symmetric trilinear terms and the Cauchy-Schwarz and Poincaré-Friedrichs inequalities to the remaining terms yields

$$(34) \quad -\Delta t b(\langle u_h \rangle_e^n, u_h^{n+1}, v_h) \leq C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| \|\nabla v_h\|,$$

$$(35) \quad -\Delta t b(u_h^n, 2u_h^n - u_h^{n-1}, v_h) \leq 2C_1 \Delta t \|\nabla u_h^n\| \left\{ \|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right\} \|\nabla v_h\|,$$

$$(36) \quad -\Delta t Pr(\nabla u_h^{n+1}, \nabla v_h) \leq Pr \Delta t \|\nabla u_h^{n+1}\| \|\nabla v_h\|,$$

$$(37) \quad \Delta t Pr Ra(\xi(2\theta_h^n - \theta_h^{n-1}), v_h) \leq Pr Ra \Delta t \|2\theta_h^n - \theta_h^{n-1}\| \|v_h\| \leq Pr Ra C_{PF,1} \Delta t \|2\theta_h^n - \theta_h^{n-1}\| \|\nabla v_h\|,$$

$$(38) \quad \Delta t Pr Ra(\xi I_h \tau, v_h) \leq Pr Ra C_{PF,1} C_I C_{tr} |\Gamma_N|^{1/2} \Delta t \|\nabla v_h\|,$$

$$(39) \quad \Delta t (f^{n+1}, v_h) \leq \Delta t \|f^{n+1}\|_{-1} \|\nabla v_h\|.$$

Apply the above estimates in (33), divide by the common factor $\|\nabla v_h\|$ on both sides, and take the supremum over all $0 \neq v_h \in V_h$. Then,

$$\begin{aligned}
(40) \quad & \frac{1}{2} \|3u_h^{n+1} - 4u_h^n + u_h^{n-1}\|_{V_h^*} \leq C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| \\
& + 2C_1 \Delta t \|\nabla u_h^n\| \left\{ \|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right\} \|\nabla v_h\| + Pr \Delta t \|\nabla u_h^{n+1}\| + Pr Ra C_{PF,1} \Delta t \|2\theta_h^n - \theta_h^{n-1}\| \\
& \quad + Pr Ra C_{PF,1} C_I C_{tr} |\Gamma_N|^{1/2} \Delta t + \Delta t \|f^{n+1}\|_{-1}.
\end{aligned}$$

Reconsider equation (15). Multiply by Δt and isolate the pressure term,

$$\begin{aligned}
(41) \quad & \Delta t (p_h^{n+1}, \nabla \cdot v_h) = \frac{1}{2} (3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) + \Delta t b(\langle u_h \rangle_e^n, u_h^{n+1}, v_h) + \Delta t b(u_h^n, 2u_h^n - u_h^{n-1}, v_h) \\
& \quad + Pr \Delta t (\nabla u_h^{n+1}, \nabla v_h) - Pr Ra \Delta t (\gamma(2\theta_h^n - \theta_h^{n-1} + I_h \tau), v_h) - \Delta t (f^{n+1}, v_h).
\end{aligned}$$

Apply (34) - (39) on the r.h.s terms. Then,

$$\begin{aligned}
(42) \quad & \Delta t (p_h^{n+1}, \nabla \cdot v_h) \leq \frac{1}{2} (3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h) + \left\{ C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| \right. \\
& \quad + 2C_1 \Delta t \|\nabla u_h^n\| \left(\|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right) + Pr \Delta t \|\nabla u_h^{n+1}\| \\
& \quad \left. + Pr Ra C_{PF,1} \Delta t \left(\|2\theta_h^n - \theta_h^{n-1}\| + C_I C_{tr} |\Gamma_N|^{1/2} \right) + \Delta t \|f^{n+1}\|_{-1} \right\} \|\nabla v_h\|.
\end{aligned}$$

Divide by $\|\nabla v_h\|$ and note that $\frac{(3u_h^{n+1} - 4u_h^n + u_h^{n-1}, v_h)}{2\|\nabla v_h\|} \leq \frac{1}{2}\|3u_h^{n+1} - 4u_h^n + u_h^{n-1}\|_{V_h^*}$. Take the supremum over all $0 \neq v_h \in X_h$,

$$(43) \quad \Delta t \sup_{0 \neq v_h \in X_h} \frac{(p_h^{n+1}, \nabla \cdot v_h)}{\|\nabla v_h\|} \leq 2 \left\{ C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| + 2C_1 \Delta t \|\nabla u_h^n\| \left(\|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right) \right. \\ \left. + Pr \Delta t \|\nabla u_h^{n+1}\| + Pr Ra C_{PF,1} \Delta t \left(\|2\theta_h^n - \theta_h^{n-1}\| + C_I C_{tr} |\Gamma_N|^{1/2} \right) + \Delta t \|f^{n+1}\|_{-1} \right\}.$$

Use the inf-sup condition,

$$(44) \quad \beta \Delta t \|p_h^{n+1}\| \leq 2 \left\{ C_1 \Delta t \|\nabla \langle u_h \rangle_e^n\| \|\nabla u_h^{n+1}\| + 2C_1 \Delta t \|\nabla u_h^n\| \left(\|\nabla u_h^n\| + \|\nabla u_h^{n-1}\| \right) \right. \\ \left. + Pr \Delta t \|\nabla u_h^{n+1}\| + Pr Ra C_{PF,1} \Delta t \left(\|2\theta_h^n - \theta_h^{n-1}\| + C_I C_{tr} |\Gamma_N|^{1/2} \right) + \Delta t \|f^{n+1}\|_{-1} \right\}.$$

Sum from $n = 1$ to $n = N - 1$, use condition 18, recall $T_h^{n+1} = \theta_h^{n+1} + I_h \tau$, and use the triangle inequality. The result follows, yielding stability of the pressure approximation, built on the stability of the temperature and velocity approximations. \square

Remark: Application of Lemma 3 in Theorem 4 allows for the loss of long time stability due to the exponential growth factor, in t^* .

4.2. Error Analysis. Denote u^n , p^n , and T^n as the true solutions at time $t^n = n\Delta t$. Assume the solutions satisfy the following regularity assumptions:

$$(45) \quad u \in L^\infty(0, t^*; X \cap H^{k+1}(\Omega)), \quad T, \tau \in L^\infty(0, t^*; W \cap H^{k+1}(\Omega)), \\ u_t, T_t \in L^\infty(0, t^*; H^{k+1}(\Omega)), \quad u_{tt}, T_{tt} \in L^\infty(0, t^*; H^{k+1}(\Omega)), \\ u_{ttt}, T_{ttt} \in L^\infty(0, t^*; H^{k+1}(\Omega)), \quad p \in L^\infty(0, t^*; Q \cap H^m(\Omega)).$$

Remark: Regularity of the auxiliary temperature solution θ follows from the above regularity assumptions. Convergence results will be proven for the error in the auxiliary variable θ which, by the triangle inequality and interpolation estimates, implies the results for the solution variable T .

The errors for the solution variables are denoted

$$e_u^n = u^n - u_h^n, \quad e_T^n = T^n - T_h^n, \quad e_p^n = p^n - p_h^n.$$

DEFINITION 5. (*Consistency error*). The consistency errors are defined as

$$\varsigma_u(u^n; v_h) = \left(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t} - u_t^{n+1}, v_h \right), \quad \varsigma_T(T^n; S_h) = \left(\frac{3T^{n+1} - 4T^n + T^{n-1}}{2\Delta t} - T_t^{n+1}, S_h \right).$$

LEMMA 6. Provided u and T satisfy the regularity assumptions 45, then $\exists C > 0$ such that $\forall r > 0$

$$|\varsigma_u(u^n; v_h)| \leq \frac{CC_{PF,1}^2 C_r \Delta t^3}{\epsilon} \|u_{ttt}\|_{L^2(t^{n-2}, t^n; L^2(\Omega))}^2 + \frac{\epsilon}{r} \|\nabla v_h\|^2, \\ |\varsigma_T(T^n; S_h)| \leq \frac{CC_{PF,2}^2 C_r \Delta t^3}{\epsilon} \|T_{ttt}\|_{L^2(t^{n-2}, t^n; L^2(\Omega))}^2 + \frac{\epsilon}{r} \|\nabla S_h\|^2.$$

Proof. These follow from the Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality, and Taylor's Theorem with integral remainder. \square

THEOREM 7. For (u, p, T) satisfying (1) - (5), suppose that $(u_h^0, p_h^0, T_h^0) \in (X_h, Q_h, W_h)$ are approximations of (u^0, p^0, T^0) to within the accuracy of the interpolant. Further, suppose that condition 18 holds. Then there exists a constant C such that

$$\begin{aligned}
& \frac{1}{2} \|e_T^N\|^2 + \frac{1}{2} \|2e_T^N - e_T^{N-1}\|^2 + \|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} (\|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2 + \|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2) \\
& + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla e_T^{n+1}\|^2 + Pr\Delta t \sum_{n=1}^{N-1} \|\nabla e_u^{n+1}\|^2 + \frac{Pr\Delta t}{2} (\|\nabla e_u^N\|^2 + \frac{1}{2} \|\nabla e_u^{N-1}\|^2) \\
\leq & \exp(Ct^*) \left\{ \Delta t \inf_{S_h \in W_{\Gamma_1, h}} (\|\theta - S_h\|_{\infty, 0}^2 + \|\theta - S_h\|_{\infty, 0} \|\nabla(\theta - S_h)\|_{\infty, 0} + \|\nabla(\theta - S_h)\|_{\infty, 0}^2 + \|(\theta - S_h)_t\|_{\infty, 0}^2 \right. \\
& + h\Delta t^2 \|(\theta - S_h)_{tt}\|_{\infty, 0}^2) + \Delta t \inf_{v_h \in X_h} (\|u - v_h\|_{\infty, 0}^2 + \|u - v_h\|_{\infty, 0} \|\nabla(u - v_h)\|_{\infty, 0} + \|\nabla(u - v_h)\|_{\infty, 0}^2 \\
& + \|(u - v_h)_t\|_{\infty, 0}^2 + h\Delta t^2 \|(u - v_h)_{tt}\|_{\infty, 0}^2) + \Delta t \inf_{q_h \in Q_h} \|p - q_h\|_{\infty, 0}^2 \\
& + \Delta t \inf_{S_h \in W_h} (\|\tau - I_h\tau\|_{\infty, 0}^2 + \|\nabla(\tau - I_h\tau)\|_{\infty, 0}^2) + h\Delta t^3 + \Delta t^4 \left. \right\} \\
& + \|\zeta_h^0\|^2 + \|2\zeta_h^1 - \zeta_h^0\|^2 + \|\eta_h^0\|^2 + \|2\eta_h^1 - \phi_h^0\|^2 + \frac{Pr\Delta t}{2} (\|\nabla\eta_h^1\|^2 + \frac{1}{2} \|\nabla\eta_h^0\|^2) \\
& + \frac{1}{2} (\|e_T^0\|^2 + \|2e_T^1 - e_T^0\|^2) + \|e_u^0\|^2 + \|2e_u^1 - e_u^0\|^2 + \frac{Pr\Delta t}{2} (\|\nabla e_u^1\|^2 + \frac{1}{2} \|\nabla e_u^0\|^2).
\end{aligned}$$

Proof. Let $T = \theta + \tau$. The true solutions satisfy for all $n = 1, \dots, N-1$:

$$\begin{aligned}
(46) \quad & \left(\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\Delta t}, v_h \right) + b(u^{n+1}, u^{n+1}, v_h) + Pr(\nabla u^{n+1}, \nabla v_h) - (p^{n+1}, \nabla \cdot v_h) \\
& = PrRa(\gamma(\theta^{n+1} + \tau), v_h) + (f^{n+1}, v_h) + \varsigma_u(u^{n+1}; v_h) \quad \forall v_h \in X_h,
\end{aligned}$$

$$(47) \quad (q_h, \nabla \cdot u^{n+1}) = 0 \quad \forall q_h \in Q_h,$$

$$\begin{aligned}
(48) \quad & \left(\frac{3\theta^{n+1} - 4\theta^n + \theta^{n-1}}{2\Delta t}, S_h \right) + b^*(u^{n+1}, \theta^{n+1}, S_h) + (\nabla\theta^{n+1}, \nabla S_h) + (\nabla\tau, \nabla S_h) \\
& = (\gamma^{n+1}, S_h) + \varsigma_T(\theta^{n+1}; S_h) \quad \forall S_h \in W_{\Gamma_1, h}.
\end{aligned}$$

Subtract (48) and (17), then the error equation for temperature is

$$\begin{aligned}
(49) \quad & \left(\frac{3e_\theta^{n+1} - 4e_\theta^n + e_\theta^{n-1}}{2\Delta t}, S_h \right) + b^*(u^{n+1}, \theta^{n+1}, S_h) - b^*(\langle u_h \rangle_e^n, \theta_h^{n+1}, S_h) - b^*(u_h^n, 2\theta_h^n - \theta_h^{n-1}, S_h) \\
& + b^*(u^{n+1}, \tau, S_h) - b^*(u_h^n, I_h\tau, S_h) + (\nabla e_\theta^{n+1}, \nabla S_h) + (\nabla(\tau - I_h\tau), \nabla S_h) = \varsigma_T(\theta^{n+1}, S_h) \quad \forall S_h \in W_{\Gamma_1, h}.
\end{aligned}$$

Letting $e_\theta^n = (\theta^n - \tilde{\theta}^n) - (\theta_h^n - \tilde{\theta}_h^n) = \zeta^n - \psi_h^n$ and rearranging give,

$$\begin{aligned}
& \left(\frac{3\psi_h^{n+1} - 4\psi_h^n + \psi_h^{n-1}}{2\Delta t}, S_h \right) + (\nabla\psi_h^{n+1}, \nabla S_h) = \left(\frac{3\zeta^{n+1} - 4\zeta^n + \zeta^{n-1}}{2\Delta t}, S_h \right) + (\nabla\zeta^{n+1}, \nabla S_h) \\
& + (\nabla(\tau - I_h\tau), \nabla S_h) + b^*(u^{n+1}, \theta^{n+1}, S_h) - b^*(2u_h^n - u_h^{n-1}, \theta_h^{n+1}, S_h) - b^*(u_h^n, -\theta_h^{n+1} + 2\theta_h^n - \theta_h^{n-1}, S_h) \\
& + b^*(u^{n+1}, \tau, S_h) - b^*(u_h^n, I_h\tau, S_h) - \varsigma_T(\theta^{n+1}, S_h) \quad \forall S_h \in W_{\Gamma_1, h}.
\end{aligned}$$

Setting $S_h = \psi_h^{n+1} \in W_{\Gamma_1, h}$ yields

$$\begin{aligned}
& \frac{1}{4\Delta t} \left\{ \|\psi_h^{n+1}\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \right\} - \frac{1}{4\Delta t} \left\{ \|\psi_h^n\|^2 + \|2\psi_h^n - \psi_h^{n-1}\|^2 \right\} + \frac{1}{4\Delta t} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 + \|\nabla\psi_h^{n+1}\|^2 \\
& = \frac{1}{2\Delta t} (3\zeta^{n+1} - 4\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) + (\nabla\zeta^{n+1}, \nabla\psi_h^{n+1}) + (\nabla(\tau - I_h\tau), \nabla\psi_h^{n+1}) + b^*(u^{n+1}, \theta^{n+1}, \psi_h^{n+1}) \\
& \quad - b^*(2u_h^n - u_h^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) - b^*(u_h^n, -\theta_h^{n+1} + 2\theta_h^n - \theta_h^{n-1}, \psi_h^{n+1}) + b^*(u^{n+1}, \tau, \psi_h^{n+1}) \\
& \quad - b^*(u_h^n, I_h\tau, \psi_h^{n+1}) - \varsigma_T(\theta^{n+1}, \psi_h^{n+1}).
\end{aligned}$$

Add and subtract $b^*(u^{n+1}, \theta_h^{n+1}, \psi_h^{n+1})$, $b^*(2u^n - u^{n-1}, \theta_h^{n+1}, \psi_h^{n+1})$, $b^*(u_h^n, -\theta^{n+1} + 2\theta^n - \theta^{n-1}, \psi_h^{n+1})$, and $b^*(2u^n - u^{n-1}, \tau - I_h\tau, \psi_h^{n+1})$ to the r.h.s. Rearrange, then

$$\begin{aligned}
(50) \quad & \frac{1}{4\Delta t} \left\{ \|\psi_h^{n+1}\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \right\} - \frac{1}{4\Delta t} \left\{ \|\psi_h^n\|^2 + \|2\psi_h^n - \psi_h^{n-1}\|^2 \right\} + \frac{1}{4\Delta t} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 \\
& + \|\nabla\psi_h^{n+1}\|^2 = \frac{1}{\Delta 2t} (3\zeta^{n+1} - 4\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) + (\nabla\zeta^{n+1}, \nabla\psi_h^{n+1}) + b^*(u^{n+1}, \zeta^{n+1}, \psi_h^{n+1}) \\
& + b^*(u^{n+1} - 2u^n + u^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) + b^*(2\eta^n - \eta^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) \\
& - b^*(2\phi_h^n - \phi_h^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) - b^*(u_h^n, \zeta^{n+1} - 2\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) + b^*(u_h^n, \psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}, \psi_h^{n+1}) \\
& + b^*(u_h^n, \theta^{n+1} - 2\theta^n + \theta^{n-1}, \psi_h^{n+1}) + b^*(u^{n+1} - 2u^n + u^{n-1}, \tau, \psi_h^{n+1}) + b^*(2u^n - u^{n-1}, \tau - I_h\tau, \psi_h^{n+1}) \\
& + b^*(2\eta^n - \eta^{n-1}, I_h\tau, \psi_h^{n+1}) - b^*(2\phi_h^n - \phi_h^{n-1}, I_h\tau, \psi_h^{n+1}) + (\nabla(\tau - I_h\tau), \nabla\psi_h^{n+1}) - \varsigma_T(T^{n+1}, \psi_h^{n+1}).
\end{aligned}$$

Follow analogously for the velocity error equation. Subtract (46) and (15), split the error into $e_h^n = (u^n - \tilde{u}^n) - (u_h^n - \tilde{u}^n) = \eta^n - \phi_h^n$, let $v_h = \phi_h^{n+1} \in V_h$, add and subtract $PrRa(\xi(2\theta^n - \theta^{n-1} + \tau), \phi_h^{n+1})$, $b(u^{n+1}, u_h^{n+1}, \phi_h^{n+1})$, $b(2u^n - u^{n-1}, u_h^{n+1}, \phi_h^{n+1})$, and $b(u_h^n, -u^{n+1} + 2u^n - u^{n-1}, \phi_h^{n+1})$. Then,

$$\begin{aligned}
(51) \quad & \frac{1}{4\Delta t} \left\{ \|\phi_h^{n+1}\|^2 + \|2\phi_h^{n+1} - \phi_h^n\|^2 \right\} - \frac{1}{4\Delta t} \left\{ \|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 \right\} + \frac{1}{4\Delta t} \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
& + Pr\|\nabla\phi_h^{n+1}\|^2 = \frac{1}{2\Delta t} (3\eta^{n+1} - 4\eta^n + \eta^{n-1}, \phi_h^{n+1}) + Pr(\nabla\eta^{n+1}, \nabla\phi_h^{n+1}) - (p^{n+1} - q_h^{n+1}, \nabla \cdot \phi_h^{n+1}) \\
& - PrRa(\xi(\theta^{n+1} - 2\theta^n + \theta^{n-1}), \phi_h^{n+1}) - PrRa(\xi(2\zeta^n - \zeta^{n-1}), \phi_h^{n+1}) + PrRa(\xi(2\psi_h^n - \psi_h^{n-1}), \phi_h^{n+1}) \\
& - PrRa(\xi(\tau - I_h\tau), \phi_h^{n+1}) + b(u^{n+1}, \eta^{n+1}, \phi_h^{n+1}) + b(u^{n+1} - 2u^n + u^{n-1}, u_h^{n+1}, \phi_h^{n+1}) \\
& + b(2\eta^n - \eta^{n-1}, u_h^{n+1}, \phi_h^{n+1}) - b(2\phi_h^n - \phi_h^{n-1}, u_h^{n+1}, \phi_h^{n+1}) - b(u_h^n, \eta^{n+1} - 2\eta^n + \eta^{n-1}, \phi_h^{n+1}) \\
& + b(u_h^n, \phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}, \phi_h^{n+1}) + b(u_h^n, u^{n+1} - 2u^n + u^{n-1}, \phi_h^{n+1}) - \varsigma_u(u^{n+1}, \phi_h^{n+1}).
\end{aligned}$$

We seek to now estimate all terms on the r.h.s. in such a way that we may subsume the terms involving unknown pieces ψ_h^k and ϕ_h^k into the l.h.s. The following estimates are formed using skew-symmetry, Lemma 1, and the Cauchy-Schwarz-Young inequality,

$$\begin{aligned}
(52) \quad & b^*(u^{n+1}, \zeta^{n+1}, \psi_h^{n+1}) \leq C_6 \|\nabla u^{n+1}\| \|\nabla\psi_h^{n+1}\| \sqrt{\|\zeta^{n+1}\| \|\nabla\zeta^{n+1}\|} \\
& \leq \frac{C_r C_6^2}{\epsilon_3} \|\nabla u^{n+1}\|^2 \|\zeta^{n+1}\| \|\nabla\zeta^{n+1}\| + \frac{\epsilon_3}{r} \|\nabla\psi_h^{n+1}\|^2,
\end{aligned}$$

$$\begin{aligned}
(53) \quad & b^*(2\eta^n - \eta^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) \leq C_4 \|\nabla\theta_h^{n+1}\| \|\nabla\psi_h^{n+1}\| \left\{ 2\sqrt{\|\eta^n\| \|\nabla\eta^n\|} + \sqrt{\|\eta^{n-1}\| \|\nabla\eta^{n-1}\|} \right\} \\
& \leq \frac{8C_r C_4^2}{\epsilon_5} \|\nabla\theta_h^{n+1}\|^2 \left\{ \|\eta^n\| \|\nabla\eta^n\| + \|\eta^{n-1}\| \|\nabla\eta^{n-1}\| \right\} + \frac{\epsilon_5}{r} \|\nabla\psi_h^{n+1}\|^2.
\end{aligned}$$

Applying Lemma 1, the Cauchy-Schwarz-Young inequality, and Taylor's theorem yields,

$$\begin{aligned}
(54) \quad & b^*(u^{n+1} - 2u^n + u^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) \leq C_3 \|\nabla(u^{n+1} - 2u^n + u^{n-1})\| \|\nabla\theta_h^{n+1}\| \|\nabla\psi_h^{n+1}\| \\
& \leq \frac{C_r C_3^2}{\epsilon_4} \|\nabla(u^{n+1} - 2u^n + u^{n-1})\|^2 \|\nabla\theta_h^{n+1}\|^2 + \frac{\epsilon_4}{r} \|\nabla\psi_h^{n+1}\|^2 \\
& \leq \frac{CC_r C_3^2 \Delta t^3}{\epsilon_4} \|\nabla\theta_h^{n+1}\|^2 \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_4}{r} \|\nabla\psi_h^{n+1}\|^2,
\end{aligned}$$

$$\begin{aligned}
(55) \quad & -b^*(u_h^n, \zeta^{n+1} - 2\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) \leq C_3 \|\nabla u_h^n\| \|\nabla\psi_h^{n+1}\| \|\nabla(\zeta^{n+1} - 2\zeta^n + \zeta^{n-1})\| \\
& \leq \frac{CC_r C_3^2 \Delta t^3}{\epsilon_7} \|\nabla u_h^n\|^2 \|\nabla\zeta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_7}{r} \|\nabla\psi_h^{n+1}\|^2,
\end{aligned}$$

$$\begin{aligned}
(56) \quad & b^*(u_h^n, \theta^{n+1} - 2\theta^n + \theta^{n-1}, \psi_h^{n+1}) \leq C_3 \|\nabla u_h^n\| \|\nabla(\theta^{n+1} - 2\theta^n + \theta^{n-1})\| \|\nabla\psi_h^{n+1}\| \\
& \leq \frac{CC_r C_3^2 \Delta t^3}{\epsilon_9} \|\nabla u_h^n\|^2 \|\nabla\theta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_9}{r} \|\nabla\psi_h^{n+1}\|^2.
\end{aligned}$$

Apply the triangle inequality, Lemma 1 and the Cauchy-Schwarz-Young inequality twice. This yields

$$\begin{aligned}
(57) \quad -b^*(2\phi_h^n - \phi_h^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) &\leq C_4 \|\nabla \theta_h^{n+1}\| \|\nabla \psi_h^{n+1}\| \sqrt{\|2\phi_h^n - \phi_h^{n-1}\| \|\nabla(2\phi_h^n - \phi_h^{n-1})\|} \\
&\leq C_4 C_\theta(j) \|\nabla \psi_h^{n+1}\| \sqrt{\|2\phi_h^n - \phi_h^{n-1}\| \|\nabla(2\phi_h^n - \phi_h^{n-1})\|} \\
&\leq \epsilon_6 \|\nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_\theta^2}{4\epsilon_6} \|2\phi_h^n - \phi_h^{n-1}\| \|\nabla(2\phi_h^n - \phi_h^{n-1})\| \\
&\leq \epsilon_6 \|\nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_\theta^2}{8\epsilon_6 \delta_6} \|2\phi_h^n - \phi_h^{n-1}\|^2 \\
&\quad + \frac{C_4^2 C_\theta^2 \delta_6}{2\epsilon_6} (\|\nabla \phi_h^n\|^2 + \|\nabla \phi_h^{n-1}\|^2),
\end{aligned}$$

$$\begin{aligned}
(58) \quad -b^*(2\phi_h^n - \phi_h^{n-1}, I_h \tau, \psi_h^{n+1}) &\leq \epsilon_{13} \|\nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_I^2 C_{tr}^2 |\Gamma_N|}{8\epsilon_{13} \delta_{13}} \|2\phi_h^n - \phi_h^{n-1}\|^2 \\
&\quad + \frac{C_4^2 C_I^2 C_{tr}^2 |\Gamma_N| \delta_{13}}{2\epsilon_{13}} (\|\nabla \phi_h^n\|^2 + \|\nabla \phi_h^{n-1}\|^2).
\end{aligned}$$

Use Lemma 1, the inverse inequality, and the Cauchy-Schwarz-Young inequality yielding

$$\begin{aligned}
(59) \quad \Delta t b^*(u_h^n, \psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}, \psi_h^{n+1}) &\leq \frac{\Delta t C_6 C_{inv,2}^{1/2}}{h^{1/2}} \|\nabla u_h^n\| \|\nabla \psi_h^{n+1}\| \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\| \\
&\leq \frac{2C_6^2 C_{inv,2} \Delta t^2}{h} \|\nabla u_h^n\|^2 \|\nabla \psi_h^{n+1}\|^2 + \frac{1}{8} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2.
\end{aligned}$$

Use the Cauchy-Schwarz-Young inequality on the first term. Apply Lemma 1, interpolant estimates, and Taylor's theorem on the remaining. Then,

$$\begin{aligned}
(60) \quad b^*(u^{n+1} - 2u^n + u^{n-1}, \tau, \psi_h^{n+1}) &\leq C_3 \|\nabla(u^{n+1} - 2u^n + u^{n-1})\| \|\nabla \tau\| \|\nabla \psi_h^{n+1}\| \\
&\leq \frac{CC_r C_3^2 C_{tr}^2 |\Gamma_N| \Delta t^3}{\epsilon_{10}} \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_{10}}{r} \|\nabla \psi_h^{n+1}\|^2,
\end{aligned}$$

$$\begin{aligned}
(61) \quad b^*(2u^n - u^{n-1}, \tau - I_h \tau, \psi_h^{n+1}) &\leq C_3 \|(2u^n - u^{n-1})\| \|\nabla \tau - I_h \tau\| \|\nabla \psi_h^{n+1}\| \\
&\leq \frac{C_r C_3^2}{\epsilon_{11}} C_3 \|(2u^n - u^{n-1})\|^2 \|\nabla \tau - I_h \tau\|^2 + \frac{\epsilon_{11}}{r} \|\nabla \psi_h^{n+1}\|^2,
\end{aligned}$$

$$\begin{aligned}
(62) \quad b^*(2\eta^n - \eta^{n-1}, I_h \tau, \psi_h^{n+1}) &\leq C_4 \|\nabla I_h \tau\| \|\nabla \psi_h^{n+1}\| \left\{ 2\sqrt{\|\eta^n\| \|\nabla \eta^n\|} + \sqrt{\|\eta^{n-1}\| \|\nabla \eta^{n-1}\|} \right\} \\
&\leq \frac{8C_r C_4^2 C_I^2 C_{tr}^2 |\Gamma_N|}{\epsilon_{12}} \left\{ \|\eta^n\| \|\nabla \eta^n\| + \|\eta^{n-1}\| \|\nabla \eta^{n-1}\| \right\} \\
&\quad + \frac{\epsilon_{12}}{r} \|\nabla \psi_h^{n+1}\|^2,
\end{aligned}$$

$$(63) \quad (\nabla(\tau - I_h \tau), \nabla \psi_h^{n+1}) \leq \frac{C_r}{\epsilon_{14}} \|\nabla(\tau - I_h \tau)\|^2 + \frac{\epsilon_{14}}{r} \|\nabla \psi_h^{n+1}\|^2.$$

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality and Taylor's theorem yield

$$(64) \quad \frac{1}{2\Delta t} (3\zeta^{n+1} - 4\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) \leq \frac{CC_{PF,2}^2 C_r}{\Delta t \epsilon_1} \|\zeta_t\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_1}{r} \|\nabla \psi_h^{n+1}\|^2.$$

Lastly, use the Cauchy-Schwarz-Young inequality,

$$(65) \quad (\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) \leq \frac{C_r}{\epsilon_2} \|\nabla \zeta^{n+1}\|^2 + \frac{\epsilon_2}{r} \|\nabla \psi_h^{n+1}\|^2.$$

Similar estimates follow for the r.h.s. terms in (51), however, we must treat an additional pressure term and

error term associated with the temperature,

$$(66) \quad -(p^{n+1} - q_h^{n+1}, \nabla \cdot \phi_h^{n+1}) \leq \sqrt{d} \|p^{n+1} - q_h^{n+1}\| \|\nabla \phi_h^{n+1}\| \leq \frac{dC_r}{\epsilon_{17}} \|p^{n+1} - q_h^{n+1}\|^2$$

$$(67) \quad + \frac{\epsilon_{17}}{r} \|\nabla \phi_h^{n+1}\|^2,$$

$$(68) \quad -PrRa(\xi(\theta^{n+1} - 2\theta^n + \theta^{n-1}), \phi_h^{n+1}) \leq \frac{CPr^2Ra^2C_{PF,1}^2C_r\Delta t^3}{\epsilon_{18}} \|\theta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2$$

$$(69) \quad + \frac{\epsilon_{18}}{r} \|\nabla \phi_h^{n+1}\|^2,$$

$$(70) \quad -PrRa(\xi(2\zeta^n - \zeta^{n-1}), \phi_h^{n+1}) \leq \frac{Pr^2Ra^2C_{PF,1}^2C_r}{\epsilon_{19}} (4\|\zeta^n\|^2 + \|\zeta^{n-1}\|^2) + \frac{\epsilon_{19}}{r} \|\nabla \phi_h^{n+1}\|^2,$$

$$(71) \quad PrRa(\xi(2\psi_h^n - \psi_h^{n-1}), \phi_h^{n+1}) \leq \frac{Pr^2Ra^2C_{PF,1}^2C_{PF,2}^2C_r}{\epsilon_{20}} \|2\psi_h^n - \psi_h^{n-1}\|^2 + \frac{\epsilon_{20}}{r} \|\nabla \phi_h^{n+1}\|^2,$$

$$(72) \quad -PrRa(\xi(\tau - I_h\tau), \phi_h^{n+1}) \leq \frac{Pr^2Ra^2C_{PF,1}^2C_r}{\epsilon_{21}} \|\tau - I_h\tau\|^2 + \frac{\epsilon_{21}}{r} \|\nabla \phi_h^{n+1}\|^2.$$

Multiply equations (50) and (51) by Δt . Apply the above estimates and Lemma 6. Then,

$$(73) \quad \frac{1}{4} \left\{ \|\psi_h^{n+1}\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\psi_h^n\|^2 + \|2\psi_h^n - \psi_h^{n-1}\|^2 \right\} + \frac{1}{4} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2$$

$$+ \Delta t \|\nabla \psi_h^{n+1}\|^2 \leq \frac{CC_{PF,2}^2C_r}{\epsilon_1} \|\zeta_t\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_1 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{C_r \Delta t}{\epsilon_2} \|\nabla \zeta^{n+1}\|^2 + \frac{\epsilon_2 \Delta t}{r} \|\nabla \psi^{n+1}\|^2$$

$$+ \frac{C_r C_6^2 \Delta t}{\epsilon_3} \|\nabla u^{n+1}\|^2 \|\zeta^{n+1}\| \|\nabla \zeta^{n+1}\| + \frac{\epsilon_3 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{CC_r C_3^2 \Delta t^4}{\epsilon_4} \|\nabla \theta_h^{n+1}\|^2 \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2$$

$$+ \frac{\epsilon_4 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{8C_r C_4^2 \Delta t}{\epsilon_5} \|\nabla \theta_h^{n+1}\|^2 \left\{ \|\eta^n\| \|\nabla \eta^n\| + \|\eta^{n-1}\| \|\nabla \eta^{n-1}\| \right\} + \frac{\epsilon_5 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2$$

$$+ \epsilon_6 \Delta t \|\nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_\theta^2 \Delta t}{8\epsilon_6 \delta_6} \|2\phi_h^n - \phi_h^{n-1}\|^2 + \frac{C_4^2 C_\theta^2 \delta_6 \Delta t}{2\epsilon_6} (\|\nabla \phi_h^n\|^2 + \|\nabla \phi_h^{n-1}\|^2)$$

$$+ \frac{CC_r C_3^2 \Delta t^4}{\epsilon_7} \|\nabla u_h^n\|^2 \|\nabla \zeta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_7 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{2C_6^2 C_{inv,2} \Delta t^2}{h} \|\nabla u_h^n\|^2 \|\nabla \psi_h^{n+1}\|^2$$

$$+ \frac{1}{8} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 + \frac{CC_r C_3^2 \Delta t^4}{\epsilon_9} \|\nabla u_h^n\|^2 \|\nabla \theta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_9 \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2$$

$$+ \frac{CC_r C_3^2 C_{tr}^2 |\Gamma_N| \Delta t^4}{\epsilon_{10}} \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_{10} \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{C_r C_3^2 \Delta t}{\epsilon_{11}} \|(2u^n - u^{n-1})\|^2 \|\nabla \tau - I_h \tau\|^2$$

$$+ \frac{\epsilon_{11} \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{8C_r C_4^2 C_I^2 C_{tr}^2 |\Gamma_N| \Delta t}{\epsilon_{12}} \left\{ \|\eta^n\| \|\nabla \eta^n\| + \|\eta^{n-1}\| \|\nabla \eta^{n-1}\| \right\} + \frac{\epsilon_{12} \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2$$

$$+ \epsilon_{13} \Delta t \|\nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_I^2 C_{tr}^2 |\Gamma_N| \Delta t}{8\epsilon_{13} \delta_{13}} \|2\phi_h^n - \phi_h^{n-1}\|^2 + \frac{C_4^2 C_I^2 C_{tr}^2 |\Gamma_N| \delta_{13} \Delta t}{2\epsilon_{13}} (\|\nabla \phi_h^n\|^2 + \|\nabla \phi_h^{n-1}\|^2)$$

$$+ \frac{C_r \Delta t}{\epsilon_{14}} \|\nabla(\tau - I_h \tau)\|^2 + \frac{\epsilon_{14} \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2 + \frac{CC_{PF,2}^2 C_r \Delta t^4}{\epsilon_{29}} \|\theta_{ttt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\epsilon_{29} \Delta t}{r} \|\nabla \psi_h^{n+1}\|^2$$

and

$$\begin{aligned}
(74) \quad & \frac{1}{4} \left\{ \|\phi_h^{n+1}\|^2 + \|2\phi_h^{n+1} - \phi_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 \right\} + \frac{1}{4} \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
& + Pr\Delta t \|\nabla\phi_h^{n+1}\|^2 \leq \frac{CC_r C_{PF,1}^2}{\epsilon_{15}} \|\eta_t\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\Delta t \epsilon_{15}}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{C_r Pr^2 \Delta t}{\epsilon_{16}} \|\nabla\eta^{n+1}\|^2 \\
& + \frac{\Delta t \epsilon_{16}}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{dC_r \Delta t}{\epsilon_{17}} \|p^{n+1} - q_h^{n+1}\|^2 + \frac{\Delta t \epsilon_{17}}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{CPr^2 Ra^2 C_{PF,1}^2 C_r \Delta t^4}{\epsilon_{18}} \|\theta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 \\
& + \frac{\epsilon_{18} \Delta t}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{Pr^2 Ra^2 C_{PF,1}^2 C_r \Delta t}{\epsilon_{19}} \left(4\|\zeta^n\|^2 + \|\zeta^{n-1}\|^2 \right) + \frac{\epsilon_{19} \Delta t}{r} \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{Pr^2 Ra^2 C_{PF,1}^2 C_{PF,2}^2 C_r \Delta t}{\epsilon_{20}} \|2\psi_h^n - \psi_h^{n-1}\|^2 + \frac{\epsilon_{20} \Delta t}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{Pr^2 Ra^2 C_{PF,1}^2 C_r \Delta t}{\epsilon_{21}} \|\tau - I_h \tau\|^2 \\
& + \frac{\epsilon_{21} \Delta t}{r} \|\nabla\phi_h^{n+1}\|^2 + \frac{C_5 C_r \Delta t}{\epsilon_{22}} \|\nabla u^{n+1}\|^2 \|\eta^{n+1}\| \|\nabla\eta^{n+1}\| + \frac{\Delta t \epsilon_{22}}{r} \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{CC_r C_1 \Delta t^4}{\epsilon_{23}} \|\nabla u_h^{n+1}\|^2 \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\Delta t \epsilon_{23}}{r} \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{8C_r C_2^2 \Delta t}{\epsilon_{24}} \|\nabla u_h^{n+1}\|^2 \left\{ \|\eta^n\| \|\nabla\eta^n\| + \|\eta^{n-1}\| \|\nabla\eta^{n-1}\| \right\} + \frac{\Delta t \epsilon_{24}}{r} \|\nabla\phi_h^{n+1}\|^2 + \epsilon_{25} \Delta t \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{C_2^2 C_u^2 \Delta t \delta_{25}}{8\epsilon_{25}} \|2\phi_h^n - \phi_h^{n-1}\|^2 + \frac{C_2^2 C_u^2 \Delta t}{2\epsilon_{25} \delta_{25}} \left(\|\nabla\phi_h^n\|^2 + \|\nabla\phi_h^{n-1}\|^2 \right) \\
& + \frac{CC_r C_1^2 \Delta t^4}{\epsilon_{26}} \|\nabla u_h^n\|^2 \|\nabla\eta_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\Delta t \epsilon_{26}}{r} \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{2C_5^2 C_{inv,1} \Delta t^2}{h} \|\nabla u_h^n\|^2 \|\nabla\phi_h^{n+1}\|^2 + \frac{1}{8} \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 \\
& + \frac{CC_r C_1 \Delta t^4}{\epsilon_{28}} \|\nabla u_h^n\|^2 \|\nabla u_{tt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\Delta t \epsilon_{28}}{r} \|\nabla\phi_h^{n+1}\|^2 \\
& + \frac{CC_{PF,1}^2 C_r \Delta t^4}{\epsilon_{30}} \|u_{ttt}\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \frac{\Delta \epsilon_{30}}{r} \|\nabla\phi_h^{n+1}\|^2.
\end{aligned}$$

Combine (73) and (74), choose free parameters appropriately, reorganize, use condition (18) and Theorem 4. Add $\|\psi_h^n\|^2$ and $\|\phi_h^n\|^2$ to the r.h.s. and take the maximum over all constants on the r.h.s. Then,

$$\begin{aligned}
(75) \quad & \frac{1}{4} \left\{ \|\psi_h^{n+1}\|^2 + \|2\psi_h^{n+1} - \psi_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\psi_h^n\|^2 + \|2\psi_h^n - \psi_h^{n-1}\|^2 \right\} + \frac{1}{8} \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 \\
& + \frac{\Delta t}{4} \|\nabla\psi_h^{n+1}\|^2 + \frac{1}{4} \left\{ \|\phi_h^{n+1}\|^2 + \|2\phi_h^{n+1} - \phi_h^n\|^2 \right\} - \frac{1}{4} \left\{ \|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 \right\} \\
& + \frac{1}{8} \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \frac{Pr\Delta t}{4} \|\nabla\phi_h^{n+1}\|^2 + \frac{Pr\Delta t}{4} \left(\|\nabla\phi_h^{n+1}\|^2 - \|\nabla\phi_h^n\|^2 \right) + \frac{Pr\Delta t}{8} \left(\|\nabla\phi_h^n\|^2 - \|\nabla\phi_h^{n-1}\|^2 \right) \\
& \leq C \left\{ \Delta t \|\zeta_t\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \Delta t \|\nabla\zeta^{n+1}\|^2 + \Delta t \|\zeta^{n+1}\| \|\nabla\zeta^{n+1}\| + \Delta t^4 \right. \\
& + \Delta t \left\{ \|\eta^n\| \|\nabla\eta^n\| + \|\eta^{n-1}\| \|\nabla\eta^{n-1}\| \right\} + \Delta t \left(\|\phi_h^n\|^2 + \|2\phi_h^n - \phi_h^{n-1}\|^2 \right) + h\Delta t^3 + \Delta t \|\nabla\tau - I_h \tau\|^2 \\
& + \Delta t \|\eta_t\|_{L^2(t^{n-1}, t^{n+1}; L^2(\Omega))}^2 + \Delta t \|\nabla\eta^{n+1}\|^2 + \Delta t \|p^{n+1} - q_h^{n+1}\|^2 + \Delta t \left(4\|\zeta^n\|^2 + \|\nabla\zeta^{n-1}\|^2 \right) \\
& \left. + \Delta t \left(\|\psi_h^n\|^2 + \|2\psi_h^n - \psi_h^{n-1}\|^2 \right) + \Delta t \|\tau - I_h \tau\|^2 + \Delta t \|\eta^{n+1}\| \|\nabla\eta^{n+1}\| \right\}.
\end{aligned}$$

Multiply by 4, sum from $n = 1$ to $n = N - 1$, apply Lemma 3, take infimums over X_h , Q_h , and \hat{W}_h , and

renorm. Then,

$$\begin{aligned}
& \|\psi_h^N\|^2 + \|2\psi_h^N - \psi_h^{N-1}\|^2 + \|\phi_h^N\|^2 + \|2\phi_h^N - \phi_h^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} (\|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2 + \|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2) \\
& + \Delta t \sum_{n=1}^{N-1} \|\nabla \psi_h^{n+1}\|^2 + Pr \Delta t \sum_{n=1}^{N-1} \|\nabla \phi_h^{n+1}\|^2 + \frac{Pr \Delta t}{2} (\|\nabla \phi_h^N\|^2 + \frac{1}{2} \|\nabla \phi_h^{N-1}\|^2) \\
\leq & \exp(Ct^*) \left\{ \Delta t \inf_{S_h \in W_{T_1, h}} (\|\theta - S_h\|_{\infty, 0}^2 + \|\theta - S_h\|_{\infty, 0} \|\nabla(\theta - S_h)\|_{\infty, 0} + \|\nabla(\theta - S_h)\|_{\infty, 0}^2 + \|(\theta - S_h)_t\|_{\infty, 0}^2 \right. \\
& + h \Delta t^2 \|(\theta - S_h)_{tt}\|_{\infty, 0}^2) + \Delta t \inf_{v_h \in X_h} (\|u - v_h\|_{\infty, 0}^2 + \|u - v_h\|_{\infty, 0} \|\nabla(u - v_h)\|_{\infty, 0} + \|\nabla(u - v_h)\|_{\infty, 0}^2 \\
& + \|(u - v_h)_t\|_{\infty, 0}^2 + h \Delta t^2 \|(u - v_h)_{tt}\|_{\infty, 0}^2) + \Delta t \inf_{q_h \in Q_h} \|p - q_h\|_{\infty, 0}^2 \\
& + \Delta t \inf_{S_h \in W_h} (\|\tau - I_h \tau\|_{\infty, 0}^2 + \|\nabla(\tau - I_h \tau)\|_{\infty, 0}^2) + h \Delta t^3 + \Delta t^4 \left. \right\} \\
& + \|\psi_h^0\|^2 + \|2\psi_h^1 - \psi_h^0\|^2 + \|\phi_h^0\|^2 + \|2\phi_h^1 - \phi_h^0\|^2 + \frac{Pr \Delta t}{2} (\|\nabla \phi_h^1\|^2 + \frac{1}{2} \|\nabla \phi_h^0\|^2).
\end{aligned}$$

The result follows by the relationship $e_T^n = e_\theta^n + \tau - I_h \tau$ and the triangle inequality. \square

COROLLARY 8. *Suppose the assumptions of Theorem 4 hold with $k = m = 2$. Further suppose that the finite element spaces (X_h, Q_h, W_h) are given by P2-P1-P2 (Taylor-Hood), then the errors in velocity and temperature satisfy*

$$\begin{aligned}
& \frac{1}{2} \|e_T^N\|^2 + \frac{1}{2} \|2e_T^N - e_T^{N-1}\|^2 + \|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} (\|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2 + \|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2) \\
& + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla e_T^{n+1}\|^2 + Pr \Delta t \sum_{n=1}^{N-1} \|\nabla e_u^{n+1}\|^2 + \frac{Pr \Delta t}{2} (\|\nabla e_u^N\|^2 + \frac{1}{2} \|\nabla e_u^{N-1}\|^2) \\
& \leq C \left\{ h^6 \Delta t + h^5 \Delta t + h^4 \Delta t + h^7 \Delta t^2 + h \Delta t^3 + h^4 \Delta t + h \Delta t^3 + \Delta t^4 \right. \\
& + \|\zeta_h^0\|^2 + \|2\zeta_h^1 - \zeta_h^0\|^2 + \|\eta_h^0\|^2 + \|2\eta_h^1 - \phi_h^0\|^2 + \frac{Pr \Delta t}{2} (\|\nabla \eta_h^1\|^2 + \frac{1}{2} \|\nabla \eta_h^0\|^2) \\
& \left. + \frac{1}{2} (\|e_T^0\|^2 + \|2e_T^1 - e_T^0\|^2) + \|e_u^0\|^2 + \|2e_u^1 - e_u^0\|^2 + \frac{Pr \Delta t}{2} (\|\nabla e_u^1\|^2 + \frac{1}{2} \|\nabla e_u^0\|^2) \right\}.
\end{aligned}$$

COROLLARY 9. *Suppose the assumptions of Theorem 4 hold with $k = m = 1$. Further suppose that the finite element spaces (X_h, Q_h, W_h) are given by P1b-P1-P1b (MINI element), then the errors in velocity and temperature satisfy*

$$\begin{aligned}
& \frac{1}{2} \|e_T^N\|^2 + \frac{1}{2} \|2e_T^N - e_T^{N-1}\|^2 + \|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2} \sum_{n=1}^{N-1} (\|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2 + \|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2) \\
& + \frac{\Delta t}{2} \sum_{n=1}^{N-1} \|\nabla e_T^{n+1}\|^2 + Pr \Delta t \sum_{n=1}^{N-1} \|\nabla e_u^{n+1}\|^2 + \frac{Pr \Delta t}{2} (\|\nabla e_u^N\|^2 + \frac{1}{2} \|\nabla e_u^{N-1}\|^2) \\
& \leq C \left\{ h^4 \Delta t + h^3 \Delta t + h^2 \Delta t + h^5 \Delta t^2 + h \Delta t^3 + h^2 \Delta t + h \Delta t^3 + \Delta t^4 \right. \\
& + \|\zeta_h^0\|^2 + \|2\zeta_h^1 - \zeta_h^0\|^2 + \|\eta_h^0\|^2 + \|2\eta_h^1 - \phi_h^0\|^2 + \frac{Pr \Delta t}{2} (\|\nabla \eta_h^1\|^2 + \frac{1}{2} \|\nabla \eta_h^0\|^2) \\
& \left. + \frac{1}{2} (\|e_T^0\|^2 + \|2e_T^1 - e_T^0\|^2) + \|e_u^0\|^2 + \|2e_u^1 - e_u^0\|^2 + \frac{Pr \Delta t}{2} (\|\nabla e_u^1\|^2 + \frac{1}{2} \|\nabla e_u^0\|^2) \right\}.
\end{aligned}$$

5. Numerical Experiments. In this section, we illustrate the stability and convergence of the numerical scheme described by (15) - (17) using Taylor-Hood (P2-P1-P2) elements to approximate the average

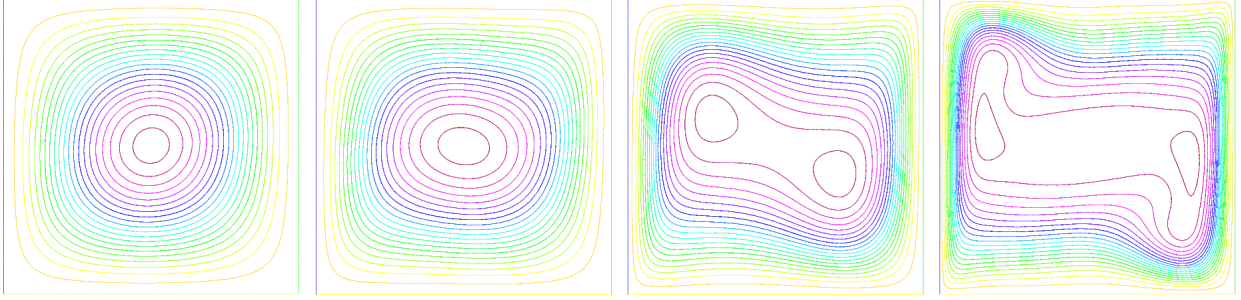


Fig. 2: Streamlines for $Ra = 10^3, 10^4, 10^5$, and 10^6 , from left to right, respectively.

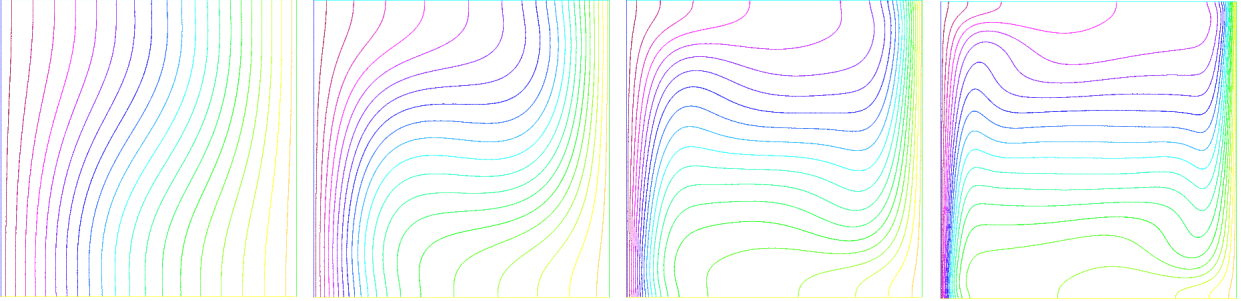


Fig. 3: Isotherms for $Ra = 10^3, 10^4, 10^5$, and 10^6 , from left to right, respectively.

velocity, pressure, and temperature. The numerical experiments include the double pane window benchmark problem of De Vahl Davis [22] and a convergence experiment with an analytical solution devised through the method of manufactured solutions. The software used for all tests is FREEFEM++ [11].

5.1. Stability condition. The constant appearing in condition 18 is estimated by pre-computations for the double pane window problem appearing below. We set $C_{\dagger} = 1$. The first condition is used and checked at each iteration. If violated, the timestep is halved and the timestep is repeated. The timestep is never increased. The condition is violated two times during the computation of the double pane window problem with $Ra = 10^6$ in Section 5.3.

5.2. Perturbation generation. The bred vector (BV) algorithm of Toth and Kalnay [21] is used to generate perturbations in the double pane window problem. The BV algorithm produces a perturbation with maximal separation rate. We set $J = 2$ in all experiments. An initial random positive/negative perturbation pair was generated $\pm\epsilon = \pm(\epsilon_1, \epsilon_2, \epsilon_3)$ with $\epsilon_i \in (0, 0.01) \forall i = 1, 2, 3$. Denote the control and perturbed numerical approximations χ_h^n and $\chi_{p,h}^n$, respectively. Then, a bred vector $bv(\chi; \epsilon_i)$ is generated via:

Step one: Given χ_h^0 and ϵ_i , put $\chi_{p,h}^0 = \chi_h^0 + \epsilon_i$. Select time reinitialization interval $\delta t \geq \Delta t$ and let $t^k = k\delta t$ with $0 \leq k \leq k^* \leq N$.

Step two: Compute χ_h^k and $\chi_{p,h}^k$. Calculate $bv(\chi^k; \epsilon_i) = \frac{\epsilon_i}{\|\chi_{p,h}^k - \chi_h^k\|} (\chi_{p,h}^k - \chi_h^k)$.

Step three: Put $\chi_{p,h}^k = \chi_h^k + bv(\chi^k; \epsilon_i)$.

Step four: Repeat **Step two** with $k = k + 1$.

Step five: Put $bv(\chi; \epsilon_i) = bv(\chi^{k^*}; \epsilon_i)$.

A positive/negative perturbed initial condition pair is generated via $\chi_{\pm} = \chi^0 + bv(\chi; \pm\epsilon_i)$. Moreover, we let $\delta t = \Delta t = 0.001$ and $k^* = 5$.

5.3. The double pane window problem. The first numerical experiment is the benchmark problem of De Vahl Davis [22]. The problem is the two-dimensional flow of a fluid in an unit square cavity with $Pr = 0.71$. Both velocity components are zero on the boundaries. The horizontal walls are insulated and the left and right vertical walls are maintained at temperatures $T(0, y, t) = 1$ and $T(1, y, t) = 0$, respectively;

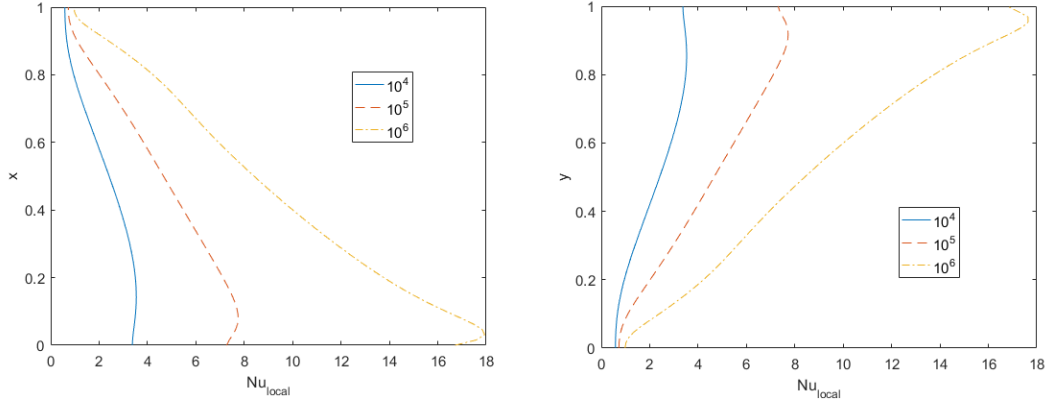


Fig. 4: Variation of the local Nusselt number at the hot (left) and cold walls (right).

recall Figure 1. We let $10^3 \leq Ra \leq 10^6$. The initial conditions for velocity and temperature are generated via the BV algorithm in Section 5.2,

$$\begin{aligned} u_{\pm}(x, y, 0) &:= u(x, y, 0; \omega_{1,2}) = (1 + bv(u; \pm\epsilon_1), 1 + bv(u; \pm\epsilon_2))^T, \\ T_{\pm}(x, y, 0) &:= T(x, y, 0; \omega_{1,2}) = 1 + bv(T; \pm\epsilon_3). \end{aligned}$$

Both $f(x, t; \omega_j)$ and $g(x, t; \omega_j)$ are identically zero for $j = 1, 2$. The finite element mesh is a division of $[0, 1]^2$ into 64^2 squares with diagonals connected with a line within each square in the same direction. The stopping condition is

$$\max_{1 \leq n \leq N-1} \left\{ \frac{\|u_h^{n+1} - u_h^n\|}{\|u_h^{n+1}\|}, \frac{\|T_h^{n+1} - T_h^n\|}{\|T_h^{n+1}\|} \right\} \leq 10^{-5}$$

and initial timestep $\Delta t = 0.001$. The first iterate was computed with the trapezoidal rule for each ensemble member. The timestep was halved twice to 0.00025 to maintain stability for $Ra = 10^6$. Several quantities are compared with benchmark solutions in the literature. These include the maximum vertical velocity at $y = 0.5$, $\max_{x \in \Omega_h} u_2(x, 0.5, t^*)$, and maximum horizontal velocity at $x = 0.5$, $\max_{y \in \Omega_h} u_1(0.5, y, t^*)$. We present our computed values for the average flow in Tables 1 and 2 alongside several of those seen in the literature. Furthermore, the local Nusselt number is calculated at the cold (+) and hot walls (-), respectively, via

$$Nu_{local} = \pm \frac{\partial T}{\partial x}.$$

The average Nusselt number on the vertical boundary at $x = 0$ is calculated via

$$Nu_{avg} = \int_0^1 Nu_{local} dy.$$

Figure 2 presents the plots of Nu_{local} at the hot and cold walls. Table 3 presents computed values of Nu_{avg} alongside several of those seen in the literature. Figures 3 and 4 present the velocity streamlines and temperature isotherms for the averages. All results appear to be in good agreement with the benchmark values in the literature [4, 16, 22, 23, 25].

Ra	Present study	Ref. [22]	Ref. [16]	Ref. [23]	Ref. [4]	Ref. [25]
10^4	16.18 (64×64)	16.18 (41×41)	16.10 (71×71)	16.10 (101×101)	15.90 (11×11)	16.18 (64×64)
10^5	34.72 (64×64)	34.81 (81×81)	34 (71×71)	34 (101×101)	33.51 (21×21)	34.74 (64×64)
10^6	64.78 (64×64)	65.33 (81×81)	65.40 (71×71)	65.40 (101×101)	65.52 (32×32)	64.81 (64×64)

Table 1: Comparison of maximum horizontal velocity at $x = 0.5$ together with mesh size used in computation for the double pane window problem.

Ra	Present study	Ref. [22]	Ref. [16]	Ref. [23]	Ref. [4]	Ref. [25]
10^4	19.60 (64×64)	19.51 (41×41)	19.90 (71×71)	19.79 (101×101)	19.91 (11×11)	19.62 (64×64)
10^5	68.53 (64×64)	68.22 (81×81)	70 (71×71)	70.63 (101×101)	70.60 (21×21)	68.48 (64×64)
10^6	215.89 (64×64)	216.75 (81×81)	228 (71×71)	227.11 (101×101)	228.12 (32×32)	220.44 (64×64)

Table 2: Comparison of maximum horizontal velocity at $y = 0.5$ together with mesh size used in computation for the double pane window problem.

Ra	Present study	Ref. [22]	Ref. [16]	Ref. [23]	Ref. [4]	Ref. [25]
10^4	2.25 (64×64)	2.24 (41×41)	2.08 (71×71)	2.25 (101×101)	2.15 (11×11)	2.25 (64×64)
10^5	4.53 (64×64)	4.52 (81×81)	4.30 (71×71)	4.59 (101×101)	4.35 (21×21)	4.53 (64×64)
10^6	8.89 (64×64)	8.92 (81×81)	8.74 (71×71)	8.97 (101×101)	8.83 (32×32)	8.87 (64×64)

Table 3: Comparison of average Nusselt number on the vertical boundary at $x = 0$ together with mesh size used in computation for the double pane window problem.

5.4. Numerical convergence study. In this section, we illustrate the convergence rates for the proposed algorithm (15) - (17). The unperturbed solution is given by

$$\begin{aligned}
u(x, y, t) &= 10(1 + 0.1t)(10x^2(x - 1)^2y(y - 1)(2y - 1), -10x(x - 1)(2x - 1)y^2(y - 1)^2)^T, \\
T(x, y, t) &= u_1(x, y, t) + u_2(x, y, t) + 1 - x, \\
p(x, y, t) &= 10(1 + 0.1t)(2x - 1)(2y - 1),
\end{aligned}$$

with $Pr = 1.0$, $Ra = 100$, and $\Omega = [0, 1]^2$. The perturbed solutions are given by

$$\begin{aligned}
u(x, y, t; \omega_{1,2}) &= (1 + \epsilon_{1,2})u(x, y, t), \\
T(x, y, t; \omega_{1,2}) &= (1 + \epsilon_{1,2})T(x, y, t), \\
p(x, y, t; \omega_{1,2}) &= (1 + \epsilon_{1,2})p(x, y, t),
\end{aligned}$$

where $\epsilon_1 = 1e - 2 = -\epsilon_2$ and both forcing and boundary terms are adjusted appropriately. The perturbed solutions satisfy the following relations,

$$\begin{aligned}
\langle u \rangle &= 0.5(u(x, y, t; \omega_1) + u(x, y, t; \omega_2)) = u(x, y, t), \\
\langle T \rangle &= 0.5(T(x, y, t; \omega_1) + T(x, y, t; \omega_2)) = T(x, y, t), \\
\langle p \rangle &= 0.5(p(x, y, t; \omega_1) + p(x, y, t; \omega_2)) = p(x, y, t).
\end{aligned}$$

The finite element mesh Ω_h is a Delaunay triangulation generated from m points on each side of Ω . We calculate errors in the approximations of the average velocity and temperature with the $L^\infty(0, t^*; L^2(\Omega))$ and $L^2(0, t^*; H^1(\Omega))$ norms and the pressure with the $L^2(0, t^*; H^1(\Omega))$ norm. Rates are calculated from the errors at two successive $\Delta t_{1,2}$ via

$$\frac{\log_2(e_\chi(\Delta t_1)/e_\chi(\Delta t_2))}{\log_2(\Delta t_1/\Delta t_2)},$$

respectively, with $\chi = u, T, p$. We set $m = 2\Delta t$ and vary Δt between 8, 16, 24, 32, and 40. Results are presented in Table 3. Second order convergence is observed for velocity and temperature in the $L^2(0, t^*; H^1(\Omega))$ norm and for pressure in the $L^2(0, t^*; L^2(\Omega))$ norm, as predicted. Moreover, third order convergence is seen for velocity and temperature $L^\infty(0, t^*; L^2(\Omega))$ norm, whereby second order convergence is predicted in Theorem 7.

$1/m$	$\ \langle u_h \rangle - u\ _{\infty,0}$	Rate	$\ \nabla \langle u_h \rangle - \nabla u\ _{2,0}$	Rate	$\ \langle T_h \rangle - T\ _{\infty,0}$	Rate	$\ \nabla \langle T_h \rangle - \nabla T\ _{2,0}$	Rate	$\ \langle p_h \rangle - p\ _{2,0}$	Rate
8	0.0005600	-	0.0206808	-	6.96E-05	-	0.0030380	-	0.0222107	-
16	6.28E-05	3.16	0.0046705	2.15	6.81E-06	3.35	0.0006157	2.30	0.0050407	2.14
24	1.82E-05	3.06	0.00209424	1.98	1.90E-06	3.14	0.0002505	2.22	0.0021921	2.05
32	6.99E-06	3.17	0.00102235	2.19	7.82E-07	3.12	0.0001264	2.23	0.0011483	2.13
40	3.99E-06	2.87	0.0007429	2.03	5.08E-07	2.59	9.25E-05	1.82	0.0007175	2.19

Table 4: Errors and rates for average velocity, temperature, and pressure in corresponding norms.

6. Conclusion. We presented an algorithm for calculating an ensemble of solutions to laminar natural convection problems. This algorithm addresses both the competition between ensemble size and resolution in simulations and the need for higher order accurate timestepping methods. In particular, the algorithm required the solution of two coupled linear systems, each involving a shared coefficient matrix, for multiple right-hand sides at each timestep. Stability and convergence of the algorithm were proven and numerical experiments were performed to illustrate these properties.

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